EIGENVALUES OF THE LAPLACIAN ON A GRAPH

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Abstract. By computing the first non-trivial eigenvalue of the Laplacian of a graph, one can understand how well a graph is connected. In this paper, we will build up to a proof of Cheeger's inequality which provides a lower and upper bound for the first non-trivial eigenvalue.

Contents

1. Introduction

In 1973, Miroslav Fiedler first described the connection between the degree of connectivity within a graph and the size of the first non-trivial eigenvalue of the graph's Laplacian in "Algebraic Connectivity of Graphs." Fiedler found that the smaller the first non-trivial eigenvalue, the more disconnected the graph. Since the discovery of this connection, Jeff Cheeger has provided an upper and lower bound for the first non-trivial eigenvalue with the aid of Cheeger's constant–the minimum ratio between the measure of a subgraph's boundary and the measure of the subgraph for all subgraphs with measure less than half the measure of the graph. By bounding the first non-trivial eigenvalue with Cheeger's constant, Cheeger has directly shown the relationship between the first non-trivial eigenvalue and the degree of a graph's connectivity as Cheeger's constant is a measure of connectivity for the subgraph with the weakest degree of connectivity within the entire graph. This connection between the first non-trivial eigenvalue and a graph's connectivity has been applied to many fields such as network security. One measure of the security of a network is how well connected a network is. If a network is not well connected, one may simply remove a few vertices or connections within the graph to prevent the flow of information within the network. Work has been done on increasing a network's security by maximizing the first non-trivial eigenvalue through the use of semi-definite programming [3]. In order to understand why maximizing the first non-trivial eigenvalue of a Laplacian strengthens a network's security, we need to understand Cheeger's Inequality. In this paper, we will establish graph theory terminology in Section 2, an intuitive understanding of the Laplacian of a graph in Section 3, an analysis of the range of eigenvalues of the Laplacian in Section 4, the connection between connected components and the first non-trivial eigenvalue in Section 5, and a proof of Cheeger's Inequality in Section 6.

2. Graphs and Adjacency Matrices

A graph is a collection of vertices and edges connecting these vertices. More precisely, a graph G is a pair of sets (V, E) where V is a set of vertices and E is a set of ordered pairs of vertices. In this paper, we consider graphs in which there is at most one edge between two distinct vertices and no edge connecting a vertex to itself. A graph $G = (V, E)$ is undirected if $(x, y) \in E$ implies $(y, x) \in E$. In other words, whether the edge is drawn from x to y or y to x does not matter for undirected graphs. We say two vertices x and y share an edge if $x \sim y$. More concretely,

$$E = \{(x, y) \mid x, y \in V \text{ and } x \sim y \text{ and } x \neq y\}$$

For the sake of simplicity, we will only work with undirected graphs in this paper. The degree $d(x)$ of a vertex $x \in V$ is the number of vertices that share an edge. We can write this as

$$\deg(x) = \sum_{y \in V} \mathbb{1}_{x \sim y}.$$

A graph is locally finite if every vertex in the graph has a finite number of neighbors. In this paper, we will mostly work with finite graphs. A common example of a finite, undirected graph is a lattice graph (V, E) where $x, y \in V \subset \mathbb{Z}^n$, $x \sim y$ if and only if $\sum_{i=1}^{n} |x_i - y_i| = 1$ where $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$. If $V \subset \mathbb{Z}$, we call the lattice graph (V, E) a path graph.

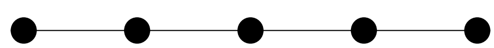

Figure 1. A path graph, $(\{1, 2, 3, 4, 5\}, E)$

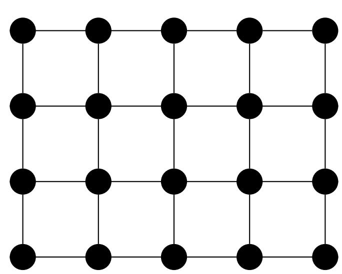

Figure 2. A lattice graph

If we would like to give more importance to certain edges within a graph, we can assign weights to edges.

Definition 2.1. A weighted graph is a pair (G, μ) where $G = (V, E)$ is a graph and $\mu : V \times V \to \mathbb{R}^+$ is a weight on G where

- $\mu_{xy} = \mu_{yx}$
- $\mu_{xy} > 0$ if and only if $(x, y) \in E$

Since μ_{xy} is positive if and only if there is an edge from x to y or y to x, μ fully describes the edge set E of an undirected graph. Therefore, we simply refer to a a graph $G = (V, E)$ with weight μ as (V, μ).

Definition 2.2. Define $\mu : V \to \mathbb{R}^+$as

$$\mu(x) = \sum_{y \sim x} \mu_{xy}.$$

We call $\mu(x)$ the weight of vertex x.

Consider $\mu : V \times V \to \mathbb{R}^+$ where $\mu_{xy} = 1$ if and only if $x \sim y$. We call this weight a simple weight and describe a graph with a simple weight as a simple graph. Note that $\mu(x) = \deg(x)$. We can also encode information of any graph using an adjacency matrix.

Definition 2.3. For $V = \{v_1, .., v_n ...\}$ and $G = (V, \mu)$, we define $A = (a_{ij})$

$$a_{ij} = \begin{cases} \mu_{ij} & v_i \sim v_j \\ 0 & v_i \not\sim v_j. \end{cases}$$

Now, we will work towards giving graphs a notion of distance. To do this, we will need the following definitions.

Definition 2.4. A path on a graph $G = (V, E)$ is a finite sequence of vertices $\{x_k\}_{k=0}^n$ where $x_{k-1} \sim x_k$ for every $k \in \{1, .., n\}$.

Definition 2.5. A graph $G = (V, E)$ is connected if for every $x, y \in V$, there exists a non-trivial path $\{x_k\}_{k=0}^n$ where$x_0 = x$ and $x_n = y$.

Definition 2.6. Let (V, E) be a connected graph and define the graph distance as $d : V \times V \to \mathbb{R}^+$ where $d(x, y)$ is the minimum length of any path between x and y.

The graph (G, V) with distance is a metric space. The name distance is justified for $d(x, y)$ because the function is symmetric, the distance between x and y is zero if and only if $x = y$, and the minimum length between x and z is at most the sum of the minimum lengths between x and y and y and z.

3. Motivation for the Laplacian of a Graph

Define the Laplace operator on the set of twice continuously differentiable functions as $\Delta : C^2(\mathbb{R}^n) \to C(\mathbb{R}^n)$ where

$$\Delta f(x_1, ..., x_n) = \frac{\partial^2 f}{\partial x_1^2} + \ldots + \frac{\partial^2 f}{\partial x_n^2}.$$

The Laplace operator on functions in Euclidean space is fundamental because of its translational and rotational invariance which makes it appear in problems like the heat equation, problems in electrostatics, and many other physical problems. So, we may wish to define a discrete analogue of the Laplace operator to see whether the Laplacian of a graph yields interesting information about a graph. In order to define a Laplace operator for functions on a discrete domain such as $\mathbb{Z}^n$, we must first define a derivative for functions on discrete domains. For $f \in C^1(\mathbb{R})$, we define the left and right derivatives of f at x as:

$$\partial_+ f(x) = \lim_{y \to x^+} \frac{f(y) - f(x)}{y - x} \approx \frac{f(y) - f(x)}{y - x} \quad for \quad y \in N_{\epsilon^+}(x).$$

$$\partial_- f(x) = \lim_{y \to x^-} \frac{f(y) - f(x)}{y - x} \approx \frac{f(y) - f(x)}{y - x} \quad for \quad y \in N_{\epsilon^-}(x).$$

Consider an infinite path graph $(\mathbb{Z}, E)$ as defined in Section 2 and a function g with domain $\mathbb{Z}$ and range $\mathbb{R}$. For $n \in \mathbb{Z}$, $n-1$ and $n+1$ are the only vertices which share an edge with n and are thus closest to n with respect to the notion of distance we gave graphs. The definitions of left and right derivatives on a continuous domain lead naturally to the discrete analogues:

$$\begin{aligned} \partial_+ g(n) &= g(n+1) - g(n). \\ \partial_- g(n) &= g(n) - g(n-1). \end{aligned}$$

To build a second derivative for a function on a discrete domain, take the composition of $\partial_+ g(n)$ and $\partial_- g(n)$ which yields:

$$\begin{aligned} \partial_-\big(\partial_+ g(n)\big) &= \partial_-\big(g(n+1) - g(n)\big), \\ &= \partial_-(g(n+1)) - \partial_-(g(n)), \\ &= \big(g(n+1) - g(n)\big) - \big(g(n) - g(n-1)\big), \\ &= g(n+1) + g(n-1) - 2g(n). \end{aligned}$$

Since $\deg(n) = 2$ for an infinite path graph and $n-1$ and $n+1$ are the only neighbors of n,

$$\partial_-\big(\partial_+ g(n)\big) = \sum_{m \sim n} g(m) - \deg(n) g(n).$$

Continuing with our evaluation of $\partial_-\big(\partial_+ g(n)\big)$,

$$\begin{aligned} \partial_-\big(\partial_+ g(n)\big) &= \big(g(n+1) - g(n)\big) - \big(g(n) - g(n-1)\big), \\ &= \partial_+(g(n)) - \partial_-(g(n-1)). \end{aligned}$$

Now, consider the infinite, lattice graph $(\mathbb{Z}^n, E)$ and a function $h : \mathbb{Z}^n \to \mathbb{R}$. We can now define the partials similarly to how we defined the derivative of g.

$$\begin{aligned} \frac{\partial_+ h(z_1, .., z_n)}{\partial_+ z_i} &= h(z_1, ..., z_i + 1, ..., z_n) - h(z_1, ..., z_i, ..., z_n). \\ \frac{\partial_- h(z_1, .., z_n)}{\partial_+ z_i} &= h(z_1, ..., z_i, ..., z_n) - h(z_1, ..., z_i - 1, ..., z_n). \end{aligned}$$

Now, we can derive the Laplace operator from the second partials with the following observation: $\deg((z_1,..,z_n)) = 2n$ for every $(z_1,...,z_n) \in \mathbb{Z}^n$.

$$\begin{aligned}\Delta h((z_1,..,z_n)) &= \frac{\partial_-\left(\frac{\partial_+ h((z_1,..,z_n))}{\partial z_1}\right)}{\partial z_1} + ... + \frac{\partial_-\left(\frac{\partial_+ h((z_1,..,z_n))}{\partial z_n}\right)}{\partial z_n},\\ &= \sum_{(y_1,..,y_n)\sim((z_1,..,z_n))} h((y_1,..,y_n)) - 2nh((z_1,..,z_n)),\end{aligned}$$

$$\Delta h((z_1,..,z_n)) = \sum_{(y_1,..,y_n)\sim(z_1,..,z_n)} h((y_1,..,y_n)) - deg((z_1,...,z_n)) \cdot h((z_1,..,z_n)). \tag{3.1}$$

We can generalize this definition of the Laplace operator for functions on the lattice graph to functions on a locally finite, connected graph (V,E).

Definition 3.2. Let (V,E) be a locally finite, connected graph. Let $\mathcal{F} = \{f : V \to \mathbb{R}\}$. For every $x \in V$ and for every $f \in \mathcal{F}$, define $\Delta : \mathcal{F} \to \mathbb{R}$ as

$$\Delta f(x) = \frac{1}{\deg(x)} \cdot \sum_{y\sim x} f(y) - f(x).$$

Remark 3.3. Notice how we first generalized equation (3.1) and then normalized $\Delta f(x)$ with the constant $deg(x)$ in Def 3.2. This normalizing constant allows us to view the the Laplace operator evaluated at $f(x)$ as the difference between the average value of the function f for vertices adjacent to x and $f(x)$.

To construct the laplace operator, we informally used another operator, the difference operator. Later on, a more formal definition of the difference operator will become necessary. Formally, the difference operator is the operation $\Delta_{xy} : \mathcal{F} \to \mathbb{R}$ where

$$\Delta_{xy} f = f(y) - f(x) \text{ for every } f \in \mathcal{F}.$$

We can further extend the Laplace operator to weighted graphs.

Definition 3.4. Let (V,μ) be a locally finite, connected, weighted graph. Define the weighted Laplace operator $\Delta\mu : \mathcal{F} \to \mathcal{F}$ as

$$\Delta_\mu f(x) = \frac{1}{\mu(x)} \sum_{y\in V} f(y)\mu_{xy} - f(x).$$

Since $\Delta_\mu(\lambda f + \alpha g)(x) = \lambda\Delta_\mu f + \alpha\Delta_\mu g$ for every $x \in V$ and every $\alpha, \lambda \in \mathbb{R}$, $\Delta_\mu : \mathcal{F} \to \mathcal{F}$ is a linear operation on $\mathcal{F}$. Typically, the Laplacian of a graph is denoted $\mathcal{L} = D - A$ where $A = (a_{ij})$ is the adjacency matrix of a graph (V,E) and D is the $|V| \times |V|$ matrix where

$$D = \begin{bmatrix} deg(v_1) & 0 & \dots & 0 \\ 0 & deg(v_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & deg(v_n) \end{bmatrix}$$

Notice that $\mathcal{L}$ is the Laplace operator adjusted by a factor of $-deg(v_i)$ for every $v_i \in V$ as

$$\begin{aligned}
\mathcal{L}(f(v_i)) &= deg(v_i)\cdot f(v_i) - \sum_{j=1}^{n} a_{ij}\cdot f(v_j),\\
&= deg(v_i)\cdot f(v_i) - \sum_{v_i\sim v_j} f(v_j),\\
&= deg(v_i)\Big(f(v_i) - \frac{1}{deg(v_i)}\sum_{v_i\sim v_j} f(v_j)\Big),\\
&= -\deg(v_i)\cdot \Delta f(v_i).
\end{aligned}$$

Thus, $-D^{-1}\times(D-A)=\Delta$. We can generalize the Laplaician from a graph with a simple weight to weighted graphs by defining $\mathcal{L}_\mu = D_\mu - A_\mu$ where

$$D_\mu = \begin{bmatrix} \mu(v_1) & 0 & \dots & 0 \\ 0 & \mu(v_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mu(v_n) \end{bmatrix}$$

$$A_\mu = \begin{bmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1n} \\ \mu_{21} & \ddots & \cdots & \mu_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n1} & \cdots & \cdots & \cdots \mu_{nn} \end{bmatrix}$$

Following the process above, we see that $\Delta_\mu = -D_\mu \times (D_\mu - A_\mu)$ and hence, $\mathcal{L}_\mu = -\Delta_\mu$.

4. Eigenvalues of the Laplace Operator

Let (V,μ) be a connected, finite, weighted, and undirected graph with $|V| = n$. Recall that $\mathcal{L}_\mu : \mathcal{F} \to \mathcal{F}$ is a linear operator in the n dimensional vector space $\mathcal{F}$. For every $f \in \mathcal{F}$ and $v \in V$, we can uniquely write $f(v)$ as $\sum_{v_i \in V} c_{v_i} \mathbb{1}_{v_i}$ where $\mathbb{1}_{v_i} \in \mathcal{F}$ for every $i \in [n]$. We now will endow $\mathcal{F}$ with an inner product.

Definition 4.1. Let (V,μ) be a finite, weighted graph. Then, for any $f, g \in \mathcal{F}$, we define the inner product of f and g by

$$\langle f, g\rangle = \sum_{i=1}^{n} f(v_i)g(v_i)\mu(v_i).$$

From Definition 4.1, we see that $\{\mathbb{1}_{v_i}\}_{i\in[n]}$ are orthogonal and thus form a basis of $\mathcal{F}$. Recall from linear algebra that any linear operator on an n dimensional vector spaces has n eigenvalues (not necessarily distinct) and n linearly independent eigenvectors. Let's recall the definitions of eigenvalues and eigenvectors.

Definition 4.2. Let A be a linear operator on an n-dimensional vector space V over a scalar field $\mathbb{R}$. A vector $v \neq 0$ and an eigenvalue $\lambda \in \mathbb{C}$ are an eigenvector and eigenvalue of A respectively if $A(v) = \lambda v$. We refer to the range of the eigenvalues as the spectrum.

We can find the eigenvalues from the roots of the characteristic polynomial $\det(A - \lambda \mathbf{I}_n)$ where A is the matrix representation of the linear operator A. By the Fundamental Theorem of Algebra, the polynomial has n complex eigenvalues, not necessarily distinct. By the Spectral Theorem, if $\mathcal{L}_\mu$ is a symmetric operator on $\mathcal{F}$, then all the eigenvalues of $\mathcal{L}_\mu$ are real[1].

Often, in analysis of graphs, eigenvalues can reveal much about the underlying structure of a graph. Through the study of eigenvalues, we can obtain a rough idea of how a graph looks without having to explicitly draw every vertex and edge, a cumbersome task which may produce a graph too detailed to glean any information from. Yet, if the number of edges and vertices of a graph are large, finding the roots of the characteristic polynomial of the Laplace operator will be impossible or computationally expensive and long. This logistical problem leads us to focus on determining which eigenvalues from the n total eigenvalues will give us the most information in order to cut down on computation time.

We will mainly focus on discovering which eigenvalue gives us some information on the connectivity of the graph. To determine which eigenvalues will yield information about a graph's connectivity, we should know the spectrum of the Laplace operator. First, we will use the Spectral Theorem to show that the eigenvalues of the Laplace operator are real. To apply the Spectral Theorem, we need to show $\mathcal{L}_\mu$ is a symmetric operator. To do this, we will need the discrete analogue of Green's Theorem.

Proposition 4.3. *Let (V, μ) be a locally finite, weighted, and connected graph and let $\Omega \subset V$ be non-empty and finite. Then, for $f, g \in \mathcal{F}$,*

$$\sum_{x\in\Omega} \Delta_\mu f(x) g(x) \mu(x) = -\frac{1}{2} \sum_{x\in\Omega} \sum_{y\in\Omega} (\Delta_{xy} f)(\Delta_{xy} g) \mu_{xy} + \sum_{x\in\Omega} \sum_{y\in V\setminus\Omega} (\Delta_{xy} f) g(x) \mu_{xy}.$$

In particular, we have

$$\sum_{x\in V} \Delta_\mu f(x) g(x) \mu(x) = -\frac{1}{2} \sum_{x\in V} \sum_{y\in V} (\Delta_{xy} f)(\Delta_{xy} g) \mu_{xy}.$$

Proof. From Definition 3.4 we have that

$$\begin{aligned}\sum_{x\in\Omega} \Delta_\mu f(x) g(x) \mu(x) &= \sum_{x\in\Omega} \frac{1}{\mu(x)} \Big(\sum_{y\in V} \big(f(y) - f(x)\big) \mu_{xy} \Big) g(x) \mu(x), \\ &= \sum_{x\in\Omega} \sum_{y\in\Omega} (\Delta_{xy} f) g(x) \mu_{xy} + \sum_{x\in\Omega} \sum_{y\in V\setminus\Omega} (\Delta_{xy} f) g(x) \mu_{xy}.\end{aligned}$$

Since $\mu_{xy} = \mu_{yx}$, the first term of the last line can be rewritten as

$$\sum_{x\in\Omega} \sum_{y\in\Omega} (\Delta_{xy} f) g(x) \mu_{xy} = \sum_{y\in\Omega} \sum_{x\in\Omega} (\Delta_{yx} f) g(y) \mu_{yx} = -\sum_{x\in\Omega} \sum_{y\in\Omega} (\Delta_{xy} f) g(y) \mu_{xy}.$$

Averaging the first and last terms from before concludes our proof [2]. □

Now, we can turn to the task of showing that the operator $\mathcal{L}_\mu$ is symmetric with respect to the inner product we defined on $\mathcal{F}$.

Theorem 4.4. *Let (V, μ) be a finite, connected, and weighted graph. Then the linear operator $\mathcal{L}_\mu : \mathcal{F} \to \mathcal{F}$ is symmetric with respect to the inner product defined on $\mathcal{F}$.*

Proof. We must show $\langle \mathcal{L}_\mu f, g\rangle = \langle f, \mathcal{L}_\mu g\rangle$ for every $f, g \in \mathcal{F}$. Indeed, from Proposition 4.3, we have that both expressions equal

$$\frac{1}{2}\sum_{x\in V}\sum_{y\in V}(\Delta_{xy} f)(\Delta_{xy} g)\mu_{xy}.$$

□

Now, we can apply the Spectral Theorem to $\mathcal{F}$, but first we will define Rayleigh quotients which appear in the Spectral Theorem and throughout the rest of this paper.

Definition 4.5. Let V be a finite dimensional vector space and let A be a linear operator on V. For any $v \in V$ where $v \neq 0$, the Rayleigh quotient is

$$\mathcal{R}(v) = \frac{\langle A(v), v\rangle}{\langle v, v\rangle}.$$

Theorem 4.6 (Spectral Theorem)**.** *Let A be a symmetric operator in a n-dimensional inner product space over $\mathbb{R}$. Then,*

- *All the eigenvalues of A are real. Hence, we can label the eigenvalues in increasing order: $\lambda_1 \leq \lambda_2 ... \leq \lambda_n$ with corresponding eigenvectors $v_1, .., v_n$.*
- *For all $k \in \{1, .., n\}$,*

$$\lambda_k = \mathcal{R}(v_k) = \inf_{v\perp v_1,..,v_{k-1}} \mathcal{R}(v) = \sup_{v\perp v_{k+1},...,v_n} \mathcal{R}(v),$$

where $v \perp u$ if $(v, u) = 0$ for $v, u \in V$.

Using Rayleigh quotients, we can show that the spectrum for any Laplacian operator $\mathcal{L}_\mu$ is $[0, 2]$. Our normalization of $\mathcal{L}_\mu(f(v))$ by the degree of each vertex v in Section 3 insured a small spectrum of $[0, 2]$.

Theorem 4.7. *For any finite, connected, weighted graph (V, μ) with $|V| = n > 1$,*

- *Zero is a simple eigenvalue of $\mathcal{L}_\mu$.*
- *All the eigenvalues are contained in $[0, 2]$.*

Proof. Proof of first claim: Consider the constant function $\mathbb{1} \in \mathcal{F}$.

$$\mathcal{L}_\mu(\mathbb{1}(x)) = \mathbb{1}(x) - \frac{1}{\mu(x)}\sum_{y\sim x}\mathbb{1}(y)\mu_{xy} = 1 - \frac{1}{\mu(x)}\sum_{y\sim x}\mu_{xy} = 1 - \frac{\mu(x)}{\mu(x)} = 0.$$

Since $\mathcal{L}_\mu(\mathbb{1}(v)) = 0 \cdot \mathbb{1}(v)$ for every $v \in V$, $\mathbb{1}$ is an eigenfunction with eigenvalue 0. To show 0 is a simple eigenvalue (i.e. with multiplicity 1), assume 0 is not simple. Then, there exists $f \in \mathcal{F}$ where f is an eigenfunction with eigenvalue 0 and f is not a constant function. Now, $\mathcal{L}_\mu f = 0$, and using Proposition 4.3 we have

$$\begin{aligned}0 = \langle \mathcal{L}_\mu f, f\rangle &= \langle -\Delta_\mu f, f\rangle,\\ &= -\sum_{x\in V} \Delta_\mu f(x) f(x)\mu_x,\\ &= \frac{1}{2}\sum_{x\in V}\sum_{y\in V}(\Delta_{xy} f)(\Delta_{xy} f)\mu_{xy} = \frac{1}{2}\sum_{x\in V}\sum_{y\in V}(f(y)-f(x))^2\mu_{xy}.\end{aligned}$$

Since $(f(y)-f(x))^2 \geq 0$ and $\mu_{xy} \geq 0$, each summand in the above sum is non-negative. Since the sum equals 0, $f(y) = f(x)$ for every $y \sim x$ because $\mu_{xy} > 0$ if $x \sim y$. Since (V, μ) is a connected graph, for any $v, u \in V$, there exists a path $\{w_k\}_{k=0}^n$ between $w_0 = v$ and $w_n = u$ where $w_k \sim w_{k+1}$ for every $k \in \{0, .., n-1\}$. Since w_k and w_{k+1} are neighbors, $f(w_k) = f(w_{k+1})$ for every $k \in \{0, ..., n-1\}$. By transitivity, $f(v) = f(u)$. Hence, f is a constant function which is a contradiction of our assumption. Therefore, 0 is a simple eigenvalue with eigenfunction $\mathbb{1}$.

Proof of second claim: Let λ and f be an eigenvalue and eigenfunction of $\mathcal{L}_\mu$ respectively. From the proof of the first claim, we know

$$\langle \mathcal{L}_\mu f, f\rangle = \frac{1}{2}\sum_{x\in V}\sum_{y\in V}(f(x)-f(y)).^2$$

Since each summand is non-negative, $\langle \mathcal{L}_\mu f, f\rangle \geq 0$. Since $\lambda\langle f, f\rangle = \langle \mathcal{L}_\mu f, f\rangle$ where $\langle f, f\rangle \geq 0$ and $\langle \mathcal{L}_\mu f, f\rangle \geq 0$, $\lambda = \frac{\langle \mathcal{L}_\mu f, f\rangle}{\langle f, f\rangle} \geq 0$. Therefore, all eigenvalues of the Laplacian operator are non-negative.

Now, to prove $\lambda \leq 2$, we need to use the following inequality for $a, b \in \mathbb{R}$:

$$(a+b)^2 \leq 2(a^2+b^2). \tag{4.8}$$

This inequality will allows us to show $\lambda\langle f, f\rangle \leq 2\langle f, f\rangle$. Indeed, using (4.8)

$$\begin{aligned}\lambda\langle f, f\rangle = \langle \mathcal{L}_\mu f, f\rangle &= \frac{1}{2}\sum_{x\in V}\sum_{y\in V}(f(y)-f(x))^2\mu_{xy},\\ &\leq \sum_{x\in V}\sum_{y\in V}(f(x)^2+f(y)^2)\mu_{xy},\\ &\leq \sum_{x\in V}\sum_{y\in V} f(x)^2\mu_{xy} + \sum_{y\in V}\sum_{x\in V} f(y)^2\mu_{xy},\\ &\leq \sum_{x\in V} f(x)^2\mu(x) + \sum_{y\in V} f(y)^2\mu(y),\\ &\leq 2\langle f, f\rangle.\end{aligned}$$

Since f is an eigenfunction, $\langle f, f\rangle \neq 0$ and we can divide the last inequality by $\langle f, f\rangle$ to obtain the desired inequality: $\lambda \leq 2$. Therefore, $0 \leq \lambda \leq 2$ and the spectrum of $\mathcal{L}_\mu$ is $[0, 2]$. □

5. Connected Components and Multiplicity of the Trivial Eigenvalue

In our proof of Theorem 4.7, we only used the assumption that (V, μ) is a connected graph to show 0 is a simple eigenvalue. For both connected and non-connected finite, weighted graphs, the spectrum is $[0, 2]$. Yet, the eigenvalue 0 will have a multiplicity strictly greater than 1 for non-connected graphs. The multiplicity of 0 for non-connected graphs will be equal to the number of connected components within the graph. A connected component of a simple graph is a subgraph in which all vertices within the subgraph are connected to each other through paths and every vertex shares no edges with the rest of the graph. One can simply remove a connected component from a simple graph without altering the rest of a graph. To define connected components for all graphs, we need to define the measure of a vertex set $\Omega \subset V$ and an edge set $S \subset E$ as well as the edge boundary of Ω.

Definition 5.1. Consider the graph (V, μ) and $\Omega \subset V$ be a vertex set of the graph. Define $\mu(\Omega)$, the measure of Ω, as $\mu(\Omega) = \sum_{x\in\Omega} \mu(x)$.

Definition 5.2. Let E be the set of edges of (V, μ) and let $S \subset E$ be an edge set of the graph. Define $\mu(S)$, the measure of S, as $\mu(S) = \sum_{e\in E} \mu_e$, where $\mu_e = \mu_{xy}$ for any edge $e = (x, y)$.

Definition 5.3. For $\Omega \subset V$, define $\partial\Omega$, the edge boundary of Ω, as

$$\partial\Omega = \{(x, y) \in E \,|\, x \in \Omega,\, y \notin \Omega\}.$$

Now, we can define a connected component using the measure of an edge boundary.

Definition 5.4. Let (V, μ) be a graph and $\Omega \subset V$ a vertex set of (V, μ). Ω is a connected component of (V, μ) if $\mu(\partial\Omega) = 0$ and Ω is connected (i.e. all vertices in Ω are connected to one another by a path).

To show the eigenvalue 0 for a finite, weighted graph with k connected components has multiplicity k, we need to find k linearly independent eigenfunctions with eigenvalue 0. Since the constant function $\mathbb{1}$ is the eigenfunction of the eigenvalue 0 for connected graphs, we would expect that the indicator functions on each connected component will be the k eigenfunctions for the eigenvalue 0. The indicator functions are linearly independent because each indicator function is only non-zero on a unique connected component and connected components do not share vertices.

Proposition 5.5. *Let (V, μ) be a finite, weighted graph with k connected components $\Omega_1 ... \Omega_k$ where $k \le |V|$. Then, 0 is an eigenvalue of $\mathcal{L}_\mu$ with multiplicity k.*

Proof. Let $\mathbb{1}_{\Omega_j}(v) = 1$ if $v \in \Omega_j$ and 0 otherwise for every $j \in \{1, .., k\}$.

$$\begin{aligned}
\langle \mathcal{L}_\mu \mathbb{1}_{\Omega_j}, \mathbb{1}_{\Omega_j}\rangle &= \langle -\Delta_\mu \mathbb{1}_{\Omega_j}, \mathbb{1}_{\Omega_j}\rangle, \\
&= \frac{1}{2}\sum_{x\in V}\sum_{y\in V} \Delta_\mu \mathbb{1}_{\Omega_j} \Delta_\mu \mathbb{1}_{\Omega_j} \mu_{xy} \text{ by Proposition 4.3}, \\
&= \frac{1}{2}\sum_{x\in V}\sum_{y\in \Omega_j} \big(\mathbb{1}_{\Omega_j}(y) - \mathbb{1}_{\Omega_j}(x)\big)^2 \mu_{xy}, \\
&= \frac{1}{2}\Big(\sum_{x\in \Omega_j}\sum_{y\in V\setminus\Omega_j} \big(\mathbb{1}_{\Omega_j}(y) - \mathbb{1}_{\Omega_j}(x)\big)^2 \mu_{xy} + \sum_{x\in V\setminus\Omega_j}\sum_{y\in \Omega_j} \big(\mathbb{1}_{\Omega_j}(y) - \mathbb{1}_{\Omega_j}(x)\big)^2 \mu_{xy}\Big), \\
&= 0 \text{ because } \mu_{xy} = 0 \text{ for } x \in V \setminus \Omega_J \text{ and } y \in \Omega_j.
\end{aligned}$$

Thus, $\mathbb{1}_{\Omega_j}$ is an eigenfunction with eigenvalue 0 for all $j \in \{1, .., k\}$. To prove linear independence, consider $i, j \in \{1, .., k\}$ where $i \neq j$

$$\langle \mathbb{1}_{\Omega_i} \mathbb{1}_{\Omega_j} \rangle = \sum_{x \in V} \mathbb{1}_{\Omega_i}(x) \mathbb{1}_{\Omega_j} \mu(x) = 0.$$

Since there are k linearly independent eigenfunctions with eigenvalue 0, the eigenvalue 0 has multiplicity k. □

Remark 5.6. Note that (V, μ) must be a finite graph for Proposition 5.6 to hold.

6. Cheeger's Inequality

From the previous proposition, we realize that the second eigenvalue tells us whether a finite, weighted graph has at least one connected component or not. We can extrapolate from the previous proposition to claim that the second eigenvalue for a connected, weighted, finite graph tells us how close the graph is to having a connected component. If the second eigenvalue is close to 0, we only have to remove a few edges from the graph to obtain a connected component. If the second eigenvalue is not close to 0, the graph is too well connected to simply remove a few edges without compromising the originial structure of the graph. The definition of Cheeger's constant and the statement of Cheeger's inequality will prove our claim.

Definition 6.1. Given a finite, weighted graph (V, μ) and $\Omega \subset V$, define its Cheeger constant by

$$h = \min \left\{ \frac{\mu(\partial\Omega)}{\mu(\Omega)} \;\middle|\; \Omega \subset V \text{ where } \mu(\Omega) \leq \frac{1}{2}\mu(V) \right\}.$$

While the definition may seem oddly narrow due to its restriction to vertex sets with measure less than half the total measure of all vertices, this specification prevents h from trivially becoming 0 or close to 0 for all graphs as seen if we let $\Omega = V$ or $V \setminus \{v\}$. Note that Cheeger's constant is 0 for graphs with connected components with measure less than half the total measure of all vertices in the graph.

Theorem 6.2. *Let (V, μ) be a connected, finite, weighted graph and λ_1 be the first non-trivial eigenvalue of $\mathcal{L}_\mu$ and h be Cheeger's constant as defined in Definition 6.1. Then,*

$$\frac{h^2}{2} \leq \lambda_1 \leq 2h.$$

To show the left-hand side of Cheeger's inequality, we will need the following lemmas.

Lemma 6.3. *Consider the graph (V, μ) with E, the set of edges for (V, μ), and $f \in \mathcal{F}$. Let $\Omega_t = \{x \in V | f(x) > t\}$. Then,*

$$\sum_{e \in E} |\Delta_e f| \, \mu_e = \int_{-\infty}^{\infty} \mu(\partial\Omega_t) dt.$$

Proof. First, consider $f \in \mathcal{F}$ and $e \in E$ where $e = (x, y)$. Without loss of generality, let $f(x) \leq f(y)$. Then, for every $e \in E$, we can define an interval $I_e \subset \mathbb{R}$ where $I_e = \big[f(x), f(y)\big)$. Let $|I_e|$ denote the Euclidean length of the interval I_e. Note, $|I_e| = |\Delta_e f|$. To integrate $\mu(\Omega_t)$ with respect to t, we need to write $\mu(\Omega_t)$ as a function of t which we can do with this interval notation and a quick verification of the following fact. For $t \in \mathbb{R}$, $t \in I_e$ if and only if $e \in \partial\Omega_t$. In other words, $f(x) \le t < f(y)$ if and only if $y \in \Omega_t$ (i.e. $f(y) > t$) and $x \in V \setminus \Omega_t$ (i.e. $f(x) \le t$). Now, consider $\mu(\Omega_t)$.

$$\mu(\partial\Omega_t) = \sum_{e\in\partial\Omega_t} \mu_e = \sum_{e\in E \text{ for } t\in I_e} \mu_e = \sum_{e\in E} \mu_e \mathbb{1}_{I_e}(t).$$

Now, we can evaluate the integral of $\mu(\Omega_t)$ with respect to t.

$$\int_{-\infty}^{\infty} \mu(\partial_t)dt = \int_{-\infty}^{\infty} \sum_{e\in E} \mu_e \mathbb{1}_{I_e}(t)dt,$$

$$= \sum_{e\in E} \int_{-\infty}^{\infty} \mu_e \mathbb{1}_{I_e}(t)dt = \sum_{e\in E} \mu_e \int_{f(x)}^{f(y)} dt = \sum_{e\in E} \mu_e \left|I_e\right| = \sum_{e\in E} \left|\Delta_e f\right| \mu_e.$$

□

Lemma 6.4. *For any non-negative function f on V where*

$$\mu\big(\{x \in V \mid f(x) > 0\}\big) \le \frac{1}{2}\mu(V),$$

the following holds,

$$h \sum_{x\in V} f(x)\mu(x) \le \sum_{e\in E} \left|\Delta_e f\right| \mu_e.$$

Remark 6.5. Notice that $\Omega_0 = \{x \in V | f(x) > 0\}$ and for every $t \in \mathbb{R}^+$, $\Omega_t \subset \Omega_0$. Hence, if $\mu(\Omega_0) \le \frac{1}{2}\mu(V)$, then $\mu(\Omega_t) \le \frac{1}{2}\mu(V)$ for $t \ge 0$ because μ is a non-negative function on V. Also note that if we let f be the indicator function on $\Omega \subset V$, the inequality in the lemma above becomes $\mu(\partial\Omega) \ge h\mu(\Omega)$.

Proof. From the previous lemma,

$$\int_0^{\infty} \mu(\partial\Omega_t)dt \le \sum_{e\in E} \left|\Delta_e f\right| \mu_e.$$

The inequality follows from the fact that μ is a non-negative function. Since our integration interval is $\mathbb{R}^+$, we can apply the definition of Cheeger's constant to the boundaries of the sets Ω_t with $t \ge 0$ due to Remark 6.5.

$$\sum_{e\in E} \left|\Delta_e f\right| \mu_e \ge h \int_0^{\infty} \mu(\Omega_t)dt \ge h \int_0^{\infty} \sum_{x\in\Omega_t} \mu(x)dt.$$

We now want to integrate $\sum_{x\in\Omega_t} \mu(x)dt$ where $f(x)$ appears and the restriction of $x \in \Omega_t$ disappears. Modifying the trick we used in the proof of Lemma 6.3, we look for a equivalent statement of $x \in \Omega_t$ that will allow us to use indicator functions on $\mathbb{R}$. Now, for $t \ge 0$, $x \in \Omega_t$ if and only if $t \in \big[0, f(x)\big)$ because $0 \le t < f(x)$. Also, the Euclidean distance of $\big[0, f(x)\big)$ is $f(x)$ which we want to appear in our sum so let's consider the indicator function $\mathbb{1}_{I_{[0,f(x))}}$.

$$\sum_{e\in E} \left|\Delta_e f\right| \mu_e \ge h \int_0^{\infty} \sum_{x\in V} \mu(x) \mathbb{1}_{[0,f(x))}(t)dt,$$

$$\ge h \sum_{x\in V} \mu(x) \int_0^{\infty} \mathbb{1}_{[0,f(x))}(t)dt, \ge h \sum_{x\in V} f(x)\mu(x).$$

□

Now, we will prove the left-hand side of Cheeger's Inequality using Lemma 6.4.

Proof of Theorem 6.2.
Recall the second statement from the Spectral theorem, $\lambda_1 = \sup_{f \perp f_2,..,f_{n-1}} \mathcal{R}(f)$ where f is the eigenfunction of the first non-trivial eigenvalue λ_1. In order to apply Lemma 6.4, we need to find a non-negative function g where $g \perp f_2, ...f_{n-1}$ and $\mu(V^+) \leq \frac{1}{2}\mu(V)$ where

$$V^+ = \{x \in V \mid f(x) \geq 0\} \text{ and } V^- = \{x \in V \mid f(x) < 0\}.$$

Let's consider $g = f_+$ where $f_+ = f$ for $f(v) \geq 0$ and $f_+ = 0$ otherwise. Since $\mu(V^+) + \mu(V^-) = \mu(V)$, either $\mu(V^+) \leq \frac{1}{2}\mu(V)$ or $\mu(V^-) \leq \frac{1}{2}\mu(V)$. Depending on which inequality holds, pick either $-f$ or f because $-f$ is also an eigenfunction of λ_1 and $-f$ is positive on V^-. Now, $g = f\mathbb{1}_{V^+}$. Since f and $\mathbb{1}$ are linearly independent to the eigenfunctions $f_2, .., f_{n-1}$, g is also also linearly independent to $f_2, .., f_{n-1}$. Thus, $\lambda_1 \geq \mathcal{R}(g)$.

Recall that $(\mathcal{L}g, g) = \frac{1}{2}\sum_{e\in E} |\Delta_e g|^2$ by Proposition 4.3. Now, we can rewrite $\mathcal{R}(g)$ to obtain a formula on which we can apply Lemma 6.4.

$$\mathcal{R}(g) = \frac{\langle \mathcal{L}g, g\rangle}{\langle g, g\rangle} = \frac{1}{2}\frac{\sum_{e\in E} |\Delta_e g|^2 \mu_e}{\sum_{x\in V} g(x)^2\mu(x)}.$$

If the following inequality holds,

$$\frac{h^2}{2}\sum_{x\in V} g(x)^2\mu(x) \leq \sum_{e\in E} |\Delta_e g|^2 \mu_e. \tag{6.6}$$

then $\frac{h^2}{2} \leq \lambda_1$ holds by transitivity. We now derive the inequality in equation (6.6).

$$\begin{aligned}
\sum_{e\in E} \left|\Delta_e(g^2)\right| \mu_e &= \frac{1}{2}\sum_{x\in V}\sum_{y\in V} \left|g^2(x) - g^2(y)\right| \mu_{xy} \text{ because } |E| = 2|V|,\\
&= \frac{1}{2}\sum_{x\in V}\sum_{y\in V} |g(x) - g(y)|\, \mu_{xy}^{\frac{1}{2}} \cdot |g(x) + g(y)|\, \mu_{xy}^{\frac{1}{2}},\\
&\leq \Big(\big(\frac{1}{2}\sum_{x\in V}\sum_{y\in V} \big(g(x) - g(y)\big)^2 \mu_{xy}\big)\big(\frac{1}{2}\sum_{x\in V}\sum_{y\in V} \big(g(x) + g(y)\big)^2\big)\Big)^{\frac{1}{2}} \text{ by Cauchy-Schwarz},\\
&= \Big(\big(\frac{1}{2}\sum_{x\in V}\sum_{y\in V} |\Delta_{xy} g|^2 \mu_{xy}\big)\big(\frac{1}{2}\sum_{x\in V}\sum_{y\in V} \big(g(x) + g(y)\big)^2\big)\Big),^{\frac{1}{2}}\\
&= \Big(\sum_{e\in E} |\Delta_e g|^2 \mu_e \sum_{x\in V}\sum_{y\in V} \big(g^2(x) + g^2(y)\big)\mu_{xy}\Big)^{\frac{1}{2}} \text{ by the inequality in (4.8)},\\
&= \Big(2\sum_{e\in E} |\Delta_e g|^2 \mu_e \sum_{x\in V}\sum_{y\in V} g^2(x)\mu_{xy}\Big).^{\frac{1}{2}}
\end{aligned}$$

Since $\mu(V^+) \leq \frac{1}{2}\mu(V)$ for both g and g^2, we can apply Lemma 6.4 to g^2

$$\sum_{e\in E} \left|\Delta_e(g^2)\right| \mu_e \leq \Big(2\sum_{e\in E} |\Delta_e g|^2 \mu_e \sum_{x\in V} g^2(x)\mu(x)\Big).^{\frac{1}{2}} \tag{6.7}$$

By Lemma 6.4,

$$\sum_{e\in E} \left|\Delta_e g^2\right| \mu_e \geq h \sum_{x\in V} g^2(x)\mu(x). \tag{6.8}$$

By transitivity, we can combine equations (6.7) and (6.8)

$$h \sum_{x\in V} g^2(x)\mu_x \leq \Big(2\sum_{e\in E} \left|\Delta_e g\right|^2 \mu_e \sum_{x\in V} g^2(x)\mu(x)\Big).^{\frac{1}{2}} \tag{6.9}$$

Squaring (6.9) and dividing by $\sum_{x\in V} g^2(x)\mu(x)$,

$$\frac{h^2}{2} \sum_{x\in V} g(x)^2\mu(x) \leq \sum_{e\in E} \left|\Delta_e g\right|^2 \mu_e. \tag{6.10}$$

Since equation (6.6) holds, $\frac{h^2}{2} \leq \lambda_1$ [2]. □

The right-hand side of Cheeger's inequality is quite simple if we view the first non-trivial eigenvalue as a Rayleigh quotient.

Proof of Theorem 6.2 continued.
Since V is a finite set, we know there exists $\Omega \subset V$ where $h = \frac{\mu(\partial\Omega)}{\mu(\Omega)}$. Let $a = \frac{\mu(\Omega)}{\mu(V\setminus\Omega)}$. Consider $f \in \mathcal{F}$ where

$$f(x) = \begin{cases} 1 & x \in \Omega \\ -a & x \in V \setminus \Omega. \end{cases}$$

Note that $\mu(\Omega) \leq \frac{1}{2}\mu(V)$. Thus, $\mu(\Omega) \leq \mu(V\setminus\Omega)$ which implies that $a \leq 1$. By construction,

$$\langle f, \mathbb{1}\rangle = \sum_{x\in V} f(x)\mu(x) = \sum_{x\in\Omega} \mu(x) - \frac{\mu(\Omega)}{\mu(V\setminus\Omega)} \sum_{x\in V\setminus\Omega} \mu(x) = \mu(\Omega) - \mu(\Omega) = 0.$$

By the Spectral Theorem, $\lambda_1 = \inf_{\mathbb{1}\perp g} \mathcal{R}(g)$. Since $\langle f, \mathbb{1}\rangle = 0$, $\lambda_1 \leq \mathcal{R}(f) = \frac{\langle \mathcal{L}_\mu f, f\rangle}{\langle f, f\rangle}$

$$\begin{aligned} \langle f, f\rangle &= \sum_{x\in V} f(x)^2\mu(x), \\ &= \sum_{x\in\Omega} \mu(x) + a^2 \sum_{x\in V\setminus\Omega} \mu(x), \\ &= \mu(\Omega) + a^2\mu(V\setminus\Omega), \end{aligned}$$

$$= \mu(\Omega) + \Big(\frac{\mu(\Omega)}{\mu(V\setminus\Omega)}\Big)^2 \mu(V\setminus\Omega), = (1+a)\mu(\Omega).$$

$$\begin{aligned} \langle \mathcal{L}_\mu f, f\rangle &= \frac{1}{2} \sum_{x\in V}\sum_{y\in V} \big(f(x) - f(y)\big)^2 \mu_{xy}, \\ &= \sum_{x\in\Omega}\sum_{y\in V\setminus\Omega} \big(f(x) - f(y)\big)^2 \mu_{xy} = (1+a)^2 \sum_{x\in\Omega}\sum_{y\in V\setminus\Omega} \mu_{xy} = (1+a)^2\mu(\partial\Omega), \end{aligned}$$

$$\mathcal{R}(f) = \frac{\langle \mathcal{L}_\mu f, f\rangle}{\langle f, f\rangle} = \frac{(1+a)^2\mu(\partial\Omega)}{(1+a)\mu(\Omega)} = (1+a)h \leq 2h.$$

Since $\mathcal{R}(f) \leq 2h$ and $\lambda_1 \leq \mathcal{R}(f)$, $\lambda_1 \leq 2h$. □

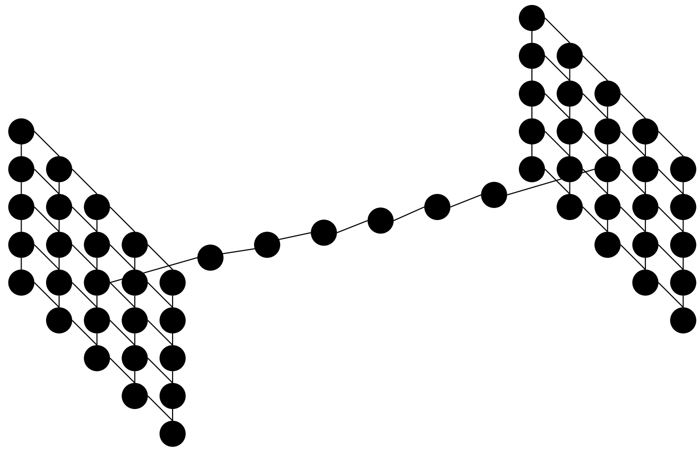

FIGURE 3. Two lattice graphs connected by a path

Example 6.11. The graph above provides a perfect example for how we can use Cheeger's inequality to understand that the first non-trivial eigenvalue of the Laplacian of a graph contains information about a graph's connectivity. The graph above consists of two lattice graphs connected by a path graph. In this example, we only need to look at how long the path graph is to establish how well-connected the graph is. The longer the path connecting the two lattices, the less connected the graph is and the smaller the first non-trivial eigenvalue is. We can see how small the first non-trivial eigenvalue becomes as the path grows longer with the aid of Cheeger's inequality. We can immediately compute Cheeger's constant by counting the number of vertices in the path connecting the two lattices. From this process, we get $h = \frac{2}{m}$ where m is the number of vertices in the path graph that are not a part of the lattices. From Cheeger's inequality, $\frac{2}{m^2} \leq \lambda_1 \leq \frac{4}{m}$. As the path grows longer (m grows larger), λ_1 approaches 0 and the graph becomes more disconnected. Once the path graph becomes infinite, the graphs basically become disconnected from one another and $\lambda_1 = 0$.

Remark 6.12. In the example above, we assumed the graph in Figure 3 had simple weights. This allowed us to simply look at the graph to determine Cheeger's constant because we only needed to count vertices to determine which subset of vertices Ω had the smallest ratio of boundary vertices to the number of vertices within the subset.

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7. BIBLIOGRAPHY

References

[1] Charles W. Curtis *Linear Algebra: An Introductory Approach.* Springer. 1984.
[2] Alexander Grigor'yan. *Introduction to Analysis on Graphs.* American Mathematical Society. 2018.
[3] Yoonsoo Kim and Mehran Mesbahi. *On Maximizing the Second Eigenvalue of a State-dependent Graph Laplacian.* Institute of Electrical and Electronics Engineers, Vol. 51, Issue 1, (2006) 116-120.