

C&O URA Spring 2017

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Contents

1	Inertia Bounds	1
1.1	Introduction on Inertia Bounds	1
2	Algorithm to Find Graphs Lacking a Tight Inertia Bound	6
2.1	Preliminary Tests to Determine if the Graph may be Suitable	6
2.1.1	Test for α -Critical	6
2.1.2	Test for Arc Transitivity	10

1 Inertia Bounds

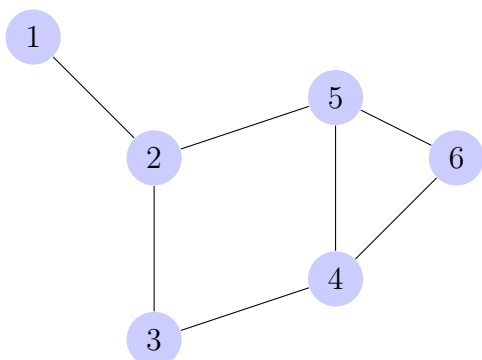
1.1 Introduction on Inertia Bounds

1.1 Definition.

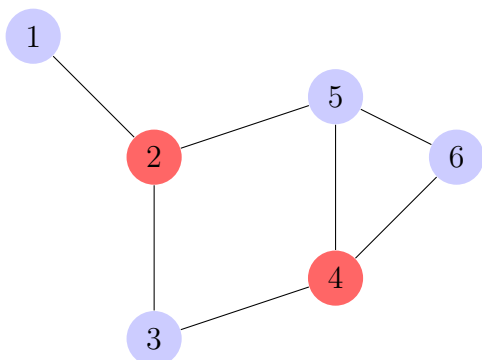
Independent Set — An independent set is a set of vertices belonging to a graph in which no two vertices are adjacent.

1.1 Example.

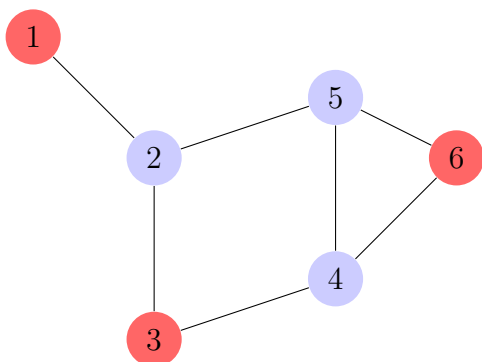
Consider the following graph:



An example of an independent set in this graph is:



However, often the independent set we are most interested in finding is the largest one:



1.2 Definition.

Independence Number — The independence number of a graph G , denoted $\alpha(G)$, is the size of the largest independent set of G .

1.3 Definition.

Weight Matrix — The weight matrix of a graph G , is a matrix defined by:

$$W_{i,j} = \begin{cases} c_{i,j} & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

with v_i a vertice of G and $c_{i,j}$, a constant.

The weight matrix of a graph, is identical to an adjacency matrix, except where there was a 1 in the matrix at entry $A_{i,j}$ if vertices v_i and v_j were adjacent, there is now a constant indicating a weighting for the edge between v_i and v_j .

For any graph G , there exists a bound on $\alpha(G)$, known as the Cvetković bound (also referred to as the Interia Bound). This bound provides a relationship between $\alpha(G)$ and the number of positive, negative, and zero eigenvalues of the weight matrix, W , associated with G . The Cvetković bound of G , is:

$$\alpha(G) \leq \min\{|G| - n_+(W), |G| - n_-(W)\} \quad (2)$$

Where $n_+(W)$ and $n_-(W)$ denote the number of positive and negative eigenvalues of W , respectively.

To prove this, we first need to introduce a result that comes from the Eigenvalue Interlacing Theorem:

1.1 Theorem.

Corollary of Eigenvalue Interlacing Theorem — Let A be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ and let C be a $k \times k$ principal submatrix of A with eigenvalues $\tau_1 \geq \tau_2 \geq \dots \geq \tau_k$. Then $\lambda_i \geq \tau_i$ for all $i \in \{1, \dots, k\}$. [2]

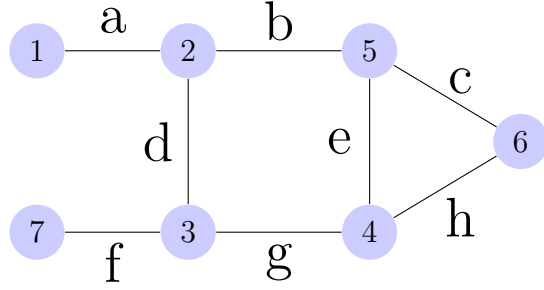
1.4 Definition.

Principal Submatrix — The principal submatrix of an $n \times n$ matrix A is the submatrix obtained where if row_i is excluded in the submatrix, then $column_i$ is excluded as well. Note that all principal submatrices of a weight matrix W , correspond to an induced subgraph in the graph represented by W .

1.2 Example.

The following is an example of a principal submatrix in relation to graph theory.

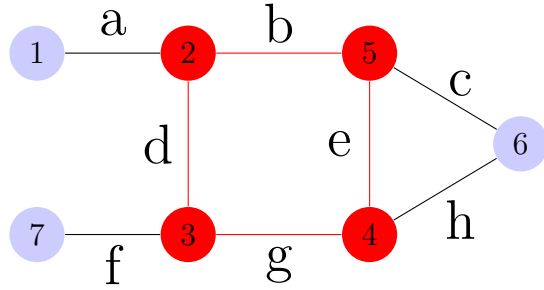
Consider the following graph:



and corresponding weight matrix:

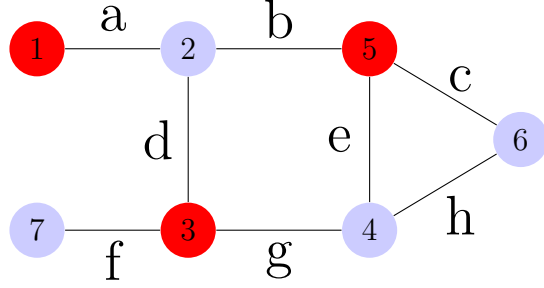
$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & d & 0 & b & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f \\ 0 & 0 & g & 0 & e & h & 0 \\ 0 & b & 0 & e & 0 & c & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can see the following principal submatrix and corresponding induced subgraph:



$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & d & 0 & b & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f \\ 0 & 0 & g & 0 & e & h & 0 \\ 0 & b & 0 & e & 0 & c & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 \end{bmatrix}$$

As well, we see the following principal submatrix of an independent set of the graph:



$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & d & 0 & b & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f \\ 0 & 0 & g & 0 & e & h & 0 \\ 0 & b & 0 & e & 0 & c & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now to prove the Cvetković Bound:

1.2 Theorem.

Cvetković Bound — Let G be a graph on n vertices, and W be the weight matrix of G . Then the following inequality holds:

$$\alpha(G) \leq \min\{|G| - n_+(W), |G| - n_-(W)\} \quad (3)$$

*Proof.*¹ Let H be the subgraph of G formed by the vertices in an independent set of size s . Then H is an induced subgraph of G and all eigenvalues of the principal submatrix $W(H)$ are 0 since the principal submatrix will just be a zero matrix. Let λ_i denote the i th largest eigenvalue of W and τ_i denote the i th largest eigenvalue of $W(H)$. Now, by interlacing, we have,

$$\lambda_i \geq \tau_i = 0 \text{ for all } i \in \{1, \dots, s\} \quad (4)$$

and so

$$n - n_-(W) = n_+(W) + n_0(W) \geq s \quad (5)$$

Also, note that by negating W , the positive eigenvalues become negative eigenvalues and vice versa. Thus,

$$n - n_+(W) = n - n_-(-W), \quad (6)$$

¹Interesting Graphs and their Colourings, unpublished lecture notes C. Godsil (2004)

However, the principal submatrix corresponding to H in $-W$ is still the zero matrix and thus has all zero eigenvalues. Thus, by interlacing, we get a similar result as above,

$$n - n_+(W) = n - n_-(-W) = n_+(-W) + n_0(-W) \geq s \quad (7)$$

Therefore, both $n - n_+(W)$ and $n - n_-(W)$ are greater than or equal to s . Since s is the size of the independent set, we can see that letting $s = \alpha(G)$, we get:

$$\alpha(G) \leq \min\{|G| - n_+(W), |G| - n_-(W)\} \quad (8)$$

□

2 Algorithm to Find Graphs Lacking a Tight Inertia Bound

2.1 Preliminary Tests to Determine if the Graph may be Suitable

2.1.1 Test for α -Critical

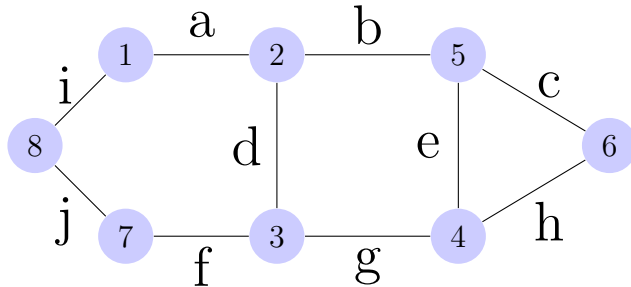
2.1 Definition.

Optimal Weight Matrix — A weight matrix, W , of a graph, G , is optimal if

$$\alpha(G) = \min\{|G| - n_+(W), |G| - n_-(W)\} \quad (9)$$

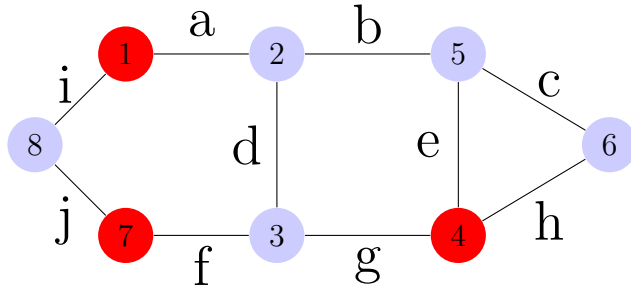
2.1 Example.

Consider the following graph, G , with corresponding weight matrix W :

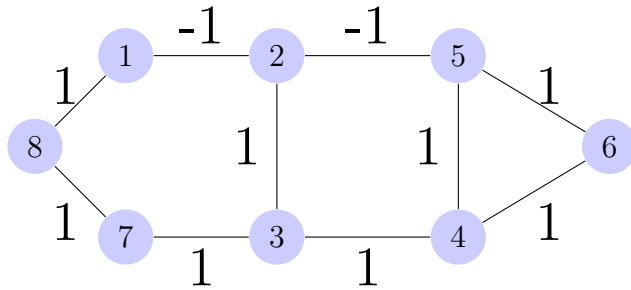


$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 & i \\ a & 0 & d & 0 & b & 0 & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f & 0 \\ 0 & 0 & g & 0 & e & h & 0 & 0 \\ 0 & b & 0 & e & 0 & c & 0 & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 & j \\ i & 0 & 0 & 0 & 0 & 0 & j & 0 \end{bmatrix}$$

We can see the independent number of G is 3:



Now, let G have the following weighting:



$$\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

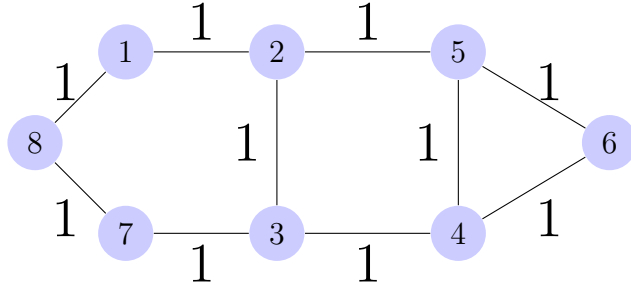
Finding the eigenvalues of W , we find there are 3 positive eigenvalues and

5 negative eigenvalues. Thus, we see for this weight matrix we have:

$$\begin{aligned}
\alpha(G) &= \min\{|G| - n_+(W), |G| - n_-(W)\} \\
&= \min\{8 - 3, 8 - 5\} \\
&= \min\{5, 3\} \\
&= 3
\end{aligned} \tag{10}$$

Therefore, this is an optimal weight matrix of G.

Now consider the following weighting for the same graph:



$$\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}$$

Finding the eigenvalues of this weight matrix, we find there are 4 positive eigenvalues and 4 negative eigenvalues. Thus, we see we get:

$$\begin{aligned}
\alpha(G) &= 3 \neq \min\{|G| - n_+(W), |G| - n_-(W)\} \\
&= \min\{8 - 4, 8 - 4\} \\
&= \min\{4, 4\} \\
&= 4
\end{aligned} \tag{11}$$

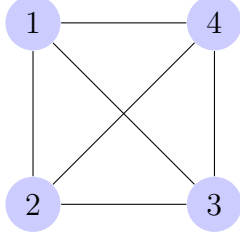
Therefore, we see that the previous weighting was not optimal for G.

2.2 Definition.

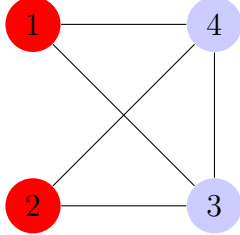
α -Critical — A graph, G , is α -critical if $\alpha(G) < \alpha(G - e)$ for all edges e .

2.2 Example.

Consider the following graph G :



we see that $\alpha(G) = 1$. But since this graph is complete, we see that if we delete any edge, we can get an independent set of size 2 by making the set include the two vertices that were connected by the edge we deleted. For example:



Thus, G is α -critical.

2.1 Lemma.

If G is α -critical, and W an optimal weight matrix of G , then $w_{i,j} \neq 0$ for all $i, j \in E(G)$

Proof. Assume for the sake of contradiction, that for some $i, j \in E(G)$, we have $w_{i,j} = 0$. Then, we know $\alpha(G - e_{i,j}) > \alpha(G)$ because G is α -critical. Thus:

$$\alpha(G) < \alpha(G - e_{i,j}) \leq \min\{|G| - n_+(W), |G| - n_-(W)\}$$

Thus, we see that the inertia bound is not tight for G , so W is not an optimal weight matrix of G , which is a contradiction. \square

Due to the complexity of needing to consider edges that could potentially be zero in the weight matrix, it is easier to consider graphs that are restricted to only non-zero edge weights in its optimal weight matrix. Thus, it makes sense to only consider α -critical graphs, because of Lemma 2.1 ensuring that all α -critical graphs have non-zero weight matrices.

2.1.2 Test for Arc Transitivity

2.3 Definition.

Arc Transitive — A graph is called arc-transitive if its automorphism group has a single orbit on the set of all ordered pairs of adjacent vertices in the graph. [1]

References

- [1] Marston DE Conder, Cai Heng Li, and Primož Potočnik. On the orders of arc-transitive graphs. *Journal of Algebra*, 421:167–186, 2015.
- [2] John Sinkovic. A graph for which the inertia bound is not tight. *arXiv preprint arXiv:1609.02826*, 2016.