The area of a triangle and applications

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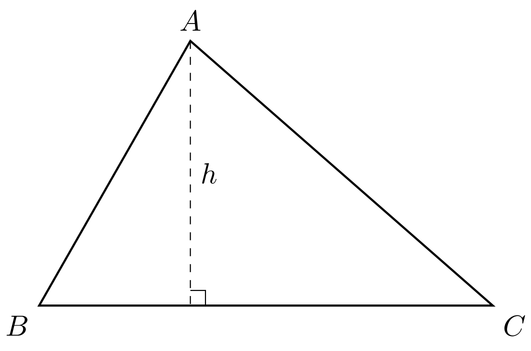
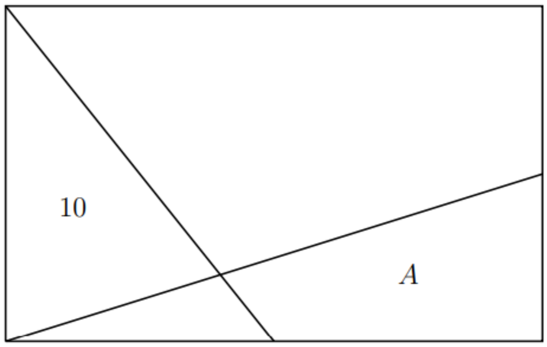
Theorem 1. The area of a triangle $\triangle ABC$, sometimes denoted $[ABC]$, is given by the formula

$$\frac{1}{2}h \cdot BC$$



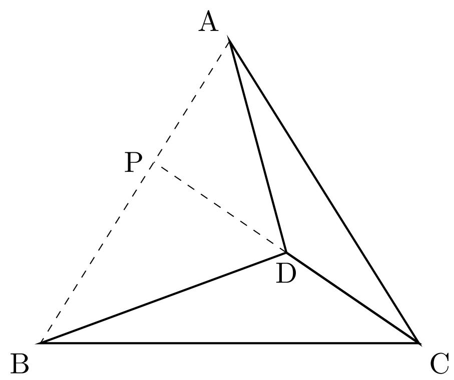
Everything we will be doing this session shall be centered around this theorem. It's middle school level easy, yet surprisingly useful.

Example 1. (2022 Basic Series, P5) In the figure below two segments have been drawn in a rectangle to connect a corner to a midpoint of a side. This gives rise to four areas. The area of the triangle on the left is 10. The area of the quadrilateral on the bottom right corner is A. Determine A.



Lemma 1. Given the segment CD, and any points A and B.
Let P be the intersection of AB and CD. The following is true.

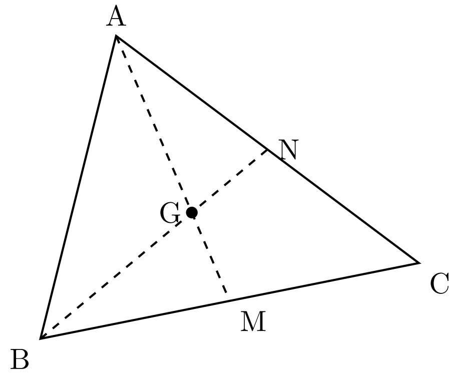
$$\frac{PA}{PB} = \frac{[CDA]}{[CDB]}$$



Notice how general this lemma is: It works with any nonparallel segments AB and CD, and allows us to convert information about areas into information about lengths. The idea is not to memorize this lemma, but instead to understand it so that it feels intuitive and obvious. A notable special case is $P = D$.

Theorem 2 (Centroid division)**.** If medians AM and BN of triangle ABC intersect at G, then $AG = 2GM$.

A median of a triangle is a line segment joining a vertex of the triangle to the midpoint of the opposite side.

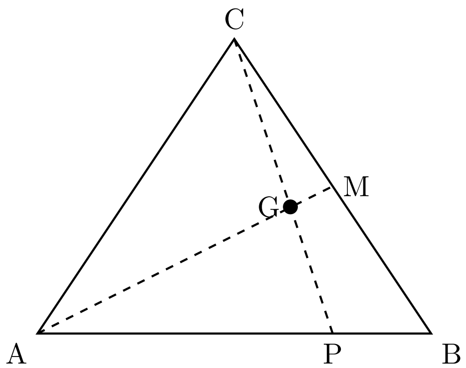


By symmetry, $BG = 2GN$ holds too. Additionally, we will prove that CG bisects the segment AB.

This observation means that the three medians of a triangle concur, i.e. meet at a common point. We call this point of concurrence the centroid, usually denoted G.

We will use the same technique for proving **Theorem 2** to solve the following problem, which was presented as a homework problem in Katja Mannila's pre-IB vector course.

Example 2. Consider a triangle ABC. Point P divides the side AB in the ratio $3 : 1$. In what ratio does the line segment CP divide the median drawn from the vertex A?



Example 3. (2018 Basic Series, P8) In the following figure, we have two right triangles. In both of them, the length of the shorter catheti are 1 and the bigger ones of the sharp angles are equal to 60 degrees.

Determine the common area.