Subject: Mathematics

Investigating patterns in the product of Euler's totient function φ and the sum of divisors function σ

Research Question: What gives rise to the patterns in the sequence

$$a_n = n^{-2}\varphi(n)\sigma(n)?$$

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Contents

1 Introduction

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2 Prerequisite Knowledge, Notation, and Definitions

The concepts investigated in this paper are strictly elementary number theory, which means we're mostly working with positive integers. Thus, for simplicity, you can assume all variables mentioned in this paper are positive integers, unless otherwise stated.

Theorem 1 (Fundamental Theorem of Arithmetic)**.** *Every positive integer has an unique prime factorization.*

This theorem is crucial in our ensuing discussion, because it allows us to express any number n uniquely in terms of its k prime divisors $p_1, p_2, ..., p_k$, along with their respective exponents $\alpha_1, \alpha_2, ..., \alpha_k$, as the product $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k} = n$.*

Some more complicated sums and products are discussed in this paper, which require set-builder-like† notation to impose restrictions on the numbers we're summing over. This notation will allow us to cleanly write formulas for the two functions that will be extensively discussed:

Definition 1 (Sum of Divisors Function σ and Euler's Totient Function φ)**.**

$$\sigma(n) = \sum_{d \in \mathbb{Z}^+,\ d|n} d$$

$$\varphi(n) = \sum_{\substack{m \in \mathbb{Z}^+, m \leq n \\ \gcd(m,n)=1}} 1$$

The comma inside the subscripts simply means 'and', so verbosely we would say that $\sigma(n)$ takes the sum of all numbers d which divide n, and $\varphi(n)$ counts the numbers $m \leq n$ which are coprime to n.

*for $n = 1$ simply set $k = 0$

†see https://en.wikipedia.org/wiki/Set-builder_notation

3 The Problem of Study

The aformentioned functions φ and σ are the subject of this study. Specifically, we will solve the following problem from a classic textbook in analytic number theory‡ and investigate it's intricacies on a deeper level.

Problem. Show that $\dfrac{6}{\pi^2} < \dfrac{\varphi(n)\sigma(n)}{n^2} < 1 \quad \text{if } n \geq 2.$

Luckily, this strange emergence of π is addressed: The textbook promises to prove, in a later chapter, that the infinite product $\prod_p (1 - p^{-2})$, extended over all primes p, converges to the value $\dfrac{1}{\zeta(2)} = \dfrac{6}{\pi^2}$. *Do I need to define zeta???*

In consideration of this hint, we should incorporate prime numbers to the statement. Equally crucially, we also need to expand our understanding on these functions, as simply substituting the summations from **Definition 1** quickly becomes hopeless. These two issues can actually be overcome simultaneously, with the following result, which characterizes φ and σ in terms of the prime factorization of n:

Lemma 1. *For any positive integer n with prime factorization $n = \prod_{i=1}^{k} p_i^{\alpha_i}$*

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1) \tag{3.1}$$

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i + 1} - 1}{p_i - 1} \tag{3.2}$$

To prove this lemma, an even more elementary result on the behaviour of φ and σ is required:

‡1, Chapter 3, Exercise 9.

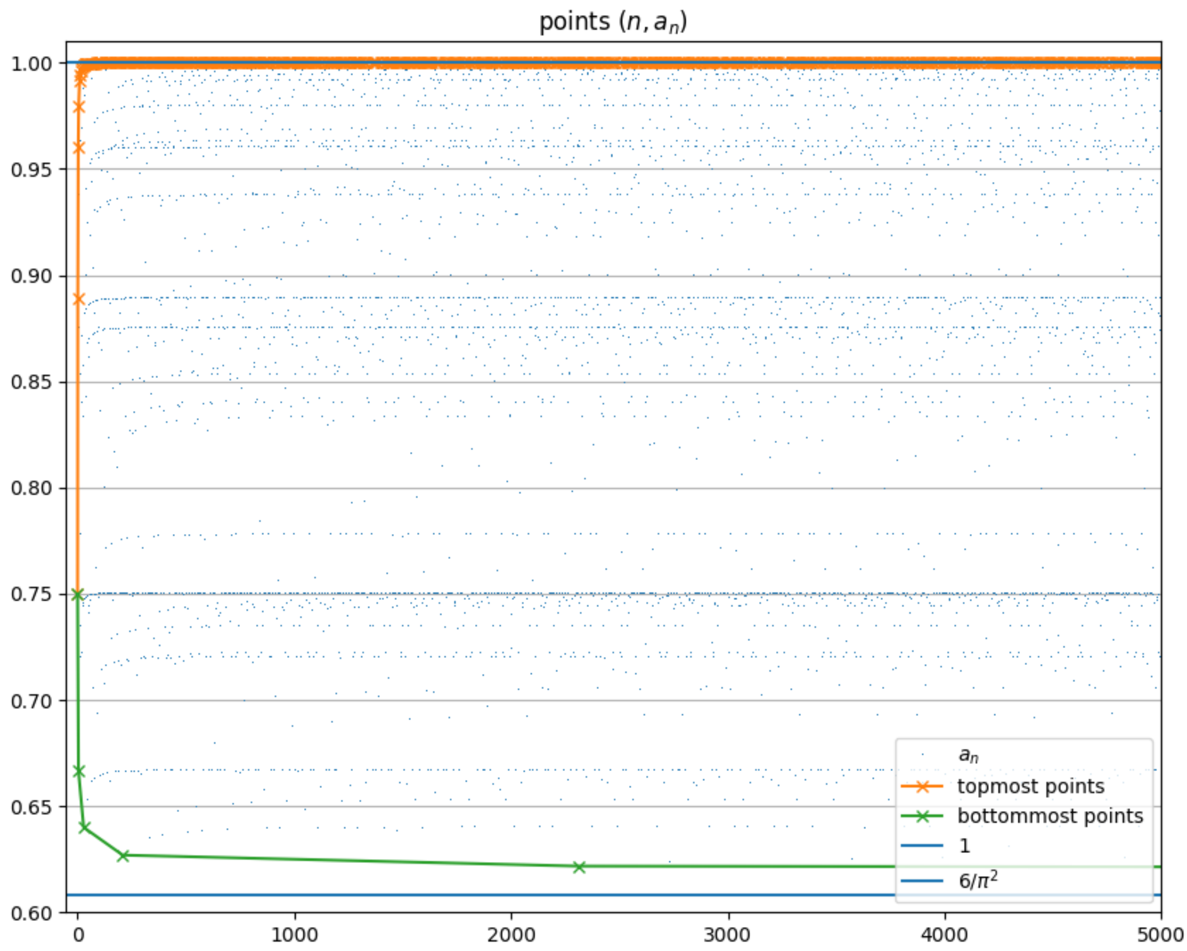
Lemma 2 (Multiplicativity)**.** *For any numbers m, n with $\gcd(m,n) = 1$, we have*

$$\varphi(mn) = \varphi(m)\varphi(n) \tag{3.3}$$

$$\sigma(mn) = \sigma(m)\sigma(n) \tag{3.4}$$

After proving **Lemma 2** and **Lemma 1**, we proceed to solve the Problem. Here we make an equivalent statement of the problem to facilitate the following discussion. Define the sequence $a_n = \dfrac{\varphi(n)\sigma(n)}{n^2}$. Then the problem is equivalent to the statement $a_n \in \left(\dfrac{6}{\pi^2}, 1\right)$ A natural question that follows is whether this interval can be made narrower.

Question 1. *Does there exist an interval $(c^-, c^+) \subsetneq \left(\dfrac{6}{\pi^2}, 1\right)$ such that $c^- < a_n < c^+$ holds for all $n \geq 2$?*



The answer turns out to be negative, which we can see with a diagram:

*Prove these statements through multiplicativity and primes.

*Whether it's before or after proving the lemma, I shall shortly hereafter motivate the product of φ and σ: the $p_i - 1$'s cancel out and we get

$$\varphi(n) * \sigma(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i^{\alpha_i + 1} - 1)$$

4 The zeta function

hello [2] hello!!

5 An Explanation for the Patterns

6 Notes on Apostol

Page 71 (pdf: 83): Main problem statement Page 230 (pdf: 242): Euler product form proof.

References

[1] Tom M. Apostol. *Introduction to Analytic Number Theory*. Accessed: 2025-03-10. New York: Springer-Verlag, 1976.

[2] Johan Wästlund. *Summing inverse squares by euclidean geometry*. https://www.math.chalmers.se/~wastlund/Cosmic.pdf. Accessed: 2025-03-10. Gothenburg, Sweden, 2010.