A Coboson Derivation of Richardson Equations for Cooper pairs

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Five years after the milestone paper[1] by Bardeen, Cooper, Schrieffer in which superconductivity is tackled within the grand canonical ensemble, Richardson has succeeded to derive the form, within the canonical ensemble, of the *exact* eigenstate of the Schrödinger equation for an arbitrary number of Cooper pairs[2, 3]. We here rederive his result using the commutation technique that we have recently developed for many-body effects between composite bosons. This procedure makes crystal clear that, within the BCS potential, interactions between Cooper pairs, which make them different from a collection of single pairs, are solely due to the Pauli exclusion principle, through electron exchanges between pairs. It also gives hints on why, as we very recently found, the interaction part of the N pairs energy depends on the pair number as $N(N-1)$ only from the dilute to the dense regime of Cooper pairs, on the single pair scale . This exact result also questions the validity of the BCS wave function ansatz, that we here discuss.

Although it has been immediately noticed that the Pauli exclusion principle plays a key role in superconductivity, it is only quite recently that the precise way it transforms a collection of single Cooper pairs into a BCS condensate, has been understood. [**?**] This understanding goes through handling Cooper pairs not within the grand canonical ensemble as done in the standard BCS theory, but through the canonical ensemble. The handling of the Pauli exclusion principle between a fixed number of fermions is known to be a formidable problem. However, adding fermion pairs one by one is the unique way to possibly follow the increasing effect of Pauli blocking when the number of pairs increases.

Five years after the milestone paper on superconductivity by Bardeen, Cooper, Schrieffer, Richardson has derived the form of the exact eigenstate of the Schrödinger equation for a fixed number of Cooper pairs. In the case of N pairs, it reads in terms of N parameters, R_1,... R_N which are solutions of N coupled non-linear equations, the energy of these N pairs reading as $\epsilon_N = R_1 + ... + R_N$. Although this result is definitely quite smart, to use it in practice, is not that easy. Indeed, these solutions have no compact analytical solution, so that they are commonly approached numerically only. This is probably why they have not had so far the attention they deserve among the superconductor community. Nowadays, they are commonly used to derive properties of small superconducting particles with an accountable number of electron pairs.

Last year, we have found an analytical way to tackle these equations by turning to their dimensionless form. We then see that these equations do have a small parameter: It is $1/N_c$ where N_c is the number of Cooper pairs over which overlap starts. This allowed us to demonstrate in the dilute limit on the single Cooper pair scale, i.e., for N/N_c small, that the energy of N Cooper pairs reads as

$$\mathcal{E}_N = N((\left[\left(2\epsilon_{F_0} + \frac{N-1}{\rho_0} \right)) - \epsilon_c \left((1 - \frac{N-1}{N_\Omega} \right) \right])) \tag{1}$$

ϵ_{F_0} is the Fermi level of the Fermi sea $|F_0\rangle$ which does not feel the attractive potential, ρ_0 is the density of states, taken as constant within the potential layer. $N_\Omega = \rho_0\Omega$ is the number of pair states in this layer, Ω being the potential layer extension. $\epsilon_c \approx 2\Omega \exp{(-2/\rho_0 V)}$ is the single pair binding energy, the potential amplitude being V.

Although our present derivation impose N/N_c small, it is quite remarkable to note that their result is also valid in the dense BCS regime, where pairs strongly overlap. Indeed the first term of Eq. (1) is the exact energy of N pairs in a normal state, since it is nothing but

$$2\epsilon_{F_0} + (2\epsilon_{F_0} + 1/\rho_0) + \cdots\ + (2\epsilon_{F_0} + (N-1)/\rho_0) = \mathcal{E}_N^{(normal)} \tag{2}$$

For a number of pairs corresponding to fill half the potential layer, which is the precise BCS configuration, Eq. (1) gives a condensation energy equal to

$$\mathcal{E}_N - \mathcal{E}_N^{(normal)} = \frac{N_\Omega}{2}\frac{\epsilon_c}{2} = \frac{1}{2}\rho_0\Omega^2 e^{-2/\rho_0 V} \tag{3}$$

This result exactly matches the one derived within the grand canonical ensemble, namely $\rho_0\Delta^2/2$ where the gap Δ reads as $2\omega_c \exp{(-1/\rho_0 V)}$ since $2\omega_c$ is the potential layer extension Ω.

The canonical approach we have used to reach Eq.(1), based on the Richardson equations, has the great advantage to prove that, by contrast to a common belief, the Cooper pair binding energy decreases when N increases due to Pauli blocking. Indeed, it is commonly said that in the dense BCS configuration, the Cooper pair energy is of the order of the gap Δ, which is far larger than ϵ_c. This understanding is obtained by splitting the condensate energy $\rho_0\Delta^2/2$ as $(\rho_0\Delta)\Delta$ within an "irrelevant" 1/2 prefactor. This deliberatly assigns a pair energy equal to the gap, the number of pairs to fit the condensation energy then being the number of pair $\rho_0\Delta$ in a gap layer. These $\rho_0\Delta$ pairs actually are "virtual pairs", as named by Schrieffer. Their number is far smaller than the number of real pairs $N_\Omega/2$ feeling the potential. This obviously makes their energy far larger than the energy $\epsilon_c/2$ of the real pairs. Actually, these virtual pairs are associated to excitations across the Fermi level of a Fermi sea $|F\rangle$ having $N+N_0$ pairs, N_0 being the number of pairs in the frozen sea $|F_0\rangle$. This Fermi sea $|F\rangle$ for sure is a well defined but somewhat mathematical concept, so as the number $\rho_0\Delta$ of these virtual pairs compared to the number of real pairs in the potential layer. This makes a pair energy of the order of the gap more mathematical than real.

A very pictural way to understand the binding energy decreases when N increases, as evidenced in eq (1), is through the so-called "moth-eaten" effect. Indeed, when pairs are added to $|F_0\rangle$, they are "eating" one by one, like little moths, the number of states in the potential layer which are available to form a bound state. As a result of this available state decrease, the bound state energy can only decrease.

Since the key role of Pauli blocking in superconductivity is enlightened by the derivation of the N pair energy we have made, based on Richardson equations, it can be of interest to precisely see the parts in these equations which directly come from the Pauli exclusion principle. In our recent works on the many body physics of composite bosons, we have proposed a "commutation technique" which allows us to evidence the effects of Pauli blocking between the fermionic components of these composite bosons. They do appear through exchange on Pauli scatterings. These dimensionally Pauli scatterings, mixed with energy-like scatterings associated to interactions between fermionic components, allow us to deal with fermion exchanges between composite bosons (cobosons in short) in an exact way. For review on this formalism, and its applications to the many-body physics of semiconductor excitons, see ref [4, 5].

In this paper, we first develop such a commutation technique for up and down electron pairs with zero total momentum. We then use it to derive in a quite compact way, the form of the exact eigenstate for N pairs within the reduced BCS potential. The Richardson equations readily follow from this approach. Its main advantage is to possibly trace back in a transparent way, the terms in these equations which directly come from the Pauli exclusion principle: they are those in $R_i - R_j$. They actually come from the non zero value of Pauli scatterings for fermion exchanges.

The paper is organized as following:

In section I, we derive the commutation technique for free electron pairs and its associated Pauli and interaction scatterings.

In section II, we use this technique to get the form of the exact equation for $N = 2, 3, \cdots$ pairs interacting through the reduced BCS potential, in order to see how the solution for general N develops. We then analyze the increasing role Pauli blocking in these solutions.

In section III, we discuss possible connection between this exact solution and the well-known BCS ansatz.

I. COMMUTATION TECHNIQUE FOR FREE FERMION PAIRS

We consider cobosons made of free fermion pairs having zero total momentum.

$$\beta^+_{\mathbf{k}} = a^+_{\mathbf{k}} b^+_{-\mathbf{k}} \tag{4}$$

So that they only have one degree of freedom by contrast to the mostgeneral fermion pairs $a^+_{\mathbf{k}_1} b^+_{\mathbf{k}_2}$ which would have two. In the case of Cooper pairs, these fermions are up and down spin electrons. The operators $(a^+_{\mathbf{k}'},a^+_{\mathbf{k}})$ and $(b^+_{\mathbf{k}'},b^+_{\mathbf{k}})$ anticommute, while $a^+_{\mathbf{k}'}$ and $b^+_{\mathbf{k}}$ commute or anticommute depending if the two fermions have the same or a different nature. However, in both cases, the resulting fermion pair operators commute,

$$\left[\beta^+_{\mathbf{k}'}, \beta^+_{\mathbf{k}}\right] = 0 \tag{5}$$

It is worth noting that while $\left(a^+_{\mathbf{k}}\right)^2 = 0$ simply follows from the anticommutation of the $a^+_{\mathbf{k}}$ operators, the cancellation of $\left(\beta^+_{\mathbf{k}}\right)^2$ does not follow from eq (5), but from the fact that $\left(\beta^+_{\mathbf{k}}\right)^2$ contains $\left(a^+_{\mathbf{k}}\right)^2$. The cancellation of $\left(\beta^+_{\mathbf{k}}\right)^2$ coming from Pauli blocking, thus seems to be lost when turning from single electron operators to pair operators, we will however see that this blocking is yet preserve in the commutaion algebra of free fermion pairs.

If we now consider creation and annihilation operators, $\left[a_{\mathbf{k}'}, a^+_{\mathbf{k}}\right] = \delta_{\mathbf{k}'\mathbf{k}}$ leads to

$$\left[\beta_{\mathbf{k}'}, \beta^+_{\mathbf{k}}\right] = \delta_{\mathbf{k}'\mathbf{k}} - D_{\mathbf{k}'\mathbf{k}} \tag{6}$$

the deviation-from-boson operator $D_{\mathbf{k}'\mathbf{k}}$ is defined as

$$D_{\mathbf{k}'\mathbf{k}} = \delta_{\mathbf{k}'\mathbf{k}} \left(a^+_{\mathbf{k}\uparrow} a_{\mathbf{k}\uparrow} + a^+_{-\mathbf{k}\downarrow} a_{-\mathbf{k}\downarrow} \right) \tag{7}$$

This operator would reduce to zero if the bosons were elementary. This operator allows to generate the Pauli scatterings in fermion exchange. They are formally defined through

$$\left[D_{\mathbf{k}'_1\mathbf{k}_1}, \beta^+_{\mathbf{k}_2} \right] = \sum_{\mathbf{k}'_2} \left\{ \lambda \begin{pmatrix} \mathbf{k}'_2 & \mathbf{k}_2 \\ \mathbf{k}'_1 & \mathbf{k}_1 \end{pmatrix} + (\mathbf{k}'_1 \leftrightarrow \mathbf{k}'_2) \right\} \beta^+_{\mathbf{k}'_2} \tag{8}$$

By noting that

$$\left[a^+_{\mathbf{k}} a_{\mathbf{k}}, \beta^+_{\mathbf{p}} \right] = \delta_{\mathbf{kp}} \beta^+_{\mathbf{p}} = \left[b^+_{-\mathbf{k}} b_{-\mathbf{k}}, \beta^+_{\mathbf{p}} \right] \tag{9}$$

it is easy to show that

$$\left[D_{\mathbf{k}'_1\mathbf{k}_1}, \beta^+_{\mathbf{k}_2} \right] = 2\beta^+_{\mathbf{k}_2} \delta_{\mathbf{k}_1\mathbf{k}_2} \delta_{\mathbf{k}'_1,\mathbf{k}_2} \tag{10}$$

So that we are led to identify the Pauli scattering with a product of Kronecker symbols

$$\lambda \begin{pmatrix} \mathbf{k}'_2 & \mathbf{k}_2 \\ \mathbf{k}'_1 & \mathbf{k}_1 \end{pmatrix} = \delta_{\mathbf{k}'_1\mathbf{k}_1} \delta_{\mathbf{k}'_2\mathbf{k}_2} \delta_{\mathbf{k}_1\mathbf{k}_2} \tag{11}$$

Actually, this is just the value we expect. In the scattering associated to fermion exchanges between $(\mathbf{k}_1, \mathbf{k}_2)$ fermion pairs, as visualized by the diagram of fig (1a). Indeed from this diagram, it is clear that we must have $(\mathbf{k}'_1 = \mathbf{k}_1, \mathbf{k}'_2 = \mathbf{k}_2)$ and $(-\mathbf{k}'_2 = -\mathbf{k}_1, -\mathbf{k}'_1 = -\mathbf{k}_2)$ which reduces to $\delta_{\mathbf{k}'_1\mathbf{k}_1} \delta_{\mathbf{k}'_2\mathbf{k}_2} \delta_{\mathbf{k}_1\mathbf{k}_2}$ in agreement with eq (11).

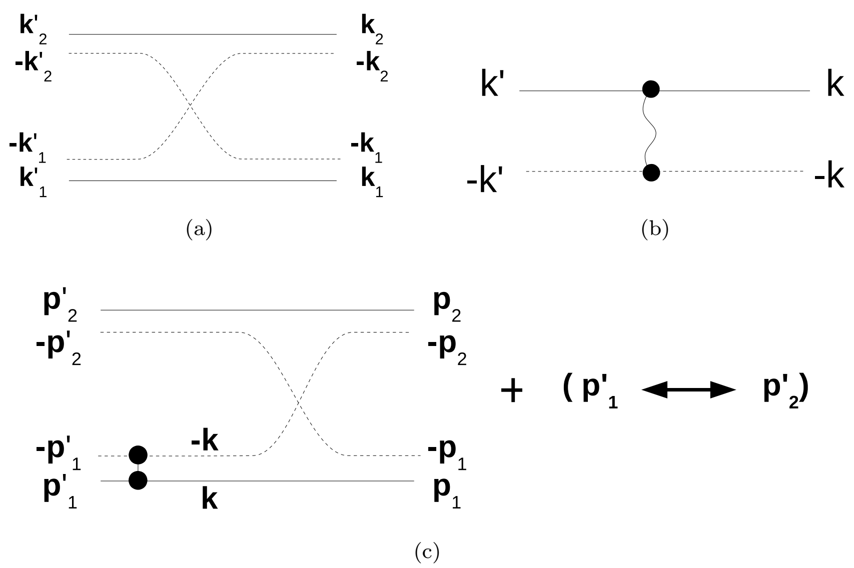

FIG. 1: Shiva diagram of free pairs

(a) Pauli scattering $\lambda \begin{pmatrix} \mathbf{p}'_2 & \mathbf{p}_2 \\ \mathbf{p}'_1 & \mathbf{p}_1 \end{pmatrix}$ for electron exchange between two free pairs $(\mathbf{p}_1, \mathbf{p}_2)$, as given by eq. (11).

(b) The BCS potential given in eq. (14) transforms a $\mathbf{k}$ pair into a $\mathbf{k}'$ pair, with a constant scattering $-V$, in the case of separable potential $v_{\mathbf{k}'\mathbf{k}} = -V\, w_{\mathbf{k}'} w_{\mathbf{k}}$. Up spin electrons are represented by solid line while down spin electrons are represented by dashed line.

(c) Interaction scattering $\chi \begin{pmatrix} \mathbf{p}'_2 & \mathbf{p}_2 \\ \mathbf{p}'_1 & \mathbf{p}_1 \end{pmatrix}$ between two free electron pairs, as given in eq (19). Since the BCS potential acts within one pair only, the interaction between two pairs come from exchange induced by the Pauli exclusion principle.

To get the interaction scatterings associated to fermion interaction, we first note that for a free hamiltonian

$$H_0 = \sum \epsilon_{\mathbf{k}} \left(a^+_{\mathbf{k}} a_{\mathbf{k}} + b^+_{\mathbf{k}} b_{\mathbf{k}} \right) \tag{12}$$

eq (9) leads to

$$\left[H_0, \beta_{\mathbf{p}}^+\right] = 2\epsilon_{\mathbf{p}}\beta_{\mathbf{p}}^+ \tag{13}$$

We have considered that these fermion pairs interact through a BCS like 1×1 potential in which the fermion $\mathbf{k}$ only interacts with one fermion $(-\mathbf{k})$ of the other species

$$V_{BCS} = \sum v_{\mathbf{k}'\mathbf{k}}\beta_{\mathbf{k}'}^+\beta_{\mathbf{k}} \tag{14}$$

as shown in the diagram of fig 1b. For this potential, we then have

$$\left[V_{BCS}, \beta_{\mathbf{p}}^+\right] = \gamma_{\mathbf{p}}^+ + V_{\mathbf{p}}^+ \tag{15}$$

$\gamma_{\mathbf{p}}^+ = \sum_{\mathbf{k}} \beta_{\mathbf{k}}^+ v_{\mathbf{kp}}$. The "creation potential" for the free fermion pair $\mathbf{p}$ appears to be

$$V_{\mathbf{p}}^+ = -\gamma_{\mathbf{p}}^+ \left(a_{\mathbf{p}}^+ a_{\mathbf{p}} + b_{-\mathbf{p}}^+ b_{-\mathbf{p}}\right) \tag{16}$$

While the $\gamma_{\mathbf{p}}^+$ part of eq (14) commutes with $\beta_{\mathbf{p}}^+$, this is not so for the creation potential $V_{\mathbf{p}}^+$. Its commutator precisely reads

$$\left[V_{\mathbf{p}_1}^+, \beta_{\mathbf{p}_2}^+\right] = -2\delta_{\mathbf{p}_1\mathbf{p}_2}\gamma_{\mathbf{p}_1}^+\beta_{\mathbf{p}_1}^+ \tag{17}$$

This allows us to identify the interaction scattering in free pairs, formally defined as

$$\left[V_{\mathbf{p}_1}^+, \beta_{\mathbf{p}_2}^+\right] = \sum \chi\begin{pmatrix} \mathbf{p}_2' & \mathbf{p}_2 \\ \mathbf{p}_1' & \mathbf{p}_1 \end{pmatrix} \beta_{\mathbf{p}_1'}^+\beta_{\mathbf{p}_2'}^+ \tag{18}$$

with a sequence of 2×2 fermion pair exchanged and 1×1 fermion pair interaction. Indeed

$$\begin{aligned} \chi\begin{pmatrix} \mathbf{p}_2' & \mathbf{p}_2 \\ \mathbf{p}_1' & \mathbf{p}_1 \end{pmatrix} &= -\sum_{\mathbf{k}} \left\{ v_{\mathbf{p}_1'\mathbf{k}}\lambda\begin{pmatrix} \mathbf{p}_2' & \mathbf{p}_2 \\ \mathbf{k} & \mathbf{p}_1 \end{pmatrix} + (\mathbf{p}_1' \leftrightarrow \mathbf{p}_2') \right\} \\ &= -\left(v_{\mathbf{p}_1',\mathbf{p}_1}\delta_{\mathbf{p}_2',\mathbf{p}_2} + v_{\mathbf{p}_2',\mathbf{p}_2}\delta_{\mathbf{p}_1',\mathbf{p}_1}\right)\delta_{\mathbf{p}_2,\mathbf{p}_1} \end{aligned} \tag{19}$$

This interaction scattering is visualized by the diagram of fig 1c: the free pairs $\mathbf{p}_1'$ and $\mathbf{p}_2'$ first exchange a fermion. As for any exchange, this brings a minus sign. In a second step, the fermions of one of the two pairs interact via the BCS potential. Note that since the potential has a 1×1 structure, the 2×2 interaction between two pairs can only result from fermion exchange, i.e., Pauli blocking.

We are now going to use this commutation formalism to derive the Richardson equations from Cooper pairs.

II. RICHARDSON EQUATION FOR COOPER PAIRS

In order to better grasp how these equations develop, let us consider an increasing number of pairs.

A. One pair

We consider a state in which one free pair $\mathbf{k}_1$ is added to a frozen Fermi sea $|F_0\rangle$ which does not feel the BCS potential. This means that the $v_{\mathbf{k}'\mathbf{k}}$ prefactors in eq (14) cancel for all $\mathbf{k}$ belonging to $|F_0\rangle$. Note that this so-called one-pair state actually contains $N_0 + 1$ fermion pairs, N_0 being the number of pairs in the frozen sea. So that this state is in fact a many-body state, but in the most simple sense since the Fermi sea $|F_0\rangle$ is just there to block states by the Pauli exclusion principle. It also to bring a finite density of state above it, as crucial to have a bound state, even for an extremely small attracting BCS potential.

Due to eqs (13,15), the hamiltonian $H = H_0 + V_{BCS}$ acting on this one free pair state gives, by taking the zero energy through $H\,|F_0\rangle = 0$

$$H\beta_{\mathbf{k}}^+\,|F_0\rangle = \left[H, \beta_{\mathbf{k}}^+\right]|F_0\rangle = \left(2\epsilon_{\mathbf{k}}\beta_{\mathbf{k}}^+ + \gamma_{\mathbf{k}}^+ + V_{\mathbf{k}}^+\right)|F_0\rangle \tag{20}$$

we then note that, due to the $v_{\mathbf{kp}}$ factor included in the $\gamma_{\mathbf{k}}^+$ part of $V_{\mathbf{k}}^+$ (see eq 16), the creation potential $V_{\mathbf{k}}^+$ acting on $|F_0\rangle$ gives zero.

If we now substitute $\epsilon_{\mathbf{k}}\beta^+_{\mathbf{k}}\,|F_0\rangle$ to the two sides of the above equation and multiply the reesult by $(2\epsilon_{\mathbf{k}}-E_1)^{-1}$, we find

$$(H-E_1)\frac{1}{2\epsilon_{\mathbf{k}}-E_1}\beta^+_{\mathbf{k}}\,|F_0\rangle=\beta^+_{\mathbf{k}}\,|F_0\rangle+\frac{1}{2\epsilon_{\mathbf{k}}-E_1}\sum\mathcal{V}_{\mathbf{kp}}\beta^+_{\mathbf{p}}\,|F_0\rangle \tag{21}$$

To go further and possibly get the one-pair eigenstate of the hamiltonian H in an analytical form, we must approximate the BCS potential by a separable potential $\mathcal{V}_{\mathbf{kp}}=-V\,w_{\mathbf{k}}w_{\mathbf{p}}$, the $w_{\mathbf{k}}$'s being in addition such that $w^2_{\mathbf{k}}=w_{\mathbf{k}}$. This yeilds

$$\gamma^+_{\mathbf{k}}=-V\,w_{\mathbf{k}}\beta^+\qquad\beta^+=\sum_{\mathbf{k}}w_{\mathbf{k}}\beta^+_{\mathbf{k}} \tag{22}$$

If we then multiply eq (21) by $w_{\mathbf{k}}$ and sum over $\mathbf{k}$, we find

$$(H-E_1)B^+(E_1)\,|F_0\rangle=\left(1-V\sum\frac{w_{\mathbf{k}}}{2\epsilon_{\mathbf{k}}-E_1}\right)\beta^+\,|F_0\rangle \tag{23}$$

in which we have set

$$B^+_{\mathbf{k}}(E)=\frac{w_{\mathbf{k}}}{2\epsilon_{\mathbf{k}}-E}\beta^+\qquad B^+(E)=\sum_{\mathbf{k}}B^+_{\mathbf{k}}(E) \tag{24}$$

Eq (22) readily shows that the linear combination of one-pair operator $B^+(E_1)$ generate the one-pair eigenstate $B^+(E_1)\,|F_0\rangle$ of the hamiltonian H into the energy E_1, provided that this energy is such that

$$1=V\sum_{\mathbf{k}}\frac{w_{\mathbf{k}}}{2\epsilon_{\mathbf{k}}-E_1} \tag{25}$$

This is nothing but the well-known equation for the simple pair energy derived by Cooper.

B. Two pairs

Let us consider two pairs. Eqs (13,15) then yield

$$\begin{aligned}H\beta^+_{\mathbf{k}_1}\beta^+_{\mathbf{k}_2}\,|F_0\rangle&=\left\{\left[H,\beta^+_{\mathbf{k}_1}\right]\beta^+_{\mathbf{k}_2}+\beta^+_{\mathbf{k}_1}\left[H,\beta^+_{\mathbf{k}_2}\right]\right\}|F_0\rangle\\&=\left(2\epsilon_{\mathbf{k}_1}+2\epsilon_{\mathbf{k}_2}\right)\beta^+_{\mathbf{k}_1}\beta^+_{\mathbf{k}_2}\,|F_0\rangle+|\mathcal{V}_{\mathbf{k}_1\mathbf{k}_2}\rangle\end{aligned} \tag{26}$$

where $|\mathcal{V}_{\mathbf{k}_1\mathbf{k}_2}\rangle$ comes from interactions among the $(\mathbf{k}_1,\mathbf{k}_2)$ pairs induced by the BCS potential. Its precise value is

$$|\mathcal{V}_{\mathbf{k}_1\mathbf{k}_2}\rangle=\left(\gamma^+_{\mathbf{k}_1}\beta^+_{\mathbf{k}_2}+\gamma^+_{\mathbf{k}_2}\beta^+_{\mathbf{k}_1}+V^+_{\mathbf{k}_1}\beta^+_{\mathbf{k}_2}\right)|F_0\rangle \tag{27}$$

Eq (19) allows us to write the last term of $|\mathcal{V}_{\mathbf{k}_1\mathbf{k}_2}\rangle$ as

$$V^+_{\mathbf{k}_1}\beta^+_{\mathbf{k}_2}\,|F_0\rangle=\left[V^+_{\mathbf{k}_1},\beta^+_{\mathbf{k}_2}\right]|F_0\rangle=\sum_{\mathbf{p}'_1\mathbf{p}'_2}\chi\begin{pmatrix}\mathbf{p}'_2&\mathbf{k}_2\\\mathbf{p}'_1&\mathbf{k}_1\end{pmatrix}\beta^+_{\mathbf{p}'_1}\beta^+_{\mathbf{p}'_2}\,|F_0\rangle \tag{28}$$

So that $|\mathcal{V}_{\mathbf{k}_1\mathbf{k}_2}\rangle$ can be visualized by the diagram of fig 2. This diagram evidences the fact that, due to the 1×1 form of the BCS potential, the two pairs $\mathbf{k}_1$ and $\mathbf{k}_2$ interact by fermion exchange only, as a result of the Pauli exclusion principle.

By using the value of the interaction scattering given in eq (17), the $|\mathcal{V}_{\mathbf{k}_1\mathbf{k}_2}\rangle$ ultimately appears

$$|\mathcal{V}_{\mathbf{k}_1\mathbf{k}_2}\rangle=-V\left(w_{\mathbf{k}_1}\beta^+_{\mathbf{k}_2}+w_{\mathbf{k}_2}\beta^+_{\mathbf{k}_1}+2\delta_{\mathbf{k}_1\mathbf{k}_2}w_{\mathbf{k}_1}\beta^+_{\mathbf{k}_1}\right)\beta^+\,|F_0\rangle \tag{29}$$

To go further, we substitute $E_2\beta^+_{\mathbf{k}_1}\beta^+_{\mathbf{k}_2}\,|F_0\rangle$ to the two sides of eq (26), with E_2 written as R_1+R_2 and we multiply the resulting equation by $w_{\mathbf{k}_1}w_{\mathbf{k}_2}/\left(2\epsilon_{\mathbf{k}_1}-R_1\right)\left(2\epsilon_{\mathbf{k}_2}-R_2\right)$. This gives

$$(H-E_2)B^+_{\mathbf{k}_1}(R_1)B^+_{\mathbf{k}_2}(R_2)\,|F_0\rangle=\left\{B^+_{\mathbf{k}_1}(R_1)\left(w_{\mathbf{k}_2}\beta^+_{\mathbf{k}_2}-\frac{Vw_{\mathbf{k}_2}}{2\epsilon_{\mathbf{k}_2}-R_2}\beta^+\right)+1\leftrightarrow2\right\}|F_0\rangle \tag{30}$$

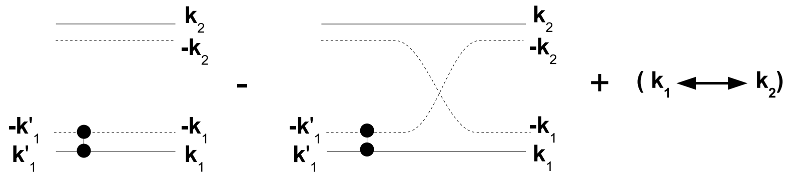

FIG. 2: Shiva diagram of two pairs

To go further, we note that $(2\epsilon_{\mathbf{k}_1}-R_1)^{-1}(2\epsilon_{\mathbf{k}_2}-R_2)^{-1}$ also reads $\left[(2\epsilon_{\mathbf{k}_1}-R_1)^{-1}-(2\epsilon_{\mathbf{k}_2}-R_2)^{-1}\right]/(R_1-R_2)$ provided that $R_1\neq R_2$. By taking sums over $\mathbf{k}_1$ and $\mathbf{k}_2$, the above equation then gives

$$(H-E_2)B^+(R_1)B^+(R_2)\left|F_0\right\rangle=\left\{B^+(R_1)\left(1-V\sum\frac{w_{\mathbf{k}}}{2\epsilon_{\mathbf{k}}-R_2}+\frac{2V}{R_1-R_2}\right)+(1\leftrightarrow 2)\right\}\beta^+\left|F_0\right\rangle \tag{31}$$

This readily shows that the two-pair state $B^+(R_1)B^+(R_2)\left|F_0\right\rangle$ is eigenstate of the hamiltonian with the energy $E_2=R_1+R_2$ provided that (R_1,R_2) fulfill two equations, known as Richardson equations for two pairs.

$$1=V\sum\frac{w_{\mathbf{k}}}{2\epsilon_{\mathbf{k}}-R_1}+\frac{2V}{R_1-R_2}=(1\leftrightarrow 2) \tag{32}$$

C. Three pairs

We now turn to three pairs to see how these equations develop for an increasing number of pairs, we start with

$$H\beta^+_{\mathbf{k}_1}\beta^+_{\mathbf{k}_2}\beta^+_{\mathbf{k}_3}\left|F_0\right\rangle=\left\{\left[H,\beta^+_{\mathbf{k}_1}\right]\beta^+_{\mathbf{k}_2}\beta^+_{\mathbf{k}_3}+\beta^+_{\mathbf{k}_1}\left[H,\beta^+_{\mathbf{k}_2}\right]\beta^+_{\mathbf{k}_3}+\beta^+_{\mathbf{k}_1}\beta^+_{\mathbf{k}_2}\left[H,\beta^+_{\mathbf{k}_3}\right]\right\}\left|F_0\right\rangle \tag{33}$$

The same eqs (13,15) give

$$H\beta^+_{\mathbf{k}_1}\beta^+_{\mathbf{k}_2}\beta^+_{\mathbf{k}_3}\left|F_0\right\rangle=(2\epsilon_{\mathbf{k}_1}+2\epsilon_{\mathbf{k}_2}+2\epsilon_{\mathbf{k}_3})\,\beta^+_{\mathbf{k}_1}\beta^+_{\mathbf{k}_2}\beta^+_{\mathbf{k}_3}\left|F_0\right\rangle+\left|v_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\right\rangle \tag{34}$$

where the part resulting from the BCS potential appears as

$$\begin{aligned}\left|v_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\right\rangle&=\left(\gamma^+_{\mathbf{k}_1}\beta^+_{\mathbf{k}_2}\beta^+_{\mathbf{k}_3}+\gamma^+_{\mathbf{k}_2}\beta^+_{\mathbf{k}_3}\beta^+_{\mathbf{k}_1}+\gamma^+_{\mathbf{k}_3}\beta^+_{\mathbf{k}_1}\beta^+_{\mathbf{k}_2}\right)\left|F_0\right\rangle\\&\quad+\left(V^+_{\mathbf{k}_1}\beta^+_{\mathbf{k}_2}\beta^+_{\mathbf{k}_3}+\beta^+_{\mathbf{k}_1}V^+_{\mathbf{k}_2}\beta^+_{\mathbf{k}_3}+\beta^+_{\mathbf{k}_1}\beta^+_{\mathbf{k}_2}V^+_{\mathbf{k}_3}\right)\left|F_0\right\rangle\end{aligned} \tag{35}$$

The last term of the second vector gives zero since $V^+_{\mathbf{k}}\left|F_0\right\rangle=0$. Using eq (18), the two remaining terms of the second vector can be rewritten as

$$\begin{aligned}&\left\{\left[V^+_{\mathbf{k}_1},\beta^+_{\mathbf{k}_2}\right]\beta^+_{\mathbf{k}_3}+\beta^+_{\mathbf{k}_2}\left[V^+_{\mathbf{k}_1},\beta^+_{\mathbf{k}_3}\right]+\beta^+_{\mathbf{k}_1}\left[V^+_{\mathbf{k}_2},\beta^+_{\mathbf{k}_3}\right]\right\}\left|F_0\right\rangle\\&=\sum_{vk'_1\mathbf{k}'_2}\beta^+_{\mathbf{k}'_1}\beta^+_{\mathbf{k}'_2}\left\{\chi\begin{pmatrix}\mathbf{k}'_2&\mathbf{k}_2\\\mathbf{k}'_1&\mathbf{k}_1\end{pmatrix}\beta^+_{\mathbf{k}_3}+\chi\begin{pmatrix}\mathbf{k}'_2&\mathbf{k}_3\\\mathbf{k}'_1&\mathbf{k}_2\end{pmatrix}\beta^+_{\mathbf{k}_1}+\chi\begin{pmatrix}\mathbf{k}'_2&\mathbf{k}_1\\\mathbf{k}'_1&\mathbf{k}_3\end{pmatrix}\beta^+_{\mathbf{k}_2}\right\}\left|F_0\right\rangle\end{aligned} \tag{36}$$

This leads to represent the vector $\left|v_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\right\rangle$ by the diagram of fig 3. This interaction term results from the interactions inside a single pair, with in addition to a possible exchange with a second pair, the third pair then staying unchanged.

If we come back to eq (34), substitute $E_3\beta^+_{\mathbf{k}_1}\beta^+_{\mathbf{k}_2}\beta^+_{\mathbf{k}_3}\left|F_0\right\rangle$ to both sides, with E_3 written as $R_1+R_2+R_3$, and multiply the resulting equation by $w_{\mathbf{k}_1}w_{\mathbf{k}_2}w_{\mathbf{k}_3}/\,(2\epsilon_{\mathbf{k}_1}-R_1)\,(2\epsilon_{\mathbf{k}_2}-R_2)\,(2\epsilon_{\mathbf{k}_3}-R_3)$, we find

$$\begin{aligned}(H-E_3)B^+_{\mathbf{k}_1}(R_1)B^+_{\mathbf{k}_2}(R_2)B^+_{\mathbf{k}_3}(R_3)\left|F_0\right\rangle&=\left\{B^+_{\mathbf{k}_1}(R_1)B^+_{\mathbf{k}_2}(R_2)\left(w_{\mathbf{k}_3}\beta^+_{\mathbf{k}_3}-\frac{Vw_{\mathbf{k}_3}}{2\epsilon_{\mathbf{k}_2}-R_3}\beta^+\right)+2\text{ perm}\right\}\left|F_0\right\rangle\\&\quad+2V\beta^+\left\{B^+_{\mathbf{k}_3}(R_3)\frac{\delta_{\mathbf{k}_1\mathbf{k}_2}w_{\mathbf{k}_1}}{(2\epsilon_{\mathbf{k}_1}-R_1)\,(2\epsilon_{\mathbf{k}_1}-R_2)}\beta^+_{\mathbf{k}_1}+2\text{ perm}\right\}\left|F_0\right\rangle\end{aligned} \tag{37}$$

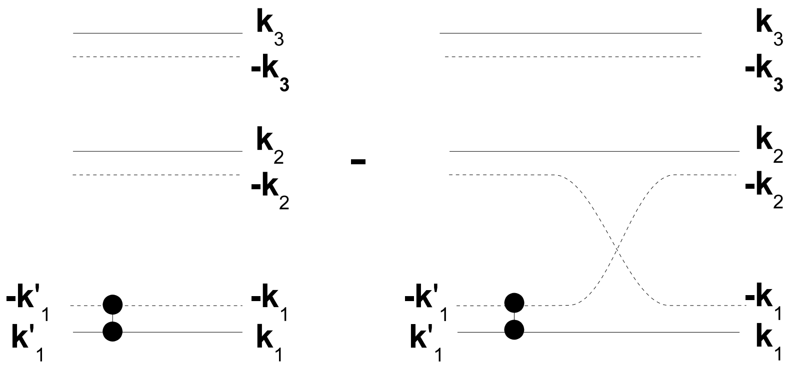

FIG. 3: Shiva diagram of two pairs
???? resulting from the BCS potential acting on three pairs, as given in eqs (35,36). This fact also contain two similar contribution as the one visualized in the figure, obtained by circular permutation.

To proceed, we rewrite $(2\epsilon_{\mathbf{k}_1}-R_1)^{-1}(2\epsilon_{\mathbf{k}_2}-R_2)^{-1}$ as $\left[(2\epsilon_{\mathbf{k}_1}-R_1)^{-1}-(2\epsilon_{\mathbf{k}_2}-R_2)^{-1}\right]/(R_1-R_2)$ provided that $R_1\neq R_2$ and do the same for the two other terms. By taking the sum over $(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$, we end with

$$\begin{aligned}(H-E_3)B^+(R_1)B^+(R_2)B^+(R_3)\left|F_0\right\rangle = \\ \beta^+\left\{B^+(R_2)B^+(R_3)\left(1-V\sum\frac{w_{\mathbf{k}_1}}{2\epsilon_{\mathbf{k}_1}-R_1}-\frac{2V}{R_1-R_2}+\frac{2V}{R_3-R_1}\right)+2\text{ perm}\right\}\beta^+\left|F_0\right\rangle\end{aligned} \quad (38)$$

This leads us to again conclude that the three-pair state $B^+(R_1)B^+(R_2)B^+(R_3)\left|F_0\right\rangle$ is eigenstate of the hamiltonian with the energy $E_3=R_1+R_2+R_3$, provided that (R_1,R_2,R_3) fulfill three equations,

$$\begin{aligned}1&=V\sum\frac{w_{\mathbf{k}}}{2\epsilon_{\mathbf{k}}-R_1}+\frac{2V}{R_1-R_2}+\frac{2V}{R_1-R_3}\\1&=V\sum\frac{w_{\mathbf{k}}}{2\epsilon_{\mathbf{k}}-R_2}+\frac{2V}{R_2-R_3}+\frac{2V}{R_2-R_1}\\1&=V\sum\frac{w_{\mathbf{k}}}{2\epsilon_{\mathbf{k}}-R_3}+\frac{2V}{R_3-R_1}+\frac{2V}{R_3-R_2}\end{aligned} \quad (39)$$

D. N pairs

The above commutator technique can be easily generated to N pairs. As nicely visualized by the diagrams of figs 2 and 3, the effect of the BCS potential on these N pairs splits into two sets of terms: In one set, one pair is affected by the 1×1 scattering while the other $N-1$ pairs stay unchanged. In the other, these pair in addition has a fermion exchange before the interaction, with an other pair, the remaining $N-2$ pairs staying unchanged. So that to increase the pair number above two, does not really change the structure of the equations since $N-2$ pairs stay unchanged, the pair exchanging its fermions using the pair scattering ???? interaction begin just one among those $(N-1)$ pairs.

Although the equations become more and more cumbersome to be explicitly written, the procedure is rather ???? once we have understood that either $(N-1)$ or $(N-2)$ pairs stay unaffected in the process. The general form of the N-pair eigenstate ultimately appears as

$$(H-E_N)B^+(R_1)\cdots B^+(R_N)\left|F_0\right\rangle=0 \quad (40)$$

with $E_3=R_1+\cdots+R_N$, these R_N being the solutions of N equations like

$$1=V\sum\frac{w_{\mathbf{k}}}{2\epsilon_{\mathbf{k}}-R_i}+\sum_{i\neq j}\frac{2V}{R_i-R_j}\qquad\text{for } i=(1,...,N) \quad (41)$$

E. Physical understanding

This new derivation of the Richardson equation has the main advantage to possibly trace back the parts in these equations which are directly linked to the Pauli exclusion principle between fermion pairs.

Form a mathematical point of view, the link is rather obvious: In the absence of terms in $(R_i - R_j)$, the N equations for R_i reduced to the same equation (25) so that the result would be $R_i^0 = E_1$ for all i. The fact that the energy of N pairs differs from N times the single pair energy thus comes from those $(R_i - R_j)$ difference.

Physically, the fact that E_N differs from NE_1 comes from interactions between pairs. Due to the 1×1 form of the BCS potential, interaction between pairs can only be mediated by fermion exchange as clear from fig 1c. Interaction between pairs thus is solely the result of the Pauli exclusion principle between pairs. This Pauli blocking mathematically appears through the varies $\delta_{\mathbf{p}'\mathbf{p}}$ factors seen in Pauli scattering $\lambda \begin{pmatrix} \mathbf{p}_2' & \mathbf{p}_2 \\ \mathbf{p}_1' & \mathbf{p}_1 \end{pmatrix}$. It is then easy to mathematically trace back the $(R_i - R_j)$ difference of the Richardson equations to these δ factors.

In short, the Kronecker symbols with Pauli scattering's of the fermion pairs result from the Pauli exclusion principle. They induce terms with $(R_i - R_j)$ differences in the Richardson equations which make the energy of N pairs different from the one of a collection of N independent single pairs.

Another very important feature of the energy E_N of N pairs, this new derivation explains in a rather clear way, is the fact that the part of the N pairs energy coming from interaction, namely $E_N - NE_1$ depends on N as $N(N-1)$ only. Indeed, diagram 3 evidences that the contribution of the 1×1 BCS potential and fermion exchanges between pairs having one degree of freedom only, ends by ????? effective scatterings which are 2×2 only. Since in order to have terms in $N(N-1)(N-2)$, we need to ????? interaction processes between 3 objects, $N(N-1)(N-2)$ terms as well as all the higher order terms, cannot exist in the energy of N Cooper pairs.

This actually is ??? we have found by solving these equations analytically in the dilute limit on the single Cooper pair size. In this limit, the energy of N pairs was shown to read as

$$E_N = NE_1 + N(N-1)\left(\frac{1}{\rho_0} + \frac{\epsilon_c}{N_\Omega}\right) \tag{42}$$

ρ_0 is the density of pair states in the potential layer, $N_\Omega = \rho_0 \Omega$ is number of states in the layer, and ϵ_c is the single pair ??? energy. By writing it as $\epsilon_c = N_\Omega \epsilon_V$, this energy also reads, for $E_1 = 2\epsilon_{F_0} - \epsilon_c$ where ϵ_{F_0} is the Fermi level of the frozen sea $|F_0\rangle$

$$E_N = 2N\left[\epsilon_{F_0} + \frac{N(N-1)}{\rho_0}\right] - N\epsilon_V\left(N_V - \frac{N-1}{N_\Omega}\right) \tag{43}$$

The first term corresponds to the kinetic energy of N pairs added to ϵ_{F_0}, i.e.,

$$2\epsilon_{F_0} + (2\epsilon_{F_0} + 1/\rho_0) + \cdots + (2\epsilon_{F_0} + (N-1)/\rho_0) \tag{44}$$

The second term evidences the fact that the Cooper pair binding energy linearly decreases with piar number, this energy being proportional to the number of empty states $N_V - (N-1)$ filling the potential.

This brings the binding energy down to $\epsilon_c/2$ in the BCS configuration, i.e., when pairs fill half the potential layer. Actually, the result energy appears with the BCS condensation energy, ??? to read

$$E_{\text{super}} - E_{\text{normal}} = \frac{1}{2}\rho_0 \Delta^2 = \frac{\rho_0 \Omega}{2} \frac{2\Omega e^{-2/\rho_0 V}}{2} \tag{45}$$

In spite of the fact that eq (42) has up to now been derived within the dilute limit only. It turns out that it is valid over the whole density range. Thus validity is a base result of the existence of 2×2 scatterings only between fermion pairs, this argument having nothing to do with the pair density layer is small on the single Cooper pair scale.

III. RICHARDSON EXACT EIGENSTATE VERSUS BCS ANSATZ

Another very interesting result the Richardson procedure generates is the *exact* form of the eigenstate, namely

$$B^+(R_1) \cdots B^+(R_N)\,|F_0\rangle \tag{46}$$

with $B^+(R)$ given in eq (24). The fact that by construction all the R_i's are different, strongly questions the standard BCS ansatz. In Cooper pair wave function $(B^+)^N\,|F_0\rangle$ with *all* the pairs condensed into the same state.

To discuss this problem on present framework, let us start with two pairs. In a previous work[6], we have shown, that the two "Richardson enegies" then read $R_1 = R + iR'$ and $R_2 = R - iR'$ with R and R' real. Their previous value being $R \approx \epsilon_c + 1/\rho_0 + \epsilon_c/N\Omega$ and $R' = \sqrt{2\epsilon_c/\rho_0}$ in the small sample limit, i.e. for $1/\rho_0$ small. By noting that

$$B^+(R_1)B^+(R_2) = \left[B^+(R) + B^+(R_1) - B^+(R)\right]\left[B^+(R) + B^+(R_2) - B^+(R)\right] \tag{47}$$

we get from eq (24)

$$B^+(R_1)B^+(R_2) = \left[B^+(R)\right]^2 + R'^2\left\{C_+^+C_-^+ - 2B^+(R)D^+\right\} \tag{48}$$

where we have set

$$C_\pm^+ = \sum \frac{w_{\mathbf{k}}}{(2\epsilon_{\mathbf{k}} - R)\,(2\epsilon_{\mathbf{k}} - R \pm iR')}\beta_{\mathbf{k}}^+ \tag{49}$$

$$D^+ = \sum \frac{w_{\mathbf{k}}}{(2\epsilon_{\mathbf{k}} - R)\left[(2\epsilon_{\mathbf{k}} - R)^2 + R'^2\right]}\beta_{\mathbf{k}}^+ \tag{50}$$

So that at first ??? is sample volume, i.e., ?? $1/\rho_0$, we find since $N_\Omega = \rho_0\Omega$

$$B^+(R_1)B^+(R_2) - \left[B^+(\frac{E_2}{2})\right]^2 \approx \frac{2\epsilon_c}{\rho_0}\left\{-2B^+(E_1)\sum\frac{w_{\mathbf{k}}}{(2\epsilon_{\mathbf{k}} - E_1)^3}\beta_{\mathbf{k}}^+ + \left[\sum\frac{w_{\mathbf{k}}}{(2\epsilon_{\mathbf{k}} - E_1)^2}\beta_{\mathbf{k}}^+\right]^2\right\} + O(\frac{1}{\rho_0^2}) \tag{51}$$

the above result shows that $B^+(R_1)B^+(R_2)$ can be written as $(B^+(E_2/2))^2$ provided that we drop all $1/\rho_0$ terms. But B^+ then reduces to $B^+(E_1)$. This corresponds to consider the two-pair eigenstate as the product of two non-interacting single pairs. If instead, we want to, into the condensed pair creative operator, include the change from one to two pairs induced by Pauli blocking while brings the energy per pair from E_1 to $E_2/2 = E_1 + 1/\rho_0 + \epsilon_c/N\Omega$, we are led to replace $B^+(R_1)B^+(R_2)$ by $(B^+(E_2/2))^2$. This however is inconsistent because we then keep in this condensed pair operator contribution in $1/\rho_0$ which are as large as the one we drop by the LHS of eq (51). In the case of two pairs, the replacement of the exact eigenstate $B^+(R_1)B^+(R_2)\,|F_0\rangle$ by a BCS-like condensed state $(B^+(E_2/2))^2\,|F_0\rangle$ thus is inconsistent.

It is actually claimed that the BCS ansatz is valid in the thermodynamical limit. Derivation of the "validity" is in fact with restrict to the energy only. We fully agree that the BCS ansatz give the correct energy since the energy obtained using this ansatz is just the one we have derived from the exact Richardson procedure. However agreement on the energy by no mean proves agreement on the wave function. Many examples have been given in the past with wave function very different from the exact one, while giving the correct energy.

The possible replacement of $B^+(R_1)\cdots B^+(R_N)\,|F_0\rangle$ by $(B^+)^N\,|F_0\rangle$ is actually crucial to support the overall picture we all have of superconductivity, with all the pairs in the same state, as an army of little solders, all walking similarly.

This question of the BCS ansatz in the wave function has the approached in a different way by Bogoliubox. Let us here reconsider his argument.

IV. CONCLUSION

We have rederived the Richardson equations using a commutation technique for free electron pairs with zero total momentum similar to the one we have developed for composite boson excitons. Almost half a century ago, Richardson has shown that the *exact* wave-function and energy for an arbitrary number N of pairs can be written in a compact form in terms of N energy-like quantities $R_1, ..., R_N$, which are solution of N coupled non-linear equations. This $2N$ many-body problem is exactly solvable provided that the interaction potential is taken as a BCS-like $|x|$ potential having a separable scattering $v_{\mathbf{k'k}} = -V\,w_{\mathbf{k'}}w_{\mathbf{k}}$ with $w_{\mathbf{k}}$?????? such that $w_{\mathbf{k}}^2 = 1$. Note that these assumptions are already those necessary to get the energy of a single pair in the compact form obtained by Cooper. Richardson manged to extend this exact solution to N pairs by decoupling them through rewriting their energy E_N as $R_1 + \cdots + R_N$.

The new composite boson derivation we have proposed allows to trace back the physical origin of the various terms of these equations. It in particular clearly shows that N pairs differ from N independent single pairs, due to Pauli exclusion principle only. This Pauli blocking also enforces the R_i energy-like parameters to be different, namely the exact N-pair eigenstate different from the BCS ansatz. ?????? the diagrammatic representation of this derivation evidences that, due to the fact that pairs with zero total momentum, do have one degree of freedom only, they only have 2×2 scatterings within the 1×1 BCS potential. This explains why the N pair energy has terms in N and $N(N-1)$ but not in $N(N-1)(N-2)$ and so on.

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