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Banach spaces generated by strongly linearly independent fuzzy numbers

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Abstract

This article introduces the notion of strong linear independence (SLI) for a set of fuzzy numbers. Based on this notion, we prove that there exist isomorphisms between $\mathbb{R}^n$ and special classes of fuzzy numbers generated by SLI sets of n fuzzy numbers. Such a bijection can be used to induce the structure of Banach space on its range. We prove that the finite SLI sets are dense in the set of finite fuzzy numbers. Moreover, we proposed two methods to produce SLI sets based on consecutive powering hedges and Zadeh extension of polynomials.

Keywords: Vector space; Banach space; Fuzzy number; Strong linear independence

1. Introduction

The main difficult to establish a calculus theory for fuzzy number-valued functions lies on the fact that the class of fuzzy numbers with the usual addition, scalar product, and the metric d_∞ (given in terms of the Hausdorff distance) forms a quasilinear metric space [1] as well as a semilinear space [2]. The algebraic structure of these spaces are weaker than the one of a vector space. In general, the properties of the existence of an inverse element of addition and the distributivity of the scalar multiplication with respect to field addition fail. In a recent article, Lupulescu and O'Regan introduce a theory of calculus for mappings taking values in a general quasilinear metric space [3] whose the notion of differentiability is defined considering four possible cases.

The most widely known and used differential theories for fuzzy number-valued functions are based on the notion of Hukuhara difference and its generalizations [4–9]. More recently, a differential calculus has been proposed for autocorrelated fuzzy processes in which the interactivity between the fuzzy variables $f(t)$ and $f(t+h)$ is explicitly considered in the definition of derivative [10,11]. Under some weak conditions, in [12], Wasques et al. showed that the generalized derivative can be viewed as particular case of the derivative introduced in [10].

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In [13,14], authors develop the notion of Fréchet derivative and Riemann integral for fuzzy number-valued functions of the form

$$f(t) = r(t) + q(t)A$$

where A is a fuzzy number and $q, r : \mathbb{R} \to \mathbb{R}$. Although this theory deals with a particular class of fuzzy functions, the calculus of the Fréchet derivative of f at t is pretty simple. More precisely, the Fréchet derivative of f at t is given in terms of the derivatives of the real-valued function q and r at t as follows:

$$f'(t) = r'(t) + q'(t)A.$$

In [14], authors solve general linear fuzzy ordinary differential equations using this notion of differentiability. The development of this theory is based on the fact that the function $\psi(r, q) = r + qA$, $r, q \in \mathbb{R}$, is a bijection from $\mathbb{R}^2$ onto the subset $\mathcal{S}(1, A) = \{r + qA \,|\, r, q \in \mathbb{R}\}$ of the set of fuzzy numbers $\mathbb{R}_{\mathcal{F}}$ whenever A is a non-symmetric fuzzy number. This bijection can be used to induce an addition $+_\psi$, a scalar product $\cdot_\psi$, and a norm $\|\cdot\|_\psi$ on $\mathcal{S}(1, A)$ and, thus, to obtain an isomorphism between $(\mathbb{R}^2, +, \cdot, \|\cdot\|)$ and the Banach space $(\mathcal{S}(1, A), +_\psi, \cdot_\psi, \|\cdot\|_\psi)$.

The main goal of this paper is to extend the results presented in [13] which establish sufficient and necessary conditions for $(\mathcal{S}(1, A), +_\psi, \cdot_\psi, \|\cdot\|_\psi)$ be a Banach space. To this end, we provide necessary and sufficient conditions on the fuzzy numbers $A_1, \ldots, A_n$ for the function $\psi : \mathbb{R}^n \to \mathbb{R}_{\mathcal{F}}$ given by

$$\psi(q_1, \ldots, q_n) = q_1 A_1 + q_2 A_2 + \ldots + q_n A_n$$

be injective, where $+$ stands for the standard addition of fuzzy numbers. The main result of this manuscript states that the function ψ is injective if, and only if, the set of the fuzzy numbers $\{A_1, \ldots, A_n\}$ satisfies the property of *strong linear independence* given in Definition 4. In this case, we have that the image of ψ, denoted by $\mathcal{S}(A_1, \ldots, A_n)$, forms a Banach space equipped with the addition, the scalar product, and the norm induced respectively by the function ψ. Following and extending the ideas presented in [13,14], the results discussed in this paper can be used as a basis to develop a calculus theory for functions with fuzzy coefficients $A_1, \ldots, A_n \in \mathbb{R}_{\mathcal{F}}$ of the form

$$f(t) = q_1(t)A_1 + q_2(t)A_2 + \ldots + q_n(t)A_n$$

where $q_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, n$.

This manuscript is organized as follows. Section 2 presents the mathematical background required for the development of this work. Section 3 introduces and characterizes the notion of *strong linear independence* (SLI) for sets of fuzzy numbers. In Section 4, we present two practical methods to yield SLI sets of fuzzy numbers considering non-symmetric trapezoidal fuzzy numbers and powering hedges and Zadeh's extension of polynomial functions. Conclusion and final remarks are given in Section 5.

2. Mathematical background

In this section, we recall some results and basic concepts which are needed throughout this paper. Specifically, the mathematical background regarding fuzzy set theory and the class of fuzzy numbers is given in Subsection 2.1. Subsection 2.2 deals with isomorphisms between Banach spaces.

2.1. Fuzzy numbers

A fuzzy set A of a non-empty universe of discourse X is identified by a function $\mu_A : X \to [0, 1]$, called membership function of A, where the value $\mu_A(x)$ denotes the degree of membership of $x \in X$ in A. Alternatively, for notational convenience, we simply write $A(x)$ instead of $\mu_A(x)$ for all $x \in X$. The symbol $\mathcal{F}(X)$ denotes the class of all fuzzy sets over the universe X. Note that every $Y \subseteq X$ can be associated with a fuzzy set whose membership function is given by its characteristic function χ_Y. Let $A, B \in \mathcal{F}(X)$, we have that $A \subseteq B$ whenever $A(x) \leq B(x)$ for all $x \in X$. The union and intersection of A and B are given, respectively, by fuzzy sets $C = A \cup B$ and $D = A \cap B$ where their membership functions are given by $C(x) = \max\{A(x), B(x)\}$ and $D(x) = \min\{A(x), B(x)\}$ for every $x \in X$.

Fuzzy sets are uniquely determined by their α-cuts [15]. For all $\alpha \in (0, 1]$, the α-cut of A is given by the set $[A]^\alpha = \{x \in X | A(x) \geq \alpha\}$. If X is also a topological space, then $[A]^0$ can be defined as the closure of the support of A, that is, $cl\{x \mid A(x) > 0\}$ [16]. The support of $A \in \mathcal{F}(X)$ is defined as the set $supp(A) = \{x \in X | A(x) > 0\}$. Moreover, A is said to be normal if $[A]^1 \neq \emptyset$. In this paper, we focus on a particular class of fuzzy sets of $\mathbb{R}$ which are called fuzzy numbers.

Definition 1. A fuzzy set $A \in \mathcal{F}(\mathbb{R})$ is called a fuzzy number if the following properties are satisfied:

1. A is normal.
2. $supp(A)$ is bounded.
3. the α-cuts of A are closed intervals.

The set of all fuzzy numbers is denoted by the symbol $\mathbb{R}_{\mathcal{F}}$.

Note that $\mathbb{R}$ can be embedded in $\mathbb{R}_{\mathcal{F}}$ since $[\chi_{\{x\}}]^\alpha = [x, x]$ for all $\alpha \in [0, 1]$. Thus, every real number x can be viewed as a fuzzy number and, in this case, we write $\chi_{\{x\}} \in \mathbb{R}_{\mathcal{F}}$ or, simply, $x \in \mathbb{R}_{\mathcal{F}}$. Well-known examples of fuzzy numbers are the trapezoidal fuzzy numbers. Let a, b, c, d be real numbers such that $a \leq b \leq c \leq d$. A trapezoidal fuzzy number is a fuzzy number A whose membership function is given by

$$A(x) = \begin{cases} 1 & \text{if } x \in [b, c] \\ \frac{x-a}{b-a} & \text{if } x \in [a, b) \\ \frac{x-d}{c-d} & \text{if } x \in (c, d] \\ 0 & \text{otherwise} \end{cases}, \; \forall x \in \mathbb{R}.$$

We denote the above trapezoidal fuzzy number A by the symbol $(a; b; c; d)$. In addition, we speak of triangular fuzzy number whenever $b = c$ and, in this case, we use the symbol $(a; b; d)$ instead of $(a; b; c; d)$.

The next definition establishes the notion of symmetry for fuzzy numbers.

Definition 2. Let A be a fuzzy number. If there exists $x \in \mathbb{R}$ such that $A(x - y) = A(x + y)$ for all $y \in \mathbb{R}$, then A is said to be symmetric with respect to x (or, simply, symmetric) and it is denoted by the symbol $(A \mid x)$. We say that A is non-symmetric if there is no $x \in \mathbb{R}$ such that this property is satisfied.

Clearly, we have that $(\chi_{\{x\}} \mid x)$ for all $x \in \mathbb{R}$. Fuzzy numbers given by closed intervals are symmetric since $(\chi_{[a,b]} \mid 0.5(a + b))$ for all $a, b \in \mathbb{R}$ satisfying $a \leq b$. Other examples may include the trapezoidal fuzzy numbers of the form $(b - \epsilon; b; c; c + \epsilon)$, which are symmetric with respect to $0.5(b + c)$ for any $b \leq c$ and $\epsilon \geq 0$.

Since each α-cut of a fuzzy number A is a closed interval, we can uniquely associate A to a pair of functions $L_A, U_A : [0, 1] \to \mathbb{R}$ such that $[A]^\alpha = [L_A(\alpha), U_A(\alpha)]$ for all $\alpha \in [0, 1]$. The set of fuzzy numbers such that the corresponding functions L_A and U_A are continuous is denoted by the symbol $\mathbb{R}^C_{\mathcal{F}}$. We use the symbol δ_A to denote the sum of L_A and U_A, that is, $\delta_A(\alpha) = L_A(\alpha) + U_A(\alpha)$ for all $\alpha \in [0, 1]$.

The class of fuzzy numbers forms a complete metric space with the metric d_∞ given in terms of the Hausdorff distance [17], but not separable [18]. Recall that the metric d_∞ between the fuzzy numbers A and B is defined as follows [19]:

$$d_\infty(A, B) = \bigvee_{\alpha \in [0,1]} \left(|L_A(\alpha) - L_B(\alpha)| \vee |U_A(\alpha) - U_B(\alpha)|\right). \tag{1}$$

Let $\mathcal{FS}(\mathbb{R}_{\mathcal{F}})$ be the set of all non-empty and finite sets of $\mathbb{R}_{\mathcal{F}}$, that is, $\mathcal{FS}(\mathbb{R}_{\mathcal{F}}) = \{\{A_1, \ldots, A_n\} \subset \mathbb{R}_{\mathcal{F}} \mid n \in \{1, 2, \ldots\}\}$. A metric $\mathcal{D}$ on $\mathcal{FS}(\mathbb{R}_{\mathcal{F}})$ can be defined in terms of the metric d_∞ as follows [20]:

$$\mathcal{D}(\{A_1, \ldots, A_n\}, \{B_1, \ldots, B_m\}) = \left(\bigvee_{i=1}^{n} \bigwedge_{j=1}^{m} d_\infty(A_i, B_j)\right) \vee \left(\bigvee_{j=1}^{m} \bigwedge_{i=1}^{n} d_\infty(A_i, B_j)\right) \tag{2}$$

It is worth noting that the metric $\mathcal{D}$ corresponds to the Hausdorff distance between finite sets of the metric space $(\mathbb{R}_{\mathcal{F}}, d_\infty)$ [17].

The next lemma is a direct consequence of Lemma 3.5 of [13] and characterizes the fuzzy numbers that are symmetric with respect to some x in terms of their corresponding endpoint functions.

Lemma 1. *A fuzzy number A is symmetric with respect to x if, and only if, $x - L_A(\alpha) = U_A(\alpha) - x$ or, equivalently, $2x = L_A(\alpha) + U_A(\alpha) = \delta_A(\alpha)$ for all $\alpha \in [0, 1]$.*

A fuzzy modifier, or a hedge, is a function $m : \mathcal{F}(X) \to \mathcal{F}(X)$. A well-known example of fuzzy modifiers are the powering hedges m_s, $s \in (0, \infty)$, that are given by [21,22]:

$$m_s(A) = A^s, \ \forall A \in \mathcal{F}(X),$$

where

$$A^s(x) = (A(x))^s, \ \forall x \in X.$$

We define $A^0 = 1$ for all $A \in \mathbb{R}_\mathcal{F}$. Note that $L_{A^s}(\alpha) = L_A(\alpha^{\frac{1}{s}})$ and $U_{A^s}(\alpha) = U_A(\alpha^{\frac{1}{s}})$ for $\alpha \in [0, 1]$ and $s \in (0, \infty)$. For example, for a trapezoidal fuzzy number $A = (a; b; c; d)$, we have, for all $\alpha \in [0, 1]$,

$$\begin{aligned} [A^s]^\alpha &= [L_{A^s}(\alpha), U_{A^s}(\alpha)] \\ &= [L_A(\alpha^{\frac{1}{s}}), U_A(\alpha^{\frac{1}{s}})] \\ &= [a + \alpha^{\frac{1}{s}}(b - a), d - \alpha^{\frac{1}{s}}(d - c)] \end{aligned} \tag{3}$$

since $L_A(\beta) = a + \beta(b - a)$ and $U_A(\beta) = d - \beta(d - c)$ for all $\beta \in [0, 1]$.

The Zadeh's extension principle can be viewed as a mathematical method to extend functions for the fuzzy domain [23].

Definition 3. Let $f : X \to Y$. The Zadeh's extension of f at $A \in \mathcal{F}(X)$ is the fuzzy set $\hat{f}(A)$ whose the membership function is given by

$$\hat{f}(A)(y) = \sup_{x \in f^{-1}(y)} A(x), \quad \forall y \in Y \tag{4}$$

where $f^{-1}(y) = \{z \in X \mid f(z) = y\}$ and, by definition, $\sup \emptyset = 0$.

The next proposition states that if $X = Y = \mathbb{R}$ and f is continuous, then the α-cuts of the Zadeh's extension of f at a fuzzy number A coincide with the image of the α-cuts of A under f [24,25].

Proposition 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $A \in \mathbb{R}_\mathcal{F}$. We have that $\hat{f}(A) \in \mathbb{R}_\mathcal{F}$ and*

$$[\hat{f}(A)]^\alpha = f([A]^\alpha), \quad \forall \alpha \in [0, 1]. \tag{5}$$

Let $A \in \mathbb{R}_\mathcal{F}$ and f_s be the function defined by $f_s(t) = t^s$ for all $t \in \mathbb{R}$ and some $s \in \{0, 1, 2, \ldots\}$. In view of Proposition 1, for all $\alpha \in [0, 1]$, we have that

$$[\hat{f}_s(A)]^\alpha = \begin{cases} \left[L_A(\alpha)^s, U_A(\alpha)^s\right] & \text{, if } s \text{ is odd} \\ \left[L_A(\alpha)^s, U_A(\alpha)^s\right] & \text{, if } 0 \leq L_A(\alpha) \text{ and } s \text{ is even} \\ \left[U_A(\alpha)^s, L_A(\alpha)^s\right] & \text{, if } U_A(\alpha) \leq 0 \text{ and } s \text{ is even} \\ \left[0, U_A(\alpha)^s \vee L_A(\alpha)^s\right] & \text{, if } L_A(\alpha) \leq 0 \leq U_A(\alpha) \text{ and } s \text{ is even} \end{cases}. \tag{6}$$

Thus, if $A = (a; b; c; d)$, then, for all $\alpha \in [0, 1]$, we have

$$[\hat{f}_s(A)]^\alpha = \begin{cases} \left[(a + \alpha(b - a))^s, (d - \alpha(d - c))^s\right] & \text{, if } s \text{ is odd} \\ \left[(a + \alpha(b - a))^s, (d - \alpha(d - c))^s\right] & \text{, if } 0 \leq L_A(\alpha) \text{ and } s \text{ is even} \\ \left[(d - \alpha(d - c))^s, (a + \alpha(b - a))^s\right] & \text{, if } U_A(\alpha) \leq 0 \text{ and } s \text{ is even} \\ \left[0, (d - \alpha(d - c))^s \vee (a + \alpha(b - a))^s\right] & \text{, if } L_A(\alpha) \leq 0 \leq U_A(\alpha) \text{ and } s \text{ is even} \end{cases}, \tag{7}$$

since $L_A(\alpha) = a + \alpha(b - a)$ and $U_A(\alpha) = d - \alpha(d - c)$.

The usual addition of two fuzzy numbers and the scalar product of a real number by a fuzzy number can be defined by means of Zadeh's extension principle. Specifically, let $A, B \in \mathbb{R}_\mathcal{F}$ and $q \in \mathbb{R}$, then

i) the sum of A and B, for all $\alpha \in [0, 1]$, is given pointwise by

$$\begin{aligned}[A+B]^\alpha &= [A]^\alpha + [B]^\alpha \\ &= [L_A(\alpha) + L_B(\alpha), U_A(\alpha) + U_B(\alpha)].\end{aligned} \tag{8}$$

ii) the scalar product of q by A, for all $\alpha \in [0, 1]$, is given pointwise by

$$q[A]^\alpha = \begin{cases} [qL_A(\alpha), qU_A(\alpha)] & , \text{if } q \geq 0 \\ [qU_A(\alpha), qL_A(\alpha)] & , \text{if } q < 0 \end{cases}. \tag{9}$$

The next proposition establishes some properties involving the usual operation of addition and scalar product [19,22].

Proposition 2. *For every $A, B, C \in \mathbb{R}_\mathcal{F}$ and $p, q \in \mathbb{R}$, the following properties hold:*

P1: $A + B = B + A$;
P2: $A + (B + C) = (A + B) + C$;
P3: $A + \chi_{\{0\}} = A$;
P4: *if $A \in \mathbb{R}_\mathcal{F} \setminus \mathbb{R}$, then there is no $B \in \mathbb{R}_\mathcal{F}$ such that $A + B = 0$;*
P5: $(p + q)A = pA + qA$ *if* $pq \geq 0$;
P6: $p(A + B) = pA + pB$;
P7: $(pq)A = p(qA)$.

Furthermore, the addition of symmetric fuzzy numbers and the scalar product of symmetric fuzzy numbers are symmetric fuzzy numbers as well. Such properties are stated in the next proposition.

Proposition 3. *Let A, B be two fuzzy numbers such that $(A \mid x)$ and $(B \mid y)$ for some $x, y \in \mathbb{R}$. Then, $(A + B \mid x + y)$ and $(qA \mid qx)$ for all $q \in \mathbb{R}$.*

Proof. The proof is straightforward from Lemma 1. □

The next lemma will be useful to prove Theorem 3.

Lemma 2. *Let $\{A_1, \ldots, A_n\} \in \mathcal{FS}(\mathbb{R}_\mathcal{F})$, $\varepsilon = (0; 0; \epsilon)$ with $1 > \epsilon > 0$, and $f_s(t) = t^s$ for all $t \in \mathbb{R}$ and $s = 1, 2, \ldots$. If $B_i = A_i$ or $B_i = A_i + \hat{f}_{s_i}(\varepsilon)$ for some $s_i \in \{1, 2, \ldots\}$ and $i = 1, \ldots, n$, then*

$$\mathcal{D}(\{A_1, \ldots, A_n\}, \{B_1, \ldots, B_n\}) \leq \epsilon.$$

Proof. Note that if $B_i = A_i$, then $d_\infty(A_i, B_i) = 0 < \epsilon$. Now, suppose that $B_i = A_i + \hat{f}_{s_i}(\varepsilon)$ for some $s_i \in \{1, 2, \ldots\}$. For all $\alpha \in [0, 1]$, from Equation (7), we have $[\hat{f}_{s_i}(\varepsilon)]^\alpha = [0, ((1-\alpha)\epsilon)^{s_i}]$ and, consequently, $[B_i]^\alpha = [L_{A_i}(\alpha), U_{A_i}(\alpha) + ((1-\alpha)\epsilon)^{s_i}]$. Since $((1-\alpha)\epsilon)^{s_i} \leq \epsilon$ for all $s_i \in \{1, 2, \ldots\}$ and $\alpha \in [0, 1]$, we obtain

$$d_\infty(A_i, B_i) = ((1-\alpha)\epsilon)^{s_i} \leq \epsilon.$$

Thus, for every $i, j \in \{1, \ldots, n\}$, we have

$$\bigwedge_{k=1}^n d_\infty(A_i, B_k) \leq d_\infty(A_i, B_i) \leq \epsilon \quad \text{and} \quad \bigwedge_{k=1}^n d_\infty(A_k, B_j) \leq d_\infty(A_j, B_j) \leq \epsilon$$

This implies that

$$\mathcal{D}(\{A_1, \ldots, A_n\}, \{B_1, \ldots, B_n\}) = \left(\bigvee_{i=1}^n \bigwedge_{k=1}^n d_\infty(A_i, B_k)\right) \vee \left(\bigvee_{j=1}^n \bigwedge_{k=1}^n d_\infty(A_k, B_j)\right) \leq \epsilon. \quad \square$$

2.2. On isomorphism

A Banach space is nothing more than a complete normed vector space (see Appendix A for a review of basic concepts such as Banach space). The class of linear and continuous operators between two Banach spaces is an important class of operators in functional analysis. For example, the notion of Fréchet derivative is given in terms of a linear and continuous operator from a Banach space to itself [20]. It is worth noting that every linear operator defined on a finite Banach space is continuous [20]. The definitions of continuity and linearity are also given in Appendix A. Let the set of all linear continuous function from X to Y be denoted by the symbol $L(X, Y)$.

Let X and Y be two arbitrary universes and let ψ be a bijection from X to Y. If d is a metric defined on X, then we can use d and ψ to induce a metric d_ψ on Y given by $d_\psi(y_1, y_2) = d(\psi^{-1}(y_1), \psi^{-1}(y_2))$ for all $y_1, y_2 \in Y$. In this case, the bijection ψ corresponds to an isometry between the metric spaces (X, d) and (Y, d_ψ). Thus, if (X, d) is complete, then (Y, d_ψ) is complete as well. Recall that any isometry is a continuous operator. On the other hand, if X is a vector space over a field K, that is, $(X, K, +, \cdot)$, then Y becomes a vector space over the same field K with the operation of addition $+_\psi$ and scalar multiplication $\cdot_\psi$ defined by

a) for all $y_1, y_2 \in Y$ as

$$y_1 +_\psi y_2 = \psi\left(\psi^{-1}(y_1) + \psi^{-1}(y_2)\right);$$

b) for all $y \in Y$ and $\lambda \in K$ as

$$\lambda \cdot_\psi y = \psi\left(\lambda\psi^{-1}(y)\right).$$

The map ψ is a linear operator from the vector space $(X, K, +, \cdot)$ to the vector space $(Y, K, +_\psi, \cdot_\psi)$. In addition, if $\|\cdot\|$ is a norm on X, then the function $\|\cdot\|_\psi : Y \to \mathbb{R}$ defined by $\| y \|_\psi = \| \psi^{-1}(y) \|$ for all $y \in Y$, is also a norm on Y. Therefore, if $\psi : X \to Y$ is a bijection and X is a Banach space, that is, a system $(X, K, +, \cdot, \|\cdot\|)$, then, from the above comments, ψ represents an isomorphism from $(X, K, +, \cdot, \|\cdot\|)$ to the Banach space $(Y, K, +_\psi, \cdot_\psi, \|\cdot\|_\psi)$.

We finish this subsection by recalling that, for a given non-symmetric $A \in \mathbb{R}_\mathcal{F}$, $\mathbb{R}^2$ is isomorphic to the space $\mathcal{S}(1, A) = \{q_1 + q_2 A \mid q_1, q_2 \in \mathbb{R}\}$ with the isomorphism $\psi : \mathbb{R}^2 \to \mathbb{R}_\mathcal{F}$ given by

$$\psi(q_1, q_2) = q_1 + q_2 A, \quad \forall q_1, q_2 \in \mathbb{R},$$

where $+$ stands for the standard addition of fuzzy numbers and, in this case, q_1 is regarded as a fuzzy number. Thus, $(\mathbb{R}^2, \mathbb{R}, +, \cdot, \|\cdot\|)$ is isomorphic to the Banach space $(\mathcal{S}(1, A), \mathbb{R}, +_\psi, \cdot_\psi, \|\cdot\|_\psi)$ if A is non-symmetric fuzzy number [13,14]. In these papers, authors also developed a integral and differential calculus for functions from $\mathbb{R}$ to $\mathcal{S}(1, A)$ for every non-symmetric $A \in \mathbb{R}_\mathcal{F}$ in the sense of Fréchet derivative and Riemann integral, establishing fundamental theorems of calculus. This approach was employed to obtain analytical solutions for certain fuzzy initial value problems (FIVPs). Theorems of existence and uniqueness of solution for certain FIVPs are also established in [14]. In addition, authors were able to extend these results for the case where A is a symmetric fuzzy number. Functions from $\mathbb{R}$ to $\mathcal{S}(1, A)$ are typically autocorrelated fuzzy process for which there is a theory of fuzzy differential equations based on the concept of interactivity that is also connected with this last one [10]. As we shall see in the next section, for a non-symmetric fuzzy number A, the space $\mathcal{S}(1, A)$ is isomorphic to $\mathbb{R}^2$ because the set $\{1, A\}$ satisfies a special property called strong linear independence given in Definition 4.

3. Banach spaces on subclasses of fuzzy numbers

This section presents the main results of this manuscript. Subsection 3.1 introduces the concept of *strong linear independence* (SLI) for a set of fuzzy numbers that can be used to obtain a Banach space in a subclass of fuzzy numbers. Subsection 3.2 characterizes the notion of SLI in terms of the notion of linear independence of a vector space of functions. Subsection 3.3 states that the set of finite SLI subsets of $\mathbb{R}_\mathcal{F}$ is dense in the set of finite subsets of $\mathbb{R}_\mathcal{F}$.

3.1. Strong linear independence

Let $\mathcal{S}(A_1, A_2, \ldots, A_n)$ be the subset of $\mathbb{R}_{\mathcal{F}}$ generated by $\{A_1, A_2, \ldots, A_n\} \subset \mathbb{R}_{\mathcal{F}}$ that is defined as follows:

$$\mathcal{S}(A_1, \ldots, A_n) = \{q_1 A_1 + \ldots + q_n A_n \,|\, q_1, \ldots, q_n \in \mathbb{R}\}. \tag{10}$$

Note that the set $\mathcal{S}(A_1, A_2, \ldots, A_n)$ is given by means of "linear combinations" using the standard operations of scalar product and addition defined in (9) and (8), respectively. Moreover, we have that $\chi_{\{0\}} \in \mathcal{S}(A_1, A_2, \ldots, A_n)$, since

$$0A_1 + 0A_2 + \ldots + 0A_n = \chi_{\{0\}}.$$

The set $\mathcal{S}(A_1, A_2, \ldots, A_n)$ is closed with respect to the scalar product, but it may not be with respect to the addition. For example, consider the set

$$\mathcal{S}((1; 2; 3)) = \{(\lambda 1; \lambda 2; \lambda 3) \,|\, \lambda \geq 0\} \cup \{(\lambda 3; \lambda 2; \lambda 1) \,|\, \lambda < 0\},$$

we have that $(1; 2; 3), (-3; -2; -1) \in \mathcal{S}((1; 2; 3))$, but $(1; 2; 3) + (-3; -2; -1) = (-2; 0; 2) \notin \mathcal{S}((1; 2; 3))$.

Next, we introduce the concept of strongly linearly independent for a set of fuzzy numbers.

Definition 4. A set of fuzzy numbers $\{A_1, A_2, \ldots, A_n\}$ is said to be strongly linearly independent (SLI) if the following implication holds for every $A \in \mathcal{S}(A_1, A_2, \ldots, A_n)$:

$$(A \,|\, 0) \Rightarrow q_1 = q_2 = \ldots = q_n = 0, \tag{11}$$

where $A = q_1 A_1 + \ldots + q_n A_n$.

From Definition 4, we have that a set of fuzzy numbers $\{A_1, \ldots, A_n\}$ is SLI if, and only if, the unique fuzzy number in $\mathcal{S}(A_1, A_2, \ldots, A_n)$ that is symmetric with respect to 0 is the real number 0 regarded as a fuzzy number. The concept of strong linear independence given in Definition 4 resembles the concept of linear independence of the linear algebra, however, they have different properties. In linear algebra, if the set of vectors $\{v_1, \ldots, v_n\}$ is linear independent and u is vector that is not a linear combination of them, then one can prove that the set $\{(v_1 + u), \ldots, (v_n + u)\}$ is linear independent as well. This property may not hold for SLI set of fuzzy numbers. For example, the set $\{(0.5; 1; 2), (0.5; 1; 1)\}$ is SLI and $-1 \equiv (-1; -1; -1) \notin \mathcal{S}((0.5; 1; 2), (0.5; 1; 1))$, but $\{(0.5; 1; 2) + (-1; -1; -1), (0.5; 1; 1) + (-1; -1; -1)\} = \{(-0.5; 0; 1), (-0.5; 0; 0)\}$ is not SLI since $(-0.5; 0; 1) + (-0.5; 0; 0) = (-1; 0; 1) \in \mathcal{S}((-0.5; 0; 1), (-0.5; 0; 0))$ is symmetric with respect to 0.

Theorem 1. *If $\{A_1, A_2, \ldots, A_n\} \subseteq \mathbb{R}_{\mathcal{F}}$ is strongly linearly independent, then the function $\psi : \mathbb{R}^n \to \mathcal{S}(A_1, A_2, \ldots, A_n)$ given by*

$$\psi(x_1, \ldots, x_n) = x_1 A_1 + \ldots + x_n A_n$$

for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$ is a bijection.

Conversely, if the function ψ is a bijection from $\mathbb{R}^n$ to $\mathcal{S}(A_1, A_2, \ldots, A_n)$, then the set $\{A_1, A_2, \ldots, A_n\}$ is strongly linearly independent.

Proof. By definition of $\mathcal{S}(A_1, A_2, \ldots, A_n)$, it is obvious that, for every $A \in \mathcal{S}(A_1, A_2, \ldots, A_n)$, there exists $(x_1, \ldots, x_n)$ such that $\psi(x_1, \ldots, x_n) = A$. Thus, it is sufficient to prove that ψ is injective in order to conclude that ψ is a bijection.

For each $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we define $I_{\mathbf{x}} = \{i \in \{1, \ldots, n\} \,|\, x_i \geq 0\}$ and $J_{\mathbf{x}} = \{1, \ldots, n\} \setminus I_{\mathbf{x}}$. Let $B \in \mathcal{S}(A_1, A_2, \ldots, A_n)$ and let $\mathbf{p} := (p_1, \ldots, p_n), \mathbf{q} := (q_1, \ldots, q_n) \in \mathbb{R}^n$ such that $B = \sum_{i=1}^n p_i A_i = \sum_{i=1}^n q_i A_i$. Using the properties of Proposition 2, we obtain

$$L_B(\alpha) = \sum_{i \in I_{\mathbf{p}}} p_i L_{A_i}(\alpha) + \sum_{i \in J_{\mathbf{p}}} p_i U_{A_i}(\alpha) = \sum_{i \in I_{\mathbf{q}}} q_i L_{A_i}(\alpha) + \sum_{i \in J_{\mathbf{q}}} q_i U_{A_i}(\alpha) \tag{12}$$

and

$$U_B(\alpha) = \sum_{i\in I_{\mathbf{p}}} p_i U_{A_i}(\alpha) + \sum_{i\in J_{\mathbf{p}}} p_i L_{A_i}(\alpha) = \sum_{i\in I_{\mathbf{q}}} q_i U_{A_i}(\alpha) + \sum_{i\in J_{\mathbf{q}}} q_i L_{A_i}(\alpha) \tag{13}$$

for all $\alpha \in [0, 1]$.

Now, let us consider

$$C = \sum_{i=1}^{n} (p_i - q_i)A_i \in \mathcal{S}(A_1, A_2, \ldots, A_n).$$

Note that, for all $\alpha \in [0, 1]$, we have

$$L_C(\alpha) = \sum_{i\in I_{\mathbf{p}-\mathbf{q}}} (p_i - q_i)L_{A_i}(\alpha) + \sum_{i\in J_{\mathbf{p}-\mathbf{q}}} (p_i - q_i)U_{A_i}(\alpha)$$

and

$$U_C(\alpha) = \sum_{i\in I_{\mathbf{p}-\mathbf{q}}} (p_i - q_i)U_{A_i}(\alpha) + \sum_{i\in J_{\mathbf{p}-\mathbf{q}}} (p_i - q_i)L_{A_i}(\alpha).$$

Using Equations (12) and (13), we obtain that

$$\begin{aligned}
L_C(\alpha) + U_C(\alpha) = & \sum_{i\in I_{\mathbf{p}-\mathbf{q}}} (p_i - q_i)L_{A_i}(\alpha) + \sum_{i\in J_{\mathbf{p}-\mathbf{q}}} (p_i - q_i)U_{A_i}(\alpha) \\
& + \sum_{i\in I_{\mathbf{p}-\mathbf{q}}} (p_i - q_i)U_{A_i}(\alpha) + \sum_{i\in J_{\mathbf{p}-\mathbf{q}}} (p_i - q_i)L_{A_i}(\alpha) = \\
& + \left(\sum_{i\in I_{\mathbf{p}}\cap I_{\mathbf{p}-\mathbf{q}}} p_i L_{A_i}(\alpha) + \sum_{i\in I_{\mathbf{p}}\cap J_{\mathbf{p}-\mathbf{q}}} p_i L_{A_i}(\alpha) \right) \\
& + \left(\sum_{i\in J_{\mathbf{p}}\cap I_{\mathbf{p}-\mathbf{q}}} p_i L_{A_i}(\alpha) + \sum_{i\in J_{\mathbf{p}}\cap J_{\mathbf{p}-\mathbf{q}}} p_i L_{A_i}(\alpha) \right) \\
& + \left(\sum_{i\in I_{\mathbf{p}}\cap I_{\mathbf{p}-\mathbf{q}}} p_i U_{A_i}(\alpha) + \sum_{i\in I_{\mathbf{p}}\cap J_{\mathbf{p}-\mathbf{q}}} p_i U_{A_i}(\alpha) \right) \\
& + \left(\sum_{i\in J_{\mathbf{p}}\cap I_{\mathbf{p}-\mathbf{q}}} p_i U_{A_i}(\alpha) + \sum_{i\in J_{\mathbf{p}}\cap J_{\mathbf{p}-\mathbf{q}}} p_i U_{A_i}(\alpha) \right) \\
& - \left(\sum_{i\in I_{\mathbf{q}}\cap I_{\mathbf{p}-\mathbf{q}}} q_i L_{A_i}(\alpha) + \sum_{i\in I_{\mathbf{q}}\cap J_{\mathbf{p}-\mathbf{q}}} q_i L_{A_i}(\alpha) \right) \\
& - \left(\sum_{i\in J_{\mathbf{q}}\cap I_{\mathbf{p}-\mathbf{q}}} q_i L_{A_i}(\alpha) + \sum_{i\in J_{\mathbf{q}}\cap J_{\mathbf{p}-\mathbf{q}}} q_i L_{A_i}(\alpha) \right) \\
& - \left(\sum_{i\in I_{\mathbf{q}}\cap I_{\mathbf{p}-\mathbf{q}}} q_i U_{A_i}(\alpha) + \sum_{i\in I_{\mathbf{q}}\cap J_{\mathbf{p}-\mathbf{q}}} q_i U_{A_i}(\alpha) \right) \\
& - \left(\sum_{i\in J_{\mathbf{q}}\cap I_{\mathbf{p}-\mathbf{q}}} q_i U_{A_i}(\alpha) + \sum_{i\in J_{\mathbf{q}}\cap J_{\mathbf{p}-\mathbf{q}}} q_i U_{A_i}(\alpha) \right) = \\
& + \left(\sum_{i\in I_{\mathbf{p}}} p_i L_{A_i}(\alpha) + \sum_{i\in J_{\mathbf{p}}} p_i U_{A_i}(\alpha) \right) + \left(\sum_{i\in I_{\mathbf{p}}} p_i U_{A_i}(\alpha) + \sum_{i\in J_{\mathbf{p}}} p_i L_{A_i}(\alpha) \right)
\end{aligned}$$

$$-\left(\sum_{i\in I_{\mathbf{q}}} q_i L_{A_i}(\alpha) + \sum_{i\in J_{\mathbf{q}}} q_i U_{A_i}(\alpha)\right) - \left(\sum_{i\in I_{\mathbf{q}}} q_i U_{A_i}(\alpha) + \sum_{i\in J_{\mathbf{q}}} q_i L_{A_i}(\alpha)\right) =$$
$$L_B(\alpha) + U_B(\alpha) - L_B(\alpha) - U_B(\alpha) = 0$$

The above equalities imply that $L_C(\alpha) = -U_C(\alpha)$ for all $\alpha \in [0, 1]$. An application of Lemma 1 ensures that C is symmetric with respect to 0. Thus, we conclude that $p_i = q_i$ for $i = 1, \ldots, n$, because $\{A_1, \ldots, A_n\}$ is SLI. Hence, ψ is a bijection.

Now, we proceed to prove the second claim of the theorem. Let us assume that the function $\psi : \mathbb{R}^n \to \mathcal{S}(A_1, \ldots, A_n)$ given by $\psi(x_1, \ldots, x_n) = x_1 A_1 + \ldots + x_n A_n$ for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$ is a bijection. Let $\psi(x_1, \ldots, x_n) = B$ be symmetric with respect to 0 for some $\mathbf{x} := (x_1, \ldots, x_n) \in \mathbb{R}^n$. Considering $C = \psi(-x_1, \ldots, -x_n)$ and $-\mathbf{x} = (-x_1, \ldots, -x_n)$, we shall show that $B = C$. Indeed, from Lemma 1 and from the above equations, we have $L_B = -U_B$ and

$$\begin{aligned}
L_B(\alpha) &= \sum_{i\in I_{\mathbf{x}}} x_i L_{A_i}(\alpha) + \sum_{i\in J_{\mathbf{x}}} x_i U_{A_i}(\alpha) \\
&= -U_B(\alpha) \\
&= -\sum_{i\in I_{\mathbf{x}}} x_i U_{A_i}(\alpha) - \sum_{i\in J_{\mathbf{x}}} x_i L_{A_i}(\alpha) \\
&= \sum_{i\in I_{\mathbf{x}}} -x_i U_{A_i}(\alpha) + \sum_{i\in J_{\mathbf{x}}} -x_i L_{A_i}(\alpha) \\
&= \sum_{i\in J_{(-\mathbf{x})}} -x_i U_{A_i}(\alpha) + \sum_{i\in I_{(-\mathbf{x})}} -x_i L_{A_i}(\alpha) \\
&= L_C(\alpha)
\end{aligned}$$

for all $\alpha \in [0, 1]$. Similarly, we can prove that $U_B(\alpha) = U_C(\alpha)$ for all $\alpha \in [0, 1]$. Now, using the fact that ψ is injective, we conclude that $\mathbf{x} = -\mathbf{x}$ which implies that $x_i = 0$ for all $i = 1, \ldots, n$. Hence, we have that $(B \mid 0) \Rightarrow x_i = 0$ for all $i = 1, \ldots, n$. Therefore $\{A_1, A_2, \ldots, A_n\}$ is SLI. □

In view of Theorem 1 and of Subsection 2.2, we obtain the following immediate corollary.

Corollary 1. *If $\{A_1, A_2, \ldots, A_n\} \subseteq \mathbb{R}_{\mathcal{F}}$ is SLI, then $(\mathcal{S}(A_1, \ldots, A_n), +_\psi, \cdot_\psi, \|\cdot\|_\psi)$ is a Banach space, where ψ is given as in Theorem 1.*

Proof. Evident. □

Corollary 1 reveals that the study of the notion of strong linear independence is central if one wants to take advantage of the algebraic structure of the Banach space in order to deal with functions from $\mathbb{R}$ to $(\mathcal{S}(A_1, \ldots, A_n), +_\psi, +_\psi, \cdot_\psi, \|\cdot\|_\psi)$. The next results are consequences of Theorem 1 and establish some properties that are satisfied by SLI sets of fuzzy numbers.

Corollary 2. *Let $\{A_1, A_2, \ldots, A_n\} \subseteq \mathbb{R}_{\mathcal{F}}$. If there exists $I \subseteq \{1, \ldots, n\}$, $I \neq \emptyset$, such that $\{A_i : i \in I\}$ is not SLI, then $\{A_1, A_2, \ldots, A_n\}$ is not SLI.*

Proof. Let $I = \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$, $0 < m \leq n$, such that $\{A_{i_1}, \ldots, A_{i_m}\}$ is not SLI.

Suppose that $\{A_1, A_2, \ldots, A_n\}$ is SLI. By Theorem 1, the function $\psi : \mathbb{R}^n \to \mathcal{S}(A_1, A_2, \ldots, A_n)$ given by $\psi(\mathbf{x}) = \sum_{i=1}^n x_i A_i$, $\forall \mathbf{x} \in \mathbb{R}^n$, is a bijection. Thus, the function $\tilde{\psi} : \mathbb{R}^m \to \mathcal{S}(A_{i_1}, \ldots, A_{i_m})$ given by the following restriction of ψ:

$$\tilde{\psi}(\mathbf{y}) = \sum_{j=1}^m y_j A_{i_j}, \ \forall \mathbf{y} \in \mathbb{R}^m,$$

is a bijection. This implies that $\{A_{i_1}, \ldots, A_{i_m}\}$ is SLI by Theorem 1, contradicting the hypothesis that it is not SLI. Therefore, $\{A_1, A_2, \ldots, A_n\}$ cannot be SLI. $\square$

For example, we have that $\{(-2; 1; 1), (-1; -1, 2)\}$ is not SLI, since $(-3; 0; 3) \in \mathcal{S}((-2; 1; 1), (-1; -1, 2))$. Thus, the set $\{(-2; 1; 1), (-1; -1, 2), (2; 7; 8)\}$ is not SLI.

Corollary 3. *Given $\{A_1, A_2, \ldots, A_n\} \subseteq \mathbb{R}_{\mathcal{F}}$. If $(A_i \mid 0)$ for some $i \in \{1, \ldots, n\}$, then $\{A_1, A_2, \ldots, A_n\}$ is not SLI.*

Proof. If $(A_i \mid 0)$, then by Proposition 3, for some $x \neq 0$, we have $(xA_i \mid x0)$, that is, $(xA_i \mid 0)$. Thus, $\{A_i\}$ is not SLI since $xA_i \in \mathcal{S}(A_i)$ for all $x \neq 0$. By Corollary 2, we conclude that $\{A_1, A_2, \ldots, A_n\}$ is not SLI. $\square$

For example, we have that the set $\{A_1 = (-1; 0; 1), A_2 = (-2; 3; 5), A_3 = (1; 3; 4)\}$ is not SLI because $(A_1 \mid 0)$.

In linear algebra, if a set of vectors is linearly independent, then there exists no vector in the set that can be defined as a linear combination of the others. Corollary 4 states that a similar property is also satisfied for SLI sets of fuzzy numbers.

Corollary 4. *Given $\{A_1, A_2, \ldots, A_n\} \subseteq \mathbb{R}_{\mathcal{F}}$. If*

$$A_i \in \mathcal{S}(A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n)$$

for some $i \in \{1, \ldots, n\}$, then $\{A_1, A_2, \ldots, A_n\}$ is not SLI.

Proof. If $\{A_1, A_2, \ldots, A_n\}$ is SLI, then Theorem 1 establishes that the function $\psi : \mathbb{R}^n \to \mathcal{S}(A_1, A_2, \ldots, A_n)$ given by $\psi(\mathbf{x}) = \sum_{i=1}^n x_i A_i$, $\forall \mathbf{x} \in \mathbb{R}^n$, is a bijection. Now, if $A_i \in \mathcal{S}(A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n)$, that is, $A_i = x_1 A_1 + \ldots + x_{i-1} A_{i-1} + x_{i+1} A_{i+1} + \ldots + x_n A_n$ for some $x_j \in \mathbb{R}$, $i \neq j$, then we obtain the following contradiction with respect to the bijection of ψ: $\psi(\mathbf{e}^i) = A_i = \psi(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$, where $\mathbf{e}^i$ denotes the ith canonical vector of $\mathbb{R}^n$. Hence, $\{A_1, A_2, \ldots, A_n\}$ is not SLI. $\square$

The next proposition states that an SLI set of fuzzy numbers can not contain two or more symmetric fuzzy numbers.

Proposition 4. *Given $\{A_1, A_2, \ldots, A_n\} \subseteq \mathbb{R}_{\mathcal{F}}$. If $(A_i \mid x_i)$ and $(A_j \mid x_j)$ for $i \neq j$, then $\{A_1, A_2, \ldots, A_n\}$ is not SLI.*

Proof. If $x_i = 0$ or $x_j = 0$, then we have that $\{A_1, A_2, \ldots, A_n\}$ is not SLI from Corollary 3. Now, assume that $x_i \neq 0$ and $x_j \neq 0$. Note that $(x_j A_i - x_i A_j) \in \mathcal{S}(A_1, \ldots, A_n)$. From Proposition 3, we have $(x_j A_i - x_i A_j \mid 0)$. By Definition 4, $\{A_1, A_2, \ldots, A_n\}$ is not SLI since $(x_j A_i - x_i A_j \mid 0)$ but $-x_i \neq 0$ and $x_j \neq 0$. $\square$

For example, we have that the set $\{A_1 = (1; 2; 3; 4), A_2 = (1; 3; 5; 6), A_3 = (-3; -1; 3; 5), A_4 = (-1; 2; 3; 5)\}$ is not SLI because $(A_1 \mid \frac{5}{2})$ and $(A_3 \mid 1)$.

As one can observe from Definition 4 and the results above, the property of symmetry of fuzzy numbers is closely related to the notion of SLI. Proposition 5 establishes a connection between the symmetry property of fuzzy numbers in the space $\mathcal{S}(A_1, \ldots, A_n)$ generated by an SLI set and the linear dependence on the Euclidean space $\mathbb{R}^n$.

Proposition 5. *Let $\{A_1, \ldots, A_n\}$ be an SLI set of fuzzy numbers. Then, for all $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$, with $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$, such that $(p_1 A_1 + \ldots + p_n A_n \mid x)$ and $(q_1 A_1 + \ldots + q_n A_n \mid y)$ for some $x, y \in \mathbb{R}$ we have that $\{\mathbf{p}, \mathbf{q}\}$ is linearly dependent.*

Proof. Let $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$ such that $(p_1 A_1 + \ldots + p_n A_n \mid x)$ and $(q_1 A_1 + \ldots + q_n A_n \mid y)$ for some $x, y \in \mathbb{R}$. If $x = 0$, then $\mathbf{p} = \mathbf{0} := (0, \ldots, 0)$ because $\{A_1, \ldots, A_n\}$ is SLI and, thus, $\{\mathbf{p}, \mathbf{q}\}$ is linearly dependent. Similarly, if $y = 0$ then $\mathbf{q} = \mathbf{0}$ which implies that $\{\mathbf{p}, \mathbf{q}\}$ is linearly dependent.

Now, we consider the case where $x \neq 0$ and $y \neq 0$. If $\{\mathbf{p}, \mathbf{q}\}$ is linearly independent then we have that $\mathbf{u} := y\mathbf{p} - x\mathbf{q} \neq \mathbf{0}$. From Lemma 1 and Proposition 3, we have that

$$(B_1 := yp_1 A_1 + \ldots + yp_n A_n \mid yx),$$

$$(B_2 := -xq_1A_1 + \ldots + -xq_nA_n \mid -xy)\,,$$

and

$$L_{B_1}(\alpha) + U_{B_1}(\alpha) = 2yx \quad \text{and} \quad L_{B_2}(\alpha) + U_{B_2}(\alpha) = -2xy.$$

Considering $C = u_1A_1 + \ldots + u_nA_n$, $I_{\mathbf{v}} = \{i \in \{1, \ldots, n\} \,|\, v_i \geq 0\}$, and $J_{\mathbf{v}} = \{1, \ldots, n\} \setminus I_{\mathbf{v}}$, for all $\mathbf{v} \in \mathbb{R}^n$, we have that

$$\begin{aligned}
L_C(\alpha) + U_C(\alpha) =& \Big(\sum_{i\in I_{\mathbf{u}}} u_i L_{A_i}(\alpha) + \sum_{i\in J_{\mathbf{u}}} u_i U_{A_i}(\alpha)\Big) + \Big(\sum_{i\in I_{\mathbf{u}}} u_i U_{A_i}(\alpha) + \sum_{i\in J_{\mathbf{u}}} u_i L_{A_i}(\alpha)\Big)\\
=& \sum_{i=1}^{n} u_i L_{A_i}(\alpha) + \sum_{i=1}^{n} u_i U_{A_i}(\alpha)\\
=& \sum_{i=1}^{n} (yp_i - xq_i) L_{A_i}(\alpha) + \sum_{i=1}^{n} (yp_i - xq_i) U_{A_i}(\alpha)\\
=& \Big(\sum_{i=1}^{n} yp_i L_{A_i}(\alpha) + \sum_{i=1}^{n} yp_i U_{A_i}(\alpha)\Big) + \Big(\sum_{i=1}^{n} -xq_i L_{A_i}(\alpha) + \sum_{i=1}^{n} -xq_i U_{A_i}(\alpha)\Big)\\
=& \Big[\Big(\sum_{i\in I_{y\mathbf{p}}} yp_i L_{A_i}(\alpha) + \sum_{i\in J_{y\mathbf{p}}} yp_i U_{A_i}(\alpha)\Big) + \Big(\sum_{i\in I_{y\mathbf{p}}} yp_i U_{A_i}(\alpha) + \sum_{i\in J_{y\mathbf{p}}} yp_i L_{A_i}(\alpha)\Big)\Big] +\\
& \Big[\Big(\sum_{i\in I_{-x\mathbf{q}}} -xq_i L_{A_i}(\alpha) + \sum_{i\in J_{-x\mathbf{q}}} -xq_i U_{A_i}(\alpha)\Big) + \Big(\sum_{i\in I_{-x\mathbf{q}}} -xq_i U_{A_i}(\alpha) + \sum_{i\in J_{-x\mathbf{q}}} -xq_i L_{A_i}(\alpha)\Big)\Big]\\
=& \Big[L_{B_1}(\alpha) + U_{B_1}(\alpha)\Big] + \Big[L_{B_2}(\alpha) + U_{B_2}(\alpha)\Big]\\
=& \Big[2xy\Big] + \Big[-2xy\Big] = 0,
\end{aligned}$$

for all $\alpha \in [0, 1]$. An application of Lemma 1 reveals that C is symmetric with respect to 0. Since $\{A_1, \ldots, A_n\}$ is SLI, we must have $\mathbf{u} = \mathbf{0}$ which contradicts the assumption of linear independence of $\{\mathbf{p}, \mathbf{q}\}$. Therefore, $\{\mathbf{p}, \mathbf{q}\}$ are linearly dependent. □

Let $\{A_1, A_2, \ldots, A_n\} \subset \mathbb{R}_{\mathcal{F}}$ be an SLI set. A consequence of Proposition 5 is that if $B_1, B_2 \in S(A_1, \ldots, A_n)$ such that $(B_1 \mid y)$ and $(B_2 \mid y)$ then $B_1 = B_2$. In fact, by Proposition 5, if $B_1 = \Psi(\mathbf{x})$ then $B_2 = \Psi(\lambda\mathbf{x})$ for some $\lambda \in \mathbb{R}$ and $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. From Proposition 2, we have that

$$\begin{aligned}
B_2 &= \lambda x_1 A_1 + \ldots \lambda x_n A_n\\
&= \lambda(x_1A_1 + \ldots x_nA_n) = \lambda B_1.
\end{aligned}$$

An application of Proposition 3 and Lemma 1 ensures that $(B_2 \mid y) = (\lambda B_1 \mid \lambda y)$, which implies that $\lambda = 1$ since $y = \lambda y$. Thus, for example, there is no an SLI set $\{A_1, A_2, \ldots, A_n\}$ such that $B_1 = \chi_{\{1\}}$, $B_2 = (0; 1; 2) \in S(A_1, A_2, \ldots, A_n)$ because $(B_1 \mid 1)$ and $(B_2 \mid 1)$ but $B_1 \neq B_2$.

3.2. Characterization of SLI via function vector space

The next theorem characterizes the notion of SLI in terms of the well-known concept of linear independence of the vector space of the functions from $[0, 1]$ to $\mathbb{R}$.

Theorem 2. *A set $\{A_1, \ldots, A_n\} \subset \mathbb{R}_{\mathcal{F}}$ is SLI if, and only if, the set of functions $\{\delta_{A_1}, \ldots, \delta_{A_n}\}$ is linearly independent (LI).*

Proof. Let $B = q_1A_1 + \ldots + q_nA_n \in \mathcal{S}(A_1, \ldots, A_n)$ and let $I_{\mathbf{q}} = \{i \in \{1, \ldots, n\} \,|\, q_i \geq 0\}$ and $J_{\mathbf{q}} = \{1, \ldots, n\} \setminus I_{\mathbf{q}}$. The following equalities hold true:

$$\begin{aligned}\delta_B(\alpha) =& L_B(\alpha)+U_B(\alpha)\\ =&\Big(\sum_{i\in I_{\mathbf{q}}} q_i L_{A_i}(\alpha)+\sum_{i\in J_{\mathbf{q}}} q_i U_{A_i}(\alpha)\Big)+\Big(\sum_{i\in I_{\mathbf{q}}} q_i U_{A_i}(\alpha)+\sum_{i\in J_{\mathbf{q}}} q_i L_{A_i}(\alpha)\Big)\\ =&\Big(\sum_{i\in I_{\mathbf{q}}} q_i (L_{A_i}(\alpha)+U_{A_i}(\alpha))\Big)+\Big(\sum_{i\in J_{\mathbf{q}}} q_i (U_{A_i}(\alpha)+L_{A_i}(\alpha))\Big)\\ =&\Big(\sum_{i\in I_{\mathbf{q}}} q_i \delta_{A_i}(\alpha)\Big)+\Big(\sum_{i\in J_{\mathbf{q}}} q_i \delta_{A_i}(\alpha)\Big)\\ =&\sum_{i=1}^{n} q_i\delta_{A_i}(\alpha)\end{aligned} \tag{14}$$

Let us suppose that $\{A_1,\dots,A_n\}\subset\mathbb{R}_{\mathcal{F}}$ is SLI. By (14) and Lemma 1, if $0=\sum_{i=1}^{n} q_i\delta_{A_i}(\alpha)=\delta_B(\alpha)$ for all $\alpha\in[0,1]$, then $(B\mid 0)$. Since $\{A_1,\dots=A_n\}$ is SLI, by Definition 4, $(B\mid 0)\Rightarrow q_1=q_2=\dots,q_n=0$ and, therefore, $\{\delta_{A_1},\dots,\delta_{A_n}\}$ is linearly independent.

Now, suppose that $\{\delta_{A_1},\dots,\delta_{A_n}\}$ is linearly independent. Moreover, consider $B=q_1A_1+\dots+q_nA_n\in\mathcal{S}(A_1,\dots,A_n)$ such that $(B\mid 0)$. By (14) and Lemma 1, we have $0=\delta_B(\alpha)=\sum_{i=1}^{n} q_i\delta_{A_i}(\alpha)$ for all $\alpha\in[0,1]$. Thus, since $\{\delta_{A_1},\dots,\delta_{A_n}\}$ is linearly independent, it follows that $q_1=q_2=\dots=q_n=0$. Therefore, by Definition 4, $\{A_1,\dots,A_n\}$ is SLI. □

For example, let $A_1=(0;0;1)$ and $A_2=(-1;-1;0)\}$. The set $\{\delta_{A_1}(\alpha)=1-\alpha,\delta_{A_2}(\alpha)=-1-\alpha\}$ is linearly independent, then, by Theorem 2, $\{(0;0;1),(-1;-1;0)\}$ is SLI. Now, consider the fuzzy numbers $B_1=(1;1;2)$ and $B_2=(2;2;3)=B_1+1$. Note that the set $\{\delta_{B_1}(\alpha)=3-\alpha,\delta_{B_2}(\alpha)=5-\alpha\}$ is linearly independent, thus, by Theorem 2, $\{(1;1;2),(2;2;3)\}$ is SLI.

In view of Theorem 2, a natural question that arises is that if $\text{span}\{\delta_{A_1},\dots,\delta_{A_n}\}=\text{span}\{\delta_{B_1},\dots,\delta_{B_n}\}$ then $\mathcal{S}(A_1,\dots,A_n)=\mathcal{S}(B_1,\dots,B_n)$, where $\text{span}\{f_1,\dots,f_m\}$ is the set of all linear combination of the functions $f_1,\dots,f_m$. Unfortunately, the answer for this question is negative as we can see in the following simple example. The sets $\{A_1=(1;2;3)\}$ and $\{A_2=(2;3;4)\}$ are SLI and $S(A_1)\neq S(A_2)$ but we have that $\delta_{A_1}(\alpha)=4$ and $\delta_{A_2}(\alpha)=6$, for all $\alpha\in[0,1]$, and $\text{span}\{\delta_{A_1}\}=\text{span}\{\delta_{A_2}\}$.

Since the notion of SLI can be described in terms of the usual notion of linear independence from Theorem 2, one can use some well-known tests for linear independence of a set of functions in order to verify if a set of fuzzy numbers is SLI. The next two corollaries state these tests.

Corollary 5. *Let $A_1,\dots,A_n\in\mathbb{R}_{\mathcal{F}}$. If there exist $\alpha_1\neq\alpha_2\neq\dots\neq\alpha_n$, $\alpha_i\in[0,1]$, $i=1,\dots,n$, such that the matrix*

$$M=\begin{pmatrix}\delta_{A_1}(\alpha_1) & \delta_{A_2}(\alpha_1) & \dots & \delta_{A_n}(\alpha_1)\\ \delta_{A_1}(\alpha_2) & \delta_{A_2}(\alpha_2) & \dots & \delta_{A_n}(\alpha_2)\\ \vdots & \vdots & \ddots & \vdots\\ \delta_{A_1}(\alpha_n) & \delta_{A_2}(\alpha_n) & \dots & \delta_{A_n}(\alpha_n)\end{pmatrix}$$

is non-singular, then $\{A_1,\dots,A_n\}$ is SLI.

Proof. Let us consider $c_1,\dots,c_n\in\mathbb{R}$ such that

$$c_1\delta_{A_1}(\alpha)+\dots+c_n\delta_{A_n}(\alpha)=0,$$

for all $\alpha\in[0,1]$. Suppose that there exist $\alpha_1\neq\alpha_2\neq\dots\neq\alpha_n$, $\alpha_i\in[0,1]$, $i=1,\dots,n$, such that the system

$$\begin{cases} c_1\delta_{A_1}(\alpha_1)+c_2\delta_{A_2}(\alpha_1)+\dots+c_n\delta_{A_n}(\alpha_1) &= 0\\ c_1\delta_{A_1}(\alpha_2)+c_2\delta_{A_2}(\alpha_2)+\dots+c_n\delta_{A_n}(\alpha_2) &= 0\\ \vdots & \\ c_1\delta_{A_1}(\alpha_n)+c_2\delta_{A_2}(\alpha_n)+\dots+c_n\delta_{A_n}(\alpha_n) &= 0\end{cases} \tag{15}$$

admits only the trivial solution, that is, $c_1=\dots=c_n=0$. Therefore, $\{\delta_{A_1},\dots,\delta_{A_n}\}$ is linearly independent and, by Theorem 2, we have $\{A_1,\dots,A_n\}$ is SLI. □

Corollary 6. *Let $A_1, \ldots, A_n$ be fuzzy numbers such that each function δ_{A_i}, for $i = 1, \ldots, n$, is $(n-1)$ times continuously differentiable. If there exists $\overline{\alpha} \in [0,1]$ such that the (Wronskian) matrix*

$$W = \begin{pmatrix} \delta_{A_1}(\overline{\alpha}) & \delta_{A_2}(\overline{\alpha}) & \ldots & \delta_{A_n}(\overline{\alpha}) \\ \delta'_{A_1}(\overline{\alpha}) & \delta'_{A_2}(\overline{\alpha}) & \ldots & \delta'_{A_n}(\overline{\alpha}) \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{(n-1)}_{A_1}(\overline{\alpha}) & \delta^{(n-1)}_{A_2}(\overline{\alpha}) & \ldots & \delta^{(n-1)}_{A_n}(\overline{\alpha}) \end{pmatrix} \tag{16}$$

is non-singular, then $\{A_1, \ldots, A_n\}$ is SLI.

Proof. The proof is similar to the proof of Corollary 5 and uses the fact that the functions $\delta_{A_1}, \ldots, \delta_{A_n}$ are LI since the Wronskian matrix W is non-singular [26]. □

In order to illustrate the practical value of Corollary 6, we consider the following example. Let $A_1 = (1; 3; 4)$ and $A_2 = (-2; 0; 5)$. The matrix

$$W = \begin{pmatrix} \delta_{A_1}(\overline{\alpha}) & \delta_{A_2}(\overline{\alpha}) \\ \delta'_{A_1}(\overline{\alpha}) & \delta'_{A_2}(\overline{\alpha}) \end{pmatrix} = \begin{pmatrix} 5+\overline{\alpha} & 3-3\overline{\alpha} \\ 1 & -3 \end{pmatrix}$$

is non-singular. From Theorem 6, we conclude $\{A_1 = (1; 3; 4), A_2 = (-2; 0; 5)\}$ is SLI.

3.3. Density of finite SLI sets in $\mathcal{FS}(\mathbb{R}_{\mathcal{F}})$

In this subsection, we establish that the set of SLI finite sets of $\mathbb{R}_{\mathcal{F}}$ is dense in the metric space $(\mathcal{FS}(\mathbb{R}_{\mathcal{F}}), \mathcal{D})$.

Theorem 3. *Let $\{B_1, \ldots, B_n\} \subset \mathbb{R}_{\mathcal{F}}$. For all $\epsilon > 0$, there exists an SLI set $\{A_1, \ldots, A_n\} \subset \mathbb{R}_{\mathcal{F}}$ such that*

$$\mathcal{D}(\{A_1, \ldots, A_n\}, \{B_1, \ldots, B_n\}) < \epsilon \tag{17}$$

with $d_\infty(A_i, B_i) < \epsilon$ for all $i = 1, \ldots, n$.

Proof. Suppose, without loss of generality, that $0 < \epsilon < 1$. Let $\varepsilon = (0; 0; \epsilon)$ and let $\hat{f}_s(\varepsilon)$ be the Zadeh's extension of f_s at ε (see Definition 3), where $f_s(t) = t^s$ for all $t \in \mathbb{R}$ and $s \in \{1, 2, \ldots\}$. We can prove this theorem using induction as follows.

For $n = 1$. If $\{B_1\}$ is SLI, then we define $A_1 = B_1$ and $\mathcal{D}(\{A_1\}, \{B_1\}) = d_\infty(A_1, B_1) = 0 < \epsilon$. Now, if $\{B_1\}$ is not SLI then $\delta_{B_1}(\alpha) = 0$ for all $\alpha \in [0,1]$ from Theorem 2. Defining $A_1 = B_1 + \varepsilon$, by Lemma 2, we obtain that $\mathcal{D}(\{A_1\}, \{B_1\}) = d_\infty(A_1, B_1) < \epsilon$ since $\varepsilon = \hat{f}_1(\varepsilon)$.

Suppose that the claim of Theorem holds for $n-1$, $n > 2$. Let $\{B_1, \ldots, B_n\} \subset \mathbb{R}_{\mathcal{F}}$. If $\{B_1, \ldots, B_n\}$ is SLI, then there is nothing to prove. Now, suppose that $\{B_1, \ldots, B_n\}$ is not SLI. By hypothesis, there exists a SLI set $\{A_1, \ldots, A_{n-1}\}$ such that $\mathcal{D}(\{A_1, \ldots, A_{n-1}\}, \{B_1, \ldots, B_{n-1}\}) < \epsilon$. On the one hand, if $\{A_1, \ldots, A_{n-1}, B_n\}$ is SLI, then we define $A_n = B_n$ and we obtain that $d_\infty(A_n, B_n) = 0 < \epsilon$. On the other hand, if $\{A_1, \ldots, A_{n-1}, B_n\}$ is not SLI, then we define $A_n = B_n + \hat{f}_r(\varepsilon))$ where $r = \inf I$ and

$$I = \{s \in \mathbb{N} \mid \{A_1, \ldots, A_{n-1}, B_n + \hat{f}_s(\varepsilon)\} \text{ is SLI and } s \geq 1\}.$$

The fuzzy number A_n is well defined since $\emptyset \neq I \subset \{1, 2, \ldots\}$. In fact, if $I = \emptyset$, then for every $s = 1, 2, \ldots$, we have that $\{A_1, \ldots, A_{n-1}, B_n + \hat{f}_s(\varepsilon)\}$ is not SLI or, equivalently, $\{\delta_{A_1}, \ldots, \delta_{A_{n-1}}, \delta_{(B_n + \hat{f}_s(\varepsilon))}\}$ is linearly dependent. Since $\{\delta_{A_1}, \ldots, \delta_{A_{n-1}}\}$ is linearly independent, we have that $\delta_{(B_n + \hat{f}_s(\varepsilon))} \in \text{span}\{\delta_{A_1}, \ldots, \delta_{A_{n-1}}\}$ for all $s \in \{1, 2, \ldots\}$. From Equation (7), for all $\alpha \in [0,1]$, we obtain

$$\begin{aligned} \delta_{(B_n + \hat{f}_s(\varepsilon))}(\alpha) &= L_{(B_n + \hat{f}_s(\varepsilon))}(\alpha) + U_{(B_n + \hat{f}_s(\varepsilon))}(\alpha) \\ &= L_{B_n}(\alpha) + L_{\hat{f}_s(\varepsilon)}(\alpha) + U_{B_n}(\alpha) + U_{\hat{f}_s(\varepsilon)}(\alpha) \\ &= (L_{B_n}(\alpha) + U_{B_n}(\alpha)) + (L_{\hat{f}_s(\varepsilon)}(\alpha) + U_{\hat{f}_s(\varepsilon)}(\alpha)) \\ &= \delta_{B_n}(\alpha) + (1-\alpha)^s \epsilon^s. \end{aligned}$$

Thus, for all $s = 1, 2, \ldots$, we have that $g_s := \delta_{(B_n+\hat{f}_s(\varepsilon))} - \delta_{B_n} \in \text{span}\{\delta_{A_1}, \ldots, \delta_{A_i}\}$ since $\delta_{B_n}, \delta_{(B_n+\hat{f}_s(\varepsilon))} \in \text{span}\{\delta_{A_1}, \ldots, \delta_{A_{n-1}}\}$ and $\text{span}\{\delta_{A_1}, \ldots, \delta_{A_m}\}$ is a vector space of dimension $n-1$. However, the fact that $g_s(\alpha) = (1-\alpha)^s\epsilon^s$, $\forall \in \alpha \in [0,1]$, is a polynomial of order s and $\{g_s | s \in \mathbb{N}\}$ is linearly independent contradicts the fact of the dimension of $\text{span}\{\delta_{A_1}, \ldots, \delta_{A_{n-1}}\}$ is finite. Therefore, we must have that $I \neq \emptyset$ and $r = \inf I$ exists and $r \in I$. This implies that $\{A_1, \ldots, A_{n-1}, B_n + \hat{f}_r(\varepsilon)\} = \{A_1, \ldots, A_n\}$ is SLI. Moreover, from Lemma 2, we have that $\mathcal{D}(\{A_n\}, \{B_n\}) = d_\infty(A_n, B_n) < \epsilon$.

Finally, from Equation (2), we obtain that

$$\begin{aligned}
\mathcal{D}(\{A_1,\ldots,A_n\},\{B_1,\ldots,B_n\}) &= \left(\bigvee_{i=1}^{n}\bigwedge_{j=1}^{n} d_\infty(A_i,B_j)\right) \vee \left(\bigvee_{j=1}^{n}\bigwedge_{i=1}^{n} d_\infty(A_i,B_j)\right)\\
&= \left(\bigvee_{i=1}^{n-1}\bigwedge_{j=1}^{n} d_\infty(A_i,B_j)\right) \vee \left(\bigwedge_{j=1}^{n} d_\infty(A_n,B_j)\right) \vee\\
&\quad \left(\bigvee_{j=1}^{n-1}\bigwedge_{i=1}^{n} d_\infty(A_i,B_j)\right) \vee \left(\bigwedge_{i=1}^{n} d_\infty(A_i,B_n)\right)\\
&\leq \left(\bigvee_{i=1}^{n-1}\bigwedge_{j=1}^{n-1} d_\infty(A_i,B_j)\right) \vee d_\infty(A_n,B_n) \vee\\
&\quad \left(\bigvee_{j=1}^{n-1}\bigwedge_{i=1}^{n-1} d_\infty(A_i,B_j)\right) \vee d_\infty(A_n,B_n)\\
&\leq \mathcal{D}(\{A_1,\ldots,A_{n-1}\},\{B_1,\ldots,B_{n-1}\}) \vee d_\infty(A_n,B_n)\\
&< \epsilon. \quad \square
\end{aligned}$$

Remark 1. Theorem 3 ensures that a fuzzy number-valued function $f : [a,b] \to \mathbb{R}_\mathcal{F}$ given by

$$f(t) = q_1(t)B_1 + \ldots + q_n(t)B_n, \forall t \in [a,b] \subset \mathbb{R},$$

where $B_i \in \mathbb{R}_\mathcal{F}$ and $q_i : [a,b] \to \mathbb{R}$ is a continuous function for $i = 1, \ldots, n$, can be approximated by a fuzzy number-valued function

$$\tilde{f}(t) = q_1(t)A_1 + \ldots + q_n(t)A_n, \forall t \in [a,b] \subset \mathbb{R}$$

with respect to the uniform metric d and $\{A_1, \ldots, A_n\}$ being an SLI set. In fact, for every $\epsilon > 0$ there exists an SLI set $\{A_1, \ldots, A_n\}$ such that

$$\mathcal{D}(\{A_1,\ldots,A_n\},\{B_1,\ldots,B_n\}) < \frac{\epsilon}{nL} \quad \text{and} \quad d_\infty(A_i,B_i) < \frac{\epsilon}{nL}, \ \forall i = 1,\ldots,n,$$

where

$$\max_{i=1,\ldots,n} \sup_{t\in[a,b]} |q_i(t)| \leq L.$$

Since $d_\infty(A + B, C + D) \leq d_\infty(A, C) + d_\infty(B, D)$, for all $A, B, C, D \in \mathbb{R}_\mathcal{F}$ [18], we have that

$$\begin{aligned}
d_\infty(f(t),\tilde{f}(t)) &= d_\infty\left(q_1(t)A_1 + \ldots + q_n(t)A_n, q_1(t)B_1 + \ldots + q_n(t)B_n\right)\\
&\leq \sum_{i=1}^{n} d_\infty\left(q_i(t)A_i, q_i(t)B_i\right)\\
&\leq \sum_{i=1}^{n} |q_i(t)| d_\infty\left(A_i, B_i\right)
\end{aligned}$$

$$\leq L \sum_{i=1}^{n} d_\infty (A_i, B_i)$$
$$\leq nL \left(\bigvee_{i=1}^{n} d_\infty (A_i, B_i) \right) < \epsilon,$$

for all $t \in [a, b]$. This implies that

$$d(f, \tilde{f}) = \sup_{t \in [a,b]} d_\infty(f(t), \tilde{f}(t)) < \epsilon.$$

Theorem 3 reveals that not only the set of SLI finite sets is dense in the metric space $(\mathcal{FS}(\mathbb{R}_\mathcal{F}), \mathcal{D})$ but also provides a procedure to obtain an SLI subset sufficiently close of any arbitrary finite set of $\mathbb{R}_\mathcal{F}$.

From the proof of Theorem 3, we can also conclude that the set of non-symmetric fuzzy numbers is dense in $\mathbb{R}_\mathcal{F}$. Intuitively, any small perturbation of a symmetric fuzzy number may lead us to a non-symmetric fuzzy number.

4. Generating SLI sets of fuzzy numbers

In this section, we shall deal with the following question: how to produce SLI sets? We provide two theorems that address this question whose proofs are based on Theorem 2.

Theorem 4 presents some conditions for a finite set composed by powering hedges of a non-symmetric trapezoidal fuzzy number be strongly linearly independent. It is worth noting that the class of trapezoidal fuzzy numbers contains the class of triangular fuzzy numbers.

Theorem 4. *Let $A = (a; b; c; d)$ be a non-symmetric trapezoidal fuzzy number. The set $\{\chi_{\{1\}}, A, A^2, A^3, \ldots, A^n\}$ is SLI for any $n \geq 1$.*

Proof. The case where $n = 1$ is proved in Theorem 3.6 of [13]. Thus, it remains to prove the case where $n \geq 2$.

First, note that the trapezoidal fuzzy set A can not represent an interval or a real number since A is non-symmetric. Thus, we have that $a \neq b$ or $c \neq d$. If $a = b$, then we have $L_A(\alpha) = a$, $U_A(\alpha) = d - \alpha(d - c)$, and $\delta_A(\alpha) = (d + a) - \alpha(d - c)$ for all $\alpha \in [0, 1]$. Analogously, if $c = d$ we have $\delta_A(\alpha) = (d + a) + \alpha(b - a)$ for all $\alpha \in [0, 1]$. Now, if $a \neq b$ and $c \neq d$, then we have $L_A(\alpha) = a + \alpha(b - a)$, $U_A(\alpha) = d - \alpha(d - c)$, and $\delta_A(\alpha) = (d + a) + \alpha[(b - a) - (d - c)]$, for all $\alpha \in [0, 1]$. Lemma 1 implies $[(b - a) - (d - c)] \neq 0$ because $\delta_A(\alpha)$ can not be a constant. Hence, we can conclude that δ_A is a function of the form

$$\delta_A(\alpha) = w\alpha + u \tag{18}$$

for all $\alpha \in [0, 1]$, where $w, u \in \mathbb{R}$ with $w \neq 0$.

For $i = 2, \ldots, n$, we have $\delta_{A_i}(\alpha) = \delta_A(\alpha^{\frac{1}{i}})$, since the following equalities hold true:

$$L_{A_i}(\alpha) = L_A(\alpha^{\frac{1}{i}}) \text{ and } U_{A_i}(\alpha) = U_A(\alpha^{\frac{1}{i}}).$$

The last comment and Equation (18) imply that $\delta_{A^i}(\alpha^{\frac{1}{i}}) = w\alpha^{\frac{1}{i}} + u$ for all $\alpha \in [0, 1]$ and $i = 2, \ldots, n$.

Suppose there exist $q_0, q_1, \ldots, q_n \in \mathbb{R}$ such that

$$\begin{aligned} 0 &= q_0\delta_{\chi_{\{1\}}}(\alpha) + q_1\delta_A(\alpha) + q_2\delta_{A^2}(\alpha) + \ldots + q_n\delta_{A^n}(\alpha) \\ &= 2q_0 + q_1(w\alpha + u) + q_2(w\alpha^{\frac{1}{2}} + u) + \ldots + q_n(w\alpha^{\frac{1}{n}} + u) \\ &= [2q_0 + u(q_1 + \ldots + q_n)] + q_1 w\alpha + q_2 w\alpha^{\frac{1}{2}} + \ldots + q_n w\alpha^{\frac{1}{n}} \end{aligned}$$

for all $\alpha \in [0, 1]$. Setting $\gamma_0 = 2q_0 + u(q_1 + q_2 + \ldots + q_n)$ and $\gamma_i = q_i w$, for $i = 1, \ldots, n$, we have that

$$0 = \gamma_0 + \gamma_1\alpha + \gamma_2\alpha^{\frac{1}{2}} + \ldots + \gamma_n\alpha^{\frac{1}{n}}, \ \forall \alpha \in [0, 1],$$

and $(q_0, q_1, \ldots, q_n)$ is the unique solution of the linear system

$$\begin{pmatrix} 2 & u & u & \dots & u \\ 0 & w & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & w \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ \vdots \\ q_n \end{pmatrix} = \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix}. \tag{19}$$

Let $\beta_0, \beta_1, \dots, \beta_{n!} \in (0, 1)$ such that $\beta_i \neq \beta_j$ for all $i \neq j$. Since $\beta_i^k \in [0, 1]$ for all $k \in \mathbb{N}$ and $i = 0, \dots, n!$, we obtain the following system:

$$\begin{cases} 0 = \gamma_0 + \gamma_1 \beta_0^{n!} + \gamma_2 \beta_0^{\frac{n!}{2}} + \dots + \gamma_n \beta_0^{\frac{n!}{n}} \\ 0 = \gamma_0 + \gamma_1 \beta_1^{n!} + \gamma_2 \beta_1^{\frac{n!}{2}} + \dots + \gamma_n \beta_1^{\frac{n!}{n}} \\ \vdots \; \vdots \; \vdots \\ 0 = \gamma_0 + \gamma_1 \beta_{n!}^{n!} + \gamma_2 \beta_{n!}^{\frac{n!}{2}} + \dots + \gamma_n \beta_{n!}^{\frac{n!}{n}} \end{cases}.$$

Thus, $\beta_0, \beta_1, \dots, \beta_{n!}$ are roots of the polynomial $p(\beta) = \gamma_0 + \gamma_1 \beta^{n!} + \gamma_2 \beta^{\frac{n!}{2}} + \dots + \gamma_n \beta^{\frac{n!}{n}}$. Using the fundamental theorem of algebra, we have that the unique polynomial of order less or equal $n!$ with $n! + 1$ distinct roots is the polynomial given by the constant zero. Therefore, we have $\gamma_0 = \gamma_1 = \dots = \gamma_n = 0$, which implies that $q_0 = q_1 = \dots = q_n = 0$ from Equation (19). Hence, we conclude that $\{\chi_{\{1\}}, A, A^2, A^3, \dots, A^n\}$ is SLI. □

For example, for $A = (0; 1; 3)$ we have that $\{\chi_{\{1\}}, A, A^2\}$ is SLI from Theorem 4 since A is non-symmetric. It is worth noting that A is a trapezoidal fuzzy number but, by Equation (3), the fuzzy numbers $A^2, A^3, \dots, A_n$ are not trapezoidal fuzzy numbers.

The next theorem states that we can also obtain an SLI set of fuzzy numbers from the Zadeh's extension of polynomial functions at non-symmetric trapezoidal fuzzy numbers.

Theorem 5. *Let $A = (a; b; c; d)$ be a non-symmetric trapezoidal fuzzy number, where $a \leq b \leq c \leq d$. If $f_i : \mathbb{R} \to \mathbb{R}$ is given by $f_i(t) = t^i$ for all $t \in \mathbb{R}$ and $i = 0, 1, 2, 3 \dots$, then the set $\{\hat{f}_0(A) = \chi_{\{1\}}, \hat{f}_1(A), \hat{f}_2(A), \hat{f}_3(A), \dots, \hat{f}_n(A)\}$ is SLI for any $n \geq 1$.*

Proof. The α-cuts of A are given by $[A]^\alpha = [a + \alpha(b - a), d - \alpha(d - c)]$, for all $\alpha \in [0, 1]$. Note that $(b - a) \neq (d - c)$ since A is non-symmetric. Moreover, note that $\delta_{\hat{f}_0(A)}(\alpha) = 1$, for all α.

If $b > 0$ then there exists $\beta \in [0, 1)$ such that $a + \alpha(b - a) > 0$. By Proposition 1, we have that

$$[\hat{f}_i(A)]^\alpha = [(a + \alpha(b - a))^i, (d + \alpha(c - d))^i],$$

for all $\alpha \in [\beta, 1]$ and $i \geq 1$. Thus, for $i \geq 1$, we have

$$\begin{aligned} \delta_{\hat{f}_i(A)}(\alpha) &= (a + \alpha(b - a))^i + (d + \alpha(c - d))^i \\ &= \alpha^i[(b - a)^i + (c - d)^i] + p_{i-1}(\alpha), \end{aligned}$$

for every $\alpha \in [\beta, 1]$, where p_{i-1} is a polynomial of order less or equal $i - 1$. Therefore, every $\delta_{\hat{f}_i(A)}$ is a polynomial of order i on $[\beta, 1]$ for $i = 0, \dots, n$, which implies that $\{\delta_{\hat{f}_0(A)}, \dots, \delta_{\hat{f}_n(A)}\}$ is LI. This implies that $\{\hat{f}_0(A), \hat{f}_1(A), \hat{f}_2(A), \hat{f}_3(A), \dots, \hat{f}_n(A)\}$ is SLI from Theorem 2.

Now, if $c < 0$ then there exists $\beta \in [0, 1)$ such that $d + \alpha(c - d) < 0$. By Proposition 1, we have that

$$[\hat{f}_i(A)]^\alpha = \begin{cases} \left[(d + \alpha(c - d))^i, (a + \alpha(b - a))^i\right] & \text{if } i \text{ is even} \\ \left[(a + \alpha(b - a))^i, (d + \alpha(c - d))^i\right] & \text{if } i \text{ is odd} \end{cases},$$

for all $\alpha \in [\beta, 1]$ and $i \geq 1$. Thus, for $i \geq 1$ and $\alpha \in [\beta, 1]$, we obtain

$$\begin{aligned} \delta_{\hat{f}_i(A)}(\alpha) &= (a + \alpha(b - a))^i + (d + \alpha(c - d))^i \\ &= \alpha^i[(b - a)^i + (c - d)^i] + p_{i-1}(\alpha), \end{aligned}$$

where p_{i-1} is a polynomial of order less or equal $i-1$. This last observation implies that every $\delta_{\hat{f}_i(A)}$ is a polynomial of order i on $[\beta, 1]$, for $i = 0, \ldots, n$ and, consequently, the set $\{\delta_{\hat{f}_0(A)}, \ldots, \delta_{\hat{f}_n(A)}\}$ is LI. Hence, by Theorem 2, $\{\hat{f}_0(A), \hat{f}_1(A), \hat{f}_2(A), \hat{f}_3(A), \ldots, \hat{f}_n(A)\}$ is SLI.

Suppose that $b \le 0 \le c$. If $-b = c$ then, for all $\alpha \in [0, 1]$, we have

$$-a - \alpha(b-a) > d - \alpha(d-c) \text{ if } (b-a) > (d-c)$$

or

$$-a - \alpha(b-a) < d - \alpha(d-c) \text{ if } (b-a) < (d-c).$$

Thus, from Proposition 1, we have

$$[\hat{f}_i(A)]^\alpha = \begin{cases} \left[(a+\alpha(b-a))^i, (d+\alpha(c-d))^i\right], & \text{if } i \text{ is odd} \\ \left[0, (-a+\alpha(a-b))^i\right], & \text{if } i \text{ is even and } (b-a) > (d-c) \\ \left[0, (d+\alpha(c-d))^i\right], & \text{if } i \text{ is even and } (b-a) < (d-c) \end{cases}.$$

Using the last equation, we obtain

$$\delta_{\hat{f}_i(A)}(\alpha) = \begin{cases} \alpha^i[(b-a)^i + (c-d)^i] + p_{i-1}(\alpha) \text{ if } i \text{ is odd} \\ \alpha^i(a-b)^i + p_{i-1}(\alpha) \text{ if } i \text{ is even and } (b-a) > (d-c) \\ \alpha^i(c-d)^i + p_{i-1}(\alpha) \text{ if } i \text{ is even and } (b-a) < (d-c) \end{cases},$$

where p_{i-1} is a polynomial of order less or equal $i-1$. Thus, for all cases above we have that every $\delta_{\hat{f}_i(A)}$ is a polynomial of order i on $[0, 1]$, for $i = 0, \ldots, n$ and, consequently, the set $\{\delta_{\hat{f}_0(A)}, \ldots, \delta_{\hat{f}_n(A)}\}$ is LI. Hence, by Theorem 2, $\{\hat{f}_0(A), \hat{f}_1(A), \hat{f}_2(A), \hat{f}_3(A), \ldots, \hat{f}_n(A)\}$ is SLI.

Finally, suppose that $b \le 0 \le c$ and $-b \ne c$. Let us assume without loss of generality that $-b < c$. Thus, in this case, there exists $\beta \in [0, 1)$ such that $-a - \alpha(b-a) < d - \alpha(d-c)$, for all $\alpha \in [\beta, 1]$. From Proposition 1, we have

$$[\hat{f}_i(A)]^\alpha = \begin{cases} \left[(a+\alpha(b-a))^i, (d+\alpha(c-d))^i\right] \text{ if } i \text{ is odd} \\ \left[0, (d+\alpha(c-d))^i\right] \text{ if } i \text{ is even} \end{cases},$$

for all $\alpha \in [\beta, 1]$. This implies that

$$\delta_{\hat{f}_i(A)}(\alpha) = \begin{cases} \alpha^i[(b-a)^i + (c-d)^i] + p_{i-1}(\alpha) \text{ if } i \text{ is odd} \\ \alpha^i(c-d)^i + p_{i-1}(\alpha) \text{ if } i \text{ is even} \end{cases},$$

for all $\alpha \in [\beta, 1]$ and $i \ge 1$. Again, for all cases above we have that every $\delta_{\hat{f}_i(A)}$ is a polynomial of order i on $[\beta, 1]$, for $i = 0, \ldots, n$. Therefore, the set $\{\delta_{\hat{f}_0(A)}, \ldots, \delta_{\hat{f}_n(A)}\}$ is LI, which implies that $\{\hat{f}_0(A), \hat{f}_1(A), \hat{f}_2(A), \hat{f}_3(A), \ldots, \hat{f}_n(A)\}$ is SLI according Theorem 2. □

The connection between the concepts of SLI and linear independence of functions revealed by Theorem 2 motivates us to extend the notion of SLI for any (finite or infinite) sets of fuzzy numbers as follows.

Definition 5. Let Y be a non-empty subset of $\mathbb{R}_\mathcal{F}$. We say that Y is strongly linearly independent (SLI) if every finite non-empty subset of Y is strongly linearly independent.

The next corollary follows immediately from Theorem 2.

Corollary 7. *A non-empty subset Y of $\mathbb{R}_\mathcal{F}$ is SLI if, and only if, the set of functions $\{\delta_A \mid A \in Y\}$ is linearly independent.*

As immediate consequences of Corollaries 2 and 7 and Theorems 4 and 5, we obtain the next two corollaries.

Corollary 8. *Let $I \subset \{1, 2, \ldots\}$ be a non-empty set and let A be a non-symmetric trapezoidal fuzzy number. The set $\{A^i : i \in I\}$ is SLI.*

Corollary 9. *Let $I \subset \{0, 1, 2, \ldots\}$ be a non-empty set, A be a non-symmetric trapezoidal fuzzy number and $f_i(t) = t^i$, for all $t \in \mathbb{R}$ and $i = 0, 1, \ldots$. The set $\{\hat{f}_i(A) \mid i \in I\}$ is SLI.*

Regarding the comments at the end of Subsection 2.2, for a non-symmetric trapezoidal fuzzy number A, we have that

$$(\mathcal{S}(\chi_{\{1\}}, A, A^2, \ldots, A^n), \mathbb{R}, +_\psi, \cdot_\psi) \tag{20}$$

(as well as $(\mathcal{S}(\chi_{\{1\}}, A, \hat{f}_2(A), \ldots, \hat{f}_n(A)), \mathbb{R}, +_\psi, \cdot_\psi)$ with $f_i(t) = t^i$) is a vector space, where $+_\psi$ and $\cdot_\psi$ is given by

a) for all $B, C \in \mathcal{S}(\chi_{\{1\}}, A, A^2, \ldots, A^n)$ as

$$B +_\psi C = \psi\left(\psi^{-1}(B) + \psi^{-1}(C)\right); \tag{21}$$

b) for all $B \in \mathcal{S}(\chi_{\{1\}}, A, A^2, \ldots, A^n)$ and $\lambda \in \mathbb{R}$ as

$$\lambda \cdot_\psi B = \psi\left(\lambda\psi^{-1}(B)\right). \tag{22}$$

Moreover, ψ represents an isomorphism from $(\mathbb{R}^{n+1}, \mathbb{R}, +, \cdot, ||\cdot||)$ to the Banach space $(\mathcal{S}(\chi_{\{1\}}, A, A^2, \ldots, A^n), \mathbb{R}, +_\psi, \cdot_\psi, ||\cdot||_\psi)$.

5. Conclusion

In this manuscript, we present the concept of strong linear independence for fuzzy numbers. This concept plays a similar role to the one of linear independence of vectors, which is the basis for the development of many mathematical tools for calculus, analysis, *etc.*, in Banach spaces. For a given strongly linearly independent set $\{A_1, \ldots, A_n\}$, one can define an isomorphism between $\mathbb{R}^n$ and the subspace $\mathcal{S}(A_1, \ldots, A_n)$ generated by this set. From this isomorphism, we can carry all algebraic and topological frameworks of $\mathbb{R}^n$ to the subclasses of fuzzy numbers $\mathcal{S}(A_1, \ldots, A_n)$. This allows us to define Banach spaces over sets of fuzzy numbers and, then, to use the well-established calculus theory for Banach spaces. To this end, we intend to extend the results presented in [13,14] and establish a calculus theory for functions with fuzzy coefficients of the form

$$f(t) = g_1(t)A_1 + g_2(t)A_2 + \ldots + g_n(t)A_n$$

where $\{A_1, \ldots, A_n\}$ is SLI and $g_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, n$. It is worth noting that a differential and integral calculus theory for functions from $\mathbb{R}$ to $\mathcal{S}(1, A)$ was introduced in [13,14].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Banach, vector, and metric spaces

In this subsection we will review some of the well-known concepts found in the large literature about functional analysis.

Definition 6. A metric space is a pair (V, d) where V is a non-empty set and d is a metric on V, that is, a function $d : V \times V \to \mathbb{R}$ such that for every $x, y, z \in V$ the following properties hold:

1. $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) \leq d(x, z) + d(z, y)$.

Definition 7. Let (X, d_X) and (Y, d_Y) be two metric spaces. A function $f : X \to Y$ is said to be continuous at $x_0 \in X$ if, for all $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \epsilon$ for every $x \in X$ satisfying $d_X(x, x_0) < \delta$. If f is continuous at all $x \in X$, then f is called a continuous function.

Definition 8. A sequence (x_n) in a metric space (V, d) converges to $x \in V$ if, for every $\epsilon > 0$, there exists $\delta > 0$ such that $d(x_n, x) < \epsilon$ for all $n > \delta$. In this case, we write $x_n \to x$ or $\lim_{n\to\infty} d(x_n, x) = 0$.

Definition 9. A sequence (x_n) in a metric space (V, d) is called a Cauchy sequence if, for every $\epsilon > 0$, there exists $\delta > 0$ such that $d(x_n, x_m) < \epsilon$ for all $n, m > \delta$.

Definition 10. A metric space (V, d) is said to be complete if every Cauchy sequence converges in V.

Definition 11. A vector space over a field K is a non-empty set V where it is defined an operation $+ : V \times V \to V$, called addition, and an operation $\cdot : K \times V \to V$, called scalar multiplication, that satisfy the following properties:

1. $x + (y + z) = (x + y) + z,\ \forall x, y, z \in V$;
2. $x + y = y + x,\ \forall x, y \in V$;
3. $\exists 0 \in V$ such that $x + 0 = x,\ \forall x \in V$;
4. for every $x \in V$ there exists $-x \in V$ such that $x + (-x) = 0$,
5. $a(bx) = (ab)x$ for all $a, b \in K$ and $x \in V$;
6. $(a + b)x = ax + bx$ for all $a, b \in K$ and $x \in V$;
7. $a(x + y) = ax + ay$ for all $a \in K$ and $x, y \in V$;
8. $1x = x$ for all $x \in V$ (where 1 denotes the multiplication identity of K).

Definition 12. Let X and Y be two vector spaces over a field K. A function $f : X \to Y$ is called linear if it satisfies the property:

$$f(\lambda x + y) = \lambda f(x) + f(y),\ \forall x, y \in X \text{ and } \forall \lambda \in K. \tag{A.1}$$

We use the symbol $L(X, Y)$ to denote the set of all continuous linear functions from X to Y.

Two vector spaces X and Y over the same field K are said to be isomorphic if there is a map $\psi : X \to Y$ called isomorphism that is a linear bijection. In addition, one can be shown that ψ^{-1} is a linear bijection as well.

Definition 13. A finite subset $\{v_i \mid i \in I\}$, $I \subseteq \mathbb{N}$, of a vector space V is called linearly independent (LI) if $\sum_{i\in I} \lambda_i v_i = 0$ implies that $\lambda_i = 0$ for all $i \in I$. A subset Y of V is said to be linearly independent if every non empty and finite subset of Y is LI.

Definition 14. A normed vector space is a pair $(V, \|\cdot\|)$ where V is a vector space over a field $\mathbb{R}$ or $\mathbb{C}$ and $\|\cdot\|$ is a norm on V, that is, a function $\|\cdot\|: V \to \mathbb{R}$ that satisfies the following properties for every $x, y, z \in V$ and $\lambda \in K$:

1. $\| x \| \geq 0$ and $\| x \| = 0 \Leftrightarrow x = 0$;
2. $\| \lambda x \| = |\lambda| \, \| x \|$;
3. $\| x + y \| \leq \| x \| + \| y \|$.

Given a normed vector space $(V, \|\cdot\|)$, the metric d on V induced by the norm $||.||$ is the function given by $d(x, y) = ||x - y||$ for all $x, y \in V$. Thus, every normed vector space is also a metric space by considering the metric induced by the corresponding norm.

Definition 15. A complete normed space or simply a Banach space is a normed vector space $(V, \|\cdot\|)$ that is a complete metric space with respect to the metric induced by the norm $\|\cdot\|$.

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