Algorithm Design Manual Solutions

Zachary William Grimm
Solutions to Selected Problems
zwgrimm@gmail.com

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Contents

1 Introduction To Algorithm Design

1.1 Finding Counter Examples

Ex:1-1. *Show that a + b can be less than min(a, b)*

Let $a = -1, b = -1$
Then $a + b = -2,\ min(a, b) = -1$
$\therefore \exists\ a, b \in \mathbb{Z} : a + b < min(a, b)$

Ex:1-2. *Show that a * b can be less than min(a, b)*

Let $a = -1, b = 5.$
Then $a * b = -5,\ min(a, b) = -1$
$\therefore \exists\ a, b \in \mathbb{Z} : a * b < min(a, b)$

Ex:1-3. *Design/draw a road network with two points a and b such that the fastest route between a and b is not the shortest route*

$$
\begin{tikzcd}
A \arrow[r, "D{=}5m,\ S{=}1m/s"] \arrow[dr, "D{=}6m,\ S{=}3m/s"'] & C \arrow[r, "D{=}5m,\ S{=}.2m/s"] & B \\
 & D \arrow[ur, "D{=}6m,\ S{=}3m/s"'] &
\end{tikzcd}
$$

Although the distance from A to B through C is shorter than goingthrough D, road constraints limit the time it takes making the route through D faster despite it **Ex:1-4.** *Design/draw a road network with two points a and b such that the shortest route between a and b is not the route with the fewest turns*

$$
\begin{tikzcd}
 & {} \arrow[d, "1"] & A \arrow[l, "1"'] \arrow[r, "6"] & C \arrow[dd, "2"] \\
{} \arrow[d, "1"] & {} \arrow[l, "1"'] & & \\
{} \arrow[rr, "2"] & & B & {} \arrow[l, "6"']
\end{tikzcd}
$$

Proofs of Correctness

The route from A through C to B has only two turns but is a total length of 14 units while the direct route from A to B (the shortest) has 4 turns and is a length of 6 units. $\therefore$ The shortest route between A and B is not the route with the fewest turns. **1-7.** *Prove the correctness of the following recursive algorithm to multiply two natural numbers, for all integer constants* $c \geq 2$

function $multiply(y, z)$

1. If $z = 0$ then return(0) else

2. return $multiply(cy, \lfloor (z/c) \rfloor) + y(z \ mod \ c))$

case 1: *c* = *z*

$$\begin{aligned}
multiply(y, z) &= multiply(cy, \lfloor (z/c) \rfloor) + y(z \ mod \ c) \\
&= multiply(zy, \lfloor (z/z) \rfloor) + y(z \ mod \ z) \\
&= multiply(zy, 1) + y(z \ mod \ z) \\
&= multiply(zy, 1) \\
&= multiply(czy, \lfloor (1/c) \rfloor) + zy(1 \ mod \ c) \\
&= multiply(zzy, \lfloor (1/z) \rfloor) + zy(1 \ mod \ z) \\
&= multiply(zzy, 0) + zy(1 \ mod \ z) \\
&= yz
\end{aligned}$$

case 2: *c* > *z*

$$\begin{aligned}
multiply(y, z) &= multiply(cy, \lfloor (z/c) \rfloor) + y(z \ mod \ c) \\
&= multiply(cy, 0) + y(z \ mod \ c) \\
&= multiply(cy, 0) + y(z \ mod \ c) \\
&= yz
\end{aligned}$$

case 3: *c* < *z*

Assumptions:
$c \geq 2$
$z \leq n$ (inductive hypothesis)
$y \geq 0$

Base Case: $z = 0,\ multiply(y, 0) = 0$, (which is true)

Lemma: we show that

$$\lfloor (z/c) \rfloor * c + (z \ mod \ c) = z$$

by the quotient remainder theorem

$$\begin{aligned} z &= cq + r \\ &= cq + z \ mod \ c \\ &= \lfloor (z/c) \rfloor * c + (z \ mod \ c) \quad \text{(1*)} \end{aligned}$$

Assuming the algorithm holds for all numbers $\leq n$, we must show that

$$multiply(y, n+1) = y(n+1)$$

Now,

$$multiply(y, n+1) = multiply(cy, \lfloor ((n+1)/c)) \rfloor + y((n+1) \ mod \ c)$$

since $c \geq 2$,

$$\lfloor ((n+1)/c) \rfloor < n+1$$

$\therefore$ the first term returns a valid result (based on our inductive hypothesis) so following the algorithm:

$$multiply(y, n+1) = multiply(cy, \lfloor ((n+1)/c)) \rfloor + y((n+1) \ mod \ c)$$

(for simplicity let $z^{'} = n+1$)

$$\begin{aligned} &= cy\lfloor ((z^{'})/c) \rfloor + y((z^{'}) \ mod \ c) \\ &= y(c\lfloor (z^{'}/c) \rfloor + (z^{'} \ mod \ c)) \end{aligned}$$

and from (1*)

$$\begin{aligned} &= yz^{'} \\ &= y(n+1) \end{aligned}$$

$\blacksquare$

1-8. *Prove the correctness of the following algorithm for evaluating a polynomial:*

$$P(x) = a_n^n + a_{n-1}^{n-1} + ... + a_1 + a_0$$

$$\textit{function } horner(A, x):$$

$$p = A_n$$

$$\text{for } i \text{ from } n - 1 \ to \ 0$$

$$p = px + A_i$$

$$\text{return } p$$

For polynomials of degree 0, $P(x) = A_0$, which the algoritm satisfies.
Assuming the algorithm holds for polyomials of degree $\leq n$ and that A is an ordered set of coefficients of siz
$horner(A, x) = P(x) = a_n^{x^n} + a_{n-1}x^{n-1} + ... + a_1x + a_0$
we must show it holds for polynomials of degree $n + 1$, ie: $A^{'}$ is an ordered set of coefficients of size $n + 1, \ [A_{n+1}, A_n, ...A_0]$

$$\textbf{horner(} A^{'}, x) :$$
$$p \Rightarrow$$
$$A^{'}_{n+1}$$
$$A^{'}_{n+1} * x + A^{'}_n$$
$$A^{'}_{n+1} * x^2 + A^{'}_n * x + A^{'}_{n-1}$$
$$.$$
$$.$$
$$.$$
$$A^{'}_{n+1} * x^{n+1} + A^{'}_n * x^n + ... + A^{'}_1 * x + A^{'}_0$$
$$=$$
$$A^{'}_{n+1} * x^{n+1} + horner(A, x)$$

■ *??? seems circular... define what we need to show better*

1-9. *Prove the correctness of the following sorting algorithm:*

function bubblesort (A : list[1 ...n])

var int $i, \ j$

for $i \ from \ n \ to \ 1$

for $j \ from \ 1 \ to \ i - 1$

if $A[j] > A[j + 1]$

swap the values of $A[j] \ and \ A[j + 1]$

Base case is a list of two elements $[a, b]$ (we are indexing from 1 not 0 as usual)

case 1: *a > b*

$$\begin{gathered} i = 2 \\ j = 1 \\ \text{if } (A[1] > A[2]) = \ true \\ \text{swap values, so } A = [b, a] \\ i = 1 \\ j = 1 \\ \text{if } (A[1] > A[2]) = \ false \end{gathered}$$

so $A = [b, a]$ and from our assumption that $a > b$ for case 1 the algorithm is true.

case 2: *a < b*

$$\begin{gathered} i = 2 \\ j = 1 \\ \text{if } (A[1] > A[2]) = \ false \\ \text{swap values, so } A = [b, a] \\ i = 1 \\ j = 1 \\ \text{if } (A[1] > A[2]) = \ false \end{gathered}$$

so $A = [a, b]$ and from our assumption that $a < b$ for case 2 the algorithm is true.

case 3: *a = b*
The list is considered sorted regardless of how the algorithm may rearange its items
Now we procede with induction, assuming that for all lists of size $\leq n$ the algorithm
returns a sorted list We must show that for a list of size n+1, the algorithm returns an ordered list
that is function bubblesort $(A : list[1...n, n+1])$

$$\begin{gathered} \text{var int } i, j \\ \text{for } \ i \ from \ n+1 \ to \ 1 \\ \text{for } \ j \ from \ 1 \ to \ i-1 \\ \text{if } \ (A[j] > A[j+1]) \\ \text{swap the values of } \ A[j] \ and \ A[j+1] \end{gathered}$$

The inner loop "bubbles up" the largest num to position i
on the last j iteration of the first i iteration the altered set has n+1 elems with the largest
element in position n+1 now we have sort the rest of the list which is of size n.
Our inductive hypothesis states we can do this. ■

Induction

1-10. *Prove that* $\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \ for \ n \geq 0$*, by induction*

Let $n = 1$, then

$$\frac{n(n+1)}{2} = 1 = \sum_{i=1}^{1} i$$

establishing our base case. We assume that $\exists n \geq 0 \in \mathbb{N}$ where

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \ for \ n \leq \ some \ k \in \mathbb{N}$$

We must show

$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

We can see that

$$\begin{aligned}\sum_{i=1}^{n+1} i &= (n+1) + \sum_{i=1}^{n} i \\ &= (n+1) + \frac{n(n+1)}{2} \\ &= \frac{[2(n+1) + n(n+1)]}{2} \\ &= \frac{(n+1)(n+2)}{2} \ \blacksquare\end{aligned}$$

1-11. *Prove that* $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \ for \ n \geq 0$, *by induction*
Let $n = 1$. Then

$$\frac{n(n+1)(2n+1)}{6} = 1 = \sum_{i=1}^{1} i^2$$

establishing our base case. We assume that $\exists n \geq 0 \in \mathbb{N}$ where

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \ for \ n \leq \ some \ k \in \mathbb{N}$$

We must show

$$\sum_{i=1}^{n+1} i^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

We can see that

$$
\begin{aligned}
\sum_{i=1}^{n+1} i^2 &= (n+1)^2 + \sum_{i=1}^{n} i^2 \\
&= (n+1)^2 + \frac{n(n+1)(2n+1)}{6} \\
&= \frac{(n+1)[6(n+1) + n(2n+1)]}{6} \\
&= \frac{(n+1)[2n^2 + 7n + 6]}{6} \\
&= \frac{(n+1)(n+2)(2n+3)}{6} \blacksquare
\end{aligned}
$$

1-12. *Prove that* $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ $for\ n \geq 0$, *by induction*