0.1 Two-Body Density Matrix in Richardson Solution

As pointed in [2], the existing of eigenvalue in order of N in the two-body density matrix of BCS solution is probably the most fundamental fact of the superfluid phenomenon. In the traditional BCS ansatz, the order parameter $F_{\mathbf{k}} = u_{\mathbf{k}} v^*_{\mathbf{k}}$ is this eigenvector with the eigenvalue of $N_0 = \frac{\pi}{4}\Delta N(0)\Omega$, [2, (5.4.32)]. Therefore it is important to check the solution with Richardson approach [1] about this quantity. What we are looking for is only zero-central-momentum eigenfunction,

$$\langle \beta^\dagger_{\mathbf{k}} \beta_{\mathbf{k}'} \rangle = \langle a^\dagger_{\mathbf{k}} b^\dagger_{-\mathbf{k}} b_{-\mathbf{k}'} a_{\mathbf{k}'} \rangle \tag{0.1}$$

Here we mostly follow the original paper's notion, with some short-hand explained as following:

$$B^\dagger_i = B^\dagger(R_i) \tag{0.2}$$

$$\langle i|\mathbf{k}\rangle = \langle \nu | B_i \beta^\dagger_{\mathbf{k}} |\nu\rangle \tag{0.3}$$

$$\langle i|j\rangle = \langle \nu | B_i B^\dagger_j |\nu\rangle = \sum_{\mathbf{k}} \langle i|\mathbf{k}\rangle \langle \mathbf{k}|j\rangle \tag{0.4}$$

In the Richardson solution, ground wave-function is $|\Psi_n\rangle = \prod_i^n B^\dagger_i |\nu\rangle$. Here different $B^\dagger_i |\nu\rangle$'s are neither orthogonal nor normalized. In fact, they have rather substantial overlap most of the time. Two-body density matrix here is further complicated by the fact that many-body wave-function is affected by the **composite** nature of those cobosons. Generally,

$$\langle \beta^\dagger_{\mathbf{k}} \beta_{\mathbf{k}'} \rangle = \frac{\langle \Psi_n | \beta^\dagger_{\mathbf{k}} \beta_{\mathbf{k}'} | \Psi_n\rangle}{\langle \Psi_n|\Psi_n\rangle} \tag{0.5}$$

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0.1.1 Two pairs

We started with a two-pair state, $|\Psi_2\rangle = B^\dagger_i B^\dagger_j |\nu\rangle$.

$$\begin{aligned} &\langle \Psi_2 | \beta^\dagger_{\mathbf{k}} \beta_{\mathbf{k}'} | \Psi_2 \rangle \\ =& \left[\left(\langle i|j\rangle \langle j|\mathbf{k}\rangle \langle \mathbf{k}'|j\rangle + \langle i|j\rangle \langle j|\mathbf{k}\rangle \langle \mathbf{k}'|i\rangle \right) + (i \leftrightarrow j) \right] \\ &- 2\left[\left(\langle j|\mathbf{k}\rangle \langle i|\mathbf{k}\rangle \langle \mathbf{k}|i\rangle \langle \mathbf{k}'|j\rangle + \langle \mathbf{k}'|i\rangle \langle \mathbf{k}'|j\rangle \langle j|\mathbf{k}\rangle \langle i|\mathbf{k}'\rangle \right) + (i \leftrightarrow j) \right] \\ &+ 4\langle j|\mathbf{k}\rangle \langle i|\mathbf{k}\rangle \langle \mathbf{k}'|i\rangle \langle \mathbf{k}'|j\rangle \delta_{\mathbf{k}\mathbf{k}'} \end{aligned} \tag{0.6}$$

Here the first four terms (fig. 1a,1b) are from the paring, while the next eight terms (fig. 1c-1f) as well as the last term (fig. 1g) are due to the Pauli exclusion of **composite** nature. They show the moth-eaten effect.

The last term is especially interseting as this is two separate pieceses (fig.2). Normally, they do not exist, but here they do show up because we need pair of $\mathbf{k}$,$\mathbf{k}'$. And the normalization factor:

$$\langle \Psi_2|\Psi_2\rangle = \langle B_j B_i B^\dagger_i B^\dagger_j \rangle = \langle i|i\rangle \langle j|j\rangle + |\langle i|j\rangle|^2 - 2\sum_{\mathbf{k}} \langle j|\mathbf{k}\rangle \langle i|\mathbf{k}\rangle \langle \mathbf{k}|i\rangle \langle \mathbf{k}|j\rangle \tag{0.7}$$

0.1.2 Three pairs

For a three-pair state $|\Psi_3\rangle = B^\dagger_i B^\dagger_j B^\dagger_m |\nu\rangle$. we have

$$\begin{aligned} \langle \Psi_3 | \beta^\dagger_{\mathbf{k}} =& \langle \nu | B_m B_j B_i \beta^\dagger_{\mathbf{k}} \\ =& -2\langle j|\mathbf{k}\rangle \langle m|\mathbf{k}\rangle \langle \nu | \beta_{\mathbf{k}} B_i - 2\langle m|\mathbf{k}\rangle \langle i|\mathbf{k}\rangle \langle \nu | \beta_{\mathbf{k}} B_j - 2\langle i|\mathbf{k}\rangle \langle j|\mathbf{k}\rangle \langle \nu | \beta_{\mathbf{k}} B_m \\ &+ \langle m|\mathbf{k}\rangle \langle \nu | B_i B_j + \langle i|\mathbf{k}\rangle \langle \nu | B_j B_m + \langle i|\mathbf{k}\rangle \langle \nu | B_m B_i \\ =& \left(-2\langle i|\mathbf{k}\rangle \langle j|\mathbf{k}\rangle \langle \nu | \beta_{\mathbf{k}} B_m + \langle i|\mathbf{k}\rangle \langle \nu | B_j B_m \right) + (\text{i,j,m rotate permutation}) \end{aligned} \tag{0.8}$$

We are lots of terms in the final expectation, like $\langle B^\dagger B^\dagger B B\rangle$, $\langle \beta^\dagger B^\dagger B B\rangle$,$\langle \beta^\dagger B^\dagger B \beta\rangle$, $\langle B^\dagger B^\dagger B \beta\rangle$.

References

[1] Monique Combescot and Guojun Zhu, **Coboson derivation of richardsons equations for cooper pairs**, 2010.

[2] Anthony J. Leggett, **Quantum Liquids**, Oxford University Press, 2006.

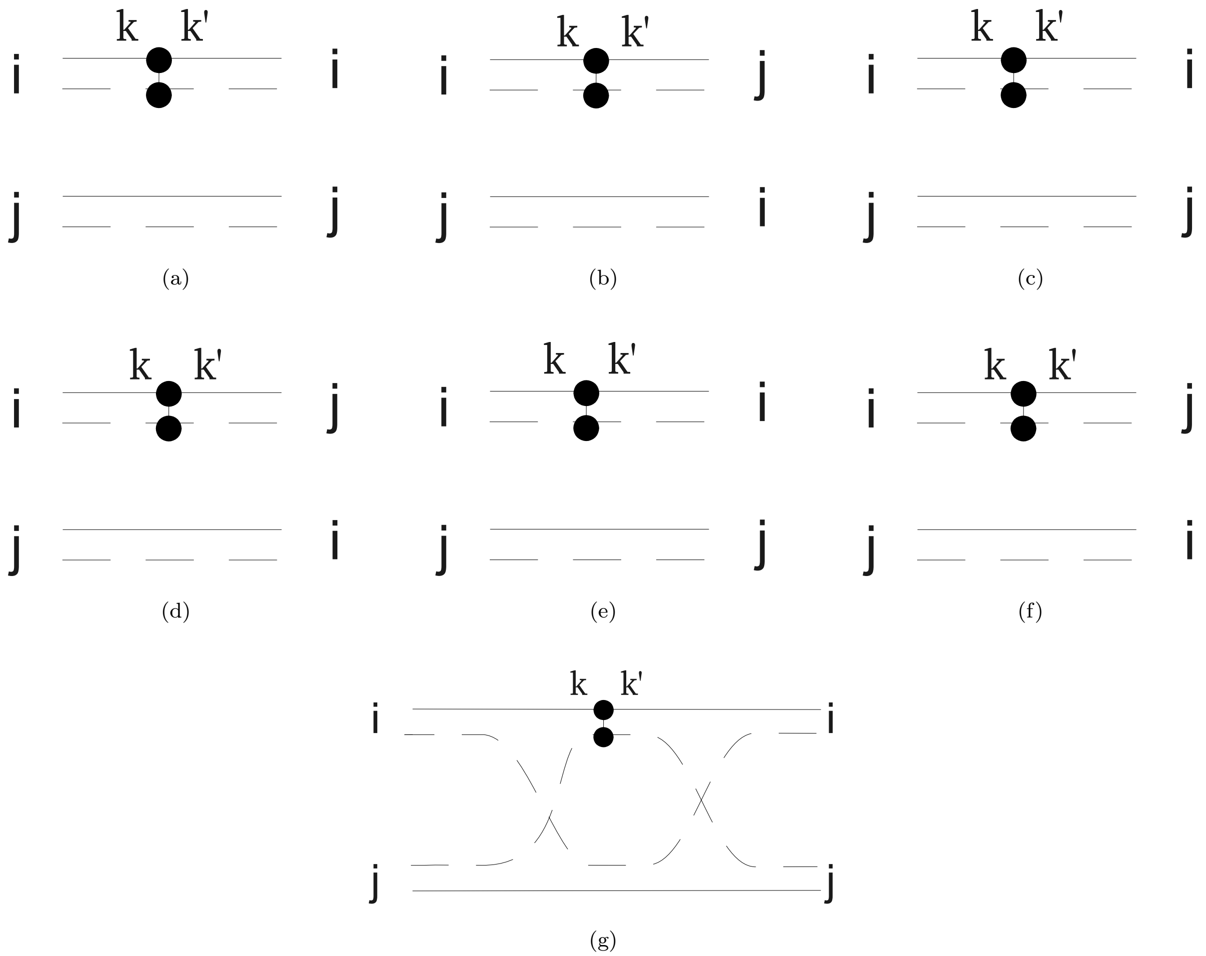

Figure 1: Shiva diagram of two pairs

(a),(b) Signle pair without Pauli scattering. There are also terms with $(i \leftrightarrow j)$. First line of the above equation.

(c),(d),(e),(f) Two-pair with Pauli scattering. There are also terms with $(i \leftrightarrow j)$. Second line of the equation.

(g) This coresponds the last term. And there are the permutation of (i,j) as other terms.

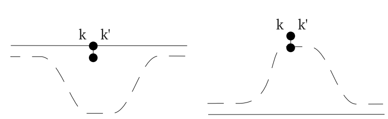

Figure 2: Two separate connection parts of the last term of (0.6)