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On the minimal solutions of max–min fuzzy relational equations

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Abstract

In this paper, the minimal solutions of max–min fuzzy relational equations are investigated. A sufficient and necessary condition, for discriminating whether a given solution is minimal or not, is shown. Furthermore, we propose a new algorithm for computing all minimal solutions less than or equal to a given one.

Keywords: Fuzzy relational equation; Minimal solution; Max–min algebra; Matrix representation; Irredundant covering

1. Introduction

Fuzzy relational equations play important roles in many applications, such as intelligence technology [5,9], image reconstruction [8,24,29,28], etc. Therefore, how to compute the solutions of fuzzy relational equations is a fundamental problem. Recently, there have been many research papers investigating the solvability of fuzzy relational equations, by generalizing and extending the original results of [32,33] in various directions [1–4,6,7,11,14–17,20,21,26,27,30–37,39]. In addition, many authors have studied the optimization problems with fuzzy relational equation constraints [10,12,13,19,18,22,23,25,38].

The notion of fuzzy relational equations was first proposed and investigated by Sanchez [32,33], and was further studied by Czogala et al. [6]. Higashi and Klir [14] derived several alternative general schemes for solving the solutions. In 1988, Lichung and Boxing [21] introduced an algebraic method for calculating all minimal solutions. Later, De Baets [7, p. 291–340] provided an analytical method. In 2002, Louh et al. [26] used the matrix pattern to compute graphically the minimal solutions. Peeva [30] proposed a universal algorithm which improves the algebraic method. In this paper, we use the covering matrix, which was first introduced by Lichung and Boxing [21], to develop a new algorithm for computing all minimal solutions of max–min fuzzy relational equations. In Section 2, some basic definitions and preliminary theorems are presented. In Section 3, a sufficient and necessary condition, for discriminating whether a given solution is minimal or not, is shown. Some relevant propositions are also proved. In Section 4, we present the results in Section 3 in terms of the splitting type of the free terms. In Section 5, we prove the main theorem which consists of a computational method for determining all minimal solutions. In Section 6, we propose a new algorithm for computing all minimal solutions less than or equal to a given one. If the given solution is maximal, we hence obtain all minimal solutions.

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2. Definitions and preliminaries

In this section, we provide some basic definitions and preliminary results, which will be greatly used in this paper. A max–min fuzzy algebra $\Re$ is a linearly ordered set $([0, 1], \leqslant, \vee, \wedge)$ with the maximum operation $\vee$ and the minimum operation $\wedge$. Suppose that m, n are two positive integers.

- M denotes the set of $\{1, 2, \ldots, m\}$, and N denotes the set of $\{1, 2, \ldots, n\}$.
- $\Re^n$ denotes the set of all n-dimensional column vectors over $\Re$.
- $\Re^{m\times n}$ denotes the set of all matrices of type $m \times n$ over $\Re$.
- For $x, y \in \Re^n$, we write $x \leqslant y$, if $x_j \leqslant y_j$ holds for all $j \in N$, and $x < y$, if $x \leqslant y$ and $x \neq y$.

The problem of solving max–min fuzzy relational equations with finite sets is defined as follows: Let $A = [A_{ij}] \in \Re^{m\times n}$ and $b = (b_1, \ldots, b_m)^{\mathrm{T}} \in \Re^m$. Determine a vector $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \Re^n$ such that

$$A \circ x = b, \tag{2.1}$$

where $\circ$ denotes the max–min composition, i.e.

$$\bigvee_{j\in N} (A_{ij} \wedge x_j) = b_i$$

for all $i \in M$.

Let $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)^{\mathrm{T}} \in \Re^n$, where $\hat{x}_j$ is defined as

$$\hat{x}_j := \begin{cases} \min\{b_i | A_{ij} > b_i\} & \text{if there is an } i \in M \text{ with } A_{ij} > b_i, \\ 1 & \text{otherwise.} \end{cases} \tag{2.2}$$

Higashi and Klir have proved the following theorem.

Theorem 2.1 (*Higashi and Klir [14, Theorem 1]*). *Eq.* (2.1) *is solvable iff $\hat{x}$ is the maximal solution.*

The *matrix representation* operator, $\Delta = [\Delta_{ij}] : \Re^n \to \Re^{m\times n}$, is defined as

$$\Delta_{ij}(x) := \begin{cases} b_i & \text{if } A_{ij} \wedge x_j = b_i, \\ 0 & \text{if } A_{ij} \wedge x_j \neq b_i, \end{cases} \tag{2.3}$$

where $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \Re^n$.

The matrix representation $\Delta(\hat{x})$ of the maximal solution $\hat{x}$ is called the *covering matrix* of Eq. (2.1), which was first introduced by Lichung and Boxing [21]. Obviously, Eq. (2.3) implies that the (i, j)-entry of any matrix representation is b_i or 0.

A matrix $T = [T_{ij}] \in \Re^{m\times n}$ is called a *submatrix* (of $\Delta(\hat{x})$) iff $T_{ij} = \Delta_{ij}(\hat{x})$ or 0, for all $i \in M$ and $j \in N$. A submatrix T is a *covering* of Eq. (2.1) iff for every $i \in M$, there exists a $j \in N$ such that $T_{ij} = b_i$. T is also called a *covering submatrix*. We now restate a theorem of Lichung and Boxing for max–min fuzzy relational equations.

Theorem 2.2 (*Lichung and Boxing [21, Theorem 5]*). *The matrix representation of any solution of Eq.* (2.1) *is a covering submatrix.*

For every submatrix $T = [T_{ij}] \in \Re^{m\times n}$, the *sup-vector* $v(T) = (v_1, \ldots, v_n)^{\mathrm{T}}$ is defined by

$$v_j = v_j(T) := \max_{i\in M} T_{ij} \tag{2.4}$$

for all $j \in N$. Now, we reformulate another theorem of Lichung and Boxing for max–min fuzzy relational equations in Theorem 2.3, which can be considered as the converse of Theorem 2.2.

Theorem 2.3 (*Lichung and Boxing [21, Theorem 4]*). *The sup-vector of any covering submatrix is a solution of Eq.* (2.1).

Let x be a column vector such that $\Delta(x)$ is a covering submatrix. Theorem 2.3 implies $v(\Delta(x))$ is a solution. We now claim that the solution $v(\Delta(x))$ is less than or equal to x. Let $v_j(\Delta(x))$ and x_j be the j-th components of $v(\Delta(x))$ and x, respectively. Eq. (2.4) implies $v_j(\Delta(x)) = \max_{i\in M} \Delta_{ij}(x)$. Obviously, $v_j(\Delta(x)) = 0$ implies $v_j(\Delta(x)) \leqslant x_j$. If $v_j(\Delta(x)) > 0$, Eq. (2.3) implies

$$v_j(\Delta(x)) = b_{i_0} = A_{i_0 j} \wedge x_j$$

for some $i_0 \in M$. This also implies $v_j(\Delta(x)) \leqslant x_j$. Thus, we obtain the following Lemma 2.4.

Lemma 2.4. *Let x be a column vector such that $\Delta(x)$ is a covering submatrix. Then $v(\Delta(x))$ is a solution with $v(\Delta(x)) \leqslant x$.*

3. A sufficient and necessary condition

Let $M_0 = \{i \in M|\ b_i = 0\}$ and $N_0 = \{j \in N|\ A_{ij} > 0 \text{ for some } i \in M_0\}$. Then every solution x of Eq. (2.1) has $x_j = 0$ for all $j \in N_0$. Therefore, it is possible to omit the equations with indices from M_0 and the columns of A with indices from N_0. In this paper, we assume $\boldsymbol{b_i > 0}$ for all $i \in M$.

Definition 3.1. Let $x = (x_1, \ldots, x_n)^{\mathrm{T}}$ be a column vector of $\Re^n$, and $j \in N$. Define

$$I_j(x) := \{i \in M | A_{ij} \geqslant b_i = x_j\}, \tag{3.1}$$

$$K_j(x) := \{i \in M|\ A_{ij} = b_i < x_j\}, \tag{3.2}$$

$$\Im(x) := \{I_j(x)|\ x_j > 0\}, \tag{3.3}$$

$$M^-(x) := M - \bigcup_{j\in N} K_j(x). \tag{3.4}$$

If x is the maximal solution of Eq. (2.1), the definitions of $I_j(x)$ and $K_j(x)$ are the same as in [11, p. 388]. Note that, the above definitions trivially imply the following properties.

Property 1. $x_j = 0$ implies $I_j(x) = \emptyset$, by our assumption $b_i > 0$.

Property 2. $I_j(x) \cap K_j(x) = \emptyset$ for all $j \in N$.

Property 3. $\{i \in M|\ A_{ij} \wedge x_j = b_i\} = I_j(x) \cup K_j(x)$ for all $j \in N$.

Let H be a subset of M, and $\Im = \{I_j|\ j \in J\}$ be a collection of subsets of M, where J is an index set. $\Im$ is called a *covering collection* of H iff

$$\bigcup_{j\in J} I_j \supseteq H.$$

A covering collection $\Im$ of H is *minimal* iff $\Im - \{I_j\}$ is not a covering collection of H for each $j \in J$. Obviously, if $\Im$ is a covering collection of H with some $I_j = \emptyset$, then $\Im$ is not minimal, since $\Im - \{I_j\}$ is also a covering collection of H. That is, every $I_j \in \Im$ is non-empty, if $\Im$ is a minimal covering collection.

Lemma 3.2. *Let x be a column vector of $\Re^n$, then $\Delta(x)$ is a covering submatrix iff $\Im(x)$ is a covering collection of $M^-(x)$.*

Proof. Recall that, $\Delta(x)$ is a covering submatrix iff for every $i \in M$, there exists a $j \in N$ such that $\Delta_{ij}(x) = b_i$. By the assumption $b_i > 0$, Eq. (2.3) implies $A_{ij} \wedge x_j = b_i$, so that $i \in I_j(x) \cup K_j(x)$ by Property 3. We hence get $\Delta(x)$ is a covering matrix iff

$$M \subseteq \bigcup_{j\in N} (K_j(x) \cup I_j(x)).$$

Property 1 implies $I_j(x) = \emptyset$ for all $x_j = 0$. By applying Eqs. (3.3) and (3.4), the above equation can be changed to

$$M^-(x) \subseteq \bigcup_{j\in N} I_j(x) = \bigcup_{j:x_j>0} I_j(x) = \bigcup_{I_j(x)\in\Im(x)} I_j(x).$$

The completes the proof. $\square$

Lemma 3.3. *A column vector x is a solution of Eq.* (2.1) *iff $x \leqslant \hat{x}$ and $\Im(x)$ is a covering collection of $M^-(x)$.*

Proof. ($\Rightarrow$) Theorem 2.1 implies $x \leqslant \hat{x}$. Theorem 2.2 implies $\varDelta(x)$ is a covering submatrix. It follows $\Im(x)$ is a covering collection of $M^-(x)$, by Lemma 3.2.

($\Leftarrow$) Suppose that $\Im(x)$ is a covering collection of $M^-(x)$. Lemma 3.2 implies $\varDelta(x)$ is a covering submatrix. By applying Lemma 2.4, we obtain a solution $v(\varDelta(x))$ with $v(\varDelta(x)) \leqslant x$. Because that $v(\varDelta(x))$ and $\hat{x}$ are solutions such that $v(\varDelta(x)) \leqslant x \leqslant \hat{x}$, x is obviously a solution. $\square$

Theorem 3.4. *A column vector x is a minimal solution of Eq.* (2.1) *iff $x \leqslant \hat{x}$ and $\Im(x)$ is a minimal covering collection of $M^-(x)$.*

Proof. ($\Rightarrow$) Let x be a minimal solution. Lemma 3.3 gives $x \leqslant \hat{x}$ and $\Im(x)$ is a covering collection of $M^-(x)$. Therefore, it suffices to prove that $\Im(x)$ is minimal. We argue by contradiction. Suppose $\Im(x)$ is not minimal. Without loss of generality, we may assume that $\Im(x) - \{I_1(x)\}$ is also a covering collection of $M^-(x)$, where $I_1(x) \in \Im(x)$. That is,

$$M \subseteq K_1(x) \cup \bigcup_{j\geqslant 2} (I_j(x) \cup K_j(x)). \tag{3.5}$$

Eq. (3.3) implies $x_1 > 0$. Let $r = \max\{b_i \mid i \in K_1(x)\}$ if $K_1(x) \neq \emptyset$, and $r = 0$ otherwise. Eq. (3.2) implies $r < x_1$ if $K_1(x) \neq \emptyset$. $r = 0$ and $x_1 > 0$, thus $r < x_1$. Hence, both cases imply $r < x_1$. Let $y = (y_1, \ldots, y_n)^{\mathrm{T}}$, where $y_1 = r$ and $y_j = x_j$ for $j \geqslant 2$. We get

$$y < x \leqslant \hat{x},$$

since $y_1 < x_1$ and x is a solution. Because that $y_j = x_j$ for all $j \geqslant 2$, Eq. (3.1) and (3.2) imply

$$I_j(x) = I_j(y) \quad \text{and} \quad K_j(x) = K_j(y),$$

for all $j \geqslant 2$. On the other hand, for every $i \in K_1(x)$, we have

$$A_{i1} = b_i \leqslant \max\{b_{i'} \mid i' \in K_1(x)\} = y_1.$$

Hence, $i \in I_1(y)$ if $b_i = y_1$, and $i \in K_1(y)$ if $b_i < y_1$. That is,

$$K_1(x) \subseteq I_1(y) \cup K_1(y).$$

From Eq. (3.5), we compute

$$\begin{aligned} M &\subseteq K_1(x) \cup \bigcup_{j\geqslant 2} (I_j(x) \cup K_j(x)) \\ &\subseteq I_1(y) \cup K_1(y) \cup \bigcup_{j\geqslant 2} (I_j(y) \cup K_j(y)) \\ &= \bigcup_{j\in N} (I_j(y) \cup K_j(y)). \end{aligned}$$

Property 1 implies $I_j(y) = \emptyset$ for all $y_j = 0$. This shows that

$$M^-(y) \subseteq \bigcup_{j\in N} I_j(y) = \bigcup_{j:y_j>0} I_j(y) = \bigcup_{I_j(y)\in\Im(y)} I_j(y).$$

Thus, $\Im(y)$ is a covering collection of $M^-(y)$. Lemma 3.3 implies y is also a solution, which is a contradiction.

($\Leftarrow$) Obviously, Lemma 3.3 implies x is a solution. We argue by contradiction again. Suppose x is not minimal. Then there exists a solution $y = (y_1, \dots, y_n)^{\mathrm{T}}$ with $y < x$. Any column vector z with $y \leqslant z \leqslant x$ is also a solution. Hence, without loss of generality, we may assume that $y_1 < x_1$ and $y_j = x_j$ for all $j \geqslant 2$. Eqs. (3.1) and (3.2) imply

$$I_j(x) = I_j(y) \quad \text{and} \quad K_j(x) = K_j(y),$$

for all $j \geqslant 2$. Because that $i \in K_1(y)$ implies $A_{i1} = b_i < y_1 < x_1$, we also get

$$K_1(y) \subseteq K_1(x).$$

If $i \in I_1(y)$ then $A_{i1} \geqslant b_i = y_1 < x_1$. This implies $A_{i1} = b_i$. Otherwise $A_{i1} \wedge x_1 > b_i$, which contradicts to x is a solution. Hence, $i \in K_1(x)$. That is,

$$I_1(y) \subseteq K_1(x).$$

We conclude the following assertions:

- $\Im(x) - \{I_1(x)\} = \Im(y) - \{I_1(y)\}$, since $I_j(x) = I_j(y)$ for all $j \geqslant 2$.
- $M^-(y) \supseteq M^-(x)$, since $K_j(y) \subseteq K_j(x)$ for all $j \in N$.
- $M^-(x) - I_1(y) = M^-(x)$, since $I_1(y) \subseteq K_1(x)$.

Because that y is a solution, Lemma 3.3 implies $\Im(y)$ is a covering collection of $M^-(y)$. This is equivalent to $\Im(y) - \{I_1(y)\}$ is a covering collection of $M^-(y) - I_1(y)$. From the above assertions, we get

$$\Im(x) - \{I_1(x)\} = \Im(y) - \{I_1(y)\} \quad \text{is a covering collection of } M^-(y) - I_1(y), \quad \text{and}$$
$$M^-(y) - I_1(y) \supseteq M^-(x) - I_1(y) = M^-(x).$$

That is to say, $\Im(x) - \{I_1(x)\}$ is a covering collection of $M^-(x)$, which contradicts to $\Im(x)$ is minimal. $\Box$

Theorem 3.4 implies the following Corollary 3.5, which is a generalization of Gavalec's theorem [11, Theorem 4.5].

Corollary 3.5. *A solution x of Eq.* (2.1) *is minimal iff $\Im(x)$ is a minimal covering collection of $M^-(x)$.*

Theorem 3.6 (*Gavalec [11, Theorem 4.5]*). *Eq.* (2.1) *has a unique solution iff $\Im(\hat{x})$ is a minimal covering collection of $M^-(\hat{x})$, where $\hat{x}$ is the maximal solution of Eq.* (2.1).

Obviously, Eq. (2.1) has a unique solution iff the maximal solution $\hat{x}$ is minimal. By Corollary 3.5, $\hat{x}$ is minimal iff $\Im(\hat{x})$ is a minimal covering collection of $M^-(\hat{x})$. Hence, Corollary 3.5 implies Theorem 3.6.

Example 3.7. Let us consider the fuzzy relational equation as follows:

$$A = \begin{bmatrix} 1 & 0.9 & 0.9 & 0.9 & 1 \\ 0.6 & 0.6 & 0.4 & 0.3 & 0.6 \\ 0.7 & 0.6 & 0.5 & 0.6 & 0.5 \\ 0 & 0.9 & 0.6 & 0.5 & 0.4 \\ 0.2 & 0.2 & 0.1 & 0.4 & 0.2 \\ 0.1 & 0.1 & 0.2 & 0.5 & 0.1 \end{bmatrix}, \quad b = \begin{pmatrix} 0.9 \\ 0.6 \\ 0.6 \\ 0.6 \\ 0.2 \\ 0.2 \end{pmatrix}, \quad u = \begin{pmatrix} 0.6 \\ 0 \\ 0.6 \\ 0.2 \\ 0.9 \end{pmatrix}, \quad \text{and} \quad v = \begin{pmatrix} 0.6 \\ 0 \\ 0.6 \\ 0 \\ 0.9 \end{pmatrix}.$$

By Eqs. (3.1) and (3.2), we compute $I_1(u) = \{2, 3\}$, $I_2(u) = \emptyset$, $I_3(u) = \{4\}$, $I_4(u) = \{5, 6\}$, $I_5(u) = \{1\}$ (see Fig. 1, marked with boxes), and $K_1(u) = \{5\}$, $K_2(u) = K_4(u) = \emptyset$, $K_3(u) = \{6\}$, $K_5(u) = \{2, 5\}$ (see Fig. 1, marked with circles). Therefore,

$$\Im(u) = \{\{2, 3\}, \{4\}, \{5, 6\}, \{1\}\} \quad \text{(see Fig. 1, marked with boxes)},$$
$$M^-(u) = \{1, 3, 4\} \quad \text{(see Fig. 1, delete the rows with indices from } K_j(u)).$$

Observe that $\Im(u)$ is a covering collection of $M^-(u)$ but not minimal, since $\Im(u) - \{I_4(u)\}$ is also a covering collection of $M^-(u)$. Eq. (2.2) implies the maximal solution is $\hat{x} = (0.6, 0.6, 1, 0.2, 0.9)^{\mathrm{T}}$. It is easily seen that $u \leqslant \hat{x}$. By applying

$$
(A \mid b) = \left(\begin{array}{ccccc|c}
1 & 0.9 & 0.9 & 0.9 & \boxed{1} & 0.9 \\
\cancel{\boxed{0.6}} & \cancel{0.6} & \cancel{0.4} & \cancel{0.3} & \cancel{0.6} & 0.6 \\
\boxed{0.7} & 0.6 & 0.5 & 0.6 & 0.5 & 0.6 \\
0 & 0.9 & \boxed{0.6} & 0.5 & 0.4 & 0.6 \\
\cancel{0.2} & \cancel{0.2} & \cancel{0.1} & \cancel{\boxed{0.4}} & \cancel{0.2} & 0.2 \\
\cancel{0.1} & \cancel{0} & \cancel{0.2} & \cancel{\boxed{0.5}} & \cancel{0.1} & 0.2
\end{array}\right)
$$

$$
u = \begin{pmatrix} 0.6 & 0 & 0.6 & 0.2 & 0.9 \end{pmatrix}^{\mathrm{T}}
$$

Fig. 1.

Lemma 3.3 and Theorem 3.4, we obtain u is a solution but not minimal. We easily verify $\Im(v) = \Im(u) - \{I_4(u)\}$ and $M^-(v) = M^-(u)$. Hence, v is a minimal solution of Eq. (2.1). In fact, v is the uniquely minimal solution with $v < u$ (see Example 5.7).

4. The splitting types

In this section, we still assume $b_i > 0$ for all $i \in M$.

Definition 4.1. Let $\{M_1, \ldots, M_l\}$ denote the unique partition of M such that

$$
\begin{cases} b_i > b_{i'} & \text{for all } i \in M_s,\ i' \in M_t,\ 1 \leqslant s < t \leqslant l, \\ b_i = b_{i'} & \text{for all } i, i' \in M_t,\ 1 \leqslant t \leqslant l, \end{cases}
$$

and $L := \{1, \ldots, l\}$, $\bar{b}_t := b_i$ for all $t \in L$ and any $i \in M_t$, and $\bar{B} := \{\bar{b}_t \mid t \in L\}$.

The next lemma follows trivially from the above definition.

Lemma 4.2. *Let $s, t \in L$ and $i \in M$.*

(1) $s < t$ *iff* $\bar{b}_s > \bar{b}_t$.
(2) $b_i = \bar{b}_t$ *iff* $i \in M_t$.

Definition 4.3. Let $x = (x_1, \ldots, x_n)^{\mathrm{T}}$ be a column vector of $\Re^n$, and $t \in L$. Define

$$M_t^-(x) := M_t - \cup_{j \in N} K_j(x), \tag{4.1}$$
$$\Im_t(x) := \{I_j(x) \mid x_j = \bar{b}_t\}. \tag{4.2}$$

Lemma 4.4. *If $I_j(x) \in \Im_t(x)$, then $I_j(x) \subseteq M_t$.*

Proof. Obviously, $I_j(x) \in \Im_t(x)$ implies $x_j = \bar{b}_t$, by Eq. (4.2). Let $i \in I_j(x)$. Eq. (3.1) gives $A_{ij} \geqslant b_i = x_j$. We hence obtain $b_i = \bar{b}_t$. Lemma 4.2(2) implies $i \in M_t$. $\square$

Obviously, Eqs. (3.4) and (4.1) together imply

$$M_t^-(x) = M_t \cap M^-(x).$$

This shows that $M_1^-(x), \ldots, M_l^-(x)$ are disjoint and form a covering collection of $M^-(x)$, since $\{M_1, \ldots, M_l\}$ is a partition of M. Notice that, this covering collection may be not minimal. This is because that there may have some $t \in L$ such that $M_t^-(x) = \emptyset$. If $M_t^-(x) = \emptyset$, then any collection (including the empty collection $\{\,\}$) is a covering collection of $M_t^-(x)$, and the covering collection $\{\emptyset\}$ is not minimal by convention.

By Eq. (3.1), $x_j \notin \bar{B}$ implies $I_j(x) = \emptyset$. From Eq. (3.3), we compute

$$\Im(x) = \{I_j(x) \mid x_j \in \bar{B}\} \cup \{I_j(x) \mid x_j \notin \bar{B}\}$$
$$= \begin{cases} \bigcup_{t \in L} \Im_t(x) & \text{if } x_j \in \bar{B} \text{ for all } x_j > 0, \\ \bigcup_{t \in L} \Im_t(x) \cup \{\emptyset\} & \text{if there exists an } 0 < x_j \notin \bar{B}. \end{cases} \tag{4.3}$$

In addition, Lemma 4.4 implies

$$\bigcup_{I_j(x) \in \Im_t(x)} I_j(x) \subseteq M_t(x) \tag{4.4}$$

for each $t \in L$. This shows that $\Im(x)$ is a covering collection of $M^-(x)$ iff $\Im_t(x)$ is a covering collection of $M_t^-(x)$ for each $t \in L$. We hence obtain the splitting type of Lemma 3.3.

Lemma 4.5. *A column vector x is a solution of Eq.* (2.1) *iff $x \leqslant \hat{x}$ and $\Im_t(x)$ is a covering collection of $M_t^-(x)$ for each $t \in L$.*

Note that, if $\Im(x)$ is a minimal covering collection of $M^-(x)$, every $I_j(x) \in \Im(x)$ is non-empty. By Eqs. (4.3) and (4.4), $\Im(x)$ is a minimal covering collection of $M^-(x)$ iff $\Im_t(x)$ is a minimal covering collection of $M_t^-(x)$ for each $t \in L$ and $x_j \in \bar{B}$ for all $x_j > 0$. Hence, we also obtain the splitting type of Theorem 3.4.

Theorem 4.6. *A column vector x is a minimal solution of Eq.* (2.1) *iff the following statements hold*:

(1) $x \leqslant \hat{x}$,
(2) $x_j \in \bar{B}$, *for all* $x_j > 0$, and
(3) $\Im_t(x)$ *is a minimal covering collection of* $M_t^-(x)$, *for each* $t \in L$.

Corollary 4.7. *A solution x of Eq.* (2.1) *is minimal iff $x_j \in \bar{B}$ for all $x_j > 0$ and $\Im_t(x)$ is a minimal covering collection of $M_t^-(x)$ for each $t \in L$.*

In fact, the condition "$x_j \in \bar{B}$ for all $x_j > 0$" is necessary. A counterexample is shown as follows:

Example 4.8 (*Continued*). Let us consider again Example 3.7, and $w = (0.6, 0, 0.6, 0.1, 0.9)^{\mathrm{T}}$. Obviously,

$$\Im(w) = \{\{2, 3\}, \{4\}, \emptyset, \{1\}\}$$

(see Fig. 2, marked with boxes) and

$$M^-(w) = \{1, 3, 4\}$$

(see Fig. 2, delete the rows with indices from $K_j(w)$, where $K_j(w)$ are marked with circles). We easily verify that $\Im(w)$ is a covering collection of $M^-(w)$ but not minimal. Lemma 3.3 and Theorem 3.4 imply w is a solution but not minimal. Observe that $M_1 = \{1\}$, $M_2 = \{2, 3, 4\}$, $M_3 = \{5, 6\}$. Eq. (4.1) implies

$$M_1^-(w) = \{1\}, \quad M_2^-(w) = \{3, 4\}, \quad M_3^-(w) = \emptyset,$$

and Eq. (4.2) implies

$$\Im_1(w) = \{\{1\}\}, \quad \Im_2(w) = \{\{2, 3\}, \{4\}\}, \quad \Im_3(w) = \{\,\}$$

(see Fig. 2, marked with boxes and divided by dotted lines). Notice that, $w \leqslant \hat{x}$ and $\Im_t(w)$ is a minimal covering collection of $M_t^-(w)$ for $t = 1, 2, 3$, but $w_4 = 0.1 \notin \bar{B}$. That is to say, in Theorem 4.6, the condition "$x_j \in \bar{B}$ for all $x_j > 0$" is necessary.

$$
\begin{array}{rcl}
(A|b)= & \left(\begin{array}{ccccc|c}
1 & 0.9 & 0.9 & 0.9 & \boxed{1} & 0.9 \\
\hline
\cancel{\boxed{0.6}} & \cancel{0.6} & \cancel{0.4} & \cancel{0.3} & \cancel{0.6} & \cancel{0.6} \\
\boxed{0.7} & 0.6 & 0.5 & 0.6 & 0.5 & 0.6 \\
0 & 0.9 & \boxed{0.6} & 0.5 & 0.4 & 0.6 \\
\hline
\cancel{0.2} & \cancel{0.2} & \cancel{0.1} & \cancel{0.4} & \cancel{0.2} & \cancel{0.2} \\
\cancel{0.1} & \cancel{0} & \cancel{0.2} & \cancel{0.5} & \cancel{0.1} & \cancel{0.2}
\end{array}\right) &
\begin{array}{l}
M_1^-(w)=\{1\} \\
\\
M_2^-(w)=\{3,4\} \\
\\
\\
M_3^-(w)=\varnothing
\end{array} \\
w= & (0.6 \quad 0 \quad 0.6 \quad 0.1 \quad 0.9)^{\mathrm{T}} &
\end{array}
$$

Fig. 2.

5. Main theorems and a computational method

Let x be a solution of Eq. (2.1), and $\check{x}$ be a minimal solution with $\check{x} \leqslant x$. If $\Delta_{ij}(\check{x}) > 0$, Eq. (2.3) implies $\Delta_{ij}(\check{x}) = b_i$. We get

$$b_i = A_{ij} \wedge \check{x}_j \leqslant A_{ij} \wedge x_j \leqslant b_i,$$

hence $\Delta_{ij}(x) = b_i$. That is to say, $\Delta(\check{x})$ is a submatrix of $\Delta(x)$ (i.e. $\Delta_{ij}(\check{x}) = \Delta_{ij}(x)$ or 0 for all i, j). This shows that, we can determine all matrix representation of the minimal solutions less than or equal to x from $\Delta(x)$. After computing the sup-vectors of those matrix representations, we will obtain the minimal solutions. If x is the maximal solution, we hence determine all minimal solutions of Eq. (2.1). The following work is to develop an algorithm for computing all minimal solutions less than or equal to a given solution x.

Let x be a solution of Eq. (2.1), and $H \subseteq M$. A subset $N' \subseteq N$ is called an *x-column covering* of H iff for every $i \in H$, there is a $j \in N'$ such that $\Delta_{ij}(x) = b_i$ (or equivalently $A_{ij} \wedge x_j = b_i$ by Eq. (2.3)). For example, N is an x-column covering of M, by Theorem 2.2. An x-column covering N' of H is *minimal* iff $N' - \{j\}$ is not an x-column covering of H for each $j \in N'$. If $H = \emptyset$, then we say the empty set $\emptyset$ is the uniquely minimal x-column covering of H.

Lemma 5.1. *Let x be a solution of Eq.* (2.1), *and N' be an x-column covering of $H \subseteq M$. If $S \subseteq N'$ satisfies $\Delta_{ij}(x) = 0$ for all $i \in H$ and $j \in S$, then $N' - S$ is also an x-column covering of H.*

Proof. Let $i \in H$. Because that N' is an x-column covering of H, there is a $j_0 \in N'$ such that $\Delta_{ij_0}(x) = b_i$. $b_i > 0$ implies $j_0 \notin S$, thus $j_0 \in N' - S$. $\square$

Lemma 5.2. *Let x be a solution of Eq.* (2.1), *N' be an x-column covering of $H \subseteq M$, $S \subseteq N'$, and let*

$$H' = \{i \in H \mid \Delta_{ij}(x) = 0, \quad \textit{for all } j \in S\}.$$

Then $N' - S$ is an x-column covering of H'.

Proof. Obviously, N' is an x-column covering of H', and $\Delta_{ij}(x) = 0$ for all $i \in H'$ and $j \in S$. Lemma 5.1 implies that $N' - S$ is an x-column covering of H'. $\square$

Let $M_1, \ldots, M_l$ be defined in Definition 4.1, and $M'_1 = M_1$. Because that x is a solution and $M'_1 \subseteq M$, Theorem 2.2 implies N is an x-column covering of M'_1. Let N_1 be an arbitrary minimal x-column covering of M'_1, and

$$H' = \{i \in M \mid \Delta_{ij}(x) = 0, \text{ for all } j \in N_1\}.$$

Lemma 5.2 implies $N - N_1$ is an x-column covering of H'. Consequently, let

$$M'_2 = \{i \in M_2 \mid \Delta_{ij}(x) = 0, \text{ for all } j \in N_1\} = M_2 \cap H',$$

and $N_2 \subseteq N - N_1$ be an arbitrary minimal x-column covering of M'_2. Hence, we can repeatedly apply Lemma 5.2 to obtain $N_1, \ldots, N_l$, that are arbitrary disjoint subsets of N such that N_t is a minimal x-column covering of M'_t for each

$t \in L$, where

$$M_1' = M_1 \quad \text{and} \quad M_t' := \left\{ i \in M_t \mid \Delta_{ij}(x) = 0, \text{ for all } j \in \bigcup_{1 \leqslant s < t} N_s \right\}.$$

In Theorem 5.6, we prove that they produce a minimal solution less than or equal to x. In order to prove Theorem 5.6, let's define

$$P_{tj}(x) := \{i \in M_t \mid A_{ij} \wedge x_j = b_i\} \tag{5.1}$$

for all $t \in L$ and $j \in N$.

Lemma 5.3. *Let x be a solution of Eq.* (2.1), *$S \subseteq N$, and $H \subseteq M_t$. Then S is an x-column covering of H iff $\{P_{tj}(x) \mid j \in S\}$ is a covering collection of H. Therefore, S is a minimal x-column covering of H iff $\{P_{tj}(x) \mid j \in S\}$ is a minimal covering collection of H.*

Proof. S is an x-column covering of H iff for every $i \in H$, so there is a $j \in S$ such that $\Delta_{ij}(x) = b_i$. That is, for every $i \in H$, there is a $j \in S$ such that $i \in P_{tj}(x)$, by Eqs. (2.3) and (5.1). This is equivalent to $H \subseteq \cup_{j \in S} P_{tj}(x)$. □

Lemma 5.4. *Let x be a solution of Eq.* (2.1), *and $y = (y_1, \ldots, y_n)^{\mathrm{T}} \leqslant x$. If $y_j = \bar{b}_t$, then $I_j(y) = P_{tj}(x)$. Therefore, $\Im_t(y) = \{P_{tj}(x) \mid y_j = \bar{b}_t\}$ for all $t \in L$.*

Proof. Let $i \in I_j(y)$. Eq. (3.1) implies $A_{ij} \geqslant b_i = y_j$, so that $b_i = \bar{b}_t$. Lemma 4.2(2) implies $i \in M_t$. Since x is a solution with $x \geqslant y$, we get

$$b_i \geqslant A_{ij} \wedge x_j \geqslant A_{ij} \wedge y_j = b_i.$$

Hence, $A_{ij} \wedge x_j = b_i$. Eq. (5.1) implies $i \in P_{tj}(x)$. Conversely, let $i \in P_{tj}(x)$. Eq. (5.1) and Lemma 4.2(2) imply

$$A_{ij} \wedge x_j = b_i = \bar{b}_t.$$

Because that $y_j = \bar{b}_t$, we get $A_{ij} \geqslant b_i = y_j$. Eq. (3.1) implies $i \in I_j(y)$. This completes the proof. □

Lemma 5.5. *Let x be a solution of Eq.* (2.1), *$N_1, \ldots, N_l$ be disjoint subsets of N,*

$$M_1' = M_1 \quad \textit{and} \quad M_t' := \left\{ i \in M_t \mid \Delta_{ij}(x) = 0, \ \textit{for all } j \in \bigcup_{1 \leqslant s < t} N_s \right\},$$

for all $1 < t \in L$. Let $y = (y_1, \ldots, y_n)^{\mathrm{T}}$, where $y_j = \bar{b}_t$ if $j \in N_t$ for some $t \in L$, and $y_j = 0$ otherwise. If $y \leqslant x$, then $M_t' = M_t^-(y)$ for all $t \in L$.

Proof. Obviously, the definition of y_j implies $y_j \leqslant \bar{b}_1$ for all $j \in N$. By Eq. (3.2), $i \in K_j(y)$ implies

$$A_{ij} = b_i < y_j \leqslant \bar{b}_1.$$

Lemma 4.2(2) implies $i \notin M_1$. From Eq. (4.1), we get

$$M_1^-(y) = M_1 - \bigcup_{j \in N} K_j(y) = M_1 = M_1'.$$

It suffices to show that $M_t' = M_t^-(y)$ for all $t > 1$. Let $s < t$ and $j \in N_s$. Since $y \leqslant x$, we get

$$K_j(y) \cap M_t \subseteq K_j(x) \cap M_t \subseteq \{i \in M_t \mid A_{ij} \wedge x_j = b_i\} = P_{tj}(x). \tag{5.2}$$

The definition of y_j and Lemma 4.2(1) imply

$$y_j = \bar{b}_s > \bar{b}_t. \tag{5.3}$$

Suppose that there is an $i \in P_{tj}(x)$ such that $A_{ij} > b_i$. Obviously, Eq. (5.1) gives $i \in M_t$, hence

$$b_i = \bar{b}_t < y_j.$$

By the assumption $x \geqslant y$, we get

$$A_{ij} \wedge x_j \geqslant A_{ij} \wedge y_j > b_i,$$

which contradicts to x is a solution. Hence, each $i \in P_{tj}(x)$ implies $A_{ij} = b_i$. Lemma 4.2(2) gives $b_i = \bar{b}_t$. By Eqs. (5.1) and (5.3), we get

$$P_{tj}(x) \subseteq \{i \in M_t \mid A_{ij} = b_i < y_j\} = K_j(y) \cap M_t. \tag{5.4}$$

Combining Eqs. (5.2) and (5.4), we obtain

$$P_{tj}(x) = K_j(y) \cap M_t$$

for all $j \in N_s$ and $s < t$. Now, let's compute

$$\begin{aligned} M_t' &= \left\{ i \in M_t \mid \Delta_{ij}(x) = 0, \text{ for all } j \in \bigcup_{1 \leqslant s < t} N_s \right\} \\ &= M_t - \bigcup_{1 \leqslant s < t} \bigcup_{j \in N_s} \{i \in M_t \mid \Delta_{ij}(x) > 0\} \\ &= M_t - \bigcup_{1 \leqslant s < t} \bigcup_{j \in N_s} \{i \in M_t \mid \Delta_{ij}(x) = b_i\} \quad \text{(by Eq. (2.3))} \\ &= M_t - \bigcup_{1 \leqslant s < t} \bigcup_{j \in N_s} P_{tj}(x) \\ &= M_t - \bigcup_{1 \leqslant s < t} \bigcup_{j \in N_s} (K_j(y) \cap M_t). \end{aligned}$$

Observe that,

$$\bigcup_{1 \leqslant s < t} \bigcup_{j \in N_s} (K_j(y) \cap M_t) \subseteq \bigcup_{j \in N} (K_j(y) \cap M_t).$$

On the other hand, $i \in K_j(y) \cap M_t$ is equivalent to $A_{ij} = b_i < y_j$ and $b_i = \bar{b}_t$. They imply $y_j > \bar{b}_t$. The definition of y_j implies $y_j = \bar{b}_s$ for some $s \in L$ and $j \in N_s$. Lemma 4.2(1) implies $s < t$. That is to say,

$$\bigcup_{j \in N} (K_j(y) \cap M_t) \subseteq \bigcup_{1 \leqslant s < t} \bigcup_{j \in N_s} (K_j(y) \cap M_t).$$

We conclude

$$\bigcup_{1 \leqslant s < t} \bigcup_{j \in N_s} (K_j(y) \cap M_t) = \bigcup_{j \in N} (K_j(y) \cap M_t).$$

Thus, $M_t' = M_t - \cup_{j \in N} (K_j(y) \cap M_t) = M_t^-(y)$, by Eq. (4.1). $\square$

Theorem 5.6. *Let x be a solution of Eq.* (2.1). *Then $y = (y_1, \ldots, y_n)^{\mathrm{T}}$ is a minimal solution with $y \leqslant x$ iff there are disjoint subsets $N_1, \ldots, N_l$ of N such that N_t is a minimal x-column covering of M_t' for each $t \in L$, where*

$$M_1' = M_1 \quad \textit{and} \quad M_t' = \{i \in M_t \mid \Delta_{ij}(x) = 0, \textit{ for all } j \in \bigcup_{1 \leqslant s < t} N_s\},$$

and $y_j = \bar{b}_t$ if $j \in N_t$ for some $t \in L$, otherwise $y_j = 0$.

Proof. ($\Rightarrow$) Theorem 4.6 implies that, $y_j \in \bar{B}$ for all $y_j > 0$ and $\Im_t(y)$ is a minimal covering collection of $M_t^-(y)$ for each $t \in L$. Let

$$N_t = \{j \in N | \; y_j = \bar{b}_t\}$$

for each $t \in L$. In other words, $y_j = \bar{b}_t$ if $j \in N_t$ for some $t \in L$, and $y_j = 0$ otherwise. Lemma 5.4 implies

$$\Im_t(y) = \{P_{tj}(x) | \; y_j = \bar{b}_t\} = \{P_{tj}(x) | \; j \in N_t\}.$$

Because that $N_1, \ldots, N_l$ are disjoint subsets and $y \leqslant x$, Lemma 5.5 implies $M_t' = M_t^-(y)$. Hence, $\{P_{tj}(x) | \; j \in N_t\}$ is a minimal covering collection of M_t'. Lemma 5.3 implies that N_t is a minimal x-column covering of M_t'. This completes the proof.

($\Leftarrow$) Let $y_j > 0$. The definition of y_j implies $y_j = \bar{b}_t$ for some $t \in L$ and $j \in N_t$. Thus,

$$y_j \in \bar{B}.$$

Suppose that $\Delta_{ij}(x) \neq b_i$ for all $i \in M_t'$, or equivalently $\Delta_{ij}(x) = 0$ for all $i \in M_t'$, by Eq. (2.3). Lemma 5.1 implies $N_t - \{j\}$ is an x-column covering of M_t', which contradicts to N_t is minimal. Hence, there exists an $i \in M_t'$ such that $\Delta_{ij}(x) = b_i$. $i \in M_t'$ implies $b_i = \bar{b}_t$ by Lemma 4.2(2), and $\Delta_{ij}(x) = b_i$ implies $A_{ij} \wedge x_j = b_i$ by Eq. (2.3). These together lead to

$$y_j = \bar{b}_t = b_i = A_{ij} \wedge x_j \leqslant x_j.$$

Thus, $y \leqslant x \leqslant \hat{x}$. By Theorem 4.6, it suffices to prove that $\Im_t(y)$ is a minimal covering collection of $M_t^-(y)$ for each $t \in L$. Because that N_t is a minimal x-column covering of M_t', Lemma 5.3 implies that $\{P_{tj}(x) | \; j \in N_t\}$ is a minimal covering collection of M_t' for each $t \in L$. Since $y \leqslant x$, Lemma 5.5 implies $M_t' = M_t^-(y)$. By the definition of y_j, we get

$$\{P_{tj}(x) | \; j \in N_t\} = \{P_{tj}(x) | \; y_j = \bar{b}_t\}.$$

Lemma 5.4 implies

$$\Im_t(y) = \{P_{tj}(x) | \; y_j = \bar{b}_t\}.$$

Hence, $\Im_t(y)$ is a minimal covering collection of $M_t^-(y)$. $\square$

Suppose b_i, $i \in M$, are all equal. Theorem 5.6 implies each minimal x-column covering of M corresponds to a minimal solution $\check{x}$ with $\check{x} \leqslant x$. Hence, in this case the problem of solving minimal solutions can be in terms of finding irredundant coverings in the table obtained by the matrix representation of x. There is an efficient algorithm of finding irredundant coverings which was proposed by Markovskii [27].

Example 5.7 (*Continued*). Let us consider again Example 3.7, and determine all minimal solutions less than or equal to $x = (0.6, 0.6, 1, 0.2, 0.9)^{\mathrm{T}}$. Note that, Eq. (2.2) implies that x is the maximal solution of Eq. (2.1). Hence, we obtain all minimal solutions of Eq. (2.1).

By Eq. (2.3), we compute the matrix representation $\Delta(x)$, as shown in Fig. 3. Since $M_1' = M_1 = \{1\}$, we obtain two minimal x-column coverings of M_1': $N_1 = \{3\}$ or $\{5\}$.

Case 1: $N_1 = \{3\}$.

Delete the ith rows with $\Delta_{i3}(x) \neq 0$, we get

$$M_2' = \{i \in M_2 | \; \Delta_{ij}(x) = 0, \text{ for all } j \in N_1\} = \{i \in M_2 | \; \Delta_{i3}(x) = 0\} = \{2, 3\},$$

as shown in Fig. 4(a). There are two minimal x-column coverings of M_2' (see Fig. 4(b)):

$N_2 = \{1\}$ (marked with a blue box), or
$\{2\}$ (marked with a red box).

Each case implies $M_3' = \emptyset$. We hence obtain two minimal solutions (see Fig. 5(a)):

$(0.6, 0, 0.9, 0, 0)^{\mathrm{T}}$ (marked with a blue arrow and a black arrow), and
$(0, 0.6, 0.9, 0, 0)^{\mathrm{T}}$ (marked with a red arrow and a black arrow).

$$\Delta(x)=\begin{array}{c}\left(\begin{array}{ccccc} 0 & 0 & 0.9 & 0 & 0.9 \\ \hline 0.6 & 0.6 & 0 & 0 & 0.6 \\ 0.6 & 0.6 & 0 & 0 & 0 \\ 0 & 0.6 & 0.6 & 0 & 0 \\ \hline 0.2 & 0.2 & 0 & 0.2 & 0.2 \\ 0 & 0 & 0.2 & 0.2 & 0 \end{array}\right)\end{array}\begin{array}{l} M_1=\{1\} \\ \\ M_2=\{2,3,4\} \\ \\ M_3=\{5,6\} \end{array}$$

Fig. 3.

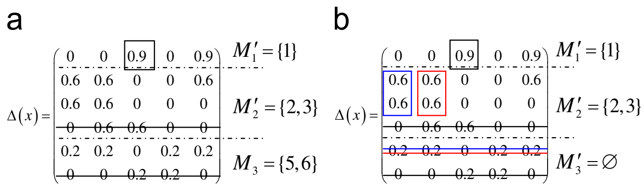

Fig. 4.

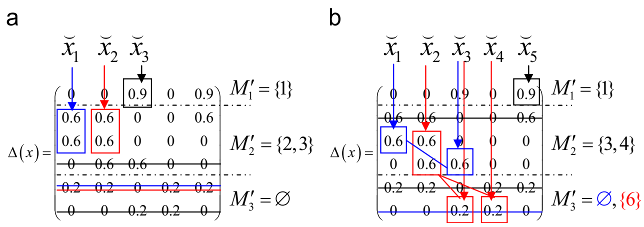

Fig. 5.

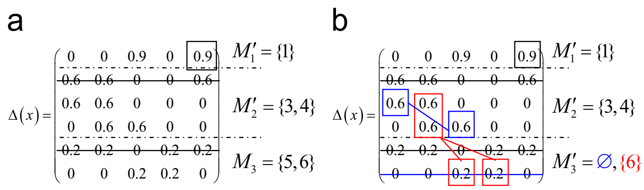

Fig. 6.

Case 2: $N_1=\{5\}$.
Delete the ith rows with $\Delta_{i5}(x)\neq 0$, we get

$$M_2'=\{i\in M_2|\ \Delta_{ij}(x)=0,\quad \text{for all } j\in N_1\}=\{i\in M_2|\ \Delta_{i5}(x)=0\}=\{3,4\},$$

as shown in Fig. 6(a). There are two minimal x-column coverings of M_2' (see Fig. 6(b)):

$$N_2=\{1,3\}\text{ (marked with blue boxes), or}$$
$$\{2\}\text{ (marked with a big red box).}$$

$N_2=\{1,3\}$ implies $M_3'=\emptyset$, we hence obtain a minimal solution (see Fig. 5(b)):

$$(0.6, 0, 0.6, 0, 0.9)^{\mathrm{T}}\quad\text{(marked with blue arrows and a black arrow).}$$

$N_2 = \{2\}$ implies

$$\begin{aligned} M_3' &= \{i \in M_3 | \varDelta_{ij}(x) = 0, \quad \text{for all } j \in N_1 \cup N_2\} \\ &= \{i \in M_3 | \varDelta_{i5}(x) = \varDelta_{i2}(x) = 0\} \\ &= \{6\}. \end{aligned}$$

Hence, there are two minimal x-column coverings of M_3': $N_3 = \{3\}$ or $\{4\}$ (see Fig. 6(b), marked with small red boxes). Each case produces a minimal solution. We hence obtain two minimal solutions:

$$(0, 0.6, 0.2, 0, 0.9)^{\mathrm{T}} \quad \text{and} \quad (0, 0.6, 0, 0.2, 0.9)^{\mathrm{T}}$$

(see Fig. 5(b), marked with red arrows and a black arrow). We conclude that there are five minimal solutions, as shown in Fig. 5.

6. Algorithms

In the previous section, Theorem 5.6 provides a computational method for determining all minimal solutions. If Eq. (2.1) has many minimal solutions, it will take a lot of time to find all minimal x-column coverings of M_t', $1 \leqslant t \leqslant l$, which is an NP-hard problem. In the following, we improve the computational method.

Lemma 6.1. *Let $M_1, \ldots, M_l$ be defined in Definition* 4.1, *x be a solution of Eq.* (2.1), *and $\mathcal{J}_t$ be the set of all minimal x-column coverings of M_t, $1 \leqslant t \leqslant l$. Assume that $N_1, \ldots, N_l$ are disjoint subsets of N such that N_t is a minimal x-column covering of M_t' for each $t \in L$, where $M_1' = M_1$ and*

$$M_t' = \left\{ i \in M_t | \ \varDelta_{ij}(x) = 0, \ \textit{for all } j \in \bigcup_{1 \leqslant s < t} N_s \right\}. \tag{6.1}$$

Then,

(1) *for each $J \in \mathcal{J}_t$, $t > 1$, $J - \bigcup_{1 \leqslant s < t} N_s$ is an x-column covering of M_t', and*
(2) *for each N_t, $t > 1$, there exists a $J \in \mathcal{J}_t$ such that $J - \bigcup_{1 \leqslant s < t} N_s = N_t$.*

Proof. (1) Let $J \in \mathcal{J}_t$, $t > 1$. By Eq. (6.1) and applying Lemma 5.2 to $N' = J$, $H = M_t$, $H' = M_t'$, and $S = \cup_{1 \leqslant s < t} N_s \cap J$, we get $J - \cup_{1 \leqslant s < t} N_s$ is an x-column covering of M_t'.

(2) Obviously, $N_1 \cup \cdots \cup N_t$ is an x-column covering of M_t. There hence exists a minimal x-column covering $J \subseteq N_1 \cup \cdots \cup N_t$, i.e. $J \in \mathcal{J}_t$. By statement (1), $J - \bigcup_{1 \leqslant s < t} N_s$ is an x-column covering of M_t'. Notice that,

$$J - \bigcup_{1 \leqslant s < t} N_s \subseteq N_t.$$

The assumption that, N_t is minimal, implies the above equality holds. This completes the proof. □

Let

$$\mathcal{J}_t^- := \left\{ K | K = J - \bigcup_{1 \leqslant s < t} N_s, \ J \in \mathcal{J}_t \right\}.$$

Lemma 6.1(1) implies that each $K \in \mathcal{J}_t^-$ is an x-column covering of M_t'. Lemma 6.1(2) shows that every N_t can be found in $\mathcal{J}_t^-$. Let $\mathfrak{N}_t$ denote the set of all minimal x-column coverings of M_t', so that $\mathfrak{N}_t \subseteq \mathcal{J}_t^-$. We now propose the following algorithm:

Algorithm 1. Input $A = [A_{ij}]$, $b = (b_i)$, and a solution $x = (x_j)$, where $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$.

Step 1: Let $M = \{i | b_i > 0\}$, $\bar{b}_1 > \bar{b}_2 > \cdots > \bar{b}_l$ be real numbers such that

$$\{\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_l\} = \{b_i | b_i > 0\},$$

and

$$M_t = \{i \in M | b_i = \bar{b}_t\}, \quad 1 \leqslant t \leqslant l,$$

$$\Delta_{\cdot j} = \{i \in M | A_{ij} \wedge x_j = b_i\}, \quad 1 \leqslant j \leqslant n.$$

Step 2: Let $\mathcal{J}_t$ be the set of all minimal x-column coverings of M_t, $1 \leqslant t \leqslant l$.

Step 3: Let $M'_1 = M_1$ and $\mathfrak{N}_1 = \mathcal{J}_1$. Execute the function **Stack**(1), and we will obtain the set Ω of all minimal solutions $\check{x}$ with $\check{x} \leqslant x$, where the function **Stack** is defined as follows:

Function Stack(t)
Case: $t < l$;
for each $N_t \in \mathfrak{N}_t$, do {
$M'_{t+1} = M_{t+1} - \bigcup_{1 \leqslant s \leqslant t} \bigcup_{j \in N_s} \Delta_{\cdot j}$;
if $M'_{t+1} = \emptyset$ then { $\mathfrak{N}_{t+1} = \{\emptyset\}$;}
else {
$\mathfrak{N}_{t+1} = \{\ \}$; (initially)
$\mathcal{J}^-_{t+1} = \{K | K = J - \bigcup_{1 \leqslant s \leqslant t} N_s,\ J \in \mathcal{J}_{t+1}\}$;
for each $K \in \mathcal{J}^-_{t+1}$, do { if K is minimal, then put K in $\mathfrak{N}_{t+1}$; }
}
Stack($t + 1$);
}

Case: $t = l$;
for each $N_l \in \mathfrak{N}_l$, do {

$$\check{x}_j = \begin{cases} \bar{b}_t & \text{if } j \in N_t, 1 \leqslant t \leqslant l, \\ 0 & \text{if } j \notin N_1 \cup \cdots \cup N_l, \end{cases}$$

put $\check{x} = (\check{x}_1, \ldots, \check{x}_l)$ in Ω;
}

To illustrate the above Algorithm 1, let's consider again Example 5.7.

Example 6.2 (*Continued*). Determine all minimal solutions of Example 5.7 by Algorithm 1.

Step 1: By Example 5.7, we obtain

$$M_1 = \{1\}, \quad M_2 = \{2, 3, 4\}, \quad M_3 = \{5, 6\}, \quad \text{and } \bar{b}_1 = 0.9, \quad \bar{b}_2 = 0.6, \bar{b}_3 = 0.2.$$

After computing $A_{ij} \wedge x_j$ for all i, j (see Fig. 3), we obtain

$$\Delta_{\cdot 1} = \{2, 3, 5\}, \quad \Delta_{\cdot 2} = \{2, 3, 4, 5\}, \Delta_{\cdot 3} = \{1, 4, 6\}, \Delta_{\cdot 4} = \{5, 6\}, \Delta_{\cdot 5} = \{1, 2, 5\}.$$

Step 2: The above M_t, $1 \leqslant t \leqslant 3$, and $\Delta_{\cdot j}$, $1 \leqslant j \leqslant 5$, imply

$$\mathcal{J}_1 = \{\{3\}, \{5\}\}, \quad \mathcal{J}_2 = \{\{1, 3\}, \{2\}\}, \quad \mathcal{J}_3 = \{\{1, 3\}, \{2, 3\}, \{3, 5\}, \{4\}\}.$$

Step 3: Obviously, $M'_1 = \{1\}$, $\mathfrak{N}_1 = \{\{3\}, \{5\}\}$, and then $N_1 = \{3\}$ or $\{5\}$. Now, execute **Stack**(1):
$N_1 = \{3\}$ implies

$$M'_2 = M_2 - \bigcup_{j \in N_1} \Delta_{\cdot j} = M_2 - \Delta_{\cdot 3} = \{2, 3\}.$$

Since $M'_2 \neq \emptyset$, compute $\mathcal{J}^-_2$ and $\mathfrak{N}_2$:

$$\mathcal{J}^-_2 = \{K | K = J - N_1,\ J \in \mathcal{J}_2\} = \{\{1\}, \{2\}\}.$$

Table 1

<table>
<tr><td colspan="6">$\Delta_{\cdot 1} = \{2,3,5\}, \Delta_{\cdot 2} = \{2,3,4,5\}, \Delta_{\cdot 3} = \{1,4,6\}, \Delta_{\cdot 4} = \{5,6\}, \Delta_{\cdot 5} = \{1,2,5\}$</td></tr>
<tr><td>$M_1 = \{1\}$
$\bar{b}_1 = 0.9$</td><td colspan="2">$M_2 = \{2,3,4\}$
$\bar{b}_2 = 0.6$</td><td colspan="2">$M_3 = \{5,6\}$
$\bar{b}_3 = 0.2$</td><td rowspan="2">minimal
solutions</td></tr>
<tr><td>$\mathcal{J}_1 : \{3\}, \{5\}$
$(\mathfrak{N}_1 = \mathcal{J}_1)$</td><td colspan="2">$\mathcal{J}_2 : \{1,3\}, \{2\}$</td><td colspan="2">$\mathcal{J}_3 : \{1,3\}, \{2,3\}, \{3,5\}, \{4\}$</td></tr>
<tr><td rowspan="2">$N_1 = \{3\}$
$(\check{x}_3 = 0.9)$</td><td rowspan="2">$M_2' = \{2,3\}$
$\mathcal{J}_2^- : \{1\}, \{2\}$
$\mathfrak{N}_2 : \{1\}, \{2\}$</td><td>$N_2 = \{1\}$
$(\check{x}_1 = 0.6)$</td><td colspan="2">$M_3' = \varnothing$
$\mathfrak{N}_3 = \{\varnothing\}$</td><td>(0.6,0,0.9,0,0)</td></tr>
<tr><td>$N_2 = \{2\}$
$(\check{x}_2 = 0.6)$</td><td colspan="2">$M_3' = \varnothing$
$\mathfrak{N}_3 = \{\varnothing\}$</td><td>(0,0.6,0.9,0,0)</td></tr>
<tr><td rowspan="3">$N_1 = \{5\}$
$(\check{x}_5 = 0.9)$</td><td rowspan="3">$M_2' = \{3,4\}$
$\mathcal{J}_2^- : \{1,3\}, \{2\}$
$\mathfrak{N}_2 : \{1,3\}, \{2\}$</td><td>$N_2 = \{1,3\}$
$(\check{x}_1 = \check{x}_3 = 0.6)$</td><td colspan="2">$M_3' = \varnothing$
$\mathfrak{N}_3 = \{\varnothing\}$</td><td>(0.6,0,0.6,0,0.9)</td></tr>
<tr><td rowspan="2">$N_2 = \{2\}$
$(\check{x}_2 = 0.6)$</td><td rowspan="2">$M_3' = \{6\}$
$\mathcal{J}_3^- : \{1,3\}, \{3\}, \{4\}$
$\mathfrak{N}_3 : \{3\}, \{4\}$</td><td>$N_3 = \{3\}$
$(\check{x}_3 = 0.2)$</td><td>(0,0.6,0.2,0,0.9)</td></tr>
<tr><td>$N_3 = \{4\}$
$(\check{x}_4 = 0.2)$</td><td>(0,0.6,0,0.2,0.9)</td></tr>
</table>

Each $K \in \mathcal{J}_2^-$ is a minimal x-column covering of M_2', since it has only one element. Hence, $\mathfrak{N}_2 = \mathcal{J}_2^- = \{\{1\}, \{2\}\}$, see Table 1. Next, execute **Stack**(2):

$N_2 = \{1\}$ implies

$$M_3' = M_3 - \bigcup_{j \in N_1 \cup N_2} \Delta_{\cdot j} = M_3 - \Delta_{\cdot 3} - \Delta_{\cdot 1} = \emptyset.$$

Hence, $\mathfrak{N}_3 = \{\emptyset\}$. Consequently, execute **Stack**(3):

Since $t = 3 = l$, the function **Stack** will run the case $t = l$. $N_1 = \{3\}$, $N_2 = \{1\}$, and $N_3 = \emptyset$ produce the minimal solution (0.6, 0, 0.9, 0, 0). Here ends **Stack**(3). Similarly, we may obtain another minimal solution (0, 0.6, 0.9, 0, 0) which is produced by $N_1 = \{3\}$, $N_2 = \{2\}$, and $N_3 = \emptyset$. Here ends **Stack**(2).

Now, run the case $N_1 = \{5\}$ in **Stack**(1). We compute

$$M_2' = M_2 - \bigcup_{j \in N_1} \Delta_{\cdot j} = M_2 - \Delta_{\cdot 5} = \{3, 4\}$$

and

$$\mathcal{J}_2^- = \{K | K = J - N_1, J \in \mathcal{J}_2\} = \{\{1, 3\}, \{2\}\} = \mathfrak{N}_2.$$

Next, execute **Stack**(2). $N_2 = \{1, 3\}$ implies $M_3' = \emptyset$, then we obtain the minimal solution (0.6,0,0.6,0,0.9), produced by $N_1 = \{5\}$, $N_2 = \{1, 3\}$, and $N_3 = \emptyset$. $N_2 = \{2\}$ implies

$$M_3' = M_3 - \Delta_{\cdot 5} - \Delta_{\cdot 2} = \{6\}$$

and

$$\mathcal{J}_3^- = \{K | K = J - N_1 - N_2, \; J \in \mathcal{J}_3\} = \{\{1, 3\}, \{3\}, \{4\}\}.$$

Notice that, the x-column covering $\{1, 3\}$ of M_3' is not minimal. Hence, $\mathfrak{N}_3 = \{\{3\}, \{4\}\}$. Each $N_3 \in \mathfrak{N}_3$ produces a minimal solution, see Table 1.

7. Conclusions

In the present paper, there are three types of covering definitions: (1) covering submatrix, (2) covering collection, and (3) x-column covering. We summarize these definitions and their relevant theorems as follows:

Covering submatrix: A submatrix T of $\Delta(\hat{x})$ is a *covering* iff for every $i \in M$, there is a $j \in N$ such that $T_{ij} = b_i$, where $\hat{x}$ is the maximal solution of Eq. (2.1).

- x is a solution $\Rightarrow$ the matrix representation $\Delta(x)$ is a covering submatrix (Lichung and Boxing [21, Theorem 5]).
- T is a covering submatrix $\Rightarrow$ the sup-vector $\upsilon(T)$ is a solution (Lichung and Boxing [21, Theorem 4]).

Covering collection: $\Im$ is a *covering collection* of H iff $H \subseteq \cup_{I \in \Im} I$.

- x is a solution
  $\Leftrightarrow x \leqslant \hat{x}$ and $\Im(x)$ is a covering collection of $M^-(x)$ (Lemma 3.3),
  $\Leftrightarrow x \leqslant \hat{x}$ and $\Im_t(x)$ is a covering collection of $M_t^-(x)$ for each $t \in L$ (Lemma 4.5).
- x is a minimal solution
  $\Leftrightarrow x \leqslant \hat{x}$ and $\Im(x)$ is a minimal covering collection of $M^-(x)$ (Theorem 3.4),
  $\Leftrightarrow x \leqslant \hat{x}$, $x_j \in \bar{B}$ for all $x_j > 0$, and $\Im_t(x)$ is a minimal covering collection of $M_t^-(x)$ for each $t \in L$ (Theorem 4.6).

x-column covering: A subset $N' \subseteq N$ is an *x-column covering* of H iff for every $i \in H$, there is a $j \in N'$ such that $\Delta_{ij}(x) = b_i$, where x is an arbitrary solution of Eq. (2.1).

- $\check{x}$ is a minimal solution with $\check{x} \leqslant x$
  $\Leftrightarrow$ there are disjoint subsets $N_1, \ldots, N_l$ of N such that N_t is a minimal x-column covering of M_t' for each $t \in L$, where

$$M_1' = M_1 \quad \text{and} \quad M_t' = \left\{ i \in M_t \,|\, \Delta_{ij}(x) = 0, \ \text{for all } j \in \bigcup_{1 \leqslant s < t} N_s \right\},$$

  and $\check{x}_j = \bar{b}_t$ if $j \in N_t$ for some $t \in L$, otherwise $\check{x}_j = 0$ (Theorem 5.6).

Up to now, we have assumed $b_i > 0$ for all $i \in M$. Let's remove the assumption now. Let

$$\tilde{M} := M - \{i \in M \,|\, b_i = 0\},$$

and $\tilde{M}_1, \ldots, \tilde{M}_l$ be the unique partition of $\tilde{M}$ such that

$$\begin{cases} b_i > b_{i'} & \text{for all } i \in \tilde{M}_s, i' \in \tilde{M}_t, 1 \leqslant s < t \leqslant l, \\ b_i = b_{i'} & \text{for all } i, i' \in \tilde{M}_t, 1 \leqslant t \leqslant l. \end{cases}$$

If the definitions of $M^-(x)$, $M_t^-(x)$, and M_t' are changed to

$$M^-(x) := \tilde{M} - \bigcup_{j \in N} K_j(x),$$

$$M_t^-(x) := \tilde{M}_t - \bigcup_{j \in N} K_j(x),$$

$$M_t' := \left\{ i \in \tilde{M}_t \,|\, \Delta_{ij}(x) = 0, \ \text{for all } j \in \bigcup_{1 \leqslant s < t} N_s \right\},$$

then the above statements still hold in general case.

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