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Laplacian spectra and spanning trees of threshold graphs

P.L. Hammer*, A.K. Kelmans

RUTCOR, Rutgers University, New Brunswick, NJ 08903, USA

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Abstract

In this paper we study the Laplacian spectra, the Laplacian polynomials, and the number of spanning trees of a special class of graphs called *threshold graphs*. We give formulas for the Laplacian spectrum, the Laplacian polynomial, and the number of spanning trees of a threshold graph, in terms of so-called *composition sequences* of threshold graphs. It is shown that the degree sequence of a threshold graph and the sequence of eigenvalues of its Laplacian matrix are “almost the same”. On this basis, formulas are given to express the Laplacian polynomial and the number of spanning trees of a threshold graph in terms of its degree sequence. Moreover, threshold graphs are shown to be uniquely defined by their spectrum, and a polynomial time procedure is given for testing whether a given sequence of numbers is the spectrum of a threshold graph.

1. Introduction and notations

The Laplacian matrix (or the matrix of admittance) of a graph is one of the classical concepts in graph theory [1, 2, 5, 20]. One of the most interesting and useful results about this matrix is the well-known matrix-tree theorem, which states that if n is the order of the matrix, then the number of spanning trees of the graph is equal to the determinant of any principal minor of order $n - 1$. It turns out that the Laplacian polynomial of the graph, i.e. the characteristic polynomial of the Laplacian matrix of the graph (introduced in [24, 25], see also [10]), is also a very useful concept, which has a natural combinatorial interpretation. It has been shown in [30] that the inclusion–exclusion formula of [33] for the number of spanning trees of a graph can be directly “read” from this polynomial.

This polynomial is also useful in evaluating the probability of connectivity of graphs, in constructing graphs having a maximum number of spanning trees, and for

* Corresponding author.

various other problems in extremal graph theory [12, 24–26, 28–30], as well as in statistical experiments (see, for example, [7]); among the papers dealing with this topic we mention [3, 9, 11, 13, 14, 16, 18, 21–23, 34–37].

For the special case of regular graphs, there is a straightforward relationship between the Laplacian polynomial and the characteristic polynomial of the adjacency matrix of a graph, allowing to translate directly results concerning one of those polynomials to results concerning the other one.

One of the natural questions about the Laplacian polynomial concerns the extent to which it characterizes the corresponding graph. There are many examples of *cospectral graphs*, i.e. non-isomorphic graphs with the same Laplacian polynomial (e.g. [12, 23, 24, 26, 28, 29, 34], see also [10]). On the other hand certain classes of graphs are uniquely defined by their spectrum. Such classes include complete graphs, regular k -partite graphs, paths, circuits, trees up to 10 vertices, stars and some other special trees (e.g. [12, 24, 26, 28–30]). It is also known that in the class of regular graphs the line graphs of some combinatorial structures (complete graphs, complete bipartite graphs, block designs, projective planes, finite affine planes, etc.) are uniquely defined under certain conditions by their Laplacian polynomials among the regular graphs (e.g. [13, 14, 21–23, 35]).

In this paper we study the Laplacian spectra, the Laplacian polynomials, and the number of spanning trees of a special class of graphs called *threshold graphs*. Threshold graphs were introduced in [6], and numerous properties of them were investigated in the literature; surveys on this topic appear in Ch. 10 of [17], in [4], and in the forthcoming monograph [32]. It is easy to see that a canonical description of a threshold graph having n vertices can be given by a sequence of at most n integer parameters, called its *composition sequence*. We give formulas for the Laplacian spectrum, the Laplacian polynomial, and the number of spanning trees of a threshold graph, in terms of its composition sequence (Theorems 4.2, 4.3, and 5.4). It is shown (Theorem 5.3) that the degree sequence of a threshold graph and the sequence of eigenvalues of its Laplacian matrix are “almost the same”. On this basis, formulas are given for the Laplacian polynomial and the number of spanning trees of a threshold graph, in terms of its degree sequence (Theorem 5.4). Moreover, threshold graphs are shown to be uniquely defined by their spectrum (Theorem 6.1), and a polynomial time procedure is given for testing whether a given sequence of numbers is the spectrum of a threshold graph.

All the graphs considered in this paper are undirected and have no loops or multiple edges [1, 2, 20]. Let $V(G)$ and $E(G)$ denote the set of vertices and edges of G , respectively. Let $v(G)$ denote the number of vertices of G .

2. Decomposable graphs

Given two disjoint graphs A and B , a graph G is called the *sum* of A and B , $G = A + B$, if $V(G) = V(A) \cup V(B)$ and $E(G) = E(A) \cup E(B)$. Similarly, the product

$A \times B$ will denote the graph obtained from $A + B$ by adding all the edges (a, b) with $a \in V(A)$ and $b \in V(B)$. In particular, if B consists of a single vertex b , we write $A + b$ and $A \times b$ instead of $A + B$ and $A \times B$; in these cases b is called an *isolated* and a *universal vertex*, respectively. Given a graph G , let $i(G)$ and $u(G)$ denote the number of isolated and universal vertices, respectively.

The graph \bar{G} is said to be the *complement* of a graph G if there is a bijection $\varphi: V(G) \rightarrow V(\bar{G})$ such that (u, v) is an edge of G if and only if $(\varphi(u), \varphi(v))$ is not an edge of \bar{G} . Let K_n be the *complete graph* on n vertices, and let its complement \bar{K}_n be the *empty graph* on n vertices.

If $V(K_n) = V(\bar{K}_n) = \{g_1, \dots, g_n\}$ then $K_n = g_1 \times g_2 \times \dots \times g_n$ and $\bar{K}_n = g_1 + g_2 + \dots + g_n$; we shall write simply $K_n = g^n$ and $\bar{K}_n = ng$.

A graph is called *decomposable* [25] if any induced subgraph having at least two vertices is the sum or the product of two graphs. It is easy to see that a graph is decomposable if and only if it can be obtained from one-vertex graphs by the above operations $\{ + \}$ and $\{ \times \}$, and if and only if it does not contain a path on 4 vertices as an induced subgraph. Let \mathcal{D} denote the set of decomposable graphs.

Clearly every disconnected graph is the (unique) sum of its connected components. Similarly, every connected decomposable graph is the (unique) product of some disconnected or one-vertex graphs (the complements of which are the connected components of the complement of the given graph). For example, the graph H in Fig. 1 is decomposable and can be represented as $H = (g_1 + g_2)(g_4(g_3 + g_5) + g_6g_7)$.

A decomposable graph G can be naturally described [25] by a labelled rooted tree $T(G)$ which will be called the *composition tree* of G . The root r will be labelled having the label $s(r) = \{ \times \}$ if G is connected, and $s(r) = \{ + \}$ if G is disconnected. This tree can be defined recursively as follows. If G is a one-vertex graph then let $T(G)$ be the trivial rooted tree consisting of one vertex which is the root of $T(G)$ and has the label $\{ \times \}$. Suppose that G_i is a decomposable graph, that $T(G_i)$ is already defined for G_i , and that r_i is the root of $T(G_i)$, $i = 1, \dots, k$. Suppose also that the label $s(r_i)$ of the root r_i of $T(G_i)$ is $\{ \times \}$ if G_i is connected, and is $\{ + \}$ if G_i is disconnected. Let T be the tree

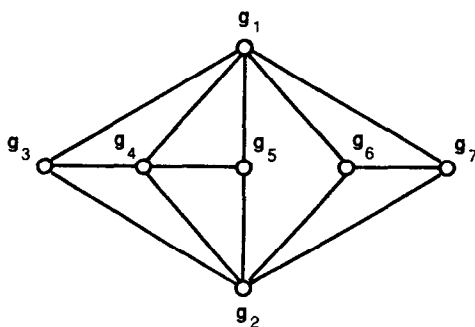


Fig. 1.

obtained from disjoint rooted trees $T(G_i)$ by adding a new vertex r , the root of T , and by connecting r with every root r_i by an edge (r, r_i) . If $G = G_1 \times \dots \times G_k$ where each G_i is not the product of two graphs (which means in our case that G_i is a one-vertex or disconnected graph) then put $T(G) = T$, and $s(r) = \{\times\}$. If $G = G_1 + \dots + G_k$ where each G_i is not the sum of two graphs (i.e. G_i is a connected graph) then put $T(G) = T$, and $s(r) = \{+\}$.

The rooted tree $T(H)$ of the graph H in Fig. 1 is shown in Fig. 2.

It is easy to see that $T(G)$ is uniquely defined by G , and that two decomposable graphs G and F are isomorphic if and only if $T(G)$ and $T(F)$ are isomorphic as rooted labelled trees. Therefore a decomposable graph is uniquely defined (up to isomorphism) by its composition tree. It is easy to see that the vertices of degree 1 in $T(G)$ correspond to the vertices of G . In other words there is a one-to-one correspondence between the set of connected (disconnected) decomposable graphs with n vertices and the set of rooted labelled trees with n leaves.

The *height* of a rooted tree is the number of edges of a longest path of T having the root as an endpoint. The *height* $h(G)$ of a decomposable graph G is the height of its tree $T(G)$.

A graph is called *1-decomposable* if it can be obtained from a one-vertex graph by “adding” or “multiplying” sequentially the current graph by a new one-vertex graph (i.e. by adding an isolated or a universal vertex).

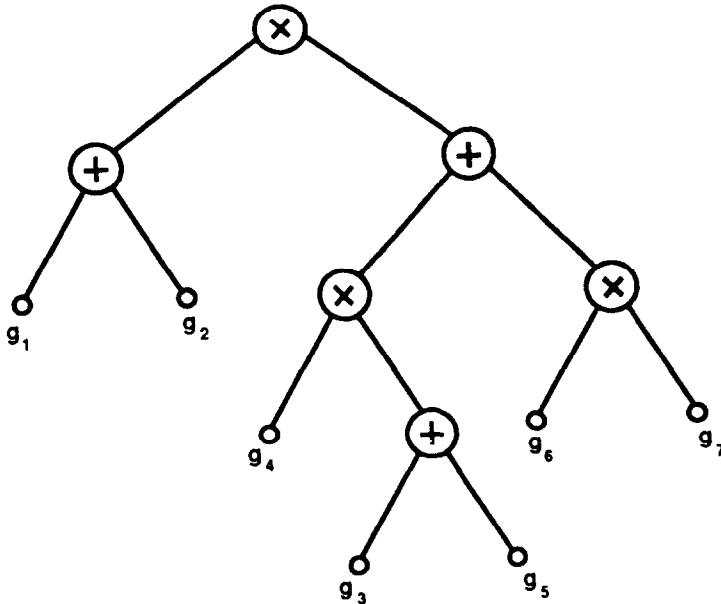


Fig. 2.

It is easy to show that the tree $T(G)$ of a 1-decomposable graph G is characterized by the following property: if $rP_1x_1e_1y_1$ and $rP_2x_2e_2y_2$ are two paths of the same length of $T(G)$ starting from the root r with the pendant edges $e_1 = (x_1, y_1)$ and $e_2 = (x_2, y_2)$, respectively, then $rP_1x_1 = rP_2x_2$.

3. Laplacian polynomial, spectrum and number of spanning trees of a graph

In this section we recall some previously established results (many of which were published in Russian) about the Laplacian polynomial, the spectrum, and the number of spanning trees of an arbitrary graph. These results will play an important role in establishing the main results of this paper.

Let $V(G) = \{v_1, \dots, v_n\}$. Let $L(G)$ be the Laplacian matrix of G , i.e. $L(G) = \{l_{ij}\}$, where

$$l_{ij} = \begin{cases} -1 & \text{if } v_i \text{ and } v_j \text{ are adjacent, } i \neq j. \\ 0 & \text{if } v_i \text{ and } v_j \text{ are not adjacent, } i \neq j, \\ d_G(v_i) & \text{if } i = j. \end{cases} \quad (1)$$

Let $L_i(G)$ denote the matrix obtained from $L(G)$ by deleting the i th row and the i th column.

Let $\tau(G)$ denote the number of spanning trees of G .

The following result is well known and is called the matrix-tree theorem [2, 10, 20].

Theorem 3.1. *If G is a graph with n vertices, then*

$$\tau(G) = \det(L_i(G))$$

for any $i \in \{1, 2, \dots, n\}$.

Let $\lambda_0, \dots, \lambda_{n-1}$ be the eigenvalues of the matrix $L(G)$. Let $Q(\lambda, G) = \det(\lambda I - L(G))$. Obviously $Q(\lambda_i, G) = 0$ for any $i \in \{0, 1, \dots, n-1\}$.

By using Theorem 3.1 it is easy to prove [25, 26] that the matrix $L(G)$ is positive semi-definite, i.e. all λ_i , $i \in \{1, 2, \dots, n-1\}$, are real non-negative numbers. We can assume that $n \geq 1$ and $0 \leq \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_{n-1}$. Obviously $\det L(G) = 0$, and so $\lambda_0 = 0$. Put $P(\lambda, G) = \lambda^{-1}Q(\lambda, G)$. Then $P(\lambda, G)$ is a polynomial, and $S(G) = \{\lambda_1, \dots, \lambda_{n-1}\}$ is the set of its roots. We call $P(\lambda, G)$ the Laplacian polynomial and $S(G)$ the spectrum of G . Let $a_1 < \dots < a_h$ be all the different numbers occurring in $S(G)$ and let m_i be the multiplicity of a_i in $S(G)$; clearly $m_i \geq 1$ and $m_1 + \dots + m_h = n - 1$. We shall sometimes call $S(G) = (a_1^{(m_1)}, \dots, a_h^{(m_h)})$ the spectrum sequence of G .

Let $\lambda_{\max}(G)$ and $\lambda_{\min}(G)$ denote the largest and the smallest number in the spectrum $S(G)$, respectively: $\lambda_{\max}(G) = \lambda_{n-1}$ and $\lambda_{\min}(G) = \lambda_1$.

From Theorem 3.1 we have the following theorem.

Theorem 3.2 (Kelmans [25, 26]). *Let G be a graph with n vertices. Then*

$$\tau(G) = n^{-1} \prod_{i=1}^{n-1} \lambda_i = n^{-1} \prod_{i=1}^h a_i^{m_i}.$$

It turns out that there is a natural interconnection between the spectra of a graph and that of its complement.

Theorem 3.3 (Kelmans [25]). *Let \bar{G} be the complement of a graph G . Let $S(G) = \{\lambda_1 \leq \dots \leq \lambda_{n-1}\}$ and $S(\bar{G}) = \{\bar{\lambda}_1 \leq \dots \leq \bar{\lambda}_{n-1}\}$. Then $\lambda_i + \bar{\lambda}_{n-i} = n$ for any $i \in \{1, 2, \dots, n-1\}$.*

For the sake of completeness we give a proof of this important theorem.

Proof. Put $L(G) = L$ and $L(\bar{G}) = \bar{L}$. Let x_0 be an n -vector with every coordinate equal to 1. Obviously x_0 is an eigenvector of L and \bar{L} with the eigenvalue 0. Since L is a symmetric matrix, there exists the list (x_0, \dots, x_{n-1}) of n pairwise orthogonal eigenvectors of L containing x_0 . Let $\lambda_i(G) = \lambda_i$ be an eigenvalue corresponding to the eigenvector x_i , and so $\lambda_0 = 0$ and $Lx_i = \lambda_i x_i$ for $i = 0, 1, \dots, n-1$. We can assume that $0 = \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_{n-1}$.

Clearly $L + \bar{L} = nI - E$, where E is the $(n \times n)$ -matrix with every entry equal to 1. Since $x_0 x_i = 0$ for every $i \in \{1, \dots, n-1\}$, we have $E x_i = 0$. Therefore

$$\bar{L}x_i = (nI - E - L)x_i = (n - \lambda_i)x_i, \quad \text{for every } i \in \{1, \dots, n-1\}.$$

Thus x_i is an eigenvector of \bar{L} with the eigenvalue $\bar{\lambda}_{n-i} = n - \lambda_i$ for $i = 0, 1, \dots, n-1$. Obviously $0 = \bar{\lambda}_0 \leq \bar{\lambda}_1 \leq \dots \leq \bar{\lambda}_{n-1}$. \square

From Theorem 3.3 we have the following theorem.

Theorem 3.4 (Kelmans [25]). *Let G, G_1 and G_2 be graphs with n, n_1 and n_2 vertices, respectively. Then*

$$P(\lambda, \bar{G}) = (-1)^{n-1} P(n - \lambda, G), \quad (2)$$

$$P(\lambda, G_1 + G_2) = \lambda P(\lambda, G_1) P(\lambda, G_2), \quad (3)$$

and

$$P(\lambda, G_1 \times G_2) = (\lambda - n_1 - n_2) P(\lambda - n_2, G_1) P(\lambda - n_1, G_2). \quad (4)$$

Equality (2) follows from Theorem 3.3, equality (3) is obvious, and equality (4) follows from (2) and (3).

Theorem 3.3 on the relation between the spectrum of a graph and the spectrum of its complement is a very useful result. This theorem together with Theorem 3.4 were used in [25] (see also [27]) to give an algorithm for finding formulas describing the

spectrum and the number of spanning trees of a decomposable graph in terms of some natural parameters of the corresponding decomposition of the graph. It follows in particular from this algorithm that the spectrum of a decomposable graph consists of integers.

Since $L(G)$ is positive semi-definite, Theorem 3.3 implies the following theorem.

Theorem 3.5 (Kelmans [26]). *Let G be a graph with n vertices, and let $\lambda_i \in S(G)$. Then $0 \leq \lambda_i \leq n$.*

Let $z(G)$ and $\bar{z}(G)$ denote the multiplicity of 0 and of n in $S(G)$, respectively (and so 0 is an eigenvalue of $L(G)$ of multiplicity $z(G) + 1$). Let $c(G)$ denote the number of components of G .

From Theorem 3.4 we have

Theorem 3.6 (Kelmans [25]). *For any graph G*

$$c(G) = z(G) + 1$$

and

$$c(\bar{G}) = \bar{z}(G) + 1.$$

4. Laplacian polynomial, spectrum and number of spanning trees of a threshold graph

A graph G is called *threshold* if there exists a “weight” function $w: V(G) \rightarrow \mathbb{R}$ defined on the set of vertices of G such that $w(X) < w(Y)$ for any stable vertex set X and any non-stable vertex set Y of G . This concept was introduced in [6].

The following characterizations of threshold graphs will be used in this paper.

Theorem 4.1 (Chvátal and Hammer [6]). *For every finite graph G the following conditions are equivalent:*

- (c₁) G is threshold;
- (c₂) \bar{G} is threshold;
- (c₃) G is 1-decomposable.

Let G be a threshold graph with n vertices. Since G is 1-decomposable, it has the following representation:

$$G = g^{k_0}(s_1g + g^{k_1}(s_2g + \dots + g^{k_{i-1}}(s_i g + \dots + g^{k_r-1}(s_r g + g^{k_r}) \dots)), \quad (5)$$

where all k_i and s_i , $i = 1, \dots, r$, are positive integers, k_0 is a non-negative integer, and $\sum_{j=1}^{r+1} k_j + \sum_{i=1}^r s_i = n$. Obviously G is connected if and only if $k_0 > 0$.

As we mentioned above a threshold graph G is uniquely defined by its composition tree $T(G)$. If $k_r > 1$ then $T(G)$ is uniquely described by formula (5). Therefore, in this

case, formulae (5) is a canonical representation of the threshold graph G . The sequence $C(G) = (k_0, k_1, \dots, k_r; s_1, s_2, \dots, s_r)$ is called the *composition sequence* of the threshold graph G .

If $k_r = 1$, then the corresponding canonical representation of G is given by

$$G = g^{k_0}(s_1g + g^{k_1}(s_2g + \dots + g^{k_{r-1}}(s_rg + \dots + g^{k_r}((s_r + 1)g) \dots)). \quad (6)$$

In this case it is convenient to put $s_r := s_r + 1$. Then $s_r > 1$, and the *composition sequence* of G is $C(G) = (k_0, k_1, \dots, k_{r-1}; s_1, s_2, \dots, s_r + 1)$.

We can apply the above-mentioned algorithm to find formulas for the spectrum of a threshold graph G as a function of its composition sequence $C(G)$. With the convention $s_0 = 0$ and $\sum_{j=1}^0 s_j = 0$ we have the following theorem.

Theorem 4.2. *Let G be a threshold graph with n vertices described by (5) with either $k_r = 0$ or $k_r \geq 2$. Let $S(G) = (a_1^{(m_1)}, \dots, a_h^{(m_h)})$ be the spectrum of G where $a_1 < \dots < a_h$, and m_i is the multiplicity of a_i . Then:*

(a1) *If $k_0 > 0$ and $k_r > 1$ then*

$$a_i = \begin{cases} \sum_{j=0}^{i-1} k_j & \text{if } i = 1, \dots, r+1, \\ n - \sum_{j=1}^{2r+1-i} s_j & \text{if } i = r+2, \dots, 2r+1 \end{cases} \quad (7)$$

(and so $a_1 > 0$ and $a_{2r+1} = n$), and

$$m_i = \begin{cases} s_i & \text{if } i = 1, \dots, r, \\ k_r - 1 & \text{if } i = r+1, \\ k_{2r+1-i} & \text{if } i = r+2, \dots, 2r+1. \end{cases} \quad (8)$$

(a2) *If $k_0 > 0$ and $k_r = 0$ then*

$$a_i = \begin{cases} \sum_{j=0}^{i-1} k_j & \text{if } i = 1, \dots, r, \\ n - \sum_{j=1}^{2r-i} s_j & \text{if } i = r+1, \dots, 2r \end{cases} \quad (9)$$

(and so $a_1 > 0$ and $a_{2r} = n$), and

$$m_i = \begin{cases} s_i & \text{if } i = 1, \dots, r-1, \\ s_r - 1 & \text{if } i = r, \\ k_{2r-i} & \text{if } i = r+1, \dots, 2r. \end{cases} \quad (10)$$

(a3) *If $k_0 = 0$ and $k_r > 1$ then*

$$a_i = \begin{cases} \sum_{j=1}^{i-1} k_j & \text{if } i = 1, \dots, r+1, \\ n - \sum_{j=1}^{2r+1-i} s_j & \text{if } i = r+2, \dots, 2r \end{cases} \quad (11)$$

(and so $a_1 = 0$ and $a_{2r} < n$), and

$$m_i = \begin{cases} s_i & \text{if } i = 1, \dots, r, \\ k_r - 1 & \text{if } i = r + 1, \\ k_{2r+1-i} & \text{if } i = r + 2, \dots, 2r. \end{cases} \quad (12)$$

(a4) If $k_0 = 0$ and $k_r = 0$ then

$$a_i = \begin{cases} \sum_{j=1}^{i-1} k_j & \text{if } i = 1, \dots, r, \\ n - \sum_{j=1}^{2r-i} s_j & \text{if } i = r + 1, \dots, 2r - 1 \end{cases} \quad (13)$$

(and so $a_1 = 0$ and $a_{2r-1} < n$), and

$$m_i = \begin{cases} s_i & \text{if } i = 1, \dots, r - 1, \\ s_r - 1 & \text{if } i = r, \\ k_{2r-i} & \text{if } i = r + 1, \dots, 2r - 1. \end{cases} \quad (14)$$

This theorem can also be proved directly by using Theorem 3.4. For this we need the following lemma.

Lemma 4.1. Let G be a graph with n vertices. Let $S(G) = (a_1^{(m_1)}, \dots, a_h^{(m_h)})$ be the spectrum of G , where $a_1 < \dots < a_h$, and m_i is the multiplicity of a_i . Then:

(a1) If \bar{G} is the complement of the graph G , then

$$S(\bar{G}) = ((n - a_h)^{(m_h)}, \dots, (n - a_1)^{(m_1)}). \quad (15)$$

(a2) If G is connected, then

$$S(G + sg) = (0^{(s)}, a_1^{(m_1)}, \dots, a_h^{(m_h)}). \quad (16)$$

(a3) If G is disconnected, then

$$S(G \times g^k) = ((a_1 + k)^{(m_1)}, \dots, (a_h + k)^{(m_h)}, (n + k)^{(k)}). \quad (17)$$

The relations (15), (16), and (17) follow from (2), (3), and (4) in Theorem 3.4, respectively.

Proof of Theorem 4.2. Let us prove the theorem by induction on the height of G . The only threshold graphs of height 1 are the complete graphs and the edge empty graphs. By Theorem 3.6, $S(K_n) = (n^{(n-1)})$ and $S(\bar{K}_n) = (0^{(n-1)})$, and so the statement of the theorem is true for the graphs of height 1.

Now let G be a threshold graph of height $h(G) = h \geq 2$. Let n be the number of vertices of G . We should consider two cases depending on whether G is connected or not.

Case 1: Suppose that G is connected. Then $G = G' \times g^{k_0}$ where G' is disconnected and $k_0 \geq 1$. Therefore G' is a threshold graph with $n' = n - k_0$ vertices, and of height $h(G') = h(G) - 1 \geq 1$. By the inductive hypothesis, the statement holds for G' .

Let us assume that $h(G') = h'$ is even. Then by (17)

$$S(G') = ((a_1 - k_0)^{(m_1)}, \dots, (a_{h'} - k_0)^{(m_{h'})})$$

where $h' = h - 1$. By the inductive hypothesis, $[S(G'), C(G')]$ satisfies the theorem. Since G' is disconnected and $h(G')$ is even, $S(G')$ satisfies (a3) in the theorem. Since $n = n' + k_0$, the spectrum $S(G)$ satisfies (a1) in the theorem.

Now let us assume that $h(G') = h'$ is odd. Then by the same argument, it follows from (17) and from (a4) in the theorem that $[S(G), C(G)]$ satisfies (a2) in the theorem.

Case 2: Now suppose that G is disconnected. Then $G = G' + s_0g$ where G' is connected and $s_0 > 1$. Therefore the complement \bar{G}' of G' is disconnected, and $\bar{G} = \bar{G}' \times g^{s_0}$. Obviously $n = n' + s_0$. Therefore \bar{G} satisfies the assumption of the previous case. Applying the arguments of the previous case and the relation (15) we get the required statement. \square

Corollary 4.1. *The spectrum of a threshold graph consists of integers.*

As mentioned before the same is true for any decomposable graph.

Let $\phi(G)$ denote the number h of different numbers in the spectrum $S(G)$ of G . It is easy to see that if G is described by (5) then

$$h(G) = \begin{cases} 2r + 1 & \text{if } k_0 > 0 \text{ and } k_r > 1, \\ 2r - 1 & \text{if } k_0 = 0 \text{ and } k_r = 1, \\ 2r & \text{otherwise.} \end{cases} \quad (18)$$

Therefore we have from Theorem 4.2 the following corollary.

Corollary 4.2. *Let G be a threshold graph described by (5). Then $\phi(G) = h(G)$.*

From Theorems 3.6 and 4.2 it can be seen that for threshold graphs there is a strong relationship between the number $i(G)$ of isolated vertices, the number $c(G)$ of components, the number $z(G)$ of zeros in the spectrum of G , and the largest number $\lambda_{\max}(G)$ in the spectrum of G .

Corollary 4.3. *For any threshold graph G*

$$i(G) = c(G) - 1 = z(G) = n - \lambda_{\max}(G).$$

With the convention that $\sum_{j=1}^0 s_j = 0$, Theorems 3.2 and 4.2 give

Theorem 4.3. *Let G be a threshold graph described by (5) with $k_r \geq 1$. Then*

$$P(\lambda, G) = \prod_{i=1}^r \left(\lambda - \sum_{j=0}^{i-1} k_j \right)^{s_i} \prod_{i=0}^{r-1} \left(\lambda - \left(n - \sum_{j=1}^i s_j \right) \right)^{k_i} \left(\lambda - \sum_{j=0}^r k_j \right)^{k_r-1}$$

and

$$\tau(G) = n^{-1} \prod_{i=1}^r \left(\sum_{j=0}^{i-1} k_j \right)^{s_i} \prod_{i=0}^{r-1} \left(n - \sum_{j=1}^i s_j \right)^{k_i} \left(\sum_{j=0}^r k_j \right)^{k_{r-1}}.$$

5. Spectra and degree sequences of threshold graphs

Let $v_1 < \dots < v_s$ be all the different vertex degrees of G , and let $D(G) = (v_1^{(n_1)}, \dots, v_s^{(n_s)})$, where n_i is the multiplicity of v_i in G ; clearly $n_i \geq 1$ and $n_1 + \dots + n_s = v(G)$. The sequence $D(G)$ is called the *degree sequence* of G . Let again $S(G) = (a_1^{(m_1)}, \dots, a_h^{(m_h)})$ be the spectrum of G ; clearly $m_i \geq 1$ and $m_1 + \dots + m_s = v(G) - 1$. Erdős and Gallai gave the following criterion for a sequence of numbers to be the degree sequence of a simple graph.

Theorem 5.1 (Erdős and Gallai [15]). *Let $Z = (z_1, \dots, z_n)$ be a sequence of integers such that $n - 1 \geq z_1 \geq \dots \geq z_n \geq 0$. Then $Z' = (z_n, \dots, z_1)$ is the degree sequence of a simple graph if and only if*

- (a) $\sum_{i=1}^n z_i$ is even, and
- (b) for every $k = 1, \dots, n - 1$,

$$\sum_{i=1}^k z_i - k(k-1) \leq \sum_{i=k+1}^n \min\{k, z_i\}. \quad (19)$$

It is easy to see that if $Z' = (z_n, \dots, z_1)$ is the degree sequence of a simple graph then conditions (a) and (b) hold. Indeed (a) is obvious, and (b) holds because the left side of (19) is the minimum possible number of edges going out of the first k vertices, and the right side of (19) is the maximum possible number of edges going out of the remaining $n - k$ vertices. So the non-trivial part of the above theorem is that conditions (a) and (b) together are sufficient for a sequence Z' to be the degree sequence of a simple graph.

Let $v = v[Z] = \max\{k : z_k \geq k - 1\}$. It was noticed in [31] that if the first v inequalities in (19) hold, then the remaining $n - v$ inequalities in (19) also hold. We write $v(G)$ instead of $v[D(G)]$.

A threshold graph can be characterized in terms of its degree sequence as follows.

Theorem 5.2 (Hammer et al. [19]). *A simple graph is threshold if and only if its degree sequence satisfies the first $v(G)$ inequalities (19) as equalities.*

In this section we shall describe the relation between the degree sequence and the spectrum of a threshold graph. As a result we shall obtain a formula for the number of spanning trees of a threshold graph in terms of its vertex degrees.

For the degree sequence of a threshold graph we obviously have the following lemma.

Lemma 5.1. *Let G be a graph with n vertices. Let $D(G) = (v_1^{(n_1)}, \dots, v_s^{(n_s)})$ where $v_1 < \dots < v_s$, and n_i is the multiplicity of v_i in G . Then:*

(a1) *If \bar{G} is the complement of G then*

$$D(\bar{G}) = ((n-1-v_s)^{(n_s)}, \dots, (n-1-v_1)^{(n_1)}). \quad (20)$$

(a2) *If G is connected then*

$$D(G + s_0g) = (0^{(s_0)}, v_1^{(n_1)}, \dots, v_s^{(n_s)}). \quad (21)$$

(a3) *If G is disconnected then*

$$D(G \times g^{k_0}) = ((v_1 + k_0)^{(n_1)}, \dots, (v_s + k_0)^{(n_s)}, (n-1+k_0)^{(k_0)}). \quad (22)$$

Theorem 5.3. *Let G be a threshold graph, and let $S(G) = (a_1^{(m_1)}, \dots, a_h^{(m_h)})$ and $D(G) = (v_1^{(n_1)}, \dots, v_s^{(n_s)})$ be its spectrum sequence and degree sequence, respectively. Then $s = h = h(G)$; moreover:*

(a1) *If G is connected and $h = 2r + 1$, then*

$$a_i = \begin{cases} v_i & \text{if } i = 1, \dots, r, \\ v_i + 1 & \text{if } i = r + 1, \dots, 2r + 1 \end{cases} \quad (23)$$

and

$$m_i = \begin{cases} n_i & \text{if } i = 1, \dots, r, r + 2, \dots, 2r + 1, \\ n_i - 1 & \text{if } i = r + 1. \end{cases} \quad (24)$$

(a2) *If G is connected and $h = 2r$, then*

$$a_i = \begin{cases} v_i & \text{if } i = 1, \dots, r, \\ v_i + 1 & \text{if } i = r + 1, \dots, 2r \end{cases} \quad (25)$$

and

$$m_i = \begin{cases} n_i & \text{if } i = 1, \dots, r-1, r+1, \dots, 2r, \\ n_i - 1 & \text{if } i = r. \end{cases} \quad (26)$$

(a3) *If G is disconnected and $h = 2r + 1$, then*

$$a_i = \begin{cases} v_i & \text{if } i = 1, \dots, r + 1, \\ v_i + 1 & \text{if } i = r + 2, \dots, 2r + 1 \end{cases} \quad (27)$$

and

$$m_i = \begin{cases} n_i & \text{if } i = 1, \dots, r, r + 2, \dots, 2r + 1, \\ n_i - 1 & \text{if } i = r + 1. \end{cases} \quad (28)$$

(a4) If G is disconnected and $h = 2r$, then

$$a_i = \begin{cases} v_i & \text{if } i = 1, \dots, r, \\ v_i + 1 & \text{if } i = r + 1, \dots, 2r \end{cases} \quad (29)$$

and

$$m_i = \begin{cases} n_i & \text{if } i = 1, \dots, r, r + 2, \dots, 2r, \\ n_i - 1 & \text{if } i = r + 1. \end{cases} \quad (30)$$

Proof. Let us prove the theorem by induction on the height of G . The only threshold graphs of height 1 are the complete graphs and the edge empty graphs. It is easy to check that for these graphs the statement is true.

Now let G be a threshold graph of height $h(G) = h \geq 2$. Let n be the number of vertices of G . We should consider two cases depending on whether G is connected or not.

Case 1: Suppose that G is connected. Then $G = G' \times g^{k_0}$ where G' is disconnected and $k_0 \geq 1$. Therefore G' is a threshold graph with $n' = n - k_0$ vertices, and of height $h(G') = h(G) - 1 \geq 1$. By the inductive hypothesis, the statement holds for G' .

Let us assume that $h(G') = h'$ is even. Then by (17),

$$S(G') = ((a_1 - k_0)^{(m_1)}, \dots, (a_{h'} - k_0)^{(m_{h'})}) \quad \text{where } h' = h - 1,$$

and by (22),

$$D(G') = ((v_1 - k_0)^{(n_1)}, \dots, (v_{s'} - k_0)^{(n_{s'})}) \quad \text{where } s' = s - 1.$$

By the inductive hypothesis, $[S(G'), D(G')]$ satisfies (a4) in the theorem with $h' = s'$. Since $n = n' + k_0$ and every universal vertex in G is of degree $n - 1$, the pair $[S(G), D(G)]$ satisfies (a1) in the theorem and $h = s = s' + 1 = h' + 1$.

Now let us assume that $h(G') = h'$ is odd. Then by the same argument it follows from (17), (22), and from (a3) in the theorem that $[S(G), D(G)]$ satisfies (a2) in the theorem, and $h = s = s' + 1 = h' + 1$.

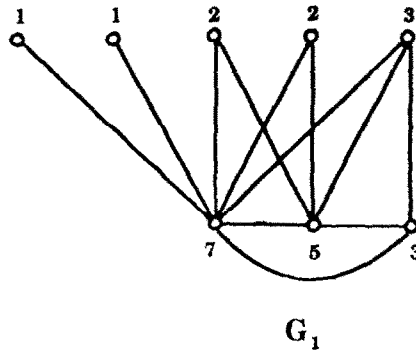
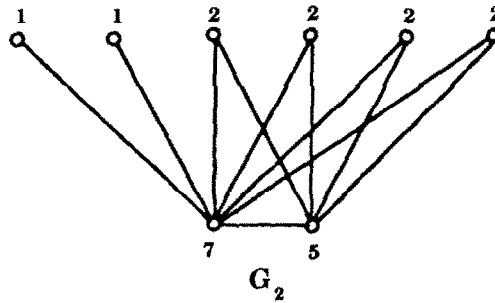
Case 2: Now suppose that G is disconnected. Then $G = G' + s_0 g$ where G' is connected and $s_0 > 1$. $\bar{G} = \bar{G}' \times g^{s_0}$. Obviously $n = n' + s_0$. Therefore \bar{G} satisfies the assumption of the previous case. Applying the arguments of the previous case and the relations (15) and (20) we get the required statement. \square

Here are some examples illustrating the above theorem.

Let $G_1 = g(2g + g(2g + g^2))$ (see Fig. 3). Then G_1 is connected, $v(G_1) = n = 8$, $h(G_1) = 2r + 1 = 5$, implying that $r = 2$ (see (a1)), and

$$S(G_1) = (1^{(2)}, 2^{(2)} | 4^{(1)}, 6^{(1)}, 8^{(1)}), \quad (31)$$

$$D(G_1) = (1^{(2)}, 2^{(2)} | 3^{(2)}, 5^{(1)}, 7^{(1)}). \quad (32)$$

Fig. 3. $G_1 = g(2g + g(2g + g^2))$.Fig. 4. $G_2 = g(2g + g(4g))$.

Let $G_2 = g(2g + g(4g))$ (see Fig. 4). Then G_2 is connected, $v(G_2) = n = 8$, $h(G_2) = 2r = 4$, implying that $r = 2$ (see (a2)), and

$$S(G_2) = (1^{(2)}, 2^{(3)} | 6^{(1)}, 8^{(1)}), \quad (33)$$

$$D(G_2) = (1^{(2)}, 2^{(4)} | 5^{(1)}, 7^{(1)}). \quad (34)$$

Let $\bar{G}_1 = g + g^2(g + g^2(2g))$ (see Fig. 5). Then \bar{G}_1 is disconnected, $v(\bar{G}_1) = n = 8$, $h(\bar{G}_1) = 2r + 1 = 5$, implying that $r = 2$ (see (a3)), and by (15) and (20) we have from (31) and (32):

$$S(\bar{G}_1) = (0^{(1)}, 2^{(1)}, 4^{(1)} | 6^{(2)}, 7^{(2)}), \quad (35)$$

$$D(\bar{G}_1) = (0^{(1)}, 2^{(1)}, 4^{(2)} | 5^{(2)}, 6^{(2)}). \quad (36)$$

Let us consider now $\bar{G}_2 = g + g^2(g + g^4)$ (see Fig. 6). Then \bar{G}_2 is disconnected, $v(\bar{G}_2) = n = 8$, $h(\bar{G}_2) = 2r = 4$, implying that $r = 2$ (see (a4)), and by (15) and (20) we have from (33) and (34):

$$S(\bar{G}_2) = (0^{(1)}, 2^{(1)} | 6^{(3)}, 7^{(2)}), \quad (37)$$

$$D(\bar{G}_2) = (0^{(1)}, 2^{(1)} | 5^{(4)}, 6^{(2)}). \quad (38)$$

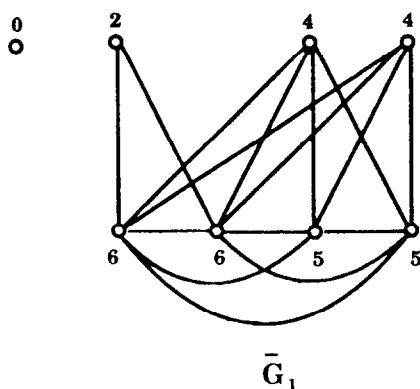


Fig. 5. $\bar{G}_1 = g + g^2(g + g^2(2g))$.

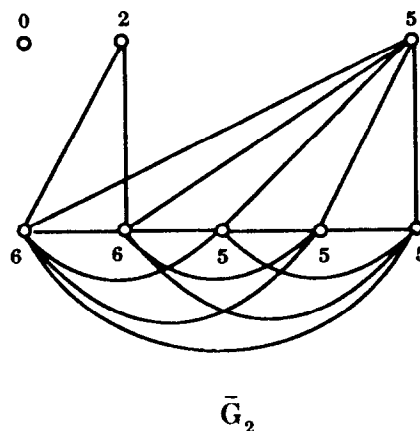


Fig. 6. $\bar{G}_2 = g + g^2(g + g^4)$.

Let $d_{\max}(G)$ and $d_{\min}(G)$ denote the largest and the smallest vertex degree in G , respectively. Similarly let $\lambda_{\max}(G)$ and $\lambda_{\min}(G)$ denote the largest and the smallest number in the spectrum of G , respectively. It was proved in [30] that $d_{\max}(G) + 1 \leq \lambda_{\max}(G)$ and $d_{\min}(G) \geq \lambda_{\min}(G)$. From Theorem 3.6 it is easy to see that $v(G) = d_{\max}(G) + 1 = \lambda_{\max}(G)$ if and only if G has a universal vertex.

From Theorem 5.3 we have the following corollary.

Corollary 5.1. *Let G be a threshold graph. Then $d_{\max} = \lambda_{\max}(G) - 1$ and $d_{\min}(G) = \lambda_{\min}(G)$.*

From Theorems 4.3 and 5.3 we get formulas for the Laplacian polynomial and the number of spanning trees of a threshold graph in terms of the degrees of the graph.

Theorem 5.4. Let G be a connected threshold graph, and let $D(G) = (v_1^{(n_1)}, \dots, v_s^{(n_s)})$ be the degree sequence of G . Then

$$P(\lambda, G) = \prod_{i=1}^k (\lambda - v_i)^{n_i} \prod_{i=k+2}^s (\lambda - v_i - 1)^{n_i} (\lambda - v_{k+1} - 1)^{n_{k+1} - 1}$$

and

$$\tau(G) = \prod_{i=1}^k v_i^{n_i} \prod_{i=k+2}^{s-1} (v_i + 1)^{n_i} (v_{k+1} + 1)^{n_{k+1} - 1} (v_s + 1)^{n_s - 1},$$

where $k = \lfloor (s - 1)/2 \rfloor$ (the maximum integer less than or equal to $(s - 1)/2$).

6. The spectrum is a complete invariant of a threshold graph

Several classes of graphs uniquely defined by their spectra are known [12–14, 21–26, 28–30, 35]. In this section we shall show that the spectrum is also a complete invariant for threshold graphs.

From Theorem 3.3 we have the following lemma.

Lemma 6.1. A graph G is uniquely defined by its spectrum if and only if \bar{G} is uniquely defined by its spectrum.

By using the results on the spectrum of a graph described in Section 3 and Lemma 6.1 we can prove the following theorem.

Theorem 6.1. A threshold graph G is uniquely defined by its spectrum $S(G)$.

Proof. Let us prove the theorem by induction on the height of G . The only threshold graphs of height 1 are the complete graphs and the empty graphs. By Theorem 3.6, these graphs are uniquely defined by their spectrum. Let $h \geq 2$. Suppose that the theorem is true for any threshold graph G' of height $h' < h$. Let G be a threshold graph of height h . Since $h(G) = h \geq 2$, it follows that G is not a complete graph and not an empty graph. We know that every threshold graph has either an isolated vertex or a universal vertex. An isolated vertex in a graph is a universal one in its complement. Therefore by Lemma 6.1, we can assume without loss of generality that G has an isolated vertex. Let $c(G) = c$ be the number of components of G . Clearly $c \geq 2$. Since $h(G) = h \geq 2$, it follows that G can be obtained from a connected threshold graph G' by adding $c - 1$ isolated vertices. Therefore $h(G') = h(G) - 1 < h$, and G' has a universal vertex (and hence \bar{G}' is disconnected). Let $v(G') = n'$. Clearly $n = n' + c - 1$. By Theorem 3.6, $n' \in S(G')$, and $S(G)$ is obtained from $S(G')$ by adding $c - 1$ zeros, i.e. $S(G) = S(G') \cup \{0^{(c-1)}\}$; in particular, $n' \in S(G)$.

Let F be a graph with the same spectrum as G , i.e. $S(F) = S(G)$. Then $v(F) = v(G) = |S(G)| = n$, $c(F) = c(G) = c = z(G) + 1$ and $n' \in S(F)$. Since $n' \in S(F)$, it

follows from Theorem 3.5 that F has a component, say F' , of at least n' vertices. If F' has more than n' vertices then F has less than c components, a contradiction. Hence F' has exactly n' vertices, and so F has exactly $c - 1$ isolated vertices. Therefore $S(F') = S(G')$. Since $h(G') < h(G) = h$, it follows from the inductive hypothesis that G' and F' are isomorphic. Therefore G and G' are also isomorphic. \square

One can prove by induction on the number of vertices (by removing an isolated or a universal vertex) that the following holds.

Theorem 6.2 (Hammer et al [19]). *A threshold graph G is uniquely defined by its degree sequence $D(G)$.*

In view of Theorem 5.3, it can be seen that Theorems 6.1 and 6.2 can be deduced from each other.

7. Recognition of Laplacian polynomials and spectra of threshold graphs

Theorem 7.1. *Given a polynomial (a sequence of numbers), it can be recognized in polynomial time whether it is the Laplacian polynomial (respectively, the spectrum) of a threshold graph.*

Proof. Let $P(\lambda)$ be a polynomial. Let $d(P)$ denote the degree of $P(\lambda)$. Let $z(P)$ and $\bar{z}(P)$ denote the multiplicities of the roots $\lambda = 0$ and $\lambda = d(P) + 1$ of $P(\lambda)$. We can assume that $d(P) \geq 1$. Let us consider the following procedure.

Procedure 1. Put $A = A(\lambda) := P(\lambda)$ and $\bar{A} = \bar{A}(\lambda) = (-1)^{d(A)} A(d(A) + 1 - \lambda)$. Put $F := g$.

S1. Find $\bar{z}(A)$. Put $F := F \times g^{\bar{z}(A)}$. Put $B = (\lambda)^{-\bar{z}(A)} \bar{A}$. If $B \equiv 1$ then stop: $P(\lambda)$ is the Laplacian polynomial of the threshold graph F . Otherwise go to **S2**.

S2. Find $\bar{z}(B)$. If $\bar{z}(B) = 0$ then stop: $P(\lambda)$ is not the Laplacian polynomial of a threshold graph. Otherwise put $F := F + \bar{z}(A)g$, and $A = (\lambda)^{-\bar{z}(B)} \bar{B}$. If $A \equiv 1$ then stop: $P(\lambda)$ is the Laplacian polynomial of the threshold graph F . Otherwise go to **S1**.

Since a threshold graph is 1-decomposable, a threshold graph has either an isolated or a universal vertex. Therefore it follows from Theorem 3.6 that the above procedure is correct. It is easy to see that this procedure is polynomial. In order to determine whether a given sequence S of numbers is the spectrum of a threshold graph it is sufficient to find the polynomial $P(\lambda)$ with the roots in S and to apply the above procedure. \square

Theorem 4.3 enables us to give the following more simple procedure for testing a sequence $A = \{a_1^{(m_1)}, \dots, a_h^{(m_h)}\}$ for being the spectrum of a threshold graph; here $h \geq 1$, $a_1 < \dots < a_h$ and m_i is the multiplicity of a_i .

Procedure 2. If at least one of the numbers a_i is not a non-negative integer then A is not the spectrum of a threshold graph. Find $n = \sum_{j=1}^h m_j + 1$.

- S1.** This step recognizes whether A is the spectrum of a connected threshold graph. If $a_1 = 0$ then A is not the spectrum of a connected threshold graph. If h is odd ($h = 2r + 1$), then find $k_0 = a_1$, $k_i = a_{i+1} - a_i$, and $s_i = a_{2r+2-i} - a_{2r+1-i}$ for any $i = 1, \dots, r$. Check whether the equalities (7) and (8) in Theorem 4.2 hold. If the answer is yes, then A is the spectrum of a connected threshold graph. Otherwise not. If h is even ($h = 2r$), then find $k_0 = a_1$, $k_{i-1} = a_i - a_{i-1}$ for any $i = 2, \dots, r$, and $s_i = a_{2r+1-i} - a_{2r-i}$ for any $i = 1, \dots, r$. Check whether the equalities (9) and (10) in Theorem 4.2 hold. If the answer is yes, then A is the spectrum of a connected threshold graph. Otherwise not.
- S2.** This step recognizes whether A is the spectrum of a disconnected threshold graph. Find $\bar{A} = \{\bar{a}_1^{(m_1)}, \dots, \bar{a}_h^{(m_h)}\}$ where $\bar{a}_i = n - a_{h-i}$, $i = 1, \dots, h$, and so $\bar{a}_1 < \dots < \bar{a}_h$. Apply **S1** to \bar{A} and determine whether \bar{A} is the spectrum of a connected threshold graph. If the answer is yes, then A is the spectrum of a disconnected threshold graph. Otherwise A is not the spectrum of a threshold graph.

From Theorems 3.3 and 4.3 it follows that the above procedure is correct. If A is the spectrum of a threshold graph G_A then G_A is described by formula (5) with the parameters k_0, k_i and s_i , $i = 1, \dots, r$, found by the above procedure. It is easy to see that this procedure is polynomial.

Because of the special relation between the degree sequence and the spectrum of a threshold graph described in Theorem 5.3, any algorithm used for recognizing the degree sequence of a threshold graph can also be used for recognizing its Laplacian spectrum.

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