Proofs for the handout "The area of a triangle and applications"

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I made this quickly so it might have small mistakes.

This document contains the proofs for **Lemma 1** on page 2, **Theorem 2** on page 4, **Example 2** on page 6, and finally **Example 3** on page 7. If you missed the session, you will likely derive benefit from this; the handout itself does not do much.

Excerpt from Mowing

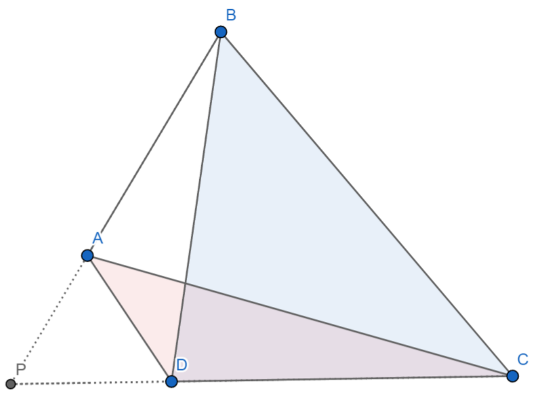
ROBERT FROST

It was no dream of the gift of idle hours,
Or easy gold at the hand of fay or elf:
Anything more than the truth would have seemed too weak
To the earnest love that laid the swale in rows,

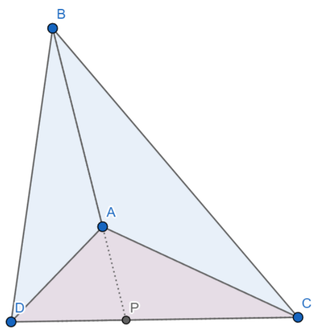
Lemma 1. Given the segment CD, and any points A and B.

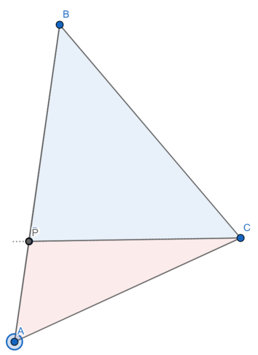
Let P be the intersection of AB and CD. The following is true.

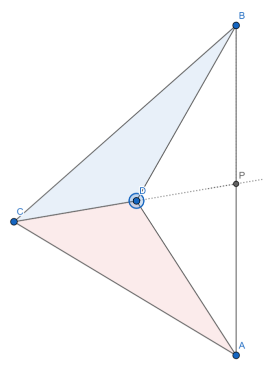
$$\frac{PA}{PB} = \frac{[CDA]}{[CDB]}$$



The lemma basically says that $\dfrac{\text{Red area}}{\text{Blue area}} = \dfrac{PA}{PB}$ for any A, B, C, D and the intersection point P. Here are a few more examples:







Proof.

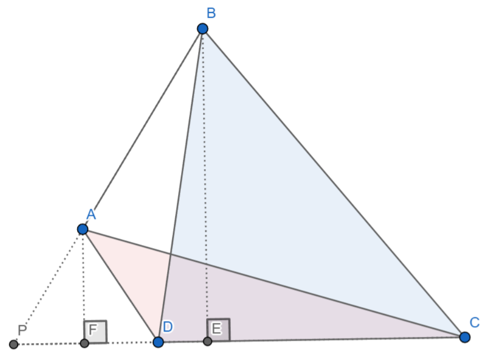
Calculating the area of a triangle gives us

$$[ADC] = \frac{1}{2}DC \cdot AF \text{and} [BDC] = \frac{1}{2}DC \cdot BE$$

Substituting this, we get

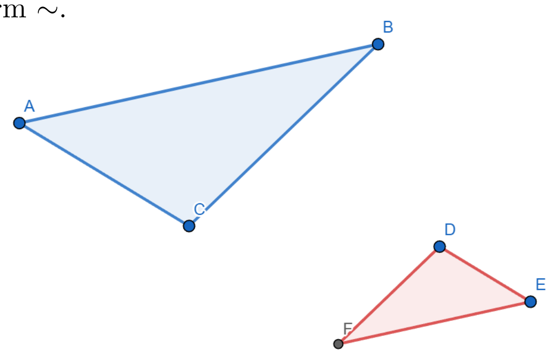
$$\frac{[ADC]}{[BDC]} = \frac{\frac{1}{2}DC \cdot AF}{\frac{1}{2}DC \cdot BE} = \frac{AF}{BE}$$

Similar triangles $\triangle AFP \sim \triangle BEP$ gives us $\dfrac{AF}{BE} = \dfrac{PA}{PB}$, so $\dfrac{[ADC]}{[BDC]} = \dfrac{PA}{PB}$. $\square$



I did not prove why the triangles are similar. Triangles $\triangle AFP$ and $\triangle BEP$ are similar because corresponding angles are equal. ($\angle AFP = 90^\circ = \angle BEP; \angle FPA = \angle EPB$)

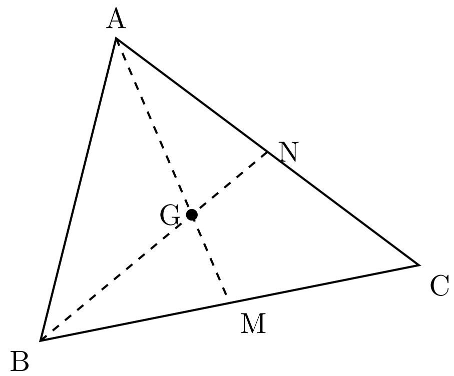
Alternatively, we may state that if two triangles have pairwise parallel sides, they must be similar. "Pairwise parallel" just means that a side in the first triangle corresponds to a side of the second triangle, such that the sides are parallel. Similarity is denoted with a worm $\sim$.



Here $\triangle ABC \sim \triangle DEF$. Try to prove it.

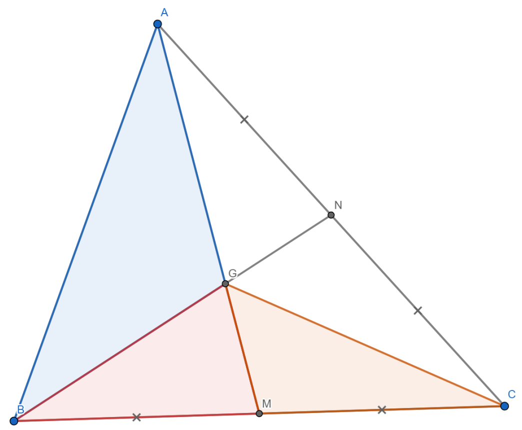
Theorem 2 (Centroid division)**.** If medians AM and BN of triangle ABC intersect at G, then $AG = 2GM$.

A median of a triangle is a line segment joining a vertex of the triangle to the midpoint of the opposite side.



Proof.

We want to prove that the value of the expression $\dfrac{GA}{GM} = 2$, since then multiplying by GM would finish the proof.



Lemma 1 states that $\dfrac{[GBA]}{[GBM]} = \dfrac{GA}{GM}$.

$[GBM] = \frac{1}{2}[GBC]$ by **Lemma 1**. (Also because the latter triangle has the same height but twice the base, hence twice the area.)

Therefore, $\dfrac{[GBA]}{[GBM]} = \dfrac{[GBA]}{\frac{1}{2}[GBC]} = 2\dfrac{[GBA]}{[GBC]} = 2\dfrac{AN}{CN}$.

The last equality above is obtained again using our lemma. Since $AN = CN$ by the definition of N, the proof is complete. $\square$

Using the exact same method we can prove that $BG = 2GN$. Try doing that if you're not convinced.

We will next prove that CG bisects the segment AB.

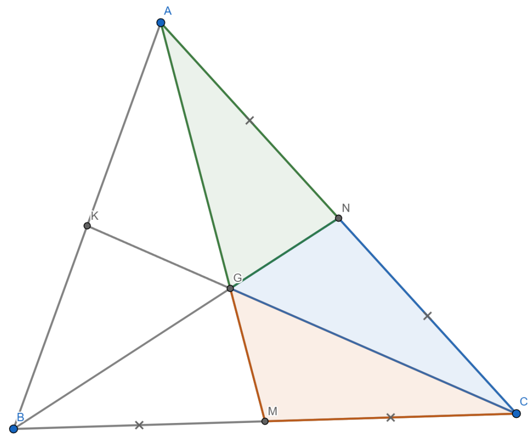
Lemma 1 and **Theorem 2** give the following:

$$\frac{[AGC]}{[MGC]} = \frac{AG}{MG} = 2$$

Since again $[AGC] = [AGN] + [CGN] = 2[CGN]$, the above expression gives $[CGN] = [CGM]$, which in turn is equal to $[BGM]$.

Therefore, $\dfrac{AK}{BK} = \dfrac{[AGC]}{[BGC]} = 1$ (I skipped a step but should be clear).

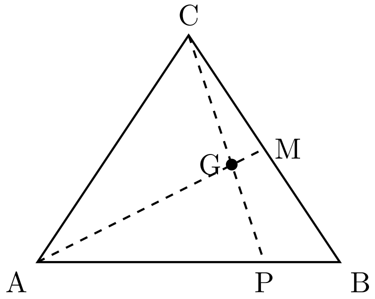
This means that K is indeed the midpoint. □



This observation means that the three medians of a triangle concur, i.e. meet at a common point. We call this point of concurrence the centroid, usually denoted G.

We will use the same technique for proving **Theorem 2** to solve the following problem, which was presented as a homework problem in Katja Mannila's pre-IB vector course.

Example 2. Consider a triangle ABC. Point P divides the side AB in the ratio $3 : 1$. In what ratio does the line segment CP divide the median drawn from the vertex A?



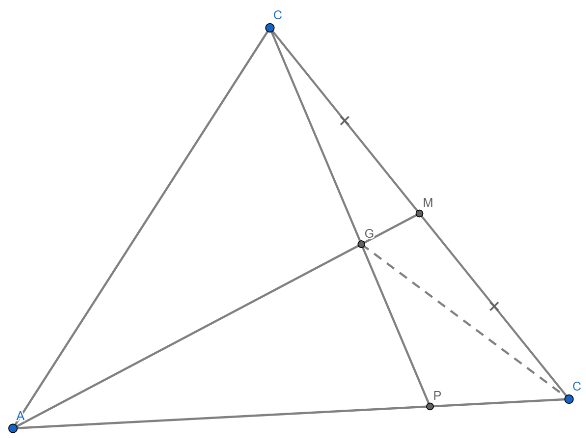
In this solution and the next one after this, I won't explicitly refer to **Lemma 1**, but it is used pretty much every time I state that a ratio of areas is equal to a ratio of line segment lengths.

Remember to tell Mannila this when you actually get this problem as homework in pre-IB term 4. I bet she'd be proud of you.

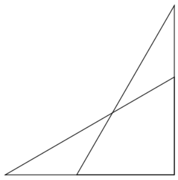
Solution for Example 2. Let M be the midpoint of BC; let G be the intersection of AM and CP.

We see that $[MCG] = [MBG] = \frac{1}{2}[CBG]$, same as in the previous solution.

$$\frac{AG}{MG} = \frac{[ACG]}{[MCG]} = \frac{[ACG]}{\frac{1}{2}[BCG]} = 2\frac{[ACG]}{[BCG]} = 2\frac{AP}{BP} = 2 \cdot 3 = 6.$$

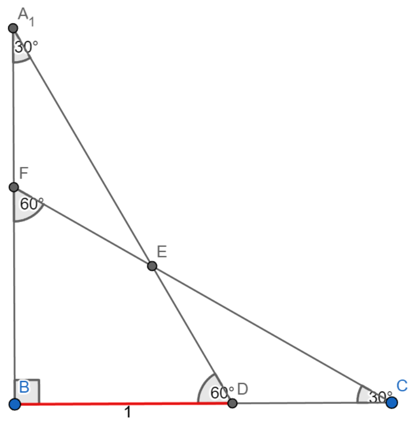


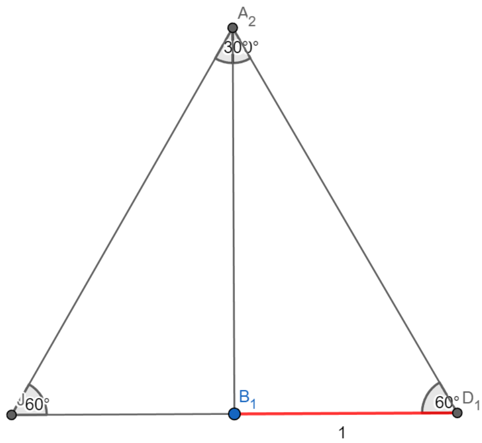
Example 3. (2018 Basic Series, P8) In the following figure, we have two right triangles. In both of them, the length of the shorter catheti are 1 and the bigger ones of the sharp angles are equal to 60 degrees.

Determine the common area. 

Solution for Example 3.

We will name the points according to the left figure below. Do not mind the subscripts.



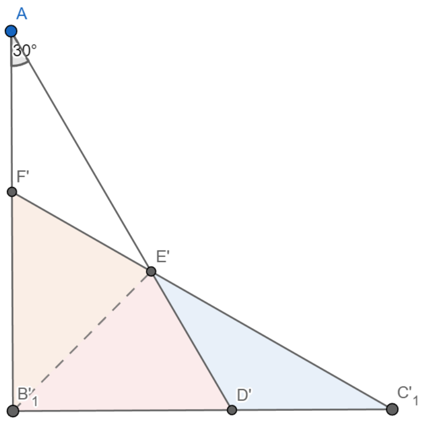


The first step is to get information regarding the length of AB. In the right figure above, we see that $\triangle ADJ$ has all angles equal to 60°, so it is equilateral. Therefore, $AD = JD = 2BD = 2$. Now applying the Pythagorean theorem on $\triangle ABD$ yields

$$AB^2 + 1^2 = 2^2 \implies AB^2 = 3 \implies AB = \sqrt{3}.$$

Consequently $BC = \sqrt{3}, AF = CD = \sqrt{3} - 1$.

But we want to calculate the area of the whole figure. Since it's a weird shape, we will instead divide it into two equal parts through BE, namely $[ABE] = [CBE]$. We'll only need to calculate the area of the triangle, and multiply the result by 2 to finish.



Clearly calculating the $[FBC]$ is much easier than $[EBC]$. It is equal to $\frac{1}{2}FB \cdot BC = \frac{\sqrt{3}}{2}$.

But $\triangle FBC$ has that extra triangle $[FBE] = [DBE]$. If we calculate that, and additionally $[DCE]$, summing the two values will give us half the total area.

The equation above can be simplified:

$$\sqrt{3} = [FBC] = [FBE] + [DBE] + [DCE] = 2[DBE] + [DCE].$$

This is still not enough; we need another equation to make it a pair of equations in two variables. Let's examine the ratio of the areas.

$$\frac{[DCE]}{[DBE]} = \frac{DC}{DB} = \frac{\sqrt{3}-1}{1} = \sqrt{3}-1.$$

Multiplying by $[DBE]$ yields $[DCE] = (\sqrt{3}-1)[DBE]$. Combining this with the other equation gives the following:

$$\begin{cases} [DCE] = (\sqrt{3}-1)[DBE] \\ 2[DBE] + [DCE] = \sqrt{3} \end{cases}$$

We can save ourselves from excess writing by substituting $x = [DBE], y = [DCE]$. Then solving the pair of linear equations and summing the solutions, and finally multiplying the sum by 2 will finish the solution.

The final value should be $\dfrac{3-\sqrt{3}}{2}$