Subject: Mathematics

Investigating patterns in the product of Euler's totient function φ and the sum of divisors function σ

Research Question: What gives rise to the patterns in the graph of $f(n) = \sqrt{\varphi(n) \cdot \sigma(x)}$, and how does this relate to the Riemann zeta function?

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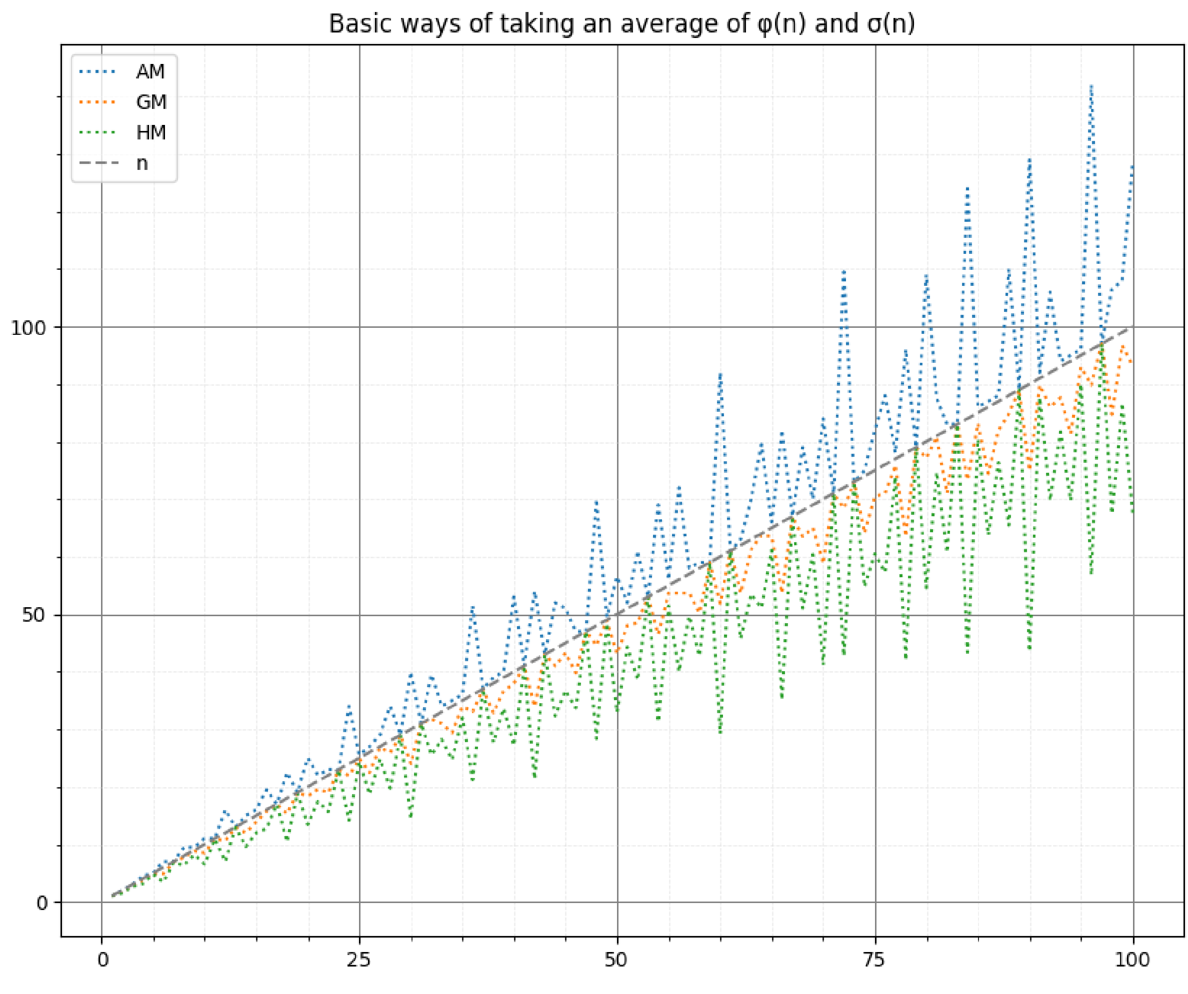
Word count: 1729

Presentation to the Class

We wish to find some way of taking an average that sends the pairs of points to the line $y = n$, i.e. discover a function that reduces the amount of chaos and erraticity in this graph.

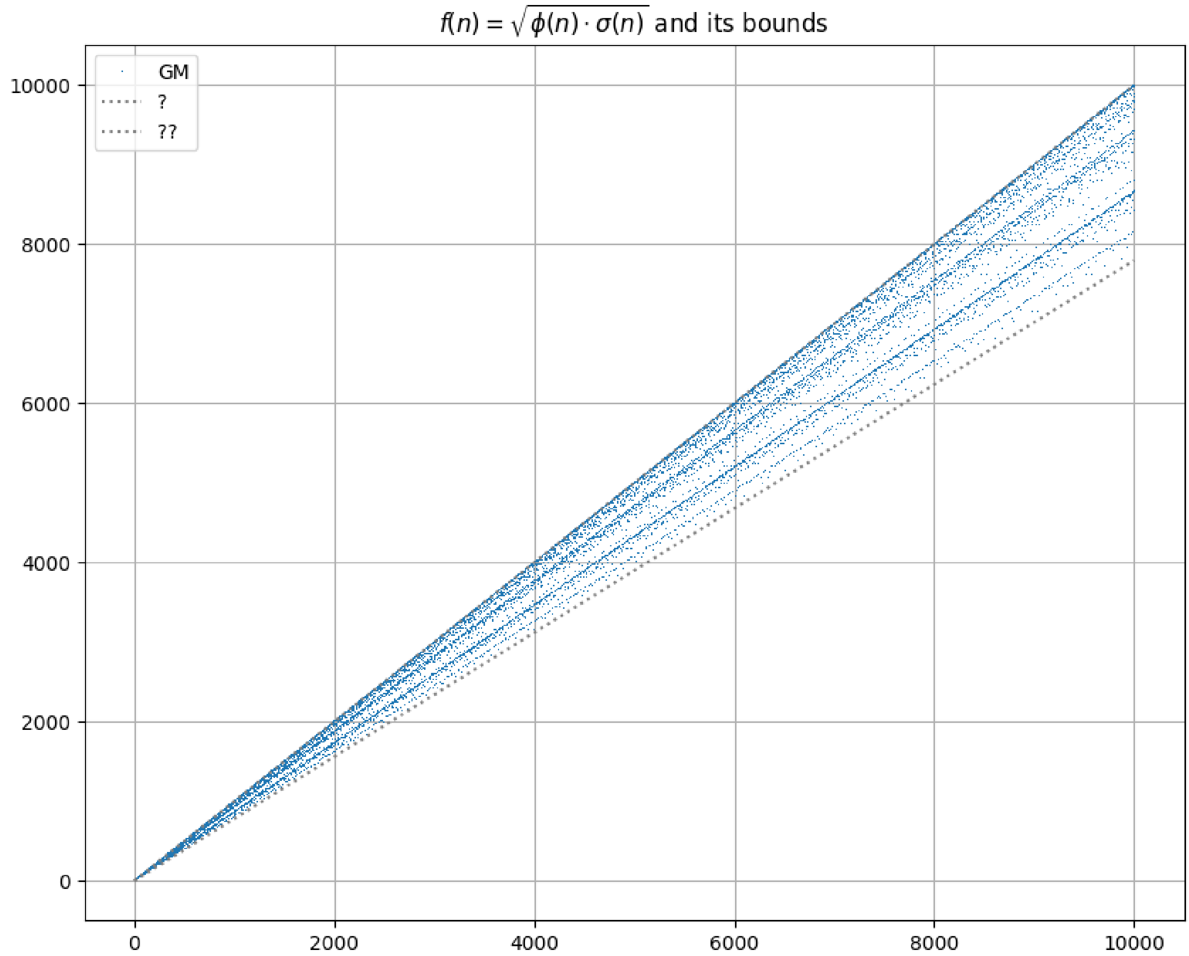
Initial guesses for ways to take an average:

$$\textbf{AM}: \ \frac{a+b}{2} \qquad \textbf{GM}: \ \sqrt{ab} \qquad \textbf{HM}: \ \frac{2}{\frac{1}{a}+\frac{1}{b}}$$

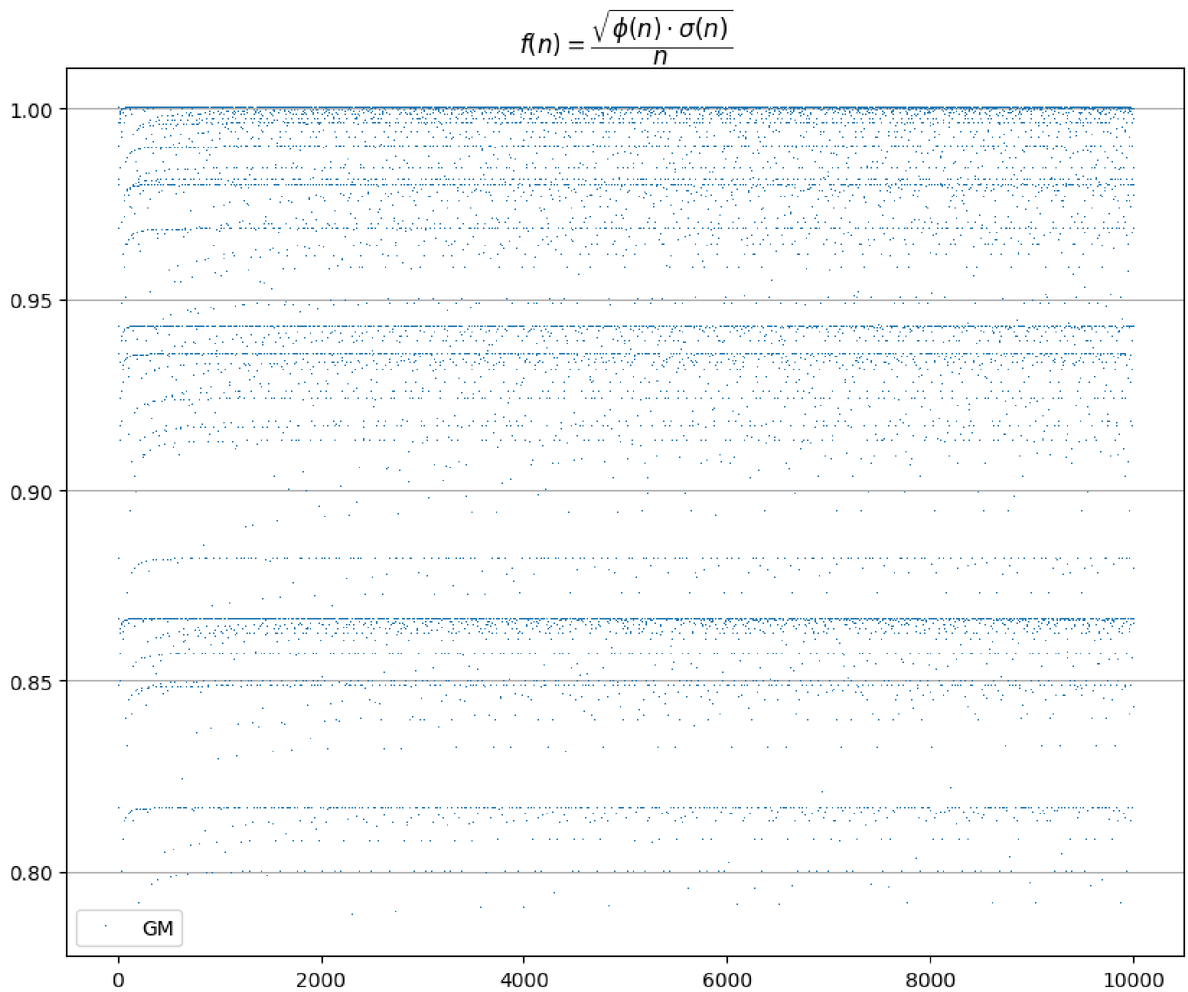


It's also possible to take some weighted version of these, to make the result closer to the target. Investigating that may be a part of my Research, if the word limit allows for that.

I hope the conclusion is that the geometric mean is the best. Then I will proceed to graph the function $f(x) = \sqrt{\varphi(n) \cdot \sigma(n)}$ in more detail and notice some lines with points packed more densely.



Since it looks linear, we are motivated to graph the function $g(n) = \dfrac{\sqrt{\varphi(n) \cdot \sigma(n)}}{n}$ in more detail, to get a clearer view of what the lines are.



In the rest of the paper, I will compute the lower and upper bounds, show that they're asymptotically strict, and lastly explain why the patterns (denser lines) occur.

Contents

1 Introduction

Write this when most of the essay is finished

2 Prerequisite Knowledge

The reader should be familiar with mathematical notation and basic definitions like 'co-prime' and 'divisor'. Additionally, understanding of *Fundamental Theorem of Arithmetic* is required:

Theorem 2.1 (Fundamental Theorem of Arithmetic) *Every positive integer has an unique prime factorization.*

In other words, for any n, there exists a unique set of distinct prime numbers $p_1, p_2, ..., p_k$, along with their respective exponents $\alpha_1, \alpha_2, ..., \alpha_k$, such that

$$n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}.$$

Representing a number in this form will play an exceptionally important role in this paper.

3 An Introduction to Arithmetic Functions

A special class of functions called *arithmetic functions* play a vital role in the study of number theory, which is the study of positive integers and their properties. The two arithmetic functions concerned in this paper are the following.

Definition 3.1 (Sum of Divisors Function) *Denoted by $\sigma(n)$, it returns the sum of all positive divisors of n.*

Definition 3.2 (Euler's Totient Function) *Denoted by $\varphi(n)$, it returns the number of positive integers $\leq n$ and coprime to n.*

Mathematics relies on abstraction, so properties like these are encoded using functions. For example, the property "n is a perfect number if it is equal to the sum of its proper divisors" can be expressed symbolically as:

$$S = \{n \in \mathbb{N} : \sigma(n) = 2n\}$$

Euler's totient function is best known as a part of *Euler's Theorem*, upon which RSA encryption relies. It turns out that both functions have neat closed forms in terms of the prime factorization of n.

Lemma 3.1 *The functions φ and σ have product forms*

$$\begin{aligned} \varphi(n) &= \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1) \\ \sigma(n) &= \prod_{i=1}^{k} \frac{p_i^{\alpha_i + 1} - 1}{p_i - 1} \\ \textit{given } n &= \prod_{i=1}^{k} p_i^{\alpha_i}. \end{aligned}$$

*Prove these statements through multiplicativity and primes.

*It might be a potential danger that I'm not addressing my research question early enough, since the proof of Lemma 3.1 will take about three pages and I plan to do it here.

*Whether it's before or after proving the lemma, I shall shortly hereafter motivate the product of φ and σ: the $p_i - 1$'s cancel out and we get

$$\varphi(n) * \sigma(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i^{\alpha_i + 1} - 1)$$

4 The zeta function

hello [2] hello!!

5 An Explanation for the Patterns

6 Appendix

References

[1] G. H. Hardy. *A Mathematician's Apology*. Accessed: 2025-05-09. Cambridge: Cambridge University Press, 1940.

[2] Johan Wästlund. *Summing inverse squares by euclidean geometry*. https://www.math.chalmers.se/~wastlund/Cosmic.pdf. Accessed: 2025-03-10. Gothenburg, Sweden, 2010.