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RELATION EQUATIONS IN RESIDUATED LATTICES

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In questa nota si affronta il problema della risoluzione di equazioni matriciali del tipo $AX = B$, dove A e B sono matrici a valori su un reticolo distributivo residuato rispetto a una moltiplicazione. In particolare, si individua la più grande soluzione di una tale equazione e si danno condizioni relative alle soluzioni minimali.

1. Introduction.

The study of equations over lattice-valued relations is a generalization of the classical theory of boolean equations, [1] [2] [3] [4]. In this note we shall be concerned with the problem of solving relation equations when the relations are valued on a lattice which is right-residuated under an isotone binary multiplication.

We extend in this framework the results stated by Luce.

In [2] he solved the equation $AX = B$ where A and B are boolean matrices and X is unknown. In [3] D. Rudeanu stressed that analogous results can be stated in a Brouwerian lattice. In this case also the greatest solution if found by E. Sanchez (see [4]).

In this note we also consider the set of solutions of the above equation.

2. Lattice relation equations.

Let L be a complete lattice. Assume L is equipped with a binary (multiplication) operation satisfying the following conditions: order-preservation i.e.

(1) $a \leq b \Rightarrow xa \leq xb$ and $ax \leq bx$ for every $a, b, x \in L$;

and right-residuation under this multiplication, i.e. for $a, b \in L$ there exists a largest x such that

(2) $$ax \leq b;$$

we shall denote such x by $a * b$.

If X is a nonempty set, $F(X) = \{A : X \to L\}$ is the set of L-sets (lattice valued sets) and if Y is another nonempty set, we define L-relation every element of $F(X \times Y)$.

Let $R(L)$ be the set of L-relations. Then in $R(L)$ we define a partial order and two partial multiplications as follows:

(3) for $A, B \in F(X \times Y)$ $A \leq B \Leftrightarrow A(x,y) \leq B(x,y)$ for every $(x,y) \in X \times Y$;

(4)
$$AB = C \Leftrightarrow C(x,z) = \bigvee_{y \in Y} (A(x,y)B(y,z))$$
$$(A \odot B) = D \Leftrightarrow \bigwedge_{y \in Y} (A(x,y) * B(y,z))$$

where $A \in F(X \times Y)$, $B \in F(Y \times Z)$.

Lastly for $A \in F(X \times Y)$ the L-relation A^{-1}, inverse of A, is defined by $A^{-1} \in F(Y \times X)$ and $A^{-1}(y,x) = A(x,y)$. From (1) and (4) it follows:

LEMMA 1. *Let $A \in F(X \times Y)$ and $B \in F(X \times Y)$ $A \leq B$. Then $AR \leq BR$ for every $R \in F(Y \times Z)$, and $R'A \leq R'B$ for every $R' \in F(Z \times X)$.* ❑

Let

$$AH \leq B \tag{5}$$

and

$$AH = B \tag{6}$$

be an L-relation inequation and an L-relation equation respectively, where $A \in F(X \times Y)$, $B \in F(X \times Z)$ and H is unknown.

Let now $I(A, B)$ and $S(A, B)$ be the set of solutions of (5) and (6), namely

$$I(A, B) = \{R \in F(Y \times Z)/AR \leq B\}$$

and

$$S(A, B) = \{R \in F(Y \times Z)/AR = B\}.$$

PROPOSITION 1. *$I(A, B)$ is nonempty and has a largest element.*

Proof. Let $M = A^{-1} * B$, we will show that $M \in I(A, B)$. For $(y, z) \in Y \times Z$:

$$M(y, z) = \bigwedge_{x \in X} (A(x, y) * (B(x, z)).$$

Hence from (1)

$$(AM)(x, z) = \bigvee_{y \in Y} (A(x, y)M(y, z) \leq \bigvee_{y \in Y} (A(x, y)(A(x, y) * B(x, z))) \leq$$
$$\leq \bigvee_{y \in Y} B(x, z) \quad \text{for every} \quad (x, z) \in X \times Z.$$

If $R \in I(A, B)$, then $AR \leq B$, therefore

$$(A(x, y)\, R(y, z)) \leq B(x, z) \quad \text{for every} \quad x \in X,\ y \in Y \quad \text{and} \quad z \in Z.$$

From (2) it follows that

$$R(y, z) \leq A(x, y) * B(x, z), \quad \text{for every} \quad x \in X,\ y \in Y,\ z \in Z.$$

Then

$$R(y,z) \leq \bigwedge_{x \in X} (A(x,y) * B(x,z)) = M(y,z). \qquad \square$$

THEOREM 1. *$S(A,B)$ is nonempty iff $M = A^{-1} * B \in S(A,B)$.*

Proof. Let $S(A,B) \neq \emptyset$. Then $R \in S(A,B) \subseteq I(A,B)$, implies $R \leq M$. From Lemma 1 it follows that

$$B = AR \leq AM. \tag{7}$$

From (7) and Proposition 1 it follows $M \in S(A,B)$.

The converse is obvious. $\square$

COROLLARY *If $S(A,B)$ is nonempty then $A^{-1} * B = \max S(A,B)$.*

Proof. Trivial. $\square$

PROPOSITION 2. *Let L be a complete lattice equipped with a binary multiplication satisfying (1) and (2). Then*

$$a \bigvee_{t \in T} x_t = \bigvee_{t \in T} (a x_t). \tag{8}$$

Proof. Let $\bar{x} = \bigvee_{t \in T} (a x_t)$. Then $a x_t \leq \bar{x}$, and hence $x_t \leq a * \bar{x}$ for all $t \in T$ and $\bigvee_{t \in T} x_t \leq a * \bar{x}$. From (1) we get

$$a \left(\bigvee_{t \in T} x_t \right) \leq a(a * \bar{x}) \leq \bar{x} = \bigvee_{t \in T} (a x_t). \tag{9}$$

We also have $a x_t \leq a \left(\bigvee_{t \in T} x_t \right)$ for all $t \in T$ and this implies

$$\bigvee_{t \in T} (a x_t) \leq a \bigvee_{t \in T} x_t. \tag{10}$$

From (9) and (10) the required identity (8) easily follows. $\square$

PROPOSITION 3. *$S(A, B)$ is a join semilattice.*

Proof. Indeed, let $R, R' \in S(A, B)$. Then for every $(x, z) \in X \times Z$ we have

$$\bigvee_{y \in Y} (A(x, y)R(y, z)) = B(x, z). \tag{11}$$

Further,

$$\bigvee_{y \in Y} (A(x, y)R'(y, z)) = B(x, z). \tag{12}$$

From (11) and (12) we obtain

$$\left(\bigvee_{y \in Y} (A(x, y)R(y, z)\right) \vee \left(\bigvee_{y \in Y} (A(x, y)R'(y, z))\right) =$$

$$= \bigvee_{y \in Y} ((A(x, y))R(y, z)) \vee (A(x, y)R'(y, z))) = B(x, z).$$

By Proposition 2 we have

$$\bigvee_{y \in Y} (A(x, y)(R(y, z) \vee R'(y, z))) = B(x, z)$$

which is equivalent to $R \vee R' \in S(A, B)$. ❑

PROPOSITION 4. *If R', $R'' \in S(A, B)$ and $R' \leq R \leq R''$ then $R \in S(A, B)$.*

Proof. Trivial from Lemma 1. ❑

3. Equations over finite relations.

From now on we suppose that $X = \{x_1, \ldots, x_n\}$ $Y = \{y_1, \ldots, y_m\}$ $Z = \{z_1, \ldots, z_p\}$ are arbitrary finite sets. Let $A \in F(X \times Y)$,

$R \in F(Y \times Z)$, $B \in F(X \times Z)$ be L-relations and $I_n = \{1, \dots, n\}$. For the sake of brevity we write $A(x_i, y_i) = a_{ij}$, $R(y_j, z_k) = r_{jk}$, $B(x_i, z_k) = b_{ik}$ for every $i \in I_n$, $j \in I_m$, $k \in I_p$.

Moreover for any $h \in I_p$ we denote by R_h and B_h the h^{th} column of R and B respectively. It is evident that $R_h \in F(Y \times \{z_h\})$ and B_h $F(X \times \{z_h\})$.

The problem of solving equation (6) is turned into the problem of solving p many equations of type

$$AH = B_h. \tag{13}$$

Then we can safely restrict to the case $p = 1$.

Every $R \in F(Y \times \{z\})$ will be denoted by $(r_j)_{j \in I_m}$ and every $B \in F(X \times \{z\})$ by $(b_i)_{i \in I_n}$.

Let $A = (a_{ij})$ $B = (b_i)$, $i \in I_n$, $j \in I_m$. For every matrix $w = (w_{ij})$ over L such that

$$\bigvee_{j \in I_m} w_{ij} = b_i \quad i \in I_n \tag{14}$$

we consider

$$H_{ij}^w = \{x / a_{ij} x = w_{ij}\}$$

and

$$H_j^w = \bigcap_{i \in I_n} H_{ij}^w.$$

THEOREM 2. *$S(A, B)$ is nonempty iff there exists a matrix (w_{ij}) satisfying (14) and such that $H_j^w \neq \emptyset$ for every $j \in I_m$.*

Proof. Let $S(A, B) \neq \emptyset$ and $R(r_1, \dots, r_m) \in S(A, B)$. Then $\bigvee_{j \in I_m} a_{ij} r_j = b_i$ for every $i \in I_n$. Set $w_{ij} = a_{ij} r_j$, the matrix (w_{ij}) satisfies (14), hence $r_j \in H_j^w$.

Conversely, let $w = (w_{ij})$ be a matrix satisfying (14) with $H_j^w \neq \emptyset$ for every $j \in I_m$. Then any L-relation $R(r_1, \dots, r_m)$, where $r_j \in H_j^w$ is obviously a solution of Equation (6). ❑

If we denote by $W(A,B)$ the set of the matrices $w = (w_{ij})$ satisfying (14) with $H_j^w \neq \emptyset$ for every $j \in I_m$, then the Theorem 2 characterizes the set of solutions of Equation (6) as

$$S(A,B) = \bigcup_{w \in W(A,B)} (H_1^w \times \ldots \times H_n^w).$$

Whenever $S(A,B)$ is nonempty, it has a largest element given by $A^{-1} * B$.

Observe that if $R = (r_1, \ldots, r_m) \in \bigcup_{w \in W(A,B)} (H_1^w \times \ldots \times H_m^w)$, then there exists a matrix w satisfying (14) such that $a_{ij} r_j = w_{ij} \leq b_i$ for every $i \in I_n$ and $j \in I_m$. It follows that $r_j \leq \bigwedge_{i \in I} (a_{ij} * b_i)$, for every $j \in I_m$, hence $R \leq (A^{-1} * B)$. We have still to prove that $(A^{-1} * B)$ belongs to $\bigcup_{w \in W(A,B)} (H_1^w \times \ldots \times H_m^w)$. To increase the readability we write $C_j = \bigwedge_{i \in I_n} (a_{ij} * b_i)$ for every $j \in I_m$.

Letting $w_{ij} = a_{ij} C_j$ and $R \in S(A,B)$, we have

$$\bigvee_{j \in I_m} a_{ij} C_j \geq \bigvee_{j \in I_m} a_{ij} r_j = b_i. \tag{15}$$

On the other hand

$$\bigvee_{j \in I_m} a_{ij} C_j \leq \bigvee_{j \in I_m} a_{ij}(a_{ij} * b_i) \leq \bigvee_{j \in I_m} b_i = b_i. \tag{16}$$

From (15) and (16) we obtain

$$\bigvee_{j \in I_m} w_{ij} = \bigvee_{j \in I_m} a_{ij} c_j = b_i$$

and, by definition of (w_{ij}), we finally obtain $C_j \in H_j^w$, for each $j \in I_m$.

Theorem 2 allows us to reduce the study of solutions of Equation (6) to the study of the set $S^w(A,B) = H_1^w \times \ldots \times H_m^w$, for every $w \in W(A,B)$.

Let us observe that if $S(A,B) \neq \emptyset$ then for every $j \in I_m$ H_j^w has the greatest element $m_j^w = \bigwedge_{i\in I_n} (a_{ij} * w_{ij})$ and hence $m^w = (m_j^w)_{j\in I_m}$ is the greatest element of $S^w(A,B)$.

PROPOSITION 4. *$S^w(A,B)$ is a join-semilattice, for every $w \in W(A,B)$.*

Proof. Trivial from Proposition 2. ❑

PROPOSITION 5. *For every $w \in W(A,B)$ $S^w(A,B)$ is a convex set.*

Proof. Let $P = (p_j)_{j\in I_m}$, $Q = (q_j)_{j\in I_m}$, $P \leq R = (r_j)_{j\in I_m} \leq Q$ and $P, Q \in S^w(A,B)$. Then for every $i \in I_n$, $j \in I_m$ we have the following chain of implications:

$$a_{ij}p_j \leq a_{ij}r_j \leq a_{ij}q_j \Rightarrow w_{ij} \leq a_{ij}r_j \leq w_{ij} \Rightarrow$$

$$a_{ij}r_j = w_{ij} \Leftrightarrow R \in S^w(A,B).$$ ❑

Let $w^* = (w_{ij}^*)$ where $w_{ij}^* = a_{ij}m_j$ for every $i \in I_n$ and $j \in I_m$.

COROLLARY *Let $R' \in S(A,B)$, $R \in S^{w*}(A,B)$ and $R \leq R'$. Then $R' \in S^{w*}(A,B)$.* ❑

PROPOSITION 6. *If for every $a, x, y \in L$ $a(x \wedge y) = ax \wedge ay$, then $s^w(A,B)$ is a lattice, for each $w \in W(A,B)$.*

Proof. Let $P = (p_j)_{j\in I_m}$, $Q = (q_j)_{j\in I_m}$ and $P, Q \in S^w(A,B)$ then for every $i \in I_n$

$$\bigvee_j a_{ij}(p_j \wedge q_j) = \bigvee_j (a_{ij}p_j \wedge a_{ij}q_j) = \bigvee_j w_{ij} = b_i,$$

so $(P \wedge Q) \in S^w(A,B)$.

From the above identity together with Proposition 4, we obtain the desired conclusion.

Let us remark that if L is a Brouwerian lattice, where xy is defined as $x \wedge y$ then for every $w \in W(A,B)$, $S^w(A,B)$ is a lattice.

4. Minimal solutions.

Let us observe that if the set $S(A, B)$ has minimal solutions, each of them is a minimal solution in the respective set $S^w(A, B)$ to which it belongs. So, the minimal solutions of $S(A, B)$, if they exist, are to be looked for among the minimal elements of the sets $S^w(A, B)$.

PROPOSITION 7. *Let* $(e_j^w)_{j \in I_m}$ *be a minimal element of* $S^w(A, B)$ *and* (w_{ij}) *a minimal element of* $W(A, B)$. *Then* $(e_j^w)_{j \in I_m}$ *is a minimal element of* $S(A, B)$.

Proof. Let $R = (r_1, \ldots, r_m) \in S(A, B)$ and $r_j \leq e_j^w$ for every $j \in I_m$. If $R \in S^w(A, B)$, then from (1) it is $w'_{ij} \leq w_{ij}$ for every $i \in I_n$ and $j \in I_m$. Because (w_{ij}) is a minimal element of $W(A, B)$ it follows that $w'_{ij} = w_{ij}$ for every $i \in I_n$ and $j \in I_m$. Then $R \in S^w$ and $r_j = e_j^w$ for any $j \in I_m$. ❑

COROLLARY *Let* (w_{ij}) *be a minimal element of* $W(A, B)$. *Then* $(e_j^w)_{j \in I_m}$ *is a minimal element of* $S^w(A, B)$ *if and only if it is a minimal element of* $S(A, B)$.

Proof. Trivial. ❑

One can wonder if the minimal solutions given by Proposition 7 and its Corollary exhaust the set of all minimal solutions of $S(A, B)$. Under the considered hypotheses we have no answer to this question. However, it is possible to give an affirmative answer under the additional assumption that multiplication distributes over $\wedge$ operation, i.e. for every $a, x, y \in L$

$$a(x \wedge y) = ax \wedge ay. \tag{17}$$

Assuming (17) we shall prove

PROPOSITION 8. *If* $(e_j^w)_{j \in I_m}$ *is a minimal element of* $S(A, B)$, *then it is a minimal element of* $S^w(A, B)$ *and* $w = (w_{ij})$ *is a minimal element of* $W(A, B)$.

Proof. The first part of the thesis is obvious, let us prove the second one. Let $w' = (w'_{ij}) \in W(A, B)$ and $w'_{ij} \leq w_{ij}$ for every $i \in I_n$

and $j \in I_m$. Let j_0 a fixed element of I_m. Then we have

$$b_i = \bigvee_{j \in I_m} w'_{ij} \leq w'_{ij_0} \vee \bigvee_{j \in I_m - \{j_0\}} w_{ij} \leq \bigvee_{j \in I_m} w_{ij} = b_i$$

for every $i \in I_n$, and

$$w'_{ij_0} \vee \bigvee_{j \in I_m - \{j_0\}} w_{ij} = b_i \quad \text{for every} \quad i \in I_n.$$

So the matrix $w'' = (w''_{ij})$ defined by

$$w''_{ij} = \begin{cases} w_{ij} & \text{if } j \in I_m - \{j_0\} \\ w'_{ij} & \text{if } j = j_0 \end{cases}$$

belongs to $W(A, B)$. $H_j^{w''} = H_j^{w}$ if $j \in I_m - \{i_0\}$ and $H_{j_0}^{w''} = H_{j_0}^{w'}$.

Furthermore, $x \in H_{j_0}^{w}$ and $y \in H_{j_0}^{w'}$ imply

$$a_{ij_0}(x \wedge y) = a_{ij_0}x \wedge a_{ij_0}\, y = w_{ij_0} \wedge w'_{ij_0} = w'_{ij_0}\ i \in I_n,$$

so $e_{j_0}^{w} \wedge r_{j_0} \in H_{j_0}^{w'}$ for every $r_{j_0} \in H_{j_0}^{w'}$. Then the relation $\bar{R} = (\bar{r}_1, \ldots, \bar{r}_m)$ defined as follows:

$$\bar{r}_j = \begin{cases} e_j^{w} & \text{if } j \in I_m - \{j_0\} \\ e_j^{w} \wedge r_{j_0} & \text{if } j = j_0 \text{ and } r_{j_0} \in H_{j_0}^{w'} \end{cases}$$

belongs to $S^{w''}(A, B) \subseteq S(A, B)$. It follows that $(\bar{r}_j) = (e_j^{w})$ because $(\bar{r}_j) \leq (e_j^{w})$ and (e_j^{w}) is a minimal element of $S(A, B)$. Thus we have $e_{j_0}^{w} \leq r_{j_0}$ and $w_{ij_0} \leq w'_{ij_0}$ i.e. $w_{ij_0} = w'_{ij_0}$ for every $i \in I_n$. Since j_0 is arbitrary, we immediately obtain the desired conclusion. ❑

COROLLARY *Let L be a complete lattice satisfying (1), (2) and (17), $R = (e_1^{w}, \ldots, e_m^{w}) \in S^{w}(A, B)$. Then R is a minimal element of $S(A, B)$ if and only if it is a minimal element of $S^{w}(A, B)$ and $w = (w_{ij})$ is a minimal element of $W(A, B)$.*

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