Subject: Mathematics

Investigating patterns in the product of Euler's totient function φ and the sum of divisors function σ

Research Question: What gives rise to the patterns in the graph of $f(n) = \sqrt{\varphi(n) \cdot \sigma(x)}$, and how does this relate to the Riemann zeta function?
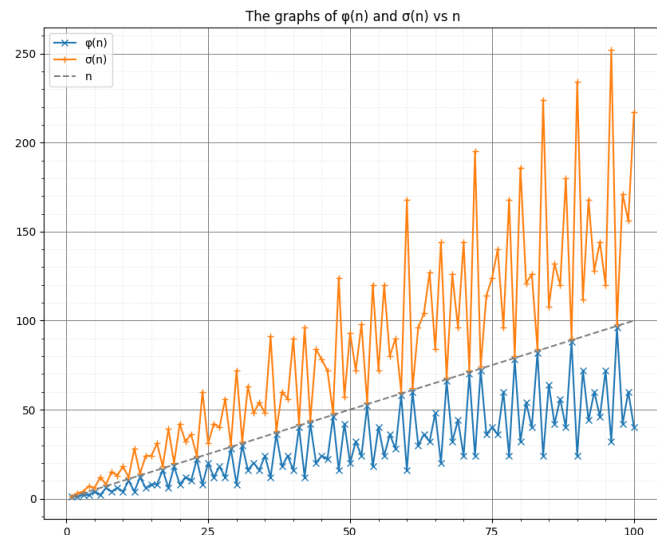
April 14, 2025

Word count: 1729

1   Presentation to the Class

$\varphi(n)$ is the "number of positive integers $\leq n$ and **coprime** to n." $\sigma(n)$ is the "sum of the positive divisors of n."

n	$\varphi(n)$ numbers		$\sigma(n)$ numbers	
1	1	1	$1 =$	1
2	1	1	$3 =$	1+2
3	2	1,2	$4 =$	1+3
4	2	1,3	$7 =$	1+2+4
5	4	1,2,3,4	$6 =$	1+5
6	2	1,5	$12 = 1+2+3+6$	
7	6	1,2,3,4,5,6	$8 =$	1+7
8	4	1,3,5,7	$15 = 1+2+4+8$	
9	6	1,2,4,5,7,8	$13 =$	1+3+9
10	4	1,3,7,9	$18 = 1+2+5+10$	

The graphs of φ(n) and σ(n) vs n

We call functions like φ and σ *"arithmetic functions"*, i.e. functions that express some property of the number.

Lemma 1.1 (Multiplicativity of φ and σ) *For any two **coprime** integers m, n, we have $\varphi(mn) = \varphi(m)\varphi(n)$ and $\sigma(mn) = \sigma(m)\sigma(n)$.*

This allows us to write closed formulas of $\varphi(n)$ and $\sigma(n)$ in terms of the prime factorization of n.
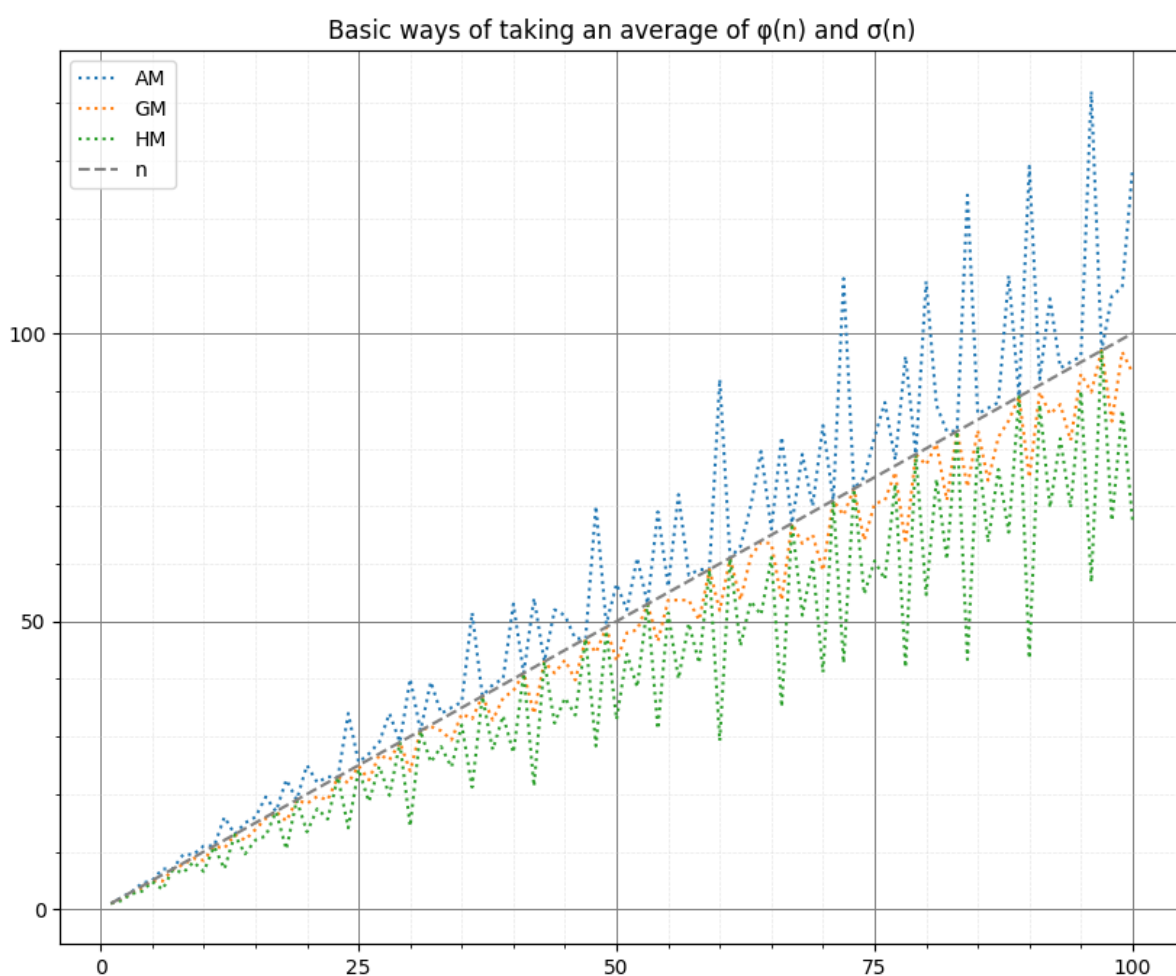
Corollary 1.1 $f(x) = \varphi(x) \cdot \sigma(x)$ *is multiplicative.*

What gives rise to the patterns in the graph of $f(n) = \sqrt{\varphi(n) \cdot \sigma(x)}$, and how does this relate to the Riemann zeta function?

We wish to find some way of taking an average that sends the pairs of points to the line $y = n$, i.e. discover a function that reduces the amount of chaos and erraticity in this graph.

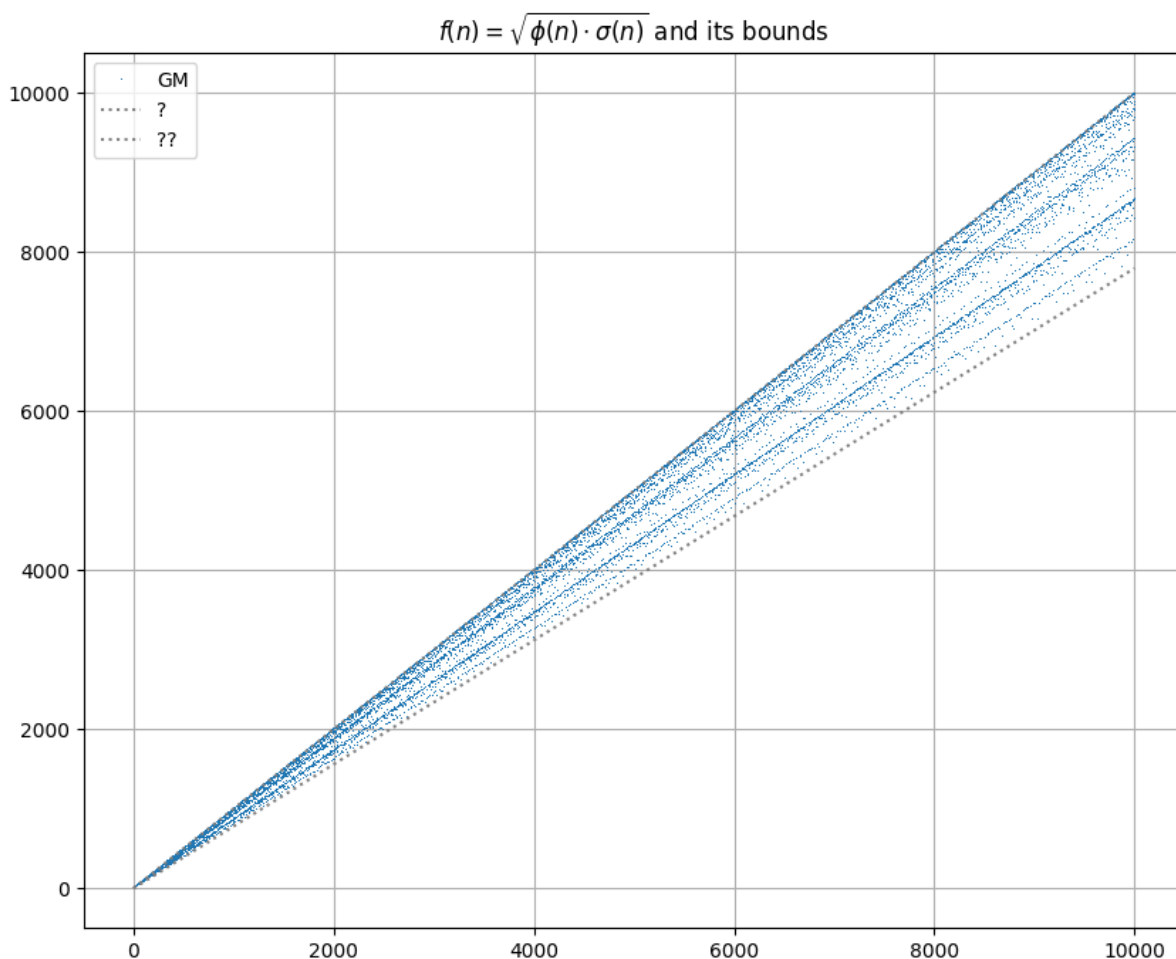Initial guesses for ways to take an average:

$$\mathbf{AM} : \frac{a+b}{2} \qquad \mathbf{GM} : \sqrt{ab} \qquad \mathbf{HM} : \frac{2}{\frac{1}{a} + \frac{1}{b}}$$

Basic ways of taking an average of φ(n) and σ(n)

It's also possible to take some weighted version of these, to make the result closer to the target. Investigating that may be a part of my Research, if the word limit allows for that.

What gives rise to the patterns in the graph2 of $f(n) = \sqrt{\varphi(n) \cdot \sigma(x)}$, and how does this relate to the Riemann zeta function?

I hope the conclusion is that the geometric mean is the best (whether 'best' may refer to most close to the target or easiest to compute). Then I will proceed to graph the function $f(x) = \sqrt{\varphi(n) \cdot \sigma(n)}$ in more detail, compute the lower and upper bounds, show that they're strict, and lastly explain why the patterns occur.

What gives rise to the patterns in the graph of $f(n) = \sqrt{\varphi(n) \cdot \sigma(x)}$, and how does this relate to the Riemann zeta function?

2   Abstract

What gives rise to the patterns in the graph of $f(n) = \sqrt{\varphi(n) \cdot \sigma(x)}$, and how does this relate to the Riemann zeta function?

Contents

3   Introduction

Broadly speaking, number theory is the branch of mathematics concerned with the properties of integers. In a school setting, this often includes topics such as divisibility rules, prime factorization, and, most importantly, the **Fundamental Theorem of Arithmetic**.

Theorem 3.1 (Fundamental Theorem of Arithmetic) *Every integer $n > 1$ can be represented as a product of prime factors in precisely one way, up to the order of the factors.*

This theorem asserts that for any integer n, there exists a unique set of distinct prime numbers $p_1, p_2, ..., p_k$, along with their respective exponents $\alpha_1, \alpha_2, ..., \alpha_k$, such that

$$n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}.$$

This form of writing numbers as a product of primes is the key to the proofs delivered in this paper. For example, we can construct all the factors (positive divisors) of a number when given its prime factorization. Namely, they are precisely the numbers of the form

$$d = p_1^{\beta_1} p_2^{\beta_2} ... p_k^{\beta_k},$$

where $\beta_i \leq \alpha_i$ for all $1 \leq i \leq k$.

4   The zeta function

hello [**basel**] hello!!

What gives rise to the patterns in the graph6 of $f(n) = \sqrt{\varphi(n) \cdot \sigma(x)}$, and how does this relate to the Riemann zeta function?

5   An Explanation for the Patterns

6   Appendices

What gives rise to the patterns in the graph of $f(n) = \sqrt{\varphi(n) \cdot \sigma(x)}$, and how does this relate to the Riemann zeta function?