Solving the Minimal Solutions of Max-min Fuzzy Relation Equation by Graph Method and Branch Method

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***Abstract*—The problem of solving max-min fuzzy relation equation is further discussed in this paper. The main matter of solving max-min fuzzy relation equation is to find its whole minimal solutions. Based on the analysis of the characteristic of minimal solution and previous results, this paper transforms the problem of seeking the whole minimal solutions into constructing a bipartite fuzzy graph and finding the various covering manners for all equations, at the same time an algorithm of generating the whole minimal solutions is given. Furthermore, to remove useless computations in the process of solving the equation, a new method, which is called branch method, is given. Both the two methods are simple, explicit for using, and are helpful for understanding the feature of the equation's minimal solution. Lastly, an example is given to illustrate the application of the two methods.**

***Keywords*—max-min fuzzy relation equation; minimal solution; bipartite fuzzy graph; branch method**

Ⅰ. INTRODUCTION

Fuzzy relation equations have been used extensively in fuzzy control, fuzzy reasoning, fuzzy logic, etc [1-4]. It was Sanchez who first introduced the concept of fuzzy relation equations and began to study them. Since then, many researchers have studied the equations extensively with many methods obtained to solve them [5-8], but these approaches usually are too complex and difficult to use. Based on the characteristic of minimal solutions of max-min relation equation and the knowledge of graph, this paper transforms the problem from finding the minimal solutions of the equation into searching the various covering manners for all equations by constructing a bipartite fuzzy graph. Besides, it puts forward branch method of obtaining the whole minimal solutions by branch and bound method in operations research to remove useless computations in the process of solving the equation. Both the two methods are simple for using and helpful for understanding the minimal solution of max-min relation equation.

Ⅱ. SOME DEFINITIONS AND PREVIOUS RESULTS

Firstly, we give some well-known definitions and previous results for the sake of convenience. They come from literature [3] mainly except Theorem 5, which is new.

Fuzzy relation equation is

$$A \circ X = B$$

where $A=(a_{ij})_{m\times n}$, $X=(x_1,x_2,\ldots,x_n)^T$ is unknown, $B=(b_1,b_2,\ldots,b_m)^T$, a_{ij}, x_j, $b_i\in[0,1]$, ($i=1,2,\ldots,m$; $j=1,2,\ldots,n$), "$\circ$" is the max-min ($\vee,\wedge$) composition.

We need also introduce two operators α and β. Let a, $b\in[0,1]$, define

$$a\alpha b=\begin{cases} b, & a>b \\ 1, & a\le b\end{cases}, \quad a\beta b=\begin{cases} b, & a\ge b \\ 0, & a<b\end{cases}.$$

Theorem 1: The necessary and sufficient condition of fuzzy vector $X=(x_1,x_2,\ldots,x_n)^T$ is the solution of equation $A\circ X=B$ is: to any i, j there is $a_{ij}\wedge x_j\le b_i$ and to any i there exists a j satisfying $a_{ij}\wedge x_j = b_i$.

Theorem 2: The necessary and sufficient condition of fuzzy relation equation $A\circ X=B$ has solutions is that $X^*=(x_1^*,x_2^*,\ldots,x_n^*)^T$ is its maximum solution, where $x_j^*=\mathop{\wedge}\limits_{i=1}^{m}(a_{ij}\alpha b_i)$, $j=1,2,\ldots,n$.

Theorem 2 gives a judging theorem of solvability of equation $A\circ X=B$ and a method of obtaining its maximum solution. The method is finding X^* firstly, then testing whether it is the solution of $A \circ X = B$. If X^* is the solution of $A \circ X = B$, then the equation has solutions and X^* is its maximum solution. Otherwise, the equation has not solution.

X^* can be obtained by doing $a_{ij} \alpha b_i$ operation to every a_{ij}, then taking the minimum to every column, which is very simple.

Let S is the solution set of $A \circ X = B$, take $X=(x_1,x_2,\ldots,x_n)^T$, $Y=(y_1,y_2,\ldots,y_n)^T \in S$ arbitrarily, denote $X \le Y$ if and only if $x_1 \le y_1$, $x_2 \le y_2$,…, $x_n \le y_n$, then $\langle S, \le \rangle$ is a partially ordered set. When $A \circ X = B$ is solvable, it has one maximum solution and a finite number of minimal solutions. Let $X^{(0)} = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})^T$ is an arbitrary minimal solution, then any vector X such that $X^{(0)} \le X \le X^*$ is also the solution of $A \circ X = B$, hence the problem of finding the solution set of $A \circ X = B$ is converted into seeking the maximum solution and the whole minimal solutions of the equation. Since the problem of finding the maximum solution of $A \circ X = B$ has been solved, the issue of searching the whole solutions of $A \circ X = B$ is transformed into seeking its all minimal solutions.

Theorem 3: To fuzzy relation equation $A \circ X = B$, let $X^* = (x_1^*, x_2^*, \ldots, x_n^*)$ be its maximum solution, denote $a_{ij}^* = x_j^* \beta (a_{ij} \beta b_i)$, $i=1,2,\ldots,m$; $j=1, 2, \ldots, n$, then there are:

1) $\forall i, \exists j$ such that $a_{ij}^* = b_i$;

2) a_{ij}^* is 0 or b_i, so its maximum must be b_i.

In fact, denote $A^* = (a_{ij}^*)_{m \times n}$, then A^* is the alternative matrix or invariant matrix of A in literatures[5-6], and $A \circ X = B$ has the same minimal solution set with $A^* \circ X = B$.

Theorem 4: To equation $A \circ X = B$, let $X^* = (x_1^*, x_2^*, \ldots, x_n^*)$ be its maximum solution, denote $a_{ij}^* = x_j^* \beta (a_{ij} \beta b_i)$ ($i=1,2,\ldots,m$; $j=1,2,\ldots,n$), $I=\{1,2,\ldots,m\}$, $I_0=\varnothing$, when I_1, I_2, …, I_{t-1} have been determined, denote $I_t^* = I - \bigcup_{s=0}^{t-1} I_s$, $b_{i_t} = \max_{i \in I_t^*}\{b_i\} = a^*_{i_t j_t}, t = 1, 2, \cdots, k$, $I_t = \{i \mid a_{i j_t}^* = b_i, i \in I^*_t\}$, where $1 \le k \le m$ is the minimum positive integer satisfying $I^*_{k+1} = \varnothing$, then

1) $I = \bigcup_{t=1}^{k} I_t$, when $s \ne t$, $I_t \cap I_s = \varnothing, i_t \in I_t$;

2) $b_{i_1} \ge b_{i_2} \ge \cdots \ge b_{i_k}, b_{i_t} = \max_{i \in I_t}\{b_i\}$;

3) Denote $J=\{j_1,j_2,\ldots,j_k\}$, $X^{(0)} = (x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)})^T$ and

$$x_j^{(0)} = \begin{cases} b_{i_t}, \ j \in J \text{ and } j = j_t \\ 0, \qquad j \notin J \end{cases}$$

, then $X^{(0)}$ is a minimal solution of $A \circ X = B$.

To $A \circ X = B$, b_i in B can be zero, but b_i which is not equal to zero has the more representation and universality, so for the sake of convenience of studying the minimal solutions we give a new theorem.

Theorem 5: To $A \circ X = B$, let $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$, $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$, $A_1 \circ X = B_1$, $A_2 \circ X = B_2$, and to any $b_i \in B_1$ there is $b_i > 0$, $B_2 = 0$, then if X is the minimal solution of $A \circ X = B$, it must be the minimal solution of $A_1 \circ X = B_1$; if X is the minimal solution of $A_1 \circ X = B_1$ and the solution of $A_2 \circ X = B_2$, it must be the minimal solution of $A \circ X = B$.

Proof: If X is the minimal solution of $A \circ X = B$, it is the solution of $A_1 \circ X = B_1$ and $A_2 \circ X = B_2$ obviously; if it is not the minimal solution of $A_1 \circ X = B_1$, there should exist an X_* such that $X_* \le X, X_* \ne X$ and $A_1 \circ X_* = B_1$. By $0 \le A_2 \circ X_* \le A_2 \circ X = 0$ we know that X_* is the solution of $A_2 \circ X = B_2$, so X_* is the solution of $A \circ X = B$, which is a contradiction to the minimality of X . If X is the minimal solution of $A_1 \circ X = B_1$ and the solution of $A_2 \circ X = B_2$, X must be the solution of $A \circ X = B$. Again, based on that X is the minimal solution of $A_1 \circ X = B_1$ we know that there will not exist an X_* such that $X_* \le X, X_* \ne X$ and it is the solution of $A \circ X = B$. The proof is completed.

Theorem 5 indicates that when finding the minimal solutions of $A \circ X = B$, we can search the minimal solutions of equations formed by equations whose b_i are not equal to zero in $A \circ X = B$, then test the solutions to equations whose b_i are equal to zero. If the equations hold, the solutions is the minimal solutions of $A \circ X = B$ .So for the sake of simplification, we can suppose that all b_i in B are not zero in the following discussion.

Ⅲ. ANALYSIS ON THE CHARACTERISTIC OF MINIMAL SOLUTION

Theorem 4 gives us a method of obtaining the minimal solutions of $A \circ X = B$, whose principle is similar to the methods of effective path and conservative path [5], but has some improvements. In fact, minimal solution implies that the solution should satisfy every equation of $A \circ X = B$, at the same time every element in it should be as small as possible, which means that non-zero elements in it should be as few and small as possible. By Theorem 1 the fewer non-zero elements means that every non-zero element of minimal solution $X^{(0)}$ should such that formula $a_{ij} \wedge x_j^{(0)} = b_i$ as many as possible. So when $A \circ X = B$ is solvable and we consider its minimal solutions we should do as this: suppose $\max\limits_{1 \le i \le m}\{b_i\} = b_{i_1}$, since minimal solution must also satisfy the equation corresponding to b_{i_1} , there must exist j_1 satisfying when $x_{j_1}^{(0)}$ takes b_{i_1} there is $a_{i_1 j_1} \wedge x_{j_1}^{(0)} = b_{i_1}$ and to any i there is $a_{ij_1} \wedge x_{j_1}^{(0)} \le b_i$. But $x_{j_1}^{(0)}$ takes b_{i_1} not only can make the equation of i_1 , which is the i_1-th row of $A \circ X = B$, satisfying $a_{i_1 j_1} \wedge x_{j_1}^{(0)} = b_{i_1}$, but also can make some other equations satisfying $a_{ij_1} \wedge x_{j_1}^{(0)} = b_i$, to others there is $a_{ij_1} \wedge x_{j_1}^{(0)} < b_i$ .Since minimal solution should make every equation of $A \circ X = B$ existing an item of $a_{ij} \wedge x_j^{(0)} = b_i$, we should select j_2 so as to make some of the other equations satisfying $a_{ij_2} \wedge x_{j_2}^{(0)} = b_i$, in which the method is the same. Continuous doing as this, when obtaining $j_1, j_2, \ldots, j_k$ which can make to every i there is $1 \le t \le k$ satisfying $a_{ij_t} \wedge x_{j_t}^{(0)} = b_i$, we should make the remained $x_j^{(0)}$ all take zero. By this way the solution obtained will be as small as possible in general meaning, and this just is the principle and method of obtaining the minimal solution in Theorem 4.

Ⅳ. BIPARTITE FUZZY GRAPH AND GRAPH METHOD

Although Theorem 4 gives an approach of finding the minimal solutions of $A \circ X = B$, it is too complex, and not intuitive. To overcome the defect, we can construct a bipartite fuzzy graph by graph theory[3][9] to generate the minimal solutions of equation $A \circ X = B$. The method of constructing the bipartite fuzzy graph G=<$X,Q;R$> is as this: X still is the set of every unknown element x_j (j=1,2,…,n), Q is the set of every equation q_i(i=1,2,…,m) , which represents the q_i-th row of $A \circ X = B$. Since $A^* \circ X = B$ has the same minimal solutions with $A \circ X = B$, and A^* is not only simpler than A, but also contains the information of A and B, we let R be A^*. By the definition of fuzzy graph, when drawing the bipartite fuzzy graph, we just need to connect x_j with q_i to every $a_{ij}^* \ne 0$ in $A^* = (a_{ij}^*)_{m \times n}$, and denote the number of a_{ij}^*, which must be b_i by Theorem 3, on the line.

By the characteristic of minimal solution and the content of Theorem 4, seeking minimal solution is to choosing some non-zero elements $x_{j_t}^{(0)}$ as few and small as possible satisfying to every equation there is item $a_{ij_t} \wedge x_{j_t}^{(0)} = b_i$, which is reflected in graph G is choosing some non-zero elements $x_{j_t}^{(0)}$ as few and small as possible to cover all q_i, so we can have a graph method of getting the solution set of equation $A \circ X = B$ by finding its minimal solutions as following:

Step 1. Calculate X^*, judge whether X^* is the solution of $A \circ X = B$. If X^* is not solution, say $A \circ X = B$ has not solution and stop; otherwise, computer $A^* = (a_{ij}^{\ *})_{m \times n}$, construct the bipartite fuzzy graph G by above method, go to the next step.

Step 2. Select the maximum b_i on all lines of the graph, which is denoted as b_{i_1}, to the x_j on the line of b_{i_1}, let $x_j^{(0)}$ take b_{i_1}, which can make every equation q_i connecting to x_j satisfying $a_{ij} \wedge x_j^{(0)} = b_i$ by Theorem 4, tick these q_i ,which also means that $x_j^{(0)}$ taking b_{i_1} covers these q_i. If there are more than one b_i whose values are the maximum of B ,hence there are more than one x_j on the lines of these b_{i_1}, select an x_j arbitrarily, but should indicate that there are other possible choosing for x_j.

Step 3. If there are q_i not ticked, to these remained q_i, go to Step 2; otherwise, go to the next step.

Step 4. If all equations q_i have been ticked or covered, let $x_j^{(0)}$ which have not been assigned values take zero, then obtain a minimal solution.

Step 5. If there are other possible selection manners for those non-zero $x_j^{(0)}$ in above process, erase the signs of all ticks, go to Step 2 and repeat above course to the non-zero $x_j^{(0)}$ which have other possible combining selecting ways respectively, so as to get other minimal solutions.

Step 6. Compounding every minimal solution with the maximum solution to obtain the solution set of $A \circ X = B$.

It is not difficult to find that this method is the reflecting of Theorem 4, so it is correct. Using the graph method to obtain the minimal solution of $A \circ X = B$, it is simple, clear and intuitive, can help us understanding the feature of minimal solution.

V. BRANCH METHOD

However, Theorem 4 has some problems. The minimal solution obtained by Theorem 4 should be called quasi-minimal solution [2], which means that it usually is a minimal solution. Whereas, when there are some b_i whose number are same and to a same b_i there are more than one x_j can be chosen in graph method, the minimal solution are often obtained repeatedly, and in some cases the quasi-minimal solution obtained really is not a minimal solution. These situations obvious add the amount of computation and the difficult degree. But using this method, all minimal solutions must be contained in the whole quasi-minimal solutions.

To obtain the whole minimal solutions of equation $A \circ X = B$ not repeatedly, we could improve above method. Consulting the branch and bound method in operations research to solve Integer programming[10], we give a branch method of generating the solution set of equation $A \circ X = B$ by finding its minimal solutions.

Step 1. Calculate X^*, judge whether X^* is the solution of $A \circ X = B$. If X^* is not solution, say $A \circ X = B$ has not solution and stop; otherwise, computer $A^* = (a_{ij}^{\ *})_{m \times n}$, construct its bipartite fuzzy graph G, , go to the next step.

Step 2. Choose the maximum b_i on all lines, which is denoted as b_{i_1}, to the x_j on the line of b_{i_1}, let $x_j^{(0)}$ take b_{i_1}. If there are more than one b_i whose values are the maximum of B ,hence there are more than one x_j on the lines of these b_{i_1}, branch by the number of these x_j. Of course, all these $x_j^{(0)}$ should take b_{i_1}. Furthermore, denote every equation q_i connecting to x_j aside.

Step 3. To every branch, if all $x_j^{(0)}$ in a branch has been found on other branch repeatedly indifferent to the order and their number are also identical, this means that the solutions obtained by this branch is the same as the other, cut this branch. To the remained branch, if there are q_i not denoted, to these q_i, go to Step 2; otherwise, go to the next step.

Step 4. To every reserved branch, if the solution of a branch is bigger than another one, this branch also should be cut, then let all $x_j^{(0)}$ which have not been assigned values take zero, all minimal solutions are obtained.

Step 5. Compounding every minimal solution with the maximum solution to obtain the solution set of $A \circ X = B$.

Ⅵ. EXAMPLE

Following we use an example to illustrate the application of these two methods. Here the fuzzy relation equation is:

$$\begin{pmatrix} 0.9 & 0.8 & 0.6 & 0.3 & 0.9 \\ 0.8 & 0.7 & 0.8 & 1 & 0.8 \\ 0.6 & 0.9 & 0.8 & 0.9 & 0.5 \\ 0.4 & 0.2 & 0.5 & 0.6 & 0.2 \\ 0.3 & 0.3 & 0.5 & 0.2 & 0.1 \\ 0.4 & 0.1 & 0.2 & 0.3 & 0.5 \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.8 \\ 0.8 \\ 0.5 \\ 0.5 \\ 0.4 \end{pmatrix}$$

This equation has much representative meaning. We can easy obtain the maximum solution of the equation is $X^* = (0.8\ \ 0.8\ \ 1\ \ 0.5\ \ 0.4)^T$,

$$A^* = \begin{array}{c} \begin{matrix} x_1 & \ x_2 & \ x_3 & \ x_4 & \ x_5 \end{matrix} \\ \begin{pmatrix} 0.8 & 0.8 & 0 & 0 & 0 \\ 0.8 & 0 & 0.8 & 0 & 0 \\ 0 & 0.8 & 0.8 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0.4 & 0 & 0 & 0 & 0.4 \end{pmatrix} \end{array} \begin{matrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{matrix},$$

In above A^*, to have a good look I add X and Q beside it, which usually does not need. The bipartite fuzzy graph constructed by A^* is shown in Fig.1.

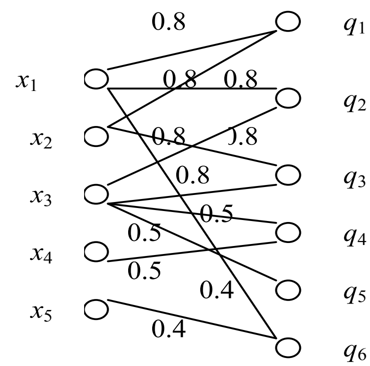

Figure1. Fuzzy graph obtained by example

We use the graph method firstly. Obviously the maximum b_i on all lines is 0.8 and x_1 is connected to one of the lines, so we choose $x_1^{(0)}$=0.8. There are equations q_1,q_2,q_6 connecting to x_1, this means that $x_1^{(0)}$=0.8 can make equations q_1,q_2,q_6 hold, ticking them. The maximum b_i in the remaining lines still is 0.8 and x_2,x_3 connected to it, we let $x_2^{(0)}$=0.8 and tick q_3. Continuous doing, $x_3^{(0)}$ will take 0.5, equations q_4, q_5 will be ticked. Now all equations have been ticked, let all remained $x_j^{(0)}$ take zero we have a minimal solution $X^{(0)}$=(0.8,0.8,0.5,0,0)^T. Repeating this course to above process that $x_j^{(0)}$ have other possible selecting combining manners, we obtain other quasi-minimal solutions as following: (0.8,0.8,0.5,0.5,0)^T, (0.8,0,0.8,0,0)^T, (0,0.8,0.8,0,0.4)^T. Obviously (0.8,0.8,0.5,0.5,0)^T is not a minimal solution, so there are three minimal solutions only, which are (0.8,0.8,0.5,0,0)^T, (0.8,0,0.8,0,0)^T, (0,0.8,0.8,0,0.4)^T. Here the process of compounding every minimal solution with the maximum solution to obtain the solution set of the equation is omitted.

We use branch method continuously. Since the maximum b_i on all lines is 0.8 and there are three x_i connecting to 0.8, which are x_1,x_2,x_3, we make branch by them. We continuous doing as it to the three branches and cutting branches which are repeated to other branch or a quasi- minimal solution is bigger than other one, the process is shown in Fig.2:

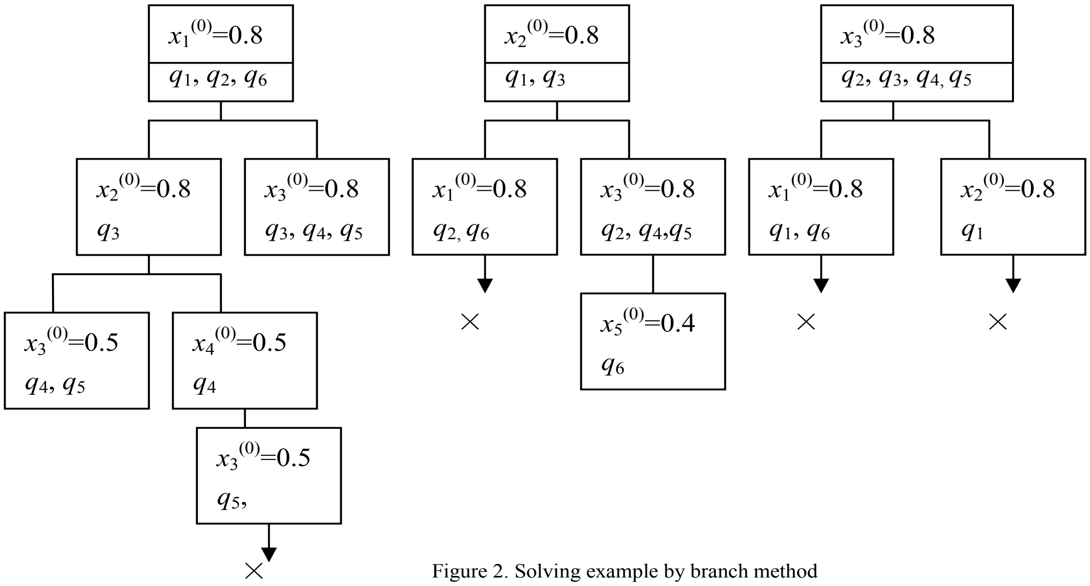

Figure 2. Solving example by branch method

By Fig. 2 we know the equation has three minimal solutions, which are $(0.8,0.8,0.5,0,0)^T$, $(0.8,0,0.8,0,0)^T$, $(0,0.8,0.8,0,0.4)^T$ and it is the same as graph method.

Ⅶ. CONCLUSIONS

Based on previous literatures，through analyzing the characteristic of minimal solution of max-min fuzzy relation equation $A \circ X = B$ ， this paper puts forward the graph method by constructing a bipartite fuzzy graph to generate the minimal solutions of the equation and branch method to reduce the computation of obtaining the minimal solutions. Both the two methods are simple and intuitive for using and helpful for understanding the minimal solutions. When obtaining the minimal solutions, if the number of branch is small and they appear late, we should use the graph method straightly; if the number of branch is big and they appear early, we should use the branch method. Furthermore, the two methods also provide new approaches of solving other types of fuzzy relation equations.

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