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Fuzzy relational calculus in land evaluation

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Abstract

In recent years, methods of fuzzy reasoning were developed for situations akin to those found in land evaluation, in which a decision about land suitability must often be based upon imprecise information. The accuracy of such land evaluations depends on the quality of weighing land qualities with respect to their effect on crop production. The advantage of the fuzzy set approach is that class boundaries are not sharply defined, thus allowing the possibility of partial membership to a class. However, the application of fuzzy set theory in land evaluation is often limited to the use of membership functions and has weaknesses with regard to the way weights are attributed to the land qualities considered. Fuzzy relational calculus is introduced to overcome these problems. This new approach is based on fuzzy relations between land qualities and land units. Such a relation mathematically describes the suitability for a particular crop. Relational calculus offers the possibility to construct new relations from those defined previously. It allows to introduce weight coefficients that account for the importance of each land quality considered.

Keywords: land evaluation; fuzzy relations; weight coefficients

1. Introduction

In recent years there has been marked interest in the use of fuzzy reasoning in land evaluation, because it is increasingly being realized that classic methods of land evaluation, based upon the Boolean logical model of mutually exclusive classes, fail to

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incorporate the inexact or fuzzy nature of much resource data. The attractions of fuzzy set methodology to land evaluation are explained by Burrough (1989) and Tang et al. (1991) and case studies are given by Wang et al. (1990), Hall et al. (1992), Burrough et al. (1992), Tang and Van Ranst (1992a,b), Davidson et al. (1994), Van Ranst et al. (1996), Lark and Bolam (1997)), Mays et al. (1997), and Dobermann and Oberthür (1997). Through the use of fuzzy set methodology the rigid Boolean logic of suitability as determined by suitable or non-suitable land qualities is replaced by fuzzy membership functions or membership values. Land qualities that exactly match the strictly defined suitable situation are assigned a membership value of 1. Land qualities which are not suitable are given a membership value between 0 and 1. The membership function of a fuzzy set defines how the grade of membership of a land quality in the different land units is determined. The choice of membership functions is a critical issue in the use of fuzzy set methodology and this is not a straightforward task since decisions have to be made on membership values according to the degree of suitability. Other critical issues are the choice of weights, which express the degree of importance of each land quality with regard to crop yield or land suitability, and the combination of all land qualities considered into a final overall land suitability classification. Although the fuzzy set approach is more powerful than the Boolean one, both methodologies are subject to data and knowledge limitations.

In this paper, fuzzy relational calculus will be introduced as a new approach to land evaluation. Relational calculus seems to offer a more natural platform to describe and deal with land evaluation problems. Moreover, relational calculus offers a strong mathematical basis for a better use of fuzzy reasoning in land evaluation. The starting point of most existing fuzzy methods is not really classical set theory and Boolean logic, but one tries to apply fuzzy reasoning directly in the conventional methods. Therefore, this paper starts by introducing classical relational calculus, highlighting the potential as well as the limitations of classical relational calculus. The classical relational calculus will be fuzzified yielding the fuzzy relational calculus which can be interpreted and used keeping in mind its classical counterpart. The paper also explains how weights can be easily used and even estimated in this new approach and how the calculation of the overall suitability can be incorporated into the model.

2. Classical relational calculus

2.1. Basic concepts and notations

In order to explain the relational calculus approach, land evaluation will be considered here as the assessment of the suitability of a set of land units $\mathbf{U} = \{u_1, u_2, \cdots, u_n\}$ for a specific crop, based on a set of land qualities $\mathbf{Q} = \{q_1, q_2, \cdots, q_m\}$. A classical relation $\mathbf{S}$ (Suitability relation) can then be defined between the set of land units $\mathbf{U}$ and the set of land qualities $\mathbf{Q}$. The relation $\mathbf{S}$ assigns to every land unit u_i those land qualities q_j that are suitable for the crop considered. The relation $\mathbf{S}$ defines a subset of the Cartesian product $\mathbf{U} \times \mathbf{Q}$, where $(u_i,q_j) \in \mathbf{S}$ if in land unit u_i the quality q_j is suitable (S) for the crop considered and $(u_i,q_j) \notin \mathbf{S}$ if in land unit u_i, land quality q_j is not suitable (N) for the crop considered. The formula $(u_i,q_j) \in \mathbf{S}$ can also be written as $u_i\mathbf{S}q_j$, showing explicitly that the relation $\mathbf{S}$ connects land unit u_i to land quality q_j.

The easiest way to represent such a suitability relation $\mathbf{S}$ is by its corresponding matrix. For such a relation $\mathbf{S}$, the row headings are the land units u_i of $\mathbf{U}$, the column headings are the land qualities q_j of $\mathbf{Q}$. The entry in row i, column j is denoted as S_{ij}. Thus, $S_{ij} = 1$ if and only if $(u_i,q_j) \in \mathbf{S}$ or $S_{ij} = 0$ if $(u_i,q_j) \notin \mathbf{S}$.

2.1.1. Example

Consider $\mathbf{U} = \{u_1,u_2,u_3,u_4,u_5,u_6\}$ and $\mathbf{Q} = \{q_1,q_2,q_3,q_4,q_5,q_6,q_7\}$ and a suitability relation $\mathbf{S}$ with the following matrix representation:

S	q_1	q_2	q_3	q_4	q_5	q_6	q_7
u_1	1	0	1	1	0	1	1
u_2	0	1	1	0	0	1	1
u_3	1	1	1	1	1	1	1
u_4	1	0	1	1	0	1	1
u_5	0	0	0	0	0	0	0
u_6	1	0	0	1	0	1	0

This matrix can be readily interpreted and shows for example that in land unit u_3 all land qualities are suitable for the crop considered (all 1's on row 3), while all land qualities are considered not suitable in land unit u_5 (all 0's on row 5). Remark also that land quality q_6 is suitable in all land units, except in land unit u_5 (column 6) and land quality q_5 is only suitable in land unit u_3 (column 3).

Some important concepts for such a relation $\mathbf{S}$ are:

(A) The $\mathbf{S}$-afterset of $u_i \in \mathbf{U}$, denoted $u_i\mathbf{S}$.

The $\mathbf{S}$-afterset of a land unit u_i is the set of all land qualities q_j, which are suitable for the considered crop in this land unit u_i. The $\mathbf{S}$-afterset of a land unit u_i can be easily obtained considering the ith row in the matrix representation of the relation.

(B) The $\mathbf{S}$-foreset of $q_j \in \mathbf{Q}$, denoted $\mathbf{S}q_j$.

The $\mathbf{S}$-foreset of a land quality q_j is the set of all land units u_i for which this quality is suitable for the crop considered. The $\mathbf{S}$-foreset of a land quality q_j can be easily obtained considering the jth column in the matrix representation of the relation.

Considering the above relation the following sets can be readily obtained:

$$u_1\mathbf{S} = \{q_1,q_3,q_4,q_6,q_7\}$$

$$\mathbf{S}q_1 = \{u_1,u_3,u_4,u_6\}$$

(C) The converse relation of $\mathbf{S}$, denoted $\mathbf{S}^{\mathrm{T}}$

The converse relation $\mathbf{S}^{\mathrm{T}}$ is a relation between $\mathbf{Q}$ and $\mathbf{U}$, i.e., from land qualities to land units, that assigns to every land quality q_i those land units u_j for which this quality is suitable for the crop considered. Considering the matrix notation of the suitability relation **S**, the converse relation $\mathbf{S}^{\mathrm{T}}$ is obtained by transposing the former matrix:

$\mathbf{S}^{\mathrm{T}}$	u_1	u_2	u_3	u_4	u_5	u_6
q_1	1	0	1	1	0	1
q_2	0	1	1	0	0	0
q_3	1	1	1	1	0	0
q_4	1	0	1	1	0	1
q_5	0	0	1	0	0	0
q_6	1	1	1	1	0	1
q_7	1	1	1	1	0	0

It is important to remark that the **S**-afterset of a land unit u_i is equal to the $\mathbf{S}^{\mathrm{T}}$-foreset of the same land unit u_i, i.e., $u_i\mathbf{S} = \mathbf{S}^{\mathrm{T}}u_i$ (transposing of the matrix).

2.2. Compositions of classical relations

The composition of relations is a simple tool to construct new relations from already existing ones. Some interesting new compositions have been introduced by Bandler and Kohout (Bandler and Kohout, 1980a,b,c, 1986; Kerre, 1993; Groenemans, 1993). These compositions are based on the concepts of after- and foresets. The power of these compositions in the context of land evaluation will be briefly explained.

2.2.1. Relations between land units

Consider the relations **S** and $\mathbf{S}^{\mathrm{T}}$, the former is a relation from **U** to **Q**, the latter is a relation from **Q** to **U**. A relational composition **S** followed by $\mathbf{S}^{\mathrm{T}}$ will be a relation from **U** to **U**, i.e., a relation between land units. In the relational compositions as defined by Bandler and Kohout, after- and foresets are compared using the inclusion and equality operator. The following relations can be defined:

(A) The subcomposition of **S** and $\mathbf{S}^{\mathrm{T}}$ is denoted $\mathbf{S} \lhd \mathbf{S}^{\mathrm{T}}$:

$$\mathbf{S} \lhd \mathbf{S}^{\mathrm{T}} = \{(u_i, u_k) | u_i \in \mathbf{U} \text{ and } u_k \in \mathbf{U} \text{ and } u_i\mathbf{S} \subseteq u_k\mathbf{S}\}$$

A land unit u_i is in relationship $\mathbf{S} \lhd \mathbf{S}^{\mathrm{T}}$ with a land unit u_k, if and only if every land quality q_j suitable in land unit u_i is also suitable in land unit u_k.

(B) The squarecomposition of **S** and $\mathbf{S}^{\mathrm{T}}$ is denoted $\mathbf{S} \diamond \mathbf{S}^{\mathrm{T}}$:

$$\mathbf{S} \diamond \mathbf{S}^{\mathrm{T}} = \{(u_i, u_k) | u_i \in \mathbf{U} \text{ and } u_k \in \mathbf{U} \text{ and } u_i\mathbf{S} = u_k\mathbf{S}\}$$

A land unit u_i is in relationship $\mathbf{S} \diamond \mathbf{S}^{\mathrm{T}}$ with a land unit u_k, if and only if the respective sets of land qualities suitable for the two land units are equal.

Thus, these compositions have the following linguistic interpretations:

$(\mathbf{S} \lhd \mathbf{S}^{\mathrm{T}})_{ik} = 1 \Leftrightarrow$ all land qualities suitable for a specific crop in land unit u_i are also suitable in land unit u_k;

$(\mathbf{S} \diamond \mathbf{S}^{\mathrm{T}})_{ik} = 1 \Leftrightarrow$ land unit u_i and u_k have the same suitable land qualities for the crop considered.

In other words $(\mathbf{S} \lhd \mathbf{S}^{\mathrm{T}})_{ik} = 1$ shows that land unit u_k is at least as suitable for the crop considered as land unit u_i, because it has at least all suitable land qualities of u_i. Constructing the relation $\mathbf{S} \lhd \mathbf{S}^{\mathrm{T}}$, the following matrix will be obtained:

$\mathbf{S} \lhd \mathbf{S}^{\mathrm{T}}$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	1	0	1	1	0	0
u_2	0	1	1	0	0	0
u_3	0	0	1	0	0	0
u_4	1	0	1	1	0	0
u_5	1	1	1	1	1	1
u_6	1	0	1	1	0	1

For example, to obtain the entry in row 3, column 4, the $\mathbf{S}$-aftersets of land unit u_3 and land unit u_4, i.e., rows 3 and 4 of the matrix representation of $\mathbf{S}$, have to be compared. Using the classical Boolean implication operator $\rightarrow$, for which $0 \rightarrow 0 = 1$, $0 \rightarrow 1 = 1$, $1 \rightarrow 0 = 0$ and $1 \rightarrow 1 = 1$, and the minimum operator, the inclusion $u_3\mathbf{S} \subseteq u_4\mathbf{S}$ can be easily validated. This inclusion is just what is needed to calculate $(\mathbf{S} \lhd \mathbf{S}^{\mathrm{T}})_{34}$.

$(\mathbf{S} \lhd \mathbf{S}^{\mathrm{T}})_{34}$	q_1	q_2	q_3	q_4	q_5	q_6	q_7	
u_3	1	1	1	1	1	1	1	
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
u_4	1	0	1	1	0	1	1	
	$\Vert$	$\Vert$	$\Vert$	$\Vert$	$\Vert$	$\Vert$	$\Vert$	
minimum	1	0	1	1	0	1	1	$=0$

Setting $\mathbf{J} = \{1,2, \cdots ,m\}$ the above process can be mathematically formulated as follows:

$$(\mathbf{S} \lhd \mathbf{S}^{\mathrm{T}})_{ik} = \min_{j \in \mathbf{J}} (\mathbf{S}_{ij} \rightarrow \mathbf{S}_{kj})$$

This formulation shows that $(\mathbf{S} \lhd \mathbf{S}^{\mathrm{T}})_{34} = 0$ or that $u_3\mathbf{S} \not\subseteq u_4\mathbf{S}$. On the other hand one can verify that $(\mathbf{S} \lhd \mathbf{S}^{\mathrm{T}})_{43} = 1$, thus $u_4\mathbf{S} \subseteq u_3\mathbf{S}$. The relation $\mathbf{S} \lhd \mathbf{S}^{\mathrm{T}}$ can be visualized by means of its Hasse-diagram (Fig. 1), in which the nodes represent the different land units, and two related land units are connected by a line. Land units more suitable for the crop considered are placed above the less suitable ones.

The squarecomposition identifies those land units which have the same land qualities suitable for the crop considered, i.e., those land units that are equally suitable. The squarecomposition identifies those rows in the matrix representation of $\mathbf{S}$ that are equal.

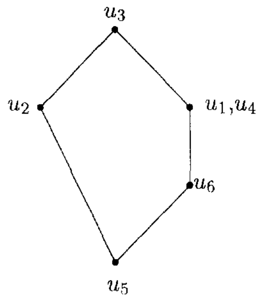

Fig. 1. Hasse-diagram of $\mathbf{S} \triangleleft \mathbf{S}^T$.

Keeping in mind that $u_i\mathbf{S} = u_k\mathbf{S}$, if and only if $u_i\mathbf{S} \subseteq u_k\mathbf{S}$ and $u_k\mathbf{S} \subseteq u_i\mathbf{S}$, it is obvious that for the squarecomposition:

$$(\mathbf{S} \diamond \mathbf{S}^{\mathrm{T}})_{ik} = \min\big((\mathbf{S} \triangleleft \mathbf{S}^{\mathrm{T}})_{ik}, (\mathbf{S} \triangleleft \mathbf{S}^{\mathrm{T}})_{ki}\big)$$

Constructing the relation $\mathbf{S} \diamond \mathbf{S}^{\mathrm{T}}$, the following matrix will be obtained:

$\mathbf{S} \diamond \mathbf{S}^{\mathrm{T}}$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	1	0	0	1	0	0
u_2	0	1	0	0	0	0
u_3	0	0	1	0	0	0
u_4	1	0	0	1	0	0
u_5	0	0	0	0	1	0
u_6	0	0	0	0	0	1

Land units u_1 and u_4 have the same land qualities suitable for the crop considered $\big((\mathbf{S} \diamond \mathbf{S}^{\mathrm{T}})_{14} = 1$ and $(\mathbf{S} \diamond \mathbf{S}^{\mathrm{T}})_{41} = 1\big)$. The relation $(\mathbf{S} \diamond \mathbf{S}^{\mathrm{T}})$ is always symmetrical.

2.2.2. Relations between land qualities

Other interesting relationships are obtained from $\mathbf{S}$ by considering the compositions $\mathbf{S}^{\mathrm{T}} \triangleleft \mathbf{S}$ and $\mathbf{S}^{\mathrm{T}} \diamond \mathbf{S}$, which are relations from $\mathbf{Q}$ to $\mathbf{Q}$. i.e., relations between land qualities.

These compositions have the following linguistic interpretation:

$(\mathbf{S}^{\mathrm{T}} \triangleleft \mathbf{S})_{ik} = 1 \Leftrightarrow$ all land units for which land quality q_i is suitable for the crop considered have also land quality q_k as suitable.

$(\mathbf{S}^{\mathrm{T}} \diamond \mathbf{S})_{ik} = 1 \Leftrightarrow$ land quality q_i is suitable for the crop considered in a certain land unit, if and only if land quality q_k is also suitable in that land unit

The relation $\mathbf{S}^{\mathrm{T}} \triangleleft \mathbf{S}$ can be seen as a sort of *inference* relation between land qualities, i.e., the condition of a land quality in a land unit can be deduced from the condition of another land quality in that land unit. For the considered relation **S**, the following matrix is obtained:

$\mathbf{S}^{\mathrm{T}} \lhd \mathbf{S}$	q_1	q_2	q_3	q_4	q_5	q_6	q_7
q_1	1	0	0	1	0	1	0
q_2	0	1	1	0	0	1	1
q_3	0	0	1	0	0	1	1
q_4	1	0	0	1	0	1	0
q_5	1	1	1	1	1	1	1
q_6	0	0	0	0	0	1	0
q_7	0	0	1	0	0	1	1

Since $(\mathbf{S}^{\mathrm{T}} \lhd \mathbf{S})_{23} = 1$, it is evident that whenever land quality q_2 is suitable for the crop considered in a certain land unit, also land quality q_3 is suitable in that land unit. If q_5 is suitable in a certain land unit, then all land qualities considered are suitable in that land unit (all 1's in row 5). If a land unit has at least one land quality suitable, then it has also land quality q_6 suitable (all 1's in column 6).

The squarecomposition $\mathbf{S}^{\mathrm{T}} \diamond \mathbf{S}$ for the considered relation is:

$\mathbf{S}^{\mathrm{T}} \diamond \mathbf{S}$	q_1	q_2	q_3	q_4	q_5	q_6	q_7
q_1	1	0	0	1	0	0	0
q_2	0	1	0	0	0	0	0
q_3	0	0	1	0	0	0	1
q_4	1	0	0	1	0	0	0
q_5	0	0	0	0	1	0	0
q_6	0	0	0	0	0	1	0
q_7	0	0	1	0	0	0	1

The squarecomposition could be used to reduce the number of land qualities needed to assess the suitability for the crop considered. $(\mathbf{S}^{\mathrm{T}} \diamond \mathbf{S})_{14} = 1$ shows that those land units having land quality q_1 suitable, also have q_4 as a suitable quality and vice versa. Those land units for which q_1 is not suitable, neither have land quality q_4 suitable. Land quality q_4 does not further contribute in the suitability assessment of the different land units. All information contained in q_1 is also contained in q_4 and vice versa.

Although classical relational calculus has an interesting linguistic interpretation and can be used to describe topics related to land evaluation, classical relations have of course serious limitations. Using crisp relations, a land quality can only be suitable (S) or not suitable (N) for a specific crop. This black or white, good or bad model is incapable of handling the imprecise information that is very often related to land evaluation. Therefore, it would be more realistic to consider partial degrees of relationship. This leads to the concept of fuzzy relations.

3. Fuzzy relational calculus

3.1. Basic concepts and notations

A fuzzy relation **S** from **U** to **Q** is a mapping from $\mathbf{U} \times \mathbf{Q}$ into [0,1]. Hence, a fuzzy relation from **U** to **Q** is a fuzzy set on the Cartesian product $\mathbf{U} \times \mathbf{Q}$. For every couple $(u_i,q_j) \in \mathbf{U} \times \mathbf{Q}$, the value $\mathbf{S}(u_i,q_j)$ is interpreted as the degree to which the fuzzy suitability relation **S** holds between land unit u_i and land quality q_j, i.e., the degree to which land quality q_j is suitable for the crop considered in land unit u_i or the degree to which the couple (u_i,q_j) belongs to the fuzzy relation **S**.

3.1.1. Example

The matrix representation of **S** can have now elements from the whole unit interval [0,1], leading to partial relationships between land units and land qualities:

S	q_1	q_2	q_3	q_4	q_5	q_6	q_7
u_1	0.9	0.4	0.7	0.8	0.5	1.0	0.8
u_2	0.3	0.9	0.7	0.2	0.6	0.9	0.9
u_3	0.9	0.9	0.7	0.9	0.8	1.0	1.0
u_4	1.0	0.3	0.8	0.8	0.2	0.8	0.7
u_5	0.2	0.2	0.3	0.4	0.3	0.4	0.3
u_6	0.8	0.2	0.4	0.7	0.4	0.9	0.6

The earlier mentioned concepts (afterset, foreset and converse) of a relation can be generalized to the fuzzy relation **S**:

(A) The **S**-afterset of u_i is denoted $u_i\mathbf{S}$:

$$u_i\mathbf{S} \;:\; \begin{array}{l} \mathbf{Q} \to [0,1] \\ q_j \to \mathbf{S}(u_i,q_j), \quad \forall q_j \in \mathbf{Q} \end{array}$$

(B) The **S**-foreset of q_j is denoted $\mathbf{S}q_j$:

$$\mathbf{S}q_j \;:\; \begin{array}{l} \mathbf{U} \to [0,1] \\ u_i \to \mathbf{S}(u_i,q_j), \quad \forall u_i \in \mathbf{U} \end{array}$$

Although the same notations are used, $u_i\mathbf{S}$ and $\mathbf{S}q_j$ are no longer classical sets when **S** is a fuzzy relation. The after- and foresets become maps from **Q**, respectively **U**, to the unit interval [0,1], i.e. fuzzy sets. Fuzzy sets allow partial degrees of membership. The **S**-afterset of $u_i \in \mathbf{U}$ is the fuzzy set of land qualities suitable for the crop considered in land unit u_i. The **S**-foreset of $q_j \in \mathbf{Q}$ is the fuzzy set of all land units having this land quality suitable for the crop considered.

(C) The converse relation of **S** is denoted $\mathbf{S}^{\mathrm{T}}$:

$$\mathbf{S}^{\mathrm{T}} \;:\; \begin{array}{l} \mathbf{Q} \times \mathbf{U} \to [0,1] \\ (q_i,u_j) \to \mathbf{S}(u_j,q_i), \quad \forall (q_i,u_j) \in \mathbf{Q} \times \mathbf{U} \end{array}$$

3.2. Composition of fuzzy relations

The compositions introduced earlier can be generalized to fuzzy relations using a triangular norm, denoted τ, and a fuzzy implication operator, denoted $\rightarrow$. A triangular norm is a mapping from $[0,1] \times [0,1]$ to $[0,1]$ which is in fact an extension of the minimum operator on $\{0,1\}$ to the unit interval (Schweizer and Sklar, 1961). A fuzzy implication operator is also a mapping from $[0,1] \times [0,1]$ to $[0,1]$, which extends the classical Boolean implication operator (Kerre, 1993). Using these operators it is possible to generalize the inclusion and equality of crisp (classical) sets to fuzzy sets, just what is needed to extend the crisp (classical) conditions $u_i\mathbf{S} \subseteq u_k\mathbf{S}$ and $u_i\mathbf{S} = u_k\mathbf{S}$, because $u_i\mathbf{S}$ and $u_k\mathbf{S}$ are fuzzy sets here. However, the exact derivation of the necessary formulas goes beyond the scope of this paper (Bandler and Kohout, 1980c; Kerre, 1993; De Baets and Kerre, 1993). The mathematical notations of the derived formulas will be explained. Examples will illustrate how the calculations are performed.

Let $\rightarrow$ be a fuzzy implication operator, τ a t-norm, and $J = \{1,2, \cdots, m\}$ then:

$$(\mathbf{S} \triangleleft \mathbf{S}^{\mathrm{T}})_{ik} = \tau_{j \in \mathrm{J}}\left(S_{ij} \rightarrow S^{\mathrm{T}}_{jk}\right)$$

$$(\mathbf{S} \Diamond \mathbf{S}^{\mathrm{T}})_{ik} = \tau\left((\mathbf{S} \triangleleft \mathbf{S}^{\mathrm{T}})_{ik}, (\mathbf{S} \triangleleft \mathbf{S}^{\mathrm{T}})_{ki}\right)$$

3.2.1. Fuzzy relations between land units

Considering $\rightarrow$ the Łukasiewicz implication operator $(a \rightarrow b) = \min(1, 1 - a + b)$ and taking the minimum as t-norm the subcomposition of $\mathbf{S}$ and $\mathbf{S}^{\mathrm{T}}$ becomes:

$$\mathbf{S} \triangleleft \mathbf{S}^{\mathrm{T}} = \begin{pmatrix} 0.9 & 0.4 & 0.7 & 0.8 & 0.5 & 1.0 & 0.8 \\ 0.3 & 0.9 & 0.7 & 0.2 & 0.6 & 0.9 & 0.9 \\ 0.9 & 0.9 & 0.7 & 0.9 & 0.8 & 1.0 & 1.0 \\ 1.0 & 0.3 & 0.8 & 0.8 & 0.2 & 0.8 & 0.7 \\ 0.2 & 0.2 & 0.3 & 0.4 & 0.3 & 0.4 & 0.3 \\ 0.8 & 0.2 & 0.4 & 0.7 & 0.4 & 0.9 & 0.6 \end{pmatrix}$$

$$\triangleleft \begin{pmatrix} 0.9 & 0.3 & 0.9 & 1.0 & 0.2 & 0.8 \\ 0.4 & 0.9 & 0.9 & 0.3 & 0.2 & 0.2 \\ 0.7 & 0.7 & 0.7 & 0.8 & 0.3 & 0.4 \\ 0.8 & 0.2 & 0.9 & 0.8 & 0.4 & 0.7 \\ 0.5 & 0.6 & 0.8 & 0.2 & 0.3 & 0.4 \\ 1.0 & 0.9 & 1.0 & 0.8 & 0.4 & 0.9 \\ 0.8 & 0.9 & 1.0 & 0.7 & 0.3 & 0.6 \end{pmatrix}$$

$$= \begin{pmatrix} 1.0 & 0.4 & 1.0 & 0.7 & 0.3 & 0.7 \\ 0.5 & 1.0 & 1.0 & 0.4 & 0.3 & 0.3 \\ 0.5 & 0.3 & 1.0 & 0.4 & 0.3 & 0.3 \\ 0.9 & 0.3 & 0.9 & 1.0 & 0.2 & 0.6 \\ 1.0 & 0.8 & 1.0 & 0.9 & 1.0 & 1.0 \\ 1.0 & 0.5 & 1.0 & 0.8 & 0.4 & 1.0 \end{pmatrix}$$

The entry in row 3 and column 4, i.e., $(\mathbf{S} \triangleleft \mathbf{S}^{\mathrm{T}})_{34}$ is again calculated by comparing (with the implication operator) one by one the degrees to which each land quality is suitable for the crop considered and then taking the minimum of those values. This process can be visualized as follows:

$(\mathbf{S} \lhd \mathbf{S}^{\mathrm{T}})_{34}$	q_1	q_2	q_3	q_4	q_5	q_6	q_7	
u_3	0.9	0.9	0.7	0.9	0.8	1.0	1.0	
	↓	↓	↓	↓	↓	↓	↓	
u_4	1.0	0.3	0.8	0.8	0.2	0.8	0.7	
	‖	‖	‖	‖	‖	‖	‖	
minimum	1.0	0.4	1.0	0.9	0.4	0.8	0.7	= 0.4

Keeping in mind that the Łukasiewicz implication operator is used, this can also be written as:

$$\begin{aligned}(\mathbf{S} \lhd \mathbf{S}^{\mathrm{T}})_{34} &= \min[\min(1, 1-0.9+1.0), \min(1, 1-0.9+0.3),\\ &\quad \min(1, 1-0.7+0.8), \min(1, 1-0.9+0.8),\\ &\quad \min(1, 1-0.8+0.2),\\ &\quad \min(1, 1-1.0+0.8), \min(1, 1-1.0+0.7)]\\ &= \min[1.0, 0.4, 1.0, 0.9, 0.4, 0.8, 0.7]\\ &= 0.4\end{aligned}$$

Moreover, the calculations can also be seen as a simple matrix multiplication, in which the operators $\times$ and $+$ are replaced by the implication operator $\rightarrow$ and the t-norm *min*, respectively.

The fuzzy relation $\mathbf{S} \lhd \mathbf{S}^{\mathrm{T}}$ is, however, difficult to interpret. How should a degree of 0.8 be interpreted? Does this indicate the relation holds between the elements considered or not? Therefore, this relation can be brought back to a simple classical relation by taking a so-called α-cut, i.e., specifying a threshold value, where $\alpha \in [0,1]$. The α-cut of the relation $\mathbf{S} \lhd \mathbf{S}^{\mathrm{T}}$ is obtained by choosing a threshold value α and restrains only those relationships between land units with a degree greater than or equal to α (Bandler and Kohout, 1991a,b; Kohout and Bandler, 1992).

(D) The α-cut of $\mathbf{S} \lhd \mathbf{S}^{\mathrm{T}}$ is denoted $(\mathbf{S} \lhd \mathbf{S}^{\mathrm{T}})^{\alpha}$:

$$(\mathbf{S} \lhd \mathbf{S}^{\mathrm{T}})^{\alpha} = \{(u_i, u_j) | u_i \in \mathbf{U} \text{ and } u_j \in \mathbf{U} \text{ and } (\mathbf{S} \lhd \mathbf{S}^{\mathrm{T}})_{ij} \geq \alpha\}$$

For the above example using $\alpha = 0.8$:

$$(\mathbf{S} \lhd \mathbf{S}^{\mathrm{T}})^{0.8} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

The above relation can be visualized by its Hasse-diagram (Fig. 2).

The diagram clearly shows that u_3 is the most suitable unit for the crop considered (u_3 has all the suitable land qualities of the other land units) and that u_5 is the least

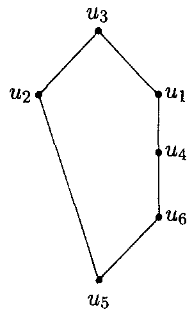

Fig. 2. Hasse-diagram of $\mathbf{S} \triangleleft \mathbf{S}^T$.

suitable land unit. The Hasse-diagram further shows that the land units u_1 and u_2 are not related to each other, i.e., the considered land qualities and the composition used do not reveal which land unit is the more suitable one.

3.2.2. *Fuzzy relations between land qualities*

The compositions $\mathbf{S}^{\mathrm{T}} \triangleleft \mathbf{S}$ and $\mathbf{S}^{\mathrm{T}} \diamond \mathbf{S}$ can be obtained by interchanging $\mathbf{S}$ and $\mathbf{S}^{\mathrm{T}}$ in the above formulas. Application of the Łukasiewicz operator and the minimum operator, results in the following:

$$\mathbf{S}^{\mathrm{T}} \diamond \mathbf{S} = \begin{pmatrix} 0.9 & 0.3 & 0.9 & 1.0 & 0.2 & 0.8 \\ 0.4 & 0.9 & 0.9 & 0.3 & 0.2 & 0.2 \\ 0.7 & 0.7 & 0.7 & 0.8 & 0.3 & 0.4 \\ 0.8 & 0.2 & 0.9 & 0.8 & 0.4 & 0.7 \\ 0.5 & 0.6 & 0.8 & 0.2 & 0.3 & 0.4 \\ 1.0 & 0.9 & 1.0 & 0.8 & 0.4 & 0.9 \\ 0.8 & 0.9 & 1.0 & 0.7 & 0.3 & 0.6 \end{pmatrix}$$

$$\diamond \begin{pmatrix} 0.9 & 0.4 & 0.7 & 0.8 & 0.5 & 1.0 & 0.8 \\ 0.3 & 0.9 & 0.7 & 0.2 & 0.6 & 0.9 & 0.9 \\ 0.9 & 0.9 & 0.7 & 0.9 & 0.8 & 1.0 & 1.0 \\ 1.0 & 0.3 & 0.8 & 0.8 & 0.2 & 0.8 & 0.7 \\ 0.2 & 0.2 & 0.3 & 0.4 & 0.3 & 0.4 & 0.3 \\ 0.8 & 0.2 & 0.4 & 0.7 & 0.4 & 0.9 & 0.6 \end{pmatrix}$$

$$= \begin{pmatrix} 1.0 & 0.3 & 0.6 & 0.8 & 0.2 & 0.4 & 0.4 \\ 0.3 & 1.0 & 0.5 & 0.3 & 0.7 & 0.3 & 0.6 \\ 0.6 & 0.5 & 1.0 & 0.5 & 0.4 & 0.5 & 0.7 \\ 0.8 & 0.3 & 0.5 & 1.0 & 0.4 & 0.3 & 0.3 \\ 0.2 & 0.7 & 0.4 & 0.4 & 1.0 & 0.4 & 0.5 \\ 0.4 & 0.3 & 0.5 & 0.3 & 0.4 & 1.0 & 0.7 \\ 0.4 & 0.6 & 0.7 & 0.3 & 0.5 & 0.7 & 1.0 \end{pmatrix}$$

Considering a threshold value(s) or α-cut(s), this relation shows a strong relationship between the land qualities q_1 and q_4 $\left((\mathbf{S}^{\mathrm{T}} \diamond \mathbf{S})_{14} = 0.8\right)$, and q_3 and q_7 $\left((\mathbf{S}^{\mathrm{T}} \diamond \mathbf{S})_{37} = 0.7\right)$. This relation is again always symmetrical.

4. Weights

The concepts of fuzzy relational composition provide a useful tool to compare the suitability of different land units. The aim of land evaluation is to select the most suitable land units for a specific land utilization type. The above compositions have still some constraints to model the relationship between different land units with respect to a specific crop. As pointed out earlier, the compositions used can not always reveal the relationship between certain land units (incomparability). For example, in land units u_1 and u_2 this is due to the fact that on the one hand land qualities q_1 and q_4 are more suitable for the crop considered in u_1, on the other hand land quality q_2 is more suitable in u_2. If land quality q_2 would be of little influence on the overall suitability, u_1 could be considered as more suitable than u_2. However, if q_2 would be of greater influence than q_1 and q_4 together, this could lead to the opposite conclusion. Therefore, it seems necessary to adjust the above compositions to reflect the special character of land evaluation problems.

4.1. Weighted compositions

Bandler and Kohout (1986) pointed out that for practical problems, it can be useful to take the arithmetic mean of all land qualities rather than using the minimum or any other t-norm:

$$(\mathbf{S} \triangleleft_m \mathbf{S}^{\mathrm{T}})_{ik} = \frac{1}{m} \sum_{j=1}^{m} \left(S_{ij} \rightarrow S^{\mathrm{T}}_{jk}\right)$$

where m is the number of land qualities. Because each land quality is considered of equal importance, it seems useful to include weights in the above formula to distinguish between the impact of each individual land quality on crop performance. Introduction of weights w_j, $j = 1, \cdots, m$ will result in the following formula:

$$(\mathbf{S} \triangleleft_w \mathbf{S}^{\mathrm{T}})_{ik} = \sum_{j=1}^{m} w_j\left(S_{ij} \rightarrow S^{\mathrm{T}}_{jk}\right)$$

where each weight is an element of the unit interval [0,1]:

$$w_j \in [0,1], \quad j = 1, \cdots, m$$

and the sum of all weights equals 1:

$$\sum_{j=1}^{m} w_j = 1$$

4.2. Calculation of weights

A critical issue in land evaluation methodologies is the prediction of the weights of the land qualities on crop performance. This section will explain how weights can be calculated, using the formalism constructed above. Hereby, the starting point is that, not only the values S_{ij} for each land unit u_i are known, i.e., the suitability of all qualities for the crop considered, but also the effective yield Y_i in this land unit. The effective yields can be rescaled to the unit interval in such a way that we get n values (one for every land unit) y_i, $i = 1, \cdots, n$. As explained earlier the value $(\mathbf{S} \triangleleft \mathbf{S}^{\mathrm{T}})_{ik}$ expresses the degree to which land unit u_k is more suitable than land unit u_i. However, knowing the yields Y_i and Y_k or equivalently y_i and y_k in u_i and u_k, respectively, this degree is known: $y_i \to y_k$

Using the least squares criterium (Stoer and Bulirsch, 1991), weights can be obtained by minimizing the differences between the observed relationships $(y_i \to y_k)$ and predicted relationships $(\mathbf{S} \triangleleft \mathbf{S}^{\mathrm{T}})_{ik}$ between each couple of land units u_i and u_k, by minimizing the following function:

$$F(w_1, w_2, \cdots, w_m) = \sum_{i,k=1}^{n} \left[(y_i \to y_k) - (\mathbf{S} \triangleleft_w \mathbf{S}^{\mathrm{T}})_{ik} \right]^2$$

or:

$$F(w_1, w_2, \cdots, w_m) = \sum_{i,k=1}^{n} \left[(y_i \to y_k) - \sum_{j=1}^{m} w_j \left(S_{ij} \to S_{jk}^{\mathrm{T}} \right) \right]^2$$

keeping in mind the constraints:

$$w_j \in [0,1], \quad j = 1, \cdots, m$$

$$\sum_{j=1}^{m} w_j = 1$$

Thus, the above optimization problem involves minimizing a function F, subject to some constraints and can be tackled by using the theory of Lagrange multipliers (Stoer and Bulirsch, 1991) or using genetic algorithms (Goldberg, 1989).

4.3. Forecasting yield in land units

The calculation of the weights can be seen as a training of the model. The model, i.e., the construction of the weighted composition, is supplied with known relationships (for every two land units it is known which is more suitable for the crop considered), and the weights are adjusted to reflect these relationships. Once the weights are calculated, the weighted composition can be used as a simple tool to forecast the yield in another land

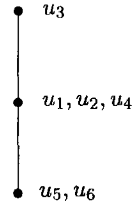

Fig. 3. Classification of land units.

unit, not used to train the model, just by comparing its land qualities with the land qualities of the *n known* land units (for which the observed yields are known). This will not give us an exact value, but nevertheless a yield range. For example, if for land unit u_p,

$$(\mathbf{S} \triangleleft_w \mathbf{S}^{\mathrm{T}})_{ip} = 0.8 \quad \text{and} \quad (\mathbf{S} \triangleleft_w \mathbf{S}^{\mathrm{T}})_{pj} = 0.9$$

and the threshold value (α-cut) 0.75 is considered, then land unit u_p is more suitable for the crop considered than u_i, but it is at most as suitable as u_j. Thus, the expected yields for land unit u_p is somewhere in between the yields of land units u_i and u_j.

5. Suitability classes

The presented approach offers a mechanism to classify land units according to their suitability for a specific crop. All land units are mutually compared and units with comparable qualities of equal importance on crop performance are grouped together. In parametric methods a land index is calculated for each land unit (Fig. 3). The land unit with the highest index is considered the most suitable one. The fuzzy relational method allows to calculate such a land index based on the following two assumptions:

(1) the more land qualities are suitable in a land unit, the higher the overall suitability of that unit;

(2) land qualities with larger weights are more important for the overall suitability than those with smaller weights.

These two assumptions can be readily translated using the earlier mentioned triangular norm τ and by introducing also the triangular conorm τ^{c}. A triangular conorm is an operator from $[0,1] \times [0,1]$ to $[0,1]$ which is in fact an extension of the maximum operator on $\{0,1\}$ to the unit interval (Schweizer and Sklar, 1961). Using these operators and putting again $\mathbf{J} = \{1,2, \cdots ,m\}$ with m being the number of land qualities, the following formula can be derived to calculate land indices:

$$I_i = \tau^{c}_{j \in \mathbf{J}}\left[\tau\left(S_{ij}, w_j\right)\right]$$

In this formula both the triangular norm and conorm can be altered. Since there is a wide variety of possible triangular norms and conorms, it is possible to adjust the formula to reflect the data.

6. Conclusion

Classical relational calculus forms a better starting point for the application of fuzzy reasoning in land evaluation. The transition from a classical, Boolean method to its fuzzy counterparts becomes more apparent. Fuzzy relational calculus offers a strong mathematical framework for the use of fuzzy logic in land evaluation. The relational compositions are one of the most powerful tools of fuzzy relational calculus. The fuzzy sub- and squarecompositions can be interpreted using land evaluation terminology. Mutual comparison of land units allows to determine the most suitable land unit. Mutual comparison of the land qualities reduces its number needed to assess the suitability of a land unit for a specific crop. Moreover, weights can be introduced into the compositions, yielding a more realistic approach to compare land units mutually. The least square method to calculate weights seems straightforward, but the underlying optimization problem can be quite complex. However, the mathematical tools for the selection of weights offer a robust approach to tackle this problem. The proposed method differs from existing methods in the way land units are classified by comparing them two by two, instead of calculating a land index for every land unit separately. However, the use of both triangular norm and conorm allows to incorporate the calculation of land indices into the fuzzy relational approach.

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