Investigating patterns in the product of Euler's totient function φ and the sum of divisors function σ

Research Question: What gives rise to the patterns and bounds in the graph of $y = \varphi(x) \cdot \sigma(x)$, and how does this relate to the Riemann zeta function and the Basel problem?

Word count: 225

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Abstract / Preliminary Research Plan

I will first define the functions φ and σ, and provide motivations and justifications to why they are important and an integral part of number theory.
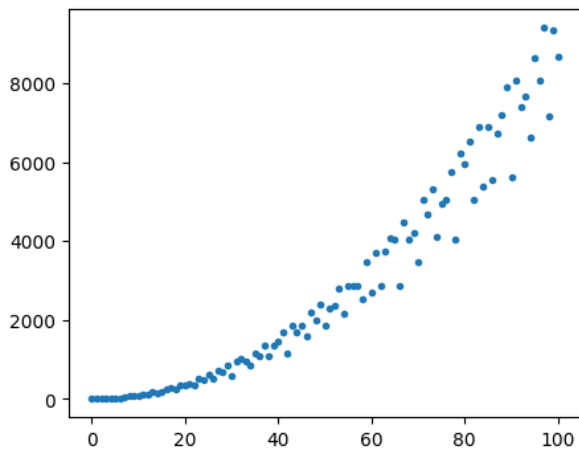

Figure 1: The graph I wish to investigate

Next I will present the graph I plan to study, $y = \varphi(x) \cdot \sigma(x)$, and suggest computing bounds for this function.

I might hint towards patterns in the graph already at this point, but I should leave the patterns until the end, for coherence. Instead, I introduce the upper bound of x^2 and lower bound of $\dfrac{x^2}{\zeta(2)} = x^2 \cdot \dfrac{6}{\pi^2}$

I should go on to prove these bounds, but perhaps before that I must allude to the famous Basel Problem.

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ... = \frac{\pi^2}{6}$$

I will discuss the relationship of this with the Riemann Zeta function, and perhaps see the need to refer to secondary sources here.

After that, the bounds will be established and we may move on to discussing the patterns in this graph.
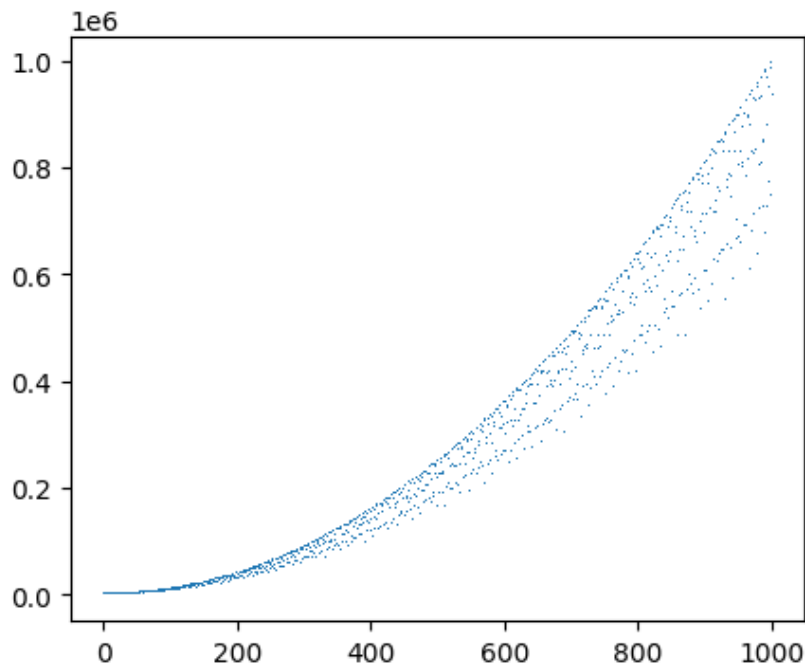
Figure 2: We see that the graph has denser regions when enough data is included

The patterns can be clearly seen and I will be showing why they are manifest. Maybe that requires me providing an exact definition on what a "dense sort of line thing" we see here is. That may require additional secondary sources.

I think this is enough content, and it will produce an analysis that I hope does not end up being too shallow nor too obstruse.

Preliminary bibliography

Introduction to Analytic Number Theory by *Tom M. Apostol*

Modern Olympiad Number Theory by *Aditya Khurmi*

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