

# C&O URA Spring 2017

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May 15, 2017

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## 1 Inertia Bounds

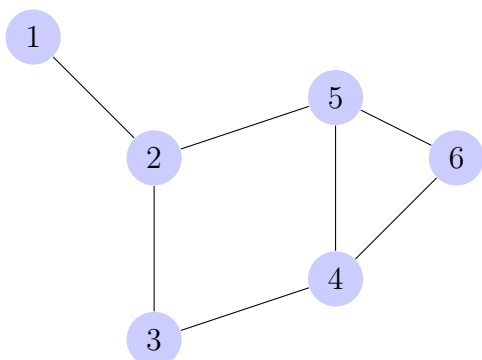
### 1.1 Introduction on Inertia Bounds

#### 1.1 Definition.

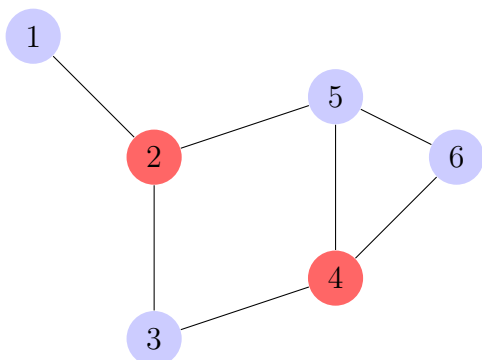
**Independent Set** — An independent set is a set of vertices belonging to a graph in which no two vertices are adjacent.

#### 1.1 Example.

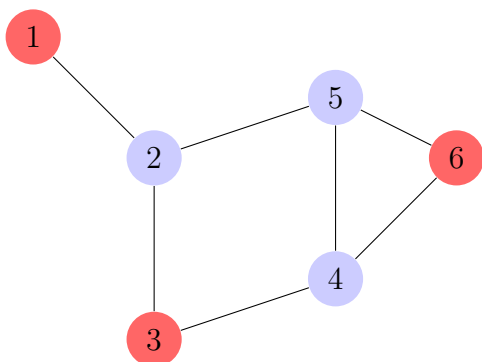
Consider the following graph:



An example of an independent set in this graph is:



However, often the independent set we are most interested in finding is the largest one:



### 1.2 Definition.

**Independence Number** — The independence number of a graph  $G$ , denoted  $\alpha(G)$ , is the size of the largest independent set of  $G$ .

### 1.3 Definition.

**Weight Matrix** — The weight matrix of a graph  $G$ , is a matrix defined by:

$$W_{i,j} = \begin{cases} c_{i,j} & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

with  $v_i$  a vertice of  $G$  and  $c_{i,j}$ , a constant.

The weight matrix of a graph, is identical to an adjacency matrix, except where there was a 1 in the matrix at entry  $A_{i,j}$  if vertices  $v_i$  and  $v_j$  were adjacent, there is now a constant indicating a weighting for the edge between  $v_i$  and  $v_j$ .

For any graph  $G$ , there exists a bound on  $\alpha(G)$ , known as the Cvetković bound (also referred to as the Interia Bound). This bound provides a relationship between  $\alpha(G)$  and the number of positive, negative, and zero eigenvalues of the weight matrix,  $W$ , associated with  $G$ . The Cvetković bound of  $G$ , is:

$$\alpha(G) \leq \min\{|G| - n_+(W), |G| - n_-(W)\} \quad (2)$$

Where  $n_+(W)$  and  $n_-(W)$  denote the number of positive and negative eigenvalues of  $W$ , respectively.

To prove this, we first need to introduce a result that comes from the Eigenvalue Interlacing Theorem:

### 1.1 Theorem.

**Corollary of Eigenvalue Interlacing Theorem** — Let  $A$  be an  $n \times n$  real symmetric matrix with eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$  and let  $C$  be a  $k \times k$  principal submatrix of  $A$  with eigenvalues  $\tau_1 \geq \tau_2 \geq \dots \geq \tau_k$ . Then  $\lambda_i \geq \tau_i$  for all  $i \in \{1, \dots, k\}$ . [1]

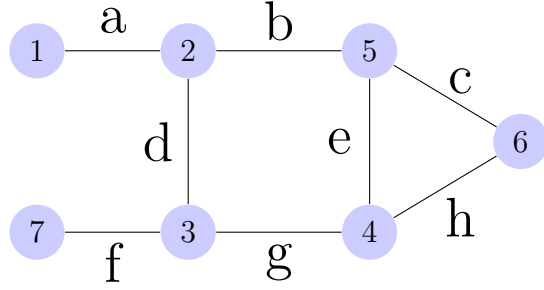
### 1.4 Definition.

**Principal Submatrix** — The principal submatrix of an  $n \times n$  matrix  $A$  is the submatrix obtained where if  $row_i$  is excluded in the submatrix, then  $column_i$  is excluded as well. Note that all principal submatrices of a weight matrix  $W$ , correspond to an induced subgraph in the graph represented by  $W$ .

### 1.2 Example.

The following is an example of a principal submatrix in relation to graph theory.

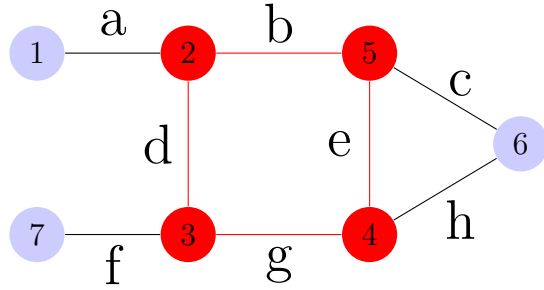
Consider the following graph:



and corresponding weight matrix:

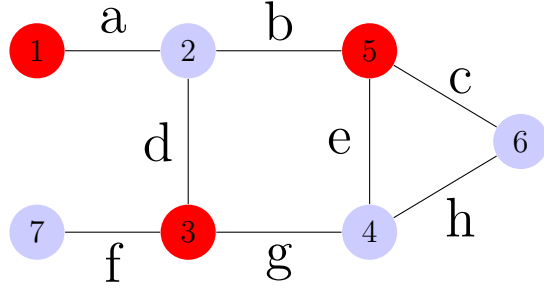
$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & d & 0 & b & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f \\ 0 & 0 & g & 0 & e & h & 0 \\ 0 & b & 0 & e & 0 & c & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can see the following principal submatrix and corresponding induced subgraph:



$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & d & 0 & b & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f \\ 0 & 0 & g & 0 & e & h & 0 \\ 0 & b & 0 & e & 0 & c & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 \end{bmatrix}$$

As well, we see the following principal submatrix of an independent set of the graph:



$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & d & 0 & b & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f \\ 0 & 0 & g & 0 & e & h & 0 \\ 0 & b & 0 & e & 0 & c & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now to prove the Cvetković Bound:

### 1.2 Theorem.

**Cvetković Bound** — Let  $G$  be a graph on  $n$  vertices, and  $W$  be the weight matrix of  $G$ . Then the following inequality holds:

$$\alpha(G) \leq \min\{|G| - n_+(W), |G| - n_-(W)\} \quad (3)$$

*Proof.*<sup>1</sup> Let  $H$  be the subgraph of  $G$  formed by the vertices in an independent set of size  $s$ . Then  $H$  is an induced subgraph of  $G$  and all eigenvalues of the principal submatrix  $W(H)$  are 0 since the principal submatrix will just be a zero matrix. Let  $\lambda_i$  denote the  $i$ th largest eigenvalue of  $W$  and  $\tau_i$  denote the  $i$ th largest eigenvalue of  $W(H)$ . Now, by interlacing, we have,

$$\lambda_i \geq \tau_i = 0 \text{ for all } i \in \{1, \dots, s\} \quad (4)$$

and so

$$n - n_-(W) = n_+(W) + n_0(W) \geq s \quad (5)$$

Also, note that by negating  $W$ , the positive eigenvalues become negative eigenvalues and vice versa. Thus,

$$n - n_+(W) = n - n_-(-W), \quad (6)$$

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<sup>1</sup>Interesting Graphs and their Colourings, unpublished lecture notes C. Godsil (2004)

However, the principal submatrix corresponding to  $H$  in  $-W$  is still the zero matrix and thus has all zero eigenvalues. Thus, by interlacing, we get a similar result as above,

$$n - n_+(W) = n - n_-(-W) = n_+(-W) + n_0(-W) \geq s \quad (7)$$

Therefore, both  $n - n_+(W)$  and  $n - n_-(W)$  are greater than or equal to  $s$ . Since  $s$  is the size of the independent set, we can see that letting  $s = \alpha(G)$ , we get:

$$\alpha(G) \leq \min\{|G| - n_+(W), |G| - n_-(W)\} \quad (8)$$

□

## 2 Algorithm to Find Graphs Lacking a Tight Inertia Bound

### 2.1 Preliminary Tests to Determine if the Graph may be Suitable

#### 2.1.1 Test for Alpha Critical

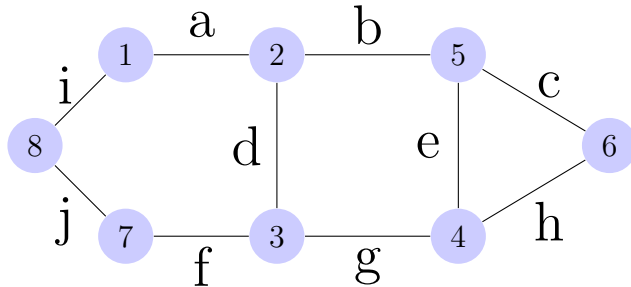
##### 2.1 Definition.

**Optimal Weight Matrix** — A weight matrix,  $W$ , of a graph,  $G$ , is optimal if

$$\alpha(G) = \min\{|G| - n_+(W), |G| - n_-(W)\} \quad (9)$$

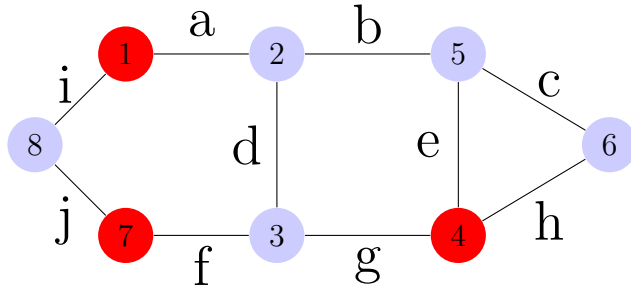
##### 2.1 Example.

Consider the following graph,  $G$ , with corresponding weight matrix  $W$ :

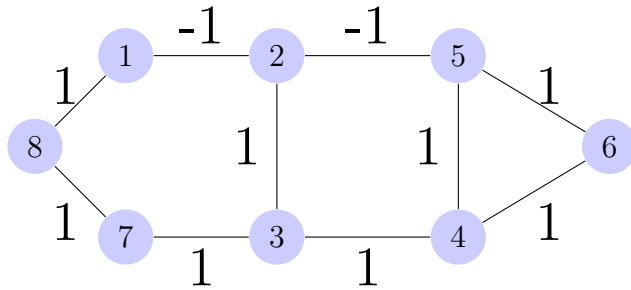


$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 & i \\ a & 0 & d & 0 & b & 0 & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f & 0 \\ 0 & 0 & g & 0 & e & h & 0 & 0 \\ 0 & b & 0 & e & 0 & c & 0 & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 & j \\ i & 0 & 0 & 0 & 0 & 0 & j & 0 \end{bmatrix}$$

We can see the independent number of  $G$  is 3:



Now, let  $G$  have the following weighting:



$$\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

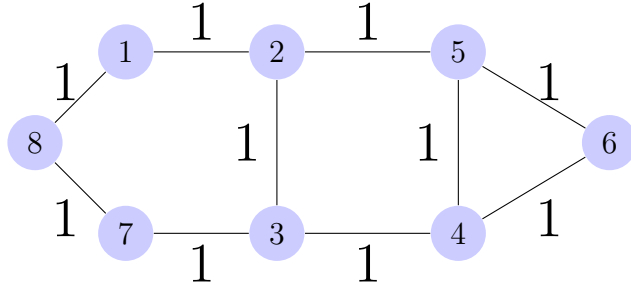
Finding the eigenvalues of  $W$ , we find there are 3 positive eigenvalues and

5 negative eigenvalues. Thus, we see for this weight matrix we have:

$$\begin{aligned}
\alpha(G) &= \min\{|G| - n_+(W), |G| - n_-(W)\} \\
&= \min\{8 - 3, 8 - 5\} \\
&= \min\{5, 3\} \\
&= 3
\end{aligned} \tag{10}$$

Therefore, this is an optimal weight matrix of G.

Now consider the following weighting for the same graph:



$$\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}$$

Finding the eigenvalues of this weight matrix, we find there are 4 positive eigenvalues and 4 negative eigenvalues. Thus, we see we get:

$$\begin{aligned}
\alpha(G) &= 3 \neq \min\{|G| - n_+(W), |G| - n_-(W)\} \\
&= \min\{8 - 4, 8 - 4\} \\
&= \min\{4, 4\} \\
&= 4
\end{aligned} \tag{11}$$

Therefore, we see that the previous weighting was not optimal for G.

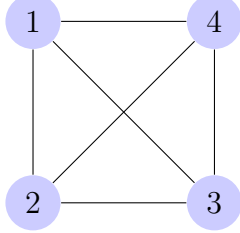


## 2.2 Definition.

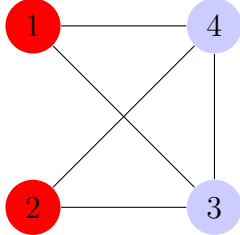
**Alpha Critical** — A graph,  $G$ , is alpha critical if  $\alpha(G) < \alpha(G - e)$  for all edges  $e$ .

## 2.2 Example.

Consider the following graph  $G$ :



we see that  $\alpha(G) = 1$ . But since this graph is complete, we see that if we delete any edge, we can get an independent set of size 2 by making the set include the two vertices that were connected by the edge we deleted. For example:



Thus,  $G$  is  $\alpha$ -critical.

## 2.1 Lemma.

If  $G$  is  $\alpha$ -critical, and  $W$  an optimal weight matrix of  $G$ , then  $w_{i,j} \neq 0$  for all  $i, j \in E(G)$

*Proof.* Assume for the sake of contradiction, that for some  $i, j \in E(G)$ , we have  $w_{i,j} = 0$ . Then, we know  $\alpha(G - e_{i,j}) > \alpha(G)$  because  $G$  is alpha critical. Thus:

$$\alpha(G) < \alpha(G - e_{i,j}) \leq \min\{|G| - n_+(W), |G| - n_-(W)\}$$

Thus, we see that the inertia bound is not tight for  $G$ , so  $W$  is not an optimal weight matrix of  $G$ , which is a contradiction.  $\square$

Due to the complexity of needing to consider edges that could potentially be zero in the weight matrix, it is easier to consider graphs that are restricted to only non-zero edge weights in its optimal weight matrix. Thus, it makes

sense to only consider alpha critical graphs, because of Lemma 2.1 ensuring that all alpha critical graphs have non-zero weight matrices.

### **2.1.2 Test for Arc Transitivity**

## **References**

- [1] John Sinkovic. A graph for which the inertia bound is not tight. *arXiv preprint arXiv:1609.02826*, 2016.