Subject: Mathematics

Investigating patterns in the product of Euler's totient function φ and the sum of divisors function σ

Research Question: What gives rise to the patterns in the graph of $n^{-2}\varphi(n)\sigma(n)$?

Word count: 1729

Contents

1 Introduction

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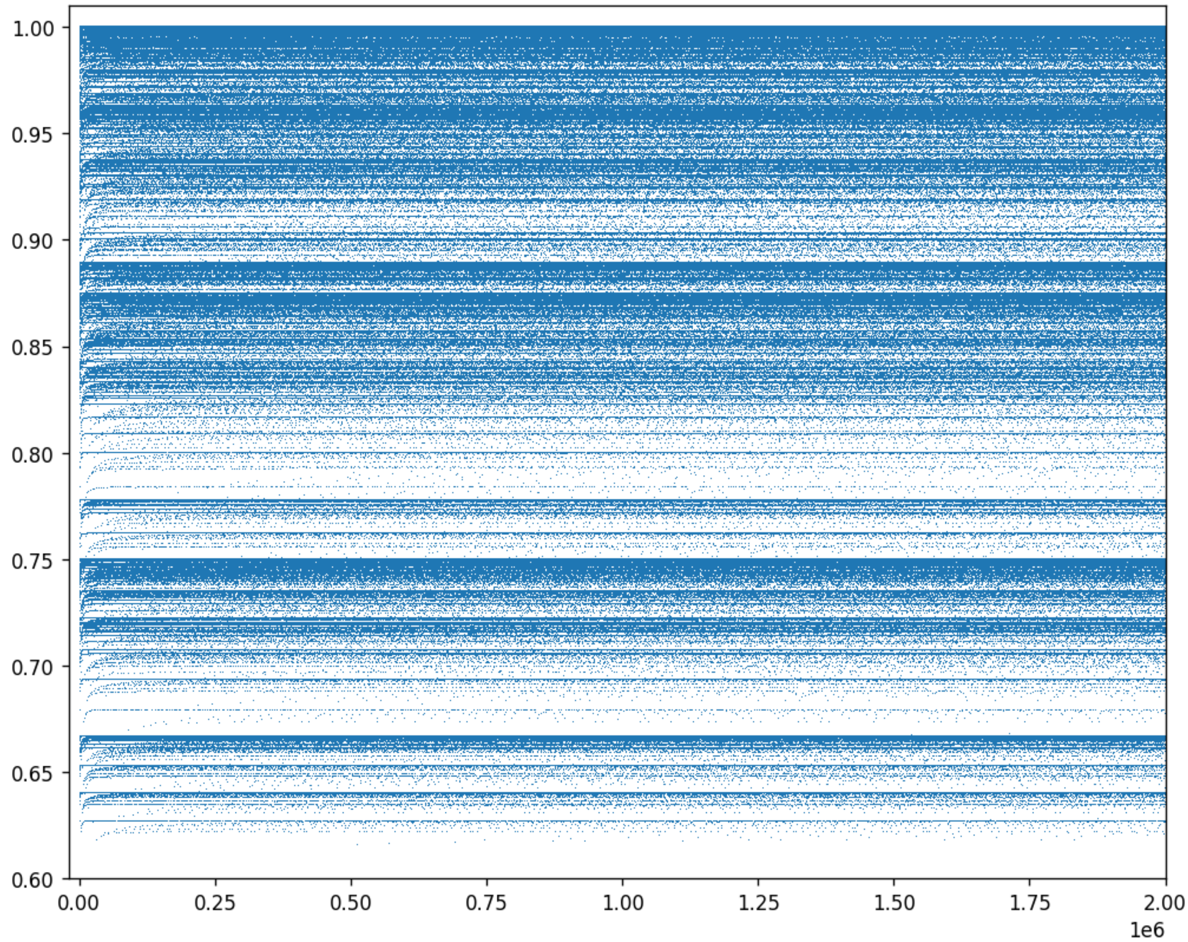

Figure 1: what?

2 Prerequisite Knowledge, Notation, and Definitions

The inarguably most important theorem in number theory is the following:

Theorem 1 (Fundamental Theorem of Arithmetic). *Every positive integer has an unique prime factorization.[1]*

This theorem is crucial in our ensuing discussion, because it allows us to express any integer $n \geq 2$ uniquely in terms of its k prime divisors $p_1, p_2, ..., p_k$, along with the positive integers $\alpha_1, \alpha_2, ..., \alpha_k$ which indicate the number of times that n is divisible by the corresponding prime factor. With this, we obtain the familiar form

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k} = \prod_{i=1}^{k} p_i^{\alpha_i} \tag{2.1}$$

In the special case $n = 1$, there are no prime factors, and by convention

$$\textit{the empty product equals 1.} \tag{2.2}$$

The reader should be comfortable with Π-notation for sequential products, which is similar to Σ but for multiplication.

2.1 Sets and Set-builder Notation

Conventionally, $\mathbb{Z}$ denotes the set of integers and $\mathbb{R}$ the set of real numbers. The notation $x \in S$ means x belongs to set S. For a finite set S, $|S|$ is used to denote the number of elements in S.

Set-builder notation is a way for mathematicians to define sets symbolically rather than verbosely. Basic set-builder notation is used frequently throughout this paper.

Notation. The set S of elements in the set A that fulfill a certain condition is defined, i.e. 'built', with the following notation:

$$S = \{a \in A \ : \ \text{conditions for } a\}$$

In this case, S is a subset of A, because every element of S is an element of A. This is denoted by $S \subseteq A$. The reader should understand the following examples before proceeding:

Example 1. Build the set of positive integers, denoted $\mathbb{Z}^+$, by choosing the integers which are positive:

$$\mathbb{Z}^+ = \{x \in \mathbb{Z} \ : \ x > 0\} \tag{2.3}$$

Example 2. Build the set of positive divisors of an integer $n \neq 0$, denoted $\mathcal{D}_n$, by choosing the positive integers that divide n:

$$\mathcal{D}_n = \{x \in \mathbb{Z}^+ \ : \ x \mid n\} \tag{2.4}$$

The notation $a \mid b$ means a divides b.

Example 3. Recall that prime numbers are positive integers greater than 1 whose only positive divisors are 1 and itself. Build the set of primes, denoted $\mathbb{P}$, using the set of positive divisors defined in (2.4):

$$\mathbb{P} = \big\{x \in \mathbb{Z}^+ \ : \ x > 1, \ \mathcal{D}_x = \{1, \ x\}\big\} \tag{2.5}$$

In this case, multiple conditions on x are necessary, and notationally we separate them with a comma.

More complicated sequential sums and products also involve require set-builder-like notation in the subscripts, used to impose restrictions on the what to take the sum or product over. (2.6) in **Definition 1** serves as an example of this.

2.2 Key Definitions

The following functions, defined for any positive integer n, are the central focus of this study.

Definition 1 (Sum of Divisors Function σ)**.** Define $\sigma(n)$ as the sum of all positive divisors of n.

Mimicking set-builder notation, write

$$\sigma(n) = \sum_{d \in \mathcal{D}_n} d \tag{2.6}$$

Example 4. Compute $\sigma(12)$: Since the positive divisors of 12 are $\mathcal{D}_{12} = \{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$, taking their sum yields $\sigma(15) = 1 + 2 + 3 + 4 + 6 + 12 = 28$

Definition 2 (Euler's Totient Function φ)**.** Define $\varphi(n)$ as the number of positive integers less than or equal to n and *coprime* to n. Formally,

$$\varphi(n) = \big|\{x \in \mathbb{Z}^+ \ :\ x \leq n,\ \gcd(x, n) = 1\}\big| \tag{2.7}$$

Terminology. We say that 'a is coprime to b', or 'a and b are coprime', if $\gcd(a, b) = 1$.

Example 5. Compute $\varphi(12)$:

The positive integers less than or equal to 15 and coprime to 15 are 1, 5, 7, 11. Since there are 4 such numbers, $\varphi(12) = 4$.

3 The Problem of Study

This investigation was inspired by the following problem from the undergraduate textbook *Introduction to Analytic Number Theory* [1].

Problem 1. Show that $\dfrac{6}{\pi^2} < \dfrac{\varphi(n)\sigma(n)}{n^2} < 1$ for all $n \geq 2$.

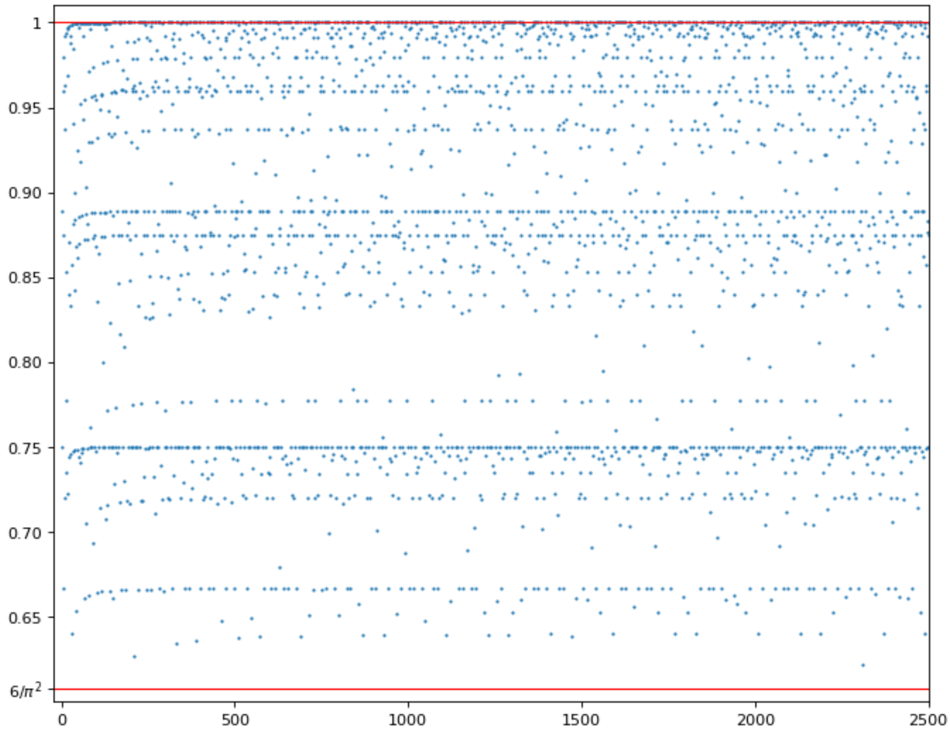

Figure 2: Graph for **Problem 1**

Figure 2 visualizes the scenario by plotting points $(n, \frac{\varphi(n)\sigma(n)}{n^2})$ (in blue) and the lower– and upper bounds (in red) on the Cartesian plane. Then, the task in **Problem 1** is to show that all blue points fall in between the red lines.

The research question of this investigation, regarding 'patterns' in the graph can now be put concretely. In addition to solving **Problem 1**, this paper will develop the necessary tools to answer the following questions.

Question 1. *Why are 'denser lines' formed?*

This question regards the horizontal lines that many of the points seem to fall in, at $y = 1,\ y = 0.75$, and $y = 0.89$ for example. This is one of the first questions arising upon seeing **Figure 2**, since the orderly patterns strike as unexpected in such a scattered and seemingly random distribution. After addressing **Question 1**, the next natural question is,

Question 2. *Where can 'denser lines' form?*

This is also a natural thing to ask. We see many dense lines between 0.95 and 1, and much fewer below 0.70.

4 Solving Problem 1

First, we must improve our undrstanding on the functions σ and φ. The following product forms for these functions in terms of the prime factorization of n can be derived from their definitions.

Lemma 1. *For any positive integer n with prime factorization $n = \prod_{i=1}^{k} p_i^{\alpha_i}$*

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1} \tag{4.1}$$

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i-1}(p_i - 1) \tag{4.2}$$

Sections (4.1) and (4.2) are proofs for these two statements.

4.1 Deriving the Product Form of σ

To do this, we first show that σ is *multiplicative*, which means

$$\sigma(mn) = \sigma(m)\sigma(n) \tag{4.3}$$

holds for all coprime positive integers m, n.

Let d be a positive divisor of mn, and write the prime factorization of d:

$$d = \prod_{i=1}^{k} p_i^{\alpha_i} \tag{4.4}$$

In this form, each term $p_i^{\alpha_i}$ either

1. divides m and is coprime to n, or
2. divides n and is coprime to m,

because d is a divisor of mn and m, n are coprime. That means we can (uniquely) write

$$d = d_m d_n \tag{4.5}$$

where d_m is the product of the terms which correspond to case 1., and d_n is the product of the terms which correspond to case 2.

Conversely, if d_m and d_n are positive divisors of m and n, respectively, then $d_m d_n$ is a divisor of mn. This means that

Lemma 2. *There is a one-to-one correspondence between elements in $\mathcal{D}_{mn}$ and pairs of elements in $\mathcal{D}_m$ and $\mathcal{D}_n$, for coprime positive integers m, n.*

Example 6. Let $m = 9,\ n = 14$. The divisors of 9 are $\mathcal{D}_9 = \{1,\ 3,\ 9\}$ and of 14 are $\mathcal{D}_{14} = \{1,\ 2,\ 7,\ 14\}$. Every divisor of $mn = 126$ is a product of one divisor of 9 and one divisor of 14:

$$\begin{aligned}
1 &= 1\cdot 1, & 2 &= 1\cdot 2, & 7 &= 1\cdot 7, & 14 &= 1\cdot 14,\\
3 &= 3\cdot 1, & 6 &= 3\cdot 2, & 21 &= 3\cdot 7, & 42 &= 3\cdot 14,\\
9 &= 9\cdot 1, & 18 &= 9\cdot 2, & 63 &= 9\cdot 7, & 126 &= 9\cdot 14.
\end{aligned}$$

Thus $\mathcal{D}_{126} = \{1,\ 2,\ 3,\ 6,\ 7,\ 9,\ 14,\ 18,\ 21,\ 42,\ 63,\ 126\}$.

Thanks to **Lemma 2**, *multiplicativity* can be established using **Definition 1**.

$$\begin{aligned}
\sigma(mn) &= \sum_{d\in\mathcal{D}_{mn}} d && (4.6)\\
&= \sum_{d_m\in\mathcal{D}_m}\sum_{d_n\in\mathcal{D}_n} d_m d_n && (4.7)\\
&= \sum_{d_m\in\mathcal{D}_m} d_m \left(\sum_{d_n\in\mathcal{D}_n} d_n\right) && (4.8)\\
&= \left(\sum_{d_m\in\mathcal{D}_m} d_m\right)\left(\sum_{d_n\in\mathcal{D}_n} d_n\right) = \sigma(m)\sigma(n). && (4.9)
\end{aligned}$$

(4.6) This is **Definition 1**.

(4.7) By **Lemma 2**, divisors of mn correspond uniquely to products $d_m d_n$ with $d_m \mid m$, $d_n \mid n$, so the sum may be rewritten as a double sum over such pairs.

(4.8) The inner and outer sums are independent, allowing the factors d_m and d_n to be separated.

(4.9) Since the inner sum depends only on n, it can be factored out of the outer sum.

This lets us express $\sigma(n)$ in terms of the prime factorization $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ for any positive integer n. Since $p_1^{\alpha_1},\ p_2^{\alpha_2},\ \ldots,\ p_k^{\alpha_k}$ are all coprime, *multiplicativity* lets us separate the terms $p_1^{\alpha_1},\ p_2^{\alpha_2},\ \ldots,\ p_k^{\alpha_k}$ out, one by one, to finally reach the form in (4.15):

$$\begin{aligned}
\sigma(n) =& \sigma(p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}) && (4.10)\\
=& \sigma(p_1^{\alpha_1}) \cdot \sigma(p_2^{\alpha_2} \ldots p_k^{\alpha_k}) && (4.11)\\
=& \sigma(p_1^{\alpha_1}) \cdot \sigma(p_2^{\alpha_2}) \cdot \sigma(p_3^{\alpha_3} \ldots p_k^{\alpha_k}) && (4.12)\\
=& \ldots && (4.13)\\
=& \sigma(p_1^{\alpha_1}) \cdot \sigma(p_2^{\alpha_2}) \cdot \cdots \cdot \sigma(p_k^{\alpha_k}) && (4.14)\\
=& \prod_{i=1}^{k} \sigma(p_i^{\alpha_i}) && (4.15)
\end{aligned}$$

To finish, we only need to compute $\sigma(p^\alpha)$ when p is prime and α is any positive integer. This is easy since

$$\mathcal{D}_{p^\alpha} = \{1,\ p,\ p^2,\ \ldots,\ p^\alpha\} \tag{4.16}$$

which implies $\sigma(p^\alpha)$ is simply a geometric series

$$\sigma(p^\alpha) = 1 + p + p^2 + \cdots + p^\alpha = \frac{p^{\alpha+1} - 1}{p - 1} \tag{4.17}$$

Finally, substitute this into (4.15) to finish:

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1} \tag{4.1}$$

This is true for all $n > 1$.

Example 7. Compute $\sigma(2025)$.

$$\begin{aligned}
\sigma(2025) =& \sigma(3^4 \cdot 5^2)\\
=& \sigma(3^4) \cdot \sigma(5^2)\\
=& \frac{3^5 - 1}{3 - 1} \frac{5^3 - 1}{5 - 1}\\
=& 121 \cdot 31 = 3751
\end{aligned}$$

4.2 Deriving the Product Form of φ

Many textbooks elect to first show that φ is *multiplicative* like the above proof for σ. Here, an alternative proof requiring less background will be presented.[2]

We will introduce a definition that is very helpful in this proof:

Definition 3. Let $\varphi_m(n)$ be the number of positive integers less than or equal to n and coprime to m. Formally,

$$\varphi_m(n) = \left|\{x \in \mathbb{Z}^+ \ : \ x \leq n, \ \gcd(x, m) = 1\}\right| \tag{4.18}$$

Example 8. Let $n = 12$. We compute $\varphi_2(12)$, $\varphi_3(12)$, and $\varphi_6(12)$ by listing the positive integers less than or equal to 12 and striking those not coprime to the subscript:

$$\varphi_2(12) = \left|\{1,\ \cancel{2},\ 3,\ \cancel{4},\ 5,\ \cancel{6},\ 7,\ \cancel{8},\ 9,\ \cancel{10},\ 11,\ \cancel{12}\}\right| = 6,$$
$$\varphi_3(12) = \left|\{1,\ 2,\ \cancel{3},\ 4,\ 5,\ \cancel{6},\ 7,\ 8,\ \cancel{9},\ 10,\ 11,\ \cancel{12}\}\right| = 8,$$
$$\varphi_6(12) = \left|\{1,\ \cancel{2},\ \cancel{3},\ \cancel{4},\ 5,\ \cancel{6},\ 7,\ \cancel{8},\ \cancel{9},\ \cancel{10},\ 11,\ \cancel{12}\}\right| = 4.$$

Observe that earlier in **Example 5** we computed $\varphi(12) = 4$, the same value as $\varphi_6(12) = 4$. As we shall see, this is not a coincidence.

Let n be a positive integer with prime factorization

$$n = \prod_{i=1}^{k} p_i^{\alpha_i} \tag{4.19}$$

Consider some positive integer $x \leq n$.

If no prime among $p_1,\ p_2,\ \ldots,\ p_k$ divides x, then

$$\gcd\left(x, \prod_{i=1}^{k} p_1\right) = \gcd(x, n) = 1 \tag{4.20}$$

since no prime divisor of n nor $\prod_{i=1}^{k} p_1$ divides x.

Otherwise, some prime among $p_1,\ p_2,\ \ldots,\ p_k$ divides x. Then it is a common divisor of x, $\prod_{i=1}^{k} p_1$, and n, which implies both

$$\gcd\left(x, \prod_{i=1}^{k} p_1\right) \neq 1 \text{ and } \gcd(x, n) \neq 1 \tag{4.21}$$

Hence we deduce that

Lemma 3. *For a positive integer n with prime factorization $n = \prod_{i=1}^{k} p_1^{\alpha_i}$, The equality $\gcd(x, \prod_{i=1}^{k} p_1) = 1$ holds precisely when $\gcd(x, n) = 1$ holds.*

This implies the following:

$$\varphi_{\prod_{i=1}^{k} p_1}(n) = \left|\left\{x \in \mathbb{Z}^+ \ :\ x \leq n,\ \gcd\left(x,\ \prod_{i=1}^{k} p_1\right) = 1\right\}\right| \tag{4.22}$$

$$= \left|\{x \in \mathbb{Z}^+ \ :\ x \leq n,\ \gcd(x, n) = 1\}\right| \tag{4.23}$$

$$= \varphi(n) \tag{4.24}$$

(4.24) This is **Definition 3**.

(4.25) By **Lemma 3**, the condition $\gcd(x, \prod_{i=1}^{k} p_1) = 1$ in (4.22) is equivalent to the condition $\gcd(x, n) = 1$, so the sets are equal.

(4.26) This is **Definition 2**.

(4.26) of this result generalizes our observation in **Example 8**. If we choose m to be the product of all distinct prime factors of n, i.e. $\prod_{i=1}^{k} p_i$, then $\varphi_m(n)$ always gives the same result as the original $\varphi(n)$.

We proceed to use induction on the prime factors $p_1,\ p_2,\ \ldots,\ p_k$ of n.

Inductive claim: For the positive integer $j < k$ holds

$$\varphi_{\prod_{i=1}^{j} p_i}(n) = n \prod_{i=1}^{j} \left(1 - \frac{1}{p_i}\right) \tag{4.25}$$

Base case, $j = 1$: Compute $\varphi_{p_1}(n)$.

Among the n positive integers less than or equal to n, there are n/p_1 multiples of p_1, namely $p_1,\ 2p_1,\ \ldots,\ \frac{n}{p_1}p_1$. All the remaining numbers are coprime to p_1, which means

$$\varphi_{p_1}(n) = n - n/p_1 = n\left(1 - \frac{1}{p_1}\right) = n \prod_{i=1}^{1} \left(1 - \frac{1}{p_i}\right) \tag{4.26}$$

This is the desired form, so the base case is done.

Inductive step: Assume that the claim holds for j, so

$$\varphi_{\prod_{i=1}^{j} p_i}(n) = n \prod_{i=1}^{j} \left(1 - \frac{1}{p_i}\right) \tag{4.25}$$

We consider p_{j+1}, the next prime divisor of n. Among the n positive integers less than or equal to n, there are n/p_{j+1} multiples of p_{j+1}, namely $p_{j+1},\ 2p_{j+1},\ \ldots,\ \frac{n}{p_{j+1}}p_{j+1}$.

These numbers need to be erased, but some of them are already erased, namely those which are

FUCKFUCKFUCK THIS IS THE MÖBIUS INVERSION PROOF

THIS IS TOO HARD! BUT IS CRT PROOF ANY EASIER??

Maybe modular inverses proo???

\section{ROUGH DRAFT TERRITORY}

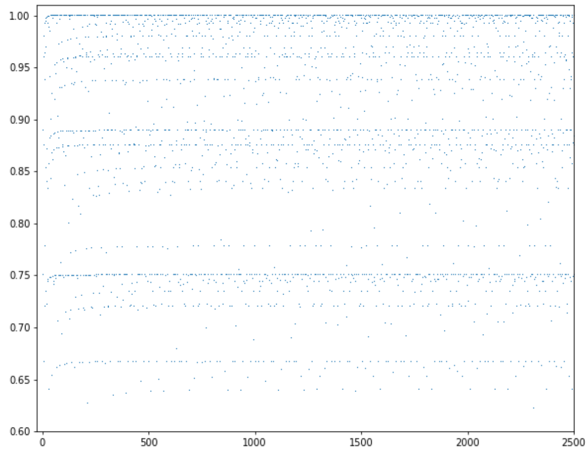

Figure 3: Plotting the Sequence

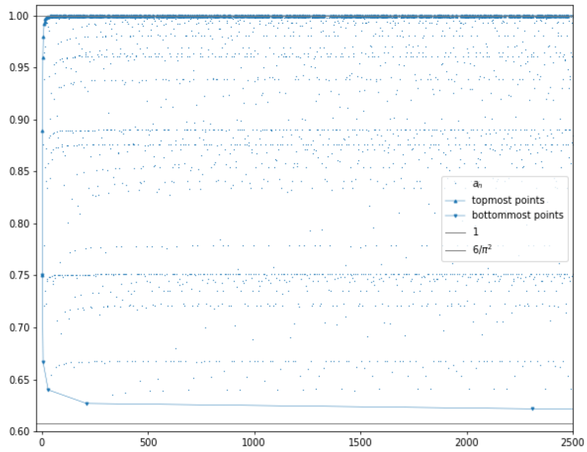

Figure 4: Highlighting Minima & Maxima

After discussing these regions with many points, we proceed to discuss the opposite: Are there any regions with no points? Formally, we may ask,

Question 3. *Does there exist a nonempty open interval* $I \subset \left(\frac{6}{\pi^2}, 1\right)$ *such that* $a_n \notin I$ *holds for all* $n \geq 2$*?*

Notice how this formulation looks familiar? Similarly to **Question 1**, the answer here is also **No.** This is actually a generalization of that question: Here we are showing that the sequence a_n gets arbitrarily close to *any* point in the interval $\left(\frac{6}{\pi^2}, 1\right)$, while **Question 1** only required us to show that the sequence gets arbitrarily close to its boundary.

Then show that the basel problem ∞ sum is the ∞ product: Page 230 (pdf: 242) of [1]. Then solve **Problem.** by writing $\dfrac{\varphi(n)\sigma(n)}{n^2}$ in terms of primes:

$$\frac{\varphi(n)\sigma(n)}{n^2} = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i^{\alpha_i+1}}$$

(we get this by plugging Lemma 1)

To finish the problem, see the maximum of each term in this product, and the minimum of each term. Each term is at least $\dfrac{p_k^2 - 1}{p_k^2}$ and at most arbitrarily close to 1 but less than 1.

Then the global maximum is strictly below 1 (we only consider $n \geq 2^*$) Then the global minimum is strictly above $\prod_{p\in\mathbb{P}} \dfrac{p^2-1}{p^2} = \dfrac{6}{\pi^2}$ (Basel Problem)

So **Problem 1.** is done.

5 Questions

Next answer Question 1. Also rewrite it not using the interval because it's IB AA HL, not Topology 101.

The answer is: no, the bounds can't be uniformly improved. For upper bound this is trivial (n $= 2^k$ for very big k)

For lower bound it's what convergence means. Now that I think about it, Question 1 isn't really worth doing since it's so fricking easy.

Question 2. The topmost 'dense line' is primes, (or prime powers? That would make the discussion more complete but also longer and not that much more impressive)

*maybe consider $n = 1$ too but that's a matter of my taste & preference

Definition 4 (‘dense line’ should I come up with a better name :()**.** We define a subsequence b_n of the sequence a_n to be a ‘dense line’ if it

- increases
- approaches some value t
- $b_n = a_g(n)$ and $g(n) < cx \ln x^{\dagger}$

Actually, we get a dense line $b_n = a_{cp_n}$ for any positive integer d, and the line is denser the smaller c is (obviously). here p_n is the nth smallest prime. Though the issue is that if I define ‘dense lines’ like this, it’s hard to verify every dense line is generated by this description. In fact, I don’t even think that’s true... Maybe it is? But how do I define it better? This is the only formal definition I could come up with!

But that’s enough knowledge about the dense lines that we can graph them! Is this a satisfying enough answer to Question 2?

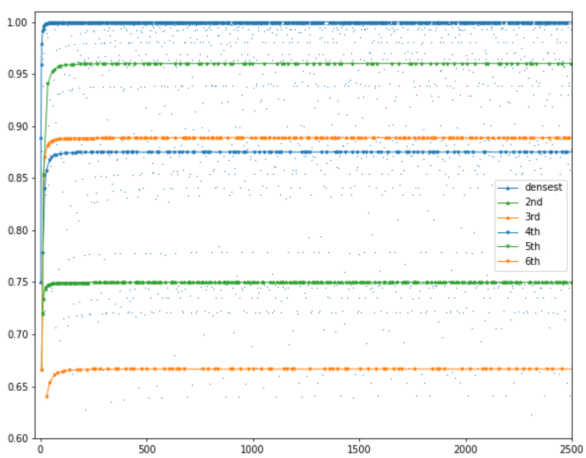

Figure 5: First 6 ‘dense lines’

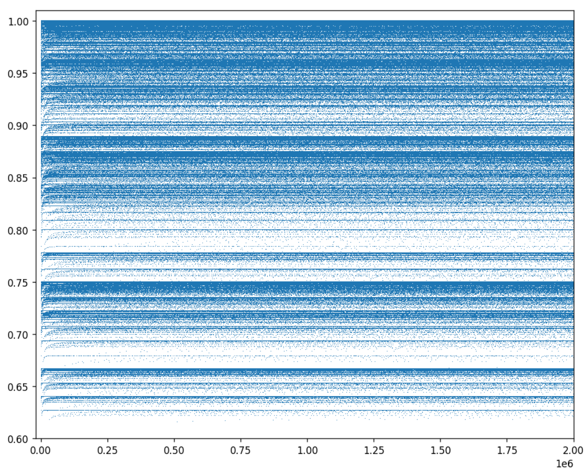

Figure 6: plotting up to $n = 2,000,000$

Now question 3. Look at figure 4. We see more blue ‘dense lines’ than previously, but also white regions with very few points. We are motivated to ask: Do dense lines

† Here $x \ln x$ is the asymptotic growth of primes (also the asymptotic growth of prime powers, cool! This condition says ‘b_n must be dense enough.’)

cover everything eventually? Will there be some strip with no points whatsoever? The answer is no, and we present an algorithm which shows this. It's really cool and relies on Nagura's bound:

Theorem 2. *For all $n \geq 25$ there is always a prime p with $n < p < 1.2n$*

And a bit of computer brute-force for $3 < n < 25$

and a tiny bit of brute-force by hand for $n = 2, n = 3$

After that though Nagura + induction can get the remaining cases. I think this is my coolest result, and I only recently come up with it

6 Concluding Remarks

I don't know what to say here. It's been fun investigating this, but seeing that the lines were just primes and constant multiples of primes is kinda anticlimactic for me; I like very hard problems and felt much more thrilled solving **Question 3.**

Also I need to write an introduction...

References

[1] Tom M. Apostol. *Introduction to Analytic Number Theory*. New York: Springer-Verlag, 1976.

[2] Eric W. Weisstein. *Totient Function*. MathWorld–A Wolfram Resource. Retrieved September 22, 2025. 2025. URL: https://mathworld.wolfram.com/TotientFunction.html.