

# C&O URA Spring 2017

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# 1 Inertia Bounds

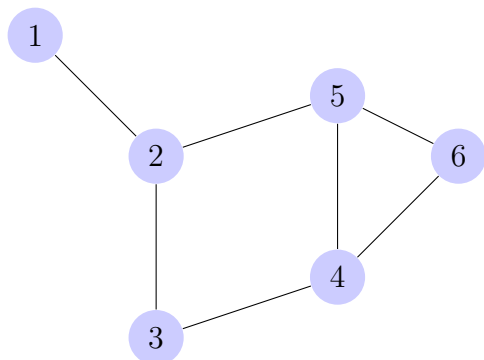
## 1.1 Introduction on Inertia Bounds

### 1.1 Definition.

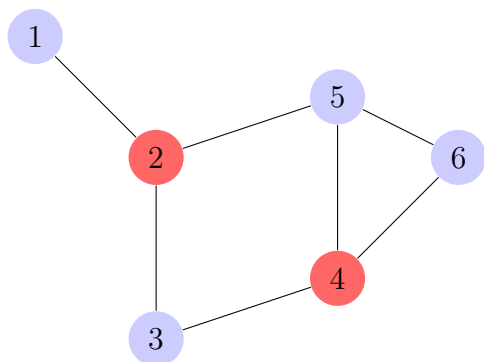
**Independent Set** — An independent set is a set of vertices belonging to a graph in which no two vertices are adjacent.

### 1.1 Example.

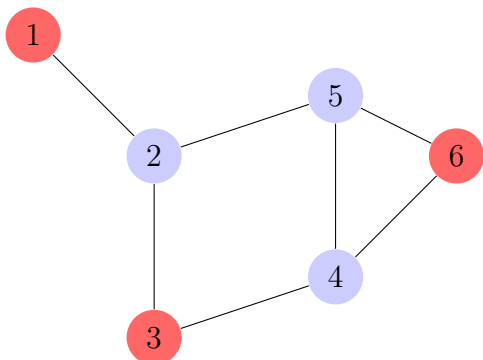
Consider the following graph:



An example of an independent set in this graph is:



However, often the independent set we are most interested in finding is the largest one:



## 1.2 Definition.

**Independence Number** — The independence number of a graph  $G$ , denoted  $\alpha(G)$ , is the size of the largest independent set of  $G$ .

## 1.3 Definition.

**Weight Matrix** — The weight matrix of a graph  $G$ , is a matrix defined by:

$$W_{i,j} = \begin{cases} c_{i,j} & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

with  $v_i$  a vertex of  $G$  and  $c_{i,j}$ , a constant.

The weight matrix of a graph, is identical to an adjacency matrix, except where there was a 1 in the matrix at entry  $A_{i,j}$  if vertices  $v_i$  and  $v_j$  were adjacent, there is now a constant indicating a weighting for the edge between  $v_i$  and  $v_j$ .

For any graph  $G$ , there exists a bound on  $\alpha(G)$ , known as the Cvetković bound (also referred to as the Interia Bound). This bound provides a relationship between  $\alpha(G)$  and the number of positive, negative, and zero eigenvalues of the weight matrix,  $W$ , associated with  $G$ . The Cvetković bound of  $G$ , is:

$$\alpha(G) \leq \min\{|G| - n_+(W), |G| - n_-(W)\} \quad (2)$$

Where  $n_+(W)$  and  $n_-(W)$  denote the number of positive and negative eigenvalues of  $W$ , respectively.

To prove this, we first need to introduce a result that comes from the Eigenvalue Interlacing Theorem:

## 1.1 Theorem.

**Corollary of Eigenvalue Interlacing Theorem** — Let  $A$  be an  $n \times n$

real symmetric matrix with eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$  and let  $C$  be a  $k \times k$  principal submatrix of  $A$  with eigenvalues  $\tau_1 \geq \tau_2 \geq \dots \geq \tau_k$ . Then  $\lambda_i \geq \tau_i$  for all  $i \in \{1, \dots, k\}$ . [5]

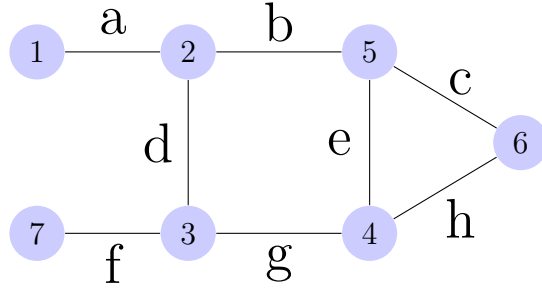
#### 1.4 Definition.

**Principal Submatrix** — The principal submatrix of an  $n \times n$  matrix  $A$  is the submatrix obtained where if  $row_i$  is excluded in the submatrix, then  $column_i$  is excluded as well. Note that all principal submatrices of a weight matrix  $W$ , correspond to an induced subgraph in the graph represented by  $W$ .

#### 1.2 Example.

The following is an example of a principal submatrix in relation to graph theory.

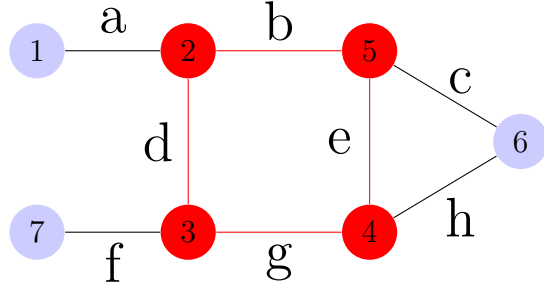
Consider the following graph:



and corresponding weight matrix:

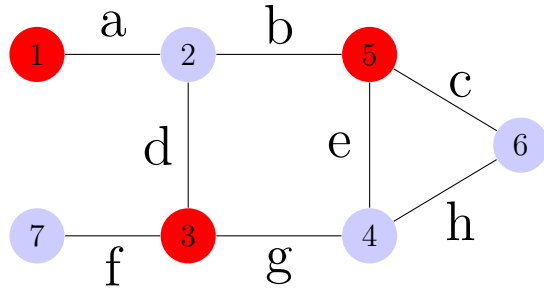
$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & d & 0 & b & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f \\ 0 & 0 & g & 0 & e & h & 0 \\ 0 & b & 0 & e & 0 & c & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can see the following principal submatrix and corresponding induced subgraph:



$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & d & 0 & b & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f \\ 0 & 0 & g & 0 & e & h & 0 \\ 0 & 0 & g & 0 & e & h & 0 \\ 0 & b & 0 & e & 0 & c & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 \end{bmatrix}$$

As well, we see the following principal submatrix of an independent set of the graph:



$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & d & 0 & b & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f \\ 0 & 0 & g & 0 & e & h & 0 \\ 0 & 0 & g & 0 & e & h & 0 \\ 0 & b & 0 & e & 0 & c & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now to prove the Cvetković Bound:

### 1.2 Theorem.

**Cvetković Bound** — Let  $G$  be a graph on  $n$  vertices, and  $W$  be the weight

matrix of  $G$ . Then the following inequality holds:

$$\alpha(G) \leq \min\{|G| - n_+(W), |G| - n_-(W)\} \quad (3)$$

*Proof.* <sup>1</sup> Let  $H$  be the subgraph of  $G$  formed by the vertices in an independent set of size  $s$ . Then  $H$  is an induced subgraph of  $G$  and all eigenvalues of the principal submatrix  $W(H)$  are 0 since the principal submatrix will just be a zero matrix. Let  $\lambda_i$  denote the  $i$ th largest eigenvalue of  $W$  and  $\tau_i$  denote the  $i$ th largest eigenvalue of  $W(H)$ . Now, by interlacing, we have,

$$\lambda_i \geq \tau_i = 0 \text{ for all } i \in \{1, \dots, s\} \quad (4)$$

and so

$$n - n_-(W) = n_+(W) + n_0(W) \geq s \quad (5)$$

Also, note that by negating  $W$ , the positive eigenvalues become negative eigenvalues and vice versa. Thus,

$$n - n_+(W) = n - n_-(-W), \quad (6)$$

However, the principal submatrix corresponding to  $H$  in  $-W$  is still the zero matrix and thus has all zero eigenvalues. Thus, by interlacing, we get a similar result as above,

$$n - n_+(W) = n - n_-(-W) = n_+(-W) + n_0(-W) \geq s \quad (7)$$

Therefore, both  $n - n_+(W)$  and  $n - n_-(W)$  are greater than or equal to  $s$ . Since  $s$  is the size of the independent set, we can see that letting  $s = \alpha(G)$ , we get:

$$\alpha(G) \leq \min\{|G| - n_+(W), |G| - n_-(W)\} \quad (8)$$

□

## 1.2 Graphs with Tight Inertia Bounds

### 1.2.1 Perfect Graphs

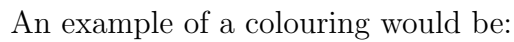
#### 1.5 Definition.

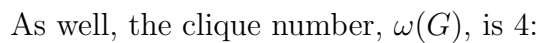
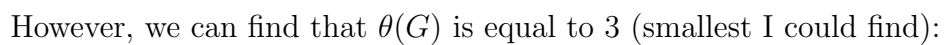
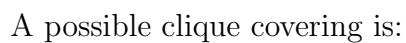
**Chromatic Number** — The chromatic number of a graph,  $\chi(G)$ , is the minimum number of colours needed in a proper colouring of  $G$ . [1]

---

<sup>1</sup>Interesting Graphs and their Colourings, unpublished lecture notes C. Godsil (2004)

Consider the following graph:





**Perfect Graph** — A graph  $G$  is perfect if  $\chi(G) = \omega(G)$  for all induced subgraphs,  $H$ , of  $G$ .

### 1.3 Theorem.



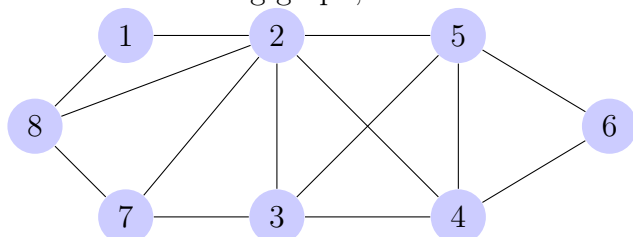
**Perfect Graph Theorem** — A Graph  $G$  is perfect if and only if its complement,  $\overline{G}$ , is also perfect.

### 1.1 Observation.

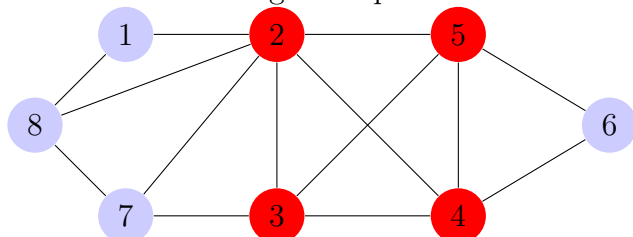
For a graph  $G$ ,  $\omega(G) = \alpha(\overline{G})$

### 1.5 Example.

Consider the following graph,  $G$ :

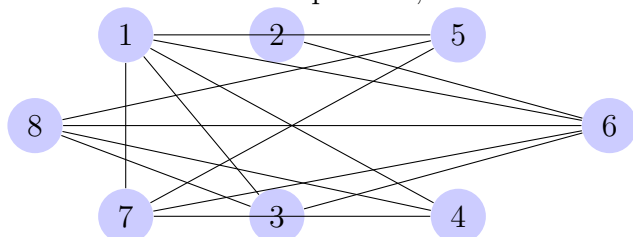


We see that the largest clique in  $G$  is:

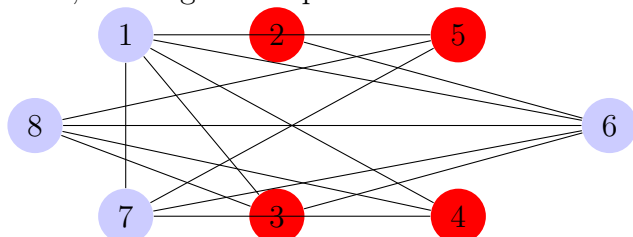


Thus,  $\omega(G)$  is 4.

Now consider  $G$ 's complement,  $\overline{G}$ :



In  $\overline{G}$ , the largest independent set is:



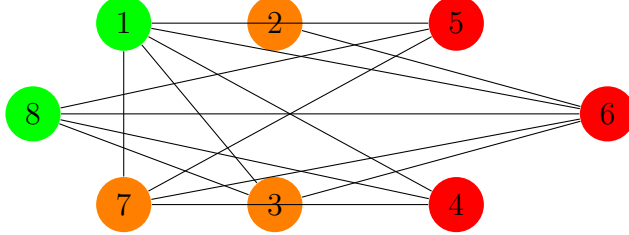
Therefore, we see  $\omega(G) = 4 = \alpha(\overline{G})$

### 1.2 Observation.

Similar to the last observation, for a graph  $G$ ,  $\theta(G) = \chi(\overline{G})$

### 1.6 Example.

Consider the same graph from the last example. Recall that we calculated  $\theta(G)$  to be 3. Now, we can find  $\chi(\overline{G})$  to be 3 as well:



Thus,  $\theta(G) = 3 = \chi(\overline{G})$

### 1.1 Lemma.

Let  $G$  be a graph. Then  $\alpha(G) \leq \min\{|G| - n_+(W), |G| - n_-(W)\} \leq \theta(G)$ . Thus, if  $\alpha(G) = \theta(G)$ ,  $G$  has a tight inertia bound. [1]

*Proof.* Consider a clique partition,  $\mathcal{C}$ , of a graph  $G$ . Let  $\hat{A}$ , denote the adjacency matrix of  $G$  where the only connected components are the cliques in  $\mathcal{C}$ .

Now if we consider the adjacency matrix of the complete graph,  $K_n$ , we see that it is equal to  $J_n - I_n$  where  $J_n$  is the all ones matrix.  $\square$

### 1.4 Theorem.

Every Perfect Graph,  $G$ , has a tight inertia bound

*Proof.* By the Perfect Graph Theorem (theorem 1.3), we know that  $\overline{G}$ , is also perfect. Thus  $\overline{G}$  satisfies that  $\chi(H) = \omega(H)$  for all subgraphs,  $H$ , of  $\overline{G}$ , by definition. Thus, since  $\chi(\overline{G}) = \omega(\overline{G})$ , we can get from the observation 1.1 and 1.2, that

$$\alpha(G) = \omega(\overline{G}) = \chi(\overline{G}) = \theta(G) \quad (9)$$

Therefore, from lemma 1.1,  $G$  has a tight inertia bound.  $\square$

### 1.5 Theorem.

**Strong Perfect Graph Theorem [1]** — A graph  $G$  is a perfect graph if and only if both  $G$  and its complement,  $\overline{G}$ , do not contain a induced odd cycle of length at least 5.

### 1.3 Observation.

Due to each perfect graph having a tight inertia bound, and the Strong Perfect Graph Theorem (theorem 1.5), every graph not containing an induced odd cycle of length 5 or greater has a tight inertia bound.

#### 1.2.2 Latin Square Graphs

#### 1.2.3 Graphs with an Eigensharp decomposition by Stars

#### 1.2.4 Summary

In summary, the following list of graphs attain a tight inertia bound:

- Graphs on 10 or fewer vertices (pg 81 [1])
- Vertex Transitive graphs on 12 or fewer vertices (pg 81 [1])
- Perfect Graphs
- Latin Square Graphs
- Graphs with an Eigensharp decomposition by stars

### 1.3 Other Bounds on Independence Number

## 2 Algorithm to Find Graphs Lacking a Tight Inertia Bound

### 2.1 Outline of Method

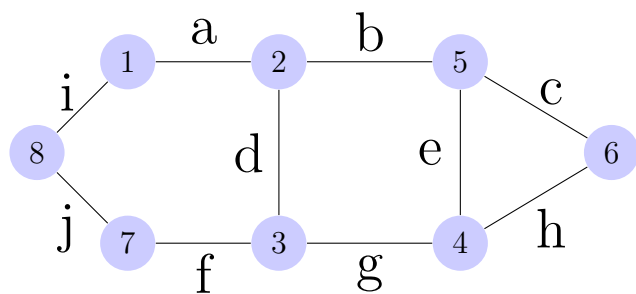
#### 2.1 Definition.

**Optimal Weight Matrix** — A weight matrix,  $W$ , of a graph,  $G$ , is optimal if

$$\alpha(G) = \min\{|G| - n_+(W), |G| - n_-(W)\} \quad (10)$$

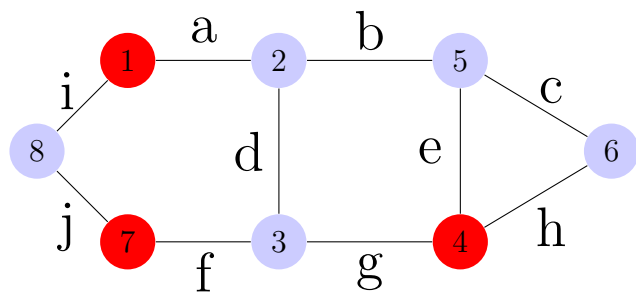
#### 2.1 Example.

Consider the following graph,  $G$ , with corresponding weight matrix  $W$ :

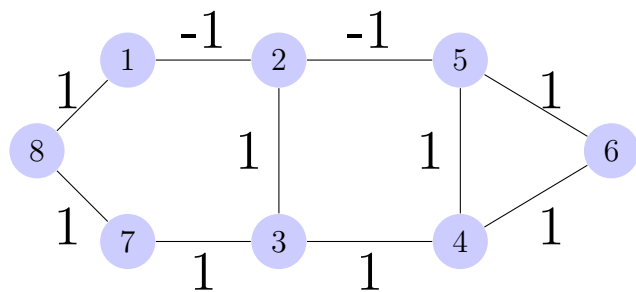


$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 & i \\ a & 0 & d & 0 & b & 0 & 0 & 0 \\ 0 & d & 0 & g & 0 & 0 & f & 0 \\ 0 & 0 & g & 0 & e & h & 0 & 0 \\ 0 & b & 0 & e & 0 & c & 0 & 0 \\ 0 & 0 & 0 & h & c & 0 & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 & j \\ i & 0 & 0 & 0 & 0 & 0 & j & 0 \end{bmatrix}$$

We can see the independent number of  $G$  is 3:



Now, let  $G$  have the following weighting:



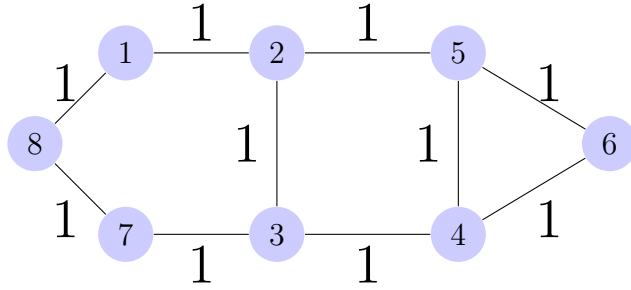
$$\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Finding the eigenvalues of  $W$ , we find there are 3 positive eigenvalues and 5 negative eigenvalues. Thus, we see for this weight matrix we have:

$$\begin{aligned} \alpha(G) &= \min\{|G| - n_+(W), |G| - n_-(W)\} \\ &= \min\{8 - 3, 8 - 5\} \\ &= \min\{5, 3\} \\ &= 3 \end{aligned} \tag{11}$$

Therefore, this is an optimal weight matrix of  $G$ .

Now consider the following weighting for the same graph:



$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Finding the eigenvalues of this weight matrix, we find there are 4 positive

eigenvalues and 4 negative eigenvalues. Thus, we see we get:

$$\begin{aligned}
\alpha(G) &= 3 \neq \min\{|G| - n_+(W), |G| - n_-(W)\} \\
&= \min\{8 - 4, 8 - 4\} \\
&= \min\{4, 4\} \\
&= 4
\end{aligned} \tag{12}$$

Therefore, we see that the previous weighting was not optimal for  $G$ .

### 2.1 Lemma.

If a graph,  $G$ , with weight matrix  $W$ , has two induced subgraphs,  $S_1$  and  $S_2$ , such that  $S_1$  has  $\alpha(G) + 1$  positive eigenvalues under the weighting of  $W$ , and  $S_2$  has  $\alpha(G) + 1$  negative eigenvalues under the weighting of  $W$ , then  $W$  is not optimal

*Proof.*

□

## 2.2 Preliminary Tests to Determine if the Graph may be Suitable

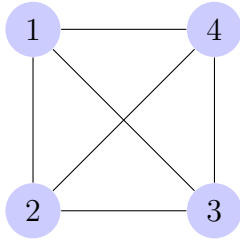
### 2.2.1 Test for $\alpha$ -Critical

#### 2.2 Definition.

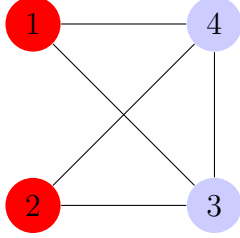
**$\alpha$ -Critical** — A graph,  $G$ , is  $\alpha$ -critical if  $\alpha(G) < \alpha(G - e)$  for all edges  $e$ .

#### 2.2 Example.

Consider the following graph  $G$ :



we see that  $\alpha(G) = 1$ . But since this graph is complete, we see that if we delete any edge, we can get an independent set of size 2 by making the set include the two vertices that were connected by the edge we deleted. For example:



Thus,  $G$  is  $\alpha$ -critical.

**2.2 Lemma.**

If  $G$  is  $\alpha$ -critical, and  $W$  an optimal weight matrix of  $G$ , then  $w_{i,j} \neq 0$  for all  $i, j \in E(G)$

*Proof.* Assume for the sake of contradiction, that for some  $i, j \in E(G)$ , we have  $w_{i,j} = 0$ . Then, we know  $\alpha(G - e_{i,j}) > \alpha(G)$  because  $G$  is  $\alpha$ -critical. Thus:

$$\alpha(G) < \alpha(G - e_{i,j}) \leq \min\{|G| - n_+(W), |G| - n_-(W)\}$$

Thus, we see that the inertia bound is not tight for  $G$ , so  $W$  is not an optimal weight matrix of  $G$ , which is a contradiction.  $\square$

Due to the complexity of needing to consider edges that could potentially be zero in the weight matrix, it is easier to consider graphs that are restricted to only non-zero edge weights in its optimal weight matrix. Thus, it makes sense to only consider  $\alpha$ -critical graphs, because of Lemma 2.2 ensuring that all  $\alpha$ -critical graphs have non-zero weight matrices.





## 2.2.2 Determining Each Triangle Must Have the Same Sign

## 2.3 Graphs Currently Found

Graph	Vertices	$\alpha$	Degree	Circulant	Strongly Regular	Arc Transitive
<sup>1</sup>	16	4	5	No	No	No
<sup>2</sup>	16	2	10	No	(16,10,6,6)	Yes
<sup>3</sup>	17	3	8	[1,2,4,8]	(17,8,3,4)	Yes
<sup>4</sup>	19	4	6	[1,7,8]	No	Yes
<sup>5</sup>	20	2	13	No	No	No
<sup>6</sup>	20	2	13	[1,3,4,7,8,9,10]	No	No
<sup>7</sup>	22	3	11	[1,2,3,5,10,11]	No	No
<sup>8</sup>	24	3	12	No	No	No
<sup>9</sup>	24	3	12	No	No	No
<sup>10</sup>	24	4	9	No	No	No
<sup>11</sup>	24	4	10	[1,2,4,8,9]	No	No
<sup>12</sup>	24	4	9	No	No	No
<sup>13</sup>	24	3	12	No	No	No
<sup>14</sup>	24	4	9	No	No	No
<sup>15</sup>	24	4	9	No	No	No
<sup>16</sup>	24	2	16	No	No	No
<sup>17</sup>	24	2	16	No	No	No
<sup>18</sup>	24	2	16	[1,2,3,4,6,7,8,10]	No	No
<sup>19</sup>	24	2	16	No	No	No

<sup>1</sup>Otr@PKoE?T.iOoOG\_dg\_m

<sup>2</sup>O~~em]uj[vmsZTUrfFwN~

<sup>3</sup>P}qtSeLUbaKeQZJabfGmmG~G

<sup>4</sup>R}ecZ@OH?oW~@gOWcL.p?hkHL?GuG

<sup>5</sup>S~~vVjjve}vmxymIG~Oi~Qm{jfxjNw{z{

<sup>6</sup>S~~vnZjvUtvimj'~nibtTP}{ffwk~wR~{

<sup>7</sup>Uv~LnbgeDShP\G}HuXmePrSemapSxqJWG|ZCVhw

<sup>8</sup>Wunneyzx~W]OwBPfcroK~S{OlogtIoyPlPFMIjWPUvaGu~

<sup>9</sup>WvrlvjZj~c\\_wBTRcroK~K{HLpGtPo[ikpImQHrWaUn'Cv^

<sup>10</sup>WvvdIJpB\_c[LEHPiH?PsE\_GAsWKcwBXhGDgOFXWIBV@CZT

<sup>11</sup>W}mKmlbqD.JJMMBYa]\_{??ucC{YKeHKXPadVXOmQqbqEDMp

<sup>12</sup>W}nS]QeoOq\_nWS]?KcPQUdPgU@\_TBG\_ug@ei?jCgCwY\_?J~

<sup>13</sup>W}{}VNBmtdyWkic?zg]gevHT\_TfGo~bPK|xHkJJMolzdz\s

<sup>14</sup>W}~SvAbp@IcjDgEaj?@BKPCgBbXP@oCz?BLdE@KwGu[?EFZ

<sup>15</sup>W~nELU\'aKkXTJ]?@cGUB@KgBSX?wG\_sS'DUCGyWO'}?@M^

<sup>16</sup>W~~~vnnv|~}gzH}'za|J^ef|~wB.JNisn[bn^@^nwez~V^~

<sup>17</sup>W~~~vnn{vT{nvFnFo^}~Dnw\{^AF|hFz[YZ~DT~wX^~n{B~

<sup>18</sup>W~~~vnn{vXyjqnnFs^}~Knw\[^QF]hiz[iznCt~wX^~n{B~

<sup>19</sup>W~~~vvu|^\\jvivTvtTyj\_~|}ibyiiF}{b{~C{~wU^~f~~



### 2.3.1 Graphs Created from Deleting a Vertex

Graph	Created From	Vertices	$\alpha$	Regular	Circulant	Strongly Regular	Arc Transitive
1	1	15	4	No	No	No	No
2	2	15	2	No	No	No	No
3	3	16	3	No	No	No	No
4	4	18	4	No	No	No	No
5	6	19	2	No	No	No	No
6	7	21	3	No	No	No	No
7	8	23	3	No	No	No	No
8	9	23	3	No	No	No	No
9	10	23	4	No	No	No	No
10	11	23	4	No	No	No	No
11	12	23	4	No	No	No	No
12	13	23	3	No	No	No	No
13	14	23	4	No	No	No	No
14	15	23	4	No	No	No	No
15	16	23	2	No	No	No	No
16	17	23	2	No	No	No	No
17	18	23	2	No	No	No	No
18	19	23	2	No	No	No	No
19	4	17	4	No	No	No	No
20	5	18	2	No	No	No	No
21	9	22	4	No	No	No	No
22	11	22	4	No	No	No	No
23	15	22	2	No	No	No	No
24	18	22	2	No	No	No	No
25	19	16	4	No	No	No	No
26	21	21	4	No	No	No	No
27	24	21	2	No	No	No	No

<sup>1</sup>Ntr@PKoE?T iOoOG\_dg

<sup>2</sup>N~~em]uj[vmsZTUrfFw

<sup>3</sup>O}qtSeLUbaKeQZJabfGmm

<sup>4</sup>Q}ecZ@OH?oW~@gOWcI\_p?hkHL?

<sup>5</sup>R~~vnZjvUtvimj~nibtTP}{ffwk~w

<sup>6</sup>Tv~LnbgeDShP\G}HuXmePrSemapSxqJWG|Z

<sup>7</sup>Vunneyzx~W]OwBPfcroK~S{OlogtIoyPIPFMIjWPUv\_

<sup>8</sup>VvrlvjZj~c\\_wBTRcroK~K{HLpGtPo{pImQHRWaUn\_

<sup>9</sup>VvdtIJpB\_c[LEHPiH?PsE\_GAsWKcwBXhGDgOFXWIBV?

<sup>10</sup>V}mKmIbqD\_JJMMBYa]\_{??ucC{YKeHKXPadVXOmQqbq?

<sup>11</sup>V}nS|QeoOq\_nWS]?KcPQUPDgU@\_TBG\_ug@ei?jCgCwY\_

<sup>12</sup>W}}VNBmtdyWkic?zg]gevHT\_TfGo~bPK|xHkJJMolozdq\s

<sup>13</sup>V}~SvAbp@IcjDgEaj?@BKPCgBbXP@oCz?BLdE@KwGu[?

<sup>14</sup>V~nELU\'aKkXTJ]?@cGUB@KgBSX?wG\_sS'DUCgyWO'}?

<sup>15</sup>V~~~vnnv|~}gzH}'za|J^ef]~wBJNisn[bn^@^nwez~\_

<sup>16</sup>V~~~vnn{vT{nvFnFo^}~Dnw\{^AF|hFz|YZ~DT~wX^~

## 3 Other Useful Information

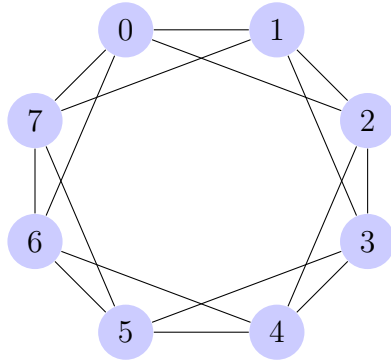
### 3.1 Cayley Graphs

#### 3.1 Definition.

**Cayley Graph** — Let  $H$  be a finite group, and  $S \subseteq H$ , be a subset of  $H$ . Then the Cayley Graph  $C(H, S)$ , has a vertex for each element in  $H$ . There exists an edge between two vertices  $g$  and  $h$ , if and only if there exists  $s \in S$  such that  $sh = g$ . If  $G$  is a graph such that there exists a group  $H$  and a generating set  $S \subseteq H$  with  $G \cong C(H, S)$ , then  $G$  is a Cayley Graph. [2]

#### 3.1 Example.

Consider the group  $\mathbb{Z}_8$  and let the generating set be  $S = \{1, 2\}$ . The vertex set will be  $\{0, 1, 2, 3, 4, 5, 6, 7\}$  and there will be an edge between two vertices,  $g$  and  $h$ , if for an  $s \in S$ ,  $g + s = h$ :



### 3.2 John's Proof

#### 3.2 Definition.

**k-saturated** — The graph  $G$  is said to be  $k$ -saturated if it does not contain a complete  $(k + 1)$ -graph, but every graph  $G'$  obtained from adding a new edge to  $G$  contains a complete  $(k + 1)$ -graph. [3]

#### 3.3 Definition.

**Conical Vertex** — A vertex,  $V$ , is a conical vertex of a graph,  $G$ , if  $V$  is adjacent to every vertex in  $G$ .

#### 3.1 Observation.

A graph,  $G$ , is  $k$ -saturated, if and only if its complement,  $\overline{G}$ , is  $\alpha$ -critical,

with  $\alpha(\overline{G}) = k$ .

This is due to the fact that since  $G$  contains a complete  $k$ -graph but not a complete  $(k + 1)$ -graph,  $\overline{G}$  will have a maximum independent set of size  $k$ . By adding any edge to  $G$  to obtain  $G'$ ,  $G'$  will contain a complete  $(k+1)$ -graph, and thus  $\overline{G'}$  will then have an independent set of size  $k + 1$ . Thus,  $\overline{G}$  is  $\alpha$ -critical and  $\alpha(G) = k$ .

### 3.1 Theorem.

Assume  $G$  is  $k$ -saturated. Then  $G$  contains at least  $2k - |G|$  conical vertices.  
[3]

### 3.2 Observation.

From theorem 3.1, thinking in terms of the complement of a graph, we get that if  $G$  is  $\alpha$ -critical and connected, then  $G$  must satisfy  $\alpha(G) \leq \frac{|G|}{2}$ .

*Proof.* Beginning with a graph  $G$ , if  $G$  is  $\alpha$ -critical with  $\alpha(G) = k$ , then due to observation 3.1,  $\overline{G}$  is  $k$ -saturated. Now, following from theorem 3.1,  $\overline{G}$  contains at least  $2k - |\overline{G}|$  conical vertices. This means that  $G$  must contain at least  $2k - |\overline{G}| = 2\alpha(G) - |G|$  isolated vertices. However,  $G$  is connected, so the number of isolated vertices must equal zero and so  $2\alpha(G) - |G| \leq 0$ . Rearranging, gives  $\alpha(G) \leq \frac{|G|}{2}$  as required.  $\square$

As well, consider the contrapositive of the statement: if  $\alpha(G) > \frac{|G|}{2}$ , then  $G$  is either not connected or not  $\alpha$ -critical.

### 3.2 Theorem.

Let  $G$  be a connected graph such that  $\alpha(G) \geq \frac{|G|}{2}$ . Then either  $G$  has a tight weight matrix, or there exists an induced subgraph,  $H$ , such that  $H$  has no tight weight matrix and  $\alpha(H) < \frac{|H|}{2}$ .

*Proof.* First off, following from [4], the only graph,  $H$ , with  $\alpha(H) = \frac{|H|}{2}$  is the complete graph,  $K_2$ . However, since  $K_2$  has less than 10 vertices, we know that from [1] that  $K_2$  does not have a tight weight matrix. Thus, the statement is true for  $K_2$  and since it is the only graph with  $\alpha(K_2) = \frac{|K_2|}{2}$ , we now only need to consider graphs,  $G$ , with  $\alpha(G) > \frac{|G|}{2}$ .

We will proceed with induction on the number of vertices.

Base case: Let  $G$  be a connected graph on 10 vertices or less with  $\alpha(G) > \frac{|G|}{2}$ . Then from [1], we know that all graphs on 10 vertices or less does not have a tight weight matrix, and so  $G$  does not have a tight weighting.

Inductive Hypothesis: Let  $G$  be a connected graph on  $n$  vertices such that  $\alpha(G) > \frac{|G|}{2}$ . Then either  $G$  has a tight weight matrix, or there exists an induced subgraph,  $H$ , such that  $H$  has no tight weight matrix and  $\alpha(H) \leq \frac{|H|}{2}$ .

Inductive Step: Consider a connected graph  $G$  on  $n + 1$  vertices such that  $\alpha(G) > \frac{|G|}{2}$ . Now if  $G$  has a tight weight matrix, we're done, so assume that  $G$  does not have a tight weight matrix. Now, from the contrapositive of observation 3.2 and due to  $G$  being connected, we know  $G$  is not  $\alpha$ -critical. Thus, there exists an edge,  $e_1$ , such that  $\alpha(G - e_1) = \alpha(G)$ . This leaves us with 2 cases:

Case 1:  $G - e_1$  is disconnected. In this case, either both of the components have a tight weighting, or at least one has a non-tight weighting. If one has a non-tight weight matrix, then  $G$  contains an induced subgraph,  $H$ , such that  $H$  has no tight weight matrix. Now, either  $H$  satisfies  $\alpha(H) \leq \frac{|H|}{2}$  in which case we're done, or  $\alpha(H) > \frac{|H|}{2}$ . If  $\alpha(H) > \frac{|H|}{2}$ , then we apply the inductive hypothesis to  $H$  and since we already know  $H$  does not have a tight weight matrix, it must be the case the  $H$  contains an induced subgraph  $F$  such that  $F$  has no tight weight matrix and  $\alpha(F) \leq \frac{|F|}{2}$ . However, since  $F$  is an induced subgraph of  $H$ , it must also be an induced subgraph of  $G$ , and so we find that  $F$  satisfies the condition that  $G$  has an induced subgraph with  $\alpha(F) \leq \frac{|F|}{2}$  and so we are done. Thus, assume that both components have a tight weighting. However, using both of these tight weightings on  $G$ , and setting  $e_1$  to a zero weighting,  $G$  then has a tight weight matrix. However, this is a contradiction as we assumed  $G$  did not have a tight weight matrix. This completes this case.

Case 2:  $G - e_1$  is connected. Now because  $G$  was not  $\alpha$ -critical,  $\alpha(G) = \alpha(G - e_1)$  and so  $\alpha(G - e_1) > \frac{|G - e_1|}{2}$ . Thus, due to observation 3.1,  $G - e_1$  is still not  $\alpha$ -critical. Therefore, we can continue to delete edges without changing the fact that the resulting graph will be  $\alpha$ -critical as long as it's connected. Now, consider the graph that results from  $G$ , which will be denoted  $G'$ , where after deleting  $k - 1$  edges,  $G$  is still connected, but after deleting the  $k$ th edge,  $G$  has now become disconnected. We can now add back all edges that do not reconnect  $G'$  as this will not decrease  $\alpha(G')$  since deleting it did not. Now, we have the same situation as Case 1, with the only difference being that if the two disconnected components of  $G'$  have a tight weighting, we use their tight weightings and put the weightings of all edges we deleted to obtain  $G'$  to 0, giving us a tight weighting for  $G$ . This

completes this case.

Therefore,  $G$  satisfies the statement, and so by induction, the statement is true for all graphs.  $\square$

### 3.3 No Bound on Inertia Bound Gap

#### 3.3 Theorem.

For a graph  $G$ , there does not exist a bound on the difference between  $\alpha(G)$  and the minimum inertia of a weight matrix corresponding to  $G$ .

*Proof.* Consider a graph,  $G$ , such that  $G$  does not have a tight weight matrix with the gap from equality in the inertia bound being  $\gamma$ , and  $G-v$  does not have a tight weight matrix with the gap also being  $\gamma$  and  $\alpha(G) = \alpha(G-v) \forall v \in V(G)$ . Paley 17 is an example of a graph that has this property so there will exist such a graph. Now, construct a graph  $H$  by connecting two copies of  $G$ ,  $G_1$  and  $G_2$ , with an edge between a vertex on each copy of  $G$  that does not belong to the independent set of either  $G_1$  or  $G_2$ .

Now, consider the induced subgraph  $H'$  obtained by deleting the vertex from the component  $G_1$  in  $H$  that connects to the cut-edge in  $H$ . This will result in  $H'$  having two components,  $G_1 - v$  and  $G_2$ . Now any weight matrix of  $H'$ ,  $W'$  will have the form:

$$W' = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}$$

With  $W_1$  and  $W_2$  being the weight matrices associated with the components  $G_1 - v$  and  $G_2$ , respectively. As a result, the eigenvalues of  $W'$  will be equal to the eigenvalues of  $W_1$  and  $W_2$ . Thus, for any weight matrix  $W'$  of  $H'$ , the inertia of  $W'$  will be equal to the sum of the inertias of  $W_1$  and  $W_2$ . Therefore,

$$\begin{aligned} \min\{|H'| - n_+(W'), |H'| - n_-(W')\} &= \min\{|G-v| - n_+(W_1), |G-v| - n_-(W_1)\} \\ &\quad + \min\{|G| - n_+(W_2), |G| - n_-(W_2)\} \\ &= 2 \min\{|G| - n_+(W_2), |G| - n_-(W_2)\} \end{aligned} \tag{13}$$

From Cauchy's Eigenvalue Interlacing Theorem, we know any weight matrix,  $W$ , associated with  $H$ , will have at least as many positive and negative

eigenvalues as  $W'$ . Thus,

$$\begin{aligned} 2 \min\{|G| - n_+(W_2), |G| - n_-(W_2)\} &= \min\{|H'| - n_+(W'), |H'| - n_-(W')\} \\ &\leq \min\{|H| - n_+(W), |H| - n_-(W)\} \end{aligned} \tag{14}$$

Due to how we created  $H$ ,  $\alpha(H) = 2\alpha(G)$  because the largest independent set of  $H$  will just equal the union of the largest independent set of  $G_1$  and  $G_2$  since we connected  $G_1$  and  $G_2$  without connecting edges between the two independent sets.

Therefore, we get:

$$\begin{aligned} \alpha(H) &= 2\alpha(G) \\ &= 2\gamma + 2 \min\{|G| - n_+(W_2), |G| - n_-(W_2)\} \\ &\leq 2\gamma + \min\{|H| - n_+(W), |H| - n_-(W)\} \end{aligned} \tag{15}$$

Therefore, the gap between equality in the inertia bound has at least doubled. Now as long as there exists a graph,  $F$ , without a tight weight matrix, such that  $F - v_1 - v_2$  for some  $v_1, v_2 \in V(F)$  also does not have a tight weight matrix with a gap in the inertia bound equal to the gap of of the inertia bound in  $F$ , and both  $F$  and  $F - v_1 - v_2$  have the same independent set, we can generalize this result to create a new graph, connecting arbitrarily many copies of  $F$  and following the same argument to multiply the gap in the inertia bound by how many copies of  $F$  we connect. (I THINK THERE DOES EXIST A GRAPH THAT WOULD SATISFY THIS PROPERTY BUT NEED TO CHECK)

□

## References

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- [4] Michael D Plummer. Some covering concepts in graphs. *Journal of Combinatorial Theory*, 8(1):91–98, 1970.
- [5] John Sinkovic. A graph for which the inertia bound is not tight. *arXiv preprint arXiv:1609.02826*, 2016.