

C&O URA Spring 2017

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1 Inertia Bounds

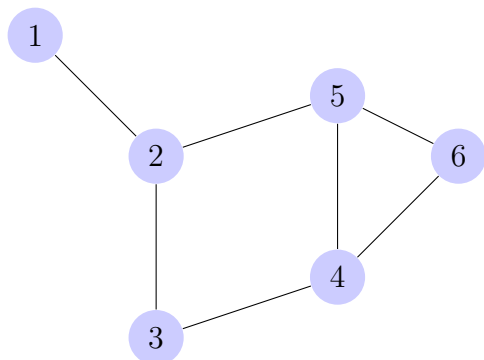
1.1 Introduction on Inertia Bounds

1.1 Definition.

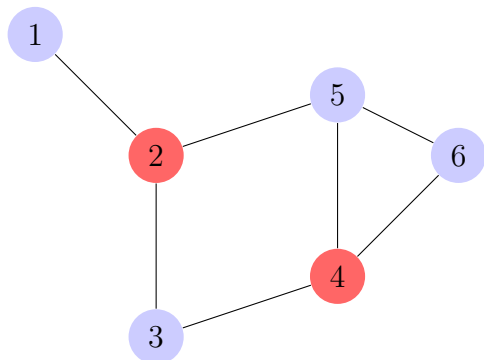
Independent Set — An independent set is a set of vertices belonging to a graph in which no two vertices are adjacent.

1.1 Example.

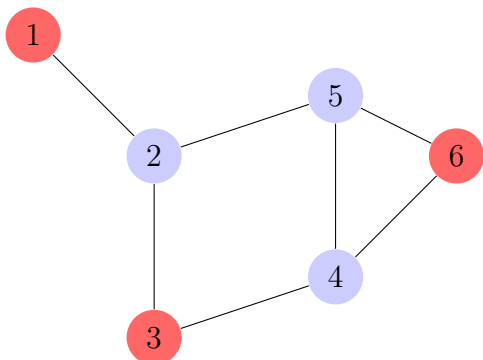
Consider the following graph:



An example of an independent set in this graph is:



However, often the independent set we are most interested in finding is the largest one:



1.2 Definition.

Independence Number — The independence number of a graph G , denoted $\alpha(G)$, is the size of the largest independent set of G .

1.3 Definition.

Weight Matrix — The weight matrix of a graph G , is a matrix defined by:

$$W_{i,j} = \begin{cases} c_{i,j} & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

with v_i a vertex of G and $c_{i,j}$, a constant.

The weight matrix of a graph, is identical to an adjacency matrix, except where there was a 1 in the matrix at entry $A_{i,j}$ if vertices v_i and v_j were adjacent, there is now a constant indicating a weighting for the edge between v_i and v_j .

For any graph G , there exists a bound on $\alpha(G)$, known as the Cvetković bound (also referred to as the Interia Bound). This bound provides a relationship between $\alpha(G)$ and the number of positive, negative, and zero eigenvalues of the weight matrix, W , associated with G . The Cvetković bound of G , is:

$$\alpha(G) \leq \min\{|G| - n_+(W), |G| - n_-(W)\} \quad (2)$$

Where $n_+(W)$ and $n_-(W)$ denote the number of positive and negative eigenvalues of W , respectively.

To prove this, we first need to introduce a result that comes from the Eigenvalue Interlacing Theorem:

1.1 Theorem.

Corollary of Eigenvalue Interlacing Theorem — Let A be an $n \times n$

real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ and let C be a $k \times k$ principal submatrix of A with eigenvalues $\tau_1 \geq \tau_2 \geq \dots \geq \tau_k$. Then $\lambda_i \geq \tau_i$ for all $i \in \{1, \dots, k\}$. [5]

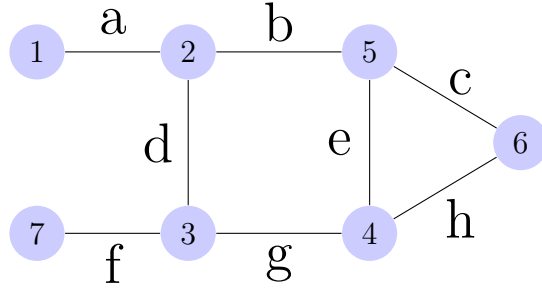
1.4 Definition.

Principal Submatrix — The principal submatrix of an $n \times n$ matrix A is the submatrix obtained where if row_i is excluded in the submatrix, then $column_i$ is excluded as well. Note that all principal submatrices of a weight matrix W , correspond to an induced subgraph in the graph represented by W .

1.2 Example.

The following is an example of a principal submatrix in relation to graph theory.

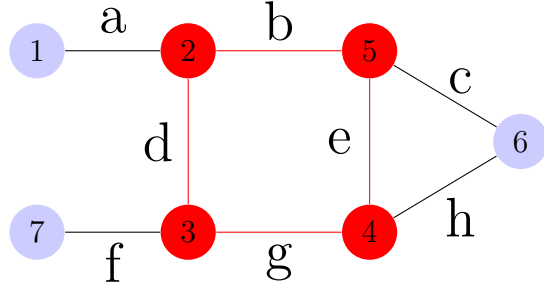
Consider the following graph:



and corresponding weight matrix:

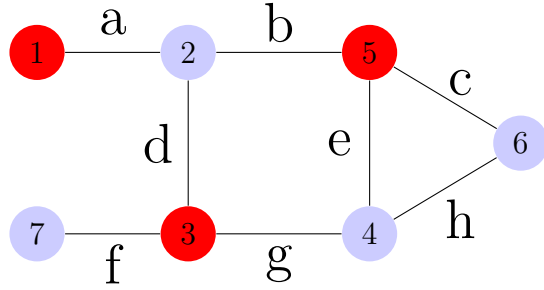
$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & d & 0 & b & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f \\ 0 & 0 & g & 0 & e & h & 0 \\ 0 & b & 0 & e & 0 & c & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can see the following principal submatrix and corresponding induced subgraph:



$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & d & 0 & b & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f \\ 0 & 0 & g & 0 & e & h & 0 \\ 0 & b & 0 & e & 0 & c & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 \end{bmatrix}$$

As well, we see the following principal submatrix of an independent set of the graph:



$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & d & 0 & b & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f \\ 0 & 0 & g & 0 & e & h & 0 \\ 0 & b & 0 & e & 0 & c & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now to prove the Cvetković Bound:

1.2 Theorem.

Cvetković Bound — Let G be a graph on n vertices, and W be the weight

matrix of G . Then the following inequality holds:

$$\alpha(G) \leq \min\{|G| - n_+(W), |G| - n_-(W)\} \quad (3)$$

Proof. ¹ Let H be the subgraph of G formed by the vertices in an independent set of size s . Then H is an induced subgraph of G and all eigenvalues of the principal submatrix $W(H)$ are 0 since the principal submatrix will just be a zero matrix. Let λ_i denote the i th largest eigenvalue of W and τ_i denote the i th largest eigenvalue of $W(H)$. Now, by interlacing, we have,

$$\lambda_i \geq \tau_i = 0 \text{ for all } i \in \{1, \dots, s\} \quad (4)$$

and so

$$n - n_-(W) = n_+(W) + n_0(W) \geq s \quad (5)$$

Also, note that by negating W , the positive eigenvalues become negative eigenvalues and vice versa. Thus,

$$n - n_+(W) = n - n_-(-W), \quad (6)$$

However, the principal submatrix corresponding to H in $-W$ is still the zero matrix and thus has all zero eigenvalues. Thus, by interlacing, we get a similar result as above,

$$n - n_+(W) = n - n_-(-W) = n_+(-W) + n_0(-W) \geq s \quad (7)$$

Therefore, both $n - n_+(W)$ and $n - n_-(W)$ are greater than or equal to s . Since s is the size of the independent set, we can see that letting $s = \alpha(G)$, we get:

$$\alpha(G) \leq \min\{|G| - n_+(W), |G| - n_-(W)\} \quad (8)$$

□

1.2 Graphs with Tight Inertia Bounds

1.2.1 Perfect Graphs

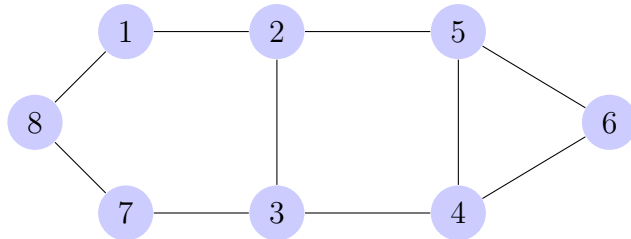
1.5 Definition.

Chromatic Number — The chromatic number of a graph, $\chi(G)$, is the minimum number of colours needed in a proper colouring of G . [1]

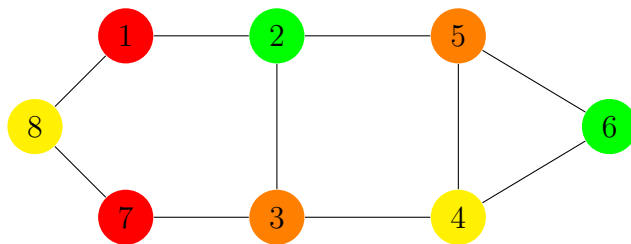
¹Interesting Graphs and their Colourings, unpublished lecture notes C. Godsil (2004)

1.3 Example.

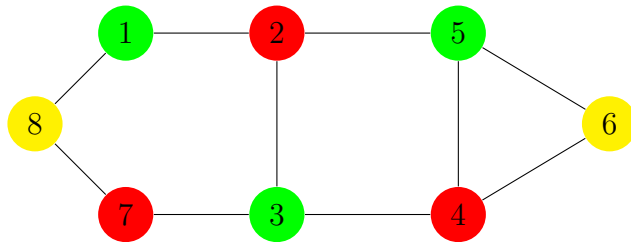
Consider the following graph:



An example of a colouring would be:



However, $\chi(G)$ for this graph is 3:



1.6 Definition.

Clique — An m -clique in a graph is a complete subgraph on m vertices. [1]

The clique number, $\omega(G)$, is the number of vertices in a maximum clique of G .

1.7 Definition.

Clique Cover — A Clique Cover of the vertex set $V(G)$ of a graph G is a set of cliques C , such that each vertex is in at least one clique in C .

The clique cover number, $\theta(G)$ is the minimum number of cliques needed in a clique cover of G . [1]

1.4 Example.

Consider the following graph:

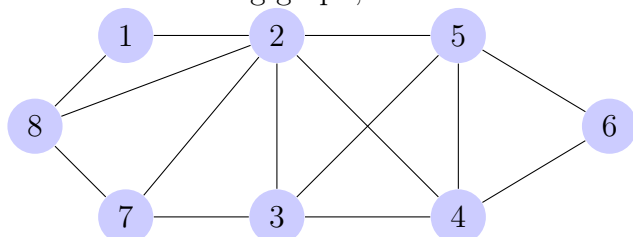
Perfect Graph Theorem — A Graph G is perfect if and only if its complement, \overline{G} , is also perfect.

1.1 Observation.

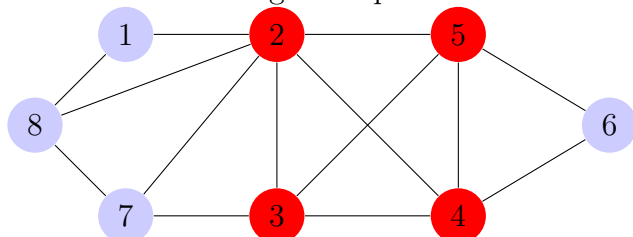
For a graph G , $\omega(G) = \alpha(\overline{G})$

1.5 Example.

Consider the following graph, G :

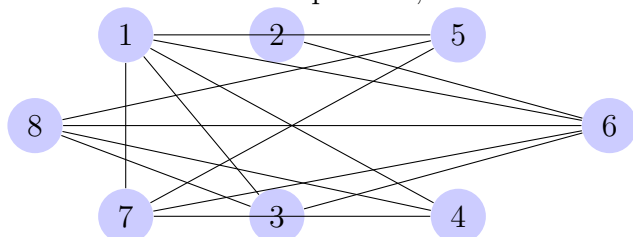


We see that the largest clique in G is:

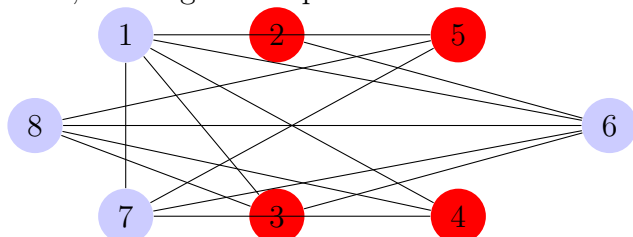


Thus, $\omega(G)$ is 4.

Now consider G 's complement, \overline{G} :



In \overline{G} , the largest independent set is:



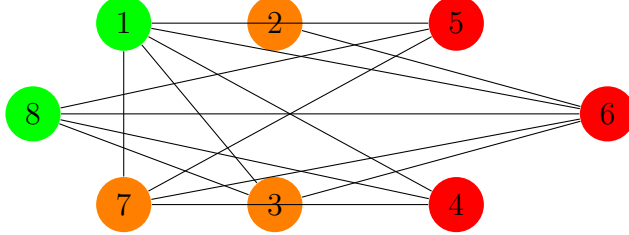
Therefore, we see $\omega(G) = 4 = \alpha(\overline{G})$

1.2 Observation.

Similar to the last observation, for a graph G , $\theta(G) = \chi(\overline{G})$

1.6 Example.

Consider the same graph from the last example. Recall that we calculated $\theta(G)$ to be 3. Now, we can find $\chi(\overline{G})$ to be 3 as well:



Thus, $\theta(G) = 3 = \chi(\overline{G})$

1.1 Lemma.

Let G be a graph. Then $\alpha(G) \leq \min\{|G| - n_+(W), |G| - n_-(W)\} \leq \theta(G)$. Thus, if $\alpha(G) = \theta(G)$, G has a tight inertia bound. [1]

Proof. Consider a clique partition, \mathcal{C} , of a graph G . Let \hat{A} , denote the adjacency matrix of G where the only connected components are the cliques in \mathcal{C} .

Now if we consider the adjacency matrix of the complete graph, K_n , we see that it is equal to $J_n - I_n$ where J_n is the all ones matrix. \square

1.4 Theorem.

Every Perfect Graph, G , has a tight inertia bound

Proof. By the Perfect Graph Theorem (theorem 1.3), we know that \overline{G} , is also perfect. Thus \overline{G} satisfies that $\chi(H) = \omega(H)$ for all subgraphs, H , of \overline{G} , by definition. Thus, since $\chi(\overline{G}) = \omega(\overline{G})$, we can get from the observation 1.1 and 1.2, that

$$\alpha(G) = \omega(\overline{G}) = \chi(\overline{G}) = \theta(G) \quad (9)$$

Therefore, from lemma 1.1, G has a tight inertia bound. \square

1.5 Theorem.

Strong Perfect Graph Theorem [1] — A graph G is a perfect graph if and only if both G and its complement, \overline{G} , do not contain a induced odd cycle of length at least 5.

1.3 Observation.

Due to each perfect graph having a tight inertia bound, and the Strong Perfect Graph Theorem (theorem 1.5), every graph not containing an induced odd cycle of length 5 or greater has a tight inertia bound.

1.2.2 Latin Square Graphs

1.2.3 Graphs with an Eigensharp decomposition by Stars

1.2.4 Summary

In summary, the following list of graphs attain a tight inertia bound:

- Graphs on 10 or fewer vertices (pg 81 [1])
- Vertex Transitive graphs on 12 or fewer vertices (pg 81 [1])
- Perfect Graphs
- Latin Square Graphs
- Graphs with an Eigensharp decomposition by stars

1.3 Other Bounds on Independence Number

2 Algorithm to Find Graphs Lacking a Tight Inertia Bound

2.1 Outline of Method

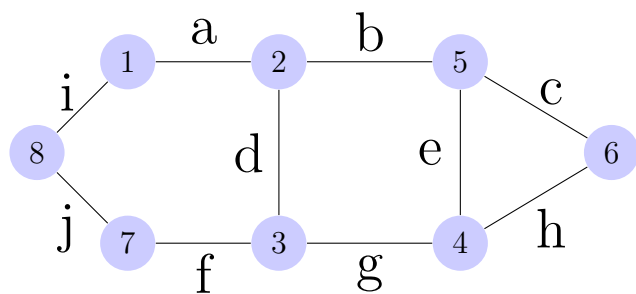
2.1 Definition.

Optimal Weight Matrix — A weight matrix, W , of a graph, G , is optimal if

$$\alpha(G) = \min\{|G| - n_+(W), |G| - n_-(W)\} \quad (10)$$

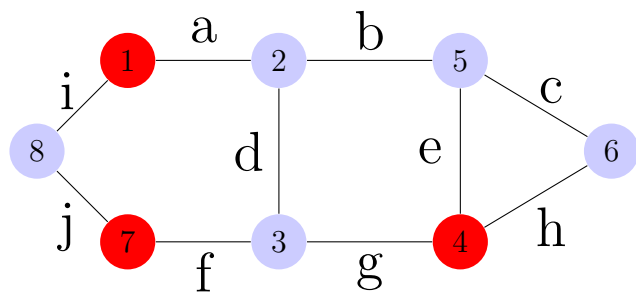
2.1 Example.

Consider the following graph, G , with corresponding weight matrix W :

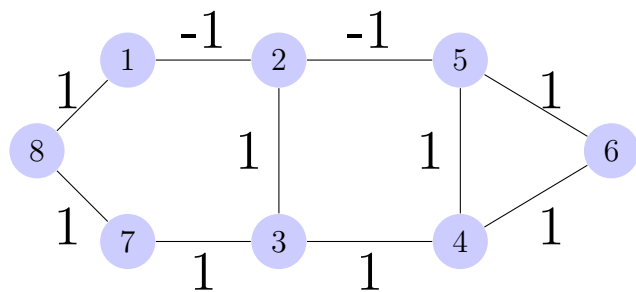


$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 & i \\ a & 0 & d & 0 & b & 0 & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f & 0 \\ 0 & 0 & g & 0 & e & h & 0 & 0 \\ 0 & b & 0 & e & 0 & c & 0 & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 & j \\ i & 0 & 0 & 0 & 0 & 0 & j & 0 \end{bmatrix}$$

We can see the independent number of G is 3:



Now, let G have the following weighting:



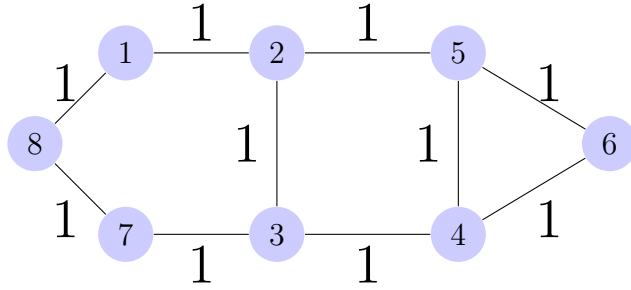
$$\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Finding the eigenvalues of W , we find there are 3 positive eigenvalues and 5 negative eigenvalues. Thus, we see for this weight matrix we have:

$$\begin{aligned} \alpha(G) &= \min\{|G| - n_+(W), |G| - n_-(W)\} \\ &= \min\{8 - 3, 8 - 5\} \\ &= \min\{5, 3\} \\ &= 3 \end{aligned} \tag{11}$$

Therefore, this is an optimal weight matrix of G .

Now consider the following weighting for the same graph:



$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Finding the eigenvalues of this weight matrix, we find there are 4 positive

eigenvalues and 4 negative eigenvalues. Thus, we see we get:

$$\begin{aligned}
\alpha(G) &= 3 \neq \min\{|G| - n_+(W), |G| - n_-(W)\} \\
&= \min\{8 - 4, 8 - 4\} \\
&= \min\{4, 4\} \\
&= 4
\end{aligned} \tag{12}$$

Therefore, we see that the previous weighting was not optimal for G .

2.1 Lemma.

If a graph, G , with weight matrix W , has two induced subgraphs, S_1 and S_2 , such that S_1 has $\alpha(G) + 1$ positive eigenvalues under the weighting of W , and S_2 has $\alpha(G) + 1$ negative eigenvalues under the weighting of W , then W is not optimal

Proof.

□

2.2 Preliminary Tests to Determine if the Graph may be Suitable

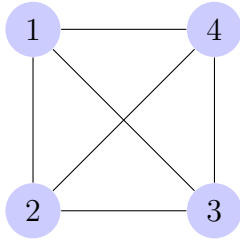
2.2.1 Test for α -Critical

2.2 Definition.

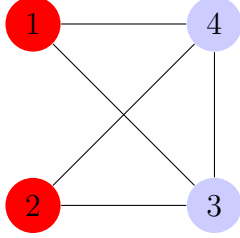
α -Critical — A graph, G , is α -critical if $\alpha(G) < \alpha(G - e)$ for all edges e .

2.2 Example.

Consider the following graph G :



we see that $\alpha(G) = 1$. But since this graph is complete, we see that if we delete any edge, we can get an independent set of size 2 by making the set include the two vertices that were connected by the edge we deleted. For example:



Thus, G is α -critical.

2.2 Lemma.

If G is α -critical, and W an optimal weight matrix of G , then $w_{i,j} \neq 0$ for all $i, j \in E(G)$

Proof. Assume for the sake of contradiction, that for some $i, j \in E(G)$, we have $w_{i,j} = 0$. Then, we know $\alpha(G - e_{i,j}) > \alpha(G)$ because G is α -critical. Thus:

$$\alpha(G) < \alpha(G - e_{i,j}) \leq \min\{|G| - n_+(W), |G| - n_-(W)\}$$

Thus, we see that the inertia bound is not tight for G , so W is not an optimal weight matrix of G , which is a contradiction. \square

Due to the complexity of needing to consider edges that could potentially be zero in the weight matrix, it is easier to consider graphs that are restricted to only non-zero edge weights in its optimal weight matrix. Thus, it makes sense to only consider α -critical graphs, because of Lemma 2.2 ensuring that all α -critical graphs have non-zero weight matrices.

2.2.2 Determining Each Triangle Must Have the Same Sign

2.3 Graphs Currently Found

Graph	Vertices	α	Degree	Circulant	Strongly Regular	Arc Transitive
¹	16	4	5	No	No	No
²	16	2	10	No	(16,10,6,6)	Yes
³	17	3	8	[1,2,4,8]	(17,8,3,4)	Yes
⁴	19	4	6	[1,7,8]	No	Yes
⁵	20	2	13	No	No	No
⁶	20	2	13	[1,3,4,7,8,9,10]	No	No
⁷	22	3	11	[1,2,3,5,10,11]	No	No
⁸	24	3	12	No	No	No
⁹	24	3	12	No	No	No
¹⁰	24	4	9	No	No	No
¹¹	24	4	10	[1,2,4,8,9]	No	No
¹²	24	4	9	No	No	No
¹³	24	3	12	No	No	No
¹⁴	24	4	9	No	No	No
¹⁵	24	4	9	No	No	No
¹⁶	24	2	16	No	No	No
¹⁷	24	2	16	No	No	No
¹⁸	24	2	16	[1,2,3,4,6,7,8,10]	No	No
¹⁹	24	2	16	No	No	No

¹Otr@PKoE?T.iOoOG_dg_m

²O~~em]uj[vmsZTUrfFwN~

³P}qtSeLUbaKeQZJabfGmmG~G

⁴R}ecZ@OH?oW~@gOWcL.p?hkHL?GuG

⁵S~~vVjjve}vmxymIG~Oi~Qm{jfxjNw{z{

⁶S~~vnZjvUtvimj'~nibtTP}{ffwk~wR~{

⁷Uv~LnbgeDShP\G}HuXmePrSemapSxqJWG|ZCVhw

⁸Wunneyzx~W]OwBPfcroK~S{OlogtIoyPlPFMIjWPUvaGu~

⁹WvrlvjZj~c_wBTRcroK~K{HLpGtPo[ikpImQHrWaUn'Cv^

¹⁰WvvdIJpB_c[LEHPiH?PsE_GAsWKcwBXhGDgOFXWIBV@CZT

¹¹W}mKmlbqD.JJMMBYa]_{??ucC{YKeHKXPadVXOmQqbqEDMp

¹²W}nS]QeoOq_nWS]?KcPQUdPgU@_TBG_ug@ei?jCgCwY_?J~

¹³W}{}VNBmtdyWkic?zg]gevHT_TfGo~bPK|xHkJJMolozdq\s

¹⁴W}~SvAbp@IcjDgEaj?@BKPCgBbXP@oCz?BLdE@KwGu[?EFZ

¹⁵W~nELU\'aKkXTJ]?@cGUB@KgBSX?wG_sS'DUCGyWO'}?@M^

¹⁶W~~~vnnv|~}gzH}'za|J^ef|~wB.JNisn[bn^@^nwez~V^~

¹⁷W~~~vnn{vT{nvFnFo^}~Dnw\{^AF|hFz[YZ~DT~wX^~n{B~

¹⁸W~~~vnn{vXyjqnnFs^}~Knw\[_QF]hiz[iznCt~wX^~n{B~

¹⁹W~~~vvu|^\\jvivTvtTyj_~|}ibyiiF}{b{~C{~wU^~f~~

2.3.1 Graphs Created from Deleting a Vertex

Graph	Created From	Vertices	α	Regular	Circulant	Strongly Regular	Arc Transitive
1	1	15	4	No	No	No	No
2	2	15	2	No	No	No	No
3	3	16	3	No	No	No	No
4	4	18	4	No	No	No	No
5	6	19	2	No	No	No	No
6	7	21	3	No	No	No	No
7	8	23	3	No	No	No	No
8	9	23	3	No	No	No	No
9	10	23	4	No	No	No	No
10	11	23	4	No	No	No	No
11	12	23	4	No	No	No	No
12	13	23	3	No	No	No	No
13	14	23	4	No	No	No	No
14	15	23	4	No	No	No	No
15	16	23	2	No	No	No	No
16	17	23	2	No	No	No	No
17	18	23	2	No	No	No	No
18	19	23	2	No	No	No	No
19	4	17	4	No	No	No	No
20	5	18	2	No	No	No	No
21	9	22	4	No	No	No	No
22	11	22	4	No	No	No	No
23	15	22	2	No	No	No	No
24	18	22	2	No	No	No	No
25	19	16	4	No	No	No	No
26	21	21	4	No	No	No	No
27	24	21	2	No	No	No	No

¹Ntr@PKoE?T iOoOG_dg

²N~~em]uj[vmsZTUrfFw

³O}qtSeLUbaKeQZJabfGmm

⁴Q}ecZ@OH?oW~@gOWcI_p?hkHL?

⁵R~~vnZjvUtvimj~nibtTP}{ffwk~w

⁶Tv~LnbgfeDShP\G}HuXmePrSemapSxqJWG|Z

⁷Vunneyzx~W]OwBPfcroK~S{OlogtIoyPIPFMIjWPUv_

⁸VvrlvjZj~c_wBTRcroK~K{HLpGtPo{pplmQHRWaUn_

⁹VvdtIJpB_c[LEHPiH?PsE_GAsWKcwBXhGDgOFXWIBV?

¹⁰V}mKmIbqD_JJMMBYa]_{??ucC{YKeHKXPadVXOmQqbq?

¹¹V}nS|QeoOq_nWS]?KcPQUPDgU@_TBG_ug@ei?jCgCwY_

¹²W}}VNBmtdyWkic?zg]gevHT_TfGo~bPK|xHkJJMolozdq\s

¹³V}~SvAbp@IcjDgEaj?@BKPCgBbXP@oCz?BLdE@KwGu[?

¹⁴V~nELU\'aKkXTJ]?@cGUB@KgBSX?wG_sS'DUCgyWO'}?

¹⁵V~~~vnnv|~}gzH}'za|J^ef]~wBJNisn[bn^@^nwez~_

¹⁶V~~~vnnv{vT{nvFnFo^}~Dnw\{^AF|hFz|YZ~DT~wX^~

3 Other Useful Information

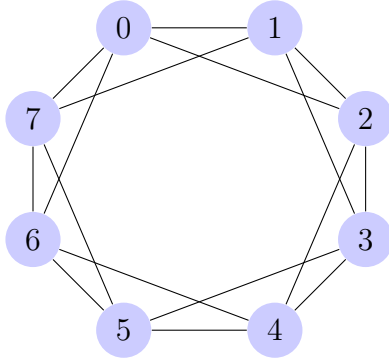
3.1 Cayley Graphs

3.1 Definition.

Cayley Graph — Let H be a finite group, and $S \subseteq H$, be a subset of H . Then the Cayley Graph $C(H, S)$, has a vertex for each element in H . There exists an edge between two vertices g and h , if and only if there exists $s \in S$ such that $sh = g$. If G is a graph such that there exists a group H and a generating set $S \subseteq H$ with $G \cong C(H, S)$, then G is a Cayley Graph. [2]

3.1 Example.

Consider the group \mathbb{Z}_8 and let the generating set be $S = \{1, 2\}$. The vertex set will be $\{0, 1, 2, 3, 4, 5, 6, 7\}$ and there will be an edge between two vertices, g and h , if for an $s \in S$, $g + s = h$:



3.2 John's Proof

3.2 Definition.

k-saturated — The graph G is said to be k -saturated if it does not contain a complete $(k + 1)$ -graph, but every graph G' obtained from adding a new edge to G contains a complete $(k + 1)$ -graph. [3]

3.3 Definition.

Conical Vertex — A vertex, V , is a conical vertex of a graph, G , if V is adjacent to every vertex in G .

3.1 Observation.

A graph, G , is k -saturated, if and only if its complement, \overline{G} , is α -critical,

with $\alpha(\overline{G}) = k$.

This is due to the fact that since G contains a complete k -graph but not a complete $(k + 1)$ -graph, \overline{G} will have a maximum independent set of size k . By adding any edge to G to obtain G' , G' will contain a complete $(k+1)$ -graph, and thus $\overline{G'}$ will then have an independent set of size $k + 1$. Thus, \overline{G} is α -critical and $\alpha(G) = k$.

3.1 Theorem.

Assume G is k -saturated. Then G contains at least $2k - |G|$ conical vertices.
[3]

3.2 Observation.

From theorem 3.1, thinking in terms of the complement of a graph, we get that if G is α -critical and connected, then G must satisfy $\alpha(G) \leq \frac{|G|}{2}$.

Proof. Beginning with a graph G , if G is α -critical with $\alpha(G) = k$, then due to observation 3.1, \overline{G} is k -saturated. Now, following from theorem 3.1, \overline{G} contains at least $2k - |\overline{G}|$ conical vertices. This means that G must contain at least $2k - |\overline{G}| = 2\alpha(G) - |G|$ isolated vertices. However, G is connected, so the number of isolated vertices must equal zero and so $2\alpha(G) - |G| \leq 0$. Rearranging, gives $\alpha(G) \leq \frac{|G|}{2}$ as required. \square

As well, consider the contrapositive of the statement: if $\alpha(G) > \frac{|G|}{2}$, then G is either not connected or not α -critical.

3.2 Theorem.

Let G be a connected graph such that $\alpha(G) \geq \frac{|G|}{2}$. Then either G has a tight weight matrix, or there exists an induced subgraph, H , such that H has no tight weight matrix and $\alpha(H) < \frac{|H|}{2}$.

Proof. First off, following from [4], the only graph, H , with $\alpha(H) = \frac{|H|}{2}$ is the complete graph, K_2 . However, since K_2 has less than 10 vertices, we know that from [1] that K_2 does not have a tight weight matrix. Thus, the statement is true for K_2 and since it is the only graph with $\alpha(K_2) = \frac{|K_2|}{2}$, we now only need to consider graphs, G , with $\alpha(G) > \frac{|G|}{2}$.

We will proceed with induction on the number of vertices.

Base case: Let G be a connected graph on 10 vertices or less with $\alpha(G) > \frac{|G|}{2}$. Then from [1], we know that all graphs on 10 vertices or less does not have a tight weight matrix, and so G does not have a tight weighting.

Inductive Hypothesis: Let G be a connected graph on n vertices such that $\alpha(G) > \frac{|G|}{2}$. Then either G has a tight weight matrix, or there exists an induced subgraph, H , such that H has no tight weight matrix and $\alpha(H) \leq \frac{|H|}{2}$.

Inductive Step: Consider a connected graph G on $n+1$ vertices such that $\alpha(G) > \frac{|G|}{2}$. Now if G has a tight weight matrix, we're done, so assume that G does not have a tight weight matrix. Now, from the contrapositive of observation 3.2 and due to G being connected, we know G is not α -critical. Thus, there exists an edge, e_1 , such that $\alpha(G - e_1) = \alpha(G)$. This leaves us with 2 cases:

Case 1: $G - e_1$ is disconnected. In this case, either both of the components have a tight weighting, or at least one has a non-tight weighting. If one has a non-tight weight matrix, then G contains an induced subgraph, H , such that H has no tight weight matrix. Now, either H satisfies $\alpha(H) \leq \frac{|H|}{2}$ in which case we're done, or $\alpha(H) > \frac{|H|}{2}$. If $\alpha(H) > \frac{|H|}{2}$, then we apply the inductive hypothesis to H and since we already know H does not have a tight weight matrix, it must be the case the H contains an induced subgraph F such that F has no tight weight matrix and $\alpha(F) \leq \frac{|F|}{2}$. However, since F is an induced subgraph of H , it must also be an induced subgraph of G , and so we find that F satisfies the condition that G has an induced subgraph with $\alpha(F) \leq \frac{|F|}{2}$ and so we are done. Thus, assume that both components have a tight weighting. However, using both of these tight weightings on G , and setting e_1 to a zero weighting, G then has a tight weight matrix. However, this is a contradiction as we assumed G did not have a tight weight matrix. This completes this case.

Case 2: $G - e_1$ is connected. Now because G was not α -critical, $\alpha(G) = \alpha(G - e_1)$ and so $\alpha(G - e_1) > \frac{|G - e_1|}{2}$. Thus, due to observation 3.1, $G - e_1$ is still not α -critical. Therefore, we can continue to delete edges without changing the fact that the resulting graph will be α -critical as long as it's connected. Now, consider the graph that results from G , which will be denoted G' , where after deleting $k - 1$ edges, G is still connected, but after deleting the k th edge, G has now become disconnected. We can now add back all edges that do not reconnect G' as this will not decrease $\alpha(G')$ since deleting it did not. Now, we have the same situation as Case 1, with the only difference being that if the two disconnected components of G' have a tight weighting, we use their tight weightings and put the weightings of all edges we deleted to obtain G' to 0, giving us a tight weighting for G . This

completes this case.

Therefore, G satisfies the statement, and so by induction, the statement is true for all graphs. \square

3.3 No Bound on Inertia Bound Gap

3.3 Theorem.

For a graph G , there does not exist a bound on the difference between $\alpha(G)$ and the minimum inertia of a weight matrix corresponding to G .

Proof. Consider a graph, G , such that G does not have a tight weight matrix with the gap from equality in the inertia bound being γ , and $G-v$ does not have a tight weight matrix with the gap also being γ and $\alpha(G) = \alpha(G-v) \forall v \in V(G)$. Paley 17 is an example of a graph that has this property so there will exist such a graph. Now, construct a graph H by connecting two copies of G , G_1 and G_2 , with an edge between a vertex on each copy of G that does not belong to the independent set of either G_1 or G_2 .

Now, consider the induced subgraph H' obtained by deleting the vertex from the component G_1 in H that connects to the cut-edge in H . This will result in H' having two components, $G_1 - v$ and G_2 . Now any weight matrix of H' , W' will have the form:

$$W' = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}$$

With W_1 and W_2 being the weight matrices associated with the components $G_1 - v$ and G_2 , respectively. As a result, the eigenvalues of W' will be equal to the eigenvalues of W_1 and W_2 . Thus, for any weight matrix W' of H' , the inertia of W' will be equal to the sum of the inertias of W_1 and W_2 . Therefore,

$$\begin{aligned} \min\{|H'| - n_+(W'), |H'| - n_-(W')\} &= \min\{|G-v| - n_+(W_1), |G-v| - n_-(W_1)\} \\ &\quad + \min\{|G| - n_+(W_2), |G| - n_-(W_2)\} \\ &= 2 \min\{|G| - n_+(W_2), |G| - n_-(W_2)\} \end{aligned} \tag{13}$$

From Cauchy's Eigenvalue Interlacing Theorem, we know any weight matrix, W , associated with H , will have at least as many positive and negative

eigenvalues as W' . Thus,

$$\begin{aligned} 2 \min\{|G| - n_+(W_2), |G| - n_-(W_2)\} &= \min\{|H'| - n_+(W'), |H'| - n_-(W')\} \\ &\leq \min\{|H| - n_+(W), |H| - n_-(W)\} \end{aligned} \tag{14}$$

Due to how we created H , $\alpha(H) = 2\alpha(G)$ because the largest independent set of H will just equal the union of the largest independent set of G_1 and G_2 since we connected G_1 and G_2 without connecting edges between the two independent sets.

Therefore, we get:

$$\begin{aligned} \alpha(H) &= 2\alpha(G) \\ &= 2\gamma + 2 \min\{|G| - n_+(W_2), |G| - n_-(W_2)\} \\ &\leq 2\gamma + \min\{|H| - n_+(W), |H| - n_-(W)\} \end{aligned} \tag{15}$$

Therefore, the gap between equality in the inertia bound has at least doubled. Now as long as there exists a graph, F , without a tight weight matrix, such that $F - v_1 - v_2$ for some $v_1, v_2 \in V(F)$ also does not have a tight weight matrix with a gap in the inertia bound equal to the gap of of the inertia bound in F , and both F and $F - v_1 - v_2$ have the same independent set, we can generalize this result to create a new graph, connecting arbitrarily many copies of F and following the same argument to multiply the gap in the inertia bound by how many copies of F we connect. (I THINK THERE DOES EXIST A GRAPH THAT WOULD SATISFY THIS PROPERTY BUT NEED TO CHECK)

□

References

- [1] Randall J Elzinga. *The Minimum Witt Index of a Graph*. PhD thesis, Queens University, 2007.
- [2] Cameron Franc. Cayley graphs.
- [3] A. Hajnal. A theorem on k-saturated graphs. *Canad. J. Math.*, 17:720–724, 1965.

- [4] Michael D Plummer. Some covering concepts in graphs. *Journal of Combinatorial Theory*, 8(1):91–98, 1970.
- [5] John Sinkovic. A graph for which the inertia bound is not tight. *arXiv preprint arXiv:1609.02826*, 2016.