2015级一元函数积分（信）

一、C B A C D

二、1. $\frac{2}{3}$

2. $\frac{\pi}{3}$

3. 5

4. $-e^{-x}+e^{2y-x}+2x-4y$

5. $\cos x^2$

三. 1. $$\int \frac{x^2}{(x-1)^7}\,dx \xlongequal{t=x-1} \int \frac{t^2+2t+1}{t^7}\,dt$$

$$= -\frac{1}{4}t^{-4} - \frac{2}{5}t^{-5} - \frac{1}{6}t^{-6} + C$$

$$= -\frac{1}{4}(x-1)^{-4} - \frac{2}{5}(x-1)^{-5} - \frac{1}{6}(x-1)^{-6} + C$$

2. $$\int \frac{x}{x^2+2x+5}\,dx$$

$$= \frac{1}{2}\int \frac{d(x^2+2x+5)}{x^2+2x+5} - \int \frac{d(x+1)}{(x+1)^2+2^2}$$

$$= \frac{1}{2}\ln(x^2+2x+5) - \frac{1}{2}\arctan\frac{x+1}{2} + C$$

3. $\int \frac{\arctan x}{x^2} dx = -\int \arctan x \, d\frac{1}{x}$

$$= -\frac{1}{x}\arctan x + \int \frac{dx}{x(1+x^2)}$$

$$= -\frac{1}{x}\arctan x + \frac{1}{2}\int \left(\frac{1}{x^2} - \frac{1}{1+x^2}\right) d(x^2)$$

$$= -\frac{1}{x}\arctan x + \frac{1}{2}\ln \frac{x^2}{1+x^2} + C$$

四. 1. $\int_{\frac{\sqrt{2}}{2}}^{1} \frac{\sqrt{1-x^2}}{x^2} dx$

$$\underline{\underline{x=\sin t}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^2 t}{\sin^2 t} dt$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\csc^2 t - 1) dt$$

$$= -\cot t \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \frac{\pi}{4} = 1 - \frac{\pi}{4}$$

2. $\int_0^1 \ln(1+\sqrt{x})\,dx$

$$= x\ln(1+\sqrt{x})\Big|_0^1 - \int_0^1 \frac{\sqrt{x}}{2(1+\sqrt{x})}\,dx$$

$$\xlongequal{t=1+\sqrt{x}} \ln 2 - \int_1^2 \frac{(t-1)^2}{t}\,dt$$

$$= \ln 2 - \left(\frac{1}{2}t^2 - 2t + \ln t\right)\Big|_1^2$$

$$= \frac{1}{2}$$

3. 令 $I = \int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\cos x + \sin x}\,dx$

$$\xlongequal{t=\frac{\pi}{2}-x} -\int_{\frac{\pi}{2}}^{0} \frac{\sin^3 t}{\sin t + \cos t}\,dt = \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\cos x + \sin x}\,dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{\cos^3 x + \sin^3 x}{\cos x + \sin x}\,dx = \int_0^{\frac{\pi}{2}} (1 - \sin x\cos x)\,dx$$

$$= \frac{\pi}{2} - \frac{1}{2}\sin^2 x\Big|_0^{\frac{\pi}{2}} = \frac{\pi-1}{2} \quad \Rightarrow I = \frac{\pi-1}{4}$$

五. $\because \lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{\sqrt{|xy|}}{x^2+y^2}\sin(x^2+y^2)$

$= \lim\limits_{\substack{x\to 0\\ y\to 0}} \sqrt{|xy|} = 0 = f(0,0)$

$\therefore$ $f(x,y)$ 在点 $(0,0)$ 处连续

$f_x(0,0) = \lim\limits_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0} = 0$

同理, $f_y(0,0)=0$

$\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{f(\Delta x,\Delta y)-f(0,0)-f_x(0,0)\Delta x - f_y(0,0)\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}$

$= \lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\sqrt{|\Delta x\Delta y|}}{\sqrt{\Delta x^2+\Delta y^2}} \frac{\sin(\Delta x^2+\Delta y^2)}{\Delta x^2+\Delta y^2}$

$= \lim\limits_{\Delta x\to 0\ \Delta y\to 0} g(\Delta x,\Delta y)$

$$\because \lim_{\substack{\Delta x \to 0 \\ \Delta y = 0}} g(\Delta x, \Delta y) = 0$$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y = \Delta x}} g(\Delta x, \Delta y) = \frac{1}{2} \neq 0$$

$$\therefore \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} g(\Delta x, \Delta y) \text{ 不存在}$$

即 $f(x,y)$ 在 $(0,0)$ 处不可微

六 证明：令 $F(x)=e^{1-x^4}f(x)$

由积分中值定理，$\exists a\in(0,1)$，$F(1)=F(a)$

由罗尔定理，$\exists \xi\in(0,a)\subset(0,1)$，$F'(\xi)=0$

即 $f'(\xi)-4\xi^3 f(\xi)=0$

七. 证明：易证 $g(t)=\frac{t}{a}+\frac{b}{t}$ 在 $[a,b]$ 上

的最大值为 $g(a)=g(b)=1+\frac{b}{a}$

左式 $=\int_0^1\left[\frac{f(x)}{a}+\frac{b}{f(x)}\right]dx$

$\leq\int_0^1(1+\frac{b}{a})dx$

$=1+\frac{b}{a}$