Coboson Approach to Cooper Pairs

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(Dated: October 24, 2009)

Abstract

I. BASIC COBOSON FORMALISM

We consider pairs of opposite spin electrons with zero total momentum. These electrons interact through the standard BCS potential that we take here as granted without discussing its oversimplification. This potential reads as

$$V_{BCS} = \sum v_{\mathbf{k}'\mathbf{k}} \beta^+_{\mathbf{k}'} \beta_{\mathbf{k}} \tag{1}$$

where $v_{\mathbf{k}'\mathbf{k}}$ is taken as a separable potential, $v_{\mathbf{k}'\mathbf{k}} = -V\, w_{\mathbf{k}'} w_{\mathbf{k}}$ with V being a small positive constant. $\beta^+_{\mathbf{k}} = a^+_{\mathbf{k}\uparrow} a^+_{-\mathbf{k}\downarrow}$ creates a pair of opposite spin electrons with momentum $(\mathbf{k}, -\mathbf{k})$ while $w_{\mathbf{k}}$ is equal to 1 for electrons in the energy layer where the potential acts. This layer lies above Fermi sea $|F_0\rangle$ full of electrons which are thus "frozen" with respect to the V_{BCS} potential. Electrons feeling this potential have an energy $\epsilon_{\mathbf{k}}$ such that, $\epsilon_{F_0} < \epsilon_{\mathbf{k}} < \epsilon_{F_0} + \Omega$. Scattering processes induced by the BCS potential are represented by the fig. (1a) of equation (1). The BCS hamiltonian for up and down spin electrons reads as $H_{BCS} = H_0 + V_{BCS}$, the free part H_0 being given by

$$H_0 = \sum \epsilon_{\mathbf{k}} \left(a^+_{\mathbf{k}\uparrow} a_{\mathbf{k}\uparrow} + a^+_{\mathbf{k}\downarrow} a_{\mathbf{k}\downarrow} \right) \tag{2}$$

The coboson formalism [1] originally developed for composite bosons made of two fermions like excitons or Hydrogen atoms, relies on an operator algebra made of commutators. This algebra has to be contrasted with the scalar algebra made of Green functions developed for elementary quantum particles. The first advantage of the operator algebra is to possibly deal with the Pauli exclusion principle between the particular fermionic component in an exact way. A few elementary commutators between free electron pairs are necessary to construct the formalism adapted to Cooper pair composite bosons. Let us calculate them first.

A. Commutators for free electron pairs

From the anticommutation of electron operators $\left\{a^+_{\mathbf{k}'s'}, a^+_{\mathbf{k}s}\right\} = a^+_{\mathbf{k}'\,s'} a^+_{\mathbf{k},s} + a^+_{\mathbf{k}s} a^+_{\mathbf{k}'s'} = 0$ and $\left\{a_{\mathbf{k}'s'}, a^+_{\mathbf{k}s}\right\} = \delta_{\mathbf{k}'\mathbf{k}} \delta_{s'\,s}$, it is easy to deduce the commutators for fermion pair creation operators as $\left[\beta^+_{\mathbf{k}'}, \beta^+_{\mathbf{k}}\right] = 0$. Note that while $\left\{a^+_{\mathbf{k},s}, a^+_{\mathbf{k},s}\right\} = 0$ implies $\left(a^+_{\mathbf{k},s}\right)^2 = 0$, the fact that the commutator $\left[\beta^+_{\mathbf{k}}, \beta^+_{\mathbf{k}}\right]$ is equal to zero does not imply $\left(\beta^+_{\mathbf{k}}\right)^2 = 0$. This is however so since $\left(\beta^+_{\mathbf{k}}\right)^2$ contains $\left(a^+_{\mathbf{k}s}\right)^2$. The fact that $\left(\beta^+_{\mathbf{k}}\right)^2$ reduces to zero seems to be lost when turning from $\left(a^+_{\mathbf{k}\uparrow}, a^+_{-\mathbf{k}\downarrow}\right)$ operators to $\beta^+_{\mathbf{k}}$ operators. We will see below that the relation $\left(\beta^+_{\mathbf{k}}\right)^2 = 0$, which comes from Pauli blocking, is in fact preserved in the commutation algebra of these electron pairs.

If we now consider creation and annihilation operators, we find

$$\left[\beta_{\mathbf{k}'}, \beta^+_{\mathbf{k}}\right] = \delta_{\mathbf{k}'\mathbf{k}} - D_{\mathbf{k}'\mathbf{k}} \tag{3}$$

$$D_{\mathbf{k}'\mathbf{k}} = \delta_{\mathbf{k}'\mathbf{k}}\left(a^+_{\mathbf{k}\uparrow}a_{\mathbf{k}\uparrow} + a^+_{-\mathbf{k}\downarrow}a_{-\mathbf{k}\downarrow}\right) \tag{4}$$

The operator $D_{\mathbf{k}'\mathbf{k}}$ differs from zero because electron pairs are not elementary bosons. By noting that

$$\left[a^+_{\mathbf{k}\uparrow}a_{\mathbf{k}\uparrow}, \beta^+_{\mathbf{p}}\right] = \delta_{\mathbf{k}\mathbf{p}}\beta^+_{\mathbf{p}} = \left[a^+_{\mathbf{k}\downarrow}a_{\mathbf{k}\downarrow}, \beta^+_{\mathbf{p}}\right] \tag{5a}$$

it is easy to show that

$$\left[D_{\mathbf{k}'_1\mathbf{k}_1}, \beta^+_{\mathbf{k}_2}\right] = 2\beta^+_{\mathbf{k}_2}\delta_{\mathbf{k}_1\mathbf{k}_2}\delta_{\mathbf{k}'_1,\mathbf{k}_2} \tag{5b}$$

This leads us to identify that the exchange scattering for free electron pairs, formally defined by

$$\left[D_{\mathbf{k}'_1\mathbf{k}_1}, \beta^+_{\mathbf{k}_2}\right] = \sum_{\mathbf{k}'_2}\left\{\lambda\left(\begin{smallmatrix}\mathbf{k}'_2 & \mathbf{k}_2\\ \mathbf{k}'_1 & \mathbf{k}_1\end{smallmatrix}\right) + (\mathbf{k}_1 \leftrightarrow \mathbf{k}_2)\right\}2\beta^+_{\mathbf{k}'_2} \tag{6}$$

with a product of Kronecker symbols

$$\lambda\left(\begin{smallmatrix}\mathbf{k}'_2 & \mathbf{k}_2\\ \mathbf{k}'_1 & \mathbf{k}_1\end{smallmatrix}\right) = \delta_{\mathbf{k}'_1\mathbf{k}_1}\delta_{\mathbf{k}'_2\mathbf{k}_2}\delta_{\mathbf{k}_1\mathbf{k}_2} \tag{7}$$

Actually, this is just the value we expect for the scattering associated to electron exchanges between $(\mathbf{k}_1, \mathbf{k}_2)$ pairs, as visualized by the diagram of fig (1b).

The link between this exchange scattering and the Pauli exclusion principle can be easier to see if we consider the scalar product of two electron pairs. Indeed, using these commutators, we find

$$\begin{aligned}\langle v|\beta_{\mathbf{k}'_1}\beta_{\mathbf{k}'_2}\beta^+_{\mathbf{k}_2}\beta^+_{\mathbf{k}_1}|v\rangle &= \left\{\delta_{\mathbf{k}'_1\mathbf{k}_1}\delta_{\mathbf{k}'_2\mathbf{k}_2} - \lambda\left(\begin{smallmatrix}\mathbf{k}'_2 & \mathbf{k}_2\\ \mathbf{k}'_1 & \mathbf{k}_1\end{smallmatrix}\right)\right\} + (\mathbf{k}'_1 \leftrightarrow \mathbf{k}'_2)\\ &= \delta_{\mathbf{k}'_1\mathbf{k}_1}\delta_{\mathbf{k}'_2\mathbf{k}_2}\left(1 - \delta_{\mathbf{k}_1\mathbf{k}_2}\right) + (\mathbf{k}'_1 \leftrightarrow \mathbf{k}'_2)\end{aligned} \tag{8}$$

The above result readily shows that the Pauli scattering $\lambda\left(\begin{smallmatrix}\mathbf{k}'_2 & \mathbf{k}_2\\ \mathbf{k}'_1 & \mathbf{k}_1\end{smallmatrix}\right)$ insure the RHS to cancel for $\mathbf{k}'_1 = \mathbf{k}'_2$, as necessary from Pauli blocking in the LHS, the state $\beta^+_{\mathbf{k}_2}\beta^+_{\mathbf{k}_1}|v\rangle$ reducing to zero for $\mathbf{k}_1 = \mathbf{k}_2$

We now consider the commutator of the free pair creator operator $\beta^+_{\mathbf{p}}$ with the BCS hamiltonian. Using eq. (5b), we find for the free part

$$\left[H_0, \beta^+_{\mathbf{p}}\right] = 2\epsilon_{\mathbf{p}}\beta^+_{\mathbf{p}} \tag{9}$$

while the potential part gives

$$\left[V_{BCS}, \beta^+_{\mathbf{p}}\right] = \gamma^+_{\mathbf{p}} + V^+_{\mathbf{p}} \tag{10}$$

where we have set $\gamma^+_{\mathbf{p}} = \sum_{\mathbf{k}} \beta^+_{\mathbf{k}} v_{\mathbf{kp}}$. The "creation potential" for the free pair $\mathbf{p}$ appears to be

$$V^+_{\mathbf{p}} = -\gamma^+_{\mathbf{p}} \left(a^+_{\mathbf{p}\uparrow} a_{\mathbf{p}\uparrow} + a^+_{-\mathbf{p}\downarrow} a_{\mathbf{p}\downarrow}\right) \tag{11}$$

While the $\gamma^+_{\mathbf{p}}$ part of $\left[V_{BCS}, \beta^+_{\mathbf{p}}\right]$ commutes with $\beta^+_{\mathbf{p}}$, the creation potential $V^+_{\mathbf{p}}$ does not. Its commutator precisely reads

$$\left[V^+_{\mathbf{p}_1}, \beta^+_{\mathbf{p}_2}\right] = -2\delta_{\mathbf{p}_1\mathbf{p}_2}\gamma^+_{\mathbf{p}_1}\beta^+_{\mathbf{p}_2} \tag{12}$$

This allows us to show that the interaction scattering for free pairs formally defined as

$$\left[V^+_{\mathbf{p}_1}, \beta^+_{\mathbf{p}_2}\right] = \sum \chi \begin{pmatrix} \mathbf{p}'_2 & \mathbf{p}_2 \\ \mathbf{p}'_1 & \mathbf{p}_1 \end{pmatrix} \beta^+_{\mathbf{p}'_1}\beta^+_{\mathbf{p}'_2} \tag{13}$$

must be identified with

$$\chi \begin{pmatrix} \mathbf{p}'_2 & \mathbf{p}_2 \\ \mathbf{p}'_1 & \mathbf{p}_1 \end{pmatrix} = -\left(v_{\mathbf{p}'_1,\mathbf{p}_1}\delta_{\mathbf{p}'_2,\mathbf{p}_2} + v_{\mathbf{p}'_2,\mathbf{p}_2}\delta_{\mathbf{p}'_1,\mathbf{p}_1}\right)\delta_{\mathbf{p}_2,\mathbf{p}_1} \tag{14}$$

This scattering corresponds to the diagram of fig 1c: the two free pairs first exchange an electron; as for any exchange, this which brings a minus sign. In a second step, one of the two pairs interact via the BCS potential.

Using these commutators, it is easy to find V_{BCS} acting on free pair states. By noting that $V^+_{\mathbf{p}} \left|F_0\right\rangle = 0$ due to the $v_{\mathbf{kp}} = -V\, w_{\mathbf{k}} w_{\mathbf{p}}$ factor included is $\gamma^+_{\mathbf{p}}$, we find for one-free-pair state:

$$V_{BCS}\beta^+_{\mathbf{p}} \left|F_0\right\rangle = \gamma^+_{\mathbf{p}} \left|F_0\right\rangle \tag{15}$$

For two-free-pair state, eqs. (10,12) give

$$\begin{aligned} V_{BCS}\beta^+_{\mathbf{p}_1}\beta^+_{\mathbf{p}_2} \left|F_0\right\rangle &= \left(\gamma^+_{\mathbf{p}_1}\beta^+_{\mathbf{p}_2} + \beta^+_{\mathbf{p}_1}\gamma^+_{\mathbf{p}_2} + \sum \chi \begin{pmatrix} \mathbf{p}'_2 & \mathbf{p}_2 \\ \mathbf{p}'_1 & \mathbf{p}_1 \end{pmatrix} \beta^+_{\mathbf{p}'_2}\beta^+_{\mathbf{p}'_2}\right) \left|F_0\right\rangle \\ &= \left(\gamma^+_{\mathbf{p}_1}\beta^+_{\mathbf{p}_2} + \beta^+_{\mathbf{p}_1}\gamma^+_{\mathbf{p}_2} - 2\delta_{\mathbf{p}_1\mathbf{p}_2}\beta^+_{\mathbf{p}'_2}\beta^+_{\mathbf{p}'_2}\right) \left|F_0\right\rangle \end{aligned} \tag{16}$$

and so on... The above equation tells us that the two pairs of a two-free-pair state do not interact except through the Pauli exclusion principle. This Pauli blocking appears through the $\delta_{\mathbf{p}_1\mathbf{p}_2}$ term in eq. (16) which just insures the RHS to cancel when $\mathbf{p}_1 = \mathbf{p}_2$ as necessary since $\beta_{\mathbf{p}}^{+2} = 0$ due to the Pauli exclusion principle between up and down spin electrons.

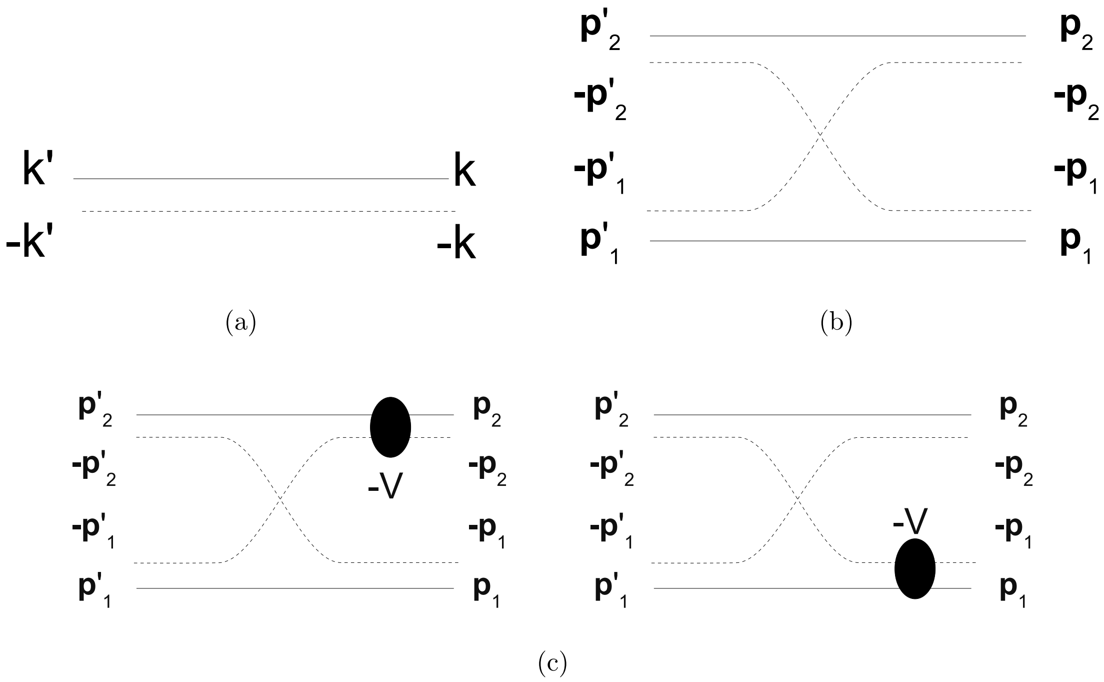


FIG. 1: Shiva diagram of free pairs

(a) The BCS potential given in eq. (1) transforms a $\mathbf{k}$ pair into a $\mathbf{k}'$ pair with a constant scattering $-V$. Up spin electrons are represented by solid line while down spin electrons are represented by dashed line.

(b) Pauli scattering $\lambda\begin{pmatrix} \mathbf{p}'_2 & \mathbf{p}_2 \\ \mathbf{p}'_1 & \mathbf{p}_1 \end{pmatrix}$ for electron exchange between two free pairs $(\mathbf{p}_1, \mathbf{p}_2)$, as given by eq. (7).

(c) Interaction scattering $\chi\begin{pmatrix} \mathbf{p}'_2 & \mathbf{p}_2 \\ \mathbf{p}'_1 & \mathbf{p}_1 \end{pmatrix}$ between two free electron pairs, as given in eq. (14)

B. Cooper pair creation operators

we now introduce the linear combinations of free pairs creation operators which are the one-pair eigenstates of the BCS hamiltonian. These operators create the Cooper pair states

$|i\rangle = B_i^+ |F_0\rangle$ where $|F_0\rangle$ is the "frozen" Fermi sea. Note that these Cooper pairs states not only include the standard ground state for single pair derived by Cooper, but also all the one-pair excited states in order to form a complete set for electron pairs with zero total momentum. The closure relations written with free pairs and Cooper pairs then read, for normalized states $|i\rangle$, i.e., states such that $\langle i|i\rangle = 1$

$$I = \sum_{\mathbf{k}} \beta_{\mathbf{k}}^+ |F_0\rangle \langle F_0| \beta_{\mathbf{k}} \tag{17}$$

$$I = \sum_{i} B_i^+ |F_0\rangle \langle F_0| B_i \tag{18}$$

By injecting eq. (17) in front of $|i\rangle$, it is then easy to show that free pair and Cooper pair creator operators, $\beta_{\mathbf{k}}^+$ and B_i are linked by

$$B_i^+ = \sum_{\mathbf{k}} \beta_{\mathbf{k}}^+ \langle \mathbf{k}|i\rangle \tag{19}$$

$$\beta_{\mathbf{k}}^+ = \sum_{i} B_i \langle i|\mathbf{k}\rangle \tag{20}$$

where $|\mathbf{k}\rangle = \beta_{\mathbf{k}}^+ |F_0\rangle$ is the single free pair state made of the frozen Fermi sea $|F_0\rangle$ plus one free electron pair $(\mathbf{k}, -\mathbf{k})$. So that $|\mathbf{k}\rangle$ differs from zero for $\epsilon_{\mathbf{k}}$ large than the frozen Fermi sea energy ϵ_{F_0}.

C. Commutation for Cooper pair operators

Using the commutator for free electron pairs given in eqs. (3,4), it is easy to show that for normalized Cooper pair eigenstate $\langle i|j\rangle = \delta_{ij}$, we do have

$$\left[B_m, B_i^+\right] = \delta_{mi} - D_{mi} \tag{21}$$

$$\begin{aligned} D_{mi} &= \sum_{\mathbf{k}} \langle m|\mathbf{k}\rangle D_{\mathbf{kp}} \langle \mathbf{p}|i\rangle \\ &= \sum_{\mathbf{k}} \langle m|\mathbf{k}\rangle \langle \mathbf{k}|i\rangle \left(a_{\mathbf{k}\uparrow}^+ a_{\mathbf{k}\uparrow} + a_{-\mathbf{k}\downarrow}^+ a_{-\mathbf{k}\downarrow}\right) \end{aligned} \tag{22}$$

Note that due to the $|\mathbf{k}\rangle$ state in the ????, free ??? $\mathbf{k}$ electrons is the ???? above $|F_0\rangle$, so that $D_{mi}|F_0\rangle = 0$. Pauli scattering for Cooper pairs follows as

$$\left[D_{mi}, B_j^+\right] = 2 \sum_{n} \lambda \left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right) B_n^+ \tag{23}$$

$$\lambda\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right)=\sum_{\mathbf{k}}\langle m|\mathbf{k}\rangle\langle n|\mathbf{k}\rangle\langle\mathbf{k}|i\rangle\langle\mathbf{k}|j\rangle \tag{24}$$

This Pauli scattering is visualized by the diagram of fig (2a). This diagram is the same structure as the Shiva diagram for exciton exchanges we introduced in the many-body theory of composite boson excitons.

From these commutators, we readily find that the scalar product of two Cooper pair states has the usual form, namely

$$\langle F_0|B_m B_n B_j^+ B_i^+|v\rangle=\delta_{mi}\delta_{nj}+\delta_{mj}\delta_{ni}-2\lambda\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right) \tag{25}$$

By noting that $\langle m|i\rangle=\sum_{\mathbf{k}}\langle m|\mathbf{k}\rangle\langle\mathbf{k}|i\rangle$, we can write this scalar product, using the expression of the Pauli scattering given in eq. (24), as

$$\langle F_0|B_m B_n B_j^+ B_i^+|v\rangle=\sum_{\mathbf{k}}\langle m|\mathbf{k}\rangle\langle\mathbf{k}|i\rangle\sum_{\mathbf{k}'\neq\mathbf{k}}\langle n|\mathbf{k}'\rangle\langle\mathbf{k}'|j\rangle+(m\leftrightarrow n) \tag{26}$$

The above expression shows in a quite clear way that the effects of the $\lambda\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right)$ term in the scalar product (25) is just to remove $\mathbf{k}'=\mathbf{k}$ terms in the sum, as a result of Pauli exclusion principle between electrons making the Cooper pairs. However, from a technical point of view, the expression of Cooper pair scalar product with restricted sum, like the one of eq (26), are less convenient to handle than expressions using $\lambda\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right)$ like the one of eq. (25). This is why the introduction of Pauli scattering to take care of the Pauli exclusion principle is indeed convenient, mostly when the number of Cooper pair at hand is large.

Let us now turn to the interaction scatterings between Cooper pairs. For that, we first note that in $|i\rangle$ using hamiltonian eigenstates, we do have

$$0=(H-E_i)|i\rangle=\sum_{\mathbf{p}}\langle\mathbf{p}|i\rangle\left[(2\epsilon_{\mathbf{p}}-E_i)\,\beta_{\mathbf{p}}^+ +\sum_{\mathbf{k}}\beta_{\mathbf{k}}^+ v_{\mathbf{kp}}\right]|F_0\rangle \tag{27}$$

By projecting the above equation on $\langle\mathbf{p}'|$, we find since $\langle\mathbf{p}'|\mathbf{p}\rangle=\delta_{\mathbf{p}'\mathbf{p}}$ that the $\langle\mathbf{p}|i\rangle$'s fulfill the Schrödinger equation

$$2\epsilon_{\mathbf{p}}\langle\mathbf{p}'|i\rangle+\sum_{\mathbf{p}}v_{\mathbf{p}'\mathbf{p}}\langle\mathbf{p}|i\rangle=E_i\langle\mathbf{p}'|i\rangle \tag{28}$$

This equation gives $\langle\mathbf{p}'|i\rangle$ in terms of a sum of $\langle\mathbf{p}|i\rangle$. In the case of a separable potential $v_{\mathbf{p}'\mathbf{p}}=-V\,w_{\mathbf{p}}w_{\mathbf{p}'}$, we recover the well-known relation of the Cooper pair energy, namely

$$\frac{1}{V}=\sum\frac{w_{\mathbf{p}}}{2\epsilon_{\mathbf{p}}-E_i} \tag{29}$$

Using eq (28), it is then easy to show, from eqs (8,9) that the commutator $\left[H_0, B_i^+\right]$ has the standard form, namely

$$\begin{aligned}\left[H, B_i^+\right] &= \sum_{\mathbf{p}} \langle \mathbf{p}|i\rangle \left(2\epsilon_{\mathbf{p}}\beta_{\mathbf{p}}^+ + \sum_{\mathbf{k}} \beta_{\mathbf{k}}^+ v_{\mathbf{kp}}\right) + V_{\mathbf{p}}^+ \\ &= E_i B_i^+ + V_i^+\end{aligned} \tag{30}$$

where $V_i^+ = \sum \langle \mathbf{p}|i\rangle V_{\mathbf{p}}^+$, using the definition of $V_{\mathbf{p}}^+$ given in eq (10), we find that

$$\left[V_i^+, B_j^+\right] = -2\sum_{\mathbf{pk}} \langle \mathbf{p}|i\rangle \langle \mathbf{p}|j\rangle \beta_{\mathbf{p}}^+ \beta_{\mathbf{k}}^+ v_{\mathbf{kp}} \tag{31}$$

If we now use eq (19) to write free pair operators in term of Cooper pair and symmetrize the result with respect to m and n, we end with

$$\left[V_i^+, B_j^+\right] = \sum_{mn} \chi \left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right) B_n^+ B_m^+ \tag{32}$$

where the interaction scattering is given by

$$\chi \left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right) = -\sum_{\mathbf{kp}} \langle m|\mathbf{k}\rangle \langle n|\mathbf{p}\rangle \langle \mathbf{p}|j\rangle \langle \mathbf{p}|i\rangle v_{\mathbf{kp}} + (m \leftrightarrow n) \tag{33}$$

This scattering is represented by the diagram of fig (2b).

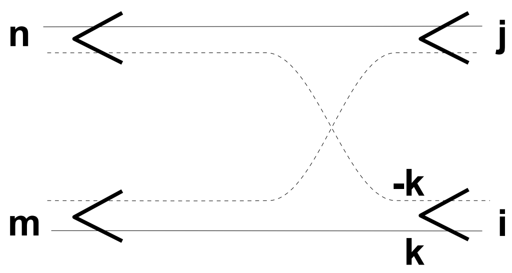
Appendix A: The possibility to ???? the BCS hamiltonian in terms of Cooper pair operators

Let us introduce the free pair hamiltonian defined as $\widetilde{H_0} = \sum 2\epsilon_{\mathbf{k}}\beta_{\mathbf{k}}^+\beta_{\mathbf{k}}$ This free pair hamiltonian and the free electron hamiltonian H_0 seems to act in the same way on n-pair states. Indeed, eq. (8) gives H_0 acting on two pairs as

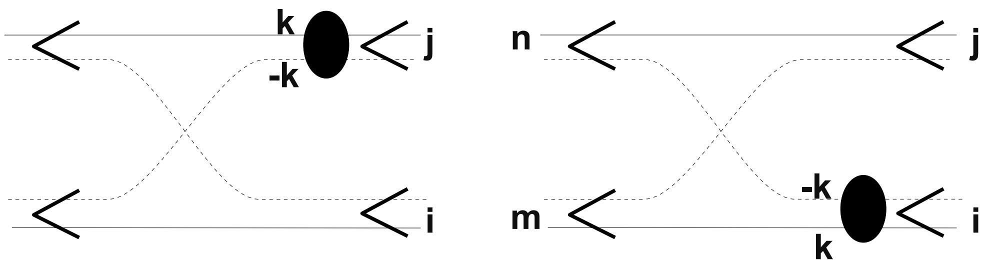
$$\begin{aligned}H_0\beta_{\mathbf{p}_1}^+\beta_{\mathbf{p}_2}|F_0\rangle &= \left(\left[H_0, \beta_{\mathbf{p}_1}^+\right] + \beta_{\mathbf{p}_1}^+ H_0\right)\beta_{\mathbf{p}_2}|F_0\rangle \\ &= \left(2\epsilon_{\mathbf{p}_1} + 2\epsilon_{\mathbf{p}_2}\right)\beta_{\mathbf{p}_1}^+\beta_{\mathbf{p}_2}^+|F_0\rangle\end{aligned} \tag{A1}$$

In the case of $\widetilde{H_0}$, the same procedure gives with eq (8) replaced by $\left[\widetilde{H_0}, \beta_{\mathbf{p}_1}^+\right] = 2\epsilon_{\mathbf{p}}\beta_{\mathbf{p}}^+ - \sum 2\epsilon_{\mathbf{k}}\beta_{\mathbf{k}}^+ D_{\mathbf{kp}}$

$$\left(\widetilde{H_0} - H_0\right)\beta_{\mathbf{p}_1}^+\beta_{\mathbf{p}_2}^+|F_0\rangle = -\sum 2\epsilon_{\mathbf{k}}\beta_{\mathbf{k}}^+ D_{\mathbf{kp}_1}\beta_{\mathbf{p}_2}^+|F_0\rangle \tag{A2}$$



(a)



(b)

FIG. 2: Shiva diagram of Cooper pairs

(a) Pauli scattering for electron exchange between Cooper pair sas given by eq. (24).

(b) Interaction scattering between Cooper pairs, as given in eq. (33). Due to the peculiar form of the BCS potential, two Cooper pairs only interact through the Pauli exclusion principle, via electron exchange.

Due to eq (4), the RHS of the above equation reduces to zero for $\mathbf{p}_1 \neq \mathbf{p}_2$. Since this condition is fulfilled for the 2-pair state of ???? for $\beta^+_{\mathbf{p}_1}\beta^+_{\mathbf{p}_2}|F_0\rangle$ to differ from zero due to the Pauli exclusion principle, we are tempted to conclude that H_0 can be replaced by $\widetilde{H_0}$.

We can be even more tempted to make such a replacement once we note that $\widetilde{H} = \widetilde{H_0} + V_{BCS}$ takes a very compact form in term of Cooper pair operators. Indeed, we get using eq (19).

$$\begin{aligned}\widetilde{H} &= \sum_{\mathbf{k}} \left(2\epsilon_{\mathbf{k}}\beta^+_{\mathbf{k}} + \sum_{\mathbf{k}'} \beta^+_{\mathbf{p}'} v_{\mathbf{p}'\mathbf{p}} \right) \beta^+_{\mathbf{k}} \\ &= \sum_{ij} B^+_i B_j \sum_{\mathbf{k}} \left[2\epsilon_{\mathbf{k}} \langle i|\mathbf{k}\rangle + \sum_{\mathbf{k}'} \langle i|\mathbf{k}'\rangle v_{\mathbf{k}'\mathbf{k}} \right] \langle \mathbf{k}|j\rangle\end{aligned} \tag{A3}$$

Since due to eq (28), the bracket reduces to $E_i\langle i|\mathbf{k}\rangle$, the summation over $\mathbf{k}$, performed through closure relation leads to

$$\widetilde{H} = \sum E_i B_i^+ B_i \tag{A4}$$

In spite of the fact that H_0 and $\widetilde{H_0}$ act in a similar way on n-pair state, the replacement of H_0 by $\widetilde{H_0}$ introduce ?????? Pauli blocking which ultimately affect all matrix elements. To see it, we can note that

$$\begin{aligned}\left[\widetilde{H_0}, \beta_{\mathbf{p}}^+\right] &= \sum 2\epsilon_{\mathbf{k}}\beta_{\mathbf{k}}^+ \left[\beta_{\mathbf{k}}, \beta_{\mathbf{p}}^+\right] \\ &= 2\epsilon_{\mathbf{p}}\beta_{\mathbf{p}}^+ - \sum 2\epsilon_{\mathbf{k}}\beta_{\mathbf{k}}^+ D_{\mathbf{kp}}\end{aligned} \tag{A5}$$

The first term in the RHS using the value of $\left[H_0, \beta_{\mathbf{p}}^+\right]$. Turning to Cooper pair operators, this leads to

$$\left[\widetilde{H}, B_i^+\right] = \left[H, B_i^+\right] + W_i^+ \tag{A6}$$

When $W_i^+ = -\sum 2\epsilon_{\mathbf{k}}\beta_{\mathbf{k}}^+ D_{\mathbf{kp}}\langle \mathbf{p}|i\rangle$. this additional operator in the commutator generates an additional contribution to the interaction scattering which then reads

$$\widetilde{\chi}\left(\begin{smallmatrix} n & j \\ m & i\end{smallmatrix}\right) = \chi\left(\begin{smallmatrix} n & j \\ m & i\end{smallmatrix}\right) - 2\sum_{\mathbf{kp}} 2\epsilon_{\mathbf{k}}\langle m|\mathbf{k}\rangle\langle n|\mathbf{p}\rangle\langle \mathbf{p}|j\rangle\langle \mathbf{p}|i\rangle v_{\mathbf{kp}} \tag{A7}$$

We now use the expression of $\chi\left(\begin{smallmatrix} n & j \\ m & i\end{smallmatrix}\right)$ given in eq (33), we find that the interaction scattering associated to $\widetilde{H}$ takes a quite compact form. Using eqs (24,28), we find

$$\begin{aligned}\widetilde{\chi}\left(\begin{smallmatrix} n & j \\ m & i\end{smallmatrix}\right) &= -\sum_{\mathbf{k}}\left(\sum_{\mathbf{p}}\langle m|\mathbf{p}\rangle v_{\mathbf{pk}} + 2\epsilon_{\mathbf{k}}\langle m|\mathbf{k}\rangle\right)\langle n|\mathbf{p}\rangle\langle \mathbf{p}|j\rangle\langle \mathbf{p}|i\rangle v_{\mathbf{kp}} \\ &= -\left(E_m + E_n\right)\lambda\left(\begin{smallmatrix} n & j \\ m & i\end{smallmatrix}\right)\end{aligned} \tag{A8}$$

Although somewhat nicer than the explicit expression of the interaction scattering $\chi\left(\begin{smallmatrix} n & j \\ m & i\end{smallmatrix}\right)$ given in eq (28), it is clear that the replacement of H_0 by $\widetilde{H}$ brings new spurious terms in the calculation.

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[1] M. Combescot, O. Betbeder-Matibet, and F. Dubin, Physics Reports **463**, 215 (2008).