A Coboson Derivation of Richardson Equations for Cooper pairs

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Five years after the milestone paper by Bardeen, Cooper, Schrieffer in which superconductivity is tackled within the grand canonical ensemble, Richardson has approached the problem within the canonical ensemble: he succeeded to write the *exact* form of the Schrödinger equation eigenstate for an arbitrary number of Cooper pairs interacting through the standard BCS potential. We here rederive his result using the commutation technique we have recently developed for many-body effects between composite bosons (cobosons in short). This derivation makes crystal clear that interactions between Cooper pairs are solely due to the Pauli exclusion principle, through electron exchanges between pairs. Our procedure also gives hints on why, as we very recently found, the interaction part of the N-pair energy depends on pair number as $N(N-1)$ only from the dilute to the dense regime of pairs. In this work, we also briefly discuss the validity of the BCS wave function ansatz in the light of Richardson exact form.

Although it has been immediately noted that the Pauli exclusion principle plays a key role in superconductivity, it is quite recently only that the precise way it transforms a collection of single Cooper pairs into a BCS condensate, has been really understood. This understanding goes through handling Cooper pairs not within the grand canonical ensemble as done in the standard BCS theory, but through the canonical ensemble. The handling of the Pauli exclusion principle between a fixed number of fermions is however known to be a formidable problem. Nevertheless, adding fermion pairs one by one is the unique way to possibly follow the increasing effect of Pauli blocking when the number of pairs increases.

Five years after the milestone paper on superconductivity by Bardeen, Cooper, Schrieffer[2], Richardson has derived the form of the exact eigenstate of the Schrödinger equation for a fixed number of Cooper pairs[3,4]. In the case of N pairs, it reads in terms of N parameters, R_1,... R_N which are solutions of N coupled non-linear equations, the energy of these N pairs reading as $\mathbf{E}_N = R_1 + ... + R_N$. Although this result is definitely quite smart, to use it in practice, is not that easy: Indeed, these equations have no compact analytical solution, so that they are commonly approached numerically only. This is probably why they have not had so far the attention they deserve among the superconductor community. Nowadays, they are commonly used to numerically study properties of small superconducting particles with an accountable number of electron pairs.

Last year, we analyzed a connection between the two classical problems: the one-pair problem solved by Cooper and the many-pair BCS configuration. In both cases, there is a "frozen" core of noninteracting electrons. There is also a potential layer above this core, where an attraction between up and down spin electrons is localized. In one-pair problem this layer contains only one correlated electron pair, while in the standard BCS theory the layer is half-filled (usually this is interpreted in terms of the extension of the potential layer symmetrically over both sides of the Fermi level, such an understanding being of course fully equivalent with our approach). By adding more and more pairs into this layer, one can continuosly reach BCS regime starting from the one-pair limit. Although manipulation of this kind can hardly been performed experimentally (at least, nowadays), it allows one to see the evolution of energetical spectrum with changing the filling of the potential layer and thus to deeper understand the role of the Pauli exclusion principle for Cooper pairs. In addition, this procedure can be considered as a simple and well-defined toy model for the BEC-BCS crossover problem, since by changing the number of pairs in the potential layer one can tune their overlap (This procedure has to be contrasted with the one of Ref. [Leggett], where the overlap was changed by tuning the interaction between pairs). The crossover thus occurs only for those pairs which are inside the potential layer. Notice that the BEC-BCS crossover problem has not been solved yet *in a fully controllable way* even for the "reduced" BCS interaction between up and down spin electrons, which couples only fermions with opposite spins and momenta. In contrast with variational BCS-like theories [Eagles, Leggett], Richardson's approach provides an exact solution to the many-pair Schrödinger equation and thus makes it possible to tackle the most interesting domain of the phase diagram, which is in the middle between the dilute and dense regimes of pairs. The interest to his domain is due to the fact that, as it was realized long time ago, the BCS ansatz for the wave function can collapse in this region [Leggett]. Notice that the crossover problem within Richardson equations was studied in Ref. [Dukelsky] by using an assumption that ground-state R's form a single arc in the complex plane and by using a continous approximation.

Although Richardson procedure drastically simplifies quantum many-body problem, the resolution of Richardson equations is a quite complicated problem. By turning to the dimensionless form of of these equations we have found an analytical way to solve these equations in the dilute regime of pairs. We then see that these equations do have a small parameter: It is $1/N_c$ where N_c is the number of Cooper pairs for which overlap between pairs starts. This allowed us to demonstrate in the dilute limit on the single Cooper pair scale, i.e., for N/N_c small, that the energy of N Cooper pairs reads as

$$\mathcal{E}_N = N\left[\left(2\epsilon_{F_0} + \frac{N-1}{\rho_0}\right)) - \epsilon_c\left(1 - \frac{N-1}{N_\Omega}\right)\right] \quad (1)$$

ϵ_{F_0} is the Fermi level of the Fermi sea $|F_0\rangle$ which does not feel the attractive potential, ρ_0 is the density of states, taken as constant within the potential layer. $N_\Omega = \rho_0\Omega$ is the number of pair states in this layer, Ω being the potential layer extension. $\epsilon_c \approx 2\Omega \exp\left(-2/\rho_0 V\right)$ is the single pair binding energy, the potential amplitude being V.

Although our actual derivation imposes N/N_c small, it is quite remarkable to note that this result is also valid in the dense BCS regime, where pairs strongly overlap. Indeed the first term of Eq.(1) is the exact energy of N pairs in a normal state, since it is nothing but

$$2\epsilon_{F_0} + (2\epsilon_{F_0} + 1/\rho_0) + \cdots + (2\epsilon_{F_0} + (N-1)/\rho_0) = \mathcal{E}_N^{(normal)} \quad (2)$$

For a number of pairs corresponding to fill half the potential layer, which is the precise BCS configuration, Eq.(1) gives a condensation energy equal to

$$\mathcal{E}_N - \mathcal{E}_N^{(normal)} = \frac{N_\Omega}{2}\frac{\epsilon_c}{2} = \frac{1}{2}\rho_0\Omega^2 e^{-2/\rho_0 V} \quad (3)$$

This result exactly matches the one derived within the grand canonical ensemble, namely $\rho_0\Delta^2/2$ where the gap Δ reads as $2\omega_c \exp\left(-1/\rho_0 V\right)$ since $2\omega_c$ is the potential layer extension Ω. By considering BCS configuration with arbitrary, but large N ($N \gg \Delta$), we have also shown that Eq.(1) remains valid here as well.

The canonical approach we have used to reach Eq.(1), based on the Richardson equations, has the great advantage to trace the evolution of the ground state energy with adding pairs one by one. We then see that it is reasonable to associate the last term in the righ-hand side of Eq.(1), i.e., $\epsilon_c\left[1 - (N-1)/N_\Omega\right]$, with the "pair binding energy" in the N-pair configuration (dressed by the Fermi sea). Indeed, at $N = 1$ this quantity matches with the single-pair binding energy as found by Cooper, while in the dense regime it gives the condensation energy per pair. Therefore, by using this quantity one can understand dilute and dense regimes of pairs on the same footing.

It is straightforward to realize that the pair binding energy, as defined above, decreases when N increases. This decrease is caused entirely by Pauli blocking that decreases the number of available one-electron states in the potential layer with the growth of N. A very pictural way to understand the binding energy decreases when N increases is through the so-called "moth-eaten" effect: when pairs are added to $|F_0\rangle$, they are "eating" one by one, like little moths, the states in the potential layer which are available to form a bound state. As a result of this available state decrease, the bound state energy can only decrease. Our understanding of the pair binding energy is in a contrast with the common belief that in the dense BCS configuration, the Cooper pair binding energy is of the order of the gap Δ, which is far larger than ϵ_c. This interpretation is obtained by splitting the condensate energy $\rho_0\Delta^2/2$ as $(\rho_0\Delta)\Delta$ within an "irrelevant" 1/2 prefactor. This deliberatly assigns a pair energy equal to the gap, the number of pairs to fit the condensation energy then being the number of pair $\rho_0\Delta$ in a gap layer. These $\rho_0\Delta$ pairs actually are "virtual pairs", as named by Schrieffer. Their number is far smaller than the number of pairs $N_\Omega/2$ feeling the potential. This obviously makes their energy far larger than the energy $\epsilon_c/2$ of the real pairs. These virtual pairs in fact correspond to excitations accross the Fermi sea $|F\rangle$ made of $N + N_0$ *non-interacting* pairs, N_0 being the number of pairs in the frozen sea $|F_0\rangle$. The concept of vurtual pairs has physical meaning in the dense regime of pairs only, since in the dilute regime the Fermi level of noninteracting electrons is completely washed out, so that all the pairs are essentially excited above this level. Thus this concept somehow masks a link with the dilute regime of pairs. This was probably one of the reasons for the Schrieffer's statement that the isolated pair concept has a little meaning in the dense regime of pairs [Schrieffer]. This point of view was however questened by Leggett who has argued that pairs in the dense limit in many aspects are similar to giant molecules [Leggett]. Notice that, in general, the problem of physical interpretation of the paired state nature has a long prehistory.

Since the key role of Pauli blocking in superconductivity is enlightened by our expression of the N-pair energy Eq.(1) obtained by solving Richardson equations, it can be of interest to precisely see the parts in these equations which directly come from the Pauli exclusion principle.

In our recent works on the many-body physics of composite bosons, we have proposed a "commutation technique" which allows us to evidence the effects of Pauli blocking between the fermionic components of these composite bosons. They appear through "Pauli scatterings" which describe fermion exchanges in the absence of fermion interaction. These dimensionless Pauli scatterings, mixed with energy-like scatterings associated to interactions between fermionic components, allow us to deal with fermion exchanges between composite bosons (cobosons in short) in an exact way. For review on this formalism and its applications to the many-body physics of semiconductor excitons, see ref[5,6].

In this paper, we first develop such a commutation technique for up and down electron pairs with zero total momentum. We then use it to derive in a quite compact way, the form of the exact eigenstate for N pairs interacting through the reduced BCS potential. The Richardson equations readily follow from this approach. Its main advantage is to possibly trace back in a transparent way, the terms in these equations which directly come from the Pauli exclusion principle: they are those in $R_i - R_j$. They actually come from the non-zero values of Pauli scatterings for fermion exchanges between cobosons made of free electron pairs.

The paper is organized as following:

In section I, we derive the commutation technique for free electron pairs and its associated Pauli and interaction scatterings.

In section II, we use this technique to get the form of the exact eigenstate for $N = 2, 3, \cdots$ pairs interacting through the reduced BCS potential, in order to see how the solution for general N develops. We then analyze the increasing role Pauli blocking in these solutions.

In section III, we discuss possible connection between this exact solution and the BCS ansatz for condensed pairs

I. COMMUTATION TECHNIQUE FOR FREE FERMION PAIRS

A. Exchange between free fermion pairs

We consider cobosons made of free fermion pairs having a zero total momentum.

$$\beta^\dagger_{\mathbf{k}} = a^\dagger_{\mathbf{k}} b^\dagger_{-\mathbf{k}} \tag{4}$$

These pairs only have one degree of freedom, namely k, by contrast to the most general fermion pairs $a^\dagger_{\mathbf{k}_1} b^\dagger_{\mathbf{k}_2}$ which have two. In the case of Cooper pairs, these fermions are up and down spin electrons. The fermion operators ($a^\dagger_{\mathbf{k}'}$,$a^\dagger_{\mathbf{k}}$) and ($b^\dagger_{\mathbf{k}'}$,$b^\dagger_{\mathbf{k}}$) anticommute while $a^\dagger_{\mathbf{k}'}$ and $b^\dagger_{\mathbf{k}}$ commute or anticommute depending if the two fermions have the same or a different nature. However, as easy to check, this does not affect the commutation relations between fermion pair operators. For two creation operators, these are

$$\left[\beta^\dagger_{\mathbf{k}'}, \beta^\dagger_{\mathbf{k}}\right] = 0 \tag{5}$$

It is worth noting that while $\left(a^\dagger_{\mathbf{k}}\right)^2 = 0$ simply follows from the anticommutation of the $a^\dagger_{\mathbf{k}}$ operators, the cancellation of $\left(\beta^\dagger_{\mathbf{k}}\right)^2$ does not follow from Eq.(5), but from the fact that $\left(\beta^\dagger_{\mathbf{k}}\right)^2$ contains $\left(a^\dagger_{\mathbf{k}}\right)^2$. The cancellation of $\left(\beta^\dagger_{\mathbf{k}}\right)^2$ which comes from Pauli blocking, seems to be lost when turning from single fermion operators to pair operators. We will however see that this blocking is yet preserve in the commutaion algebra of free fermion pairs we are developping.

For creation and annihilation operators, $\left[a_{\mathbf{k}'}, a^\dagger_{\mathbf{k}}\right] = \delta_{\mathbf{k}'\mathbf{k}}$ leads to

$$\left[\beta_{\mathbf{k}'}, \beta^\dagger_{\mathbf{k}}\right] = \delta_{\mathbf{k}'\mathbf{k}} - D_{\mathbf{k}'\mathbf{k}} \tag{6}$$

the deviation-from-boson operator $D_{\mathbf{k}'\mathbf{k}}$ being defined as

$$D_{\mathbf{k}'\mathbf{k}} = \delta_{\mathbf{k}'\mathbf{k}} \left(a^\dagger_{\mathbf{k}\uparrow} a_{\mathbf{k}\uparrow} + a^\dagger_{-\mathbf{k}\downarrow} a_{-\mathbf{k}\downarrow}\right) \tag{7}$$

This operator which would reduce to zero if the fermion pairs were elementary bosons, allows us to generate the Pauli scatterings for fermion exchanges. They are formally defined through

$$\left[D_{\mathbf{k}'_1\mathbf{k}_1}, \beta^\dagger_{\mathbf{k}_2}\right] = \sum_{\mathbf{k}'_2} \left\{\lambda\begin{pmatrix} \mathbf{k}'_2 & \mathbf{k}_2 \\ \mathbf{k}'_1 & \mathbf{k}_1 \end{pmatrix} + (\mathbf{k}'_1 \leftrightarrow \mathbf{k}'_2)\right\} \beta^\dagger_{\mathbf{k}'_2} \tag{8}$$

By noting that

$$\left[a^\dagger_{\mathbf{k}} a_{\mathbf{k}}, \beta^\dagger_{\mathbf{p}}\right] = \delta_{\mathbf{k}\mathbf{p}} \beta^\dagger_{\mathbf{p}} = \left[b^\dagger_{-\mathbf{k}} b_{-\mathbf{k}}, \beta^\dagger_{\mathbf{p}}\right] \tag{9}$$

it is easy to show that

$$\left[D_{\mathbf{k}'_1\mathbf{k}_1}, \beta^\dagger_{\mathbf{k}_2}\right] = 2\beta^\dagger_{\mathbf{k}_2} \delta_{\mathbf{k}_1\mathbf{k}_2} \delta_{\mathbf{k}'_1,\mathbf{k}_2} \tag{10}$$

So that we are led to identify the Pauli scattering with the following product of Kronecker symbols

$$\lambda\begin{pmatrix} \mathbf{k}'_2 & \mathbf{k}_2 \\ \mathbf{k}'_1 & \mathbf{k}_1 \end{pmatrix} = \delta_{\mathbf{k}'_1\mathbf{k}_1} \delta_{\mathbf{k}'_2\mathbf{k}_2} \delta_{\mathbf{k}_1\mathbf{k}_2} \tag{11}$$

Actually, this is just the value we expect for the scattering associated to fermion exchanges between $(\mathbf{k}_1, \mathbf{k}_2)$ pairs, as visualized by the diagram of fig (2a). Indeed from this diagram, it is clear that we must have $(\mathbf{k}'_1 = \mathbf{k}_1, \mathbf{k}'_2 = \mathbf{k}_2)$ and $(-\mathbf{k}'_2 = -\mathbf{k}_1, -\mathbf{k}'_1 = -\mathbf{k}_2)$: this just gives $\delta_{\mathbf{k}'_1\mathbf{k}_1} \delta_{\mathbf{k}'_2\mathbf{k}_2} \delta_{\mathbf{k}_1\mathbf{k}_2}$ in agreement with Eq.(11).

B. Interaction between free fermion pairs

To get the interaction scatterings associated to fermion interaction, we first note that for a free fermion hamiltonian

$$H_0 = \sum \epsilon_{\mathbf{k}} \left(a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + b^\dagger_{\mathbf{k}} b_{\mathbf{k}}\right) \tag{12}$$

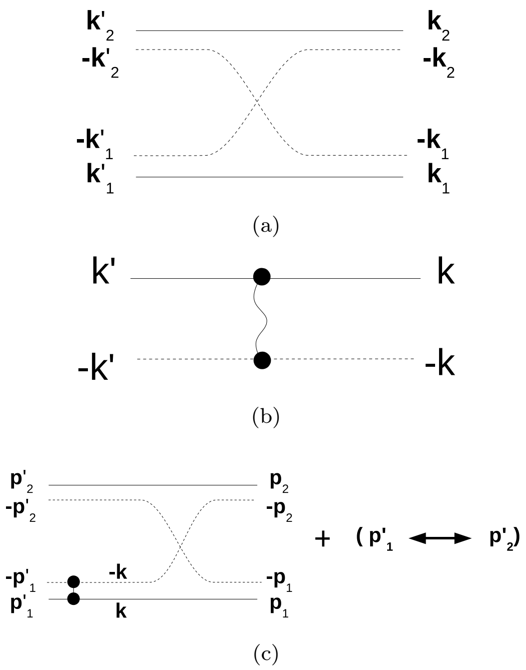
Eq.(9) readily gives

$$\left[H_0, \beta^\dagger_{\mathbf{p}}\right] = 2\epsilon_{\mathbf{p}} \beta^\dagger_{\mathbf{p}} \tag{13}$$

In the case of present interest, these fermion pairs interact through the standard BCS-like (1x1) potential in which the fermion $\mathbf{k}$ only interacts with the fermion $(-\mathbf{k})$ of the other species. This potential reads

$$V_{BCS} = \sum v_{\mathbf{k}'\mathbf{k}} \beta^\dagger_{\mathbf{k}'} \beta_{\mathbf{k}} \tag{14}$$

FIG. 1: Shiva diagram of free pairs



(a) Pauli scattering $\lambda\left(\begin{smallmatrix} \mathbf{k}'_2 & \mathbf{k}_2 \\ \mathbf{k}'_1 & \mathbf{k}_1 \end{smallmatrix}\right)$ for electron exchange between two free pairs $(\mathbf{k}_1, \mathbf{k}_2)$, as given by Eq.(11). Up spin electrons are represented by solid lines, down spin electrons by dashed lines.

(b) The BCS potential given in Eq.(14) transforms a $\mathbf{k}$ pair into a $\mathbf{k}'$ pair, with a constant scattering $-V$, in the case of a separable potential $v_{\mathbf{k}'\mathbf{k}} = -V\, w_{\mathbf{k}'} w_{\mathbf{k}}$.

(c) Interaction scattering $\chi\left(\begin{smallmatrix} \mathbf{p}'_2 & \mathbf{p}_2 \\ \mathbf{p}'_1 & \mathbf{p}_1 \end{smallmatrix}\right)$ between two free pairs, as given in Eq.(19). Since the BCS potential acts within one pair only, the interaction between two pairs can only come from exchange induced by the Pauli exclusion principle.

It is represented by the diagram of Fig. 2b. For this (1x1) potential, we do have

$$\left[V_{BCS}, \beta^\dagger_{\mathbf{p}}\right] = \gamma^\dagger_{\mathbf{p}} + V^\dagger_{\mathbf{p}} \tag{15}$$

in which we have $\gamma^\dagger_{\mathbf{p}} = \sum_{\mathbf{k}} \beta^\dagger_{\mathbf{k}} v_{\mathbf{kp}}$. The "creation potential" for the free fermion pair $\mathbf{p}$ appears to be

$$V^\dagger_{\mathbf{p}} = -\gamma^\dagger_{\mathbf{p}} \left(a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + b^\dagger_{-\mathbf{p}} b_{-\mathbf{p}}\right) \tag{16}$$

While the $\gamma^\dagger_{\mathbf{p}}$ part of Eq.(14) commutes with $\beta^\dagger_{\mathbf{p}}$, this is not so for the creation potential $V^\dagger_{\mathbf{p}}$. Its commutator precisely reads

$$\left[V^\dagger_{\mathbf{p}_1}, \beta^\dagger_{\mathbf{p}_2}\right] = -2\delta_{\mathbf{p}_1\mathbf{p}_2} \gamma^\dagger_{\mathbf{p}_1} \beta^\dagger_{\mathbf{p}_1} \tag{17}$$

This allows us to identify the interaction scattering for free pairs, formally defined as

$$\left[V^\dagger_{\mathbf{p}_1}, \beta^\dagger_{\mathbf{p}_2}\right] = \sum \chi\left(\begin{smallmatrix} \mathbf{p}'_2 & \mathbf{p}_2 \\ \mathbf{p}'_1 & \mathbf{p}_1 \end{smallmatrix}\right) \beta^\dagger_{\mathbf{p}'_1} \beta^\dagger_{\mathbf{p}'_2} \tag{18}$$

with a sequence of one (2x2) fermion pair exchange and one (1x1) fermion pair interaction. Indeed

$$\begin{aligned} \chi\left(\begin{smallmatrix} \mathbf{p}'_2 & \mathbf{p}_2 \\ \mathbf{p}'_1 & \mathbf{p}_1 \end{smallmatrix}\right) &= -\sum_{\mathbf{k}} \left\{ v_{\mathbf{p}'_1\mathbf{k}} \lambda\left(\begin{smallmatrix} \mathbf{p}'_2 & \mathbf{p}_2 \\ \mathbf{k} & \mathbf{p}_1 \end{smallmatrix}\right) + (\mathbf{p}'_1 \leftrightarrow \mathbf{p}'_2) \right\} \\ &= -\left(v_{\mathbf{p}'_1,\mathbf{p}_1} \delta_{\mathbf{p}'_2,\mathbf{p}_2} + v_{\mathbf{p}'_2,\mathbf{p}_2} \delta_{\mathbf{p}'_1,\mathbf{p}_1}\right) \delta_{\mathbf{p}_2,\mathbf{p}_1} \end{aligned} \tag{19}$$

This interaction scattering is visualized by the diagram of Fig 2c: the free pairs $\mathbf{p}'_1$ and $\mathbf{p}'_2$ first exchange a fermion. As for any exchange, this brings a minus sign. In a second step, the fermions of one of the two pairs interact via the BCS potential. It is of importance to note that since the potential has a (1x1) structure, the (2x2) interaction between two pairs can only result from fermion exchange between pairs, i.e., Pauli blocking.

We are now going to use this commutation formalism to derive the Richardson equations for Cooper pairs.

II. RICHARDSON EQUATIONS FOR COOPER PAIRS

In order to better grasp how these equations develop, let us consider an increasing number of pairs.

A. One pair

We consider a state in which one free pair $\mathbf{k}_1$ is added to a frozen Fermi sea $|F_0\rangle$ which does not feel the BCS potential. This means that the $v_{\mathbf{k}'\mathbf{k}}$ prefactors in Eq.(14) cancel for all $\mathbf{k}$ belonging to $|F_0\rangle$. Note that such a "one-pair" state actually contains $N_0 + 1$ fermion pairs, N_0 being the number of pairs in the frozen sea. So that this state is in fact a many-body state, but in the most simple sense since the Fermi sea $|F_0\rangle$ is just there to block states by the Pauli exclusion principle. This Fermi sea also brings a finite density of state for all the states above it. This is actually crucial in order to have a bound state, even for an extremely small attracting BCS potential as evidenced below.

Due to Eqs.(13,15), the hamiltonian $H = H_0 + V_{BCS}$ acting on this one free pair state gives, by taking the zero energy such that $H\,|F_0\rangle = H_0\,|F_0\rangle = 0$

$$H\beta^\dagger_{\mathbf{k}}\,|F_0\rangle = \left[H, \beta^\dagger_{\mathbf{k}}\right]|F_0\rangle = \left(2\epsilon_{\mathbf{k}}\beta^\dagger_{\mathbf{k}} + \gamma^\dagger_{\mathbf{k}} + V^\dagger_{\mathbf{k}}\right)|F_0\rangle \tag{20}$$

We then note that, due to the $v_{\mathbf{kp}}$ factor included in the $\gamma^\dagger_{\mathbf{k}}$ part of $V^\dagger_{\mathbf{k}}$ (see Eq.(16)), the creation potential $V^\dagger_{\mathbf{k}}$ acting on $|F_0\rangle$ gives zero.

If we now substract $\mathbf{E}_1\beta^\dagger_{\mathbf{k}}\,|F_0\rangle$ to the two sides of the above equation and divide the resulting equation by $(2\epsilon_{\mathbf{k}} - \mathbf{E}_1)$, we find

$$)(H - \mathbf{E}_1)\frac{1}{2\epsilon_{\mathbf{k}} - \mathbf{E}_1}\beta^\dagger_{\mathbf{k}}\,|F_0\rangle = \beta^\dagger_{\mathbf{k}}\,|F_0\rangle + \frac{1}{2\epsilon_{\mathbf{k}} - \mathbf{E}_1}\gamma^\dagger_{\mathbf{k}}\,|F_0\rangle \tag{21}$$

To go further and possibly get the one-pair eigenstate of the hamiltonian H in an analytical form, it is necessary to approximate the BCS potential coupling by $v_{\mathbf{kp}} = -V\, w_{\mathbf{k}} w_{\mathbf{p}}$, the $w_{\mathbf{k}}$'s being moreover such that $w_{\mathbf{k}}^2 = w_{\mathbf{k}}$. This yields

$$\gamma_{\mathbf{k}}^{\dagger} = -V\, w_{\mathbf{k}} \beta^{\dagger} \qquad \beta^{\dagger} = \sum_{\mathbf{p}} w_{\mathbf{p}} \beta_{\mathbf{p}}^{\dagger} \tag{22}$$

If we then multiply Eq.(21) by $w_{\mathbf{k}}$ and sum over $\mathbf{k}$, we find

$$(H - \mathbf{E}_1) B^{\dagger}(\mathbf{E}_1)\,|F_0\rangle = \left(1 - V \sum \frac{w_{\mathbf{k}}}{2\epsilon_{\mathbf{k}} - \mathbf{E}_1}\right) \beta^{\dagger}\,|F_0\rangle \tag{23}$$

in which we have set

$$B_{\mathbf{k}}^{\dagger}(E) = \frac{w_{\mathbf{k}}}{2\epsilon_{\mathbf{k}} - E} \beta^{\dagger} \qquad B^{\dagger}(E) = \sum_{\mathbf{k}} B_{\mathbf{k}}^{\dagger}(E) \tag{24}$$

Eq.(23) readily shows that the linear combination $B^{\dagger}(\mathbf{E}_1)$ of one-pair operators generates the one-pair eigenstate $B^{\dagger}(\mathbf{E}_1)\,|F_0\rangle$ of the hamiltonian H with the energy $\mathbf{E}_1$, provided that this energy is such that

$$1 = V \sum_{\mathbf{k}} \frac{w_{\mathbf{k}}}{2\epsilon_{\mathbf{k}} - \mathbf{E}_1} \tag{25}$$

This is nothing but the well-known equation for the single pair energy derived by Cooper.

B. Two pairs

Let us now consider two pairs. Eqs.(13,15) yield

$$\begin{aligned} H\beta_{\mathbf{k}_1}^{\dagger}\beta_{\mathbf{k}_2}^{\dagger}\,|F_0\rangle &= \left(\left[H, \beta_{\mathbf{k}_1}^{\dagger}\right]\beta_{\mathbf{k}_2}^{\dagger} + \beta_{\mathbf{k}_1}^{\dagger}\left[H, \beta_{\mathbf{k}_2}^{\dagger}\right]\right)|F_0\rangle \\ &= (2\epsilon_{\mathbf{k}_1} + 2\epsilon_{\mathbf{k}_2})\,\beta_{\mathbf{k}_1}^{\dagger}\beta_{\mathbf{k}_2}^{\dagger}\,|F_0\rangle + |v_{\mathbf{k}_1\mathbf{k}_2}\rangle \end{aligned} \tag{26}$$

where $|v_{\mathbf{k}_1\mathbf{k}_2}\rangle$ comes from interactions among the $(\mathbf{k}_1, \mathbf{k}_2)$ pairs induced by the BCS potential. Its precise value is

$$|v_{\mathbf{k}_1\mathbf{k}_2}\rangle = \left(\gamma_{\mathbf{k}_1}^{\dagger}\beta_{\mathbf{k}_2}^{\dagger} + \gamma_{\mathbf{k}_2}^{\dagger}\beta_{\mathbf{k}_1}^{\dagger} + V_{\mathbf{k}_1}^{\dagger}\beta_{\mathbf{k}_2}^{\dagger}\right)|F_0\rangle \tag{27}$$

Eq. (19) allows us to write the last term of $|v_{\mathbf{k}_1\mathbf{k}_2}\rangle$ as

$$V_{\mathbf{k}_1}^{\dagger}\beta_{\mathbf{k}_2}^{\dagger}\,|F_0\rangle = \left[V_{\mathbf{k}_1}^{\dagger}, \beta_{\mathbf{k}_2}^{\dagger}\right]|F_0\rangle = \sum_{\mathbf{p}'_1\mathbf{p}'_2} \chi\begin{pmatrix}\mathbf{p}'_2 & \mathbf{k}_2 \\ \mathbf{p}'_1 & \mathbf{k}_1\end{pmatrix} \beta_{\mathbf{p}'_1}^{\dagger}\beta_{\mathbf{p}'_2}^{\dagger}\,|F_0\rangle \tag{28}$$

So that $|v_{\mathbf{k}_1\mathbf{k}_2}\rangle$ can be visualized by the diagram of Fig. 2. This diagram evidences the fact that, due to the (1x1) form of the BCS potential, the two pairs $\mathbf{k}_1$ and $\mathbf{k}_2$ interact by fermion exchange only, as a result of the Pauli exclusion principle.

By using the value of the interaction scattering given in Eq.(17), we find that $|v_{\mathbf{k}_1\mathbf{k}_2}\rangle$ is given by

$$|v_{\mathbf{k}_1\mathbf{k}_2}\rangle = -V\left(w_{\mathbf{k}_1}\beta_{\mathbf{k}_2}^{\dagger} + w_{\mathbf{k}_2}\beta_{\mathbf{k}_1}^{\dagger} - 2\delta_{\mathbf{k}_1\mathbf{k}_2} w_{\mathbf{k}_1}\beta_{\mathbf{k}_1}^{\dagger}\right)\beta^{\dagger}\,|F_0\rangle \tag{29}$$

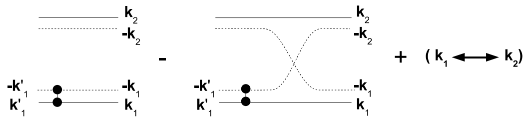


FIG. 2: Shiva diagram of two pairs

To go further, we substract $\mathbf{E}_2\beta_{\mathbf{k}_1}^{\dagger}\beta_{\mathbf{k}_2}^{\dagger}\,|F_0\rangle$ to the two sides of Eq.(26), with $\mathbf{E}_2$ written as $R_1 + R_2$ and we multiply the resulting equation by $w_{\mathbf{k}_1} w_{\mathbf{k}_2} / \left(2\epsilon_{\mathbf{k}_1} - R_1\right)\left(2\epsilon_{\mathbf{k}_2} - R_2\right)$. This gives

$$\begin{aligned} &(H - \mathbf{E}_2) B_{\mathbf{k}_1}^{\dagger}(R_1) B_{\mathbf{k}_2}^{\dagger}(R_2)\,|F_0\rangle = \\ &\left\{B_{\mathbf{k}_1}^{\dagger}(R_1)\left(w_{\mathbf{k}_2}\beta_{\mathbf{k}_2}^{\dagger} - \frac{V w_{\mathbf{k}_2}}{2\epsilon_{\mathbf{k}_2} - R_2}\beta^{\dagger}\right) + (1 \leftrightarrow 2)\right\}|F_0\rangle \end{aligned} \tag{30}$$

To go further, we note that $\left(2\epsilon_{\mathbf{k}_1} - R_1\right)^{-1}\left(2\epsilon_{\mathbf{k}_2} - R_2\right)^{-1}$ also reads $\left[\left(2\epsilon_{\mathbf{k}_1} - R_1\right)^{-1} - \left(2\epsilon_{\mathbf{k}_2} - R_2\right)^{-1}\right] / \left(R_1 - R_2\right)$ provided that $R_1 \neq R_2$. By taking sums over $\mathbf{k}_1$ and $\mathbf{k}_2$, Eq. (30) then gives

$$\begin{aligned} &(H - \mathbf{E}_2) B^{\dagger}(R_1) B^{\dagger}(R_2)\,|F_0\rangle = \\ &\left\{B^{\dagger}(R_1)\left(1 - V\sum \frac{w_{\mathbf{k}}}{2\epsilon_{\mathbf{k}} - R_2} + \frac{2V}{R_1 - R_2}\right) + (1 \leftrightarrow 2)\right\} \\ &\beta^{\dagger}\,|F_0\rangle \end{aligned} \tag{31}$$

This readily shows that the two-pair state $B^{\dagger}(R_1)B^{\dagger}(R_2)\,|F_0\rangle$ is eigenstate of the hamiltonian H with the energy $\mathbf{E}_2 = R_1 + R_2$ provided that (R_1, R_2) fulfill two equations, known as Richardson equations for two pairs.

$$1 = V\sum \frac{w_{\mathbf{k}}}{2\epsilon_{\mathbf{k}} - R_1} + \frac{2V}{R_1 - R_2} = (1 \leftrightarrow 2) \tag{32}$$

C. Three pairs

We now turn to three pairs to see how these equations develop for an increasing number of pairs. We start with

$$\begin{aligned} &H\beta_{\mathbf{k}_1}^{\dagger}\beta_{\mathbf{k}_2}^{\dagger}\beta_{\mathbf{k}_3}^{\dagger}\,|F_0\rangle = \\ &\left\{\left[H, \beta_{\mathbf{k}_1}^{\dagger}\right]\beta_{\mathbf{k}_2}^{\dagger}\beta_{\mathbf{k}_3}^{\dagger} + \beta_{\mathbf{k}_1}^{\dagger}\left[H, \beta_{\mathbf{k}_2}^{\dagger}\right]\beta_{\mathbf{k}_3}^{\dagger} + \beta_{\mathbf{k}_1}^{\dagger}\beta_{\mathbf{k}_2}^{\dagger}\left[H, \beta_{\mathbf{k}_3}^{\dagger}\right]\right\} \\ &|F_0\rangle \end{aligned} \tag{33}$$

The same eqs (13,15) give

$$\begin{aligned} H\beta_{\mathbf{k}_1}^{\dagger}\beta_{\mathbf{k}_2}^{\dagger}\beta_{\mathbf{k}_3}^{\dagger}\,|F_0\rangle &= (2\epsilon_{\mathbf{k}_1} + 2\epsilon_{\mathbf{k}_2} + 2\epsilon_{\mathbf{k}_3})\,\beta_{\mathbf{k}_1}^{\dagger}\beta_{\mathbf{k}_2}^{\dagger}\beta_{\mathbf{k}_3}^{\dagger}\,|F_0\rangle \\ &+ |v_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\rangle \end{aligned} \tag{34}$$

where the part resulting from the BCS potential appears as

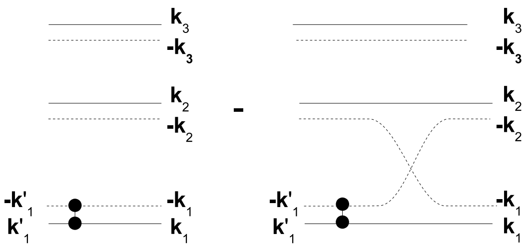
$$|v_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\rangle = \left(\gamma^\dagger_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_2}\beta^\dagger_{\mathbf{k}_3} + \gamma^\dagger_{\mathbf{k}_2}\beta^\dagger_{\mathbf{k}_3}\beta^\dagger_{\mathbf{k}_1} + \gamma^\dagger_{\mathbf{k}_3}\beta^\dagger_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_2}\right)|F_0\rangle + \left(V^\dagger_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_2}\beta^\dagger_{\mathbf{k}_3} + \beta^\dagger_{\mathbf{k}_1}V^\dagger_{\mathbf{k}_2}\beta^\dagger_{\mathbf{k}_3} + \beta^\dagger_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_2}V^\dagger_{\mathbf{k}_3}\right)|F_0\rangle \tag{35}$$

The last term gives zero since $V^\dagger_{\mathbf{k}}|F_0\rangle = 0$. Using Eq. (18), the two remaining terms of the second bracket can be rewritten as

$$\left\{\left[V^\dagger_{\mathbf{k}_1}, \beta^\dagger_{\mathbf{k}_2}\right]\beta^\dagger_{\mathbf{k}_3} + \beta^\dagger_{\mathbf{k}_2}\left[V^\dagger_{\mathbf{k}_1}, \beta^\dagger_{\mathbf{k}_3}\right] + \beta^\dagger_{\mathbf{k}_1}\left[V^\dagger_{\mathbf{k}_2}, \beta^\dagger_{\mathbf{k}_3}\right]\right\}|F_0\rangle = \sum_{vk'_1\mathbf{k}'_2}\beta^\dagger_{\mathbf{k}'_1}\beta^\dagger_{\mathbf{k}'_2} \left\{\chi\begin{pmatrix}\mathbf{k}'_2 & \mathbf{k}_2\\ \mathbf{k}'_1 & \mathbf{k}_1\end{pmatrix}\beta^\dagger_{\mathbf{k}_3} + \chi\begin{pmatrix}\mathbf{k}'_2 & \mathbf{k}_3\\ \mathbf{k}'_1 & \mathbf{k}_2\end{pmatrix}\beta^\dagger_{\mathbf{k}_1} + \chi\begin{pmatrix}\mathbf{k}'_2 & \mathbf{k}_1\%\\ vk'_1 & \mathbf{k}_3\end{pmatrix}\beta^\dagger_{\mathbf{k}_2}\right\}|F_0\rangle \tag{36}$$

This leads to represent the vector $|v_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\rangle$ by the diagram of Fig. 3. This interaction term corresponds to interactions inside a single pair, two pairs staying unchanged, with in addition a possible fermion exchange with a second pair, the third pair staying unchanged.

FIG. 3: Shiva diagram of two pairs



Part resulting from the BCS potential acting on three pairs, as given in Eqs. (35,36). $|v_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\rangle$ also contains two similar contributions as the one visualized in this figure, obtained by circular permutation.

If we come back to Eq.(34), substract $\mathbf{E}_3\beta^\dagger_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_2}\beta^\dagger_{\mathbf{k}_3}|F_0\rangle$ to both sides, with $\mathbf{E}_3$ written as $R_1 + R_2 + R_3$, and multiply the resulting equation by $w_{\mathbf{k}_1}w_{\mathbf{k}_2}w_{\mathbf{k}_3}/(2\epsilon_{\mathbf{k}_1} - R_1)(2\epsilon_{\mathbf{k}_2} - R_2)(2\epsilon_{\mathbf{k}_3} - R_3)$, we find

$$(H - \mathbf{E}_3)B^\dagger_{\mathbf{k}_1}(R_1)B^\dagger_{\mathbf{k}_2}(R_2)B^\dagger_{\mathbf{k}_3}(R_3)|F_0\rangle = \left\{B^\dagger_{\mathbf{k}_1}(R_1)B^\dagger_{\mathbf{k}_2}(R_2)\left(w_{\mathbf{k}_3}\beta^\dagger_{\mathbf{k}_3} - \frac{Vw_{\mathbf{k}_3}}{2\epsilon_{\mathbf{k}_2} - R_3}\beta^\dagger\right) + 2\text{ perm}\right\}|F_0\rangle + 2V\left\{B^\dagger_{\mathbf{k}_3}(R_3)\frac{\delta_{\mathbf{k}_1\mathbf{k}_2}w_{\mathbf{k}_1}}{(2\epsilon_{\mathbf{k}_1} - R_1)(2\epsilon_{\mathbf{k}_1} - R_2)}\beta^\dagger_{\mathbf{k}_1} + 2\text{ perm}\right\}\beta^\dagger|F_0\rangle \tag{37}$$

To proceed, we rewrite $(2\epsilon_{\mathbf{k}_1} - R_1)^{-1}(2\epsilon_{\mathbf{k}_2} - R_2)^{-1}$ as $\left[(2\epsilon_{\mathbf{k}_1} - R_1)^{-1} - (2\epsilon_{\mathbf{k}_2} - R_2)^{-1}\right]/(R_1 - R_2)$ provided that $R_1 \neq R_2$ and do the same for the two other products. By taking the sum over $(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$, we end with

$$(H - \mathbf{E}_3)B^\dagger(R_1)B^\dagger(R_2)B^\dagger(R_3)|F_0\rangle = \{B^\dagger(R_2)B^\dagger(R_3)\left(1 - V\sum\frac{w_{\mathbf{k}_1}}{2\epsilon_{\mathbf{k}_1} - R_1} - \frac{2V}{R_1 - R_2} + \frac{2V}{R_3 - R_1}\right) + 2\text{ perm}\}\beta^\dagger|F_0\rangle \tag{38}$$

This leads us to again conclude that the three-pair state $B^\dagger(R_1)B^\dagger(R_2)B^\dagger(R_3)|F_0\rangle$ is eigenstate of the hamiltonian with the energy $\mathbf{E}_3 = R_1 + R_2 + R_3$, provided that (R_1, R_2, R_3) fulfill the three equations,

$$\begin{aligned} 1 &= V\sum\frac{w_{\mathbf{k}}}{2\epsilon_{\mathbf{k}} - R_1} + \frac{2V}{R_1 - R_2} + \frac{2V}{R_1 - R_3} \\ 1 &= V\sum\frac{w_{\mathbf{k}}}{2\epsilon_{\mathbf{k}} - R_2} + \frac{2V}{R_2 - R_3} + \frac{2V}{R_2 - R_1} \\ 1 &= V\sum\frac{w_{\mathbf{k}}}{2\epsilon_{\mathbf{k}} - R_3} + \frac{2V}{R_3 - R_1} + \frac{2V}{R_3 - R_2} \end{aligned} \tag{39}$$

D. N pairs

The above commutation technique can be easily extended to N pairs. As nicely visualized by the diagrams of Figs. 2 and 3, the effect of the BCS potential on these N pairs splits into two sets of terms: In one set, one pair is affected by the (1x1) scattering while the other $N-1$ pairs stay unchanged. In the other, this pair in addition has a fermion exchange before the interaction, with another pair, the remaining $N-2$ pairs staying unchanged. This readily shows that an increase of the number of pairs above two, does not really change the structure of the equations since $N-2$ pairs stay unchanged, the pair exchanging its fermions with the pair suffering the interaction being just one among $(N-1)$ pairs.

Although the equations become more and more cumbersome to be explicitly written, the procedure is rather straightforward once we have understood that either $(N-1)$ or $(N-2)$ pairs stay unaffected in the process. The general form of the N-pair eigenstate ultimately appears as

$$(H - \mathbf{E}_N)B^\dagger(R_1)\cdots B^\dagger(R_N)|F_0\rangle = 0 \tag{40}$$

with $\mathbf{E}_3 = R_1 + \cdots + R_N$, these R_N's being solutions of N equations like

$$1 = V\sum\frac{w_{\mathbf{k}}}{2\epsilon_{\mathbf{k}} - R_i} + \sum_{i\neq j}\frac{2V}{R_i - R_j} \qquad \text{for } i = (1, ..., N) \tag{41}$$

E. Physical understanding

This new derivation of the Richardson equation has the main advantage to possibly trace back the parts in these equations which are directly linked to the Pauli exclusion principle between fermion pairs.

From a mathematical point of view, the link is rather obvious: In the absence of terms in $V/(R_i - R_j)$, the N equations for R_i reduced to the same equation (25), so that the result would be $R_i^0 = \mathbf{E}_1$ for all i. The fact that the energy of N pairs differs from N times the single pair energy thus comes from those $(R_i - R_j)$ differences.

Physically, the fact that $\mathbf{E}_N$ differs from $N\mathbf{E}_1$ comes from interactions between pairs. Due to the (1x1) form of the BCS potential, interaction between pairs can only be mediated by fermion exchanges as clear from Fig. 2c. Interaction between pairs thus is solely the result of the Pauli exclusion principle between pairs. This Pauli blocking mathematically appears through the various $\delta_{\mathbf{p}'\mathbf{p}}$ factors appearing in Pauli scatterings $\lambda\begin{pmatrix} \mathbf{p}_2' & \mathbf{p}_2 \\ \mathbf{p}_1' & \mathbf{p}_1 \end{pmatrix}$. It is then easy to mathematically trace back the $(R_i - R_j)$ differences appearing in the Richardson equations to these δ factors.

In short, the Kronecker symbols in the Pauli scatterings of fermion pairs result from the Pauli exclusion principle. They induce terms in $V/(R_i - R_j)$ in the Richardson equations which make the energy of N pairs different from the one of a collection of N independent single pairs.

Another very interesting feature of the energy $\mathbf{E}_N$ of N pairs, this new derivation explains in a rather clear way, is the fact that the part of the N pairs energy coming from interaction, namely $\mathbf{E}_N - N\mathbf{E}_1$ depends on N as $N(N-1)$ only. Indeed, diagram 3 evidences that the contribution of the (1x1) BCS potential mixed with fermion exchanges between pairs having one degree of freedom only, ends by producing effective scatterings which are (2x2) only. Since, in order to have terms in $N(N-1)(N-2)$, we need topologically connected interaction processes between 3 objects, $N(N-1)(N-2)$ terms as well as all the higher order terms, cannot exist in the energy of N Cooper pairs.

This actually is what we have found by solving these equations analytically in the dilute limit on the single Cooper pair scale. In this limit, the energy of N pairs was shown to read as

$$\mathbf{E}_N = N\mathbf{E}_1 + N(N-1)\left(\frac{1}{\rho_0} + \frac{\epsilon_c}{N_\Omega}\right) \tag{42}$$

ρ_0 is the density of pair states in the potential layer, $N_\Omega = \rho_0\Omega$ is number of states in the layer, and ϵ_c is the single pair binding energy. By writing it as $\epsilon_c = N_\Omega \epsilon_V$, this energy also reads, for $\mathbf{E}_1 = 2\epsilon_{F_0} - \epsilon_c$ where ϵ_{F_0} is the Fermi level of the frozen sea $|F_0\rangle$

$$\mathbf{E}_N = 2N\left[\epsilon_{F_0} + \frac{N(N-1)}{\rho_0}\right] - N\epsilon_V\left(N_V - \frac{N-1}{N_\Omega}\right) \tag{43}$$

The first term corresponds to the kinetic energy of N pairs added to ϵ_{F_0}, i.e.,

$$2\epsilon_{F_0} + (2\epsilon_{F_0} + 1/\rho_0) + \cdots + (2\epsilon_{F_0} + (N-1)/\rho_0) \tag{44}$$

The second term evidences the fact that the Cooper pair binding energy linearly decreases with pair number, this energy being proportional to the number of empty states $N_V - (N-1)$ filling the potential.

This brings the binding energy down to $\epsilon_c/2$ in the BCS configuration, i.e., when pairs fill half the potential layer. Actually, the result fully agrees with the BCS condensation energy, ??? to read

$$E_{\text{super}} - E_{\text{normal}} = \frac{1}{2}\rho_0\Delta^2 = \frac{\rho_0\Omega}{2}\frac{2\Omega e^{-2/\rho_0 V}}{2} \tag{45}$$

In spite of the fact that Eq. (42) has up to now been derived within the dilute limit only, it turns out that it is valid over the whole density range. This validity is a bare result of the existence of (2×2) effective scatterings only between fermion pairs, this argument having nothing to do with the pair density large or small on the single Cooper pair scale.

III. RICHARDSON EXACT EIGENSTATE VERSUS BCS ANSATZ

Another very interesting result the Richardson procedure generates is the *exact* form of the eigenstate, namely

$$B^\dagger(R_1)\cdots B^\dagger(R_N)\,|F_0\rangle \tag{46}$$

with $B^\dagger(R)$ given by Eq. (24). The fact that by construction all the R_i's are different, strongly questions the standard BCS ansatz. For Cooper pair wave function $\left(B^\dagger\right)^N|F_0\rangle$ with *all* the pairs condensed into the same state.

To discuss this problem on precise grounds, let us start with two pairs. In a previous work[7], we have shown, that the two "Richardson energies" then read $R_1 = R + iR'$ and $R_2 = R - iR'$ with R and R' real, their precise values being $R \approx \epsilon_c + 1/\rho_0 + \epsilon_c/N\Omega$ and $R' = \sqrt{2\epsilon_c/\rho_0}$ in the large sample limit, i.e. for $1/\rho_0$ small. By noting that

$$\begin{aligned} &B^\dagger(R_1)B^\dagger(R_2) = \\ &\left[B^\dagger(R) + B^\dagger(R_1) - B^\dagger(R)\right]\left[B^\dagger(R) + B^\dagger(R_2) - B^\dagger(R)\right] \end{aligned} \tag{47}$$

we get from eq (24)

$$B^\dagger(R_1)B^\dagger(R_2) = \left[B^\dagger(R)\right]^2 + R'^2\left\{C_+^\dagger C_-^\dagger - 2B^\dagger(R)D^\dagger\right\} \tag{48}$$

where we have set

$$C_\pm^\dagger = \sum \frac{w_{\mathbf{k}}}{(2\epsilon_{\mathbf{k}} - R)\,(2\epsilon_{\mathbf{k}} - R \pm iR')}\beta_{\mathbf{k}}^\dagger \tag{49}$$

$$D^\dagger = \sum \frac{w_{\mathbf{k}}}{(2\epsilon_{\mathbf{k}} - R)\left[(2\epsilon_{\mathbf{k}} - R)^2 + R'^2\right]}\beta_{\mathbf{k}}^\dagger \tag{50}$$

So that at first order in sample volume, i.e., in $1/\rho_0$, we find since $N_\Omega = \rho_0 \Omega$

$$B^\dagger(R_1)B^\dagger(R_2) - \left[B^\dagger(\frac{\mathbf{E}_2}{2})\right]^2 \approx \frac{2\epsilon_c}{\rho_0}$$
$$\left\{-2B^\dagger(\mathbf{E}_1)\sum \frac{w_{\mathbf{k}}}{(2\epsilon_{\mathbf{k}} - \mathbf{E}_1)^3}\beta^\dagger_{\mathbf{k}} + \left[\sum \frac{w_{\mathbf{k}}}{(2\epsilon_{\mathbf{k}} - \mathbf{E}_1)^2}\beta^\dagger_{\mathbf{k}}\right]^2\right\}$$
$$+ O(\frac{1}{\rho_0^2}) \quad (51)$$

The above result shows that $B^\dagger(R_1)B^\dagger(R_2)$ can be written as $\left(B^\dagger(\mathbf{E}_2/2)\right)^2$ provided that we drop all $1/\rho_0$ terms. However $B^\dagger$ then reduces to $B^\dagger(\mathbf{E}_1)$. This barely corresponds to consider the two-pair eigenstate as the product of two non-interacting single pairs. If instead, we want, in the condensed pair creation operator, include the change from one to two pairs induced by Pauli blocking which brings the energy per pair from $\mathbf{E}_1$ to $\mathbf{E}_2/2 = \mathbf{E}_1 + 1/\rho_0 + \epsilon_c/N_\Omega$, we are led to replace $B^\dagger(R_1)B^\dagger(R_2)$ by $\left(B^\dagger(\mathbf{E}_2/2)\right)^2$. This however is inconsistent because we then keep in this condensed pair operator, contribution in $1/\rho_0$ which are as large as the ones we drop in the LHS of eq (51). In the case of two pairs, the replacement of the exact eigenstate $B^\dagger(R_1)B^\dagger(R_2)\,|F_0\rangle$ by a BCS-like condense state $\left(B^\dagger(\mathbf{E}_2/2)\right)^2|F_0\rangle$ th us is inconsistent.

It is actually claimed that the BCS ansatz is valid in the thermodynamical limit, i.e., for N and V both very large. Derivation of the "validity" is in fact with restrict to the energy only. We fully agree that the BCS ansatz give the correct energy since the energy obtained using this ansatz is just the one we have derived from the exact Richardson procedure. However agreement on the energy by no mean proves agreement on the wave function. Many examples have been given in the past with wave function very different from the exact one, while giving the correct energy.

The possible replacement of $B^\dagger(R_1)\cdots B^\dagger(R_N)\,|F_0\rangle$ by $\left(B^\dagger\right)^N|F_0\rangle$ is actually crucial to support the overall picture we all have of superconductivity, with all the pairs in the same state, as an army of little solders, all walking similarly.

IV. CONCLUSION

We have rederived the Richardson equations using a commutation technique for free electron pairs with zero total momentum similar to the one we have developed for composite boson excitons. Almost half a century ago, Richardson has succeeded to write the *exact* eigenstate for an arbitrary number N of pairs in a compact form in terms of N energy-like quantities $R_1, ..., R_N$, which are solution of N coupled non-linear equations. This $2N$ many-body problem is exactly solvable provided that the interaction potential is taken as a BCS-like potential having a separable scattering $v_{\mathbf{k}'\mathbf{k}} = -V\, w_{\mathbf{k}'} w_{\mathbf{k}}$ with $w_{\mathbf{k}}$ moreover such that $w_{\mathbf{k}}^2 = 1$. Note that these assumptions are already those necessary to get the energy of a single pair in the compact form obtained by Cooper. Richardson managed to extend Cooper exact solution to N pairs by decoupling them through rewriting their energy E_N as $R_1 + \cdots + R_N$.

The new composite boson derivation we here propose, allows us to trace back the physical origin of the various terms of these equations. It in particular clearly shows that N pairs differ from N independent single pairs, due to Pauli exclusion principle only. This Pauli blocking also enforces the R_i energy-like parameters to be different, namely the exact N-pair eigenstate different from the BCS ansatz. ?????? the diagrammatic representation of this derivation evidences that, due to the fact that pairs with zero total momentum, do have one degree of freedom only, they only have 2×2 scatterings within the 1×1 BCS potential. This explains why the N pair energy has terms in N and $N(N-1)$ but not in $N(N-1)(N-2)$ and so on.

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