

Algorithm Design Manual Solutions

Zachary William Grimm

Solutions to Selected Problems

zwgrimm@gmail.com

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1 Introduction To Algorithm Design

Finding Counter Examples

1-1. Show that $a + b$ can be less than $\min(a, b)$

Let $a = -1, b = -1$

Then $a + b = -2, \min(a, b) = -1$

$\therefore \exists a, b \in \mathbb{Z} : a + b < \min(a, b)$

1-2. Show that $a * b$ can be less than $\min(a, b)$

Let $a = -1, b = 5$.

Then $a * b = -5, \min(a, b) = -1$

$\therefore \exists a, b \in \mathbb{Z} : a * b < \min(a, b)$

Proofs of Correctness

1-7. Prove the correctness of the following recursive algorithm to multiply two natural numbers, for all integer constants $c \geq 2$

function *multiply*(y, z)

1. If $z = 0$ then return(0) else

2. return *multiply*($cy, \lfloor (z/c) \rfloor$) + $y(z \bmod c)$)

case 1: $c = z$

$$\begin{aligned} \text{multiply}(y, z) &= \text{multiply}(cy, \lfloor (z/c) \rfloor) + y(z \bmod c) \\ &= \text{multiply}(zy, \lfloor (z/z) \rfloor) + y(z \bmod z) \\ &= \text{multiply}(zy, 1) + y(z \bmod z) \\ &= \text{multiply}(zy, 1) \\ &= \text{multiply}(czy, \lfloor (1/c) \rfloor) + zy(1 \bmod c) \\ &= \text{multiply}(zzy, \lfloor (1/z) \rfloor) + zy(1 \bmod z) \\ &= \text{multiply}(zzy, 0) + zy(1 \bmod z) \\ &= yz \end{aligned}$$

case 2: $c > z$

$$\begin{aligned} \text{multiply}(y, z) &= \text{multiply}(cy, \lfloor (z/c) \rfloor) + y(z \bmod c) \\ &= \text{multiply}(cy, 0) + y(z \bmod c) \\ &= \text{multiply}(cy, 0) + y(z \bmod c) \\ &= yz \end{aligned}$$

case 3: $c < z$

Assumptions:

$$c \geq 2$$

$$z \leq n \text{ (inductive hypothesis)}$$

$$y \geq 0$$

Base Case: $z = 0$, $multiply(y, 0) = 0$, (which is true)

Lemma: we show that

$$\lfloor (z/c) \rfloor * c + (z \bmod c) = z$$

by the quotient remainder theorem

$$\begin{aligned} z &= cq + r \\ &= cq + z \bmod c \\ &= \lfloor (z/c) \rfloor * c + (z \bmod c) \quad (1^*) \end{aligned}$$

Assuming the algorithm holds for all numbers $\leq n$, we must show that

$$multiply(y, n + 1) = y(n + 1)$$

Now,

$$multiply(y, n + 1) = multiply(cy, \lfloor ((n + 1)/c) \rfloor) + y((n + 1) \bmod c)$$

$$\text{since } c \geq 2,$$

$$\lfloor ((n + 1)/c) \rfloor < n + 1$$

\therefore the first term returns a valid result (based on our inductive hypothesis) so following the algorithm:

$$multiply(y, n + 1) = multiply(cy, \lfloor ((n + 1)/c) \rfloor) + y((n + 1) \bmod c)$$

(for simplicity let $z' = n + 1$)

$$= cy \lfloor ((z')/c) \rfloor + y((z') \bmod c)$$

$$= y(c \lfloor (z'/c) \rfloor + (z' \bmod c))$$

and from (1*)

$$= yz'$$

$$= y(n + 1)$$

■

1-8. Prove the correctness of the following algorithm for evaluating a polynomial:

$$P(x) = a_n^n + a_{n-1}^{n-1} + \dots + a_1 + a_0$$

function horner(A, x) :

$$p = A_n$$

for i from $n - 1$ to 0

$$p = px + A_i$$

return p

For polynomials of degree 0, $P(x) = A_0$, which the algorithm satisfies.

Assuming the algorithm holds for polynomials of degree $\leq n$ and that A is an ordered set of coefficients of size

$$\text{horner}(A, x) = P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

we must show it holds for polynomials of degree $n + 1$, ie: A' is an ordered set of coefficients of size $n + 1$, $[A_{n+1}, A_n, \dots, A_0]$

horner(A', x) :

$$p \Rightarrow$$

$$A'_{n+1}$$

$$A'_{n+1} * x + A'_n$$

$$A'_{n+1} * x^2 + A'_n * x + A'_{n-1}$$

.

.

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$$A'_{n+1} * x^{n+1} + A'_n * x^n + \dots + A'_1 * x + A'_0$$

=

$$A'_{n+1} * x^{n+1} + \text{horner}(A, x)$$

■ ??? seems circular... define what we need to show better

1-9. Prove the correctness of the following sorting algorithm:

function bubblesort (A : list[1 ...n])

var int i, j

for i from n to 1

for j from 1 to $i - 1$

if $A[j] > A[j + 1]$

swap the values of $A[j]$ and $A[j + 1]$

Base case is a list of two elements $[a, b]$ (we are indexing from 1 not 0 as usual)

case 1: $a > b$

```

        i = 2
        j = 1
        if (A[1] > A[2]) = true
        swap values, so A = [b, a]
        i = 1
        j = 1
        if (A[1] > A[2]) = false

```

so $A = [b, a]$ and from our assumption that $a > b$ for case 1 the algorithm is true.

case 2: $a < b$

```

        i = 2
        j = 1
        if (A[1] > A[2]) = false
        swap values, so A = [b, a]
        i = 1
        j = 1
        if (A[1] > A[2]) = false

```

so $A = [a, b]$ and from our assumption that $a < b$ for case 2 the algorithm is true.

case 3: $a = b$

The list is considered sorted regardless of how the algorithm may rearrange its items
 Now we proceed with induction, assuming that for all lists of size $\leq n$ the algorithm returns a sorted list
 We must show that for a list of size $n+1$, the algorithm returns an ordered list that is function `bubblesort (A : list[1...n, n + 1])`

```

        var int i, j
        for i from n + 1 to 1
        for j from 1 to i - 1
        if (A[j] > A[j + 1])
        swap the values of A[j] and A[j + 1]

```

The inner loop "bubbles up" the largest num to position i
 on the last j iteration of the first i iteration the altered set has $n+1$ elems with the largest element in position $n+1$ now we have sort the rest of the list which is of size n .
 Our inductive hypothesis states we can do this. ■

Induction

1-10. Prove that $\sum_{i=1}^n i = \frac{n(n+1)}{2}$ for $n \geq 0$, by induction

Let $n = 1$, then

$$\frac{n(n+1)}{2} = 1 = \sum_{i=1}^1 i$$

establishing our base case. We assume that $\exists n \geq 0 \in \mathbb{N}$ where

$$\sum_{i=1}^n i = \frac{n(n+1)}{2} \text{ for } n \leq \text{some } k \in \mathbb{N}$$

We must show

$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

We can see that

$$\begin{aligned} \sum_{i=1}^{n+1} i &= (n+1) + \sum_{i=1}^n i \\ &= (n+1) + \frac{n(n+1)}{2} \\ &= \frac{[2(n+1) + n(n+1)]}{2} \\ &= \frac{(n+1)(n+2)}{2} \blacksquare \end{aligned}$$

1-11. Prove that $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$ for $n \geq 0$, by induction

Let $n = 1$. Then

$$\frac{n(n+1)(2n+1)}{6} = 1 = \sum_{i=1}^1 i^2$$

establishing our base case. We assume that $\exists n \geq 0 \in \mathbb{N}$ where

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6} \text{ for } n \leq \text{some } k \in \mathbb{N}$$

We must show

$$\sum_{i=1}^{n+1} i^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

We can see that

$$\begin{aligned}\sum_{i=1}^{n+1} i^2 &= (n+1)^2 + \sum_{i=1}^n i^2 \\&= (n+1)^2 + \frac{n(n+1)(2n+1)}{6} \\&= \frac{(n+1)[6(n+1) + n(2n+1)]}{6} \\&= \frac{(n+1)[2n^2 + 7n + 6]}{6} \\&= \frac{(n+1)(n+2)(2n+3)}{6} \blacksquare\end{aligned}$$

1-12. Prove that $\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$ for $n \geq 0$, by induction