C&O URA Spring 2017

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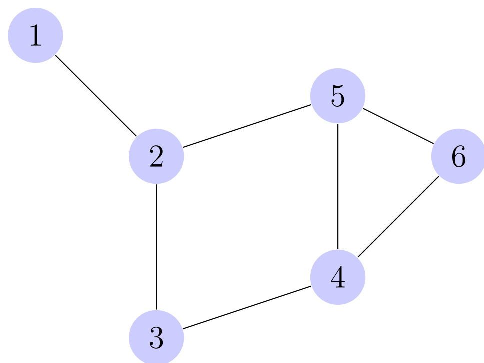
Contents

1 Inertia Bounds

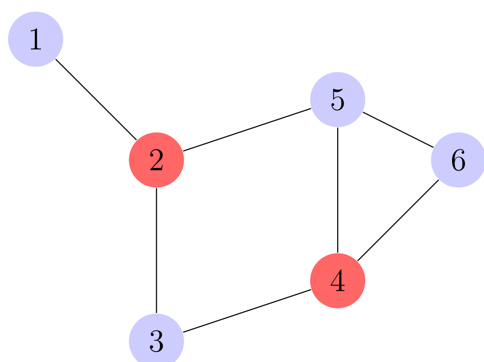
1.1 Introduction on Inertia Bounds

1.1 Definition.
Independent Set — An independent set is a set of vertices belonging to a graph in which no two vertices are adjacent.

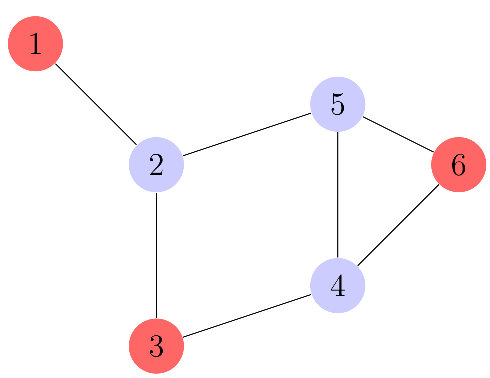
1.1 Example.
Consider the following graph:



An example of an independent set in this graph is:



However, often the independent set we are most interested in finding is the largest one:



1.2 Definition.
Independence Number — The independence number of a graph G, denoted $\alpha(G)$, is the size of the largest independent set of G.

1.3 Definition.
Weight Matrix — The weight matrix of a graph G, is a matrix defined by:

$$W_{i,j} = \begin{cases} c_{i,j} & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

with v_i a vertice of G and $c_{i,j}$, a constant.

The weight matrix of a graph, is identical to an adjacency matrix, except where there was a 1 in the matrix at entry $A_{i,j}$ if vertices v_i and v_j were adjacent, there is now a constant indicating a weighting for the edge between v_i and v_j.

For any graph G, there exists a bound on $\alpha(G)$, known as the Cvetković bound (also referred to as the Interia Bound). This bound provides a relationship between $\alpha(G)$ and the number of positive, negative, and zero eigenvalues of the weight matrix, W, associated with G. The Cvetković bound of G, is:

$$\alpha(G) \leq \min\{|G| - n_+(W), |G| - n_-(W)\} \tag{2}$$

Where $n_+(W)$ and $n_-(W)$ denote the number of positive and negative eigenvalues of W, respectively.

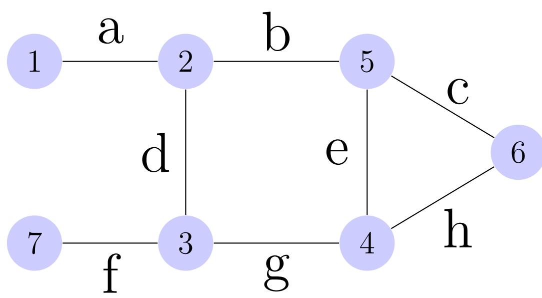
To prove this, we first need to introduce a result that comes from the Eigenvalue Interlacing Theorem:

1.1 Theorem.
Corollary of Eigenvalue Interlacing Theorem — Let A be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and let C be a $k \times k$ principal submatrix of A with eigenvalues $\tau_1 \geq \tau_2 \geq \ldots \geq \tau_k$. Then $\lambda_i \geq \tau_i$ for all $i \in \{1, \ldots, k\}$. [2]

1.4 Definition.
Principal Submatrix — The principal submatrix of an $n \times n$ matrix A is the submatrix obtained where if row_i is excluded in the submatrix, then $column_i$ is excluded as well. Note that all principal submatrices of a weight matrix W, correspond to an induced subgraph in the graph represented by W.

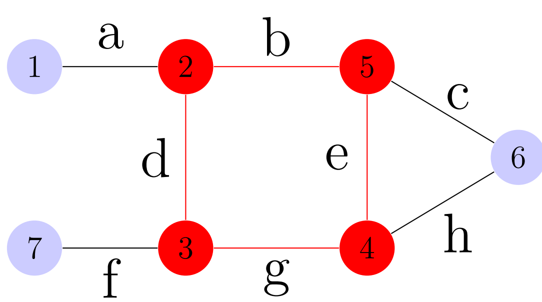
1.2 Example.
The following is an example of a principal submatrix in relation to graph theory.

Consider the following graph:



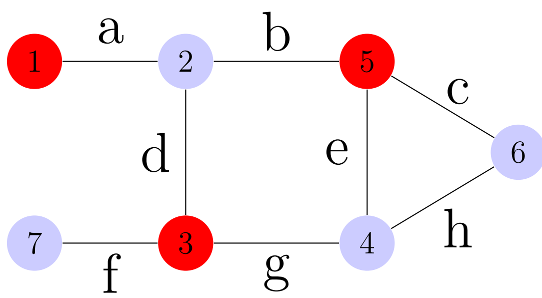
and corresponding weight matrix:
$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & d & 0 & b & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f \\ 0 & 0 & g & 0 & e & h & 0 \\ 0 & b & 0 & e & 0 & c & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can see the following principal submatrix and corresponding induced subgraph:



$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & \color{red}{0} & \color{red}{d} & \color{red}{0} & \color{red}{b} & 0 & 0 \\ 0 & \color{red}{d} & \color{red}{0} & \color{red}{g} & \color{red}{0} & 0 & f \\ 0 & \color{red}{0} & \color{red}{g} & \color{red}{0} & \color{red}{e} & h & 0 \\ 0 & \color{red}{b} & \color{red}{0} & \color{red}{e} & \color{red}{0} & c & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 \end{bmatrix}$$

As well, we see the following principal submatrix of an independent set of the graph:



$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & d & 0 & b & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f \\ 0 & 0 & g & 0 & e & h & 0 \\ 0 & b & 0 & e & 0 & c & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now to prove the Cvetković Bound:

1.2 Theorem.

Cvetković Bound — Let G be a graph on n vertices, and W be the weight matrix of G. Then the following inequality holds:

$$\alpha(G) \leq \min\{|G| - n_+(W), |G| - n_-(W)\} \tag{3}$$

Proof. [1] Let H be the subgraph of G formed by the vertices in an independent set of size s. Then H is an induced subgraph of G and all eigenvalues of the principal submatrix $W(H)$ are 0 since the principal submatrix will just be a zero matrix. Let λ_i denote the ith largest eigenvalue of W and τ_i denote the ith larest eigenvalue of $W(H)$. Now, by interlacing, we have,

$$\lambda_i \geq \tau_i = 0 \text{ for all i } \in \{1, \ldots, s\} \tag{4}$$

and so

$$n - n_-(W) = n_+(W) + n_0(W) \geq s \tag{5}$$

Also, note that by negating W, the positive eigenvalues become negative eigenvalues and vice versa. Thus,

$$n - n_+(W) = n - n_-(-W), \tag{6}$$

[1] Interesting Graphs and their Colourings, unpublished lecture notes C. Godsil (2004)

However, the principal submatrix corresponding to H in $-W$ is still the zero matrix and thus has all zero eigenvalues. Thus, by interlacing, we get a similar result as above,

$$n - n_+(W) = n - n_-(-W) = n_+(-W) + n_0(-W) \geq s \tag{7}$$

Therefore, both $n - n_+(W)$ and $n - n_-(W)$ are greater than or equal to s. Since s is the size of the idependent set, we can see that letting s $= \alpha(G)$, we get:

$$\alpha(G) \leq \min\{|G| - n_+(W), |G| - n_-(W)\} \tag{8}$$

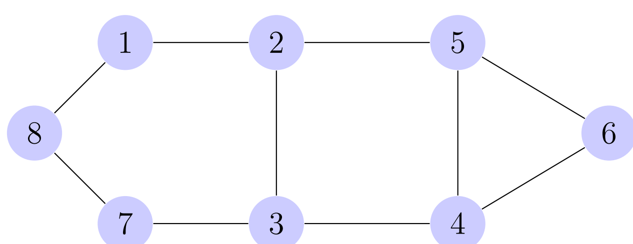
□

1.2 Graphs with Tight Inertia Bounds

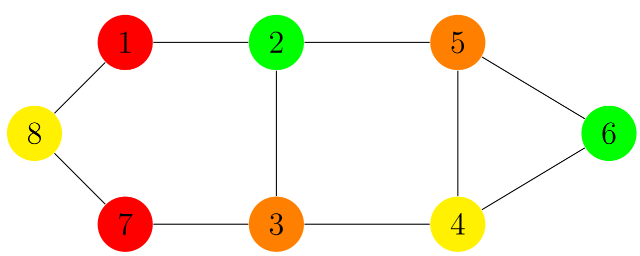
1.2.1 Perfect Graphs

1.5 Definition.
Chromatic Number — The chromatic number of a graph, $\chi(G)$, is the minimum number of colours needed in a proper colouring of G. [1]

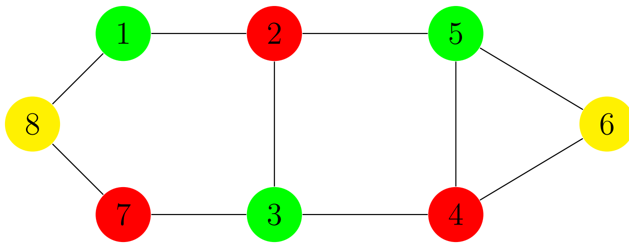
1.3 Example.
Consider the following graph:



An example of a colouring would be:



However, $\chi(G)$ for this graph is 3:



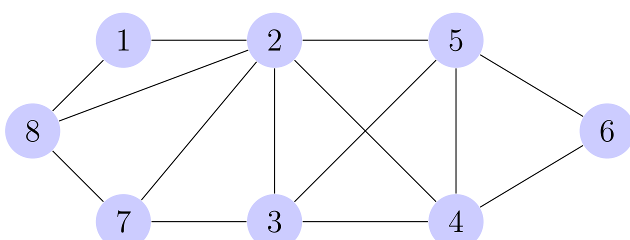
1.6 Definition.
Clique — An m-clique in a graph is a complete subgraph on m vertices. [1]

The clique number, $\omega(G)$, is the number of vertices in a maximum clique of G.

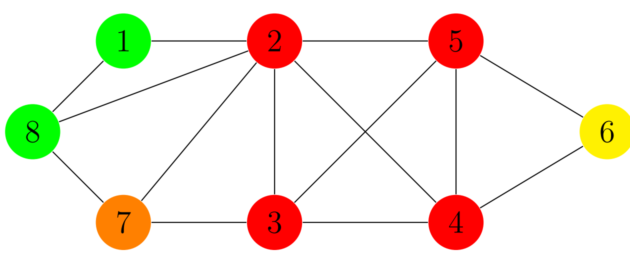
1.7 Definition.
Clique Cover — A Clique Cover of the vertex set $V(G)$ of a graph G is a set of cliques C, such that each vertex is in at least one clique in C.

The clique cover number, $\theta(G)$ is the minimum number of cliques needed in a clique cover of G. [1]

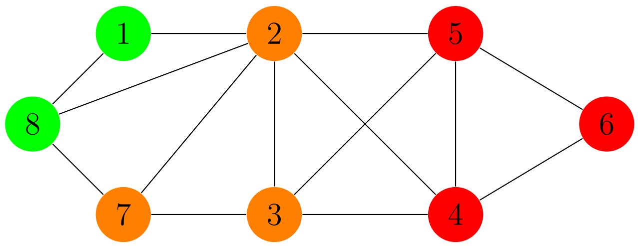
1.4 Example.
Consider the following graph:



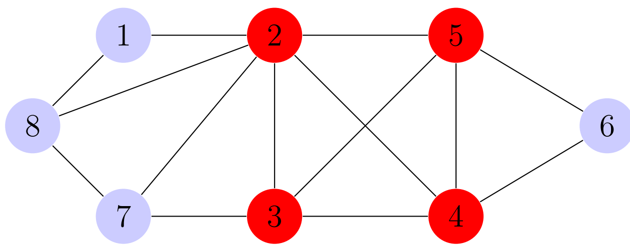
A possible clique covering is:



However, we can find that $\theta(G)$ is equal to 3 (smallest I could find):



As well, the clique number, $\omega(G)$, is 4:



1.8 Definition.
Perfect Graph — A graph G is perfect if $\chi(G) = \omega(G)$ for all induced subgraphs, H, of G.

1.2.2 Graphs with an Eigensharp decomposition by Stars

1.3 Other Bounds on Independence Number

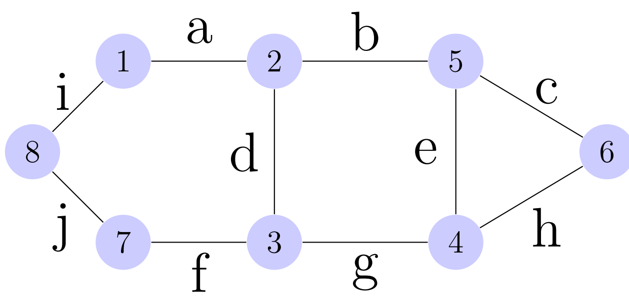
2 Algorithm to Find Graphs Lacking a Tight Inertia Bound

2.1 Outline of Method

2.1 Definition.
Optimal Weight Matrix — A weight matrix, W, of a graph, G, is optimal if

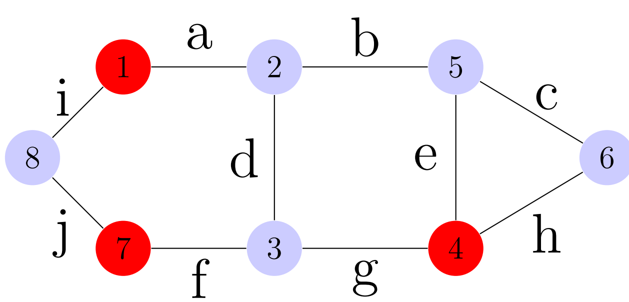
$$\alpha(G) = \min\{|G| - n_+(W), |G| - n_-(W)\} \tag{9}$$

2.1 Example.
Consider the following graph, G, with corresponding weight matrix W:

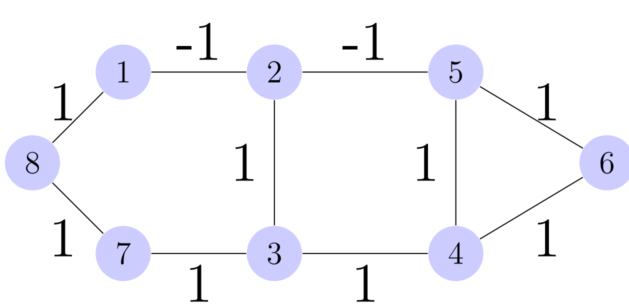


$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 & i \\ a & 0 & d & 0 & b & 0 & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f & 0 \\ 0 & 0 & g & 0 & e & h & 0 & 0 \\ 0 & b & 0 & e & 0 & c & 0 & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 & j \\ i & 0 & 0 & 0 & 0 & 0 & j & 0 \end{bmatrix}$$

We can see the independent number of G is 3:



Now, let G have the following weighting:



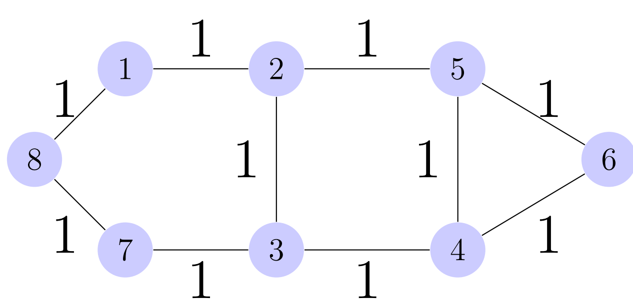
$$\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Finding the eigenvalues of W, we find there are 3 positive eigenvalues and 5 negative eigenvalues. Thus, we see for this weight matrix we have:

$$\begin{aligned} \alpha(G) &= \min\{|G| - n_+(W), |G| - n_-(W)\} \\ &= \min\{8-3, 8-5\} \\ &= \min\{5, 3\} \\ &= 3 \end{aligned} \tag{10}$$

Therefore, this is an optimal weight matrix of G.

Now consider the following weighting for the same graph:



$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Finding the eigenvalues of this weight matrix, we find there are 4 positive eigenvalues and 4 negative eigenvalues. Thus, we see we get:

$$\begin{aligned}\alpha(G) = 3 &\neq \min\{|G| - n_+(W), |G| - n_-(W)\} \\ &= \min\{8-4, 8-4\} \\ &= \min\{4, 4\} \\ &= 4\end{aligned} \tag{11}$$

Therefore, we see that the previous weighting was not optimal for G.

2.1 Lemma.
If a graph, G, with weight matrix W, has two induced subgraphs, S_1 and S_2, such that S_1 has $\alpha(G)+1$ positive eigenvalues under the weighting of W, and S_2 has $\alpha(G)+1$ negative eigenvalues under the weighting of W, then W is not optimal

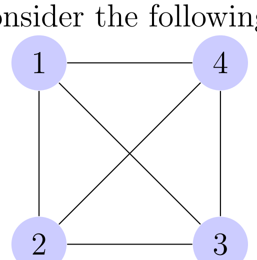
Proof. □

2.2 Preliminary Tests to Determine if the Graph may be Suitable

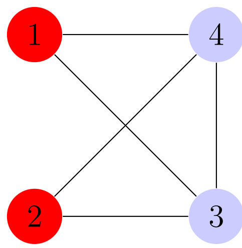
2.2.1 Test for $\boldsymbol{\alpha}-$Critical

2.2 Definition.
$\boldsymbol{\alpha}-$Critical — A graph, G, is $\alpha-$critical if $\alpha(G) < \alpha(G-e)$ for all edges e.

2.2 Example.
Consider the following graph G:



we see that $\alpha(G) = 1$. But since this graph is complete, we see that if we delete any edge, we can get an independent set of size 2 by making the set include the two vertices that were connected by the edge we deleted. For example:



Thus, G is $\alpha-$critical.

2.2 Lemma.
If G is $\alpha-$critical, and W an optimal weight matrix of G, then $w_{i,j} \neq 0$ for all $i, j \in E(G)$

Proof. Assume for the sake of contradiction, that for some $i, j \in E(G)$, we have $w_{i,j} = 0$. Then, we know $\alpha(G - e_{i,j}) > \alpha(G)$ because G is $\alpha-$critical. Thus:

$\alpha(G) < \alpha(G - e_{i,j}) \leq \min\{|G| - n_+(W), |G| - n_-(W)\}$

Thus, we see that the inertia bound is not tight for G, so W is not an optimal weight matrix of G, which is a contradiction. □

Due to the complexity of needing to consider edges that could potentially be zero in the weight matrix, it is easier to consider graphs that are restricted to only non-zero edge weights in its optimal weight matrix. Thus, it makes sense to only consider $\alpha-$critical graphs, because of Lemma 2.2 ensuring that all $\alpha-$critical graphs have non-zero weight matrices.

2.2.2 Determining Each Triangle Must Have the Same Sign

References

[1] Randall J Elzinga. *The Minimum Witt Index of a Graph.* PhD thesis, Queens University, 2007.

[2] John Sinkovic. A graph for which the inertia bound is not tight. *arXiv preprint arXiv:1609.02826*, 2016.