Practice Olympiad 1 solutions

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1. Given that a, b, c, g satisfy

$$(g-a)\cdot\frac{3}{2}+a=\frac{b+c}{2}$$

Find a value for $\dfrac{g-b}{\frac{a+c}{2}-g}$ (5)

Solution 1. Multiply the equation by 2 and simplify:

$$\begin{aligned}
&(g-a)\cdot\frac{3}{2}+a=\frac{b+c}{2}\\
\iff &(g-a)\cdot 3+2a=b+c\\
\iff &3g-a=b+c\\
\iff &c=3g-a-b
\end{aligned}$$

Isolating one of the variables a, b, c, g, or $a+c$, are all valid ways to do this, I just chose c as an example. Next we substitute the newly obtained value into the second expression and simplify:

$$\begin{aligned}
&\frac{g-b}{\frac{a+c}{2}-g}\\
=&\frac{g-b}{\frac{a+3g-a-b}{2}-g}\\
=&\frac{g-b}{\frac{3g-b}{2}-g}\\
=&\frac{g-b}{\frac{g-b}{2}}\\
=&\frac{1}{\frac{1}{2}}=2
\end{aligned}$$

2. Consider the following system of equations:

$$\begin{cases}3y-xy=7-3x\\ 5y+2xy=-7-5x\end{cases}$$

a) find $x + y$ (3)

b) find $\frac{1}{x} + \frac{1}{y}$ (2)

Solution 2. We test our luck by moving all terms to the left-hand side of the equation:

$$\begin{cases} 3y + 3x - xy - 7 = 0 \\ 5y + 5x + 2xy + 7 = 0 \end{cases}$$

Here it's easy to see that x, y have the same coefficients in both equations. That invites a substitution $s = x + y$. Making another substitution $t = xy$ we notice that the system of equations is just linear with the new variables:

$$\begin{cases} 3y + 3x - xy - 7 = 3s - t - 7 = 0 \\ 5y + 5x + 2xy + 7 = 5s + 2t + 7 = 0 \end{cases}$$

Solving it yields $s = \frac{7}{11}, t = -\frac{7}{56}$.
Part a is now done. For part b, just note that $\frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy} = \frac{s}{t} = -\frac{1}{8}$.

3. a, b, c, d, e, f are numbers that satisfy the equation

$$ax^2 + bx + c = dx^2 + ex + f$$

for <u>all values</u> of x.

a) Prove $a + b + c = d + e + f$. (1)

b) Prove that $a = d$, $b = e$, $c = f$. (4)

Solution 3. For part a, substitute $x = 1$. Alternatively solve part b first and part a will obviously follow from the result.

For part b, start by substituting $x = 0$ to obtain $c = f$.

That means we can substitute f as the value of c, and cancel the f's out in the equation to obtain

$$ax^2 + bx = dx^2 + ex$$

Now substitute $x = 1$ and $x = -1$:

$$\begin{cases} a + b = d + e \\ a - b = d - e \end{cases}$$

Add the two equations to get $a = d$; subtract them to get $b = e$.

Actually substituting precisely $0, 1, -1$ isn't needed. Just substitute any three different values for x and solve a three-variable linear system of equations, and you'll have the same result.

4. a) Expand $(a+b)(a^2-ab+b^2)$. (2)

b) Find a prime divisor of 8027. (1)

Solution 4. a) We get $(a+b)(a^2-ab+b^2) = (a(a^2-ab+b^2)+b(a^2-ab+b^2) = a^3-a^2b+ab^2+a^2b-ab^2+b^3 = a^3+b^3$

b) The intended solution here was to note that the 27 in 8027 is a cube, and so is 8000, so just write $8027 = 8000+27 = 20^3+3^3 = (20+3)(20^2-3^20+3^2)$, by the previous identity. Calculating the values, we get $8027 = 23 \cdot 349$, and we see that 23 is a prime divisor of 8027.

NOTE FOR SOME STUDENTS THAT 1 IS NOT PRIME: A prime number is a positive integer greater than 1 whose only positive divisors are 1 and itself.