# C&O URA Spring 2017

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## Contents

## 1 Inertia Bounds
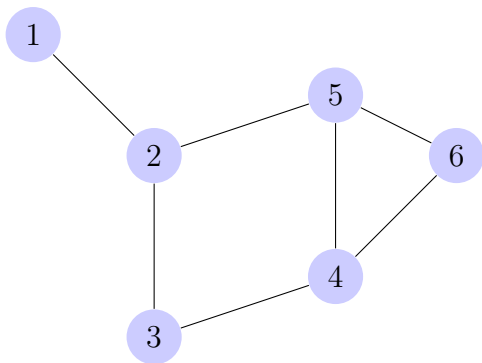
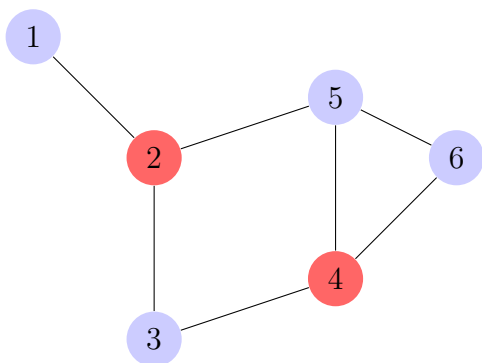### 1.1 Introduction on Inertia Bounds

**1.1 Definition.**
**Independent Set** — An independent set is a set of vertices belonging to a graph in which no two vertices are adjacent.
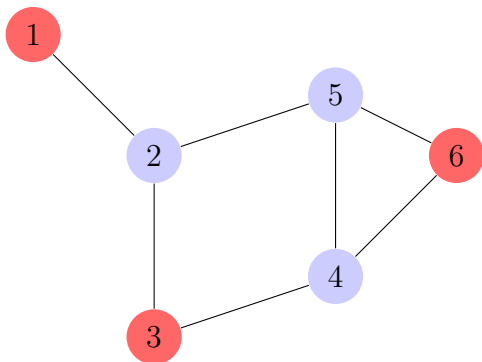
## 1.1 Example.

Consider the following graph:

An example of an independent set in this graph is:

However, often the independent set we are most interested in finding is the largest one:

## 1.2 Definition.

**Independence Number** — The independence number of a graph $G$, denoted $\alpha(G)$, is the size of the largest independent set of $G$.

**1.3 Definition.**
**Weight Matrix** — The weight matrix of a graph $G$, is a matrix defined by:

$$W_{i,j} = \begin{cases} c_{i,j} & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

with $v_i$ a vertice of $G$ and $c_{i,j}$, a constant.

The weight matrix of a graph, is identical to an adjacency matrix, except where there was a 1 in the matrix at entry $A_{i,j}$ if vertices $v_i$ and $v_j$ were adjacent, there is now a constant indicating a weighting for the edge between $v_i$ and $v_j$.

For any graph $G$, there exists a bound on $\alpha(G)$, known as the Cvetković bound (also referred to as the Interia Bound). This bound provides a relationship between $\alpha(G)$ and the number of positive, negative, and zero eigenvalues of the weight matrix, $W$, associated with $G$. The Cvetković bound of $G$, is:

$$\alpha(G) \leq \min\{|G| - n_+(W), |G| - n_-(W)\} \tag{2}$$

Where $n_+(W)$ and $n_-(W)$ denote the number of positive and negative eigenvalues of $W$, respectively.

To prove this, we first need to introduce a result that comes from the Eigenvalue Interlacing Theorem:
**1.1 Theorem.**
**Corollary of Eigenvalue Interlacing Theorem** — Let $A$ be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and let $C$ be a $k \times k$ principal submatrix of $A$ with eigenvalues $\tau_1 \geq \tau_2 \geq \ldots \geq \tau_k$. Then $\lambda_i \geq \tau_i$ for all $i \in \{1, \ldots, k\}$. [2]

**1.4 Definition.**
**Principal Submatrix** — The principal submatrix of an $n \times n$ matrix $A$ is the submatrix obtained where if $row_i$ is excluded in the submatrix, then $column_i$ is excluded as well. Note that all principal submatrices of a weight matrix $W$, correspond to an induced subgraph in the graph represented by $W$.

## 1.2 Example.

The following is an example of a principal submatrix in relation to graph theory.
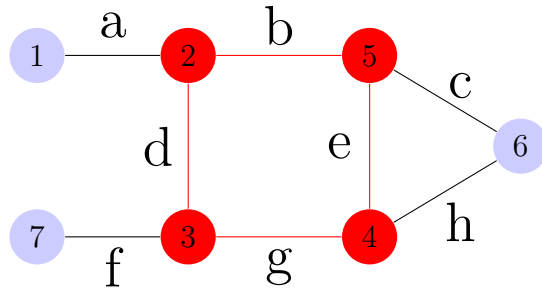
Consider the following graph:
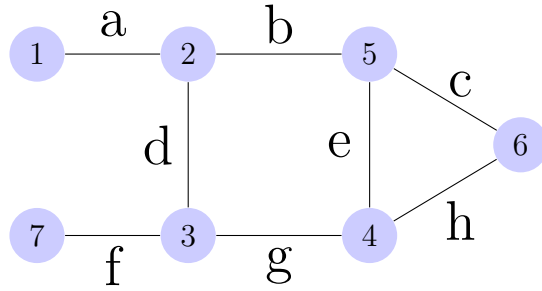


and corresponding weight matrix:

$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & d & 0 & b & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f \\ 0 & 0 & g & 0 & e & h & 0 \\ 0 & b & 0 & e & 0 & c & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can see the following principal submatrix and corresponding induced subgraph:

$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & d & 0 & b & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f \\ 0 & 0 & g & 0 & e & h & 0 \\ 0 & b & 0 & e & 0 & c & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 \end{bmatrix}$$

As well, we see the following principal submatrix of an independent set of the graph:
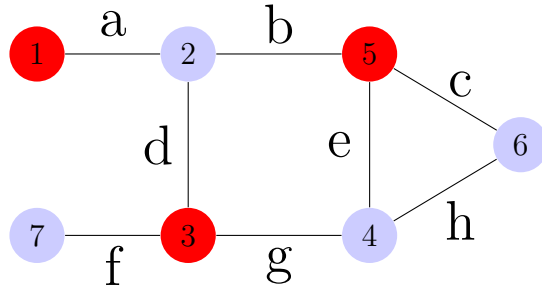


$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & d & 0 & b & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f \\ 0 & 0 & g & 0 & e & h & 0 \\ 0 & b & 0 & e & 0 & c & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now to prove the Cvetković Bound:

**1.2 Theorem.**

**Cvetković Bound** — Let $G$ be a graph on n vertices, and $W$ be the weight matrix of $G$. Then the following inequality holds:

$$\alpha(G) \leq \min\{|G| - n_+(W), |G| - n_-(W)\} \tag{3}$$

*Proof.* [1] Let $H$ be the subgraph of $G$ formed by the vertices in an independent set of size s. Then $H$ is an induced subgraph of $G$ and all eigenvalues of the principal submatrix $W(H)$ are 0 since the principal submatrix will just be a

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[1]Interesting Graphs and their Colourings, unpublished lecture notes C. Godsil (2004)

zero matrix. Let $\lambda_i$ denote the $i$th largest eigenvalue of $W$ and $\tau_i$ denote the $i$th larest eigenvalue of $W(H)$. Now, by interlacing, we have,

$$\lambda_i \geq \tau_i = 0 \text{ for all } i \in \{1, \ldots, s\} \tag{4}$$

and so

$$n - n_-(W) = n_+(W) + n_0(W) \geq s \tag{5}$$

Also, note that by negating $W$, the positive eigenvalues become negative eigenvalues and vice versa. Thus,

$$n - n_+(W) = n - n_-(-W), \tag{6}$$

However, the principal submatrix corresponding to $H$ in $-W$ is still the zero matrix and thus has all zero eigenvalues. Thus, by interlacing, we get a similar result as above,

$$n - n_+(W) = n - n_-(-W) = n_+(-W) + n_0(-W) \geq s \tag{7}$$

Therefore, both $n - n_+(W)$ and $n - n_-(W)$ are greater than or equal to s. Since s is the size of the idependent set, we can see that letting s $= \alpha(G)$, we get:

$$\alpha(G) \leq \min\{|G| - n_+(W), |G| - n_-(W)\} \tag{8}$$
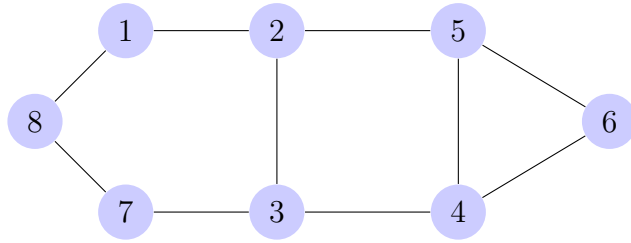
$\square$

## 1.2 Graphs with Tight Inertia Bounds

### 1.2.1 Perfect Graphs

**1.5 Definition.**
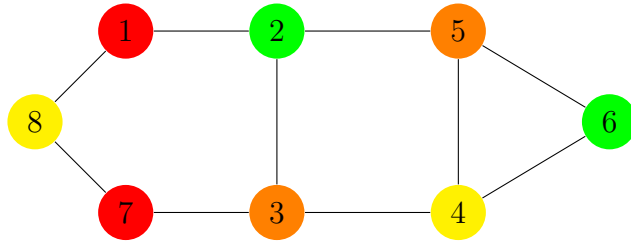**Chromatic Number** — The chromatic number of a graph, $\chi(G)$, is the minimum number of colours needed in a proper colouring of G. [1]

**1.3 Example.**
Consider the following graph:

An example of a colouring would be:



However, $\chi(G)$ for this graph is 3:
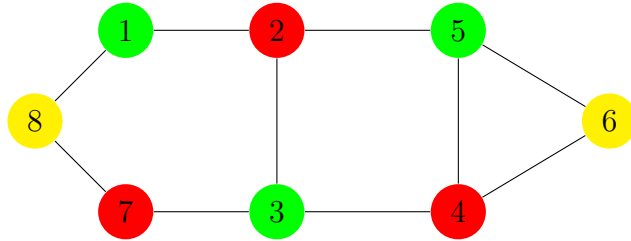


## 1.6 Definition.
**Clique** — An m-clique in a graph is a complete subgraph on m vertices. [1]

The clique number, $\omega(G)$, is the number of vertices in a maximum clique of $G$.

## 1.7 Definition.
**Clique Cover** — A Clique Cover of the vertex set $V(G)$ of a graph $G$ is a set of cliques $C$, such that each vertex is in at least one clique in $C$.

The clique cover number, $\theta(G)$ is the minimum number of cliques needed in a clique cover of $G$. [1]

## 1.4 Example.
Consider the following graph:
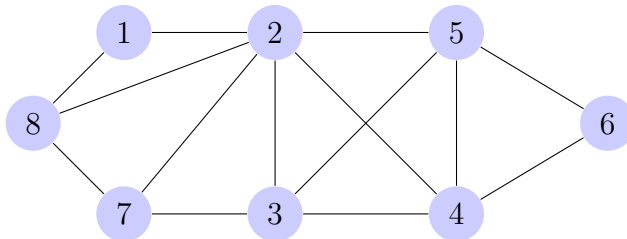
A possible clique covering is:
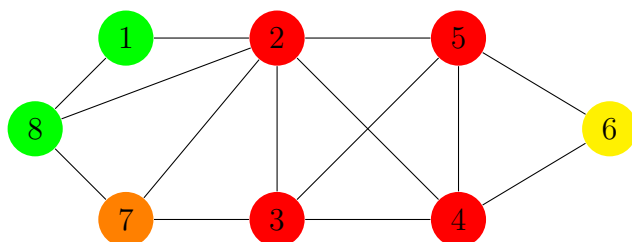


However, we can find that $\theta(G)$ is equal to 3 (smallest I could find):
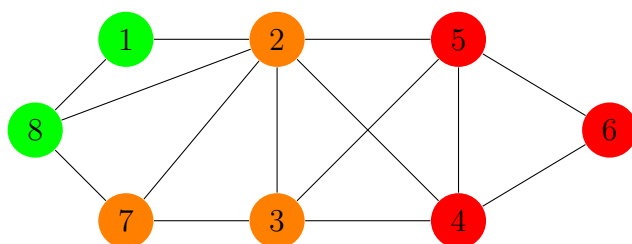


As well, the clique number, $\omega(G)$, is 4:
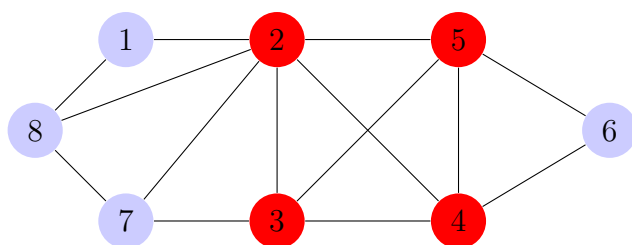


## 1.8 Definition.
**Perfect Graph** — A graph G is perfect if $\chi(G) = \omega(G)$ for all induced subgraphs, H, of G.

# 2 Algorithm to Find Graphs Lacking a Tight Inertia Bound

## 2.1 Outline of Method

**2.1 Definition.**
**Optimal Weight Matrix** — A weight matrix, $W$, of a graph, $G$, is optimal if

$$\alpha(G) = \min\{|G| - n_+(W), |G| - n_-(W)\} \tag{9}$$

**2.1 Example.**
Consider the following graph, $G$, with corresponding weight matrix $W$:
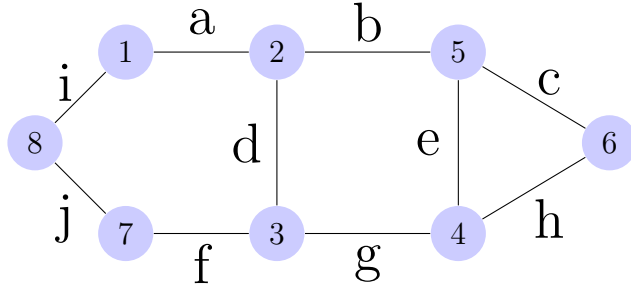


$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 & i \\ a & 0 & d & 0 & b & 0 & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f & 0 \\ 0 & 0 & g & 0 & e & h & 0 & 0 \\ 0 & b & 0 & e & 0 & c & 0 & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 & j \\ i & 0 & 0 & 0 & 0 & 0 & j & 0 \end{bmatrix}$$

We can see the independent number of $G$ is 3:
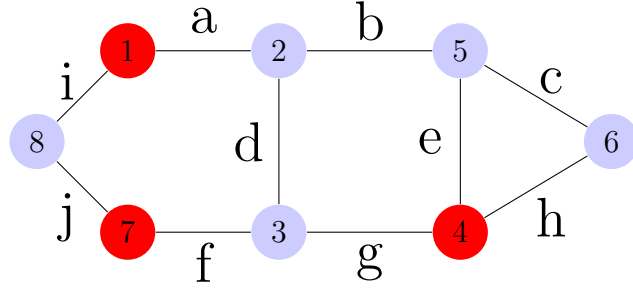
Now, let G have the following weighting:



$$\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Finding the eigenvalues of $W$, we find there are 3 positive eigenvalues and 5 negative eigenvalues. Thus, we see for this weight matrix we have:

$$\begin{aligned} \alpha(G) &= \min\{|G| - n_+(W), |G| - n_-(W)\} \\ &= \min\{8 - 3, 8 - 5\} \\ &= \min\{5, 3\} \\ &= 3 \end{aligned} \tag{10}$$

Therefore, this is an optimal weight matrix of G.
Now consider the following weighting for the same graph:
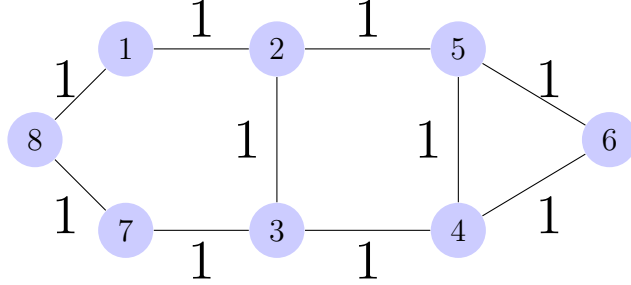
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$
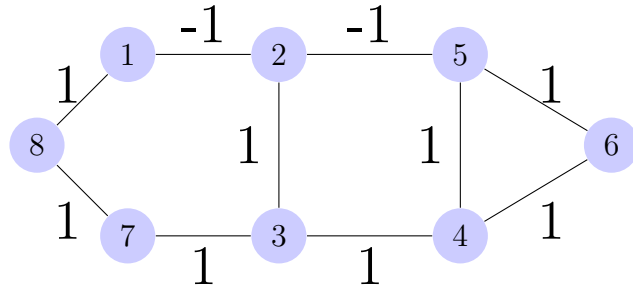
Finding the eigenvalues of this weight matrix, we find there are 4 positive eigenvalues and 4 negative eigenvalues. Thus, we see we get:

$$\begin{aligned} \alpha(G) = 3 &\neq \min\{|G| - n_+(W), |G| - n_-(W)\} \\ &= \min\{8 - 4, 8 - 4\} \\ &= \min\{4, 4\} \\ &= 4 \end{aligned} \tag{11}$$

Therefore, we see that the previous weighting was not optimal for G.

### 2.1 Lemma.

If a graph, $G$, with weight matrix $W$, has two induced subgraphs, $S_1$ and $S_2$, such that $S_1$ has $\alpha(G) + 1$ positive eigenvalues under the weighting of $W$, and $S_2$ has $\alpha(G) + 1$ negative eigenvalues under the weighting of $W$, then $W$ is not optimal

*Proof.* □

## 2.2 Preliminary Tests to Determine if the Graph may be Suitable

### 2.2.1 Test for $\alpha-$Critical

**2.2 Definition.**
$\boldsymbol{\alpha-}$**Critical** — A graph, G, is $\alpha-$critical if $\alpha(G) < \alpha(G-e)$ for all edges e.

**2.2 Example.**
Consider the following graph G:
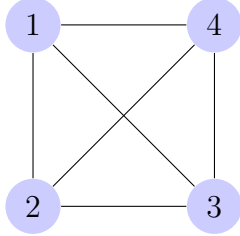


we see that $\alpha(G) = 1$. But since this graph is complete, we see that if we delete any edge, we can get an independent set of size 2 by making the set include the two vertices that were connected by the edge we deleted. For example:
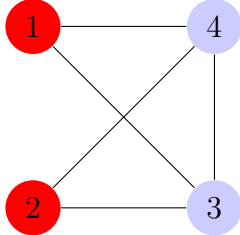


Thus, G is $\alpha-$critical.

**2.2 Lemma.**
If G is $\alpha-$critical, and $W$ an optimal weight matrix of G, then $w_{i,j} \neq 0$ for all $i, j \in E(G)$

*Proof.* Assume for the sake of contradiction, that for some $i, j \in E(G)$, we have $w_{i,j} = 0$. Then, we know $\alpha(G - e_{i,j}) > \alpha(G)$ because G is $\alpha-$critical. Thus:

$$\alpha(G) < \alpha(G - e_{i,j}) \leq \min\{|G| - n_+(W), |G| - n_-(W)\}$$

Thus, we see that the inertia bound is not tight for G, so W is not an optimal weight matrix of G, which is a contradiction. $\square$

Due to the complexity of needing to consider edges that could potentially be zero in the weight matrix, it is easier to consider graphs that are restricted to only non-zero edge weights in its optimal weight matrix. Thus, it makes sense to only consider $\alpha-$critical graphs, because of Lemma 2.2 ensuring that all $\alpha-$critical graphs have non-zero weight matrices.

## 2.2.2 Determining Each Triangle Must Have the Same Sign

## 2.3 Graphs Currently Found

| Graph | Vertices | $\alpha$ | Degree | Circulant | Strongly Regular | Arc Transitive |
|---|---|---|---|---|---|---|
| 1 | 16 | 4 | 5 | No | No | No |
| 2 | 16 | 2 | 10 | No | (16,10,6,6) | Yes |
| 3 | 17 | 3 | 8 | [1,2,4,8] | (17,8,3,4) | Yes |
| 4 | 19 | 4 | 6 | [1,7,8] | No | Yes |
| 5 | 20 | 2 | 13 | No | No | No |
| 6 | 20 | 2 | 13 | [1,3,4,7,8,9,10] | No | No |
| 7 | 22 | 3 | 11 | [1,2,3,5,10,11] | No | No |
| 8 | 24 | 3 | 12 | No | No | No |
| 9 | 24 | 3 | 12 | No | No | No |
| 10 | 24 | 4 | 9 | No | No | No |
| 11 | 24 | 4 | 10 | [1,2,4,8,9] | No | No |
| 12 | 24 | 4 | 9 | No | No | No |
| 13 | 24 | 3 | 12 | No | No | No |
| 14 | 24 | 4 | 9 | No | No | No |
| 15 | 24 | 4 | 9 | No | No | No |
| 16 | 24 | 2 | 16 | No | No | No |
| 17 | 24 | 2 | 16 | No | No | No |
| 18 | 24 | 2 | 16 | [1,2,3,4,6,7,8,10] | No | No |
| 19 | 24 | 2 | 16 | No | No | No |

[1]Otr@PKoE?T_iOoOG_dg_m

[2]O~~em]uj[vmsZTUrfFwN~

[3]P}qtSeLUbaKeQZJabfGmmG~G

[4]R}ecZ@OH?oẀ@gOWcI_p̀?hkHL?GuG

[5]S~~vVjjve}vmxymlG~Oi~Qm{jfxjNw{z{

[6]S~~vnZjvUtvimj'~nibtTP}[ffwk~wR~{

[7]Uv~LnbgfeDShP\G}HuXmePrSemapSxqJWG|ZCVhw

[8]Wunneyzx~W]OwBPfcroK~S{OlogtIoyPlPFMIIjWPUvaGu~

[9]WvrlvjZj~c\_wBTRcroK~K{HLpGtPo[ikpImQHrWaUn'Cv^

[10]WvvdtIJpB_c[LEHPiH?PsE_GAsWKcwBXhGDgOFXWIBV@CZT

[11]W}mKmIbqD_JJMMBYa]_{??ucC{YKeHKXPadVXOmqQbqEDMp

[12]W}nS|QeoOq_nWS]?KcPQUPDgU@_TBG_ug@ei?jCgCwY_?J~

[13]W}}VNbMtdyWkic?zg]gevHT_TfGo~bPK|xHkJJMolozdq\s

[14]W}~SvAbp@IcjDgEaj?@BKPCgBbXP@oCz?BLdE@KwGu[?EFZ

[15]W~nELU\'aKkXTJ]?@cGUB@KgBSX?wG_sS'DUCGyWO'}?@M^

[16]W~~~vnnv|~}gzH}'za|J^ef| ~wBJNish5pn^@^nwez~'V^~

[17]W~~~vnn{vT{nvFnFo^}~Dnw\{^AF|hFz[YZ~DT~wX^~n{B~

[18]W~~~vnn{vXyjqnnFs^}~Knw\[^QF|hiz[iznCt~wX^~n{B~

[19]W~~~vvu|^\\jvivTvtTyj_~ |}ibyiiF}[b{~C{~wU^~_f~~

## 2.3.1 Graphs Created from Deleting a Vertex

| Graph | Created From | Vertices | $\alpha$ | Regular | Circulant | Strongly Regular | Arc Transitive |
|-------|--------------|----------|----------|---------|-----------|------------------|----------------|
| [1]   | [2]          | 15       | 4        | No      | No        | No               | No             |
| [3]   | [4]          | 15       | 2        | No      | No        | No               | No             |
| [5]   | [6]          | 16       | 3        | No      | No        | No               | No             |
| [7]   | [8]          | 18       | 4        | No      | No        | No               | No             |
| [9]   | [10]         | 19       | 2        | No      | No        | No               | No             |
| [11]  | [12]         | 21       | 3        | No      | No        | No               | No             |
| [13]  | [14]         | 23       | 3        | No      | No        | No               | No             |
| [15]  | [16]         | 23       | 3        | No      | No        | No               | No             |
| [17]  | [18]         | 23       | 4        | No      | No        | No               | No             |
| [19]  | [20]         | 23       | 4        | No      | No        | No               | No             |
| [21]  | [22]         | 23       | 4        | No      | No        | No               | No             |
| [23]  | [24]         | 15       |          |         |           |                  |                |
| [25]  | [26]         | 15       |          |         |           |                  |                |
| [27]  | [28]         | 15       |          |         |           |                  |                |
| [29]  | [30]         | 15       |          |         |           |                  |                |

[1] Ntr@PKoE?T␣iOoOG␣dg
[2] Otr@PKoE?T␣iOoOG␣dg␣m
[3] N~~em]uj[vmsZTUrfFw
[4] O~~em]uj[vmsZTUrfFwN~
[5] O}qtSeLUbaKeQZJabfGmm
[6] P}qtSeLUbaKeQZJabfGmmG~G
[7] Q}ecZ@OH?oẀ@gOWcI␣p̀?hkHL?
[8] R}ecZ@OH?oẀ@gOWcI␣p̀?hkHL?GuG
[9] R~~vnZjvUtvimj'~nibtTP}[ffwk~w
[10] S~~vnZjvUtvimj'~nibtTP}[ffwk~wR~{
[11] Tv~LnbgfeDShP\G}HuXmePrSemapSxqJWG|Z
[12] Uv~LnbgfeDShP\G}HuXmePrSemapSxqJWG|ZCVhw
[13] Vunneyzx~W]OwBPfcroK~S{OlogtIoyPlPFMIIjWPUv␣
[14] Wunneyzx~W]OwBPfcroK~S{OlogtIoyPlPFMIIjWPUvaGu~
[15] VvrlvjZj~c\␣wBTRcroK~K{HLpGtPo[ikpImQHrWaUn␣
[16] WvrlvjZj~c\␣wBTRcroK~K{HLpGtPo[ikpImQHrWaUn'Cv^
[17] VvvdtIJpB␣c[LEHPiH?PsE␣GAsWKcwBXhGDgOFXWIBV?
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[23]
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[30]

# 3 Other Useful Information

## 3.1 Cayley Graphs

**3.1 Definition.**
**Cayley Graph** —

# References

[1] Randall J Elzinga. *The Minimum Witt Index of a Graph.* PhD thesis, Queens University, 2007.

[2] John Sinkovic. A graph for which the inertia bound is not tight. *arXiv preprint arXiv:1609.02826*, 2016.