

1 note of 2009.09.09

1.1 Time dependence of anomalous Green function

In the normal Green function, where annihilator and creator paired, only relative time difference matter, the absolute time postilion does not matter and cancels out for a time-independent Hamiltonian. The anomalous Green function has two annihilators or two creators, and the absolute time does bring in an extra phase factor. (See [1]), they explicitly pull the factor $e^{(\mu_1+\mu_2)t}$ out.

1.2 Thought about Skip two 4-operator terms in the Motion of Equation of equal-time-pair wave-function

In Zhang's thesis ([9]), he omit two 4-operator terms with the argument that they introduce extra real coordinators and therefore introduce extra $\mathbf{k}_F r_0$ terms, which is small in dilute system. I am still a bit skeptical about whether it is appropriate in my problem. But it is probably OK. We cannot make progress with some quite dangerous looking assumption, such throwing away all Hatree breaking of the four-operator term.

According to Zhang, the equation (Eq. 5.37) obtained by omit these two 4-operator terms still carries information about the many-body physics through the normalization of them.

1.3

χ_0 in Zhang's thesis is not the two-body close-channel bound statfe, but should coincide it in the short length-scale except the normalization. But then what? Tony's suggestion is always to express everything back to the quantities measureable, such as a_s . But it is unlikely only a_s is enough. What will be the other quantities that I can use?

2 note of 2009.09.10

2.1

If we start from real space of pair wave function. There is no obvious Δ , i.e. gap. Or we can define it as the $\sum U_{\mathbf{k},\mathbf{k}'} F'_{\mathbf{k}}$ and it is relative slowly-varied in k-space because of the slow-change of $U_{\mathbf{k},\mathbf{k}'}$.

2.2 Pair-Wave-Function in Real Space

How should pair wave function $F(r)$ looks like? Let us look in two extremes. In BCS, it falls exponentially with Piccard Coherence length (Eq. 2.62 in [9],[6]) which should be larger than $1/\mathbf{k}_F$; while in BEC, it also falls exponentially but with a characteristic length smaller than $1/\mathbf{k}_F$. And in principle, between potential range r_0 and this characteristic length, or $1/\mathbf{k}_F$ if it is smaller, it has the s-wave scattering like wave-function, $\frac{1}{r} \left(1 - \frac{r}{a_s}\right)$ characted by a_s , which is determined by two-body physics.

2.3 What is the Difference between the Many-body Motion of Equation and the Two-Body Schrödinger Equation

The equation 5.37 in [9] is the same as the two-body equation. We know in k-space, the difference is that the Fermi sea is filled for many-body case; in real space, two-body limit means that density approaches 0, so many-body means that potential is modified in the $1/\mathbf{k}_F$ scale and beyond. So maybe we can breaks the length scale by $1/\mathbf{k}_F$. For equation 5.36 (4.2) in [9], we can throw away those two four-operator terms for $r < 1/\mathbf{k}_F$ and must keep them for $r > 1/\mathbf{k}_F$. For $r > 1/\mathbf{k}_F$, we can choose r' within the range of either r_1 or r_2 , and this might be sufficient to include the many-body physics into the problem.

In Abrikosov's Green function treatment over BCS, it is exactly these four-operator terms work toward the gap equation. By breaking r into two regions, I might reconcile the two-body solution with the many-body one.

This complication however prevents a simple Fourier transformation of the equations as there are two different equations for different regions of r .

One obvious problem is whether the original Abrikosov treatment can be solved directly in real space and therefore can be used here. The short r region certainly can be solved in real space and real space indeed is more suitable for solving. But the long r region, i.e. the many-body, is easier to solve in k -space.

One possible method may be using the Pseudopotential Δ instead, just as in Bogolubov-deGennes equation.

One odd thing about Shizhong's treatment is this two-operator term, that emerges only because of the equal-time of $\Psi_\alpha(t)\Psi_\beta(t)$. They do not show in Abrikosov's treatment and needs more careful investigation.

2.4 Meet Tony

Tony has some doubts on Abrikosov's treatment over BCS ([1]). He is not convinced by their throwing the Coulomb interaction and other terms. But this is also true in the variation method. Anyway, he agrees that the four-operators thrown away by Shizhong might be important for my problem. I should explore my idea in 2.3.

My goal is to find out when we take the Pauli principle in the close-channel, open-channel due to the common species seriously, what kind of correction, no matter how small it is, I can get. In most approach, Pauli principle does not matter as close-channel is treated as independently condensate and no Pauli principle on this.

3 note of 2009.09.13

3.1 AGD in Real Space

It is not trivial to look at normal green's function in real space either. However, for the equal-time one like Shizhong's. It is just the normal density and should be just ρ .

4 note of 2009.09.15

4.1 Several Equations in Shizhong's Thesis

Equation 5.35 of [9], the Hamiltonian

$$\begin{aligned} \mathcal{H} = & \sum_{\alpha} \int d\mathbf{r} \psi_{\alpha}^{\dagger}(\mathbf{r}) \left(-\frac{\hbar^2}{2m} \nabla^2 - \mu_{\alpha} + E_{\alpha} \right) \psi_{\alpha}(\mathbf{r}) \\ & + \frac{1}{2} \sum_{\alpha\beta\gamma\delta} \int d\mathbf{r}_1 d\mathbf{r}_2 \psi_{\alpha}^{\dagger}(\mathbf{r}_1) \psi_{\beta}^{\dagger}(\mathbf{r}_2) U_{\alpha\beta\gamma\delta}(\mathbf{r}_1 - \mathbf{r}_2) \psi_{\gamma}(\mathbf{r}_2) \psi_{\delta}(\mathbf{r}_1) \end{aligned} \quad (4.1)$$

Equation 5.36, the equation of motion of the equal-time abnormal green's function, $\psi_\alpha(\mathbf{r}_1, t) \psi_\beta(\mathbf{r}_2, t)$:

$$\begin{aligned}
& i\hbar \frac{\partial \psi_\alpha(\mathbf{r}_1, t) \psi_\beta(\mathbf{r}_2, t)}{\partial t} \\
&= \left(-\frac{\hbar^2}{2m} \nabla_1^2 - \mu_\alpha + E_\alpha - \frac{\hbar^2}{2m} \nabla_2^2 - \mu_\beta + E_\beta \right) \psi_\alpha(\mathbf{r}_1, t) \psi_\beta(\mathbf{r}_2, t) \\
&+ \sum_{\gamma\delta} U_{\alpha\beta\delta\gamma}(\mathbf{r}_1 - \mathbf{r}_2) \psi_\gamma(\mathbf{r}_1) \psi_\delta(\mathbf{r}_2) \\
&+ \sum_{\beta'\gamma\delta} \int d\mathbf{r}' U_{\alpha\beta'\gamma\delta}(\mathbf{r}_1 - \mathbf{r}') \psi_{\beta'}^\dagger(\mathbf{r}') \psi_\gamma(\mathbf{r}') \psi_\delta(\mathbf{r}_1) \psi_\beta(\mathbf{r}_2) \\
&+ \sum_{\beta'\gamma\delta} \int d\mathbf{r}' U_{\beta\beta'\gamma\delta}(\mathbf{r}_2 - \mathbf{r}') \psi_{\beta'}^\dagger(\mathbf{r}') \psi_\gamma(\mathbf{r}') \psi_\alpha(\mathbf{r}_1) \psi_\delta(\mathbf{r}_2)
\end{aligned} \tag{4.2}$$

I have verified this equation and all the subscripts are correct. (See the paper notebook) It uses the fact that

$$U_{\alpha\beta\delta\gamma}(\mathbf{r}_1, \mathbf{r}_2) = U_{\beta\alpha\gamma\delta}(\mathbf{r}_2, \mathbf{r}_1) \tag{4.3}$$

Equation (4.2) is similar as the equations in AGD [1], which is for non-equal-time quantities, but with several difference. There is no two-operator term there, and there is only one four operator term because there are two terms for differentiation in the equal-time quantities. In another respect, for abnormal Green's function $F_{\alpha\beta}(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2) = \psi_\alpha(\mathbf{r}_1, t_1) \psi_\beta(\mathbf{r}_2, t_2)$, the one worked out in AGD is for $t = t_2 - t_1$ while the equal-time one here is more close to the central position $t = (t_1 + t_2)/2$. **(I am not very sure about this.)**

5 note of 2009.09.16

5.1 More Discussion on the Equal-Time Abnormal Green Function

I continue to think about the Shizhong's approach [9] with equal-time abnormal Green's Function from last section 4.1. Well, the equal-time function seems better for most of the physical application. For those non-equal-time function, one usually has to take the limit of $t \rightarrow 0$ to get back to the real physical quantities. Well, the four-vector, which includes time as the special dimension, is natural in the high-energy application where Lorentz invariance put time into a quite equal footing as space coordinators. But in statistical physics, time is always special. Is the non-equal-time sufficient to handle the problems, why do we need the non-equal time Green's function?

Normal Green's function has non-dependence on the absolute value of the time, only the time difference $t_2 - t_1$. And all the $\omega \neq 0$ quantities describe the dynamics. By fixing $t_2 = t_1$, we only have the static quantities. First of all, these equal-time quantity is not 0. Go to the Lehmann representation, we can find out that the equal-time operator pairs have no time dependence at all, at least for the normal case, no matter in zero temperature $\langle 0 | \dots | 0 \rangle$ or finite temperature $\sum_m e^{-\beta E_m} \langle m | \dots | m \rangle$. I am not sure whether the speciality of the BCS ground state makes any difference. Especially, for the abnormal Green's function, you have $\langle N | \dots | N \pm 2 \rangle$, which gives you some extra $\exp[i(2\mu t/\hbar)]$.

6 note of 2009.09.17

6.1 Thermodynamics of BCS Wave Function

The more I think about the equal-time Green's function, the weird I felt. Golden Baym comments on the problem. In the order parameter setting, $\psi\psi$ in BCS or ψ in BEC, they do have dependence on absolute time related to chemical potential, 2μ in BCS and μ in BEC. In BCS, $F(t_1, t_2) = e^{-i2\mu t_1/\hbar} F(0, t_2 - t_1)$. So equation (4.2) equals 2μ .

6.2 Baym's Comment on 3-Species Problem

He comment on the 3-species problem. In Landau Fermi liquid, short-range correlation becomes the equivalent repulsion, so maybe it is the same here. Maybe it can be dedued to some equivalent repulsion.

6.3 Decoupling Four-Operator Terms in Equal-Time Equation of Motion

It is not clear how to decouple four-operator terms in two-channel problems, such as Shizhong's equation(4.2). It is not obvious how to decouple these two terms. There are several possibilities. We may follow the similar way as in normal BCS/AGD methods. For example, for the third term in equation (4.2), with $U_{2113}(\mathbf{r}_1 - \mathbf{r}')$ and $F_{21}(\mathbf{r}_1 - \mathbf{r}_2) = \langle \psi_2(\mathbf{r}_1) \psi_1(\mathbf{r}_2) \rangle$, there are three different possible decoupling.

$$\begin{aligned}
& \int d\mathbf{r}' U_{2113}(\mathbf{r}_1 - \mathbf{r}') \langle \psi_1^\dagger(\mathbf{r}') \psi_1(\mathbf{r}') \psi_3(\mathbf{r}_1) \psi_1(\mathbf{r}_2) \rangle \\
& = + \int d\mathbf{r}' U_{2113}(\mathbf{r}_1 - \mathbf{r}') \langle \psi_1^\dagger(\mathbf{r}') \psi_1(\mathbf{r}') \rangle \langle \psi_3(\mathbf{r}_1) \psi_1(\mathbf{r}_2) \rangle \\
& \quad - \int d\mathbf{r}' U_{2113}(\mathbf{r}_1 - \mathbf{r}') \langle \psi_1^\dagger(\mathbf{r}') \psi_3(\mathbf{r}_1) \rangle \langle \psi_1(\mathbf{r}') \psi_1(\mathbf{r}_2) \rangle \\
& \quad - \int d\mathbf{r}' U_{2113}(\mathbf{r}_1 - \mathbf{r}') \langle \psi_1^\dagger(\mathbf{r}') \psi_1(\mathbf{r}_2) \rangle \langle \psi_3(\mathbf{r}_1) \psi_1(\mathbf{r}') \rangle
\end{aligned} \tag{6.1}$$

- The second term is ignored because $\langle \psi_1(\mathbf{r}') \psi_1(\mathbf{r}_2) \rangle = 0$ for the BCS type wave function.
- The first term is a Hatree term where the cloud of species 1 around species 3 to make the transition from (1,3) to (1,2). And $\psi_1^\dagger(\mathbf{r}') \psi_1(\mathbf{r}')$ is nothing but the number operator n_1 . And this term gives only in order of $n \cdot a^3 \ll 1$ due to the short-range nature of the interaction, only $|\mathbf{r}_1 - \mathbf{r}| \lesssim a$ is useful part of integration, where a is the range of the interaction. This term is usually ignored by the BCS theory because it gives the same boost for the normal state as the BCS state.
- The third term is the term as $G \cdot F$ in the AGD approach. However, here it involves both $\mathbf{r}_1 - \mathbf{r}'$ and $\mathbf{r}_2 - \mathbf{r}'$, and at least one of them is larger than the interaction range a . In simple BCS, $\langle \psi_1^\dagger(\mathbf{r}') \psi_1(\mathbf{r}_2) \rangle = \sum \exp[-i\mathbf{k} \cdot (\mathbf{r}_1 - \mathbf{r}_2)] |v_k|^2$, for the short-range in order of a , it is close to n, but **for larger than a , it is not clear.**

7 note of 2009.09.18

7.1 Continuing on the Decoupling of Four-Operator Term

Back to AGD[1], the Hamiltonian used there has the same r for four operators in the interaction terms, such as $\frac{1}{2} \sum_{\alpha\beta\gamma\delta} \int d\mathbf{r}_1 d\mathbf{r} \psi_\alpha^\dagger(\mathbf{r}) \psi_\beta^\dagger(\mathbf{r}) \lambda_{\alpha\beta\gamma\delta} \psi_\gamma(\mathbf{r}) \psi_\delta(\mathbf{r})$, this indeed simplifies quite a bit. But the justification of it is not that clear to me and to this problem. *Tony's comment: It is common to use contact interaction in place of the real one. And it works for many purpose, but one needs to be careful when pushing to higher order correction with this.*

The key is comparing the four-operator terms with the two-operator term in equation (4.2). For large $\mathbf{r}_1 - \mathbf{r}_2$, the two-operator term is actually small because the shor-range of interaction $U(\mathbf{r}_1 - \mathbf{r}_2)$, and the four-operator can be at least in the same order by taking \mathbf{r}' around $\mathbf{r}_1(\mathbf{r}_2)$. And the normal Green's function $G_1(\mathbf{r}_1 - \mathbf{r}_2) = \langle \psi_1^\dagger(\mathbf{r}') \psi_1(\mathbf{r}_2) \rangle$ relates to abnormal one. And the third term in equation (6.1) is approximately $-\int d\mathbf{r}' U_{2113}(\mathbf{r}_1 - \mathbf{r}') F_{31}(\mathbf{r}_1 - \mathbf{r}') G_1(\mathbf{r}' - \mathbf{r}_2) \approx -(\int d\mathbf{r}' U_{2113}(\mathbf{r}_1 - \mathbf{r}') F_{31}(\mathbf{r}_1 - \mathbf{r}')) G_1(\mathbf{r}_1 - \mathbf{r}_2) = \Gamma_{32} G_1(\mathbf{r}_1 - \mathbf{r}_2)$ if assuming G_1 varies slowly in the range of a . Here

$$\Gamma_{32} = \Gamma_{32}(\mathbf{r}_1) \equiv - \left(\int d\mathbf{r}' U_{2113}(\mathbf{r}_1 - \mathbf{r}') F_{31}(\mathbf{r}_1 - \mathbf{r}') \right) \tag{7.1}$$

In the integration above, only the small r is relevant, and we can supply the F_{31} with the two-body quantities. *This is a bit odd. Can this be reconciled with the simple BCS, where, the integration is over the shell around Fermi energy.* Actually, this is the same as AGD, where they define the $\Delta^2 = \lambda^2 |F^+(r \rightarrow 0+)|$ (34.16 in [1]).

In simple BCS, $G(r) = \sum \exp[-i\mathbf{k} \cdot \mathbf{r}] |v_k|^2$, v_k is like a step function and it oscillates fast for $r \ll 1/k_F$ and $G(r) \approx 0$. Similarly, $F(k) = v_k u_k^*$, v_k centres around k_F with width Δ , so its Fourier transformation $F(r)$ goes to 0 when $r \ll \hbar v_F / \Delta$. On the other hand, $F(r)$ is demonstrated that $F(r)$ decays exponentially in the scale of Pippard coherent length $\xi = \hbar v_F / \pi \Delta$, (5.4.36 in [6]) these two are consistent with each other.

7.2 Tony's Comment

He still feels puzzled by the fact that I cannot reduce the k-space equations with the 2-body quantities such as s-wave scattering length and maybe close-channel interaction strength, ... He wants me to write them up what is wrong there.

He does not think the time-differential of abnormal Green's function gives simply 2μ .

8 note of 2009.09.19

8.1 Renormalization of Gap Equation with 2-Body Quantities

This can be done formally with a 2 by 2 T matrix, which does not mean much. In 2-body, the close-channel is treated in 1-dimension Hilbert space. This in my mind is one of the key feature in Feshbach resonance, at least in 2-body treatment. It leads to reducing from a formal 2 channel problem to single parameter, s-wave scattering length, a_s .

8.2 Discussion with Shizhong

I had a discussion over skype with Shizhong tonight. He agreed that for a normal Green's function with time-independent Hamiltonian, there is no dependence on the absolute time shift. And for abnormal one, it should be chemical potential. But in this case, BCS wave function is not exactly eigenstate of the BCS Hamiltonian, and the differentiation of equation (4.2) is not exactly chemical potential, as the operators does not connect two eigenstates.

He suggested me to do the renormalization first with a formal 2 by 2 hamiltonian or T-matrix, and go from there to some simplification to a_s or δ_c . Maybe my problem is not how to renormalization, but how to reduce the 2 by 2 matrix in 2-body to these observable 2-body quantities.

He recommend R. Combescot's paper ([5, 4]) again. He suggested me to think about the non-zero central-motion close channel molecules.

9 note of 2009.09.22

9.1 Equation of Three Species

With the simplification discussed in previous, the equation of motion of order parameters (4.2) for three species can be written as (1 is the common species): for short-range,

$$\begin{aligned} i\hbar \frac{\partial F_{12}}{\partial t} &= \left(-\frac{\hbar^2}{2m} \nabla_{12}^2 - \mu_{12} + E_{12} \right) F_{12} + U F_{12} + Y F_{13} \\ i\hbar \frac{\partial F_{13}}{\partial t} &= \left(-\frac{\hbar^2}{2m} \nabla_{13}^2 - \mu_{13} + E_{13} \right) F_{13} + V F_{13} + Y F_{12} \end{aligned} \quad (9.1)$$

and for long-range,

$$\begin{aligned}
i\hbar \frac{\partial F_{12}}{\partial t} &= \left(-\frac{\hbar^2}{2m} \nabla_{12}^2 - \mu_{12} + E_{12} \right) F_{12} + U F_{12} + Y F_{13} + \Delta_{12} (G_1 + G_2) + \Gamma_{23} G_1 \\
i\hbar \frac{\partial F_{13}}{\partial t} &= \left(-\frac{\hbar^2}{2m} \nabla_{13}^2 - \mu_{13} + E_{13} \right) F_{13} + V F_{13} + Y F_{12} + \Delta_{13} (G_1 + G_3) + \Gamma_{23} G_1 \\
U &= U_{1221}, \quad V = U_{1331}, \quad Y = U_{1231} \\
\Delta_{1x} &= - \left(\int d\mathbf{r}' U_{x11x}(\mathbf{r} - \mathbf{r}') F_{x1}(\mathbf{r} - \mathbf{r}') \right) \\
\Gamma_{xy} &= - \left(\int d\mathbf{r}' U_{x11y}(\mathbf{r} - \mathbf{r}') F_{1y}(\mathbf{r} - \mathbf{r}') \right)
\end{aligned} \tag{9.2}$$

It is obvious that F is not eigenstate of the hamiltonian. However, unless something really dramatic, its short-range is very much like the two-body eigenstate.

10 note of 2009.09.23

10.1

By retrospection, this dividing over long-range and short-range is not necessary to be limited for Shizhong's equal-time equation. Maybe it can also be applied to the original AGD approach. however, there is no two-operator terms there.

10.2

Back to simple BCS, F , G takes simple forms for the contact interaction. How do they varies as the non-contact feature of potential is turned on? The guess is that the short-range behaviour is changed to be more like real two-body bound state while long-range behaviour does not change much. And how does other properties change? For example the energy gap?

In AGD of single channel simple BCS, $i \frac{\partial F_k}{\partial t} \big|_{t=0}$ is equal to $E_k F_k \big|_{t=0}$. (Note that F is not eigenstate of the hamiltonian and it happens to be E_k only at $t = 0$.)

In the original contact potential, or maybe more realistic short-range potential, is it possible to write down equations for short and long-range and in short range use a_s , then derive the relation between a_s and potential strength and the many-body quantities, such as gap. Is it possible to derive everything in real space? That would be good parallel to Feshbach case with two channel, whose two-body physics is more easily described in real-space.

If equations (9.1) (9.2) are like the single-channel case in AGD, the time derivative is not zero and not μ either. So the equation (9.1) is not exactly for 0 energy where a_s is defined. The time-derivative is probably in the order of the Fermi energy or chemical potential, (not sure whether they are the same in our case, a problem to be determined). Actually in the narrow resonance, the order parameter can be significant different from that of 0 energy solution where two-body solution is derived. While in the broad resonance, the many body energy is smaller than this level difference, and maybe indeed can take $\omega \approx 0$. Another possible complication is what should be compared with two-body energy is probably the difference of chemical potential μ and the time derivative ω , which might be small in both cases.

11 note of 2009.09.29

11.1

If the abnormal Greens function, or simply the order parameter $\psi\psi$ is literally taken as the macroscopic eigenstate of the two-body density matrix, ([6]), then its module should not depend on the time as the

quantities relating to the two-body density matrix. It probably can allow some imaginary phase, which should cancel out each other. The time-derivative of that gives a semi-energy quantity.

How to bind the short-range part $1 - r/a_s$ with the long-range part (many-body)?

11.2

How does the two-body scattering solution $\chi = 1 - r/a_s$ normalize ($\chi = r\psi_r$)?

It is normalized against constant flowing of the particles in the scattering problem, not the usual integrating to 1 over the space. But in our case, as it does not extent to ∞ , there is no such normalization problem. Its normalization is determined with the many-body part in the long-range.

This is just the solution for a Shrödinger equation with no external potential.

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \chi}{\partial r^2} = 0$$

It is OK for the $\chi \neq 0$ when $r \rightarrow 0$ as this is only for the long-range (long compare to potential range, but short compare to the many-body particle distance) behaviour where real potential is negligible.

12 note of 2009.09.30

12.1 Connexion between Two-Body Wave Function and Many-Body Wave Function

I talked with Shizhong about how he reduce the order parameter in the s-wave scattering wave function in short-distance at real space. (equation 2.61 in [9]) He does not get the a_s directly from 2.60, but by split $\frac{1}{E_k} = \left(\frac{1}{E_k} - \frac{1}{\xi_k}\right) + \frac{1}{\xi_k}$ and the first difference term is in order of $\frac{1}{k^6}$ for large k and gives negligible contribution in integral. We can neglect the large k part, approximate $\sin kr$ in the nominator as kr and that gives us $k^2 r$ in the nominator, which is the same as the the normalized gap equation. By comparing with the gap equation, we have $-2r/\pi a_s$ and the second part $\int \frac{k \sin kr}{\xi_k} dk = \frac{2m}{\hbar^2} \frac{\pi}{2}$. It does have a $2/\pi$ difference, (maybe just my calculation mistake somewhere), but the structure is correct. However, you cannot get a_s by comparing with the normalized gap equation. On the other hand, even without that, it still suggests the correct short-range behavior $\frac{1}{r} \left(1 - \frac{r}{a_s}\right)$. Presumably, $F(r)$ is a solution valid for all the region, and the above arguments demonstrates that it reduces to the correct short-range form where long-term effects, i.e. the many-body effects, can be neglected. And it gives the normalization factor which cannot be obtained by solving short-range version of equation.

So the structure of $F(r)$ is correct, and the factor happens to be a_s because the gap equation and that comes from the two-body description of T matrix. On the other hand, if working backward, we can identify the gap equation and a_s by reasoning that the short-range behavior of $F(r)$ must be proportional to the two-body solution.

Related to this, the question to ask is what type of the many-body solution reduces to the two-body solution for the Feshbach resonance, especially, the close channel, as exponentially decaying solution, $e^{-r/a^{(c)}}$. Considering that $a^{(c)}$ is much smaller than the open-channel scattering length a_s , the region where it can be treated in the form of $1 - r/a$ is very small, $r \ll a^{(c)}$; while the two-body solution is expected to be correct for the region much larger $r \ll 1/k_F$. But we need to find out how is $F(r)$ and how is $F(k, t = 0)$. The formal solution of $F(k, \omega)$ is in principle available although this needs solving third-order equation and probably only hopeless complicated analytic solution, and it needs to be integrated over both k and ω .

13 Bare Gap Equation

We start from a general Hamiltonian, (As usual, the particle number is relaxed and we use grand ensemble.)

$$\begin{aligned}
H - \mu N = & \sum_{\alpha} \int d\mathbf{r} \psi_{\alpha}^{\dagger}(\mathbf{r}) \left(-\frac{\hbar^2}{2m} \nabla^2 - \mu_{\alpha} + \eta_{\alpha} \right) \psi_{\alpha}(\mathbf{r}) \\
& + \frac{1}{2} \sum_{\alpha\beta\gamma\delta} \int d\mathbf{r}_1 d\mathbf{r}_2 \psi_{\alpha}^{\dagger}(\mathbf{r}_1) \psi_{\beta}^{\dagger}(\mathbf{r}_2) U_{\alpha\beta\gamma\delta}(\mathbf{r}_1 - \mathbf{r}_2) \psi_{\gamma}(\mathbf{r}_2) \psi_{\delta}(\mathbf{r}_1)
\end{aligned} \tag{13.1}$$

where η_{α} is the Zeeman energy of species α and μ_{α} is chemical potential. In the three-species two-channel problem, $\alpha = (a, b, c)$, where (a, b) is the open channel and (c, b) is the close channel. There are four different interaction terms, U_{abba} , V_{cbbc} and Y_{abbc} (Y_{cbba}^*). As a and c can interchange, their chemical potential is the same, $\mu_a = \mu_c$. Here we work with the finite-temperature Matsubara formalism. We can define the normal and abnormal Green's functions.

$$\begin{aligned}
G_{\alpha}(\mathbf{r}_1 - \mathbf{r}_2, \tau_1 - \tau_2) &= -\langle T_{\tau} \psi_{\alpha}(\mathbf{r}_1, \tau_1) \psi_{\alpha}^{\dagger}(\mathbf{r}_2, \tau_2) \rangle \\
&= -\langle T_{\tau} \psi_{\alpha}(\mathbf{r}_1 - \mathbf{r}_2, \tau_1 - \tau_2) \psi_{\alpha}^{\dagger}(0, 0) \rangle \\
F^+(\mathbf{r}_1 - \mathbf{r}_2, \tau_1 - \tau_2) &= -\langle T_{\tau} \psi_a^{\dagger}(\mathbf{r}_1, \tau_1) \psi_b^{\dagger}(\mathbf{r}_2, \tau_2) \rangle \\
R^+(\mathbf{r}_1 - \mathbf{r}_2, \tau_1 - \tau_2) &= -\langle T_{\tau} \psi_c^{\dagger}(\mathbf{r}_1, \tau_1) \psi_b^{\dagger}(\mathbf{r}_2, \tau_2) \rangle
\end{aligned} \tag{13.2}$$

Using the relation $\frac{\partial}{\partial \tau} \chi(\tau) = [H - \mu N, \chi(\tau)]$, we can obtain their equations of motion

$$\begin{aligned}
& \left(-\frac{\partial}{\partial \tau} + \frac{\hbar^2}{2m} \nabla^2 + \mu_b \right) G_b(x, \tau) - \frac{1}{2} \int dr [U(r-x)F(r-x, 0) + Y(r-x)R(r-x, 0)] F^+(r, \tau) \\
& - \frac{1}{2} \int dr [V(r-x)R(r-x, 0) + Y^*(r-x)F(r-x, 0)] R^+(r, \tau) = \delta(\tau)\delta(x) \\
& \left(-\frac{\partial}{\partial \tau} - \frac{\hbar^2}{2m} \nabla^2 - \mu_a \right) F^+(x, \tau) - \frac{1}{2} \int dr [U(r-x)F(r-x, 0) + Y(r-x)R(r-x, 0)] G_b(r, \tau) = 0 \\
& \left(-\frac{\partial}{\partial \tau} - \frac{\hbar^2}{2m} \nabla^2 - \mu_a + \eta \right) R^+(x, \tau) - \frac{1}{2} \int dr [V(r-x)R(r-x, 0) + Y^*(r-x)F(r-x, 0)] G_b(r, \tau) = 0
\end{aligned} \tag{13.3}$$

Here we take $\eta_a = \eta_b = 0$ and $\eta_c = \eta$. As in the normal BCS approach, we neglect Hatree terms as $\int dr U(x-r)n(r)F(x, \tau)$ because such terms can be absorbed into the chemical potential. These equations can be solved with Fourier transformation

$$\begin{aligned}
f(x, \tau) &= \frac{1}{\beta} \sum e^{-i\tau p_n} \int dk \frac{V}{(2\pi)^3} e^{ikx} f(k, ip_n) \\
f(k, ip_n) &= \int_0^{\beta} e^{i\tau p_n} \int dx e^{-ikx} f(x, \tau) \\
& \left(ip_n - \frac{\hbar^2}{2m} k^2 + \mu_b \right) G_b(k, ip_n) + \Delta_{-k}^F F^+(k, ip_n) + \Delta_{-k}^R R^+(k, ip_n) = 1 \\
& \left(ip_n + \frac{\hbar^2}{2m} k^2 - \mu_a \right) F^+(k, ip_n) + \Delta_k^F G_b(k, ip_n) = 0 \\
& \left(ip_n + \frac{\hbar^2}{2m} k^2 + \eta - \mu_a \right) R^+(k, ip_n) + \Delta_k^R G_b(k, ip_n) = 0
\end{aligned} \tag{13.4}$$

where

$$\begin{aligned}
\Delta_k^F &= -\frac{1}{2} \int dx e^{-ikx} [U(x)F^+(x, \tau=0) + Y^*(x)R^+(x, \tau=0)] \\
&= -\frac{1}{2} \sum [U(p+k)F^+(-p, \tau=0) + Y^*(p+k)R^+(-p, \tau=0)] \\
\Delta_k^R &= -\frac{1}{2} \int dx e^{-ikx} [V(x)R^+(x, \tau=0) + Y(x)F^+(x, \tau=0)] \\
&= -\frac{1}{2} \sum [V(p+k)R^+(-p, \tau=0) + Y(p+k)F^+(-p, \tau=0)]
\end{aligned} \tag{13.5}$$

Equation (13.4) is a simple algebra equation and can be solved easily. Here combine the kinetic energy with the chemical potential as $\xi_k = \frac{\hbar^2}{2m}k^2 - \mu$.

$$\begin{aligned}
G_b(k, ip_n) &= \frac{1}{A_k(ip_n)} (ip_n + \xi_k^a) (ip_n + \xi_k^a + \eta) \\
F^+(k, ip_n) &= -\frac{1}{A_k(ip_n)} \Delta_k^F (ip_n + \xi_k^a + \eta) \\
R^+(k, ip_n) &= -\frac{1}{A_k(ip_n)} \Delta_k^R (ip_n + \xi_k^a)
\end{aligned} \tag{13.6}$$

where

$$A_k(ip_n) = (ip_n - \xi_k^b) (ip_n + \xi_k^a) (ip_n + \xi_k^a + \eta) - \Delta_k^{F^2} (ip_n + \xi_k^a + \eta) - \Delta_k^{R^2} (ip_n + \xi_k^a) \tag{13.7}$$

Now the root for $A(z) = 0$ gives the pole of Green's function and the excitation spectrum of the system. This equation does have analytic solution as a third order algebraic equation. There are three different roots, corresponding three different types of excitation? Unlike in single-channel problem, where there is second-order equation and two roots degenerate ($\pm E_k$). However the roots are extremely tedious and serves little practical usage. One simple observation is that for large k , the last two terms in (13.7) can be neglected and therefore the pole is close to $\xi^a, -\xi^b$ (degenerate with ξ^a), and $\xi^c = \eta + \xi^a$. In principle, we can sum the solution over p_n to get the quantities at $\tau = 0$ which is in the gap equation (13.5).

$$F(k, \tau=0) = \frac{1}{\beta} \sum_{p_n} F(k, ip_n)$$

And the poles of $A(z) = 0$ determines this quantity. ¹

14 Renormalization with Two-Channel T-Matrix

One observation is that both abnormal Green's functions approach $\Delta/2\epsilon$ for large k ($\epsilon = \frac{\hbar^2}{2m}k^2$). This fact makes it possible to renormalize the gap equation in the similar fashion as in [6].

Formally, the gap equation (13.5) can be written in matrix form

$$\begin{pmatrix} \Delta^F \\ \Delta^R \end{pmatrix} = \begin{pmatrix} U & Y \\ Y^* & V \end{pmatrix} \begin{pmatrix} F \\ R \end{pmatrix} \tag{14.1}$$

or in a more compact way,

$$\Delta = UF \tag{14.2}$$

¹ $F(k, \tau=0)$ summation can be found using $\oint F(k, z) n_F(z) \frac{dz}{2\pi i} = 0$, where $n_F(z) = \frac{1}{e^{\beta z} + 1}$ is the Fermi distribution function with poles $p_n = i(2n+1)\pi/\beta$. The residual associated with the poles of $F(k, z)$ gives $F(k, \tau=0)$.

Note that there is really summation over k between U and F. For two-body system, we have

$$T = U + TGU = (1 + TG)U$$

notice $T = \begin{pmatrix} T^{oo} & T^{oc} \\ T^{co} & T^{cc} \end{pmatrix}$, $U, G = \begin{pmatrix} 1/2\epsilon & 0 \\ 0 & 1/(2\epsilon+\eta) \end{pmatrix}$ are 2×2 matrices. Apply $(1 + TG)$ on both side of gap equation (14.2).

$$(1 + TG)\Delta = TF$$

and

$$\Delta = T(F - G\Delta) \quad (14.3)$$

or in more elaborate form

$$\begin{pmatrix} \Delta^F \\ \Delta^R \end{pmatrix} = \begin{pmatrix} T^{oo} & T^{oc} \\ T^{co} & T^{cc} \end{pmatrix} \begin{pmatrix} \sum F - \frac{\Delta^F}{2\epsilon} \\ \sum R - \frac{\Delta^F}{2\epsilon+\eta} \end{pmatrix} \quad (14.4)$$

Now the summation of k is inside the last vector. T matrix is just four numbers which can be related to 0-energy scattering, therefore experiments. And the summation has no divergence at high k .

Several problems present here

- How is T-matrix related to the familiar quantities in literature? T^{oo} is probably directly related to open-channel s-wave scattering length a_s . More subtle is T^{co} , T^{oc} . How does that related the observed quantities?
- Although equation (14.4) is well-behaving without any divergence, $F(R)$ lacks the simple form in the single channel case. How to simplify it is not clear.
- So far, two channels are treated equally. However, two channels are quite different in reality, especially the close-channel. In two-body problem, it can be treated as only 1-dimensional, just the bound-state close to threshold. One only needs to determine its normalization. How to consider that in the many-body solution and simplify the gap equation?
- The high- k is related to the short-range behavior of the system. And the free-particle type k corresponding the free particles, not include the feature of the short-range bound state feature of the close-channel. That should be remedy somehow to simplify the computation and make progress.

15 note of 2009.10.01

15.1

A simple decay of $e^{-r/a}$'s Fourier transformation (in 3D) is $\frac{8\pi a^3}{(1+a^2 k^2)^2}$. ²But this may not be the case in the problem. The short-range part, (r within potential range r_0), does not affect the high- k , as $kr \ll 1$ and it only contribute the same constant for Fourier component unless for the really large $k \sim 1/r_0$. But the really-long-range part, ($r > 1/k_F$), many-body effects set in and it may not be the decay as such, and it might set a new scale. But due to the fast oscilation, it might be small.

Length scale in the problem

- r_0 , potential range, it does not change, and usually is the shortest length-scale.
- a_c , the close-channel bound-state size, it is also small. It is rigid with fix size when not considering the channel-coupling. It is probably in the same order as r_0
- a_s , open-channel s-wave scattering length. It varies from $-\infty$ to $+\infty$.
- $1/k_F$, the average atome distance, very large for the dilute system.

2

$$\int f(r) e^{i\mathbf{k} \cdot \mathbf{r}} d^3 \mathbf{r} = \iiint f(r) e^{i\mathbf{k} \cdot \mathbf{r}} \sin \theta r^2 dr d\phi d\theta = \frac{4\pi}{k} \int_0^\infty dr r \sin kr f(r) \quad (15.1)$$

15.2 Tony's comment

- Tony agrees the problem to bridge between the momentum space approach in many-body and the real space approach in two-body. He feels the integration is probably in the momentum space. He suggest to think about how the two-body in momentum space.
- He also suggest to think about the problem that how w.f changes when we ramp up density of diatomic molecules. This is more like the single-channel problem, but just on one side (BEC side). But we know that it might cross over the $\mu = 0$ point.
- He suggest to take the close-channel molecule also loosely bound, so $1/k_F \gg a \gg r_0$. He feels that may simplify things in momentum space.
- To take the narrow resonance, but far away from the resonance, maybe we can apply the perturbation somehow.

16 note of 2009.10.02

In two-body problem, what is the essential part for close-channel state? Bound state, or what? In normal treatment, such as Tony's [6], the only thing matter is the feedback from close-channel to open channel. And it ceases to have any importance after be capsulated into one parameter. It is the case not only in the Feshbach resonance, but also the shape resonance. The only thing that really useful for Feshbach is that the detuning is tunable and the feedback is tunable.

So key features of close-channel state:

- bound state, loose or tight?
- large level spacing between different levels. Or is it important?

In Tony's approach, only this κ affect things. And maybe it is the only thing matter for close channel. So is it possible to just tune other things in bound state to make the same κ , then you have the same effect?

17 note of 2009.10.05

It is actually not clear how to do it in many-body theory for two-channel. No one knows. I have a trial wave-function and that is my assumption. I can just work on it. If I can make it self-consistently, it is probably good enough.

In another sense, what I really want to know is what happens when many-body sets-in, how interaction changes. No one know exactly what happens. Maybe I can just use the simple ansatz and that is probably good enough if self-consistent.

18 note of 2009.10.06

18.1 Two-Body

One confusing point for two-channel is: 1. two particles has no interaction when they are far away. No matter there is any chemical potential shift or the Zeeman energy shift. This is comparing to two non-interacting free-moving particles. 2. The close-channel always has the shift due to the Zeeman energy difference, comparing 0-energy, i.e., free-moving particle in open-channel.

The 2nd point is correct.

18.2

Let us review the two-body problem treatment by Leggett[6]. We have the equations:

$$\frac{\hbar^2}{2m} \frac{\partial^2}{\partial r^2} \chi^o + U \chi^o + f \chi^c = E \chi^o \quad (18.1a)$$

$$\frac{\hbar^2}{2m} \frac{\partial^2}{\partial r^2} \chi^c + V \chi^o + f \chi^o = E \chi^c \quad (18.1b)$$

And the s-wave non-interaction scattering equation

$$\frac{\hbar^2}{2m} \frac{\partial^2}{\partial r^2} \bar{\chi}^o + U \bar{\chi}^o = 0 \quad (18.2)$$

Take E in (18.2) as 0 for the s-wave equation for a_s^0 . Multiply equation (18.2) with χ^o and equation (18.1a) with $\bar{\chi}$, integrate their difference from 0 to r_c where is outside the potential.

$$\frac{\hbar^2}{2m} \int_0^{r_c} dr \left(\bar{\chi} \chi^{o''} - \bar{\chi}'' \chi^o \right) + \int_0^{r_c} dr f \chi^c \bar{\chi} = \int_0^{r_c} dr E \chi^o \bar{\chi}$$

We take $E = 0$ and the first term in the integral can be taken as $\bar{\chi} \chi^{o'} - \bar{\chi}' \chi^o \big|_0^{r_c}$, $\chi = 0$ at 0 and $\bar{\chi} = 1 - \frac{r}{a_s^0}$, $\chi^o = 1 - \frac{r}{a_s}$ at r_c barring the normalization. Put all these together,

$$\begin{aligned} \frac{\hbar^2}{2m} \left[-\frac{1}{a_s} \left(1 - \frac{r}{a_s^0} \right) + \frac{1}{a_s^0} \left(1 - \frac{r}{a_s} \right) \right] + \int_0^{r_c} dr f \chi^c \bar{\chi} = 0 \\ \frac{1}{a_s^0} - \frac{1}{a_s} + \kappa = 0 \end{aligned} \quad (18.3)$$

where

$$\kappa = \frac{2m}{\hbar^2} \int_0^\infty dr f \chi^c \bar{\chi} \quad (18.4)$$

as the f is short-range, so we can extend r_c to ∞ .

From above, especially equation (18.3) and (18.4), we can see that what exactly χ^c does not really matter. What really matter is that it varies in large extend and provides a way to vary κ , therefore modify a_s .

19 note of 2009.10.08

So ultimately, it is just build a model for the two channel in many-body with BCS-like condensation. Most straightforward is to put close channel still as the point-molecule with no change tight bounded to open-channel. However, even that may have some subtlety. As the point-like molecule wave function (for the central mass) spread and overlap each other. And it might not be just such.

19.1

One thing maybe worthes some time is relation between the particle's own momuntum and the relative momentum between two. It is often confusing. In the two-body case, we talk about relative momentum most of the time. But it is more common to talk about individual mometum in many-bpdy cases.

20 note of 2009.10.12

The key problem of the momentum space is that the condition that many-body w.f is like the two-body one in short-distance is not readily expressed in momentum space.

In another word, we can ask the question from the other side, what happens for those system that the two-body physics is clear to us when we pack them more and more densely.

20.1 Tony's Comment

- Large close-channel bound state. What I am trying to look at is the close-channel bound state affected by Pauli principle. But the open-channel is not affected at least within the potential range. If the close-channel bound state is small and comparable to the potential range, we cannot safely assume the open-channel part within this range is not affected as well. So there is the Pauli-principle for $r > a_c \gg r_0$, no Pauli-principle consideration for short range $r \sim r_0$.
Beside, the problem is simplified by the fact no need to think about the oscillation for the bound-state within the potential range.
- another correction (not the one I am most interested) is the fact the non-zero close-channel state deplete the open channel and affect other stuff. This is simply the result of number equation.
- far away from the resonance, the close-channel can be treated perturbatively.

21 note of 2009.10.13

How does Pauli principle apply to the system? For one thing, it seems to apply something like repulsion? But how to put it in a more formal way? It is probably not the same as feel the interaction at interparticle distance. Probably more than that.

In principle you take care of it by adopting the second quantization.

How can you take care of at certain length scale while not other?

Is Gross-Pitaevskii equation a way? Check out [7] and [6]

22 note of 2009.10.14

The Gross-Pitaevskii equation is the Hartree model. So does the Bogolubov approach, they are just the assumption of the BEC form for the many-body equation. The close-channel molecule is in some way bosons, which is not affected by Pauli-principle. We have to go one-level deeper to take the Pauli-principle in. That is kind of related to the two-fermion-composite-molecule gas, and the cobosons idea. But I am not sure whether ideas there [3, 2] help.

So for the dilute coboson gas, the question is that how does the two-body correlation, which is the same as two-body molecule solution in the lowest order, affected by the many-body nature. In the normal language, only the center of mass coordinator is considered. We are more interested in the opposite side, how does the relatively coordinator changes? In terms of real-space, probably only long-scale is modified. But in terms of the momentum-space, a molecule has much more spread momentum, with no way to define things like Fermi surface. How does Pauli-principle work here?

22.1 Talk with Gordon

He pointed out that I am looking for is as such, open- \rightarrow close- \rightarrow open. And he mentioned Petrov's work. [8] That is also looking into two cobosons, but that is not exactly what I am looking forward. See figure 1.

22.2

In the perturbation suggested by Tony, the nature how close-channel molecule involved in the many-body is also needed. So either way, how two-fermion cobosons gas evolves are important. Petrov [8] gives $0.6a_s$ for two molecules interacted. But does not give the information about how the w.f. changes, which is probably more interested for my problem as this affects the open-channel component. Furthermore, he does not specify how the many-body set in. So the first thing to do is to find out how many-body change the close-channel.

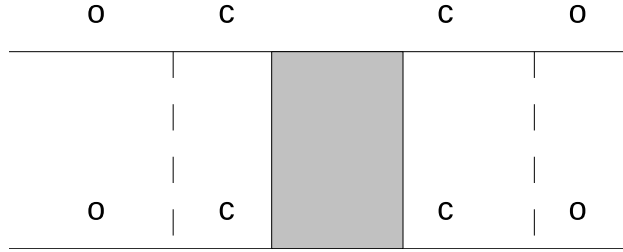


Figure 1: Open-Close Interaction

23 note of 2009.10.15

What we are more interested is how the internal structure of the coboson changes in the many-body situation. In some sense, given the fact that w.f. shape within the potential range does not change, but its normalization changes and so does the open-channel w.f. normalization in the region.

A related question is that when this effect is large?

Simple answer is that in the ultra-narrow part where close-channel is important. How about the center of mass movement of close-channel.

23.1 Shizhong's Suggestion

Shizhong: I have a very naive suggestion you might try

Shizhong: it looks like the first step you might try is to first solve the open channel problem and obtain momentum distribution for both spin state, this step is not hard and in fact is done in some papers

me: what do you mean open channel problem? you mean single-channel problem?

Shizhong: then in the second step, you freeze the open channel momentum distribution and now consider the molecular formation in the closed channel with those states blocked by the open channel state

yes. I mean single channel problem. this is exact the cooper problem

instead of the Fermi sea, you now have something a bit more complicated but the general philosophy is the same!

I am not sure how to handle the occupation of some fraction of the state as the case in the smearing around the Fermi sea. We can write down the equation in terms of the operator. The vacuum is the single-channel BCS instead of the normal vacuum, or in the 0th order, maybe the simple Fermi sea. Then we can look into what happens in the close-channel.

23.2 Tony's Comment

He suggested me to stick with the naive ansatz and ignore the non-zero central momentum molecules. He suggested me to try maybe variation method to look for how internal structure be modified. He still think the diluteness plays important role.

So it is probably still the same argument that to find out the momentum distribution in the two-body problem and compare that with the many-body ansatz.

24 note of 2009.10.20

24.1 Baym's Comment

I talked with Baym this afternoon. It took me some time to explain all the notation and convention to him. He suggested me to think it in the coordinator space and formulate it in Green's function. Then considering expand the abnormal Green's function around the simple close-channel bound state solution in the two-body problem. He mentioned that I might not be able to solve it analytically, but that is similar to Bogoliubov-de Gennes equation and I should be able to do things similarly and solve it numerically at worst. Not sure... The main difficulty is the really thing corresponding two-body wave function is the equal-time abnormal Green's function $F(t=0)$. It is not clear for me how to get it.

Maybe we can use the single pole approximation as G_0 ? No, it is not the same thing. The G_0 is for the relative coordinator of two-body wave function.

25 note of 2009.10.22

25.1 More from Baym

I talked with him more today. He looked into my solution of Green's function. He suggested me to think in the limit, $\Delta^F = 0$ (one-channel with uneven chemical potential) or $\Delta^R = 0$ (simple one-channel). And he showed me how to relate from the k-space to r-space in simple one-channel.

26 note of 2009.10.25

If we are looking for the many-body effects on the close-channel. One possible way is that mediated by the open-channel. In that situation, it seems not much difference between the large close-channel bound-state and the small close-channel bound-state. The key part in such situation is the coupling between two channel which happens in the small potential range a_0 anyway. And in that sense, the open-channel's part that within a_0 also can be modified by the many-body part at least due to the normalization.

If the many-body effects of the close-channel is restricted to the coupling of the open-close channel within a_0 , the many-body effect probably only affects the normalization without modify the wave function, at least for the short-range. But what we really modified is the long-range behavior.

In our assumption, the wave-functions in both channels are fixed within a_0 . And there are two possibilities, fixing the ratio between channels as well or letting that change. It seems reasonable to fix the ratio with the same argument as the wave-function is fixed as the two-body solution within a_0 .

However, in that case, the only many-body contribution is come from the open-channel large r . If the many-body effects show up only in the range $r \sim 1/k_F$, probably only the open-channel feels it. And this is probably just the broad resonance. (????) The close-channel feels no many-body effects at all. The only thing is the normalization. On the other hand, if the many-body effects show on $r > a_0$, both the portion of close-channel bound state outside a_0 and the open-channel part is affected. This seems corresponding to the case where each of wave function can varies independently.

What does broad/narrow resonance mean in the real-space? In the model of two-body, the close-channel part is fixed and only changes in normalization. Broad/narrow are still meaningful here.

27 note of 2009.10.26

In the two-body problem, other bound-state levels can be ignored because that energy difference between levels are large. But how about the the mixture with the continuous state.

A related question. What happens for two channels coupled at the two continuum?

What exactly is the difference between the 1-D Fourier transformation vs. the r-coordinator of the 3-d transformation?

It is different. For a function $f(\vec{x}) = f(x)$, the 3-D Fourier transformation is

$$\begin{aligned} f(\mathbf{p}) &= f(p) \\ &= 2\sqrt{2\pi} \int_0^\infty dx f(x) \frac{x}{p} \sin px \end{aligned} \quad (27.1)$$

28 note of 2009.10.29

I have tried the idea of a bound level with the Fermi-sea background. It seems that is the exact reason for many-body effects. For a bound wave-function $\chi_{\mathbf{k}_1, \mathbf{k}_2}$, we have equation

$$\frac{\hbar^2}{2m} (k_1^2 + k_2^2) \chi_{\mathbf{k}_1, \mathbf{k}_2} - \int d\mathbf{p} v(p) \chi_{\mathbf{k}_1 - \mathbf{p}, \mathbf{k}_2 + \mathbf{p}} = E \chi_{\mathbf{k}_1, \mathbf{k}_2} \quad (28.1)$$

Here the interaction $v(p)$ is only about the relative coordinator between two particles. If we only consider only the central momentum 0 solution, $\mathbf{k}_1 = -\mathbf{k}_2 = -\mathbf{k}$, the above equation can be reduced to 1-particle equation

$$\frac{\hbar^2}{2m} (2k^2) \chi_{\mathbf{k}} - \int d\mathbf{p} v(p) \chi_{\mathbf{k}\mathbf{p}} = E \chi_{\mathbf{k}} \quad (28.2)$$

Now let us apply the restriction that Fermi sea is filled and not available. $\chi = \overline{\chi^0} + \delta\chi$, where χ^0 is the solution to the above equation (28.2) in free space without the filled Fermi sea. And $\overline{\chi^0}$ is simply the same wave-function removed all the component below Fermi level.

$$\overline{\chi_{\mathbf{k}}^0} = \begin{cases} 0 & k < k_F \\ \chi_{\mathbf{k}}^0 & k \geq k_F \end{cases} \quad (28.3)$$

And eq (28.2) becomes

$$2\epsilon_k \delta\chi_{\mathbf{k}} - \int_{|\mathbf{k}+\mathbf{p}| < k_F} d\mathbf{p} v(p) \chi_{\mathbf{p}+\mathbf{k}}^0 + \int_{|\mathbf{k}+\mathbf{p}| > k_F} d\mathbf{p} v(p) \delta\chi_{\mathbf{p}+\mathbf{k}} = E \delta\chi_{\mathbf{k}} \quad (28.4)$$

Here $\chi^0(\mathbf{k})$ is much smaller than $1/a_0$, in that region, $v(p) \approx v(p=0)$ because it is short-range potential within a_0 .

$$\int_{|\mathbf{k}+\mathbf{p}| < k_F} d\mathbf{p} v(p) \chi_{\mathbf{p}+\mathbf{k}}^0 = v(p=0) \int_{|\mathbf{k}'| < k_F} d\mathbf{k}' \chi_{\mathbf{k}'}^0$$

furthermore, as $\chi^0(r)$ is much smaller than $1/k_F$, in the region $|\mathbf{k}'| < k_F$, $\chi_{\mathbf{k}'}^0 \approx \chi_{\mathbf{k}'=0}^0$. So the above equation can be further simplified into

$$2\epsilon_k \delta\chi_{\mathbf{k}} - \frac{4\pi}{3} v(p=0) \chi_{\mathbf{k}=0}^0 k_F^3 + v(p=0) \int_{|\mathbf{k}| > k_F} d\mathbf{k} \delta\chi_{\mathbf{k}} = E \delta\chi_{\mathbf{k}}$$

And we have

$$\delta\chi_{\mathbf{k}} = \frac{\frac{4\pi}{3} v(p=0) \chi_{\mathbf{k}=0}^0 k_F^3 - v(p=0) \int_{|\mathbf{k}| > k_F} d\mathbf{k} \delta\chi_{\mathbf{k}}}{-E + 2\epsilon_k} \quad (28.5)$$

We noticed that the nominator is a constant. The first term is negative, while the second is positive. Also $E = -\hbar^2/2ma_c^2 < 0$, and $a_0 \ll a_c \ll 1/k_F$, in the region where we are most interested, $|E| \sim \epsilon_k$. This is very much like the F_k in the BCS, and its Fourier transformation is a bound-state fall off at the scale of a_c as in E . So it is more like just the normalization of the close-channel bound state are modified without the wave-function changes.

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