C&O URA Spring 2017

Zach Dockstader

June 29, 2017

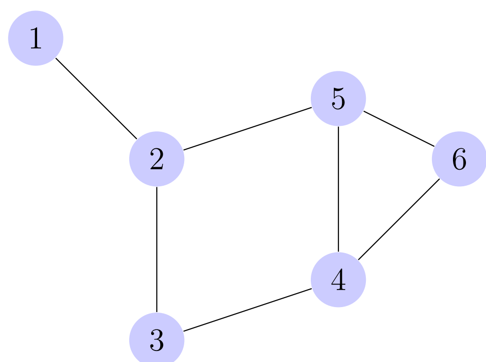
Contents

1 Inertia Bounds

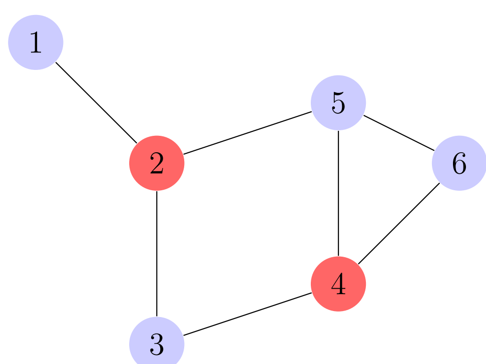
1.1 Introduction on Inertia Bounds

1.1 Definition.
Independent Set — An independent set is a set of vertices belonging to a graph in which no two vertices are adjacent.

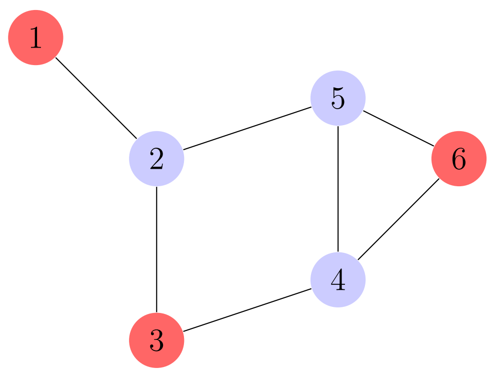
1.1 Example.
Consider the following graph:



An example of an independent set in this graph is:



However, often the independent set we are most interested in finding is the largest one:



1.2 Definition.
Independence Number — The independence number of a graph G, denoted $\alpha(G)$, is the size of the largest independent set of G.

1.3 Definition.
Weight Matrix — The weight matrix of a graph G, is a matrix defined by:

$$W_{i,j} = \begin{cases} c_{i,j} & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

with v_i a vertice of G and $c_{i,j}$, a constant.

The weight matrix of a graph, is identical to an adjacency matrix, except where there was a 1 in the matrix at entry $A_{i,j}$ if vertices v_i and v_j were adjacent, there is now a constant indicating a weighting for the edge between v_i and v_j.

For any graph G, there exists a bound on $\alpha(G)$, known as the Cvetković bound (also referred to as the Interia Bound). This bound provides a relationship between $\alpha(G)$ and the number of positive, negative, and zero eigenvalues of the weight matrix, W, associated with G. The Cvetković bound of G, is:

$$\alpha(G) \leq \min\{|G| - n_+(W), |G| - n_-(W)\} \tag{2}$$

Where $n_+(W)$ and $n_-(W)$ denote the number of positive and negative eigenvalues of W, respectively.

To prove this, we first need to introduce a result that comes from the Eigenvalue Interlacing Theorem:

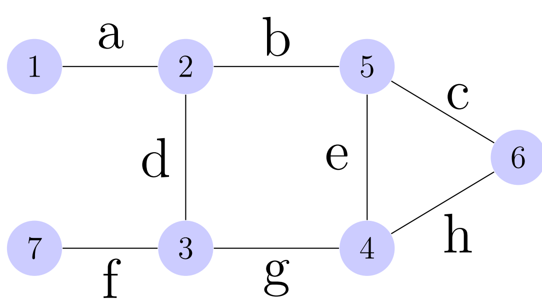
1.1 Theorem.
Corollary of Eigenvalue Interlacing Theorem — Let A be an $n \times n$

real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and let C be a $k \times k$ principal submatrix of A with eigenvalues $\tau_1 \geq \tau_2 \geq \ldots \geq \tau_k$. Then $\lambda_i \geq \tau_i$ for all $i \in \{1, \ldots, k\}$. [5]

1.4 Definition.
Principal Submatrix — The principal submatrix of an $n \times n$ matrix A is the submatrix obtained where if row_i is excluded in the submatrix, then $column_i$ is excluded as well. Note that all principal submatrices of a weight matrix W, correspond to an induced subgraph in the graph represented by W.

1.2 Example.
The following is an example of a principal submatrix in relation to graph theory.

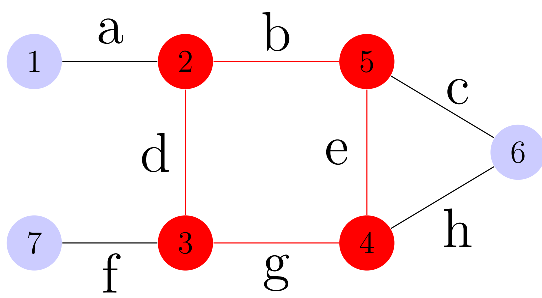
Consider the following graph:



and corresponding weight matrix:

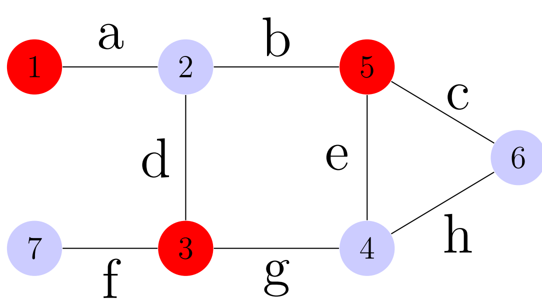
$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & d & 0 & b & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f \\ 0 & 0 & g & 0 & e & h & 0 \\ 0 & b & 0 & e & 0 & c & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can see the following principal submatrix and corresponding induced subgraph:



$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & d & 0 & b & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f \\ 0 & 0 & g & 0 & e & h & 0 \\ 0 & b & 0 & e & 0 & c & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 \end{bmatrix}$$

As well, we see the following principal submatrix of an independent set of the graph:



$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & d & 0 & b & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f \\ 0 & 0 & g & 0 & e & h & 0 \\ 0 & b & 0 & e & 0 & c & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now to prove the Cvetković Bound:

1.2 Theorem.

Cvetković Bound — Let G be a graph on n vertices, and W be the weight matrix of G. Then the following inequality holds:

$$\alpha(G) \leq \min\{|G| - n_+(W), |G| - n_-(W)\} \tag{3}$$

Proof. [1] Let H be the subgraph of G formed by the vertices in an independent set of size s. Then H is an induced subgraph of G and all eigenvalues of the principal submatrix $W(H)$ are 0 since the principal submatrix will just be a zero matrix. Let λ_i denote the ith largest eigenvalue of W and τ_i denote the ith larest eigenvalue of $W(H)$. Now, by interlacing, we have,

$$\lambda_i \geq \tau_i = 0 \text{ for all i } \in \{1, \ldots, s\} \tag{4}$$

and so

$$n - n_-(W) = n_+(W) + n_0(W) \geq s \tag{5}$$

Also, note that by negating W, the positive eigenvalues become negative eigenvalues and vice versa. Thus,

$$n - n_+(W) = n - n_-(-W), \tag{6}$$

However, the principal submatrix corresponding to H in $-W$ is still the zero matrix and thus has all zero eigenvalues. Thus, by interlacing, we get a similar result as above,

$$n - n_+(W) = n - n_-(-W) = n_+(-W) + n_0(-W) \geq s \tag{7}$$

Therefore, both $n - n_+(W)$ and $n - n_-(W)$ are greater than or equal to s. Since s is the size of the idependent set, we can see that letting s $= \alpha(G)$, we get:

$$\alpha(G) \leq \min\{|G| - n_+(W), |G| - n_-(W)\} \tag{8}$$

□

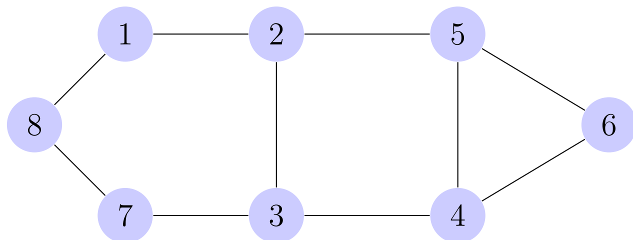
1.2 Graphs with Tight Inertia Bounds

1.2.1 Perfect Graphs

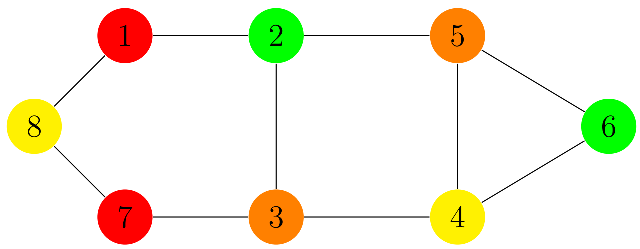
1.5 Definition.
Chromatic Number — The chromatic number of a graph, $\chi(G)$, is the minimum number of colours needed in a proper colouring of G. [1]

[1] Interesting Graphs and their Colourings, unpublished lecture notes C. Godsil (2004)

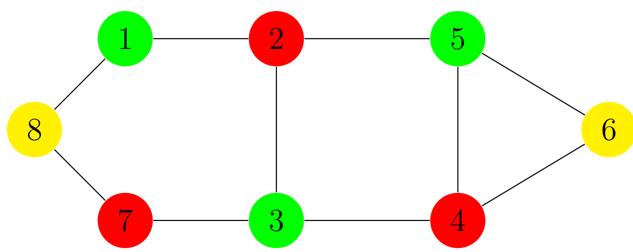
1.3 Example.
Consider the following graph:



An example of a colouring would be:



However, $\chi(G)$ for this graph is 3:



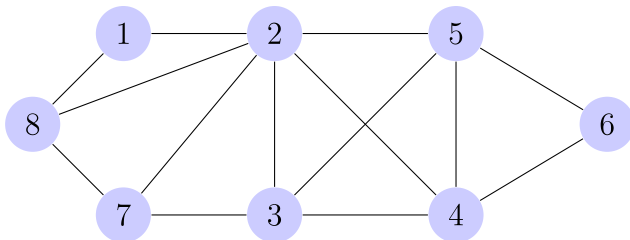
1.6 Definition.
Clique — An m-clique in a graph is a complete subgraph on m vertices. [1]

The clique number, $\omega(G)$, is the number of vertices in a maximum clique of G.

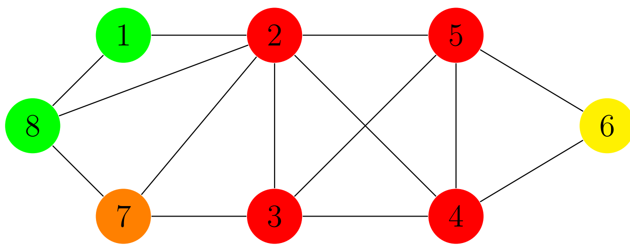
1.7 Definition.
Clique Cover — A Clique Cover of the vertex set $V(G)$ of a graph G is a set of cliques C, such that each vertex is in at least one clique in C.

The clique cover number, $\theta(G)$ is the minimum number of cliques needed in a clique cover of G. [1]

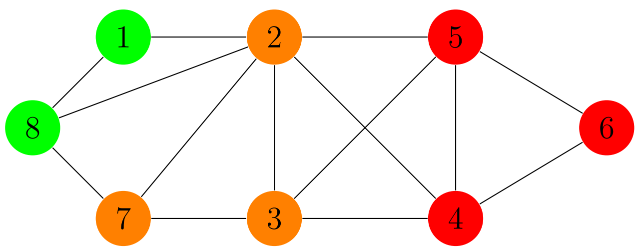
1.4 Example.
Consider the following graph:



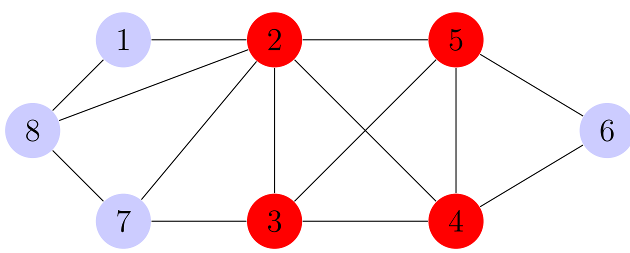
A possible clique covering is:



However, we can find that $\theta(G)$ is equal to 3 (smallest I could find):



As well, the clique number, $\omega(G)$, is 4:



1.8 Definition.
Perfect Graph — A graph G is perfect if $\chi(G) = \omega(G)$ for all induced subgraphs, H, of G.

1.3 Theorem.

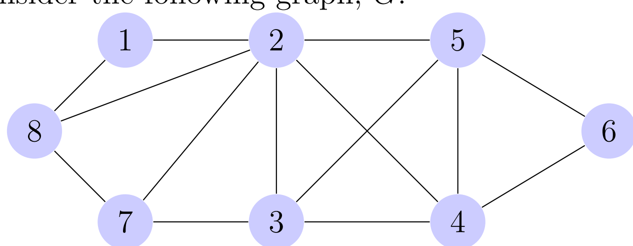
Perfect Graph Theorem — A Graph G is perfect if and only if its compliment, $\overline{G}$, is also perfect.

1.1 Observation.

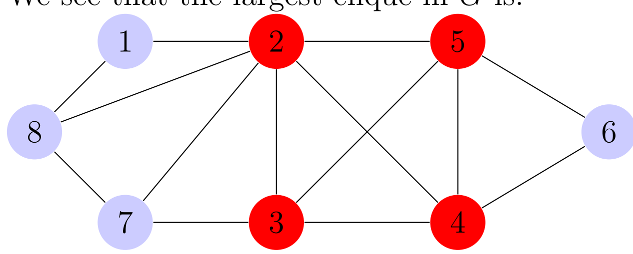
For a graph G, $\omega(G) = \alpha(\overline{G})$

1.5 Example.

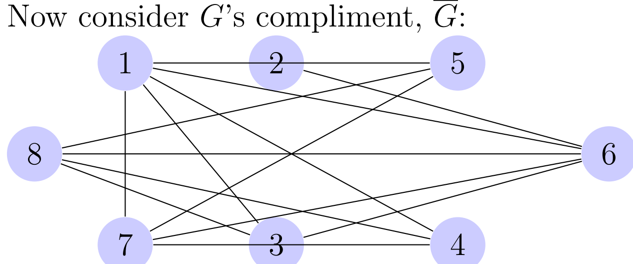
Consider the following graph, G:



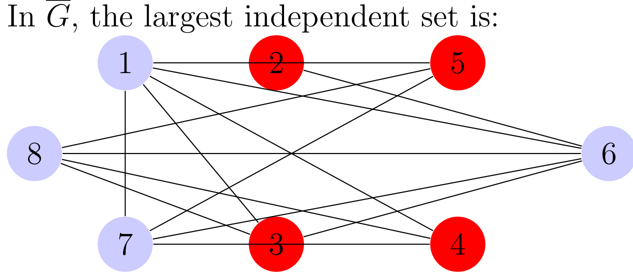
We see that the largest clique in G is:



Thus, $W(G)$ is 4.
Now consider G's compliment, $\overline{G}$:



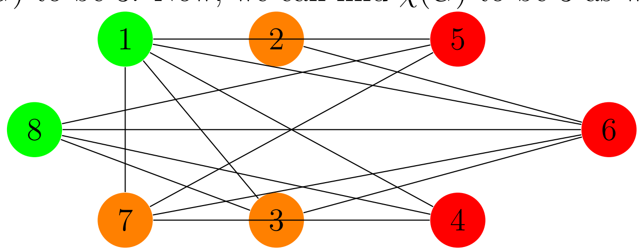
In $\overline{G}$, the largest independent set is:



Therefore, we see $\omega(G) = 4 = \alpha(\overline{G})$

1.2 Observation.
Similar to the last observation, for a graph G, $\theta(G) = \chi(\overline{G})$

1.6 Example.
Consider the same graph from the last example. Recall that we calculated $\theta(G)$ to be 3. Now, we can find $\chi(\overline{G})$ to be 3 as well:



Thus, $\theta(G) = 3 = \chi(\overline{G})$

1.1 Lemma.
Let G be a graph. Then $\alpha(G) \leq \min\{|G| - n_+(W), |G| - n_-(W)\} \leq \theta(G)$. Thus, if $\alpha(G) = \theta(G)$, G has a tight inertia bound. [1]

Proof. Consider a clique partition, $\mathcal{C}$, of a graph G. Let $\hat{A}$, denote the adjacency matrix of G where the only connected components are the cliques in $\mathcal{C}$.

Now if we consider the adjacency matrix of the complete graph, K_n, we see that it is equal to $J_n - I_n$ where J_n is the all ones matrix. $\square$

1.4 Theorem.
Every Perfect Graph, G, has a tight inertia bound

Proof. By the Perfect Graph Theorem (theorem 1.3), we know that $\overline{G}$, is also perfect. Thus $\overline{G}$ satisies that $\chi(H) = \omega(H)$ for all subgraphs, H, of $\overline{G}$, by definition. Thus, since $\chi(\overline{G}) = \omega(\overline{G})$, we can get from the observation 1.1 and 1.2, that

$$\alpha(G) = \omega(\overline{G}) = \chi(\overline{G}) = \theta(G) \tag{9}$$

Therefore, from lemma 1.1, G has a tight inertia bound. $\square$

1.5 Theorem.
Strong Perfect Graph Theorem [1] — A graph G is a perfect graph if and only if both G and its complement, $\overline{G}$, do not contain a induced odd cycle of length at least 5.

1.3 Observation.
Due to each perfect graph having a tight inertia bound, and the Strong Perfect Graph Theorem (theorem 1.5), every graph not containing an induced odd cycle of length 5 or greater has a tight inertia bound.

1.2.2 Latin Square Graphs

1.2.3 Graphs with an Eigensharp decomposition by Stars

1.2.4 Summary

In summary, the following list of graphs attain a tight inertia bound:

- Graphs on 10 or fewer vertices (pg 81 [1])
- Vertex Transitive graphs on 12 or fewer vertices (pg 81 [1])
- Perfect Graphs
- Latin Square Graphs
- Graphs with an Eigensharp decomposition by stars

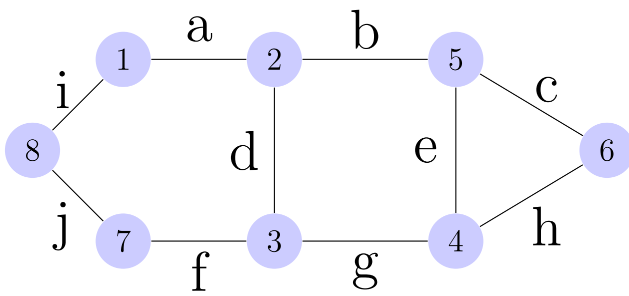
1.3 Other Bounds on Independence Number

2 Algorithm to Find Graphs Lacking a Tight Inertia Bound

2.1 Outline of Method

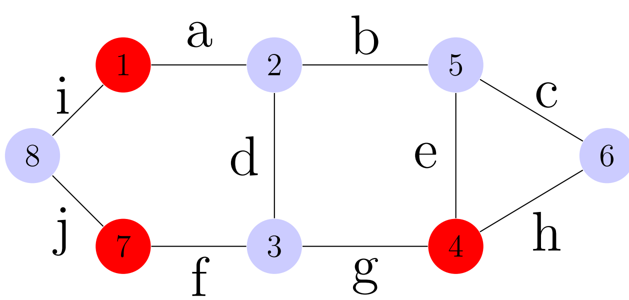
2.1 Definition.
Optimal Weight Matrix — A weight matrix, W, of a graph, G, is optimal if
$$\alpha(G) = \min\{|G| - n_+(W), |G| - n_-(W)\} \tag{10}$$

2.1 Example.
Consider the following graph, G, with corresponding weight matrix W:

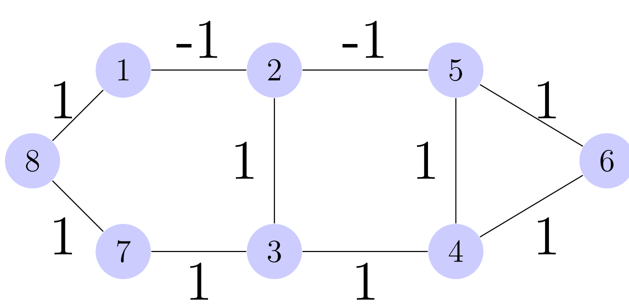


$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 & i \\ a & 0 & d & 0 & b & 0 & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f & 0 \\ 0 & 0 & g & 0 & e & h & 0 & 0 \\ 0 & b & 0 & e & 0 & c & 0 & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 & j \\ i & 0 & 0 & 0 & 0 & 0 & j & 0 \end{bmatrix}$$

We can see the independent number of G is 3:



Now, let G have the following weighting:



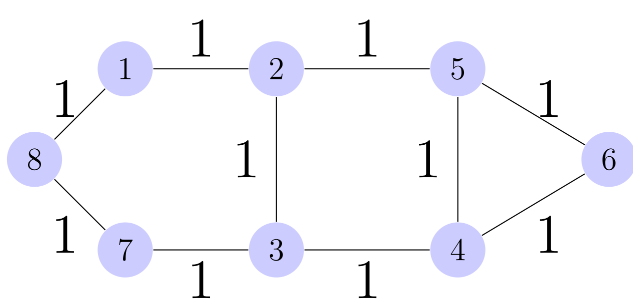
$$\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Finding the eigenvalues of W, we find there are 3 positive eigenvalues and 5 negative eigenvalues. Thus, we see for this weight matrix we have:

$$\begin{aligned} \alpha(G) &= \min\{|G| - n_+(W), |G| - n_-(W)\} \\ &= \min\{8-3, 8-5\} \\ &= \min\{5, 3\} \\ &= 3 \end{aligned} \tag{11}$$

Therefore, this is an optimal weight matrix of G.

Now consider the following weighting for the same graph:



$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Finding the eigenvalues of this weight matrix, we find there are 4 positive eigenvalues and 4 negative eigenvalues. Thus, we see we get:

$$\begin{aligned}\alpha(G) = 3 &\neq \min\{|G| - n_+(W), |G| - n_-(W)\} \\ &= \min\{8-4, 8-4\} \\ &= \min\{4,4\} \\ &= 4\end{aligned} \tag{12}$$

Therefore, we see that the previous weighting was not optimal for G.

2.1 Lemma.
If a graph, G, with weight matrix W, has two induced subgraphs, S_1 and S_2, such that S_1 has $\alpha(G)+1$ positive eigenvalues under the weighting of W, and S_2 has $\alpha(G)+1$ negative eigenvalues under the weighting of W, then W is not optimal

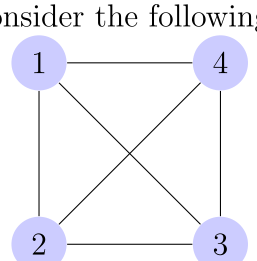
Proof. □

2.2 Preliminary Tests to Determine if the Graph may be Suitable

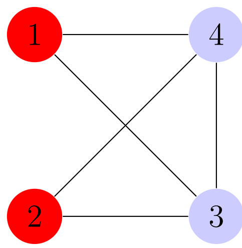
2.2.1 Test for $\alpha-$Critical

2.2 Definition.
$\alpha-$Critical — A graph, G, is $\alpha-$critical if $\alpha(G) < \alpha(G-e)$ for all edges e.

2.2 Example.
Consider the following graph G:



we see that $\alpha(G) = 1$. But since this graph is complete, we see that if we delete any edge, we can get an independent set of size 2 by making the set include the two vertices that were connected by the edge we deleted. For example:



Thus, G is $\alpha-$critical.

2.2 Lemma.
If G is $\alpha-$critical, and W an optimal weight matrix of G, then $w_{i,j} \neq 0$ for all $i, j \in E(G)$

Proof. Assume for the sake of contradiction, that for some $i, j \in E(G)$, we have $w_{i,j} = 0$. Then, we know $\alpha(G - e_{i,j}) > \alpha(G)$ because G is $\alpha-$critical. Thus:

$\alpha(G) < \alpha(G - e_{i,j}) \leq \min\{|G| - n_+(W), |G| - n_-(W)\}$

Thus, we see that the inertia bound is not tight for G, so W is not an optimal weight matrix of G, which is a contradiction. □

Due to the complexity of needing to consider edges that could potentially be zero in the weight matrix, it is easier to consider graphs that are restricted to only non-zero edge weights in its optimal weight matrix. Thus, it makes sense to only consider $\alpha-$critical graphs, because of Lemma 2.2 ensuring that all $\alpha-$critical graphs have non-zero weight matrices.

2.2.2 Determining Each Triangle Must Have the Same Sign

2.3 Graphs Currently Found

Graph	Vertices	α	Degree	Circulant	Strongly Regular	Arc Transitive
[1]	16	4	5	No	No	No
[2]	16	2	10	No	(16,10,6,6)	Yes
[3]	17	3	8	[1,2,4,8]	(17,8,3,4)	Yes
[4]	19	4	6	[1,7,8]	No	Yes
[5]	20	2	13	No	No	No
[6]	20	2	13	[1,3,4,7,8,9,10]	No	No
[7]	22	3	11	[1,2,3,5,10,11]	No	No
[8]	24	3	12	No	No	No
[9]	24	3	12	No	No	No
[10]	24	4	9	No	No	No
[11]	24	4	10	[1,2,4,8,9]	No	No
[12]	24	4	9	No	No	No
[13]	24	3	12	No	No	No
[14]	24	4	9	No	No	No
[15]	24	4	9	No	No	No
[16]	24	2	16	No	No	No
[17]	24	2	16	No	No	No
[18]	24	2	16	[1,2,3,4,6,7,8,10]	No	No
[19]	24	2	16	No	No	No

[1] Otr@PKoE?T_iOoOG_dg_m
[2] O~~em]uj[vmsZTUrfFwN~
[3] P}qtSeLUbaKeQZJabfGmmG~G
[4] R}ecZ@OH?oẀ@gOWcI_p̀?hkHL?GuG
[5] S~~vVjjve}vmxymlG~Oi~Qm{jfxjNw{z{
[6] S~~vnZjvUtvimj‘~nibtTP}[ffwk~wR~{
[7] Uv~LnbgfeDShP\G}HuXmePrSemapSxqJWG|ZCVhw
[8] Wunneyzx~W]OwBPfcroK~S{OlogtIoyPlPFMIIjWPUvaGu~
[9] WvrlvjZj~c_wBTRcroK~K{HLpGtPo[ikpImQHrWaUn‘Cvˆ
[10] WvvdtIJpB_c[LEHPiH?PsE_GAsWKcwBXhGDgOFXWIBV@CZT
[11] W}mKmIbqD_JJMMBYa]_{??ucC{YKeHKXPadVXOmqQbqEDMp
[12] W}nS|QeoOq_nWS]?KcPQUPDgU@_TBG_ug@ei?jCgCwY_?J~
[13] W}}VNbMtdyWkic?zg]gevHT_TfGo~bPK|xHkJJMolozdq\s
[14] W}~SvAbp@IcjDgEaj?@BKPCgBbXP@oCz?BLdE@KwGu[?EFZ
[15] W~nELU\‘aKkXTJ]?@cGUB@KgBSX?wG_sS‘DUCGyWO‘}?@Mˆ
[16] W~~~vnnv|~}gzH}‘za|Jˆef| ~wBJNisn[bnˆ@ˆnwez~‘Vˆ~
[17] W~~~vnn{vT{nvFnFoˆ}~Dnw\{ˆAF|hFz[YZ~DT~wXˆ~n{B~
[18] W~~~vnn{vXyjqnnFsˆ}~Knw\[ˆQF|hiz[iznCt~wXˆ~n{B~
[19] W~~~vvu|ˆ\\jvivTvtTyj_~ |}ibyiiF}[b{~C{~wUˆ~_f~~

2.3.1 Graphs Created from Deleting a Vertex

Graph	Created From	Vertices	α	Regular	Circulant	Strongly Regular	Arc Transitive
[1]	1	15	4	No	No	No	No
[2]	2	15	2	No	No	No	No
[3]	3	16	3	No	No	No	No
[4]	4	18	4	No	No	No	No
[5]	6	19	2	No	No	No	No
[6]	7	21	3	No	No	No	No
[7]	8	23	3	No	No	No	No
[8]	9	23	3	No	No	No	No
[9]	10	23	4	No	No	No	No
[10]	11	23	4	No	No	No	No
[11]	12	23	4	No	No	No	No
[12]	13	23	3	No	No	No	No
[13]	14	23	4	No	No	No	No
[14]	15	23	4	No	No	No	No
[15]	16	23	2	No	No	No	No
[16]	17	23	2	No	No	No	No
[17]	18	23	2	No	No	No	No
[18]	19	23	2	No	No	No	No
[19]	[4]	17	4	No	No	No	No
[20]	[5]	18	2	No	No	No	No
[21]	[9]	22	4	No	No	No	No
[22]	[11]	22	4	No	No	No	No
[23]	[15]	22	2	No	No	No	No
[24]	[18]	22	2	No	No	No	No
[25]	[19]	16	4	No	No	No	No
[26]	[21]	21	4	No	No	No	No
[27]	[24]	21	2	No	No	No	No

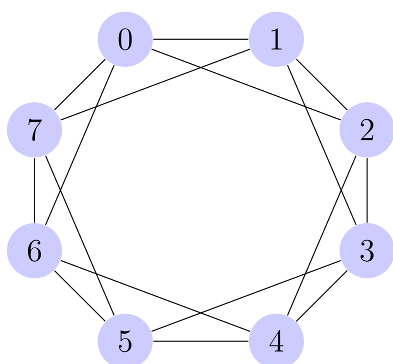
[1] Ntr@PKoE?T_iOoOG_dg
[2] N~~em]uj[vmsZTUrfFw
[3] O}qtSeLUbaKeQZJabfGmm
[4] Q}ecZ@OH?oẀ@gOWcI_p̀?hkHL?
[5] R~~vnZjvUtvimj‘~nibtTP}[ffwk~w
[6] Tv~LnbgfeDShP\G}HuXmePrSemapSxqJWG|Z
[7] Vunneyzx~W]OwBPfcroK~S{OlogtIoyPlPFMIIjWPUv_
[8] VvrlvjZj~c_wBTRcroK~K{HLpGtPo[ikpImQHrWaUn_
[9] VvvdtIJpB_c[LEHPiH?PsE_GAsWKcwBXhGDgOFXWIBV?
[10] V}mKmIbqD_JJMMBYa]_{??ucC{YKeHKXPadVXOmqQbq?
[11] V}nS|QeoOq_nWS]?KcPQUPDgU@_TBG_ug@ei?jCgCwY_
[12] W}}VNbMtdyWkic?zg]gevHT_TfGo~bPK|xHkJJMolozdq\s
[13] V}~SvAbp@IcjDgEaj?@BKPCgBbXP@oCz?BLdE@KwGu[?
[14] V~nELU\‘aKkXTJ]?@cGUB@KgBSX?wG_sS‘DUCGyWO‘}?
[15] V~~~vnnv|~}gzH}‘za|J^ef| ~wBJNisn[bn^@^nwez~_
[16] V~~~vnn{vT{nvEnFo^}~Dnw\{^AF|hEz[YZ~DT~wX^~

3 Other Useful Information

3.1 Cayley Graphs

3.1 Definition.
Cayley Graph — Let H be a finite group, and $S \subseteq H$, be a subset of H. Then the Cayley Graph $C(H, S)$, has a vertex for each element in H. There exists an edge between two vertices g and h, if and only if there exists $s \in S$ such that $sh = g$. If G is a graph such that there exists a group H and a generating set $S \subseteq H$ with $G \cong C(H, S)$, then G is a Cayley Graph. [2]

3.1 Example.
Consider the group $\mathbb{Z}_8$ and let the generating set be $S = \{1, 2\}$. The vertex set will be $\{0, 1, 2, 3, 4, 5, 6, 7\}$ and there will be an edge between two vertices, g and h, if for an $s \in S$, $g + s = h$:



3.2 John's Proof

3.2 Definition.
k-saturated — The graph G is said to be k-saturated if it does not contain a complete $(k+1)$-graph, but every graph G' obtained from adding a new edge to G contains a complete $(k+1)$-graph. [3]

3.3 Definition.
Conical Vertex — A vertex, V, is a conical vertex of a graph, G, if V is adjacent to every vertex in G.

3.1 Observation.
A graph, G, is k-saturated, if and only if its complement, $\overline{G}$, is α-critical,

with $\alpha(\overline{G}) = k$.
This is due to the fact that since G contains a complete k-graph but not a complete $(k+1)$-graph, $\overline{G}$ will have a maximum independent set of size k. By adding any edge to G to obtain G', G' will contain a complete (k+1)-graph, and thus $\overline{G'}$ will then have an independent set of size $k+1$. Thus, $\overline{G}$ is α-critical and $\alpha(G) = k$.

3.1 Theorem.
Assume G is k-saturated. Then G contains at least $2k - |G|$ conical vertices. [3]

3.2 Observation.
From theorem 3.1, thinking in terms of of the complement of a graph, we get that if G is α-critical and connected, then G must satisfy $\alpha(G) \leq \frac{|G|}{2}$.

Proof. Beginning with a graph G, if G is α-critical with $\alpha(G) = k$, then due to observation 3.1, $\overline{G}$ is k-saturated. Now, following from theorem 3.1, $\overline{G}$ contains at least $2k - |\overline{G}|$ conical vertices. This means that G must contain at least $2k - |\overline{G}| = 2\alpha(G) - |G|$ isolated vertices. However, G is connected, so the number of isolated vertices must equal zero and so $2\alpha(G) - |G| \leq 0$. Rearranging, gives $\alpha(G) \leq \frac{|G|}{2}$ as required. □

As well, consider the contrapositive of the statement: if $\alpha(G) > \frac{|G|}{2}$, then G is either not connected or not α-critical.
3.2 Theorem.
Let G be a connected graph such that $\alpha(G) \geq \frac{|G|}{2}$. Then either G has a tight weight matrix, or there exists an induced subgraph, H, such that H has no tight weight matrix and $\alpha(H) < \frac{|H|}{2}$.

Proof. First off, following from [4], the only graph, H, with $\alpha(H) = \frac{|H|}{2}$ is the complete graph, K_2. However, since K_2 has less than 10 vertices, we know that from [1] that K_2 does not have a tight weight matrix. Thus, the statement is true for K_2 and since it is the only graph with $\alpha(K_2) = \frac{|K_2|}{2}$, we now only need to consider graphs, G, with $\alpha(G) > \frac{|G|}{2}$.

We will proceed with induction on the number of vertices.

Base case: Let G be a connected graph on 10 vertices or less with $\alpha(G) > \frac{|G|}{2}$. Then from [1], we know that all graphs on 10 vertices or less does not have a tight weight matrix, and so G does not have a tight weighting.

Inductive Hypothesis: Let G be a connected graph on n vertices such that $\alpha(G) > \frac{|G|}{2}$. Then either G has a tight weight matrix, or there exists an induced subgraph, H, such that H has no tight weight matrix and $\alpha(H) \leq \frac{|H|}{2}$.

Inductive Step: Consider a connected graph G on n+1 vertices such that $\alpha(G) > \frac{|G|}{2}$. Now if G has a tight weight matrix, we're done, so assume that G does not have a tight weight matrix. Now, from the contrapositive of observation 3.2 and due to G being connected, we know G is not α-critical. Thus, there exists an edge, e_1, such that $\alpha(G - e_1) = \alpha(G)$. This leaves us with 2 cases:

Case 1: $G - e_1$ is disconnected. In this case, either both of the components have a tight weighting, or at least one has a non-tight weighting. If one has a non-tight weight matrix, then G contains an induced subgraph, H, such that H has no tight weight matrix. Now, either H satisfies $\alpha(H) \leq \frac{|H|}{2}$ in which case we're done, or $\alpha(H) > \frac{|H|}{2}$. If $\alpha(H) > \frac{|H|}{2}$, then we apply the inductive hypothesis to H and since we already know H does not have a tight weight matrix, it must be the case the H contains an induced subgraph F such that F has no tight weight matrix and $\alpha(F) \leq \frac{|F|}{2}$. However, since F is an induced subgraph of H, it must also be an induced subgraph of G, and so we find that F satisfies the condition that G has an induced subgraph with $\alpha(F) \leq \frac{|F|}{2}$ and so we are done. Thus, assume that both components have a tight weighting. However, using both of these tight weightings on G, and setting e_1 to a zero weighting, G then has a tight weight matrix. However, this is a contradiction as we assumed G did not have a tight weight matrix. This completes this case.

Case 2: $G - e_1$ is connected. Now because G was not α-critical, $\alpha(G) = \alpha(G - e_1)$ and so $\alpha(G - e_1) > \frac{|G - e_1|}{2}$. Thus, due to observation 3.1, $G - e_1$ is still not α-critical. Therefore, we can continue to delete edges without changing the fact that the resulting graph will be α-critical as long as it's connected. Now, consider the graph that results from G, which will be denoted G', where after deleting $k - 1$ edges, G is still connected, but after deleting the kth edge, G has now become disconnected. We can now add back all edges that do not reconnect G' as this will not decrease $\alpha(G')$ since deleting it did not. Now, we have the same situation as Case 1, with the only difference being that if the two disconnected components of G' have a tight weighting, we use their tight weightings and put the weightings of all edges we deleted to obtain G' to 0, giving us a tight weighting for G. This completes this case.

Therefore, G satisfies the statement, and so by induction, the statement is true for all graphs. $\square$

3.3 No Bound on Inertia Bound Gap

3.3 Theorem.
For a graph G, there does not exist a bound on the difference between $\alpha(G)$ and the minimum inertia of a weight matrix corresponding to G.

Proof. Consider a graph, G, such that G does not have a tight weight matrix with the gap from equality in the inertia bound being γ, and G-v does not have a tight weight matrix and $\alpha(G) = \alpha(G - v)\ \forall v \in V(G)$. Paley 17 is an example of a graph that has this property so there will exist such a graph. Now, construct a graph H by connecting two copies of G, G_1 and G_2, with an edge between a vertex on each copy of G that does not belong to the independent set of either G_1 or G_2.

Now, consider the induced subgraph H' obtained by deleting the vertex from the component G_1 in H that connects to the cut-edge in H. This will result in H' having two components, $G_1 - v$ and G_2. Now any weight matrix of H', W' will have the form:

$$W' = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}$$

With W_1 and W_2 being the weight matrices associated with the components $G_1 - v$ and G_2, respectively. As a result, the eigenvalues of W' will be equal to the eigenvalues of W_1 and W_2. Thus, for any weight matrix W' of H', the inertia of W' will be equal to the sum of the inertias of W_1 and W_2. Therefore,

$$\begin{aligned}
&\min\{|H'| - n_+(W'), |H'| - n_-(W')\} \\
&= \min\{|G-v| - n_+(W_1), |G-v| - n_-(W_1)\} + \min\{|G| - n_+(W_2), |G| - n_-(W_2)\} \\
&\leq 2\min\{|G| - n_+(W_2), |G| - n_-(W_2)\} \quad (13)
\end{aligned}$$

From Cauchy's Eigenvalue Interlacing Theorem , we know any weight matrix associated with H, W, will have at least as many positive and negative eigenvalues as W'. Thus,

$$\begin{aligned}\min\{|H'| - n_+(W'), |H'| - n_-(W')\} \\ \leq 2\min\{|G| - n_+(W_2), |G| - n_-(W_2)\} \\ \leq \min\{|H| - n_+(W), |H| - n_-(W)\} \quad (14)\end{aligned}$$

Due to how we created H, $\alpha(H) = 2\alpha(G)$ because the largest independent set of H will just equal the union of the largest independent set of G_1 and G_2 since we connected G_1 and G_2 without connecting edges between the two independent sets.

Therefore, we get:

$$\begin{aligned}\alpha(H) \\ = 2\alpha(G) \\ = 2\gamma + 2\min|G| - n_+(W_2), |G| - n_-(W_2) \\ \leq 2\gamma + \min\{|H| - n_+(W), |H| - n_-(W)\} \quad (15)\end{aligned}$$

Therefore, the gap between equality in the inertia bound has at least doubled. Now as long as there exists a graph without a tight weight matrix, F, such that $F - v_1 - v_2$ also does not have a tight weight matrix, we can generalize this result to create a new graph, connecting arbitrarily many copies of F and following the same argument to multiply the gap in the inertia bound by how many copies of F we connect.

□

References

[1] Randall J Elzinga. *The Minimum Witt Index of a Graph.* PhD thesis, Queens University, 2007.

[2] Cameron Franc. Cayley graphs.

[3] A. Hajnal. A theorem on k-saturated graphs. *Canad. J. Math.*, 17:720–724, 1965.

[4] Michael D Plummer. Some covering concepts in graphs. *Journal of Combinatorial Theory*, 8(1):91–98, 1970.

[5] John Sinkovic. A graph for which the inertia bound is not tight. *arXiv preprint arXiv:1609.02826*, 2016.