C&O URA Spring 2017

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1 Areas of Focus

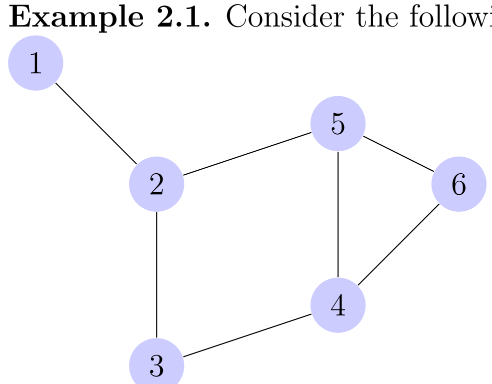
1. Inertia Bounds
   1.1. Algorithm to Find Graphs Lacking a Tight Interia Bound

2 Inertia Bounds

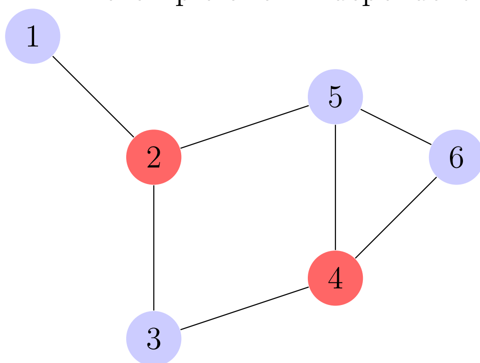
2.1 Introduction on Inertia Bounds

Definition 2.1. Independent Set An independent set is a set of vertices belonging to a graph in which no two vertices are adjacent.

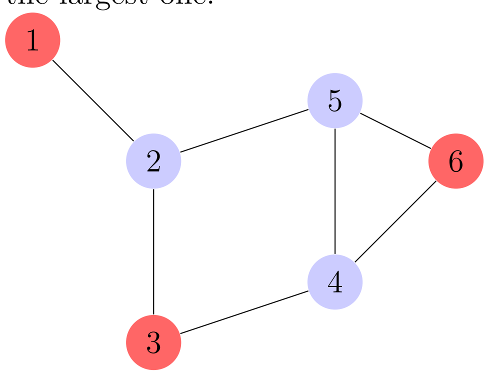
Example 2.1. Consider the following graph:



An example of an independent set in this graph is:



However, often the independent set we are most interested in finding is the largest one:



Definition 2.2. Independence Number The independence number of a graph G, denoted α(G), is the size of the largest independent set of G.

Definition 2.3. Weight Matrix The weight matrix of a graph G, is a matrix defined by:

$$W_{i,j} = \begin{cases} c_{i,j} & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

with v_i a vertice of G and $c_{i,j}$, a constant.

The weight matrix of a graph, is identical to an adjacency matrix, except where there was a 1 in the matrix at entry $A_{i,j}$ if vertices v_i and v_j were adjacent, there is now a constant indicating a weighting for the edge between v_i and v_j.

For any graph G, there exists a bound on α(G), known as the Cvetković bound (also referred to as the Interia Bound). This bound provides a relationship between α(G) and the number of positive, negative, and zero eigenvalues of the weight matrix, W, associated with G. The Cvetković bound of G, is:

$$\alpha(G) \leq min\{|G| - n_+(W), |G| - n_-(W)\} \tag{2}$$

Where $n_+(W)$ and $n_-(W)$ denote the number of positive and negative eigenvalues of W, respectively.

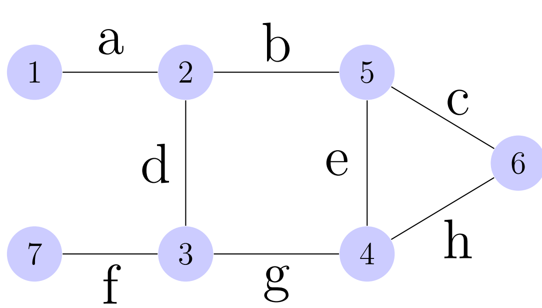
To prove this, we first need to introduce a result that comes from the Eigenvalue Interlacing Theorem:

Theorem 1. Corollary of Eigenvalue Interlacing Theorem Let A be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and let C be a $k \times k$ principal submatrix of A with eigenvalues $\tau_1 \geq \tau_2 \geq \ldots \geq \tau_k$. Then $\lambda_i \geq \tau_i$ for all i $\in \{1, \ldots, k\}$. [1]

Definition 2.4. Principal Submatrix The principal submatrix of an $n \times n$ matrix A is the submatrix obtained where if row_i is excluded in the submatrix, then $column_i$ is excluded as well. Note that all principal submatrices of a weight matrix W, correspond to an induced subgraph in the graph represented by W.

Example 2.2. The following is an example of a principal submatrix in relation to graph theory.

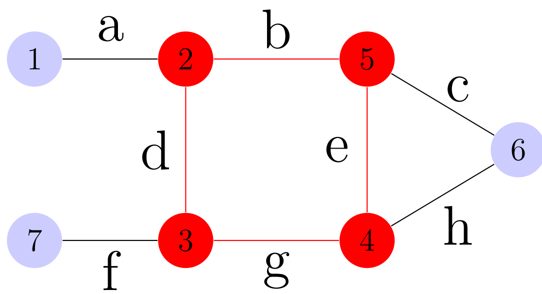
Consider the following graph:



and corresponding weight matrix:

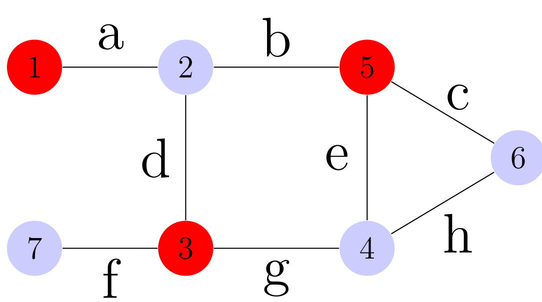
$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & d & 0 & b & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f \\ 0 & 0 & g & 0 & e & h & 0 \\ 0 & b & 0 & e & 0 & c & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can see the following principal submatrix and corresponding induced subgraph:



$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & \color{red}{0} & \color{red}{d} & \color{red}{0} & \color{red}{b} & 0 & 0 \\ 0 & \color{red}{d} & \color{red}{0} & \color{red}{g} & \color{red}{0} & 0 & f \\ 0 & \color{red}{0} & \color{red}{g} & \color{red}{0} & \color{red}{e} & h & 0 \\ 0 & \color{red}{b} & \color{red}{0} & \color{red}{e} & \color{red}{0} & c & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 \end{bmatrix}$$

As well, we see the following principal submatrix of an independent set of the graph:



$$\begin{bmatrix} \color{red}{0} & a & \color{red}{0} & 0 & \color{red}{0} & 0 & 0 \\ a & 0 & d & 0 & b & 0 & 0 \\ \color{red}{0} & d & \color{red}{0} & g & \color{red}{0} & 0 & f \\ 0 & 0 & g & 0 & e & h & 0 \\ \color{red}{0} & b & \color{red}{0} & e & \color{red}{0} & c & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now to prove the Cvetković Bound:

Theorem 2. Cvetković Bound Let G be a graph on n vertices, and W be the weight matrix of G. Then the following inequality holds:

$$\alpha(G) \leq min\{|G| - n_+(W), |G| - n_-(W)\} \tag{3}$$

Proof. [1] Let H be the subgraph of G formed by the vertices in an independent set of size s. Then H is an induced subgraph of G and all eigenvalues of the principal submatrix W(H) are 0 since the principal submatrix will just be a zero matrix. Let λ_i denote the ith largest eigenvalue of W and τ_i denote the ith larest eigenvalue of W(H). Now, by interlacing, we have,

$$\lambda_i \geq \tau_i = 0 \text{ for all i } \in \{1, \dots, s\} \tag{4}$$

and so

$$n - n_-(W) = n_+(W) + n_0(W) \geq s \tag{5}$$

Also, note that by negating W, the positive eigenvalues become negative eigenvalues and vice versa. Thus,

$$n - n_+(W) = n - n_-(-W), \tag{6}$$

However, the principal submatrix corresponding to H in -W is still the zero matrix and thus has all zero eigenvalues. Thus, by interlacing, we get a similar result as above,

$$n - n_+(W) = n - n_-(-W) = n_+(-W) + n_0(-W) \geq s \tag{7}$$

Therefore, both $n - n_+(W)$ and $n - n_-(W)$ are greater than or equal to s. Since s is the size of the idependent set, we can see that letting s = α(G), we get:

$$\alpha(G) \leq min\{|G| - n_+(W), |G| - n_-(W)\} \tag{8}$$

□

References

[1] John Sinkovic. A graph for which the inertia bound is not tight. *arXiv preprint arXiv:1609.02826*, 2016.

[1] Interesting Graphs and their Colourings, unpublished lecture notes C. Godsil (2004)