Coboson Derivation of Richardson's Equations for Cooper pairs

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Five years after the milestone paper by Bardeen, Cooper, Schrieffer (BCS) in which superconductivity is tackled within the grand canonical ensemble, Richardson found a smart way to approach the problem within the canonical ensemble: He succeeded to write down the *exact analytical form* of the Schrödinger equation eigenstate for an arbitrary number of Cooper pairs interacting through the standard BCS potential. We here rederive his result using a commutation technique similar to the one we have recently developed for many-body effects between composite bosons (cobosons in short). This derivation makes crystal clear the physical origin of the various terms, in particular the fact that difference between a collection of single Cooper pairs and the BCS condensate are solely due to the Pauli exclusion principle through electron exchanges between pairs. Our procedure gives also hints on why, as we very recently found, the interaction part of the N-pair ground state energy increases with pair number as $N(N-1)$ only from the dilute to the dense regime of pairs. Finally, in this work, we briefly question the validity of the BCS wave function ansatz in the light of Richardson's exact solution.

It is known for quite a long time that the Pauli exclusion principle plays a key role in superconductivity. None the less, the precise way Pauli blocking transforms a collection of single Cooper pairs into a BCS condensate, has been understood quite recently only. This precise understanding goes through the study of Cooper pairs not within the grand canonical ensemble as done in the standard BCS theory, but within the canonical ensemble. It is however known that to handle the Pauli exclusion principle between a fixed number of interacting fermions is quite difficult, especially when these fermions are paired. Turning to the grand canonical ensemble makes the task far easier. Yet, adding fermion pairs one by one is the best way to precisely follow the increasing effect of Pauli blocking from the dilute to the dense regime of pairs.

Five years after the milestone paper on superconductivity by Bardeen, Cooper, Schrieffer[1], Richardson succeeded to solve this N-body problem exactly and to formally write the exact form of the Schrödinger equation eigenstates for an arbitrary number N of Cooper pairs[2,3]. This exact solution is expressed in terms of N parameters, R_1,... R_N which are shown to be solutions of N coupled non-linear equations, the energy of these N pairs reading as $E_N = R_1 + ... + R_N$. Although this exact form is definitely quite smart, to use it in practice is not that easy: Except in the infinite N limit for which the BCS energy can be recovered[6], these equations for R_1,... R_N did not had up to now analytical solution for arbitrary N and arbitrary interaction strength, so that they were mostly approached through numerical procedures[4,5]. This is probably why the Richardson's equations have not had so far the attention they deserve among the superconductor community. Nowadays, these equations are commonly addressed numerically to study superconducting granules with a small number of pairs[4].

Last year, we reconsidered these Richarson's equations because we wanted to reveal the deep connection which exists between two well-known problems, namely the one-pair problem solved by Cooper and the many-pair problem considered by Bardeen, Cooper and Schrieffer. These two problems have intrinsic similarities: In both cases, there is a "frozen" core of non-interacting electrons. Above this core, there is a potential layer with attraction between up and down spin electrons. In the one-pair problem, this layer contains one electron pair only, while in the standard BCS configuration, the layer is half-filled - the potential layer is said to extend symmetrically on both sides of the Fermi level which is just equivalent to half filling. It is clear that, by adding more and more pairs into the potential layer, we can continuously go from the one-pair problem studied by Cooper to the dense regime studied by Bardeen, Cooper and Schrieffer.

Although, at the present time, such a continuous pair increase does not seem easy to achieve experimentally, this increase can at least be seen as a gedanken experiment to study the evolution of the energy spectrum when the filling of the potential layer is changed, in order to understand the exact role of the Pauli exclusion principle in superconductivity. This procedure can also be seen as a simple but well-defined toy model to tackle the BEC-BCS crossover since, by changing the number of pairs, we change their overlap. Such an overlap change has already been considered by Eagles[8], and also by Leggett[9], but through the change of the interaction strength between pairs. The number of pairs being then fixed, Pauli blocking does not change in this overlap change while it does when we increase the pair number.

Since the Richardson's procedure allows one to fix the pair number and vary this number at will from one to half filling, we seriously reconsidered solving these equations analytically in order to evidence the effect of Pauli blocking. By turning to their dimensionless form, we succeeded to find an analytical way to solve these equations

in the dilute regime of pairs[7]. Indeed, these equations do have a small dimensionless parameter, namely $1/N_c$ where N_c is the number of pairs from which overlap between single pairs would start. This allowed us to demonstrate in the dilute limit on the single Cooper pair scale, i.e., for N arbitrary large but N/N_c small, that the energy of N Cooper pairs reads in the large sample limit as

$$E_N = N\left[\left(2\epsilon_{F_0} + \frac{N-1}{\rho_0}\right) - \epsilon_c\left(1 - \frac{N-1}{N_\Omega}\right)\right] \quad (1)$$

ϵ_{F_0} is the Fermi level energy of the frozen sea. ρ_0 is the density of states within the potential layer, taken as constant. $N_\Omega = \rho_0\Omega$ is the number of free pair states in this layer, Ω being the potential layer extension. $\epsilon_c \approx 2\Omega\exp(-2/\rho_0 V)$ is the single pair binding energy for a small potential amplitude V (weak-coupling limit).

Although our actual derivation imposes N/N_c small, it turns out that this result is also valid in the dense BCS regime, where pairs strongly overlap. Indeed, the first term of Eq.(1) is the exact energy of N pairs in the normal state since it is nothing but

$$\mathcal{E}_N^{(normal)} = 2\left[\epsilon_{F_0} + (\epsilon_{F_0} + 1/\rho_0) + \cdots + (\epsilon_{F_0} + (N-1)/\rho_0)\right] \quad (2)$$

For a number of pairs corresponding to fill half the potential layer, which precisely is the BCS configuration, Eq.(1) gives a condensation energy equal to

$$\mathcal{E}_N^{(normal)} - \mathcal{E}_N = \frac{N_\Omega}{2}\frac{\epsilon_c}{2} = \frac{1}{2}\rho_0\Omega^2 e^{-2/\rho_0 V} \quad (3)$$

This result exactly matches the one derived by Bardeen, Cooper, Schrieffer within the grand canonical ensemble using the BCS wave function ansatz, namely $\rho_0\Delta^2/2$ since the gap Δ reads as $2\omega_c\exp(-1/\rho_0 V)$ where $2\omega_c$ is nothing but the potential layer extension Ω. It also is remarkable to note that if we extend the BCS grand canonical derivation originally performed for half filling, to non-symmetrical configurations, Eq.(1) remains valid.

The canonical approach we have used to reach Eq.(1), based on solving the Richardson's equations analytically, has the great advantage to follow the evolution of the ground state energy when adding pairs one by one. This leads us to, in a natural way, associate the second term in the RHS of Eq.(1), namely $\epsilon_c\left[1 - (N-1)/N_\Omega\right]$, with the average "pair binding energy" in the N-pair configuration: for $N = 1$, this quantity exactly matches the single-pair binding energy found by Cooper, while in the dense regime, it exactly gives the condensation energy per pair. This is of importance for physical understanding because this pair energy allows us to connect the dilute and dense regimes of pairs within the same framework.

We see that the pair binding energy, as defined above, decreases when N increases. This decrease is entirely due to Pauli blocking, the number of pair states in the potential layer, available to form correlated states, decreasing when N increases. A pictorial way to understand the binding energy decrease when N increases is through the so-called "moth-eaten" effect: when pairs are added to the frozen Fermi sea $|F_0\rangle$, they "eat" one by one, like little moths, the states in the potential layer available to form a bound state. As a result of this available state decrease, the bound state energy can only decrease. Note that this pair binding energy decrease is at odd from the common belief that in the dense BCS configuration, the Cooper pair binding energy is of the order of the excitation gap since Δ is far larger than ϵ_c. This common belief is obtained by splitting the condensation energy $\rho_0\Delta^2/2$ as $(\rho_0\Delta)\Delta$ within an "irrelevant" 1/2 prefactor. This procedure deliberately assigns to each pair an energy equal to the gap, the number of pairs to fit the condensation energy then being $\rho_0\Delta$, i.e., the number of pairs in a gap layer. These $\rho_0\Delta$ pairs have been called "virtual pairs" by Schrieffer. Their number is definitly far smaller than the number of pairs $N_\Omega/2$ feeling the potential. As a direct consequence, their energy is far larger than the average energy $\epsilon_c/2$ of the pairs which feel the potential. These virtual pairs in fact correspond to excitations across the Fermi sea $|F\rangle$ made of $N + N_0$ *noninteracting* pairs, N_0 being the number of pairs in the frozen core $|F_0\rangle$. It is of importance to note that the concept of virtual pairs is physically relevant in the dense regime only; in the dilute regime, the Fermi level for noninteracting electrons is completely washed out, all the pairs feeling the potential being essentially excited above this level. Another unpleasant aspect of this virtual pair concept is that it tends to mask the obvious link which exists between the dilute and dense regimes of pairs. This probably is one of the reasons for Schrieffer's statement that the isolated pair picture has little meaning in the dense regime[10]. This statement was already questioned by Leggett who showed that, in many respects, pairs in the dense limit are very similar to giant molecules made of two opposite spin electrons[9]. The result of Eq.(1) confirms Leggett's understanding.

Since the key role of Pauli blocking in superconductivity is enlightened by the expression of the N-pair energy reported in Eq.(1), through the "moth-eaten effect" it contains, while this expression has been obtained by solving the Richardson's equations analytically, it can be of interest to precisely see the parts in these equations which directly come from the Pauli exclusion principle.

In our recent works on the many-body physics of composite bosons - which was mostly concentrated on semiconductor excitons - we have proposed a "commutation technique" which allows us to evidence the effects of Pauli blocking between the fermionic components of these composite bosons (cobosons in short). They appear through "Pauli scatterings" which describe fermion exchanges in the absence of fermion interaction. These dimensionless scatterings, when mixed with energy-like scatterings coming from interactions between the coboson fermionic components, allow us to deal with fermion exchanges between any number of composite particles in an exact way.

For a review on this formalism and its applications to the many-body physics of semiconductor excitons, see Refs.[11,12].

In the present paper, we first develop such a commutation technique for up and down spin electron pairs with zero total momentum, these being the elementary composite bosons on which Cooper pairs are constructed. We then use this commutation technique to derive in a quite compact way, the form of the exact eigenstate for N pairs interacting through the reduced potential used by Bardeen, Cooper and Schrieffer. The Richardson's equations readily follow from this approach. Its main advantage is to possibly trace back the physical origin of the terms in $1/(R_i - R_j)$. These differences actually come from Pauli scatterings for fermion exchanges between up and down spin electron pairs. This leads us to conclude that the Richardson's energies R_i have N different values just because of Pauli blocking between the Cooper pair components.

The paper is organized as follow:

In section I, we present the commutation technique to handle many-body effects between free electron pairs and we derive their associated Pauli and interaction scatterings.

In section II, we use this technique to recover the Richardson's form of the exact eigenstates for $N = 1, 2, 3, \cdots$ pairs interacting through the reduced BCS potential, in order to see how the solution for general N develops. We then analyze the precise role of Pauli blocking in this solution.

In section III, we briefly question the BCS wave function ansatz for condensed pairs in the light of the exact form provided by the Richardson's procedure.

I. COMMUTATION TECHNIQUE FOR FREE FERMION PAIRS

A. Exchange between free fermion pairs

We consider cobosons made of free fermion pairs having a zero total momentum.

$$\beta^\dagger_{\mathbf{k}} = a^\dagger_{\mathbf{k}} b^\dagger_{-\mathbf{k}} \tag{4}$$

These pairs have one degree of freedom only, namely $\mathbf{k}$, by contrast to the most general fermion pairs $a^\dagger_{\mathbf{k}_1} b^\dagger_{\mathbf{k}_2}$ which have two[13]. In the case of Cooper pairs, $a^\dagger_{\mathbf{k}}$ creates a spin up electron with momentum $\mathbf{k}$ while $b^\dagger_{-\mathbf{k}}$ creates a down spin electron with momentum $-\mathbf{k}$. The fermion operators ($a^\dagger_{\mathbf{k}'}$,$a^\dagger_{\mathbf{k}}$) and ($b^\dagger_{\mathbf{k}'}$,$b^\dagger_{\mathbf{k}}$) anticommute. For ($a^\dagger_{\mathbf{k}'}$, $b^\dagger_{\mathbf{k}}$), they anticommute in the case of opposite spin electrons, but in general they commute or anticommute depending if the corresponding fermions have the same or a different nature. It however is easy to check that this does not affect the commutation relations between fermion pair operators that we are going to derive. This is why, for simplicity, we can consider that all fermion operators anticommute.

It is straightforward to show that the creation operators of free fermion pairs commute

$$\left[\beta^\dagger_{\mathbf{k}'}, \beta^\dagger_{\mathbf{k}}\right] = 0 \tag{5}$$

We can however note that while $\left(a^\dagger_{\mathbf{k}}\right)^2 = 0$ simply follows from the anticommutation of $a^\dagger_{\mathbf{k}}$ operators, the cancellation of $\left(\beta^\dagger_{\mathbf{k}}\right)^2$ does not follow from Eq.(5), but from the fact that $\left(\beta^\dagger_{\mathbf{k}}\right)^2$ contains$\left(a^\dagger_{\mathbf{k}}\right)^2 = 0$. The$\left(\beta^\dagger_{\mathbf{k}}\right)^2$ cancellation which comes from Pauli blocking, can seem to be lost when turning from single fermion operators to pair operators. We will see that this Pauli blocking is yet preserve in the commutaion algebra of free fermion pairs we are going to develop.

For creation and annihilation operators, $\left[a_{\mathbf{k}'}, a^\dagger_{\mathbf{k}}\right] = \delta_{\mathbf{k}'\mathbf{k}}$ leads to

$$\left[\beta_{\mathbf{k}'}, \beta^\dagger_{\mathbf{k}}\right] = \delta_{\mathbf{k}'\mathbf{k}} - D_{\mathbf{k}'\mathbf{k}} \tag{6}$$

where the deviation-from-boson operator $D_{\mathbf{k}'\mathbf{k}}$ is defined as

$$D_{\mathbf{k}'\mathbf{k}} = \delta_{\mathbf{k}'\mathbf{k}} \left(a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + b^\dagger_{-\mathbf{k}} b_{-\mathbf{k}}\right) \tag{7}$$

This operator which would reduce to zero for fermion pairs taken as elementary bosons, allows us to generate the Pauli scatterings for fermion exchanges between cobosons in the absence of fermion interaction. These are formally defined through[11,12]

$$\left[D_{\mathbf{k}'_1\mathbf{k}_1}, \beta^\dagger_{\mathbf{k}_2}\right] = \sum_{\mathbf{k}'_2} \left\{\lambda \begin{pmatrix} \mathbf{k}'_2 & \mathbf{k}_2 \\ \mathbf{k}'_1 & \mathbf{k}_1 \end{pmatrix} + (\mathbf{k}'_1 \leftrightarrow \mathbf{k}'_2)\right\} \beta^\dagger_{\mathbf{k}'_2} \tag{8}$$

By noting that

$$\left[a^\dagger_{\mathbf{k}} a_{\mathbf{k}}, \beta^\dagger_{\mathbf{p}}\right] = \delta_{\mathbf{k}\mathbf{p}} \beta^\dagger_{\mathbf{p}} = \left[b^\dagger_{-\mathbf{k}} b_{-\mathbf{k}}, \beta^\dagger_{\mathbf{p}}\right] \tag{9}$$

it is then easy to show that

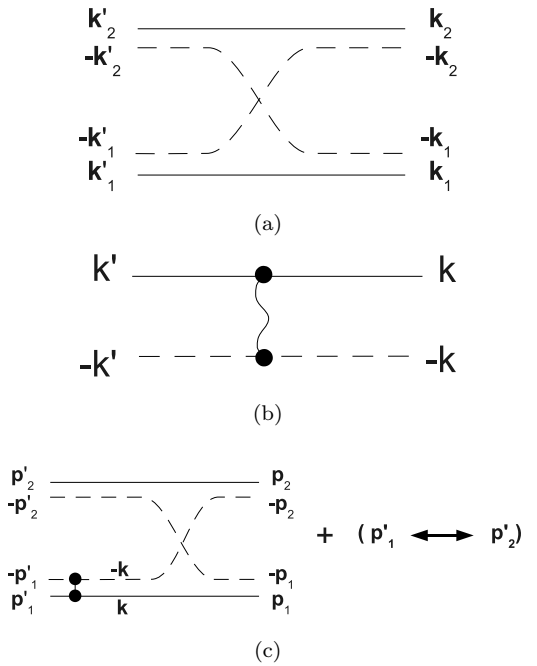
$$\left[D_{\mathbf{k}'_1\mathbf{k}_1}, \beta^\dagger_{\mathbf{k}_2}\right] = 2\beta^\dagger_{\mathbf{k}_2} \delta_{\mathbf{k}_1\mathbf{k}_2} \delta_{\mathbf{k}'_1,\mathbf{k}_2} \tag{10}$$

This leads us to identify the Pauli scattering appearing in Eq.(8) with a product of Kronecker symbols

$$\lambda \begin{pmatrix} \mathbf{k}'_2 & \mathbf{k}_2 \\ \mathbf{k}'_1 & \mathbf{k}_1 \end{pmatrix} = \delta_{\mathbf{k}'_1\mathbf{k}_1} \delta_{\mathbf{k}'_2\mathbf{k}_2} \delta_{\mathbf{k}_1\mathbf{k}_2} \tag{11}$$

Actually, this is just the value we expect for the scattering associated to fermion exchanges between $(\mathbf{k}_1, \mathbf{k}_2)$ pairs, as visualized by the diagram of Fig.(1a). Indeed, from this diagram, it is clear that we must have $(\mathbf{k}'_1 = \mathbf{k}_1, \mathbf{k}'_2 = \mathbf{k}_2)$ and $(-\mathbf{k}'_2 = -\mathbf{k}_1, -\mathbf{k}'_1 = -\mathbf{k}_2)$: this just gives $\delta_{\mathbf{k}'_1\mathbf{k}_1} \delta_{\mathbf{k}'_2\mathbf{k}_2} \delta_{\mathbf{k}_1\mathbf{k}_2}$ in agreement with Eq.(11).

FIG. 1: Shiva diagram of free pairs



(a) Pauli scattering $\lambda \begin{pmatrix} \mathbf{k}_2' & \mathbf{k}_2 \\ \mathbf{k}_1' & \mathbf{k}_1 \end{pmatrix}$ for electron exchange between two free pairs $(\mathbf{k}_1, \mathbf{k}_2)$, as given by Eq.(11). Up spin electrons are represented by solid lines, down spin electrons by dashed lines.

(b) The BCS potential given in Eq.(14) transforms a $\mathbf{k}$ pair into a $\mathbf{k}'$ pair, with a constant scattering $-V$, in the case of a separable potential $v_{\mathbf{k}'\mathbf{k}} = -V\, w_{\mathbf{k}'} w_{\mathbf{k}}$.

(c) Interaction scattering $\chi \begin{pmatrix} \mathbf{p}_2' & \mathbf{p}_2 \\ \mathbf{p}_1' & \mathbf{p}_1 \end{pmatrix}$ between two free pairs, as given in Eq.(19). Since the BCS potential acts within one pair only, the interaction between two pairs can only come from exchange induced by the Pauli exclusion principle.

B. Interaction between free fermion pairs

To get the interaction scatterings associated to fermion interaction, we first note that, for a free fermion hamiltonian

$$H_0 = \sum \epsilon_{\mathbf{k}} \left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \right) \tag{12}$$

Eq.(9) readily gives

$$\left[H_0, \beta_{\mathbf{p}}^\dagger \right] = 2\epsilon_{\mathbf{p}} \beta_{\mathbf{p}}^\dagger \tag{13}$$

In the case of interest in standard BCS superconductivity, these fermion pairs interact through the reduced potential

$$V_{BCS} = \sum v_{\mathbf{k}'\mathbf{k}} \beta_{\mathbf{k}'}^\dagger \beta_{\mathbf{k}} \tag{14}$$

This potential basically is a (1x1) potential in the fermion pair subspace since fermion $\mathbf{k}$ interacts with one fermion only of the other species, namely fermion $(-\mathbf{k})$. This BCS potential is represented by the diagram of Fig.(1b). For this (1x1) potential, we do have

$$\left[V_{BCS}, \beta_{\mathbf{p}}^\dagger \right] = \gamma_{\mathbf{p}}^\dagger + V_{\mathbf{p}}^\dagger \tag{15}$$

in which we have $\gamma_{\mathbf{p}}^\dagger = \sum_{\mathbf{k}} \beta_{\mathbf{k}}^\dagger v_{\mathbf{kp}}$. The "creation potential" $V_{\mathbf{p}}^\dagger$ for the free fermion pair $\mathbf{p}$ appears to be

$$V_{\mathbf{p}}^\dagger = -\gamma_{\mathbf{p}}^\dagger \left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + b_{-\mathbf{p}}^\dagger b_{-\mathbf{p}} \right) \tag{16}$$

While the $\gamma_{\mathbf{p}}^\dagger$ part of Eq.(14) commutes with $\beta_{\mathbf{p}'}^\dagger$, this is not so for the creation potential $V_{\mathbf{p}}^\dagger$. Its commutator precisely reads

$$\left[V_{\mathbf{p}_1}^\dagger, \beta_{\mathbf{p}_2}^\dagger \right] = -2\delta_{\mathbf{p}_1 \mathbf{p}_2} \gamma_{\mathbf{p}_1}^\dagger \beta_{\mathbf{p}_1}^\dagger \tag{17}$$

This allows us to identify the interaction scattering for free pairs, formally defined as

$$\left[V_{\mathbf{p}_1}^\dagger, \beta_{\mathbf{p}_2}^\dagger \right] = \sum \chi \begin{pmatrix} \mathbf{p}_2' & \mathbf{p}_2 \\ \mathbf{p}_1' & \mathbf{p}_1 \end{pmatrix} \beta_{\mathbf{p}_1'}^\dagger \beta_{\mathbf{p}_2'}^\dagger \tag{18}$$

with a sequence of one (2x2) fermion exchange between two pairs and one (1x1) fermion interaction inside one pair. Indeed

$$\begin{aligned} \chi \begin{pmatrix} \mathbf{p}_2' & \mathbf{p}_2 \\ \mathbf{p}_1' & \mathbf{p}_1 \end{pmatrix} &= -\sum_{\mathbf{k}} \left\{ v_{\mathbf{p}_1' \mathbf{k}} \lambda \begin{pmatrix} \mathbf{p}_2' & \mathbf{p}_2 \\ \mathbf{k} & \mathbf{p}_1 \end{pmatrix} + (\mathbf{p}_1' \leftrightarrow \mathbf{p}_2') \right\} \\ &= -\left(v_{\mathbf{p}_1', \mathbf{p}_1} \delta_{\mathbf{p}_2', \mathbf{p}_2} + v_{\mathbf{p}_2', \mathbf{p}_2} \delta_{\mathbf{p}_1', \mathbf{p}_1} \right) \delta_{\mathbf{p}_2, \mathbf{p}_1} \end{aligned} \tag{19}$$

This interaction scattering is visualized by the diagram of Fig.(1c): the free pairs $\mathbf{p}_1$ and $\mathbf{p}_2$ first exchange a fermion. As for any exchange, this brings a minus sign. In a second step, the fermions of one of the two pairs interact via the BCS potential. It is of importance to realize that, since the BCS potential has a (1x1) structure within the pair subspace, the interaction between two pairs can only result from fermion exchange between pairs, i.e., Pauli blocking, as readily seen from this diagram.

We are now going to use this commutation formalism to derive the equations that Richardson has obtained for the eigenstates of N Cooper pairs through a totally different procedure.

II. RICHARDSON'S EQUATIONS FOR COOPER PAIRS

In order to better grasp how these equations develop, we are going to increase the number of pairs feeling the potential one by one, starting from a single pair.

A. One pair

Let us first consider a state in which one free pair $(\mathbf{k}, -\mathbf{k})$ is added to a "frozen" Fermi sea $|F_0\rangle$, i.e. a sea which does not feel the BCS potential. This means that the $v_{\mathbf{k'k}}$ prefactors in Eq.(14) cancel for all $\mathbf{k}$ belonging to $|F_0\rangle$ in order to have $V_{BCS}|F_0\rangle = 0$. This "one-pair" state actually contains $N_0 + 1$ fermion pairs, N_0 being the number of pairs in the frozen sea; so that this state is a many-body state already, but in the most simple sense since the Fermi sea $|F_0\rangle$ is just there to block states by the Pauli exclusion principle. This Fermi sea mainly brings a finite density of state for all the states above it which is crucial to have a bound state whatever the weakness of the attracting BCS potential, as evidenced below.

By choosing the zero energy such that $H_0|F_0\rangle = 0$, Eqs.(13,15) allows us to write the hamiltonian $H = H_0 + V_{BCS}$ acting on a one-free-pair state as

$$H\beta^\dagger_{\mathbf{k}}|F_0\rangle = \left[H, \beta^\dagger_{\mathbf{k}}\right]|F_0\rangle = \left(2\epsilon_{\mathbf{k}}\beta^\dagger_{\mathbf{k}} + \gamma^\dagger_{\mathbf{k}} + V^\dagger_{\mathbf{k}}\right)|F_0\rangle \tag{20}$$

Due to the $v_{\mathbf{kp}}$ factor included in the $\gamma^\dagger_{\mathbf{k}}$ part of $V^\dagger_{\mathbf{k}}$, we do have $V^\dagger_{\mathbf{k}}|F_0\rangle = 0$. If we now subtract $E_1\beta^\dagger_{\mathbf{k}}|F_0\rangle$ to the two sides of the above equation, with E_1 yet undefined, and if we then divide the resulting equation by $(2\epsilon_{\mathbf{k}} - E_1)$, we find

$$(H - E_1)\frac{1}{2\epsilon_{\mathbf{k}} - E_1}\beta^\dagger_{\mathbf{k}}|F_0\rangle = \beta^\dagger_{\mathbf{k}}|F_0\rangle + \frac{1}{2\epsilon_{\mathbf{k}} - E_1}\gamma^\dagger_{\mathbf{k}}|F_0\rangle \tag{21}$$

To go further and possibly get the one-pair eigenstate of the hamiltonian H in a compact analytical form, it is necessary to approximate the BCS potential by a separable potential $v_{\mathbf{kp}} = -V\, w_{\mathbf{k}} w_{\mathbf{p}}$. This leads us to set $\gamma^\dagger_{\mathbf{k}} = -V\, w_{\mathbf{k}}\beta^\dagger$ where $\beta^\dagger$ is defined as

$$\beta^\dagger = \sum_{\mathbf{p}} w_{\mathbf{p}}\beta^\dagger_{\mathbf{p}} \tag{22}$$

If we then multiply Eq.(21) by $w_{\mathbf{k}}$ and sum over $\mathbf{k}$, we end with

$$(H - E_1)B^\dagger(E_1)|F_0\rangle = \left(1 - V\sum_{\mathbf{k}}\frac{w^2_{\mathbf{k}}}{2\epsilon_{\mathbf{k}} - E_1}\right)\beta^\dagger|F_0\rangle \tag{23}$$

where the operator $B^\dagger(E)$ is defined as

$$B^\dagger(E) = \sum_{\mathbf{k}} B^\dagger_{\mathbf{k}}(E) \qquad B^\dagger_{\mathbf{k}}(E) = \frac{w_{\mathbf{k}}}{2\epsilon_{\mathbf{k}} - E}\beta^\dagger_{\mathbf{k}} \tag{24}$$

Eq.(23) readily shows that $B^\dagger(E_1)|F_0\rangle$ is the one-pair eigenstate of the hamiltonian H with energy E_1, provided that this energy fulfills

$$1 = V\sum_{\mathbf{k}}\frac{w^2_{\mathbf{k}}}{2\epsilon_{\mathbf{k}} - E_1} \tag{25}$$

This is nothing but the well-known equation for the single pair energy derived by Cooper.

B. Two pairs

Let us now add two pairs to the frozen sea $|F_0\rangle$. Eqs.(13,15) yield

$$\begin{aligned} H\beta^\dagger_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_2}|F_0\rangle &= \left(\left[H, \beta^\dagger_{\mathbf{k}_1}\right]\beta^\dagger_{\mathbf{k}_2} + \beta^\dagger_{\mathbf{k}_1}\left[H, \beta^\dagger_{\mathbf{k}_2}\right]\right)|F_0\rangle \\ &= \left(2\epsilon_{\mathbf{k}_1} + 2\epsilon_{\mathbf{k}_2}\right)\beta^\dagger_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_2}|F_0\rangle + |v_{\mathbf{k}_1\mathbf{k}_2}\rangle \end{aligned} \tag{26}$$

$|v_{\mathbf{k}_1\mathbf{k}_2}\rangle$ comes from interactions among the $(\mathbf{k}_1, \mathbf{k}_2)$ pairs induced by the BCS potential possibly mixed with electron exchanges. Its precise value reads

$$|v_{\mathbf{k}_1\mathbf{k}_2}\rangle = \left(\gamma^\dagger_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_2} + \gamma^\dagger_{\mathbf{k}_2}\beta^\dagger_{\mathbf{k}_1} + V^\dagger_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_2}\right)|F_0\rangle \tag{27}$$

Since $V^\dagger_{\mathbf{k}}$ acting on the frozen sea $|F_0\rangle$ gives zero, Eq. (19) allows us to rewrite the last term of $|v_{\mathbf{k}_1\mathbf{k}_2}\rangle$ as

$$\begin{aligned} V^\dagger_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_2}|F_0\rangle &= \left[V^\dagger_{\mathbf{k}_1}, \beta^\dagger_{\mathbf{k}_2}\right]|F_0\rangle \\ &= \sum_{\mathbf{p}'_1\mathbf{p}'_2}\chi\begin{pmatrix}\mathbf{p}'_2 & \mathbf{k}_2\\ \mathbf{p}'_1 & \mathbf{k}_1\end{pmatrix}\beta^\dagger_{\mathbf{p}'_1}\beta^\dagger_{\mathbf{p}'_2}|F_0\rangle \end{aligned} \tag{28}$$

The three terms of $|v_{\mathbf{k}_1\mathbf{k}_2}\rangle$ are visualized by the diagram of Fig. 2. In the first process, one pair stays unchanged while in the other, the two pairs exchange an electron. This diagram evidences the fact that, due to the (1x1) structure of the BCS potential, the two pairs $\mathbf{k}_1$ and $\mathbf{k}_2$ interact by fermion exchange only, as a result of the Pauli exclusion principle.

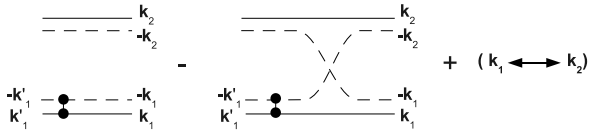


FIG. 2: Shiva diagram for the interaction part $|v_{\mathbf{k}_1\mathbf{k}_2}\rangle$ of the Hamiltonian H acting on two free pairs, as given in Eq.(27)

By using the value of the interaction scattering given in Eq.(17), we find that $|v_{\mathbf{k}_1\mathbf{k}_2}\rangle$ ultimatly reads as

$$|v_{\mathbf{k}_1\mathbf{k}_2}\rangle = -V\left(w_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_2} + w_{\mathbf{k}_2}\beta^\dagger_{\mathbf{k}_1} - 2\delta_{\mathbf{k}_1\mathbf{k}_2}w_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_1}\right)\beta^\dagger|F_0\rangle \tag{29}$$

To go further, we subtract $E_2\beta^\dagger_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_2}|F_0\rangle$ to the two sides of Eq.(26), with E_2 yet undefined. We split E_2 as $R_1 + R_2$ and we multiply the resulting equation by $w_{\mathbf{k}_1}w_{\mathbf{k}_2}/\left(2\epsilon_{\mathbf{k}_1} - R_1\right)\left(2\epsilon_{\mathbf{k}_2} - R_2\right)$. This yields

$$\begin{aligned} &(H - E_2)B^\dagger_{\mathbf{k}_1}(R_1)B^\dagger_{\mathbf{k}_2}(R_2)|F_0\rangle = \\ &\left\{B^\dagger_{\mathbf{k}_1}(R_1)\left(w_{\mathbf{k}_2}\beta^\dagger_{\mathbf{k}_2} - \frac{Vw^2_{\mathbf{k}_2}}{2\epsilon_{\mathbf{k}_2} - R_2}\beta^\dagger\right) + (1 \leftrightarrow 2)\right\}|F_0\rangle \\ &\qquad + 2V\delta_{\mathbf{k}_2\mathbf{k}_1}\frac{w^3_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_1}}{\left(2\epsilon_{\mathbf{k}_1} - R_1\right)\left(2\epsilon_{\mathbf{k}_1} - R_2\right)}\beta^\dagger|F_0\rangle \end{aligned} \tag{30}$$

To go further, we split $(2\epsilon_{\mathbf{k}_1}-R_1)^{-1}(2\epsilon_{\mathbf{k}_1}-R_2)^{-1}$ as $\left[(2\epsilon_{\mathbf{k}_1}-R_1)^{-1}-(2\epsilon_{\mathbf{k}_1}-R_2)^{-1}\right]/(R_1-R_2)$ provided that $R_1 \neq R_2$, a condition that we can always inforce since the unique requirement is to have $R_1+R_2=E_2$. This allows us to write the last term of (30) as $\delta_{\mathbf{k}_2\mathbf{k}_1}\frac{2V}{R_1-R_2}[B^\dagger_{\mathbf{k}_1}(R_1)-B^\dagger_{\mathbf{k}_1}(R_2)]\beta^\dagger|F_0\rangle$ provided that $w^2_{\mathbf{k}}=w_{\mathbf{k}}$. By taking sums over $\mathbf{k}_1$ and $\mathbf{k}_2$, Eq. (30) then gives

$$(H-E_2)B^\dagger(R_1)B^\dagger(R_2)|F_0\rangle = \left\{B^\dagger(R_1)\left(1-V\sum\frac{w^2_{\mathbf{k}}}{2\epsilon_{\mathbf{k}}-R_2}+\frac{2V}{R_1-R_2}\right)+(1\leftrightarrow 2)\right\}\beta^\dagger|F_0\rangle \quad (31)$$

The above equation readily shows that $B^\dagger(R_1)B^\dagger(R_2)|F_0\rangle$ is the two-pair eigenstate of the hamiltonian H with the energy $E_2=R_1+R_2$ provided that (R_1,R_2) fulfill two equations, known as Richardson's equations for two pairs

$$1=V\sum\frac{w^2_{\mathbf{k}}}{2\epsilon_{\mathbf{k}}-R_1}+\frac{2V}{R_1-R_2}=(1\leftrightarrow 2) \quad (32)$$

C. Three pairs

We now turn to three pairs in order to see how these equations develop for an increasing number of pairs. We start with

$$\begin{aligned} & H\beta^\dagger_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_2}\beta^\dagger_{\mathbf{k}_3}|F_0\rangle \\ = & \left\{\left[H,\beta^\dagger_{\mathbf{k}_1}\right]\beta^\dagger_{\mathbf{k}_2}\beta^\dagger_{\mathbf{k}_3}+\beta^\dagger_{\mathbf{k}_1}\left[H,\beta^\dagger_{\mathbf{k}_2}\right]\beta^\dagger_{\mathbf{k}_3}\right. \\ & \left.+\beta^\dagger_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_2}\left[H,\beta^\dagger_{\mathbf{k}_3}\right]\right\}|F_0\rangle \end{aligned} \quad (33)$$

The same Eqs.(13,15) split the above equation into a kinetic part and an interaction part

$$H\beta^\dagger_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_2}\beta^\dagger_{\mathbf{k}_3}|F_0\rangle = (2\epsilon_{\mathbf{k}_1}+2\epsilon_{\mathbf{k}_2}+2\epsilon_{\mathbf{k}_3})\beta^\dagger_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_2}\beta^\dagger_{\mathbf{k}_3}|F_0\rangle + |v_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\rangle \quad (34)$$

The contribution resulting from the BCS potential reads as

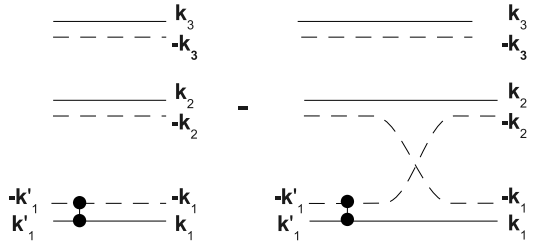
$$\begin{aligned} |v_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\rangle = & \left(\gamma^\dagger_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_2}\beta^\dagger_{\mathbf{k}_3}+\gamma^\dagger_{\mathbf{k}_2}\beta^\dagger_{\mathbf{k}_3}\beta^\dagger_{\mathbf{k}_1}+\gamma^\dagger_{\mathbf{k}_3}\beta^\dagger_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_2}\right)|F_0\rangle \\ & +\left(V^\dagger_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_2}\beta^\dagger_{\mathbf{k}_3}+\beta^\dagger_{\mathbf{k}_1}V^\dagger_{\mathbf{k}_2}\beta^\dagger_{\mathbf{k}_3}+\beta^\dagger_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_2}V^\dagger_{\mathbf{k}_3}\right)|F_0\rangle \end{aligned} \quad (35)$$

Let us concentrate on the second part of $|v_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\rangle$. Its last term gives zero since $V^\dagger_{\mathbf{k}}|F_0\rangle=0$. Using Eq. (18), the first two terms can be rewritten in a more symmetrical form as

$$\begin{aligned} & \left\{\left[V^\dagger_{\mathbf{k}_1},\beta^\dagger_{\mathbf{k}_2}\right]\beta^\dagger_{\mathbf{k}_3}+\beta^\dagger_{\mathbf{k}_2}\left[V^\dagger_{\mathbf{k}_1},\beta^\dagger_{\mathbf{k}_3}\right]+\beta^\dagger_{\mathbf{k}_1}\left[V^\dagger_{\mathbf{k}_2},\beta^\dagger_{\mathbf{k}_3}\right]\right\}|F_0\rangle \\ = & \sum_{\mathbf{k}'_1\mathbf{k}'_2}\beta^\dagger_{\mathbf{k}'_1}\beta^\dagger_{\mathbf{k}'_2} \\ & \left\{\chi\begin{pmatrix}\mathbf{k}'_2\ \mathbf{k}_2\\ \mathbf{k}'_1\ \mathbf{k}_1\end{pmatrix}\beta^\dagger_{\mathbf{k}_3}+\chi\begin{pmatrix}\mathbf{k}'_2\ \mathbf{k}_3\\ \mathbf{k}'_1\ \mathbf{k}_2\end{pmatrix}\beta^\dagger_{\mathbf{k}_1}+\chi\begin{pmatrix}\mathbf{k}'_2\ \mathbf{k}_1\\ \mathbf{k}'_1\ \mathbf{k}_3\end{pmatrix}\beta^\dagger_{\mathbf{k}_2}\right\}|F_0\rangle \end{aligned} \quad (36)$$

This leads us to represent the vector $|v_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\rangle$ by the diagram of Fig.3. It contains interactions inside a single pair, two pairs staying unchanged. It also contains processes in which the pair suffering the potential exchange one fermion with a second pair, the third pair staying unchanged.

FIG. 3: Shiva diagram for the interaction part $|v_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\rangle$ of the Hamiltonian H acting on three pairs



as given in Eqs. (35,36). $|v_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\rangle$ also contains two similar contributions as the one visualized in this figure, obtained by circular permutation.

Using Eq. (19) for the interaction scattering, the RHS of the above equation reduces to

$$\begin{aligned} & -2(\delta_{\mathbf{k}_1\mathbf{k}_2}\gamma^\dagger_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_3}+2\text{ perm.})|F_0\rangle \\ & =2V(\delta_{\mathbf{k}_1\mathbf{k}_2}w_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_3}+2\text{ perm.})\beta^\dagger|F_0\rangle \end{aligned}$$

If we now come back to Eq.(34), subtract $E_3\beta^\dagger_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_2}\beta^\dagger_{\mathbf{k}_3}|F_0\rangle$ to both sides, with E_3 written as $R_1+R_2+R_3$, and multiply the resulting equation by $w_{\mathbf{k}_1}w_{\mathbf{k}_2}w_{\mathbf{k}_3}/(2\epsilon_{\mathbf{k}_1}-R_1)(2\epsilon_{\mathbf{k}_2}-R_2)(2\epsilon_{\mathbf{k}_3}-R_3)$, we end with

$$\begin{aligned} & (H-E_3)B^\dagger_{\mathbf{k}_1}(R_1)B^\dagger_{\mathbf{k}_2}(R_2)B^\dagger_{\mathbf{k}_3}(R_3)|F_0\rangle \\ = & \left\{B^\dagger_{\mathbf{k}_1}(R_1)B^\dagger_{\mathbf{k}_2}(R_2)\big(w_{\mathbf{k}_3}\beta^\dagger_{\mathbf{k}_3}-\frac{Vw^2_{\mathbf{k}_3}}{2\epsilon_{\mathbf{k}_2}-R_3}\beta^\dagger\big)\right. \\ & \left.+2\text{ perm.}\right\}|F_0\rangle \\ & +2V\left\{B^\dagger_{\mathbf{k}_3}(R_3)\frac{\delta_{\mathbf{k}_1\mathbf{k}_2}w^3_{\mathbf{k}_1}}{(2\epsilon_{\mathbf{k}_1}-R_1)(2\epsilon_{\mathbf{k}_1}-R_2)}\beta^\dagger_{\mathbf{k}_1}\right. \\ & \left.+2\text{ perm.}\right\}\beta^\dagger|F_0\rangle \end{aligned} \quad (37)$$

To go further, we again split $(2\epsilon_{\mathbf{k}_1}-R_1)^{-1}(2\epsilon_{\mathbf{k}_1}-R_2)^{-1}$ as $\left[(2\epsilon_{\mathbf{k}_1}-R_1)^{-1}-(2\epsilon_{\mathbf{k}_1}-R_2)^{-1}\right]/(R_1-R_2)$ provided that $R_1\neq R_2$ and do the same for the two other products. By summing over $(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$, we end with

$$(H-E_3)B^\dagger(R_1)B^\dagger(R_2)B^\dagger(R_3)\,|F_0\rangle = \{B^\dagger(R_2)B^\dagger(R_3) \left(1-V\sum\frac{w_{\mathbf{k}_1}^2}{2\epsilon_{\mathbf{k}_1}-R_1}-\frac{2V}{R_1-R_2}+\frac{2V}{R_3-R_1}\right) + 2 \text{ perm.}\}\beta^\dagger\,|F_0\rangle \quad (38)$$

This leads us to again conclude that the three-pair state $B^\dagger(R_1)B^\dagger(R_2)B^\dagger(R_3)\,|F_0\rangle$ is eigenstate of the hamiltonian H with the energy $E_3=R_1+R_2+R_3$, provided that (R_1,R_2,R_3) fulfill the three equations,

$$\begin{aligned}1&=V\sum\frac{w_{\mathbf{k}}^2}{2\epsilon_{\mathbf{k}}-R_1}+\frac{2V}{R_1-R_2}+\frac{2V}{R_1-R_3}\\ 1&=V\sum\frac{w_{\mathbf{k}}^2}{2\epsilon_{\mathbf{k}}-R_2}+\frac{2V}{R_2-R_3}+\frac{2V}{R_2-R_1}\\ 1&=V\sum\frac{w_{\mathbf{k}}^2}{2\epsilon_{\mathbf{k}}-R_3}+\frac{2V}{R_3-R_1}+\frac{2V}{R_3-R_2}\end{aligned} \quad (39)$$

D. N pairs

The above commutation technique can be easily extended to N pairs. As nicely visualized by the diagrams of Figs.2 and 3, the effect of the BCS potential on these N pairs splits into two sets of processes: In one set, one pair is affected by the (1x1) scattering while the other $N-1$ pairs stay unchanged. In the other set, this pair in addition has, before the interaction, a fermion exchange with another pair, the remaining $N-2$ pairs staying unchanged. This understanding readily shows that an increase of pair number above two, does not really change the structure of the equations since $N-2$ pairs stay unchanged, the pair exchanging its fermions with the pair suffering the interaction being just one among $(N-1)$ pairs.

Although the equations become more and more cumbersome to be explicitly written, the procedure is rather straightforward once we have understood that either $(N-1)$ or $(N-2)$ pairs stay unaffected in the interaction process. The general form of the N-pair eigenstate ultimately appears as

$$(H-E_N)B^\dagger(R_1)\cdots B^\dagger(R_N)\,|F_0\rangle=0 \quad (40)$$

with $E_N=R_1+\cdots+R_N$, these R_N's being solutions of N coupled equations

$$1=V\sum\frac{w_{\mathbf{k}}^2}{2\epsilon_{\mathbf{k}}-R_i}+\sum_{i\neq j}\frac{2V}{R_i-R_j} \qquad \text{for } i=(1,...,N) \quad (41)$$

E. Physical understanding

This new derivation of the Richardson's equations has the main advantage to possibly trace back the parts in these equations which are directly linked to the Pauli exclusion principle between fermion pairs.

From a mathematical point of view, the link is rather obvious: In the absence of terms in $V/(R_i-R_j)$, the N equations for R_i reduced to the same equation (25), so that the result would be $R_i^{(0)}=E_1$ for all i: The fact that the energy of N pairs differs from N times the single pair energy E_1 thus comes from the set of (R_i-R_j) different from zero.

Physically, the fact that E_N differs from NE_1 comes from interactions between Cooper pairs. Due to the (1x1) form of the BCS potential within the pair subspace, interaction between pairs can only be mediated by fermion exchanges as clear from Fig.(1c) Interaction between pairs thus is solely the result of the Pauli exclusion principle between pairs. This Pauli blocking mathematically appears through the various $\delta_{\mathbf{p'p}}$ factors in Pauli scatterings $\lambda(^{\mathbf{p}_2'\ \mathbf{p}_2}_{\mathbf{p}_1'\ \mathbf{p}_1})$. It is then easy to mathematically trace back the (R_i-R_j) differences in the Richardson's equations to these δ factors.

In short, the Kronecker symbols in the Pauli scatterings of fermion pairs come from states which are excluded by the Pauli exclusion principle. They induce the $V/(R_i-R_j)$ terms of the Richardson's equations which ultimately makes the energy of N pairs different from the energy of N independent pairs.

Another important feature of the energy E_N for N pairs that this new derivation explains in a rather clear way, is the fact that the part of the N pairs energy coming from interaction, namely E_N-NE_1 depends on N as $N(N-1)$ only. Indeed, the diagram of Fig.3 evidences the fact that, because the electron pairs of interest have one degree of freedom only, the (1x1) BCS potential mixed with fermion exchanges between pairs, ends by producing effective scatterings which are (2x2) only. In order to have terms in the energy in $N(N-1)(N-2)$, we need topologically connected diagrams between 3 pairs. Since these do not exist, terms in $N(N-1)(N-2)$ and above, cannot exist in the energy of N Cooper pairs, in agreement with Eq.(1) which also reads

$$E_N=NE_1+N(N-1)\left(\frac{1}{\rho_0}+\frac{\epsilon_c}{N_\Omega}\right) \quad (42)$$

III. RICHARDSON'S EXACT EIGENSTATE VERSUS BCS ANSATZ

A last very smart aspect of the Richardson's procedure is that it provides the *exact* form of the eigenstate, namely

$$B^\dagger(R_1)\cdots B^\dagger(R_N)\,|F_0\rangle \quad (43)$$

with $B^\dagger(R)$ given by Eq.(24). The fact that by construction all the R_i's are different, strongly questions the standard BCS ansatz for the condensed pair wave function which reduces to $\left(B^\dagger\right)^N|F_0\rangle$ when projected into the N-pair subspace, *all* the pairs being condensed in the same state.

To discuss this problem on precise grounds, let us again start with two pairs. In a previous work[14], we have shown, that the "Richardson's energies" in the case of two pairs read as $R_1 = R + iR'$ and $R_2 = R - iR'$ with R and R' real. In the large sample limit, i.e. for $1/\rho_0$ small, the dominant terms of these real and imaginary parts are given by $R \approx \epsilon_c + 1/\rho_0 + \epsilon_c/\rho_0\Omega$ and $R' \approx \sqrt{2\epsilon_c/\rho_0}$. By writting $B^\dagger(R_1)B^\dagger(R_2)$ as

$$\left[B^\dagger(R) + B^\dagger(R_1) - B^\dagger(R)\right]\cdot \left[B^\dagger(R) + B^\dagger(R_2) - B^\dagger(R)\right] \quad (44)$$

we get from eq (24)

$$B^\dagger(R_1)B^\dagger(R_2) = \left[B^\dagger(R)\right]^2 + R'^2\left\{C_+^\dagger C_-^\dagger - 2B^\dagger(R)D^\dagger\right\} \quad (45)$$

where we have set

$$C_\pm^\dagger = \sum \frac{w_{\mathbf{k}}}{(2\epsilon_{\mathbf{k}} - R)(2\epsilon_{\mathbf{k}} - R \pm iR')}\beta_{\mathbf{k}}^\dagger \quad (46)$$

$$D^\dagger = \sum \frac{w_{\mathbf{k}}}{(2\epsilon_{\mathbf{k}} - R)\left[(2\epsilon_{\mathbf{k}} - R)^2 + R'^2\right]}\beta_{\mathbf{k}}^\dagger \quad (47)$$

Eq.(45) shows that, in order to possibly replace $B^\dagger(R_1)B^\dagger(R_2)$ by $\left[B^\dagger(R)\right]^2$ as in the BCS ansatz, we must neglect terms in R'^2, i.e., in $1/\rho_0$. We in particular find at the lowest order in the inverse sample volume, i.e., in $1/\rho_0$, that

$$B^\dagger(R_1)B^\dagger(R_2) - \left[B^\dagger(\frac{E_2}{2})\right]^2 \approx \frac{2\epsilon_c}{\rho_0}\Big\{-2B^\dagger(E_1)\sum \frac{w_{\mathbf{k}}}{(2\epsilon_{\mathbf{k}} - E_1)^3}\beta_{\mathbf{k}}^\dagger + \Big[\sum \frac{w_{\mathbf{k}}}{(2\epsilon_{\mathbf{k}} - E_1)^2}\beta_{\mathbf{k}}^\dagger\Big]^2\Big\} \quad (48)$$

This shows that $B^\dagger(R_1)B^\dagger(R_2)$ can be replaced by $\left(B^\dagger(E_2/2)\right)^2$ provided that we consider as negligeable the RHS of the above equation. This imposes to neglect terms in $1/\rho_0$. In this limit, E_2 would reduce to $2E_1$, so that $B^\dagger(E_2/2)$ would reduce to $B^\dagger(E_1)$: The two-pair eigenstate is then just the product of two non-interacting single pairs. If instead, we want to include change induced by Pauli blocking which brings the energy per pair from E_1 to $E_2/2 = E_1 + 1/\rho_0 + \epsilon_c/\rho_0\Omega$, we are led to replace $B^\dagger(R_1)B^\dagger(R_2)$ by the "condensed two-pair state" $\left(B^\dagger(E_2/2)\right)^2$. This however is inconsistent because we then keep in this two-pair operator, contribution in $1/\rho_0$ which are as large as the ones we drop by neglecting the RHS of Eq.(48). In the case of two pairs, the replacement of the exact eigenstate $B^\dagger(R_1)B^\dagger(R_2)|F_0\rangle$ by a BCS-like condensed state $\left(B^\dagger(E_2/2)\right)^2|F_0\rangle$ is fully inconsistent.

Actually, it is claimed that the validity of the BCS ansatz is restricted to the thermodynamical limit, i.e., to N very large. When N increases, the R_i's stay two by two complex conjugate but preliminary calculations in the dilute limit on the single pair scale, show that the $(R_i - R_j)$ differences get larger and larger. This is why we hardly see how, starting from the exact form of the N-pair eigenstate $B^\dagger(R_1)\cdots B^\dagger(R_N)|F_0\rangle$ obtained by Richardson, we can possibly recover the BCS ansatz with the same creation operator for all the pairs.

let us stress that, to the best of our knowledge, derivations of the "validity" of the BCS antsatz for the ground state of N pairs mainly concentrate on the energy it provides (see, e.g.,[10] and references therein). We fully agree with the fact that the BCS ansatz indeed gives the correct ground state energy for N pairs because the energy obtained using this ansatz is just the one we derived from the exact Richardson's procedure. However agreement on the energy by no mean proves agreement on the wave function. Many examples have been given in the past with wave functions very different from the exact one, while giving the correct energy. Direct experiments supporting the form of the ground state wave function seems to be even harder to achieve than the ones possibly checking the ground state energy given in Eq.(1). At this stage, it however seems to us necessary to carefully reconsider agreement with experiments in the light of the exact Richardson's wave function.

The reduced potential used in standart BCS superconductivity has the great advantage to allow an analytical resolution of the N-body Schrodinger equation - which is not that frequent. It is however clear that this potential is highly simplified. A certain amount of corrections are needed to make this potential more realistic. Nevertheless, the BCS ansatz for the wave function which is commonly considered as one of the essential features of superconductivity, has been worked out within this reduced BCS potential. To precisely compare this conventional ansatz with the exact solution of the model within the canonical ensemble is definitly of importance.

Finally, let us stress that the possible replacement of $B^\dagger(R_1)\cdots B^\dagger(R_N)|F_0\rangle$ by $\left(B^\dagger\right)^N|F_0\rangle$ is actually crucial to support the overall picture of superconductivity we commonly have in mind, with all the pairs in the same state, "as an army of little soldiers, all walking similarly". In a forthcoming paper, we are going to come back to the validity of the BCS ansatz in the thermodynamical limit, in the light of our recent analytical results on the Richardson's exact procedure.

IV. CONCLUSION

We have rederived the Richardson's equations for N Cooper pairs using a commutation technique for free elec-

tron pairs with zero total momentum, similar to the one we have developed for composite boson excitons. Almost half a century ago, Richardson has succeeded to write down the *exact form* of the eigenstate for an arbitrary number N of pairs. It reads in terms of N energy-like quantities $R_1, ..., R_N$ which are solution of N coupled non-linear equations. This many-body problem is exactly solvable for an interaction potential between $2N$ electrons with up and down spins taken as a BCS-like reduced potential provided that scattering is separable, $V_{\mathbf{k}'\mathbf{k}} = -V\, w_{\mathbf{k}'} w_{\mathbf{k}}$ with $w_{\mathbf{k}}^2 = w_{\mathbf{k}}$. Such a separable potential is also required to get the energy of a single pair in a compact form, as obtained by Cooper. Richardson managed to extend the exact one-Cooper pair solution to N pairs by decoupling them: This is done in a quite smart manner by rewriting the N-pair energy E_N as $R_1 + \cdots + R_N$.

The new derivation we here propose for the equations fulfilled by $R_1, \cdots, R_N$, allows us to trace back the physical origin of the various terms in a transparent way. In particular, this derivation clearly shows that N pairs differ from N independent pairs, due to Pauli blocking only. This Pauli blocking enforces the R_i energy-like parameters of the Richardson's equations to be all different. As a direct consequence, the exact wave function for N interacting pairs is definitely different from the BCS ansatz, although the N-pair energy this ansatz gives is the same in the large sample limit.

The diagrammatic representation of this derivation nicely evidences that, because electron pairs with zero total momentum have one degree of freedom only, they scatter within the (1×1) BCS potential in the pair subspace, through (2×2) scatterings only. This makes clearer why the N-pair interaction energy that we have previously found, has interaction terms in $N(N-1)$ but not in $N(N-1)(N-2)$ and so on... as commonly expected in N-body problems.

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