

C&O URA Spring 2017

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1 Inertia Bounds

1.1 Introduction on Inertia Bounds

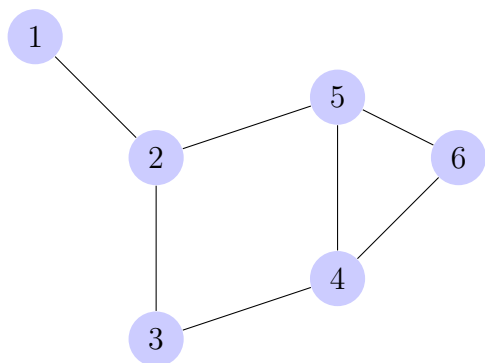
1.1 Definition.

Independent Set — An independent set is a set of vertices belonging to a

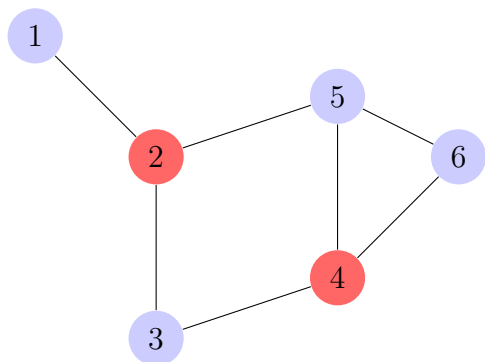
graph in which no two vertices are adjacent.

1.1 Example.

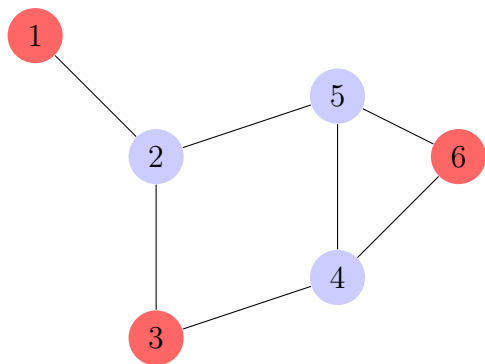
Consider the following graph:



An example of an independent set in this graph is:



However, often the independent set we are most interested in finding is the largest one:



1.2 Definition.

Independence Number — The independence number of a graph G , denoted $\alpha(G)$, is the size of the largest independent set of G .

1.3 Definition.

Weight Matrix — The weight matrix of a graph G , is a matrix defined by:

$$W_{i,j} = \begin{cases} c_{i,j} & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

with v_i a vertice of G and $c_{i,j}$, a constant.

The weight matrix of a graph, is identical to an adjacency matrix, except where there was a 1 in the matrix at entry $A_{i,j}$ if vertices v_i and v_j were adjacent, there is now a constant indicating a weighting for the edge between v_i and v_j .

For any graph G , there exists a bound on $\alpha(G)$, known as the Cvetković bound (also referred to as the Interia Bound). This bound provides a relationship between $\alpha(G)$ and the number of positive, negative, and zero eigenvalues of the weight matrix, W , associated with G . The Cvetković bound of G , is:

$$\alpha(G) \leq \min\{|G| - n_+(W), |G| - n_-(W)\} \quad (2)$$

Where $n_+(W)$ and $n_-(W)$ denote the number of positive and negative eigenvalues of W , respectively.

To prove this, we first need to introduce a result that comes from the Eigenvalue Interlacing Theorem:

1.1 Theorem.

Corollary of Eigenvalue Interlacing Theorem — Let A be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ and let C be a $k \times k$ principal submatrix of A with eigenvalues $\tau_1 \geq \tau_2 \geq \dots \geq \tau_k$. Then $\lambda_i \geq \tau_i$ for all $i \in \{1, \dots, k\}$. [3]

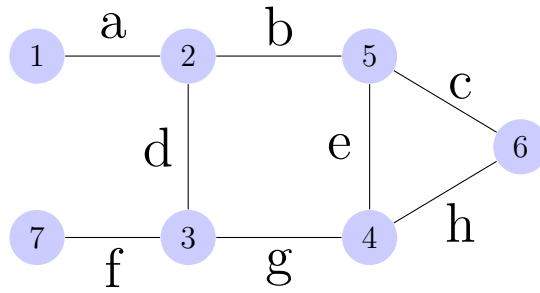
1.4 Definition.

Principal Submatrix — The principal submatrix of an $n \times n$ matrix A is the submatrix obtained where if row_i is excluded in the submatrix, then $column_i$ is excluded as well. Note that all principal submatrices of a weight matrix W , correspond to an induced subgraph in the graph represented by W .

1.2 Example.

The following is an example of a principal submatrix in relation to graph theory.

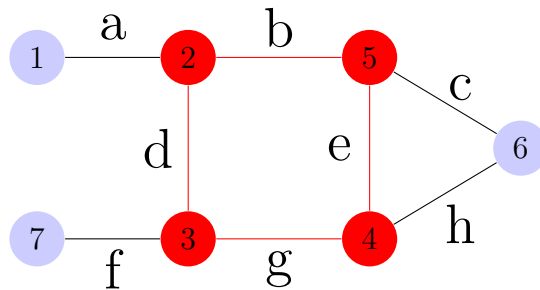
Consider the following graph:



and corresponding weight matrix:

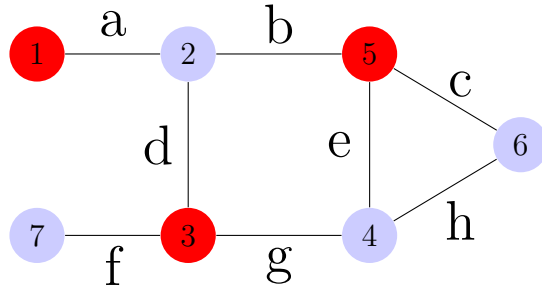
$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & d & 0 & b & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f \\ 0 & 0 & g & 0 & e & h & 0 \\ 0 & b & 0 & e & 0 & c & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can see the following principal submatrix and corresponding induced subgraph:



$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & d & 0 & b & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f \\ 0 & 0 & g & 0 & e & h & 0 \\ 0 & b & 0 & e & 0 & c & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 \end{bmatrix}$$

As well, we see the following principal submatrix of an independent set of the graph:



$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & d & 0 & b & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f \\ 0 & 0 & g & 0 & e & h & 0 \\ 0 & b & 0 & e & 0 & c & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now to prove the Cvetković Bound:

1.2 Theorem.

Cvetković Bound — Let G be a graph on n vertices, and W be the weight matrix of G . Then the following inequality holds:

$$\alpha(G) \leq \min\{|G| - n_+(W), |G| - n_-(W)\} \quad (3)$$

Proof. ¹ Let H be the subgraph of G formed by the vertices in an independent set of size s . Then H is an induced subgraph of G and all eigenvalues of the principal submatrix $W(H)$ are 0 since the principal submatrix will just be a

¹Interesting Graphs and their Colourings, unpublished lecture notes C. Godsil (2004)

zero matrix. Let λ_i denote the i th largest eigenvalue of W and τ_i denote the i th largest eigenvalue of $W(H)$. Now, by interlacing, we have,

$$\lambda_i \geq \tau_i = 0 \text{ for all } i \in \{1, \dots, s\} \quad (4)$$

and so

$$n - n_-(W) = n_+(W) + n_0(W) \geq s \quad (5)$$

Also, note that by negating W , the positive eigenvalues become negative eigenvalues and vice versa. Thus,

$$n - n_+(W) = n - n_-(-W), \quad (6)$$

However, the principal submatrix corresponding to H in $-W$ is still the zero matrix and thus has all zero eigenvalues. Thus, by interlacing, we get a similar result as above,

$$n - n_+(W) = n - n_-(-W) = n_+(-W) + n_0(-W) \geq s \quad (7)$$

Therefore, both $n - n_+(W)$ and $n - n_-(W)$ are greater than or equal to s . Since s is the size of the independent set, we can see that letting $s = \alpha(G)$, we get:

$$\alpha(G) \leq \min\{|G| - n_+(W), |G| - n_-(W)\} \quad (8)$$

□

1.2 Graphs with Tight Inertia Bounds

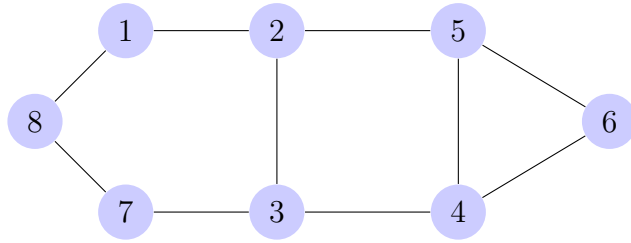
1.2.1 Perfect Graphs

1.5 Definition.

Chromatic Number — The chromatic number of a graph, $\chi(G)$, is the minimum number of colours needed in a proper colouring of G . [1]

1.3 Example.

Consider the following graph:



```

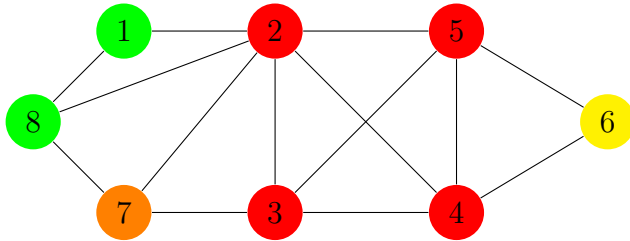
graph LR
    1((1)) --- 2((2))
    2 --- 5((5))
    5 --- 6((6))
    6 --- 4((4))
    4 --- 3((3))
    3 --- 2
    3 --- 7((7))
    7 --- 8((8))
    8 --- 1
    4 --- 5
  
```

Clique — An m -clique in a graph is a complete subgraph on m vertices. [1]
 The clique number, $\omega(G)$, is the number of vertices in a maximum clique of G .

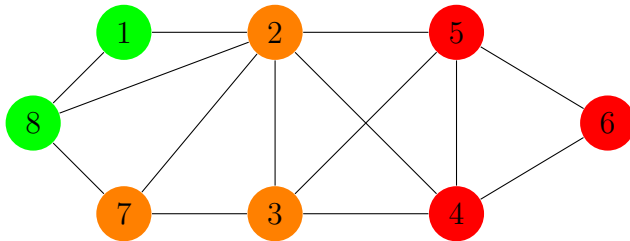
Clique Cover — A Clique Cover of the vertex set $V(G)$ of a graph G is a set of cliques C , such that each vertex is in at least one clique in C .

1.4 Example.

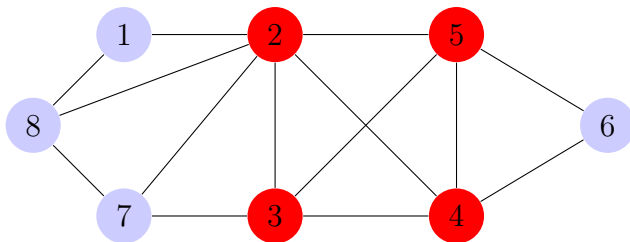
A possible clique covering is:



However, we can find that $\theta(G)$ is equal to 3 (smallest I could find):



As well, the clique number, $\omega(G)$, is 4:



1.8 Definition.

Perfect Graph — A graph G is perfect if $\chi(G) = \omega(G)$ for all induced subgraphs, H , of G .

1.3 Theorem.

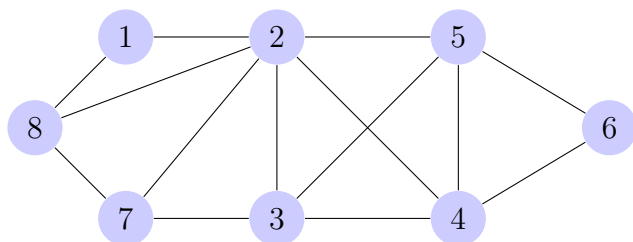
Perfect Graph Theorem — A Graph G is perfect if and only if its complement, \overline{G} , is also perfect.

1.1 Observation.

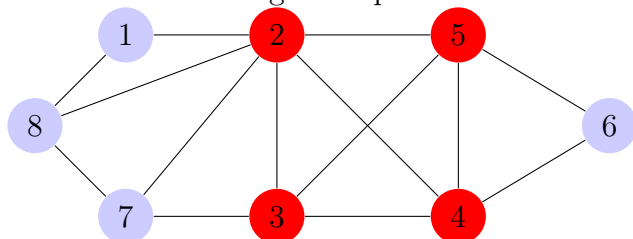
For a graph G , $\omega(G) = \alpha(\overline{G})$

1.5 Example.

Consider the following graph, G :

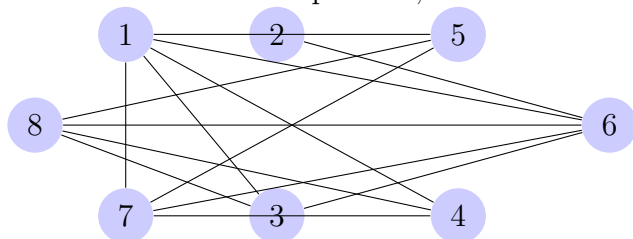


We see that the largest clique in G is:

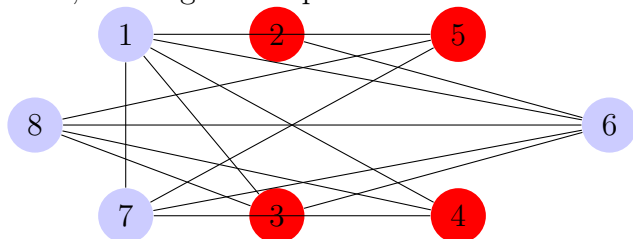


Thus, $\omega(G)$ is 4.

Now consider G 's complement, \overline{G} :



In \overline{G} , the largest independent set is:



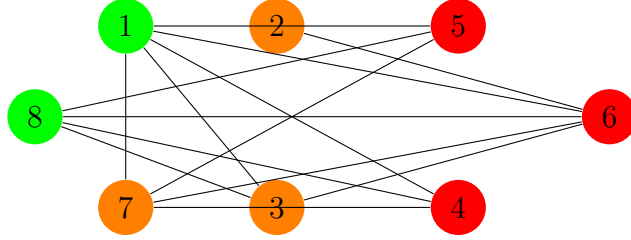
Therefore, we see $\omega(G) = 4 = \alpha(\overline{G})$

1.2 Observation.

Similar to the last observation, for a graph G , $\theta(G) = \chi(\overline{G})$

1.6 Example.

Consider the same graph from the last example. Recall that we calculated $\theta(G)$ to be 3. Now, we can find $\chi(\overline{G})$ to be 3 as well:



Thus, $\theta(G) = 3 = \chi(\overline{G})$

1.1 Lemma.

Let G be a graph. Then $\alpha(G) \leq \min\{|G| - n_+(W), |G| - n_-(W)\} \leq \theta(G)$. Thus, if $\alpha(G) = \theta(G)$, G has a tight inertia bound. [1]

Proof. Consider a clique partition, \mathcal{C} , of a graph G . Let \hat{A} , denote the adjacency matrix of G where the only connected components are the cliques in \mathcal{C} .

Now if we consider the adjacency matrix of the complete graph, K_n , we see that it is equal to $J_n - I_n$ where J_n is the all ones matrix. \square

1.4 Theorem.

Every Perfect Graph, G , has a tight inertia bound

Proof. By the Perfect Graph Theorem (theorem 1.3), we know that \overline{G} , is also perfect. Thus \overline{G} satisfies that $\chi(H) = \omega(H)$ for all subgraphs, H , of \overline{G} , by definition. Thus, since $\chi(\overline{G}) = \omega(\overline{G})$, we can get from the observation 1.1 and 1.2, that

$$\alpha(G) = \omega(\overline{G}) = \chi(\overline{G}) = \theta(G) \quad (9)$$

Therefore, from lemma 1.1, G has a tight inertia bound. \square

1.5 Theorem.

Strong Perfect Graph Theorem [1] — A graph G is a perfect graph if and only if both G and its complement, \overline{G} , do not contain a induced odd cycle of length at least 5.

1.3 Observation.

Due to each perfect graph having a tight inertia bound, and the Strong Perfect Graph Theorem (theorem 1.5), every graph not containing an induced odd cycle of length 5 or greater has a tight inertia bound.

1.2.2 Latin Square Graphs

1.2.3 Graphs with an Eigensharp decomposition by Stars

1.3 Other Bounds on Independence Number

2 Algorithm to Find Graphs Lacking a Tight Inertia Bound

2.1 Outline of Method

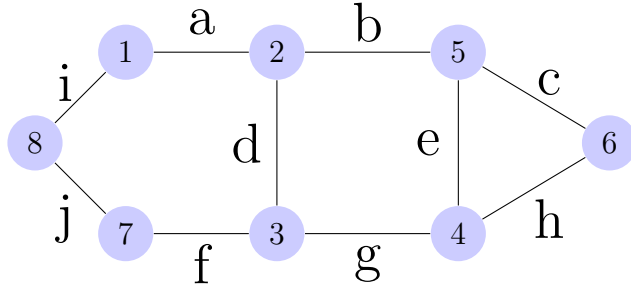
2.1 Definition.

Optimal Weight Matrix — A weight matrix, W , of a graph, G , is optimal if

$$\alpha(G) = \min\{|G| - n_+(W), |G| - n_-(W)\} \quad (10)$$

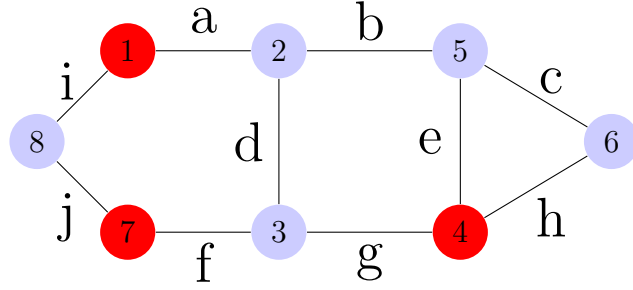
2.1 Example.

Consider the following graph, G , with corresponding weight matrix W :

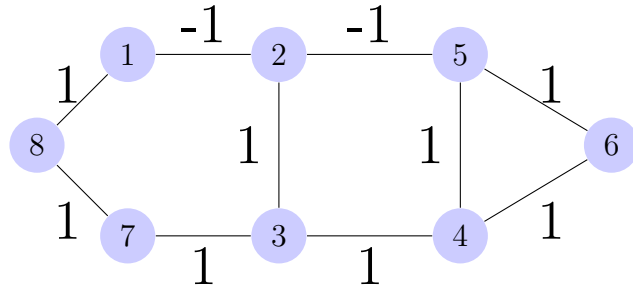


$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 & i \\ a & 0 & d & 0 & b & 0 & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f & 0 \\ 0 & 0 & g & 0 & e & h & 0 & 0 \\ 0 & b & 0 & e & 0 & c & 0 & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 & j \\ i & 0 & 0 & 0 & 0 & 0 & j & 0 \end{bmatrix}$$

We can see the independent number of G is 3:



Now, let G have the following weighting:



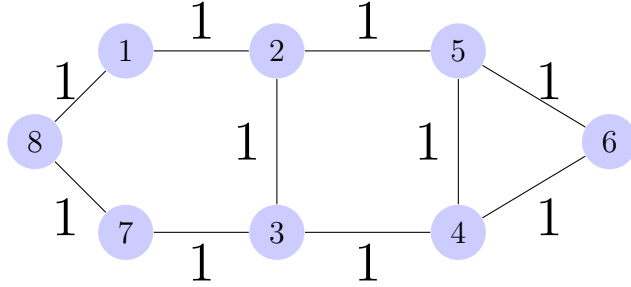
$$\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Finding the eigenvalues of W , we find there are 3 positive eigenvalues and 5 negative eigenvalues. Thus, we see for this weight matrix we have:

$$\begin{aligned} \alpha(G) &= \min\{|G| - n_+(W), |G| - n_-(W)\} \\ &= \min\{8 - 3, 8 - 5\} \\ &= \min\{5, 3\} \\ &= 3 \end{aligned} \tag{11}$$

Therefore, this is an optimal weight matrix of G .

Now consider the following weighting for the same graph:



$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Finding the eigenvalues of this weight matrix, we find there are 4 positive eigenvalues and 4 negative eigenvalues. Thus, we see we get:

$$\begin{aligned}\alpha(G) &= 3 \neq \min\{|G| - n_+(W), |G| - n_-(W)\} \\ &= \min\{8 - 4, 8 - 4\} \\ &= \min\{4, 4\} \\ &= 4\end{aligned}\tag{12}$$

Therefore, we see that the previous weighting was not optimal for G .

2.1 Lemma.

If a graph, G , with weight matrix W , has two induced subgraphs, S_1 and S_2 , such that S_1 has $\alpha(G) + 1$ positive eigenvalues under the weighting of W , and S_2 has $\alpha(G) + 1$ negative eigenvalues under the weighting of W , then W is not optimal

Proof.

☐

2.2 Preliminary Tests to Determine if the Graph may be Suitable

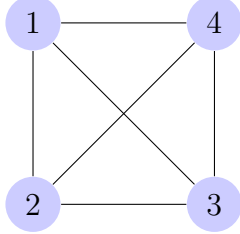
2.2.1 Test for α -Critical

2.2 Definition.

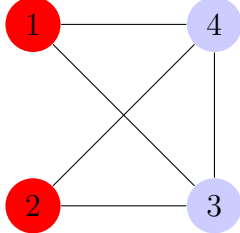
α -Critical — A graph, G , is α -critical if $\alpha(G) < \alpha(G - e)$ for all edges e .

2.2 Example.

Consider the following graph G :



we see that $\alpha(G) = 1$. But since this graph is complete, we see that if we delete any edge, we can get an independent set of size 2 by making the set include the two vertices that were connected by the edge we deleted. For example:



Thus, G is α -critical.

2.2 Lemma.

If G is α -critical, and W an optimal weight matrix of G , then $w_{i,j} \neq 0$ for all $i, j \in E(G)$

Proof. Assume for the sake of contradiction, that for some $i, j \in E(G)$, we have $w_{i,j} = 0$. Then, we know $\alpha(G - e_{i,j}) > \alpha(G)$ because G is α -critical. Thus:

$$\alpha(G) < \alpha(G - e_{i,j}) \leq \min\{|G| - n_+(W), |G| - n_-(W)\}$$

Thus, we see that the inertia bound is not tight for G , so W is not an optimal weight matrix of G , which is a contradiction. \square

Due to the complexity of needing to consider edges that could potentially be zero in the weight matrix, it is easier to consider graphs that are restricted to only non-zero edge weights in its optimal weight matrix. Thus, it makes sense to only consider α -critical graphs, because of Lemma 2.2 ensuring that all α -critical graphs have non-zero weight matrices.

2.2.2 Determining Each Triangle Must Have the Same Sign

2.3 Graphs Currently Found

Graph	Vertices	α	Degree	Circulant	Strongly Regular	Arc Transitive
¹	16	4	5	No	No	No
²	16	2	10	No	(16,10,6,6)	Yes
³	17	3	8	[1,2,4,8]	(17,8,3,4)	Yes
⁴	19	4	6	[1,7,8]	No	Yes
⁵	20	2	13	No	No	No
⁶	20	2	13	[1,3,4,7,8,9,10]	No	No
⁷	22	3	11	[1,2,3,5,10,11]	No	No
⁸	24	3	12	No	No	No
⁹	24	3	12	No	No	No
¹⁰	24	4	9	No	No	No
¹¹	24	4	10	[1,2,4,8,9]	No	No
¹²	24	4	9	No	No	No
¹³	24	3	12	No	No	No
¹⁴	24	4	9	No	No	No
¹⁵	24	4	9	No	No	No
¹⁶	24	2	16	No	No	No
¹⁷	24	2	16	No	No	No
¹⁸	24	2	16	[1,2,3,4,6,7,8,10]	No	No
¹⁹	24	2	16	No	No	No

¹Otr@PKoE?T.iOoOG_dg_m

²O~~em]uj[vmsZTUrfFwN~

³P}qtSeLUbaKeQZJabfGmmG~G

⁴R}ecZ@OH?oW~@gOWcL.p?hkHL?GuG

⁵S~~vVjjve}vmxymG~Oi~Qm{jfxjNw{z{

⁶S~~vnZjvUtvimj'~nibtTP}{ffwk~wR~{

⁷Uv~LnbgeDShP\G}HuXmePrSemapSxqJWG|ZCVhw

⁸Wunneyzx~W]OwBPfcroK~S{OlogtIoyPlPFMIjWPUvaGu~

⁹WvrlvjZj~c_wBTRcroK~K{HLpGtPo[ikpImQHrWaUn'Cv^

¹⁰WvvdIjPb_c[LEHPiH?PsE_GAsWKcwBXhGDgOFXWIBV@CZT

¹¹W}mKmlbqD.JJMMBYa]_{??ucC{YKeHKXPadVXOmQqbqEDMp

¹²W}nS]QeoOq_nWS]?KcPQUdPgU@_TBG_ug@ei?jCgCwY_?J~

¹³W}{}VNBmtdyWkic?zg]gevHT_TfGo~bPK|xHkJJMolozdq\s

¹⁴W}~SvAbp@IcjDgEaj?@BKPCgBbXP@oCz?BLdE@KwGu[?EFZ

¹⁵W~nELU\'aKkXTJ]?@cGUB@KgBSX?wG_sS'DUCGyWO' }?@M^

¹⁶W~~~vnnv|~}gzH}'za|J^ef| ~wB.JNisn[bn^@^nwez~V^~

¹⁷W~~~vnn{vT{nvFnFo^}~Dnw\{^AF|hFz[YZ~DT~wX^~n{B~

¹⁸W~~~vnn{vXyjqnnFs^}~Knw\[^QF]hiz[iznCt~wX^~n{B~

¹⁹W~~~vvu|^\\jvivTvtTyj_~|}ibyiiF}{b{~C{~wU^~f~~

2.3.1 Graphs Created from Deleting a Vertex

Graph	Created From	Vertices	α	Regular	Circulant	Strongly Regular	Arc Transitive
1	1	15	4	No	No	No	No
2	2	15	2	No	No	No	No
3	3	16	3	No	No	No	No
4	4	18	4	No	No	No	No
5	6	19	2	No	No	No	No
6	7	21	3	No	No	No	No
7	8	23	3	No	No	No	No
8	9	23	3	No	No	No	No
9	10	23	4	No	No	No	No
10	11	23	4	No	No	No	No
11	12	23	4	No	No	No	No
12	13	23	3	No	No	No	No
13	14	23	4	No	No	No	No
14	15	23	4	No	No	No	No
15	16	23	2	No	No	No	No
16	17	23	2	No	No	No	No
17	18	23	2	No	No	No	No
18	19	23	2	No	No	No	No
19	4	17	4	No	No	No	No
20	5	18	2	No	No	No	No
21	9	22	4	No	No	No	No
22	11	22	4	No	No	No	No
23	15	22	2	No	No	No	No
24	18	22	2	No	No	No	No
25	19	16	4	No	No	No	No
26	21	21	4	No	No	No	No
27	24	21	2	No	No	No	No

¹Ntr@PKoE?T iOoOG_dg

²N~~em]uj[vmsZTUrfFw

³O}qtSeLUbaKeQZJabfGmm

⁴Q}ecZ@OH?oW~@gOWcI_p?hkHL?

⁵R~~vnZjvUtvimj~nibtTP}{ffwk~w

⁶Tv~LnbgfeDShP\G}HuXmePrSemapSxqJWG|Z

⁷Vunneyzx~W]OwBPfcroK~S{OlogtIoyPIPFMIjWPUv_

⁸VvrlvjZj~c_wBTRcroK~K{HLpGtPo{lpImQHRWaUn_

⁹VvdtIJpB_c[LEHPiH?PsE_GAsWKcwBXhGDgOFXWIBV?

¹⁰V}mKmIbqD_JJMMBYa]_{??ucC{YKeHKXPadVXOmQqbq?

¹¹V}nS|QeoOq_nWS]?KcPQUPDgU@_TBG_ug@ei?jCgCwY_

¹²W}}VNBmtdyWkic?zg]gevHT_TfGo~bPK|xHkJJMolozdq\s

¹³V}~SvAbp@IcjDgEaj?@BKPCgBbXP@oCz?BLdE@KwGu[?

¹⁴V~nELU\'aKkXTJ]?@cGUB@KgBSX?wG_sS'DUCgyWO'}?

¹⁵V~~~vnnv|~}gzH}'za|J^ef]~wBJNisn[bn^@^nwez~_

¹⁶V~~~vnn{vT{nvFnFo^}~Dnw\{^AF|hFz|YZ~DT~wX^~

3 Other Useful Information

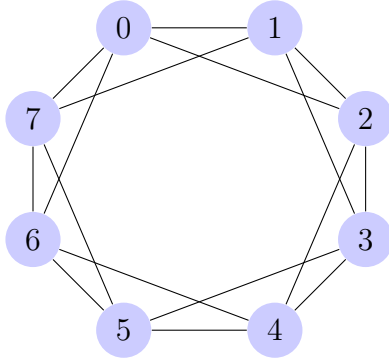
3.1 Cayley Graphs

3.1 Definition.

Cayley Graph — Let H be a finite group, and $S \subseteq H$, be a subset of H . Then the Cayley Graph $C(H, S)$, has a vertex for each element in H . There exists an edge between two vertices g and h , if and only if there exists $s \in S$ such that $sh = g$. If G is a graph such that there exists a group H and a generating set $S \subseteq H$ with $G \cong C(H, S)$, then G is a Cayley Graph. [2]

3.1 Example.

Consider the group \mathbb{Z}_8 and let the generating set be $S = \{1, 2\}$. The vertex set will be $\{0, 1, 2, 3, 4, 5, 6, 7\}$ and there will be an edge between two vertices, g and h , if for an $s \in S$, $g + s = h$:



References

- [1] Randall J Elzinga. *The Minimum Witt Index of a Graph*. PhD thesis, Queens University, 2007.
- [2] Cameron Franc. Cayley graphs.
- [3] John Sinkovic. A graph for which the inertia bound is not tight. *arXiv preprint arXiv:1609.02826*, 2016.