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System of fuzzy relation equations with inf-$\rightarrow$ composition: Complete set of solutions[☆]

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Abstract

The problem of solvability of a system of equations with inf-$\rightarrow$ composition is considered on finite universes. Equations are expressed using operations of a BL-algebra. We study complete set of solutions of the respective system in the particular case (one equation) and in the general case. In both cases, various conditions of solvability are found and proved. We characterize all maximal solutions and prove that under certain conditions each solution of the system (an equation) is less than or equal to a respective maximal one. As a result, we are able to characterize the complete set of solutions. Examples of single equations and their systems are considered and complete sets of solutions are found for each.

Keywords: System of fuzzy relation equations; inf-$\rightarrow$ composition; Maximal solution; Solution set

1. Introduction

Traditionally, the behavior of a fuzzy system is characterized by a system of fuzzy IF–THEN rules which can be considered as a partially given fuzzy function. The following two problems arise in this correlation: how this function can be represented and how it can be used in calculations. It is a common practice to represent a system of fuzzy IF–THEN rules (and thus, the respective fuzzy function) by a fuzzy relation so that the mentioned calculation is performed using the Compositional Rule of Inference [22]. If we require that for each input which coincides with one of the antecedents of the underlying IF–THEN rules, the computed output should coincide with the respective consequence, then we come to the problem of solvability of a system of fuzzy relation equations.

The described approach and technique are very natural and therefore, the problem of solvability has been widely investigated in the literature on fuzzy sets and systems. The first formulation of the problem and fundamental results were obtained by Sanchez [19] in connection with medical diagnosis. He considered fuzzy relation equations with various types of composition: max–min, min–max, min-α and found necessary and sufficient conditions for solvability. In each case, the condition is given in terms of existence of a certain extremal solution. Since then, many authors have contributed to various aspects of the problem of solvability, concentrating on e.g. new types of composition, complete sets of solutions, algorithms for obtaining minimal/maximal solutions, etc. (See e.g. [6,8,11,14–16,20].) A majority of

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publications, however, deals with solvability of fuzzy relation equations with sup-$*$ composition where $*$ is an operation of some residuated lattice. Little attention is paid to solvability of fuzzy relation equations with inf-$\longrightarrow$ composition which has been introduced by Bandler and Kohout in [1]. Similarly, solvability of a system of fuzzy relation equations is also among less popular problems. In our investigation we focus on characterization of a set of solutions of a system of fuzzy relation equations with inf-$\longrightarrow$ composition.

Let us recall some publications which consider particular aspects of the problem specified above, i.e. either solvability of a system of fuzzy relation equations or solvability of equations with inf-$\longrightarrow$ composition:

- Necessary and sufficient condition of having unique solution of a system of fuzzy relation equations with max–min composition is proved [3].
- Fuzzy relation equations and their systems are studied from an order-theoretical point of view; various types of compositions are investigated; exact solutions and complete sets of solutions are described; however, proofs are not given. For a single equation with inf-$\longrightarrow$ composition, the complete set of solutions (called *crown*) is presented under the following conditions: the universe of discourse is a complete distributive lattice where second partial mappings of $\rightarrow$ are homomorphisms, the right-hand side element in the equation is meet-irreducible or meet-decomposable. For a system of fuzzy relation equations with inf-$\longrightarrow$ composition, only the least solution has been presented [5].
- A comprehensive study of fuzzy relational equations and their systems is presented; various types of compositions are investigated; necessary and sufficient conditions for solvability of equations and solvability of systems are proved. However, full characterization of the solution set of a system of fuzzy relational equations (including characterization of minimal solutions) is given only for the case of max–min composition [6].
- A necessary and sufficient condition (in terms of existence of the greatest solution) of solvability of a system of fuzzy equations with sup-$*$ composition is given [8,9].
- For a fuzzy relation equation with inf-$\longrightarrow$ composition, the least solution and the necessary and sufficient condition of solvability have been presented [12].
- Systems of fuzzy relation equations with sup-$*$ composition ($*$ is a continuous t-norm) are considered and minimal solutions are described. Everything is done under the assumption that each solution is greater than or equal to a respective minimal solution [13].
- Necessary and sufficient conditions for solvability of a system of fuzzy relation equations in terms of input and output data are proven; the best approximate solutions to a system of fuzzy relation equations under different criteria of optimality are presented [17,18].

From this short overview it is clear that the chosen problem (characterization of the solution set of a system of fuzzy relation equations with inf-$\longrightarrow$ composition) has not been seriously investigated. If we take into account the results from [5,12] (the least solution of one equation and the criterion for its solvability) then theoretically, the problem can be reduced to the characterization of all maximal solutions. However, no results in that direction have been published for the abovementioned system of equations.

In this paper we found conditions which make this reduction possible, i.e. conditions which assure that each solution is greater than the least solution and less than or equal to a respective maximal one. Moreover, we found conditions which guarantee that each solution of a system is a minimum of a set of solutions of separate equations.

Let us stress that it is a very nontrivial task to find the abovementioned conditions which make characterization of the solution set possible. It turned out that it is necessary to restrict the operation $\rightarrow$ inside composition. The restricted algebra is a linearly ordered BL-algebra which fulfils the so-called conditional cancellation law (see Section 2.4 for justification of the restriction).

Throughout the paper we fix a general BL-algebra as an algebra of operations. This algebra is more specific than a residuated lattice, and we employed this specificity on many places.

Let us declare that in the case where a BL-algebra is the underlying algebra of operations we obtain the following principal results for a system of fuzzy relation equations with the inf-$\longrightarrow$ composition on a finite universe:

- necessary and sufficient conditions of solvability of a single equation (Section 3);
- characterization of the complete set of solutions of a single equation via characterization of the complete set of its maximal solutions (Section 3.2);
- constructive way of getting all solutions of a consistent system of equations (Section 4);

- characterization of all maximal solutions of the system (Section 4.1);
- the proof of the fact that each solution of the system is less than or equal to a respective maximal solution (Section 4.1).

Moreover, a brief introduction to BL-algebras is given in Section 2, and many numerical examples of single equations and systems are considered at the ends of Sections 3 and 4.

2. Preliminaries

2.1. BL-algebra

BL-algebra has been introduced in [10] as the algebra of operations which correspond to connectives of basic fuzzy logic (BL). In the same sense as BL generalizes Boolean logic, we can say that BL-algebra generalizes Boolean algebra. This appears in the extension of the set of Boolean operations by two operations which constitute the so-called *adjoint couple*. The following definition summarizes definitions originally introduced in [10].

Definition 1. A *BL-algebra* is an algebra

$$\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, 0, 1\rangle$$

with four binary operations and two constants such that

(i) $(L, \vee, \wedge, 0, 1)$ is a lattice with 0 and 1,
(ii) $(L, *, 1)$ is a commutative semigroup with unit 1,
(iii) $*$ and $\rightarrow$ (implication) establish an adjoint couple, i.e.
$z \leqslant (x \rightarrow y)$ iff $x*z \leqslant y$ for all $x, y, z \in L$,
(iv) for all $x, y \in L$
$x*(x \rightarrow y) = x \wedge y$,
$(x \rightarrow y) \vee (y \rightarrow x) = 1$.

Another operation of $\mathcal{L}$ is the unary $\neg$ (negation) which is defined by

$$\neg x = x \rightarrow 0.$$

Besides Boolean algebra, the following algebras are other examples of BL-algebra.

Example 1 (*Gödel algebra*).

$$\mathcal{L}_G = \langle [0, 1], \vee, \wedge, \rightarrow_G, 0, 1\rangle,$$

where the multiplication $* = \wedge$ and

$$x \rightarrow_G y = \begin{cases} 1 & \text{if } x \leqslant y, \\ y & \text{if } y < x, \end{cases} \qquad \neg_G x = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x > 0. \end{cases}$$

Example 2 (*Product (Goguen) algebra*).

$$\mathcal{L}_P = \langle [0, 1], \vee, \wedge, \odot, \rightarrow_P, 0, 1\rangle,$$

where the multiplication $\odot = \cdot$ is the ordinary product of reals and

$$x \rightarrow_P y = \begin{cases} 1 & \text{if } x \leqslant y, \\ \dfrac{y}{x} & \text{if } y < x, \end{cases} \qquad \neg_P x = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x > 0. \end{cases}$$

The following two laws specify Product algebras among BL ones:

$$\text{if } z \neq 0 \text{ then } x * z = y * z \Rightarrow x = y \quad \text{(cancellation)},$$

$$x \wedge \neg x = 0 \quad \text{(contradiction)}.$$

Example 3 (*Standard Łukasiewicz algebra*).

$$\mathcal{L}_Ł = \langle [0,1], \vee, \wedge, \otimes, \rightarrow_Ł, 0, 1\rangle,$$

where

$$x \otimes y = 0 \vee (x + y - 1), \quad x \rightarrow_Ł y = 1 \wedge (1 - x + y), \quad \neg_Ł x = 1 - x.$$

In the sequel, we will often use the following properties of BL-algebra:

$$x \leqslant y \ \text{ iff } \ (x \rightarrow y) = 1, \tag{1}$$

$$x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow y).$$

2.2. *Fuzzy sets, fuzzy relations and their compositions*

More than 40 years have passed since Zadeh published the first paper about fuzzy sets [21]. At present, the theory of fuzzy sets demonstrates its maturity and penetrates almost all mathematical subjects. According to the original concept, a fuzzy set is identified with its membership function which maps a universe of discourse into the unit interval. Goguen [7] extended the original construction by using an arbitrary bounded distributive lattice instead of the unit interval.

Let us fix a BL-algebra $\mathcal{L}$ with a support L and notice that by (1), L is partially ordered. Assume that fuzzy sets are identified with their membership functions, whose range is L endowed with operations from $\mathcal{L}$. We also fix two finite universes $U = \{u_1, \ldots, u_n\}$, $n \geqslant 1$, and $V = \{v_1, \ldots, v_m\}$, $m \geqslant 1$, so that a fuzzy set P on U is identified with its membership function $P : U \longrightarrow L$ and a fuzzy set R on V is identified with its membership function $R : V \longrightarrow L$. Two fuzzy sets on the same universe are equal (one is less than the other one) if the equality (respective inequality) holds pointwise for their membership functions.

A fuzzy set on the Cartesian product $U \times V$ is called a (binary) *fuzzy relation*. Any fuzzy relation $A \in L^{U\times V}$ determines a mapping h_A from L^U to L^V via the so-called sup-$*$ composition:

$$h_A(P)(v_j) = (A \circ P)(v_j) = \bigvee_{i=1}^{n} (A(u_i, v_j) * P(u_i)), \quad j = 1, \ldots, m.$$

The mapping $h_A : L^U \longrightarrow L^V$ has the *residual* [2,4] $g_A : L^V \longrightarrow L^U$ being determined via the inf-$\rightarrow$ composition [1]:

$$g_A(R)(u_i) = (A \rhd R)(u_i) = \bigwedge_{j=1}^{m} (A(u_i, v_j) \rightarrow R(v_j)), \quad i = 1, \ldots, n.$$

Two systems of fuzzy relation equations are connected with h_A or g_A:

$$A \circ X = B \quad (\text{or } A \rhd X = B), \tag{2}$$

where A, B are known and X is unknown. In the literature on fuzzy sets and systems, Eq. (2) are referred to as *a system of fuzzy relation equations with* sup-$*$ *composition* and *a* system of fuzzy relation equations with inf-$\rightarrow$ *composition*, respectively.

The aim of this paper is to consider systems with inf-$\rightarrow$ composition on finite universes and under different assumptions on $\mathcal{L}$. We will show that each solution of a system of this type is less than or equal to a respective maximal solution. Moreover, we will characterize complete sets of solutions of those systems by characterizing their least and maximal solutions.

2.3. *System of fuzzy relation equations with* inf-$\rightarrow$ *composition in a matrix representation*

We will take advantage of finiteness of universes U and V. Hence, in the sequel we will identify a fuzzy set P on U (a fuzzy set R on V) with the vector of values of its membership function $P : U \longrightarrow L$ ($R : V \longrightarrow L$) so that P will be identified with the vector $(p_1, \ldots, p_n)$ where $p_i = P(u_i)$, $i = 1, \ldots, n$, and R will be identified with the vector

$(r_1, \dots, r_m)$ where $r_j = R(v_j)$, $j = 1, \dots, m$. On the basis of this agreement, the system of fuzzy relation equations $A \rhd X = B$ (cf. (2)) with inf-$\rightarrow$ composition can be represented by

$$\begin{aligned}(a_{11} \rightarrow x_1) \wedge \cdots \wedge (a_{1m} \rightarrow x_m) &= b_1,\\ &\vdots\\ (a_{n1} \rightarrow x_1) \wedge \cdots \wedge (a_{nm} \rightarrow x_m) &= b_n,\end{aligned}$$

or

$$\bigwedge_{j=1}^{m} (a_{ij} \rightarrow x_j) = b_i, \quad i = 1, \dots, n, \tag{3}$$

for short. In (3), a_{ij} is the value of the membership function of the fuzzy relation A considered at point (u_i, v_j). The vector $\mathbf{b} = (b_1, \dots, b_n)$ is the representation of the known fuzzy set B and the vector $\mathbf{x} = (x_1, \dots, x_m)$ is the representation of the unknown fuzzy set X. In matrix notation, (3) can be rewritten as follows:

$$A \rhd \mathbf{x} = \mathbf{b}.$$

We will further refer to A and $\mathbf{b}$ as a matrix of coefficients and a right-hand side of (3), respectively. We say that the vector $\mathbf{x}^0 = (x_1^0, \dots, x_m^0)$ is a solution of system (3) if, after substitution of $\mathbf{x}^0$ for $\mathbf{x}$, each equality in (3) becomes true. The solution set of (3) will be denoted by S.

2.4. Solution set as a subsemilattice

Our goal is to find the solution set S of (3). According to the above accepted agreement, solutions of (3) are identified with the respective vectors from L^V (or L^m where m is the cardinality of V). L^m is a partially ordered set where the order is inherited from L, i.e. for all $(x_1, \dots, x_m), (y_1, \dots, y_m) \in L^m$ it is defined by

$$(x_1, \dots, x_m) \leqslant (y_1, \dots, y_m) \text{ iff } x_1 \leqslant y_1, \dots, x_m \leqslant y_m.$$

The solution set S is a subset of the partially ordered L^m, and for characterization of S we will use minimal, maximal, least and greatest elements of S in their standard meaning. Moreover, we will omit trivial restrictions on components of vectors from S, i.e. we will write $a_j \leqslant x_j$ $(x_j \leqslant a_j)$ and not $a_j \leqslant x_j \leqslant 1$ $(0 \leqslant x_j \leqslant a_j$, respectively).

It has been mentioned in Preliminaries that systems of equations similar to (3) have been considered in the literature (see, e.g. [12,6]). From these sources we took the following result.

Proposition 1. *System* (3) *is solvable if and only if the vector* $\check{\mathbf{x}} = (\check{x}_1, \dots, \check{x}_m)$, *where*

$$\check{x}_j = \bigvee_{i=1}^{n} (a_{ij} * b_i), \quad j = 1, \dots, m \tag{4}$$

is its solution. Then it is its least solution.

Therefore, $\check{\mathbf{x}}$ is the least element of S with respect to the order in L^m. Moreover, if $\mathbf{x}^1$ and $\mathbf{x}^2$ are solutions of (3) then $\mathbf{x}^1 \wedge \mathbf{x}^2$ is a solution too, where the operation $\wedge$ is taken componentwise. Consequently, the set S is a $\wedge$-subsemilattice with the least element. Two questions arise: Does S have maximal elements and if yes, then whether any solution is less than or equal to a respective maximal solution?

Both questions have been positively answered in [5] under the following restrictions: (3) contains one equation, $\mathcal{L}$ is a complete distributive lattice where second partial mappings of $\rightarrow$ are homomorphisms, the right-hand side element is meet-irreducible or meet-decomposable.

On the other hand, negative answers to the above questions can be obtained from the following example: $\mathcal{L}$ is a residuated lattice on [0, 1] where $*$ is the operation of nilpotent minimum and $\rightarrow$ is defined as follows:

$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leqslant b, \\ \neg a \vee b & \text{if } b < a, \end{cases}$$

where $\neg a = 1 - a$. The following equation

$$(0.5 \to x_1) \wedge (0.5 \to x_2) = 0.5$$

is solvable (vector (0, 0) is the least solution), but has no maximal solutions. Indeed, the solution set is equal to $\{(x_1, x_2) \mid 0 \leqslant x_1 < 0.5, 0 \leqslant x_2 \leqslant 1 \text{ or } 0 \leqslant x_1 \leqslant 1, 0 \leqslant x_2 < 0.5\}$, but neither (0.5, 1) nor (1, 0.5) is a solution.

In the case of a system of fuzzy relation equations with inf-$\to$ composition, no answers to the above questions are known. Aiming at finding positive answers to both questions we restrict possible algebras to linearly ordered BL-algebras (note that the residuated lattice on [0, 1] with nilpotent minimum is not a BL-algebra).

We continue as follows: in Section 3, we characterize the solution set of one equation (special case of system (3)) and in Section 4, we characterize the solution set of the system.

3. The case of a single equation

A particular case of (3) comes when the system has one equation. This is our starting point and the subject of this section. The following is a precise formulation of what will be studied.

We consider the following equation with respect to an unknown vector $\mathbf{x} = (x_1, \ldots, x_m) \in L^m$:

$$(a_1 \to x_1) \wedge \cdots \wedge (a_m \to x_m) = b$$

or

$$\bigwedge_{j=1}^{m} (a_j \to x_j) = b \tag{5}$$

for short. The vector $\mathbf{a} = (a_1, \ldots, a_m) \in L^m$ and $b \in L$ are supposed to be known. Unlike [5], we do not require that b is meet-irreducible or meet-decomposable. Our purpose is to find all solutions of (5), i.e. all vectors $\mathbf{x}^0 = (x_1^0, \ldots, x_m^0) \in L^m$ that make (5) true after substitution for $\mathbf{x}$.

Eq. (5) is a particular case of system (3). Therefore, by Proposition 1, it is solvable if and only if the vector $\check{\mathbf{x}} = (\check{x}_1, \ldots, \check{x}_m)$ is its least solution where

$$\check{x}_j = a_j * b, \quad j = 1, \ldots, m. \tag{6}$$

Throughout this section, $\check{\mathbf{x}}$ will always denote the vector whose components are given by (6).

As in the case of the system of equations (3), the set S of solutions of (5) is a $\wedge$-subsemilattice with the bottom element (the vector $\check{\mathbf{x}}$). Thus, to obtain S we will find (when it is possible) its maximal elements (i.e. maximal solutions of (5)) and prove that each solution is less than or equal to a respective maximal one. We will proceed as follows:

- investigate conditions on the given data $(a_1, \ldots, a_m)$, b which guarantee or exclude solvability of (5);
- explicitly characterize conditions formulated in terms of the given data $(a_1, \ldots, a_m)$, b that guarantee existence of maximal solutions;
- under the conditions above, characterize constructively all maximal solutions and prove that each solution is less than or equal to a respective maximal one.

3.1. Necessary and sufficient conditions for solvability of a single equation

We may easily obtain the following necessary condition for solvability of (5).

Lemma 1. *If Eq.* (5) *is solvable then the condition*

$$b \geqslant \bigwedge_{j=1}^{m} \neg a_j \tag{7}$$

necessarily holds.

Proof. By the assumption, (5) is solvable, and let $\mathbf{x} = (x_1, \ldots, x_m)$ be its solution. Then we obtain

$$b = \bigwedge_{j=1}^{m} (a_j \to x_j) \geqslant \bigwedge_{j=1}^{m} (a_j \to 0) = \bigwedge_{j=1}^{m} \neg a_j. \qquad \square$$

As can be seen, solvability of (5) leads to a certain relation between b and $\bigwedge_{j=1}^{m} \neg a_j$. We prove the following.

Theorem 1.

(i) *Eq.* (5) *is unsolvable if* $b < \bigwedge_{j=1}^{m} \neg a_j$;
(ii) *Eq.* (5) *is solvable if* $b = \bigwedge_{j=1}^{m} \neg a_j$.

Proof. (i) Assume that $b < \bigwedge_{j=1}^{m} \neg a_j$, but Eq. (5) is solvable. Then according to Proposition 1, $\check{\mathbf{x}} = (\check{x}_1, \ldots, \check{x}_m)$ given by (6), is its least solution. Then for each $j = 1, \ldots, m$, $\check{x}_j = a_j * b \leqslant a_j * \neg a_j = 0$ or $\check{x}_j = 0$. Therefore,

$$b = \bigwedge_{j=1}^{m} (a_j \to \check{x}_j) = \bigwedge_{j=1}^{m} (a_j \to 0) = \bigwedge_{j=1}^{m} \neg a_j > b.$$

This is an obvious contradiction and thus, Eq. (5) is unsolvable.

(ii) Suppose that $b = \bigwedge_{j=1}^{m} \neg a_j$ holds true. Then we easily verify that for each $k = 1, \ldots, m$, $\check{x}_k = a_k * b = 0$. Indeed, $\check{x}_k = a_k * b = a_k * \bigwedge_{j=1}^{m} \neg a_j \leqslant a_k * \neg a_k = 0$. It turns out that $\check{\mathbf{x}} = (0, \ldots, 0)$ is a solution of (5). Indeed,

$$\bigwedge_{j=1}^{m} (a_j \to \check{x}_j) = \bigwedge_{j=1}^{m} (a_j \to 0) = \bigwedge_{j=1}^{m} \neg a_j = b.$$

Therefore, by Proposition 1, (5) is solvable. $\square$

However, we cannot combine Lemma 1 and Theorem 1 and obtain the criterion for solvability of (5). The reason is that the set L is partially ordered and so, the following does not hold in general: if $b \geqslant \bigwedge_{j=1}^{m} \neg a_j$ then (5) is solvable. The assertion in this formulation holds true under special conditions on the lattice ordering given in Theorem 4.

3.2. Solution set of a single equation

By Proposition 1, Eq. (5) is solvable if and only if the vector $\check{\mathbf{x}}$ given by (6) is its solution and moreover, it is its least solution. Therefore, in order to obtain the solution set of (5) it would be sufficient to characterize all maximal solutions and all infinite chains of solutions. However, for a general BL-algebra, this problem seems to be very complicated and requires a ramified study of a relationship between coefficients of the equation and its right-hand side. Therefore, we decided to restrict our further investigation to the case where (5) is considered over a *linearly ordered* BL-algebra. We will show that in this case the set of maximal solutions is finite and each solution is less than or equal to a respective maximal one.

Let us fix some linearly ordered BL-algebra $\mathcal{L}$ and assume that the necessary condition of solvability (Lemma 1) is fulfilled, i.e. $b \geqslant \bigwedge_{j=1}^{m} \neg a_j$. Therefore, we will distinguish the following three cases depending on the value of b:

1. $b = 1$,
2. $b < 1$ and $b = \bigwedge_{j=1}^{m} \neg a_j$,
3. $b < 1$ and $b > \bigwedge_{j=1}^{m} \neg a_j$.

3.2.1. Solution set if $b = 1$

It is easy to see that in this case Eq. (5) is always solvable. The least solution $\check{\mathbf{x}}$ and the greatest solution $\hat{\mathbf{x}}$ are as follows:

$$\check{\mathbf{x}} = (a_1, \ldots, a_m), \quad \hat{\mathbf{x}} = (1, \ldots, 1).$$

All vectors $\mathbf{x} = (x_1, \ldots, x_m) \in L^m$ fulfilling $a_j \leqslant x_j$, $j = 1, \ldots, m$, are solutions.

3.2.2. Solution set if $b < 1$ and $b = \bigwedge_{j=1}^m \neg a_j$

Due to linearity of the fixed BL-algebra, the condition $b = \bigwedge_{j=1}^m \neg a_j$ implies that there exists a nonempty subset $A' = \{a_{k_1}, \dots, a_{k_l}\}$ where $1 \leqslant l \leqslant m$, such that $b = \neg a_{k_s}$ for each $a_{k_s} \in A'$. Then by Theorem 1, Eq. (5) is solvable. The following theorem characterizes maximal solutions of (5).

Theorem 2. *Let BL-algebra $\mathcal{L}$ be linearly ordered and coefficients of Eq.* (5) *fulfil the following conditions*:

- $b < 1$,
- $b = \bigwedge_{j=1}^m \neg a_j$.

Let a nonempty subset $A' = \{a_{k_1}, \dots, a_{k_l}\}$, $1 \leqslant l \leqslant m$, be such that $b = \neg a_{k_s}$ for each $a_{k_s} \in A'$. Then the finite set $\hat{\mathbf{X}} = \{\hat{\mathbf{x}}^1, \dots, \hat{\mathbf{x}}^l\}$ is a set of maximal solutions of (5), *where*

$$\hat{\mathbf{x}}^s = (1, \dots, \underbrace{0}_{k_s}, \dots, 1), \quad s = 1, \dots, l.$$

Moreover, for each solution $\mathbf{x}$ *of* (5) *there exists a maximal solution from* $\hat{\mathbf{X}}$ *greater than or equal to* $\mathbf{x}$.

Proof. Let s be one of $1, \dots, l$ and $\hat{\mathbf{x}}^s = (\hat{x}_1^s, \dots, \hat{x}_m^s) = (1, \dots, \underbrace{0}_{k_s}, \dots, 1)$.

At first, we will prove that $\hat{\mathbf{x}}^s$ is a solution of (5). This follows from:

$$\bigwedge_{j=1}^m (a_j \to \hat{x}_j^s) = a_{k_s} \to 0 = \neg a_{k_s} = b.$$

Second, we will prove that $\hat{\mathbf{x}}^s$ is a maximal solution of (5). For this we assume that there exists a solution: $\mathbf{x} = (x_1, \dots, x_m)$ of (5) such that $\mathbf{x} \geqslant \hat{\mathbf{x}}^s$. This implies that for all $j = 1, \dots, m$ and $j \neq k_s$ we have $x_j = 1$. Therefore,

$$b = \bigwedge_{j=1}^m (a_j \to x_j) = a_{k_s} \to x_{k_s}.$$

By the assumptions of the theorem, $b = \neg a_{k_s}$ and L is a linearly ordered BL-algebra. Therefore, $a_{k_s} \wedge x_{k_s} = a_{k_s} * (a_{k_s} \to x_{k_s}) = a_{k_s} * \neg a_{k_s} = 0$ and by the inequality $a_{k_s} > 0$ (the latter follows from $b < 1$), $x_{k_s} = 0$. This implies $\mathbf{x} = \hat{\mathbf{x}}^s$. Thus, $\hat{\mathbf{x}}^s$ is a maximal solution.

Finally, we will prove that for each solution $\mathbf{x}$ of (5) there exists a maximal solution from $\hat{\mathbf{X}}$ greater than or equal to $\mathbf{x}$. Let $\mathbf{x} = (x_1, \dots, x_m)$ be a solution. Then

$$b = \bigwedge_{j=1}^m (a_j \to x_j).$$

By the assumption that if $a_{k_s} \in A'$ then $b = \bigwedge_{j=1}^m \neg a_j = \neg a_{k_s}$, we have: if $a_j \notin A'$ then $\neg a_j > b$ and therefore, $a_j \to x_j \geqslant a_j \to 0 > b$. It follows that:

$$b = \bigwedge_{j=1}^m (a_j \to x_j) = \bigwedge_{s=1}^l (a_{k_s} \to x_{k_s}) = a_{k_p} \to x_{k_p}$$

for some p such that $1 \leqslant p \leqslant l$. Moreover, $b = \neg a_{k_p}$ and hence, $a_{k_p} \to x_{k_p} = \neg a_{k_p}$. In the same way as above, equality $a_{k_p} \to x_{k_p} = \neg a_{k_p}$ implies that $x_{k_p} = 0$ and therefore, $\mathbf{x} \leqslant \hat{\mathbf{x}}^p$. □

The corollary below characterizes the complete set of solutions of (5) under the assumptions which are accepted in this subsection and described in Theorem 2.

Corollary 1. *Suppose that the assumptions of Theorem 2 hold true and* $\mathbf{x} = (x_1, \ldots, x_m) \in L^m$ *is a solution of* (5). *Then there exists one* $k \in \{k_1, \ldots, k_l\}$, *such that for any* $j = 1, \ldots, m$, *it is true that*

(i) *if* $j = k$ *then* $x_j = 0$,
(ii) *if* $j \neq k$ *then* $x_j \geqslant a_j * b$.

Proof. Let $\mathbf{x}$ be a solution of (5). The least solution of (5) is $\check{\mathbf{x}} = (\check{x}_1, \ldots, \check{x}_m)$ with $\check{x}_j = a_j * b,\ j = 1, \ldots, m$. Therefore, for each $j = 1, \ldots, m$, it is true that $x_j \geqslant a_j * b$.

By the assumptions of Theorem 2, for $j = k_1, \ldots, k_l$ we easily have

$$\check{x}_j = a_j * b = 0.$$

By the conclusion of Theorem 2, for the given $\mathbf{x}$ there exists a maximal solution from $\hat{\mathbf{X}}$ greater than or equal to $\mathbf{x}$. Let this maximal solution have the following representation:

$$\hat{\mathbf{x}}^s = (1, \ldots, \underbrace{0}_{k_s}, \ldots, 1),$$

where $k_s \in \{k_1, \ldots, k_l\}$. Then $k = k_s$ proves the corollary. □

Example 4.

(a) Let $L = [0, 1]$, $m = 3$ and $\mathcal{L}$ be the standard Łukasiewicz algebra (see Example 3). The following equation

$$(1.0 \to_{Ł} x_1) \wedge (0.1 \to_{Ł} x_2) \wedge (0.3 \to_{Ł} x_3) = 1.0$$

is solvable according to the assumptions accepted in Section 3.2.1. The least solution $\check{\mathbf{x}} = (1, 0.1, 0.3)$, the largest solution $\hat{\mathbf{x}} = (1.0, 1.0, 1.0)$ and therefore, each vector $\mathbf{x} = \{x_1, x_2, x_3\} \in [0, 1]^3$ with components: $x_1 = 1, x_2 \geqslant 0.1, x_3 \geqslant 0.3$ is a solution.

(b) In the same space $[0, 1]^3$, equipped with the standard Łukasiewicz algebra, the following equation

$$(0.3 \to_{Ł} x_1) \wedge (0.5 \to_{Ł} x_2) \wedge (0.2 \to_{Ł} x_3) = 0.5$$

is considered. In this case, conditions of Theorem 2 are fulfilled. Therefore, the least solution $\check{\mathbf{x}} = (0, 0, 0)$, the largest solution $\hat{\mathbf{x}} = (1, 0, 1)$ and each solution $\mathbf{x} = \{x_1, x_2, x_3\} \in [0, 1]^3$ satisfies: $x_1 \geqslant 0, x_2 = 0, x_3 \geqslant 0$.

(c) In the same space $[0, 1]^3$, equipped with the Product algebra (see Example 2), the following equation

$$(0.3 \to_{\mathrm{P}} x_1) \wedge (0.5 \to_{\mathrm{P}} x_2) \wedge (0.2 \to_{\mathrm{P}} x_3) = 0$$

is considered. In this case, conditions of Theorem 2 are fulfilled. The least solution $\check{\mathbf{x}} = (0, 0, 0)$, the maximal solutions are: $\hat{\mathbf{x}}^1 = (0, 1, 1)$, $\hat{\mathbf{x}}^2 = (1, 0, 1)$, $\hat{\mathbf{x}}^3 = (1, 1, 0)$.

Each solution $\mathbf{x} = \{x_1, x_2, x_3\} \in [0, 1]^3$ satisfies: either $x_1 = 0, x_2 \geqslant 0, x_3 \geqslant 0$ or $x_1 \geqslant 0, x_2 = 0, x_3 \geqslant 0$, or $x_1 \geqslant 0, x_2 \geqslant 0, x_3 = 0$.

3.2.3. Solution set if $b < 1$ and $b > \bigwedge_{j=1}^m \neg a_j$

Due to linearity of the fixed BL-algebra, the conditions $b < 1$ and $b > \bigwedge_{j=1}^m \neg a_j$ imply that there exist a nonempty subset $A' = \{a_{k_1}, \ldots, a_{k_l}\}$, $1 \leqslant l \leqslant m$, such that $b > \neg a_{k_s}$ for each $a_{k_s} \in A'$ and a (possibly empty) subset $A'' = \{a_{k_{l+1}}, \ldots, a_{k_{l+r}}\}$, $0 \leqslant r \leqslant m - l$, such that $b = \neg a_{k_s}$ for each $a_{k_s} \in A''$.

Let, moreover, the chosen linearly ordered BL-algebra $\mathcal{L}$ fulfil the *conditional cancellation law*, i.e.

$$0 < a * x \leqslant a * y \quad \Rightarrow \quad x \leqslant y.$$

Typical examples are the Product algebra $\mathcal{L}_{\mathrm{P}}$ (see Example 2) and the standard Łukasiewicz algebra $\mathcal{L}_{Ł}$ (see Example 3).

We will characterize all maximal solutions of (5) under the assumptions given above.

Theorem 3. *Let the BL-algebra $\mathcal{L}$ be linearly ordered and fulfil the conditional cancellation law. Let the coefficients of Eq.* (5) *fulfill the assumptions of Section* 3.2.3 *so that there exists a nonempty subset* $A' = \{a_{k_1}, \ldots, a_{k_l}\}$, $1 \leqslant l \leqslant m$, *such that* $b > \neg a_{k_s}$ *for each* $a_{k_s} \in A'$, *and a* (*possibly empty*) *subset* $A'' = \{a_{k_{l+1}}, \ldots, a_{k_{l+r}}\}$, $0 \leqslant r \leqslant m - l$, *such that* $b = \neg a_{k_s}$ *for each* $a_{k_s} \in A''$. *Then the finite set* $\hat{\mathbf{X}} = \{\hat{\mathbf{x}}^1, \ldots, \hat{\mathbf{x}}^l, \hat{\mathbf{x}}^{l+1}, \ldots, \hat{\mathbf{x}}^{l+r}\}$ *is a set of maximal solutions of* (5), *where*

$$\hat{\mathbf{x}}^s = (1, \ldots, \underbrace{a_{k_s} * b}_{k_s}, \ldots, 1), \quad s = 1, \ldots, l$$

and

$$\hat{\mathbf{x}}^s = (1, \ldots, \underbrace{0}_{k_s}, \ldots, 1), \quad s = l+1, \ldots, l+r.$$

Moreover, *for each solution* $\mathbf{x}$ *of* (5) *there exists a maximal solution from* $\hat{\mathbf{X}}$ *greater than or equal to* $\mathbf{x}$.

Proof. Let s be one of $1, \ldots, l+r$ and $\hat{\mathbf{x}}^s = (\hat{x}_1^s, \ldots, \hat{x}_m^s) = (1, \ldots, \underbrace{a_{k_s} * b}_{k_s}, \ldots, 1)$.

At first, we will prove that $\hat{\mathbf{x}}^s$ is a solution of (5). For $s = 1, \ldots, l$ this follows from

$$\bigwedge_{j=1}^{m} (a_j \to \hat{x}_j^s) = a_{k_s} \to a_{k_s} * b = \sup\{c \mid a_{k_s} * c \leqslant a_{k_s} * b\} = \sup\{c \mid c \leqslant b\} = b,$$

where we used the conditional cancellation law and the fact that $a_{k_s} * b > 0$.

For the rest $s = l+1, \ldots, l+r$, $a_{k_s} * b = 0$ so that $\hat{\mathbf{x}}^s = (\hat{x}_1^s, \ldots, \hat{x}_m^s) = (1, \ldots, \underbrace{0}_{k_s}, \ldots, 1)$, and then

$$\bigwedge_{j=1}^{m} (a_j \to \hat{x}_j^s) = a_{k_s} \to 0 = \neg a_{k_s} = b.$$

Furthermore, we will prove that $\hat{\mathbf{x}}^s$ is a maximal solution. For this we assume that there exists a solution $\mathbf{x} = (x_1, \ldots, x_m)$ of (5) such that $\mathbf{x} \geqslant \hat{\mathbf{x}}^s$ or

$$\mathbf{x} = (1, \ldots, \underbrace{x_{k_s}}_{k_s}, \ldots, 1) \quad \text{and} \quad x_{k_s} \geqslant a_{k_s} * b.$$

We have

$$\bigwedge_{j=1}^{m} (a_j \to x_j) = a_{k_s} \to x_{k_s} = b$$

which implies $a_{k_s} > x_{k_s}$ (because L is linear and $b < 1$). Thus,

$$a_{k_s} * b = a_{k_s} * (a_{k_s} \to x_{k_s}) = a_{k_s} \wedge x_{k_s} = x_{k_s}$$

which implies that $\mathbf{x} = \hat{\mathbf{x}}^s$.

Finally, we will prove that for each solution $\mathbf{x}$ of (5) there exists a maximal solution from $\hat{\mathbf{X}}$ greater than or equal to $\mathbf{x}$. Let $\mathbf{x} = (x_1, \ldots, x_m)$ be a solution. Then

$$b = \bigwedge_{j=1}^{m} (a_j \to x_j).$$

By the assumption that if $a_{k_s} \in A'$ then $b > \neg a_{k_s}$ and if $a_{k_s} \in A''$ then $b = \neg a_{k_s}$, we have: if $a_j \notin (A' \cup A'')$ then $\neg a_j > b$ and therefore, $a_j \to x_j \geqslant a_j \to 0 > b$. It follows that

$$b = \bigwedge_{j=1}^{m} (a_j \to x_j) = \bigwedge_{s=1}^{l+r} (a_{k_s} \to x_{k_s}) = a_{k_p} \to x_{k_p}$$

for some p such that $1 \leqslant p \leqslant l+r$. Equality $b = a_{k_p} \to x_{k_p}$ implies that $a_{k_p} * b = a_{k_p} * (a_{k_p} \to x_{k_p}) = a_{k_p} \wedge x_{k_p} = x_{k_p}$ and therefore, $\mathbf{x} \leqslant \hat{\mathbf{x}}^p$. □

Similar to Corollary 1, we will obtain a characterization of the complete set of solutions of (5) under the assumptions which are accepted in this subsection and described in Theorem 3.

Corollary 2. *Suppose that the assumptions of Theorem* 3 *hold true and* $\mathbf{x} = (x_1, \ldots, x_m)$ *is a solution of* (5). *Then there exists one* $k \in \{k_1, \ldots, k_{l+r}\}$, *such that for any* $j = 1, \ldots, m$, *it is true that*

(i) *if* $j = k$ *and* $k \in \{k_1, \ldots, k_l\}$ *then* $x_k = a_k * b$,
(ii) *if* $j = k$ *and* $k \in \{k_{l+1}, \ldots, k_{l+r}\}$ *then* $x_k = 0$,
(iii) *if* $j \neq k$ *then* $x_j \geqslant a_j * b$.

Remark 1. It is easy to see that if the assumptions of Theorem 3 are fulfilled so that the subset $A' = \{a_k\}$ has exactly one element and the subset A'' is empty then the set of solutions of (5) is linearly ordered and $\hat{\mathbf{x}}^k = (1, \ldots, \underbrace{a_k * b}_{k}, \ldots, 1)$ is the largest solution.

Summarizing the results which were proven in Section 3 (Lemma 1, Theorems 2, 3, and their corollaries), we come to the following criterion.

Theorem 4. *Assume that BL-algebra* $\mathcal{L}$ *is linearly ordered and fulfills the conditional cancellation law. Then Eq.* (5) *is solvable if and only if* $b \geqslant \bigwedge_{j=1}^{m} \neg a_j$. *Moreover, each solution of* (5) *is less than or equal to a respective maximal solution and*

(i) *if* $b = 1$ *then* $\hat{\mathbf{x}} = (1, \ldots, 1)$ *is the only maximal solution*,
(ii) *if* $b < 1$ *and* $b = \bigwedge_{j=1}^{m} \neg a_j$ *then all maximal solutions are described in Theorem* 2,
(iii) *if* $b < 1$ *and* $b > \bigwedge_{j=1}^{m} \neg a_j$ *then all maximal solutions are described in Theorem* 3.

Example 5.

(a) Let $L = [0, 1]$, $m = 3$ and $\mathcal{L}$ be the Product algebra (see Example 2). The following equation

$$(0.3 \to_{\mathrm{P}} x_1) \wedge (0.5 \to_{\mathrm{P}} x_2) \wedge (0.8 \to_{\mathrm{P}} x_3) = 0.5$$

is solvable according to the assumptions accepted in Section 3.2.3. In this case, the conditions of Theorem 3 are fulfilled and $A' = \{a_1, a_2, a_3\}$, $A'' = \emptyset$. Therefore, there are three maximal solutions to this equation: (0.15, 1, 1), (1, 0.25, 1) and (1, 1, 0.4). The least solution is $\check{\mathbf{x}} = (0.15, 0.25, 0.4)$.
Each solution $\mathbf{x} = \{x_1, x_2, x_3\} \in [0, 1]^3$ satisfies: either $\check{\mathbf{x}} \leqslant \mathbf{x} \leqslant (0.15, 1, 1)$ or $\check{\mathbf{x}} \leqslant \mathbf{x} \leqslant (1, 0.25, 1)$ or $\check{\mathbf{x}} \leqslant \mathbf{x} \leqslant (1, 1, 0.4)$.

(b) The same equation as above is considered in the space $[0, 1]^3$ equipped with the standard Łukasiewicz algebra $\mathcal{L}_{Ł}$ (see Example 3):

$$(0.3 \to_{Ł} x_1) \wedge (0.5 \to_{Ł} x_2) \wedge (0.8 \to_{Ł} x_3) = 0.5.$$

Again conditions of Theorem 3 are fulfilled, but differently to the above given example, $A' = \{a_3\}$, $A'' = \{a_2\}$. There are two maximal solutions to this equation: (1, 0, 1) and (1, 1, 0.3). The least solution is $\check{\mathbf{x}} = (0, 0, 0.3)$.
Each solution $\mathbf{x} = \{x_1, x_2, x_3\} \in [0, 1]^3$ satisfies: either $\check{\mathbf{x}} \leqslant \mathbf{x} \leqslant (1, 0, 1)$ or $\check{\mathbf{x}} \leqslant \mathbf{x} \leqslant (1, 1, 0.3)$.

4. System of equations: a constructive way to obtain all solutions

In this section, the system of equations (3) is investigated and the complete set of its solutions is characterized. Here we will not assume that $\mathcal{L}$ is linearly ordered. Our goal is to find all solutions of (3). Recall that by Proposition 1, system (3) is solvable if and only if the vector $\check{\mathbf{x}} = (\check{x}_1, \ldots, \check{x}_m)$, given by (4), is its least solution.

If system (3) is solvable then each of its equations is solvable as well. Therefore, the following lemma (analogous to Lemma 1) establishes the necessary condition of solvability.

Lemma 2. *Let system* (3) *be solvable, then for each* $i = 1, \ldots n$, *the following condition*

$$b_i \geqslant \bigwedge_{j=1}^{m} \neg a_{ij} \tag{8}$$

necessarily holds.

We will continue with one necessary (Lemma 3) and one sufficient (Lemma 4) conditions of solvability.

Lemma 3. *System* (3) *is unsolvable if* $b_i < \bigwedge_{j=1}^{m} \neg a_{ij}$ *for some* $i,\ i = 1, \ldots, n$.

Proof. The proof easily follows from Theorem 1. □

Lemma 4. *System* (3) *is solvable if* $b_i = \bigwedge_{j=1}^{m} \neg a_{ij}$ *for each* $i = 1, \ldots, n$.

Proof. Suppose that $b_i = \bigwedge_{j=1}^{m} \neg a_{ij}$ holds true for each $i = 1, \ldots, n$. Then we easily verify that components $\check{x}_k$, $k = 1, \ldots, m$, of the vector $\check{\mathbf{x}}$ are zero. Indeed,

$$\check{x}_k = \bigvee_{i=1}^{n} (a_{ik} * b_i) = \bigvee_{i=1}^{n} \left(a_{ik} * \bigwedge_{j=1}^{m} \neg a_{ij} \right) \leqslant \bigvee_{i=1}^{n} (a_{ik} * \neg a_{ik}) = 0.$$

Moreover, the zero vector is a solution of (3). Indeed,

$$\bigwedge_{j=1}^{m} (a_{ij} \to 0) = \bigwedge_{j=1}^{m} \neg a_{ij} = b_i.$$

Therefore, $\check{\mathbf{x}} = (0, \ldots, 0)$ is a solution of (3) and by this, (3) is solvable. □

The following theorem is crucial for obtaining a solution of system (3) from solutions of its individual equations. It also gives a constructive way for obtaining all solutions of (3).

Theorem 5. *Let system* (3) *be solvable. Then a vector* $\mathbf{x} \in L^m$ *is a solution of* (3) *if and only if there exist solutions* $\mathbf{x}_1, \ldots, \mathbf{x}_n$ *of respective equations of* (3) *such that*

$$\mathbf{x} = \bigwedge_{i=1}^{n} \mathbf{x}_i$$

and

$$\bigwedge_{i=1}^{n} \mathbf{x}_i \geqslant \check{\mathbf{x}}. \tag{9}$$

Proof. Let system (3) be solvable and $\mathbf{x} \in L^m$ be a solution of (3). Then by Proposition 1, $\mathbf{x} \geqslant \check{\mathbf{x}}$. If we put $\mathbf{x}_i = \mathbf{x}$, $i = 1, \ldots, n$, then $\mathbf{x}_i$ solves the ith equation in (3) and by this, we easily obtain the required statement.

Let system (3) be solvable, $\mathbf{x}_1, \dots, \mathbf{x}_n$ be solutions to respective equations of (3) and (9) hold true. We will prove that $\mathbf{x} = \bigwedge_{i=1}^n \mathbf{x}_i$ is a solution of (3).

Denote $\mathbf{x} = (x_1, \dots, x_m)$ and $\mathbf{x}_i = (x_{i1}, \dots, x_{im})$, $i = 1, \dots, n$, so that

$$x_j = \bigwedge_{i=1}^n x_{ij}, \quad j = 1, \dots, m.$$

We will prove that $\mathbf{x}$ is a solution of each ith equation in (3), $i = 1, \dots, n$. Indeed,

$$\begin{aligned}\bigwedge_{j=1}^m (a_{ij} \to x_j) &= \bigwedge_{j=1}^m \left(a_{ij} \to \bigwedge_{l=1}^n x_{lj}\right) = \bigwedge_{l=1}^n \bigwedge_{j=1}^m (a_{ij} \to x_{lj}) \\ &= b_i \wedge \bigwedge_{\substack{l=1 \\ l \neq i}}^n \bigwedge_{j=1}^m (a_{ij} \to x_{lj}) \leqslant b_i.\end{aligned}$$

On the other side by (9),

$$\bigwedge_{j=1}^m (a_{ij} \to x_j) \geqslant \bigwedge_{j=1}^m (a_{ij} \to \check{x}_j) = b_i.$$

Both inequalities confirm that

$$\bigwedge_{j=1}^m (a_{ij} \to x_j) = b_i$$

so that $\mathbf{x}$ solves each ith equation in (3). □

4.1. Maximal solutions of a system of equations

In this subsection, we will characterize all maximal solutions of the system (3). Moreover, we will show that under reasonable conditions, each solution of the system (3) is less than or equal to a respective maximal solution. In the latter case, the least solution and all maximal solutions of (3) fully determine the complete set of solutions.

Lemma 5. *Let system* (3) *be solvable. Moreover, each solution of the ith equation, $i = 1, \dots, n$, is less than or equal to a respective maximal solution of the same equation. Then any maximal solution $\hat{\mathbf{x}}$ of* (3) *is represented by*

$$\hat{\mathbf{x}} = \bigwedge_{i=1}^n \hat{\mathbf{x}}_i,$$

where $\hat{\mathbf{x}}_1, \dots, \hat{\mathbf{x}}_n$ are maximal solutions of the respective equations of the system.

Proof. Suppose that system (3) is solvable and $\hat{\mathbf{x}}$ is its maximal solution. Then $\hat{\mathbf{x}}$ is a solution of each equation of (3). By the assumption, there exist maximal solutions $\hat{\mathbf{x}}_1, \dots, \hat{\mathbf{x}}_n$ of the respective equations of (3) such that $\hat{\mathbf{x}} \leqslant \hat{\mathbf{x}}_1, \dots, \hat{\mathbf{x}} \leqslant \hat{\mathbf{x}}_n$. Then

$$\hat{\mathbf{x}} \leqslant \bigwedge_{i=1}^n \hat{\mathbf{x}}_i$$

and therefore,

$$\check{\mathbf{x}} \leqslant \bigwedge_{i=1}^n \hat{\mathbf{x}}_i.$$

By Theorem 5, $\bigwedge_{i=1}^n \hat{\mathbf{x}}_i$ is a solution of (3). Therefore, $\hat{\mathbf{x}} = \bigwedge_{i=1}^n \hat{\mathbf{x}}_i$. □

Theorem 6. *Let system* (3) *be solvable and each solution of the ith equation*, $i = 1, \dots, n$, *be less than or equal to a respective maximal solution of the same equation. Denote by* S^i *the set of maximal solutions of the ith equation*, $i = 1, \dots, n$. *Then*

(i) *a solution of system* (3) *is maximal if and only if it is greater than or equal to* $\check{\mathbf{x}}$ *and it is a maximal element of the following set*:

$$\hat{\mathbf{S}} = S^1 \wedge \cdots \wedge S^n = \{\hat{\mathbf{x}}_1 \wedge \cdots \wedge \hat{\mathbf{x}}_n \mid \hat{\mathbf{x}}_1 \in S^1, \dots, \hat{\mathbf{x}}_n \in S^n\},$$

(ii) *each solution of system* (3) *is less than or equal to a respective maximal solution.*

Proof. Suppose that all assumptions of the theorem are fulfilled and S^i is the set of maximal solutions of the ith equation, $i = 1, \dots, n$.

(i) Let $\hat{\mathbf{x}}$ be a maximal solution of (3). Then according to Lemma 5, $\hat{\mathbf{x}} \in \hat{\mathbf{S}}$. Let us prove that $\hat{\mathbf{x}}$ is a maximal element of $\hat{\mathbf{S}}$. Assume that there exists $\hat{\mathbf{y}} \in \hat{\mathbf{S}}$ such that $\hat{\mathbf{x}} \leqslant \hat{\mathbf{y}}$. Then by Theorem 5, $\hat{\mathbf{y}}$ is a solution of (3) and thus, $\hat{\mathbf{y}} = \hat{\mathbf{x}}$. Therefore, $\hat{\mathbf{x}}$ is a maximal element of $\hat{\mathbf{S}}$.

Conversely, let $\hat{\mathbf{x}}$ be a maximal element of $\hat{\mathbf{S}}$ greater than or equal to $\check{\mathbf{x}}$. Then by Theorem 5, $\hat{\mathbf{x}}$ is a solution of (3). Let us prove that $\hat{\mathbf{x}}$ is a maximal solution of (3). Assume that a solution $\mathbf{x}$ of (3) is such that $\hat{\mathbf{x}} \leqslant \mathbf{x}$. Then $\mathbf{x}$ is a solution of each ith equation, $i = 1, \dots, n$, and by the assumption, is less than or equal to a respective maximal solution $\hat{\mathbf{x}}^i$ of the same equation. Therefore, $\mathbf{x} \leqslant \bigwedge_{i=1}^n \hat{\mathbf{x}}^i$ and hence $\hat{\mathbf{x}} \leqslant \bigwedge_{i=1}^n \hat{\mathbf{x}}^i$. Obviously, $\bigwedge_{i=1}^n \hat{\mathbf{x}}^i \in \hat{\mathbf{S}}$. Then the last inequality plus the fact that $\hat{\mathbf{x}}$ is a maximal element of $\hat{\mathbf{S}}$ imply that $\hat{\mathbf{x}} = \bigwedge_{i=1}^n \hat{\mathbf{x}}^i$. Therefore, $\mathbf{x} = \hat{\mathbf{x}}$ and thus, $\hat{\mathbf{x}}$ is a maximal solution of (3).

(ii) Let $\mathbf{x}$ be an arbitrary solution of (3). Denote $S^i_{\mathbf{x}}$ the set of maximal solutions of the ith equation, $i = 1, \dots, n$, greater than or equal to $\mathbf{x}$. Let us consider the following set of vectors:

$$S^1_{\mathbf{x}} \wedge \cdots \wedge S^n_{\mathbf{x}} = \{\hat{\mathbf{x}}_1 \wedge \cdots \wedge \hat{\mathbf{x}}_n \mid \hat{\mathbf{x}}_1 \in S^1_{\mathbf{x}}, \dots, \hat{\mathbf{x}}_n \in S^n_{\mathbf{x}}\}.$$

By Theorem 5, each element from $S^1_{\mathbf{x}} \wedge \cdots \wedge S^n_{\mathbf{x}}$ is a solution of (3). In the same way as above, we can prove that a maximal element of $S^1_{\mathbf{x}} \wedge \cdots \wedge S^n_{\mathbf{x}}$ is a maximal solution of (3). The latter is greater than or equal to $\mathbf{x}$. □

The following corollary combines results about solvability of a single equation with conditions of solvability for a system of equations.

Corollary 3. *Let BL-algebra* $\mathcal{L}$ *be linearly ordered and fulfill the conditional cancellation law. Moreover, let system* (3) *be solvable. Then the conclusion of Theorem* 6 *holds true.*

Proof. By the assumption, system (3) is solvable. Therefore, each equation in (3) is solvable as well. By Theorem 4, for all $i = 1, \dots, n$, $b_i \geqslant \bigwedge_{j=1}^m \neg a_{ij}$. Then by the same theorem, each solution of the ith equation, $i = 1, \dots, n$, is less than or equal to a respective maximal solution of the same equation. Hence, all assumptions of Theorem 6 are fulfilled and its conclusion follows. □

4.2. Algorithm for construction of maximal solutions of the system of equations

Theorem 6 provides the criterion for a solution to be a maximal solution of the system of equations (3) under the two assumptions: the system is solvable, and each solution of the ith equation, $i = 1, \dots, n$, is less than or equal to a respective maximal solution of the same equation. If the first assumption is true then the second one is automatically true in any BL-algebra which is linearly ordered and fulfill the conditional cancellation law (Corollary 3). For this case we propose the following algorithm for constructing all maximal solutions of (3).

Algorithm. *Input*: $(n \times m)$—matrix of coefficients A with elements from L, right-hand side vector $\mathbf{b} = (b_1, \dots, b_n) \in L^n$.

Output: The set $S_{\max}$ of maximal solutions of (3).
Step 1: Check whether the vector $\check{\mathbf{x}}$ computed by (4) is a solution of (3). If not, put $S_{\max} = \emptyset$ and exit.
Step 2: For each $i = 1, \ldots, n$, construct the set S^i of maximal solutions of the ith equation (cf. Theorem 4).
Step 3: Construct the set

$$\hat{\mathbf{S}} = \{\hat{\mathbf{x}}_1 \wedge \cdots \wedge \hat{\mathbf{x}}_n \mid \hat{\mathbf{x}}_1 \in S^1, \ldots, \hat{\mathbf{x}}_n \in S^n\}.$$

Step 4: Obtain $S_{\max}$ as a set of all maximal elements of $\hat{\mathbf{S}}$ which are greater than or equal to $\check{\mathbf{x}}$.

The complexity of the above given Algorithm (after a rough estimation) is $O(m^n)$.

Example 6. Let $L = [0, 1]$, $m = 3$ and $\mathcal{L}$ be Łukasiewicz algebra (see Example 3). The following system:

$$(1.0 \rightarrow_{Ł} x_1) \wedge (0.0 \rightarrow_{Ł} x_2) \wedge (0.4 \rightarrow_{Ł} x_3) = 1.0,$$

$$(0.3 \rightarrow_{Ł} x_1) \wedge (0.5 \rightarrow_{Ł} x_2) \wedge (0.8 \rightarrow_{Ł} x_3) = 0.5$$

is given by the matrix

$$A = \begin{pmatrix} 1.0 & 0.0 & 0.4 \\ 0.3 & 0.5 & 0.8 \end{pmatrix}$$

and the vector

$$\mathbf{b} = \begin{pmatrix} 1.0 \\ 0.5 \end{pmatrix}.$$

The first equation

$$(1.0 \rightarrow_{Ł} x_1) \wedge (0.0 \rightarrow_{Ł} x_2) \wedge (0.4 \rightarrow_{Ł} x_3) = 1.0$$

is analogous (but not the same) to that which has been considered in Example 4, case (a). The least solution is $\check{\mathbf{x}}_1 = (1, 0, 0.4)$ and the maximal solution is $\hat{\mathbf{x}}_1 = (1, 1, 1)$.

The second equation

$$(0.3 \rightarrow_{Ł} x_1) \wedge (0.5 \rightarrow_{Ł} x_2) \wedge (0.8 \rightarrow_{Ł} x_3) = 0.5$$

has been considered in Example 5, case (b). The least solution $\check{\mathbf{x}}_2 = (0, 0, 0.3)$, the maximal solutions are: $\hat{\mathbf{x}}_2^1 = (1, 0, 1)$ and $\hat{\mathbf{x}}_2^2 = (1, 1, 0.3)$.

We will proceed in accordance with the above given Algorithm.

Step 1: Vector $\check{\mathbf{x}} = (1, 0, 0.4)$, componentwise obtained by $\check{\mathbf{x}} = \check{\mathbf{x}}_1 \vee \check{\mathbf{x}}_2$, gives the least solution of the system.
Step 2: $S^1 = \{(1, 1, 1)\}$, $S^2 = \{(1, 0, 1), (1, 1, 0.3)\}$.
Step 3:

$$\hat{\mathbf{S}} = \{\hat{\mathbf{x}}_1 \wedge \hat{\mathbf{x}}_2^1, \hat{\mathbf{x}}_1 \wedge \hat{\mathbf{x}}_2^2\} = \{(1, 0, 1), (1, 1, 0.3)\}.$$

Step 4: Vector $(1, 1, 0.3)$ is a maximal element of $\hat{\mathbf{S}}$, but is not greater than or equal to $\check{\mathbf{x}} = (1, 0, 0.4)$. Therefore, it is not a maximal solution of the considered system.
Vector $\hat{\mathbf{x}} = (1, 0, 1)$ is a maximal element of $\hat{\mathbf{S}}$ and is greater than $\check{\mathbf{x}} = (1, 0, 0.4)$. Therefore, it is a maximal solution of the considered system. Moreover, in this particular case, $(1, 0, 1)$ is the only maximal solution and therefore, is the largest solution of the system.

Thus, each solution $\mathbf{x} = \{x_1, x_2, x_3\} \in [0, 1]^3$ of the considered system satisfies: $x_1 = 1$, $x_2 = 0$, $x_3 \geqslant 0.4$.

5. Conclusions

We have considered systems of fuzzy relation equations with inf-$\longrightarrow$ composition on finite universes. Equations are expressed using operations of a BL-algebra. We studied complete sets of solutions of respective systems. We considered two cases: the system comprises one equation, and it comprises a finite number of equations.

In the first case, we proved various conditions for solvability and the criterion for solvability (Theorem 4). The latter has been obtained under the assumption that the underlying BL-algebra is linearly ordered. In that case, we characterized all maximal solutions and proved that each solution is less than or equal to a respective maximal one. As a result, we were able to characterize the complete set of solutions (Theorem 4). Examples have been considered.

In the second case, we proposed a constructive way of getting all solutions (Theorem 5). Then we characterized maximal solutions (Lemma 5) and proved the criterion for a solution to be a maximal solution (Theorem 6). Moreover, we showed under which conditions a solution of the system is less than or equal to a respective maximal one (Theorem 6). We considered examples where complete sets of solutions have been obtained.

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References

[1] W. Bandler, L. Kohout, Semantics of implication operators and fuzzy relational products, Internat. J. Man-Mach. Studies 12 (1980) 89–116.
[2] T.S. Blyth, M.F. Janowitz, Residuation Theory, Pergamon, Oxford, 1972.
[3] K. Cechlárová, Unique solvability of max–min fuzzy equations and strong regularity of matrices over fuzzy algebras, Fuzzy Sets and Systems 75 (1995) 165–177.
[4] R.A. Cuninghame-Green, K. Cechlárová, Residuation in fuzzy algebra and some applications, Fuzzy Sets and Systems 71 (1995) 227–239.
[5] B. De Baets, Analytical solution methods for fuzzy relation equations, in: D. Dubois, H. Prade (Eds.), The Handbooks of Fuzzy Sets Series, Vol. 1, Kluwer Academic Publishers, Dordrecht, 2000, pp. 291–340.
[6] A. Di Nola, S. Sessa, W. Pedrycz, E. Sanchez, Fuzzy Relation Equations and Their Applications to Knowledge Engineering, Kluwer Academic Publishers, Boston, 1989.
[7] J.A. Goguen, L-fuzzy sets, J. Math. Anal. Appl. 18 (1967) 145–174.
[8] S. Gottwald, On the existence of solutions of systems of fuzzy equations, Fuzzy Sets and Systems 12 (1984) 301–302.
[9] S. Gottwald, Fuzzy Sets and Fuzzy Logic. The Foundations of Application—from a Mathematical Point of View, Vieweg, Braunschweig, Wiesbaden and Teknea, Toulouse, 1993.
[10] P. Hájek, Metamathematics of Fuzzy Logic, Kluwer Academic Publishers, Dordrecht, 1998.
[11] M. Higashi, G. Klir, Resolution of finite fuzzy relational equations, Fuzzy Sets and Systems 13 (1984) 65–82.
[12] M. Miyakoshi, M. Shimbo, Solutions of composite fuzzy relation equations with triangular norms, Fuzzy Sets and Systems 16 (1985) 53–63.
[13] M. Miyakoshi, M. Shimbo, Lower solutions of systems of fuzzy equations, Fuzzy Sets and Systems 19 (1986) 37–46.
[14] C. Pappis, M. Sugeno, Fuzzy relational equations and the inverse problem, Fuzzy Sets and Systems 15 (1985) 79–90.
[15] W. Pedrycz, Inverse problem in fuzzy relational equations, Fuzzy Sets and Systems 36 (1990) 277–291.
[16] W. Pedrycz, s–t fuzzy relational equations, Fuzzy Sets and Systems 59 (1993) 189–195.
[17] I. Perfilieva, Fuzzy function as an approximate solution to a system of fuzzy relation equations, Fuzzy Sets and Systems 147 (2004) 363–383.
[18] I. Perfilieva, A. Tonis, Compatibility of systems of fuzzy relation equations, Internat. J. General Systems 29 (2000) 511–528.
[19] E. Sanchez, Resolution of composite fuzzy relation equations, Inform. and Control 30 (1976) 38–48.
[20] X. Wang, Infinite fuzzy relational equations on a complete Brouwerian lattice, Fuzzy Sets and Systems 138 (2003) 657–666.
[21] L.A. Zadeh, Fuzzy sets, Inform. and Control 8 (1965) 338–353.
[22] L.A. Zadeh, Outline of a new approach to the analysis of complex systems and decision processes, IEEE Trans. Systems Man Cybernet. SMC-3 (1973) 28–44.