Coboson Derivation of Richardson's Equations for Cooper pairs

Monique Combescot[1,2] and Guojun Zhu[2]
[1]Institut des NanoSciences de Paris, Universite Pierre et Marie Curie,
CNRS, Campus Boucicaut, 140 rue de Lourmel, 75015 Paris
[2]Department of Physics, University of Illinois at Urbana-Champaign, 1110 W Green St, Urbana, IL, 61801
(Dated: January 26, 2010)

Five years after the milestone paper by Bardeen, Cooper, Schrieffer (B.C.S.) in which superconductivity is tackled within the grand canonical ensemble, Richardson found a way to approach the problem within the canonical ensemble: He succeeded to write down the *exact analytical form* of the Schrödinger equation eigenstate for an arbitrary number of Cooper pairs interacting through the standard BCS potential. We here rederive his result using a commutation technique similar to the one we have recently developed for many-body effects between composite bosons (cobosons in short). This derivation makes crystal clear the fact that difference between a collection of single Cooper pairs and the BCS condensate are solely due to the Pauli exclusion principle through electron exchanges between pairs. Our procedure gives hints on why, as we very recently found, the interaction part of the N-pair energy depends on pair number as $N(N-1)$ only from the dilute to the dense regime of pairs. In this work, we also briefly discuss the validity of the BCS wave function ansatz in the light of Richardson's exact form.

It is known for quite a long time that the Pauli exclusion principle plays a key role in superconductivity. None the less, the precise way Pauli blocking transforms a collection of single Cooper pairs into a BCS condensate, has been understood quite recently only. This understanding goes through the study of Cooper pairs not within the grand canonical ensemble as done in the standard BCS theory, but within the canonical ensemble. To handle the Pauli exclusion principle between a fixed number of interacting fermions is known to be a quite difficult task, especially when these fermions form paired states. Turning to the grand canonical ensemble makes it far easier. Yet, adding fermion pairs one by one is the best way to precisely follow the increasing effect of Pauli blocking from the dilute to the dense regime of pairs.

Five years after the milestone paper on superconductivity by Bardeen, Cooper, Schrieffer[1], Richardson succeeded to derive the exact form of the Schrödinger equation eigenstates for N Cooper pairs[2,3]. It is expressed in terms of N parameters, R_1,... R_N which are solutions of N coupled non-linear equations, the energy of these N pairs reading as $E_N = R_1 + ... + R_N$. Although this exact form is definitely quite smart, to use it in practice is not that easy: Indeed, up to now, the equations for R_1,... R_N had no known analytical solution for arbitrary N and interaction strength, so that they were mostly approached through numerical procedures[4,5]. This is probably why these Richardson's equations have not had so far the attention they deserve among the superconductor community, even if Richardson managed to recover the BCS result in the infinite N limit[6]. Nowadays, they are commonly solved numerically to study superconducting granules having a rather small number of pairs[4].

Last year, we came back to these Richarson's equations because we wanted to reveal the connection between two well-known problems, namely the one-pair problem solved by Cooper and the many-pair problem considered by Bardeen, Cooper and Schrieffer. These two problems have intrinsic similarities: In both cases, there is a "frozen" core of non-interacting electrons and above this core, a potential layer where the attraction between up and down spin electrons acts. In the one-pair problem, the layer contains one electron pair only, while in the standard BCS configuration, the layer is half-filled: It is usually said that the potential layer extends symmetrically on both sides of the Fermi level, but this is just equivalent to half filling. It is clear that, by adding more and more pairs into the potential layer, we can continuously go from one pair to the dense BCS regime.

Although, at the present time, such a pair increase does not seem easy to achieve experimentally, this increase can at least be seen as a gedanken experiment to study the evolution of the energy spectrum when the filling of the potential layer is changed, in order to understand the exact role of the Pauli exclusion principle in superconducting state. This procedure can also be seen as a simple but well-defined toy model to study the BEC-BCS crossover problem since, by changing the number of pairs, we change their overlap. Such an overlap change has been considered by Eagles[8], and also by Leggett[9] through the change of the interaction strength between pairs.

Since the Richardson's procedure allows one to fix the pair number and thus to vary this number at will from one to half filling, we seriously reconsidered solving these equations analytically. By turning to their dimensionless form, we succeeded to find an analytical way to solve these equations in the dilute regime of pairs[7]. Indeed, these equations do have a small parameter, namely $1/N_c$ where N_c is the number of pairs from which overlap between single pairs would start. This allowed us to demonstrate in the dilute limit on the single Cooper pair scale, i.e., for N arbitrary large but N/N_c small, that the energy of N Cooper pairs reads in the large sample limit as

$$E_N = N\left[\left(2\epsilon_{F_0} + \frac{N-1}{\rho_0}\right) - \epsilon_c\left(1 - \frac{N-1}{N_\Omega}\right)\right] \quad (1)$$

ϵ_{F_0} is the Fermi level energy of the frozen sea. ρ_0 is the density of states, taken as constant within the potential layer. $N_\Omega = \rho_0\Omega$ is the number of free pair states in this layer, Ω being the potential layer extension. $\epsilon_c \approx 2\Omega\exp(-2/\rho_0 V)$ is the single pair binding energy, the potential amplitude V being taken as small (weak-coupling limit).

Although our actual derivation imposes N/N_c small, it is quite remarkable to note that this result is also valid in the dense BCS regime, where pairs strongly overlap. Indeed the first term of Eq.(1) is the exact energy of N pairs in the normal state since it is nothing but

$$2\epsilon_{F_0} + (2\epsilon_{F_0} + 1/\rho_0) + \cdots + (2\epsilon_{F_0} + (N-1)/\rho_0) = \mathcal{E}_N^{(normal)} \quad (2)$$

For a number of pairs corresponding to fill half the potential layer, which precisely is the BCS configuration, Eq.(1) gives a condensation energy equal to

$$|\mathcal{E}_N - \mathcal{E}_N^{(normal)}| = \frac{N_\Omega}{2}\frac{\epsilon_c}{2} = \frac{1}{2}\rho_0\Omega^2 e^{-2/\rho_0 V} \quad (3)$$

This result exactly matches the one derived by Bardeen, Cooper, Schrieffer within the grand canonical ensemble, namely $\rho_0\Delta^2/2$ since the gap Δ reads as $2\omega_c\exp(-1/\rho_0 V)$ where $2\omega_c$ is nothing but the potential layer extension Ω. It also is of interest to note that if we extend the BCS grand canonical derivation originally performed for half filling, to other non-symmetrical configurations, Eq.(1) remains valid.

The canonical approach we have used to reach Eq.(1), based on solving the Richardson's equations analytically, has the great advantage to follow the evolution of the ground state energy when adding pairs one by one. This leads us to, in a natural way, associate the last term in the RHS of Eq.(1), namely $\epsilon_c[1-(N-1)/N_\Omega]$, with the average "pair binding energy" in the N-pair configuration. Indeed, for $N=1$, this quantity exactly matches the single-pair binding energy as found by Cooper, while in the dense regime it exactly gives the condensation energy per pair. Therefore, this pair energy allows us to understand the dilute and dense regimes of pairs on the same footing.

We see that the pair binding energy, as defined above, decreases when N increases. This decrease is entirely due to Pauli blocking, the number of electron states in the potential layer, available to form correlated states, decreasing when N increases. A pictorial way to understand the binding energy decrease when N increases is through the so-called "moth-eaten" effect: when pairs are added to the frozen Fermi sea $|F_0\rangle$, they "eat" one by one, like little moths, the states in the potential layer which are available to form a bound state. As a result of this available state decrease, the bound state energy can only decrease. Note that this pair binding energy decrease is in a contrast with the common belief that in the dense BCS configuration, the Cooper pair binding energy is of the order of the excitation gap since Δ is far larger than ϵ_c. This understanding is obtained by splitting the condensation energy $\rho_0\Delta^2/2$ as $(\rho_0\Delta)\Delta$ within an "irrelevant" 1/2 prefactor. This deliberately assigns to each pair an energy equal to the gap, the number of pairs to fit the condensation energy then being $\rho_0\Delta$, i.e., the number of pairs in a gap layer. These $\rho_0\Delta$ pairs are called "virtual pairs" by Schrieffer. Their number is far smaller than the number of pairs $N_\Omega/2$ feeling the potential. As a direct consequence, their energy is far larger than the average energy $\epsilon_c/2$ of the pairs which feel the potential. These virtual pairs in fact correspond to excitations across the Fermi sea $|F\rangle$ made of $N+N_0$ *noninteracting* pairs, N_0 being the number of pairs in the frozen core $|F_0\rangle$. It however is of interest to note that the concept of virtual pairs can have some physical meaning in the dense regime only because in the dilute regime, the Fermi level of noninteracting electrons is completely washed out, all the pairs feeling the potential being essentially excited above this level. One rather negative aspect of this virtual pair understanding is that it tends to mask the obvious link which exists between the dilute and dense regimes of pairs. This probably is one of the reasons for the Schrieffer's statement that the isolated pair picture has little meaning in the dense regime[10]. This statement was already questioned by Leggett who showed that, in many respects, pairs in the dense limit are very similar to giant molecules made of two opposite spin electrons[9].

Since the key role of Pauli blocking in superconductivity is enlightened by our expression of the N-pair energy Eq.(1) through the "moth-eaten effect" it contains, while this expression has been obtained by solving the Richardson's equations analytically, it can be of interest to precisely see the parts in these equations which directly come from the Pauli exclusion principle.

In our recent works on the many-body physics of composite bosons - mostly concentrated on semiconductor excitons, we have proposed a "commutation technique" which allows us to evidence the effects of Pauli blocking between the fermionic components of these composite bosons (cobosons in short). They appear through "Pauli scatterings" which describe fermion exchanges in the absence of fermion interaction. These dimensionless scatterings, when mixed with energy-like scatterings coming from interactions between the coboson fermionic components, allow us to deal with fermion exchanges between any number of composite particles in an exact way. For a review on this formalism and its applications to the many-body physics of semiconductor excitons, see Reference[11,12].

In the present paper, we first develop such a commutation technique for up and down spin electron pairs with zero total momentum. We then use it to derive in a quite compact way, the form of the exact eigenstate for N pairs interacting through the reduced BCS potential.

The Richardson's equations readily follow from this approach. Its main advantage is to possibly trace back in a transparent way, the terms in these equations which directly come from the Pauli exclusion principle: they are those in $1/(R_i - R_j)$. They actually come from the nonzero values of the Pauli scatterings for fermion exchanges between up and down spin electron pairs. This leads us to conclude that the Richardson's energies R_i have N different values just because of Pauli blocking between the Cooper pair components.

The paper is organized as follow:

In section I, we present the commutation technique for free electron pairs and derive their associated Pauli and interaction scatterings.

In section II, we use this technique to get the form of the exact eigenstates for $N = 1, 2, 3, \cdots$ pairs interacting through the reduced BCS potential, in order to see how the solution for general N develops. We then analyze the precise role of Pauli blocking in this solution.

In section III, we briefly discuss possible connection between this exact wave function and the BCS ansatz for condensed pairs.

I. COMMUTATION TECHNIQUE FOR FREE FERMION PAIRS

A. Exchange between free fermion pairs

We consider cobosons made of free fermion pairs having a zero total momentum.

$$\beta^\dagger_{\mathbf{k}} = a^\dagger_{\mathbf{k}} b^\dagger_{-\mathbf{k}} \tag{4}$$

These pairs only have one degree of freedom[13] , namely $\mathbf{k}$, by contrast to the most general fermion pairs $a^\dagger_{\mathbf{k}_1} b^\dagger_{\mathbf{k}_2}$ which have two. In the case of Cooper pairs, $a^\dagger_{\mathbf{k}}$ creates a spin up electron with momentum $\mathbf{k}$ while $b^\dagger_{-\mathbf{k}}$ creates a down spin electron with momentum $-\mathbf{k}$. The fermion operators ($a^\dagger_{\mathbf{k}'}$,$a^\dagger_{\mathbf{k}}$) and ($b^\dagger_{\mathbf{k}'}$,$b^\dagger_{\mathbf{k}}$) anticommute. While $a^\dagger_{\mathbf{k}'}$ and $b^\dagger_{\mathbf{k}}$ anticommute in the case of opposite spin electrons, they can commute or anticommute depending if the corresponding fermions have the same or a different nature. It however is easy to check that this does not affect the commutation relations between the fermion pair operators that we are going to derive. This is why, for simplicity, we can consider that all fermion operators anticommute.

It is straightforward to show that the creation operators of these free fermion pairs commute

$$\left[\beta^\dagger_{\mathbf{k}'}, \beta^\dagger_{\mathbf{k}}\right] = 0 \tag{5}$$

It is worth noting that while $\left(a^\dagger_{\mathbf{k}}\right)^2 = 0$ simply follows from the anticommutation of the $a^\dagger_{\mathbf{k}}$ operators, the cancellation of $\left(\beta^\dagger_{\mathbf{k}}\right)^2$ does not follow from Eq.(5), but from the fact that $\left(\beta^\dagger_{\mathbf{k}}\right)^2$ contains $\left(a^\dagger_{\mathbf{k}}\right)^2$. The $\left(\beta^\dagger_{\mathbf{k}}\right)^2$ cancellation which comes from Pauli blocking, may seem to be lost when turning from single fermion operators to pair operators. We will see that this Pauli blocking is yet preserve in the commutaion algebra of free fermion pairs we are developing.

For creation and annihilation operators, $\left[a_{\mathbf{k}'}, a^\dagger_{\mathbf{k}}\right] = \delta_{\mathbf{k}'\mathbf{k}}$ leads to

$$\left[\beta_{\mathbf{k}'}, \beta^\dagger_{\mathbf{k}}\right] = \delta_{\mathbf{k}'\mathbf{k}} - D_{\mathbf{k}'\mathbf{k}} \tag{6}$$

where the deviation-from-boson operator $D_{\mathbf{k}'\mathbf{k}}$ is defined as

$$D_{\mathbf{k}'\mathbf{k}} = \delta_{\mathbf{k}'\mathbf{k}} \left(a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + b^\dagger_{-\mathbf{k}} b_{-\mathbf{k}}\right) \tag{7}$$

This operator which would reduce to zero if the fermion pairs were elementary bosons, allows us to generate the Pauli scatterings for fermion exchanges between cobosons in the absence of fermion interaction. They are formally defined through

$$\left[D_{\mathbf{k}'_1\mathbf{k}_1}, \beta^\dagger_{\mathbf{k}_2}\right] = \sum_{\mathbf{k}'_2} \left\{\lambda \begin{pmatrix} \mathbf{k}'_2 & \mathbf{k}_2 \\ \mathbf{k}'_1 & \mathbf{k}_1 \end{pmatrix} + (\mathbf{k}'_1 \leftrightarrow \mathbf{k}'_2)\right\} \beta^\dagger_{\mathbf{k}'_2} \tag{8}$$

By noting that

$$\left[a^\dagger_{\mathbf{k}} a_{\mathbf{k}}, \beta^\dagger_{\mathbf{p}}\right] = \delta_{\mathbf{k}\mathbf{p}} \beta^\dagger_{\mathbf{p}} = \left[b^\dagger_{-\mathbf{k}} b_{-\mathbf{k}}, \beta^\dagger_{\mathbf{p}}\right] \tag{9}$$

it is easy to show that

$$\left[D_{\mathbf{k}'_1\mathbf{k}_1}, \beta^\dagger_{\mathbf{k}_2}\right] = 2\beta^\dagger_{\mathbf{k}_2} \delta_{\mathbf{k}_1\mathbf{k}_2} \delta_{\mathbf{k}'_1,\mathbf{k}_2} \tag{10}$$

This leads us to identify the Pauli scattering appearing in Eq.(8) with the following product of Kronecker symbols

$$\lambda \begin{pmatrix} \mathbf{k}'_2 & \mathbf{k}_2 \\ \mathbf{k}'_1 & \mathbf{k}_1 \end{pmatrix} = \delta_{\mathbf{k}'_1\mathbf{k}_1} \delta_{\mathbf{k}'_2\mathbf{k}_2} \delta_{\mathbf{k}_1\mathbf{k}_2} \tag{11}$$

Actually, this is just the value we expect for the scattering associated to fermion exchanges between $(\mathbf{k}_1, \mathbf{k}_2)$ pairs, as visualized by the diagram of Fig.(1a). .Indeed, from this diagram, it is clear that we must have $(\mathbf{k}'_1 = \mathbf{k}_1, \mathbf{k}'_2 = \mathbf{k}_2)$ and $(-\mathbf{k}'_2 = -\mathbf{k}_1, -\mathbf{k}'_1 = -\mathbf{k}_2)$: this just gives $\delta_{\mathbf{k}'_1\mathbf{k}_1} \delta_{\mathbf{k}'_2\mathbf{k}_2} \delta_{\mathbf{k}_1\mathbf{k}_2}$ in agreement with Eq.(11).

B. Interaction between free fermion pairs

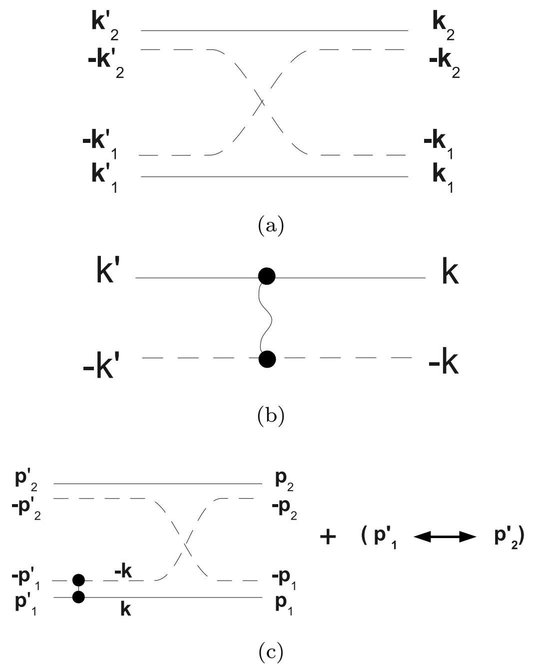
To get the interaction scatterings associated to fermion interaction, we first note that for a free fermion hamiltonian

$$H_0 = \sum \epsilon_{\mathbf{k}} \left(a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + b^\dagger_{\mathbf{k}} b_{\mathbf{k}}\right) \tag{12}$$

Eq.(9) readily gives

$$\left[H_0, \beta^\dagger_{\mathbf{p}}\right] = 2\epsilon_{\mathbf{p}} \beta^\dagger_{\mathbf{p}} \tag{13}$$

FIG. 1: Shiva diagram of free pairs



(a) Pauli scattering $\lambda\left(\begin{smallmatrix}\mathbf{k}'_2 & \mathbf{k}_2\\ \mathbf{k}'_1 & \mathbf{k}_1\end{smallmatrix}\right)$ for electron exchange between two free pairs $(\mathbf{k}_1,\mathbf{k}_2)$, as given by Eq.(11). Up spin electrons are represented by solid lines, down spin electrons by dashed lines.

(b) The BCS potential given in Eq.(14) transforms a $\mathbf{k}$ pair into a $\mathbf{k}'$ pair, with a constant scattering $-V$, in the case of a separable potential $v_{\mathbf{k}'\mathbf{k}} = -V\,w_{\mathbf{k}'}w_{\mathbf{k}}$.

(c) Interaction scattering $\chi\left(\begin{smallmatrix}\mathbf{p}'_2 & \mathbf{p}_2\\ \mathbf{p}'_1 & \mathbf{p}_1\end{smallmatrix}\right)$ between two free pairs, as given in Eq.(19). Since the BCS potential acts within one pair only, the interaction between two pairs can only come from exchange induced by the Pauli exclusion principle.

In the case of present interest, these fermion pairs interact through the standard BCS potential, which basically is a (1x1) potential in the fermion pair subspace since the fermion $\mathbf{k}$ interacts with one fermion only of the other species, namely the fermion $(-\mathbf{k})$. This BCS potential, which reads as

$$V_{BCS} = \sum v_{\mathbf{k}'\mathbf{k}}\beta^\dagger_{\mathbf{k}'}\beta_{\mathbf{k}} \tag{14}$$

is represented by the diagram of Fig.(1b). For this (1x1) potential, we do have

$$\left[V_{BCS},\beta^\dagger_{\mathbf{p}}\right] = \gamma^\dagger_{\mathbf{p}} + V^\dagger_{\mathbf{p}} \tag{15}$$

in which we have $\gamma^\dagger_{\mathbf{p}} = \sum_{\mathbf{k}}\beta^\dagger_{\mathbf{k}}v_{\mathbf{k}\mathbf{p}}$. The "creation potential" $V^\dagger_{\mathbf{p}}$ for the free fermion pair $\mathbf{p}$ appears to be

$$V^\dagger_{\mathbf{p}} = -\gamma^\dagger_{\mathbf{p}}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + b^\dagger_{-\mathbf{p}}b_{-\mathbf{p}}\right) \tag{16}$$

While the $\gamma^\dagger_{\mathbf{p}}$ part of Eq.(14) commutes with $\beta^\dagger_{\mathbf{p}'}$, this is not so for the creation potential $V^\dagger_{\mathbf{p}}$. Its commutator precisely reads

$$\left[V^\dagger_{\mathbf{p}_1},\beta^\dagger_{\mathbf{p}_2}\right] = -2\delta_{\mathbf{p}_1\mathbf{p}_2}\gamma^\dagger_{\mathbf{p}_1}\beta^\dagger_{\mathbf{p}_1} \tag{17}$$

This allows us to identify the interaction scattering for free pairs, formally defined as

$$\left[V^\dagger_{\mathbf{p}_1},\beta^\dagger_{\mathbf{p}_2}\right] = \sum\chi\left(\begin{smallmatrix}\mathbf{p}'_2 & \mathbf{p}_2\\ \mathbf{p}'_1 & \mathbf{p}_1\end{smallmatrix}\right)\beta^\dagger_{\mathbf{p}'_1}\beta^\dagger_{\mathbf{p}'_2} \tag{18}$$

with a sequence of one (2x2) fermion exchange between two pairs and one (1x1) fermion interaction inside one pair. Indeed

$$\begin{aligned}\chi\left(\begin{smallmatrix}\mathbf{p}'_2 & \mathbf{p}_2\\ \mathbf{p}'_1 & \mathbf{p}_1\end{smallmatrix}\right) &= -\sum_{\mathbf{k}}\left\{v_{\mathbf{p}'_1\mathbf{k}}\lambda\left(\begin{smallmatrix}\mathbf{p}'_2 & \mathbf{p}_2\\ \mathbf{k} & \mathbf{p}_1\end{smallmatrix}\right) + (\mathbf{p}'_1\leftrightarrow\mathbf{p}'_2)\right\}\\ &= -\left(v_{\mathbf{p}'_1,\mathbf{p}_1}\delta_{\mathbf{p}'_2,\mathbf{p}_2} + v_{\mathbf{p}'_2,\mathbf{p}_2}\delta_{\mathbf{p}'_1,\mathbf{p}_1}\right)\delta_{\mathbf{p}_2,\mathbf{p}_1}\end{aligned} \tag{19}$$

This interaction scattering is visualized by the diagram of Fig.(1c) : the free pairs $\mathbf{p}_1$ and $\mathbf{p}_2$ first exchange a fermion. As for any exchange, this brings a minus sign. In a second step, the fermions of one of the two pairs interact via the BCS potential. It is of importance to realize that, since the potential has a (1x1) structure within the pair subspace, the interaction between two pairs can only result from fermion exchange between pairs, i.e., Pauli blocking, as readily seen from this diagram.

We are now going to use this commutation formalism to derive the equations that Richardson has obtained for the eigenstates of N Cooper pairs through a totally different procedure.

II. RICHARDSON'S EQUATIONS FOR COOPER PAIRS

In order to better grasp how these equations develop, we are going to consider a number of pairs increasing one by one, starting from one pair.

A. One pair

We start with a state in which one free pair $(\mathbf{k},-\mathbf{k})$ is added to a frozen Fermi sea $|F_0\rangle$ which does not feel the BCS potential. This means that the $v_{\mathbf{k}'\mathbf{k}}$ prefactors in Eq.(14) cancel for all $\mathbf{k}$ belonging to $|F_0\rangle$. Note that such a "one-pair" state actually contains N_0+1 fermion pairs, N_0 being the number of pairs in the frozen sea; so that this state is a many-body state already, but in the most simple sense since the Fermi sea $|F_0\rangle$ is just there to block states by the Pauli exclusion principle. This Fermi sea also brings a finite density of state for all the states above it. This actually is crucial in order to always have a bound state, even if the attracting BCS potential is extremely small, as evidenced below.

By choosing the zero energy such that $H\,|F_0\rangle = H_0\,|F_0\rangle = 0$, Eqs.(13,15) allows us to write the hamiltonian $H = H_0 + V_{BCS}$ acting on this one-free-pair state as

$$H\beta^\dagger_{\mathbf{k}}\,|F_0\rangle = \left[H, \beta^\dagger_{\mathbf{k}}\right]|F_0\rangle = \left(2\epsilon_{\mathbf{k}}\beta^\dagger_{\mathbf{k}} + \gamma^\dagger_{\mathbf{k}} + V^\dagger_{\mathbf{k}}\right)|F_0\rangle \tag{20}$$

We first note that, due to the $v_{\mathbf{kp}}$ factor included in the $\gamma^\dagger_{\mathbf{k}}$ part of $V^\dagger_{\mathbf{k}}$, the creation potential $V^\dagger_{\mathbf{k}}$ acting on $|F_0\rangle$ gives zero. If we now subtract $E_1\beta^\dagger_{\mathbf{k}}\,|F_0\rangle$ to the two sides of the above equation, with E_1 yet undefined, and if we then divide the resulting equation by $(2\epsilon_{\mathbf{k}} - E_1)$, we find

$$(H - E_1)\frac{1}{2\epsilon_{\mathbf{k}} - E_1}\beta^\dagger_{\mathbf{k}}\,|F_0\rangle = \beta^\dagger_{\mathbf{k}}\,|F_0\rangle + \frac{1}{2\epsilon_{\mathbf{k}} - E_1}\gamma^\dagger_{\mathbf{k}}\,|F_0\rangle \tag{21}$$

To go further and possibly get the one-pair eigenstate of the hamiltonian H in a compact analytical form, it is necessary to approximate the BCS potential by a separable potential: its coupling then reduces to $v_{\mathbf{kp}} = -V\,w_{\mathbf{k}}w_{\mathbf{p}}$. This leads us to set $\gamma^\dagger_{\mathbf{k}} = -V\,w_{\mathbf{k}}\beta^\dagger$ where $\beta^\dagger$ is defined as

$$\beta^\dagger = \sum_{\mathbf{p}} w_{\mathbf{p}}\beta^\dagger_{\mathbf{p}} \tag{22}$$

If we then multiply Eq.(21) by $w_{\mathbf{k}}$ and sum over $\mathbf{k}$, we end with

$$(H - E_1)B^\dagger(E_1)\,|F_0\rangle = \left(1 - V\sum_{\mathbf{k}}\frac{w^2_{\mathbf{k}}}{2\epsilon_{\mathbf{k}} - E_1}\right)\beta^\dagger\,|F_0\rangle \tag{23}$$

where the operator $B^\dagger(E)$ is defined as

$$B^\dagger(E) = \sum_{\mathbf{k}} B^\dagger_{\mathbf{k}}(E) \qquad B^\dagger_{\mathbf{k}}(E) = \frac{w_{\mathbf{k}}}{2\epsilon_{\mathbf{k}} - E}\beta^\dagger_{\mathbf{k}} \tag{24}$$

Eq.(23) readily shows that the linear combination $B^\dagger(E_1)$ of the one-pair operators $\beta^\dagger_{\mathbf{k}}$ generates the one-pair eigenstate $B^\dagger(E_1)\,|F_0\rangle$ of the hamiltonian H having E_1 as energy, provided that this energy is such that

$$1 = V\sum_{\mathbf{k}}\frac{w^2_{\mathbf{k}}}{2\epsilon_{\mathbf{k}} - E_1} \tag{25}$$

This is nothing but the well-known equation for the single pair energy derived by Cooper.

B. Two pairs

Let us now consider two pairs. Eqs.(13,15) yield

$$\begin{aligned} H\beta^\dagger_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_2}\,|F_0\rangle &= \left(\left[H, \beta^\dagger_{\mathbf{k}_1}\right]\beta^\dagger_{\mathbf{k}_2} + \beta^\dagger_{\mathbf{k}_1}\left[H, \beta^\dagger_{\mathbf{k}_2}\right]\right)|F_0\rangle \\ &= (2\epsilon_{\mathbf{k}_1} + 2\epsilon_{\mathbf{k}_2})\,\beta^\dagger_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_2}\,|F_0\rangle + |v_{\mathbf{k}_1\mathbf{k}_2}\rangle \end{aligned} \tag{26}$$

where $|v_{\mathbf{k}_1\mathbf{k}_2}\rangle$ comes from interactions among the $(\mathbf{k}_1, \mathbf{k}_2)$ pairs induced by the BCS potential possibly mixed with electron exchanges. Its precise value reads

$$|v_{\mathbf{k}_1\mathbf{k}_2}\rangle = \left(\gamma^\dagger_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_2} + \gamma^\dagger_{\mathbf{k}_2}\beta^\dagger_{\mathbf{k}_1} + V^\dagger_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_2}\right)|F_0\rangle \tag{27}$$

Since $V^\dagger_{\mathbf{k}}$ acting on the frozen sea gives zero, Eq. (19) allows us to rewrite the last term of $|v_{\mathbf{k}_1\mathbf{k}_2}\rangle$ as

$$V^\dagger_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_2}\,|F_0\rangle = \left[V^\dagger_{\mathbf{k}_1}, \beta^\dagger_{\mathbf{k}_2}\right]|F_0\rangle = \sum_{\mathbf{p}'_1\mathbf{p}'_2}\chi\begin{pmatrix}\mathbf{p}'_2 & \mathbf{k}_2\\ \mathbf{p}'_1 & \mathbf{k}_1\end{pmatrix}\beta^\dagger_{\mathbf{p}'_1}\beta^\dagger_{\mathbf{p}'_2}\,|F_0\rangle \tag{28}$$

The three terms of $|v_{\mathbf{k}_1\mathbf{k}_2}\rangle$ are visualized by the diagram of Fig. 2. In the first process, one pair stays unchanged while in the other, the two pairs exchange an electron. This diagram evidences the fact that, due to the (1x1) form of the BCS potential, the two pairs $\mathbf{k}_1$ and $\mathbf{k}_2$ interact by fermion exchange only, as a result of the Pauli exclusion principle.

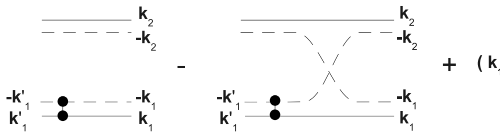


FIG. 2: Shiva diagram for the interaction part $|v_{\mathbf{k}_1\mathbf{k}_2}\rangle$ of the Hamiltonian H acting on two free pairs, as given in Eq.(27)

By using the value of the interaction scattering given in Eq.(17), we find that $|v_{\mathbf{k}_1\mathbf{k}_2}\rangle$ ultimatly reads as

$$|v_{\mathbf{k}_1\mathbf{k}_2}\rangle = -V\left(w_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_2} + w_{\mathbf{k}_2}\beta^\dagger_{\mathbf{k}_1} - 2\delta_{\mathbf{k}_1\mathbf{k}_2}w_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_1}\right)\beta^\dagger\,|F_0\rangle \tag{29}$$

To go further, we subtract $E_2\beta^\dagger_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_2}\,|F_0\rangle$ to the two sides of Eq.(26), with E_2 yet undefined. We split E_2 as $R_1 + R_2$ and we multiply the resulting equation by $w_{\mathbf{k}_1}w_{\mathbf{k}_2}/\,(2\epsilon_{\mathbf{k}_1} - R_1)\,(2\epsilon_{\mathbf{k}_2} - R_2)$. This yields

$$\begin{aligned} &(H - E_2)B^\dagger_{\mathbf{k}_1}(R_1)B^\dagger_{\mathbf{k}_2}(R_2)\,|F_0\rangle = \\ &\left\{B^\dagger_{\mathbf{k}_1}(R_1)\left(w_{\mathbf{k}_2}\beta^\dagger_{\mathbf{k}_2} - \frac{Vw^2_{\mathbf{k}_2}}{2\epsilon_{\mathbf{k}_2} - R_2}\beta^\dagger\right) + (1 \leftrightarrow 2)\right\}|F_0\rangle \\ &\qquad + 2V\delta_{\mathbf{k}_2\mathbf{k}_1}\frac{w^3_{\mathbf{k}_1}\beta^\dagger_{\mathbf{k}_1}}{(2\epsilon_{\mathbf{k}_1} - R_1)\,(2\epsilon_{\mathbf{k}_1} - R_2)}\beta^\dagger\,|F_0\rangle \end{aligned} \tag{30}$$

To go further, we split $(2\epsilon_{\mathbf{k}_1} - R_1)^{-1}\,(2\epsilon_{\mathbf{k}_1} - R_2)^{-1}$ as $\left[(2\epsilon_{\mathbf{k}_1} - R_1)^{-1} - (2\epsilon_{\mathbf{k}_1} - R_2)^{-1}\right]/\,(R_1 - R_2)$ provided that $R_1 \neq R_2$, a condition that we can always accept since the unique requirement is to have $R_1 + R_2 = E_2$. This allows us to write the last term of (30) as $\delta_{\mathbf{k}_2\mathbf{k}_1}\frac{2V}{R_1 - R_2}[B^\dagger_{\mathbf{k}_1}(R_1) - B^\dagger_{\mathbf{k}_1}(R_2)]\beta^\dagger\,|F_0\rangle$ provided that $w^2_{\mathbf{k}} = w_{\mathbf{k}}$. By taking sums over $\mathbf{k}_1$ and $\mathbf{k}_2$, Eq. (30) then gives

$$(H-E_2)B^\dagger(R_1)B^\dagger(R_2)\left|F_0\right\rangle =$$
$$\left\{B^\dagger(R_1)\left(1-V\sum\frac{w_{\mathbf{k}}^2}{2\epsilon_{\mathbf{k}}-R_2}+\frac{2V}{R_1-R_2}\right)+(1\leftrightarrow 2)\right\}$$
$$\beta^\dagger\left|F_0\right\rangle \quad (31)$$

The above equation readily shows that the two-pair state $B^\dagger(R_1)B^\dagger(R_2)\left|F_0\right\rangle$ is eigenstate of the hamiltonian H with the energy $E_2=R_1+R_2$ provided that (R_1,R_2) fulfill two equations, known as Richardson's equations for two pairs

$$1=V\sum\frac{w_{\mathbf{k}}^2}{2\epsilon_{\mathbf{k}}-R_1}+\frac{2V}{R_1-R_2}=(1\leftrightarrow 2) \qquad (32)$$

C. Three pairs

We now turn to three pairs in order to see how these equations develop for an increasing number of pairs. We start with

$$H\beta_{\mathbf{k}_1}^\dagger\beta_{\mathbf{k}_2}^\dagger\beta_{\mathbf{k}_3}^\dagger\left|F_0\right\rangle =$$
$$\left\{\left[H,\beta_{\mathbf{k}_1}^\dagger\right]\beta_{\mathbf{k}_2}^\dagger\beta_{\mathbf{k}_3}^\dagger+\beta_{\mathbf{k}_1}^\dagger\left[H,\beta_{\mathbf{k}_2}^\dagger\right]\beta_{\mathbf{k}_3}^\dagger+\beta_{\mathbf{k}_1}^\dagger\beta_{\mathbf{k}_2}^\dagger\left[H,\beta_{\mathbf{k}_3}^\dagger\right]\right\}$$
$$\left|F_0\right\rangle \qquad (33)$$

The same Eqs.(13,15) split the above equation into a kinetic part and an interaction part

$$H\beta_{\mathbf{k}_1}^\dagger\beta_{\mathbf{k}_2}^\dagger\beta_{\mathbf{k}_3}^\dagger\left|F_0\right\rangle=(2\epsilon_{\mathbf{k}_1}+2\epsilon_{\mathbf{k}_2}+2\epsilon_{\mathbf{k}_3})\,\beta_{\mathbf{k}_1}^\dagger\beta_{\mathbf{k}_2}^\dagger\beta_{\mathbf{k}_3}^\dagger\left|F_0\right\rangle$$
$$+\left|v_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\right\rangle \qquad (34)$$

The part resulting from the BCS potential reads as

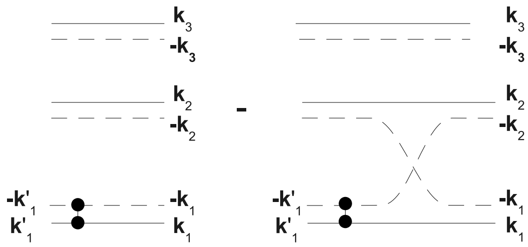
$$\left|v_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\right\rangle=\left(\gamma_{\mathbf{k}_1}^\dagger\beta_{\mathbf{k}_2}^\dagger\beta_{\mathbf{k}_3}^\dagger+\gamma_{\mathbf{k}_2}^\dagger\beta_{\mathbf{k}_3}^\dagger\beta_{\mathbf{k}_1}^\dagger+\gamma_{\mathbf{k}_3}^\dagger\beta_{\mathbf{k}_1}^\dagger\beta_{\mathbf{k}_2}^\dagger\right)\left|F_0\right\rangle$$
$$+\left(V_{\mathbf{k}_1}^\dagger\beta_{\mathbf{k}_2}^\dagger\beta_{\mathbf{k}_3}^\dagger+\beta_{\mathbf{k}_1}^\dagger V_{\mathbf{k}_2}^\dagger\beta_{\mathbf{k}_3}^\dagger+\beta_{\mathbf{k}_1}^\dagger\beta_{\mathbf{k}_2}^\dagger V_{\mathbf{k}_3}^\dagger\right)\left|F_0\right\rangle \qquad (35)$$

Its last term gives zero since $V_{\mathbf{k}}^\dagger\left|F_0\right\rangle=0$. Using Eq. (18), the two remaining terms of the second bracket can be rewritten in a more symmetrical form as

$$\left\{\left[V_{\mathbf{k}_1}^\dagger,\beta_{\mathbf{k}_2}^\dagger\right]\beta_{\mathbf{k}_3}^\dagger+\beta_{\mathbf{k}_2}^\dagger\left[V_{\mathbf{k}_1}^\dagger,\beta_{\mathbf{k}_3}^\dagger\right]+\beta_{\mathbf{k}_1}^\dagger\left[V_{\mathbf{k}_2}^\dagger,\beta_{\mathbf{k}_3}^\dagger\right]\right\}\left|F_0\right\rangle$$
$$=\sum_{\mathbf{k}_1'\mathbf{k}_2'}\beta_{\mathbf{k}_1'}^\dagger\beta_{\mathbf{k}_2'}^\dagger$$
$$\left\{\chi\begin{pmatrix}\mathbf{k}_2' & \mathbf{k}_2\\ \mathbf{k}_1' & \mathbf{k}_1\end{pmatrix}\beta_{\mathbf{k}_3}^\dagger+\chi\begin{pmatrix}\mathbf{k}_2' & \mathbf{k}_3\\ \mathbf{k}_1' & \mathbf{k}_2\end{pmatrix}\beta_{\mathbf{k}_1}^\dagger+\chi\begin{pmatrix}\mathbf{k}_2' & \mathbf{k}_1\\ \mathbf{k}_1' & \mathbf{k}_3\end{pmatrix}\beta_{\mathbf{k}_2}^\dagger\right\}\left|F_0\right\rangle \qquad (36)$$

This leads us to represent the vector $\left|v_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\right\rangle$ by the diagram of Fig.3. This interaction term contains interactions inside a single pair, two pairs staying unchanged, with in addition a possible fermion exchange with a second pair, the third pair staying unchanged.

FIG. 3: Shiva diagram for the interaction part $\left|v_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\right\rangle$ of the Hamiltonian H acting on three pairs



as given in Eqs. (35,36). $\left|v_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\right\rangle$ also contains two similar contributions as the one visualized in this figure, obtained by circular permutation.

Using Eq. (19), the RHS of the above equation reduces to

$$-2(\delta_{\mathbf{k}_1\mathbf{k}_2}\gamma_{\mathbf{k}_1}^\dagger\beta_{\mathbf{k}_1}^\dagger\beta_{\mathbf{k}_3}^\dagger+2\text{ perm.})\left|F_0\right\rangle$$
$$=2(\delta_{\mathbf{k}_1\mathbf{k}_2}w_{\mathbf{k}_1}\beta_{\mathbf{k}_1}^\dagger\beta_{\mathbf{k}_3}^\dagger+2\text{ perm.})\beta^\dagger\left|F_0\right\rangle$$

If we now come back to Eq.(34), subtract $E_3\beta_{\mathbf{k}_1}^\dagger\beta_{\mathbf{k}_2}^\dagger\beta_{\mathbf{k}_3}^\dagger\left|F_0\right\rangle$ to both sides, with E_3 written as $R_1+R_2+R_3$, and multiply the resulting equation by $w_{\mathbf{k}_1}w_{\mathbf{k}_2}w_{\mathbf{k}_3}/\left(2\epsilon_{\mathbf{k}_1}-R_1\right)\left(2\epsilon_{\mathbf{k}_2}-R_2\right)\left(2\epsilon_{\mathbf{k}_3}-R_3\right)$, we end with

$$(H-E_3)B_{\mathbf{k}_1}^\dagger(R_1)B_{\mathbf{k}_2}^\dagger(R_2)B_{\mathbf{k}_3}^\dagger(R_3)\left|F_0\right\rangle=$$
$$\left\{B_{\mathbf{k}_1}^\dagger(R_1)B_{\mathbf{k}_2}^\dagger(R_2)\left(w_{\mathbf{k}_3}\beta_{\mathbf{k}_3}^\dagger-\frac{Vw_{\mathbf{k}_3}^2}{2\epsilon_{\mathbf{k}_2}-R_3}\beta^\dagger\right)+2\text{ perm.}\right\}$$
$$\left|F_0\right\rangle$$
$$+2V\left\{B_{\mathbf{k}_3}^\dagger(R_3)\frac{\delta_{\mathbf{k}_1\mathbf{k}_2}w_{\mathbf{k}_1}^3}{\left(2\epsilon_{\mathbf{k}_1}-R_1\right)\left(2\epsilon_{\mathbf{k}_1}-R_2\right)}\beta_{\mathbf{k}_1}^\dagger+2\text{ perm.}\right\}$$
$$\beta^\dagger\left|F_0\right\rangle \qquad (37)$$

To go further, we again split $\left(2\epsilon_{\mathbf{k}_1}-R_1\right)^{-1}\left(2\epsilon_{\mathbf{k}_1}-R_2\right)^{-1}$ as $\left[\left(2\epsilon_{\mathbf{k}_1}-R_1\right)^{-1}-\left(2\epsilon_{\mathbf{k}_1}-R_2\right)^{-1}\right]/\left(R_1-R_2\right)$ provided that $R_1\neq R_2$ and do the same for the two other products. By summing over $(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$, we end with

$$(H-E_3)B^\dagger(R_1)B^\dagger(R_2)B^\dagger(R_3)\left|F_0\right\rangle=$$
$$\{B^\dagger(R_2)B^\dagger(R_3)$$
$$\left(1-V\sum\frac{w_{\mathbf{k}_1}^2}{2\epsilon_{\mathbf{k}_1}-R_1}-\frac{2V}{R_1-R_2}+\frac{2V}{R_3-R_1}\right)$$
$$+2\text{ perm.}\}\beta^\dagger\left|F_0\right\rangle \qquad (38)$$

This leads us to again conclude that the three-pair state $B^\dagger(R_1)B^\dagger(R_2)B^\dagger(R_3)\left|F_0\right\rangle$ is eigenstate of the hamiltonian H with the energy $E_3 = R_1 + R_2 + R_3$, provided that (R_1, R_2, R_3) fulfill the three equations,

$$\begin{aligned} 1 &= V \sum \frac{w_{\mathbf{k}}^2}{2\epsilon_{\mathbf{k}} - R_1} + \frac{2V}{R_1 - R_2} + \frac{2V}{R_1 - R_3} \\ 1 &= V \sum \frac{w_{\mathbf{k}}^2}{2\epsilon_{\mathbf{k}} - R_2} + \frac{2V}{R_2 - R_3} + \frac{2V}{R_2 - R_1} \\ 1 &= V \sum \frac{w_{\mathbf{k}}^2}{2\epsilon_{\mathbf{k}} - R_3} + \frac{2V}{R_3 - R_1} + \frac{2V}{R_3 - R_2} \end{aligned} \tag{39}$$

D. N pairs

The above commutation technique can be easily extended to N pairs. As nicely visualized by the diagrams of Figs.2 and 3, the effect of the BCS potential on these N pairs splits into two sets of processes: In one set, one pair is affected by the (1x1) scattering while the other $N-1$ pairs stay unchanged. In the other set, this pair in addition has, before the interaction, a fermion exchange with another pair, the remaining $N-2$ pairs staying unchanged. This understanding readily shows that an increase of the pair number above two, does not really change the structure of the equations since $N-2$ pairs stay unchanged, the pair exchanging its fermions with the pair suffering the interaction being just one among $(N-1)$ pairs.

Although the equations become more and more cumbersome to be explicitly written, the procedure is rather straightforward once we have understood that either $(N-1)$ or $(N-2)$ pairs stay unaffected in the interaction process. The general form of the N-pair eigenstate ultimately appears as

$$(H - E_N) B^\dagger(R_1) \cdots B^\dagger(R_N) |F_0\rangle = 0 \tag{40}$$

with $E_N = R_1 + \cdots + R_N$, these R_N's being solutions of N coupled equations

$$1 = V \sum \frac{w_{\mathbf{k}}^2}{2\epsilon_{\mathbf{k}} - R_i} + \sum_{i \neq j} \frac{2V}{R_i - R_j} \qquad \text{for } i = (1, ..., N) \tag{41}$$

E. Physical understanding

This new derivation of the Richardson's equations has the main advantage to possibly trace back the parts in these equations which are directly linked to the Pauli exclusion principle between fermion pairs.

From a mathematical point of view, the link is rather obvious: In the absence of terms in $V/(R_i - R_j)$, the N equations for R_i reduced to the same equation (25), so that the result would be $R_i^{(0)} = E_1$ for all i: The fact that the energy of N pairs differs from N times the single pair energy E_1 thus comes from the $(R_i - R_j)$ differences.

Physically, the fact that E_N differs from NE_1 comes from interactions between pairs. Due to the (1x1) form of the BCS potential within the pair subspace, interaction between pairs can only be mediated by fermion exchanges as clear from Fig.(1c) Interaction between pairs thus is solely the result of the Pauli exclusion principle between pairs. This Pauli blocking mathematically appears through the various $\delta_{\mathbf{p}'\mathbf{p}}$ factors in Pauli scatterings $\lambda\begin{pmatrix} \mathbf{p}_2' & \mathbf{p}_2 \\ \mathbf{p}_1' & \mathbf{p}_1 \end{pmatrix}$. It is then easy to mathematically trace back the $(R_i - R_j)$ differences in the Richardson's equations to these δ factors.

In short, the Kronecker symbols in the Pauli scatterings of fermion pairs come from states which are excluded by the Pauli exclusion principle. They induce the $V/(R_i - R_j)$ terms of the Richardson's equations which ultimately makes the energy of N pairs different from the energy of N independent pairs.

Another very interesting feature of the energy E_N of N pairs that this new derivation explains in a rather clear way, is the fact that the part of the N pairs energy coming from interaction, namely $E_N - NE_1$ depends on N as $N(N-1)$ only. Indeed, the diagram of Fig.3 evidences the fact that, because the electron pairs of interest have one degree of freedom only, the (1x1) BCS potential mixed with fermion exchanges between pairs, ends by producing effective scatterings which are (2x2) only. In order to have terms in $N(N-1)(N-2)$, we would need topologically connected interaction processes between 3 pairs. This is why terms in $N(N-1)(N-2)$ and above, cannot exist in the energy of N Cooper pairs, in agreement with Eq.(1) which also reads

$$E_N = NE_1 + N(N-1)\left(\frac{1}{\rho_0} + \frac{\epsilon_c}{N\Omega}\right) \tag{42}$$

III. RICHARDSON'S EXACT EIGENSTATE VERSUS BCS ANSATZ

A last very smart aspect of the Richardson's procedure is that it provides the *exact* form of the eigenstate, namely

$$B^\dagger(R_1) \cdots B^\dagger(R_N) |F_0\rangle \tag{43}$$

with $B^\dagger(R)$ given by Eq.(24). The fact that by construction all the R_i's are different, strongly questions the standard BCS ansatz for the Cooper pair wave function which corresponds to take $\left(B^\dagger\right)^N |F_0\rangle$, with *all* the pairs condensed into the same state.

To discuss this problem on precise grounds, let us again start with two pairs. In a previous work[14], we have shown, that the "Richardson's energies" in the case of two pairs then read as $R_1 = R + iR'$ and $R_2 = R - iR'$ with R and R' real, the dominant terms of these real and imaginary parts in the large sample limit, i.e. for $1/\rho_0$ small, being $R \approx \epsilon_c + 1/\rho_0 + \epsilon_c/\rho_0\Omega$ and $R' \approx \sqrt{2\epsilon_c/\rho_0}$. By writting $B^\dagger(R_1)B^\dagger(R_2)$ as

$$\left[B^\dagger(R) + B^\dagger(R_1) - B^\dagger(R)\right]\left[B^\dagger(R) + B^\dagger(R_2) - B^\dagger(R)\right] \tag{44}$$

we get from eq (24)

$$B^\dagger(R_1)B^\dagger(R_2) = \left[B^\dagger(R)\right]^2 + R'^2\left\{C_+^\dagger C_-^\dagger - 2B^\dagger(R)D^\dagger\right\} \tag{45}$$

where we have set

$$C_\pm^\dagger = \sum \frac{w_{\mathbf{k}}}{(2\epsilon_{\mathbf{k}} - R)\,(2\epsilon_{\mathbf{k}} - R \pm iR')}\beta_{\mathbf{k}}^\dagger \tag{46}$$

$$D^\dagger = \sum \frac{w_{\mathbf{k}}}{(2\epsilon_{\mathbf{k}} - R)\left[(2\epsilon_{\mathbf{k}} - R)^2 + R'^2\right]}\beta_{\mathbf{k}}^\dagger \tag{47}$$

So that at lowest order in inverse sample volume, i.e., in $1/\rho_0$, we end with

$$B^\dagger(R_1)B^\dagger(R_2) - \left[B^\dagger(\frac{E_2}{2})\right]^2 \approx \frac{2\epsilon_c}{\rho_0}$$
$$\left\{-2B^\dagger(E_1)\sum\frac{w_{\mathbf{k}}}{(2\epsilon_{\mathbf{k}} - E_1)^3}\beta_{\mathbf{k}}^\dagger + \left[\sum\frac{w_{\mathbf{k}}}{(2\epsilon_{\mathbf{k}} - E_1)^2}\beta_{\mathbf{k}}^\dagger\right]^2\right\} \tag{48}$$

This shows that $B^\dagger(R_1)B^\dagger(R_2)$ can be replaced by $\left(B^\dagger(E_2/2)\right)^2$ provided that we drop the RHS of the above equation. This imposes to neglect terms in $1/\rho_0$. In this limit, E_2 would reduce to $2E_1$, so that $B^\dagger(E_2/2)$ would reduce to $B^\dagger(E_1)$: This simply corresponds to see the two-pair eigenstate as the product of two non-interacting single pairs. If instead, we want, in the two-pair state, to include the change induced by Pauli blocking which brings the energy per pair from E_1 to $E_2/2 = E_1 + 1/\rho_0 + \epsilon_c/\rho_0\Omega$, we are led to replace $B^\dagger(R_1)B^\dagger(R_2)$ by $\left(B^\dagger(E_2/2)\right)^2$. This however is inconsistent because we then keep in this two-pair operator, contribution in $1/\rho_0$ which are as large as the ones we drop by neglecting the RHS of Eq.(48). In the case of two pairs, the replacement of the exact eigenstate $B^\dagger(R_1)B^\dagger(R_2)\,|F_0\rangle$ by a BCS-like condense state $\left(B^\dagger(E_2/2)\right)^2|F_0\rangle$ is thus inconsistent.

Actually, it is claimed that the validity of the BCS ansatz is restricted to the thermodynamical limit, i.e., to N very large. When N increases, the R_i's stay two by two complex conjugate but differences between their real parts get larger and larger. This is why we hardly see how, starting from the exact form of the N-pair eigenstate $B^\dagger(R_1)\cdots B^\dagger(R_N)\,|F_0\rangle$ obtained by Richardson, it can be possible to recover the BCS ansatz with the same creation operator for all the pairs. It however is of importance to note that, to the best of our knowledge, derivations of the "validity" of the BCS antsatz for the ground state of N pairs mainly concentrate to the energy it provides (see, e.g.,[10] and references therein). We fully agree with the fact that the BCS ansatz indeed gives the correct ground state energy for N pairs because the energy obtained using this ansatz is just the one we derived from the exact Richardson's procedure. However agreement on the energy by no mean proves agreement on the wave function. Many examples have been given in the past with wave functions very different from the exact one, while giving the correct energy. Direct experiments supporting the form of the ground state wave function seems to be even harder to achieve than the ones possibly checking the ground state energy given in Eq.(1). At this point, it however seems to us necessary to carefully reconsider agreement with experiments in the light of the exact Richardson's wave function.

Let us note that the possible replacement of $B^\dagger(R_1)\cdots B^\dagger(R_N)\,|F_0\rangle$ by $\left(B^\dagger\right)^N|F_0\rangle$ is actually crucial to support the overall picture of superconductivity we have commonly in mind, with all the pairs in the same state, "as an army of little soldiers, all walking similarly". In a forthcoming paper, we are going to come back to the validity of the BCS ansatz in the thermodynamical limit, in the light of our recent analytical results on the Richardson's exact procedure.

IV. CONCLUSION

We have rederived the Richardson's equations using a commutation technique for free electron pairs with zero total momentum, similar to the one we have developed for composite boson excitons. Almost half a century ago, Richardson has succeeded to write down the *exact form* of the eigenstate for an arbitrary number N of pairs. It reads in terms of N energy-like quantities $R_1, ..., R_N$ which are solution of N coupled non-linear equations. This many-body problem is exactly solvable for an interaction potential between $2N$ electrons with up and down spins taken as a BCS-like reduced potential provided that scattering is separable, $V_{\mathbf{k}'\mathbf{k}} = -V\,w_{\mathbf{k}'}w_{\mathbf{k}}$ with $w_{\mathbf{k}}^2 = w_{\mathbf{k}}$. Note that a separable potential is also required to get the energy of a single pair in a compact form, as obtained by Cooper. Richardson managed to extend the Cooper exact solution to N pairs by decoupling them: This is done in a quite smart manner by rewriting the N-pair energy E_N as $R_1 + \cdots + R_N$.

The BCS-potential used in the calculation is highly simplified, therefore leaves out certain correction. Nevertheless, it led to the BCS ansatz with the essential feature of superconductivity. And it is of importantce to obtain its solution in canonical ensemble and compare it to the conventional ansatz in grand-canonical ensemble. The new derivation we here propose for the equations fulfilled by $R_1, \cdots, R_N$, allows us to trace back the physical origin of the various terms in a transparent way. In particular, this derivation clearly shows that N pairs differ from N independent pairs, due to Pauli blocking only. This Pauli blocking enforces the R_i energy-like parameters of the Richardson's equations to be all different. As a direct consequence, the exact wave function for N interacting pairs is definitely different from the BCS ansatz, although the N-pair energy this ansatz gives is the same in the large sample limit.

The diagrammatic representation of this derivation nicely evidences that, because electron pairs with zero total momentum have one degree of freedom only, they scatter within the (1×1) BCS potential in the pair subspace, through (2×2) scatterings only. This explains why the N-pair interaction energy that we have previously found, has terms in $N(N-1)$ but not in $N(N-1)(N-2)$ and so on...

One of us (M.C.) wishes to thank the University of Illinois at Urbana-Champaign, and Tony Leggett in particular, for enlightening discussions during a month invitation at the Institute for Condensed Matter Physics where most of the present work has been performed. We also wish to thank Walter Pogosov for his constructive comments on the manuscript.

---

[1] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Physical Review **106**, 162 (1957).
[2] R. W. Richardson, physics letters **3**, 277 (1963).
[3] R. W. Richardson and N. Sherman, Nucl. Phys. **52**, 221 (1964).
[4] J. Dukelsky, S. Pittel, and G. Sierra, Rev. Mod. Phys. **76**, 643 (2004).
[5] F. Braun and J. von Delft, Phys. Rev. Lett. **81**, 4712 (1998).
[6] R. W. Richardson, J. Math. Phys. **18**, 1802 (1977).
[7] W. Pogosov, M. Combescot, ArXiv. 0911.0849
[8] D. M. Eagles, Physical Review **186**, 456 (1969).
[9] A. J. Leggett, in *Proceedings of the XVIth Karpacz Winter School of Theoretical Physics, Karpacz, Poland* (Springer-Verlag, 1980), pp. 13–27.
[10] J. R. Schrieffer, *Theory Of Superconductivity* (Perseus Books, 1999), revised edition ed., ISBN 0738201200.
[11] M. Combescot, O. Betbeder-Matibet, and F. Dubin, Physics Reports **463**, 215 (2008).
[12] M. Combescot and O. Betbeder-Matibet, The European Physical Journal B **55**, 63 (2007).
[13] Wannier excitons have two degrees of freedom while Frenkel excitons have one only. This is why the Frenkel exciton many-body physics has similarities with the one of Cooper pairs (see W. Pogosov, M. Combescot, Euro. Phys. J. B, Vol. 68, 161(2009), Vol 68, 183 (2009)).
[14] W. Pogosov, M. Combescot, M. Crouzeix ArXiv. 0911.1688