Subject: Mathematics

Investigating patterns in the product of Euler's totient function φ and the sum of divisors function σ

Research Question: What gives rise to the patterns in the sequence

$$a_n = n^{-2}\varphi(n)\sigma(n)?$$

summary: interesting math in your document; lots of work still to do

you can find detailed comments below; here are the most important things (in addition to what you wrote yourself, like "needs an intro"):

1. you need to figure out what your research question actually is, state it clearly, and address it clearly; the research question needs to be introduced, in an intuitive form, as early as possible

2. quite a few steps need to be explained in more detail

3. some concepts need to be introduced to the reader

4. you want to provide examples to make this concrete

~~September 4, 2025~~

Word count: 1729

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1 Introduction

Write this when most of the essay is finished

2 Prerequisite Knowledge, Notation, and Definitions

The inarguably most important theorem in number theory is the following:

Theorem 1 (Fundamental Theorem of Arithmetic)**.** *Every positive integer has an unique prime factorization.*

some source

This theorem is crucial in our ensuing discussion, because it allows us to express any number n uniquely in terms of its k prime divisors $p_1, p_2, ..., p_k$, along with their respective exponents $\alpha_1, \alpha_2, ..., \alpha_k$, as the product $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k} = n$.*

pos integer

domains

The reader should be comfortable with Π-notation for sequential products, which is similar to the familiar Σ but for multiplication. For example,

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k} = \prod_{i=1}^{k} p_i^{\alpha_i}$$

In a certain sense, every integer n is simply a collection of prime divisors and their respective exponents. Im mathematics, we call such collections 'sets'. This paper will deal with sets a lot, so it is necessary to introduce the relevant notation:

Notation (Set-builder)**.** Most commonly, a set S is defined, i.e. 'built', with the following notation:

$$S = \{a \in A \ : \ \text{conditions for } a\}$$

Sometimes, like in (2.4) of the following example, the domain of the set is put into the condition of a, which allows the set to take an expression in a.

* for $n = 1$ simply set $k = 0$

this is slightly tricky, so elaborate

Here are some relevant examples of set-builder notation:

$$\text{Positive Integers} \qquad \mathbb{Z}^+ = \{x \in \mathbb{Z} \ : \ x > 0\} \tag{2.1}$$

introduce notation shortly

$$\text{Divisors of } n \qquad \mathcal{D}_n = \left\{x \in \mathbb{Z}^+ \ : \ x \mid n\right\} \tag{2.2}$$

$$\text{Prime Numbers} \qquad \mathbb{P} = \left\{x \in \mathbb{Z}^+ \ : \ \mathcal{D}_x = \{1, x\}\right\} \tag{2.3}$$

is 1 \in P?

$$\text{Prime Powers} \qquad \mathbb{P}^* = \left\{x^\alpha : \ x \in \mathbb{P}, \ \alpha \in \mathbb{Z}^+\right\} \tag{2.4}$$

this notation is empoyed below, so you might want to mention its interpretation as "and"

Some more complicated sums and products are discussed in this paper, which require set-builder-like notation to impose restrictions on the numbers to sum over. This notation will allow us to cleanly write formulas for the two functions that will be extensively discussed:

Definition 1 (Sum of Divisors Function σ and Euler's Totient Function φ)**.**

$$\sigma(n) = \sum_{d \in \mathbb{Z}^+, \ d \mid n} d$$

to be on the safe side: we don't have sums like this, so explain (shortly)

$$\varphi(n) = \sum_{\substack{k \in \mathbb{Z}^+, \, k \leq n \\ \gcd(k,n)=1}} 1$$

domains of these functions

Verbosely we would say that $\sigma(n)$ takes the sum of all numbers d which divide n, and $\varphi(n)$ counts the numbers $k \leq n$ which are *coprime* to n.

at least one example here

Terminology. We say that a is coprime to b, or a, b are coprime, if $\gcd(a, b) = 1$.

3 The Problem of Study

The aformentioned functions φ and σ are the subject of this study. Specifically, we will solve the following problem from a classic textbook in analytic number theory [1] and investigate it's intricacies on a deeper level.

provide a diagram here; remember, you want to give the reader an intuitive understanding of what this is about as quickly as possible

Problem. Show that $\dfrac{6}{\pi^2} < \dfrac{\varphi(n)\sigma(n)}{n^2} < 1$ for all $n \geq 2$.

give this a symbolic name, e.g. "Problem 1"

in literature?

Luckily, this strange emergence of π is addressed: The textbook promises to prove, in a later chapter, that the infinite product $\prod_p (1 - p^{-2})$, extended over all primes p, converges to the value $\dfrac{6}{\pi^2}$.

so it addresses the emergence of \pi in the first place; but how is this "lucky," that is, how does it help?

The result we are hinted to use is known as the Basel Problem. The proof that the infinite sum and infinite product are equal will be presented in **Section 4**, but showing that they are equal to $\frac{\pi^2}{6}$ is more involved and not relevant to this study. [2]

so best not to mention it at all?

Theorem 2 (Basel Problem)**.**

$$\prod_{p \in \mathbb{P}} \frac{1}{(1 - p^{-2})} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

which statement?

In consideration of this hint, we should incorporate prime numbers to the statement. Equally crucially, we also need to expand our understanding on these functions, as simply substituting the summations from **Definition 1** quickly becomes hopeless. These two issues can be overcome simultaneously with the following result, which characterizes φ and σ in terms of the prime factorization of n:

into what?

Lemma 1. *For any positive integer n with prime factorization $n = \prod_{i=1}^{k} p_i^{\alpha_i}$*

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1) \tag{3.1}$$

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i + 1} - 1}{p_i - 1} \tag{3.2}$$

To prove this lemma, an even more elementary result on the behaviour of φ and σ is required:

Lemma 2 (Multiplicativity). *For any numbers m, n with $\gcd(m,n) = 1$, we have*

$$\varphi(mn) = \varphi(m)\varphi(n) \tag{3.3}$$

$$\sigma(mn) = \sigma(m)\sigma(n) \tag{3.4}$$

After proving **Lemma 2** and **Lemma 1**, we proceed to solve the Problem. Then, we make an equivalent statement of the problem to better facilitate the following discussion. Define the sequence $a_n = \dfrac{\varphi(n)\sigma(n)}{n^2}$. Then the problem is equivalent to the statement $a_n \in \left(\frac{6}{\pi^2}, 1\right)$ A natural question that follows is whether this interval can be made narrower.

Question 1. *Does there exist an open interval $I \subsetneq \left(\frac{6}{\pi^2}, 1\right)$ such that $a_n \in I$ holds for all $n \geq 2$?*

Using a computer to calculate the first 5000 terms of the sequence a_n and visualize them as points (n, a_n) on the Cartesian plane, we notice that the upper bound probably can't be improved. The same is suspected for the lower bound.

larger diagrams (no page count limit)

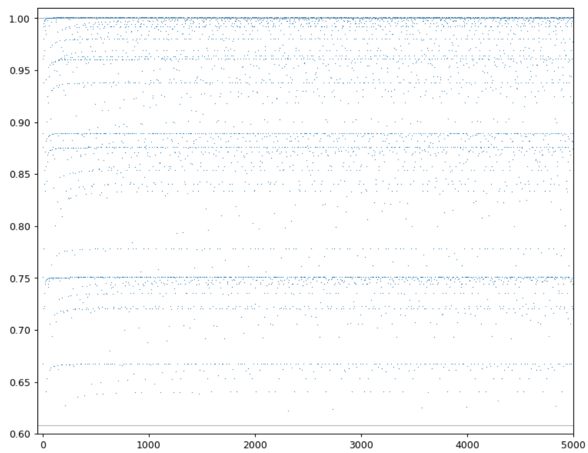

Figure 1: Plotting the Sequence

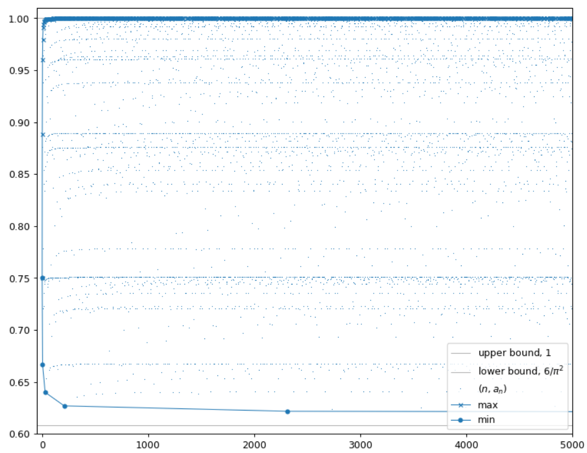

Figure 2: Highlighting Minima & Maxima

More interestingly, however, we notice some intriguing patterns in Figure 1. The points around certain special values 1.00 and 0.75 look denser and seem to form horizontal lines.

Question 2. *Why are 'denser lines' formed?*

A major section of this study will be dedicated to **Question 2**, in which we will precisely define what is meant by a 'denser line' and explain why such a pattern emerges.

After discussing these regions with many points, we proceed to discuss the opposite: Are there any regions with no points? Formally, we may ask,

Question 3. *Does there exist a nonempty open interval* $I \subset \left(\frac{6}{\pi^2}, 1\right)$ *such that* $a_n \notin I$ *holds for all* $n \geq 2$*?*

Notice how this formulation looks familiar? Similarly to **Question 1**, the answer here is also **No.** This is actually a generalization of that question: Here we are showing that the sequence a_n gets arbitrarily close to *any* point in the interval $\left(\frac{6}{\pi^2}, 1\right)$, while **Question 1** only required us to show that the sequence gets arbitrarily close to its boundary.

you need to clearly highlight the research question

also, I think formally discussing the machinery you need to address your RQ is not really relevant here yet; I think it would be better to try to

1. first quickly lead the reader to an intuitive understanding of the RQ

2. then shortly introduce what follows (the path) at a relatively high level of abstraction

4 Solving the Problem

We work from the bottom up and start by showing **Lemma 2**.

Proof of Lemma 2

We are give numbers m, n with $gcd(m, n) = 1$, and wish to show that

(3.2) $\varphi(mn) = \varphi(m)\varphi(n)$

(3.3) $\sigma(mn) = \sigma(m)\sigma(n)$

Recall that $\varphi(n)$ counts the numbers $k \leq n$ with which are coprime to n, so in set-builder notation,

$$\varphi(n) = \left|\left\{k \in \mathbb{Z}^+ : \; k \leq n, \; \gcd(k, n) = 1\right\}\right|.$$

you had a different (sigma) definition above; perhaps just use one?

Then the desired equality $\varphi(mn) = \varphi(m)\varphi(n)$ is equivalent to showing that

this differs from the one above in multiple ways: Z vs Z^+, < vs <=; why?

$$|\{k \in \mathbb{Z} : \; 0 \leq k < mn, \; \gcd(k, mn) = 1\}| \tag{4.1}$$

$$= |\{k \in \mathbb{Z} : \; 0 \leq k < m, \; \gcd(k, m) = 1\}| \cdot |\{k \in \mathbb{Z} : \; 0 \leq k < n, \; \gcd(k, n) = 1\}| \tag{4.2}$$

$$= |\{(k_m, k_n) \in \mathbb{Z} \times \mathbb{Z} : \; 0 \leq k_m < m, \; 0 \leq k_n < n, \; \gcd(k_m, m) = \gcd(k_n, n) = 1\}| \tag{4.3}$$

elaborate

for all of these theorems and lemmas, you (in terms of showing understanding) and the reader will benefit from example(s)

To prove this, we use the following fact:

which is...

Theorem 3 (Special Case of CRT[†]). *If m, n are coprime, and for integers k_m, k_n hold $0 \leq k_m < m, \; 0 \leq k_n < n$, then there exists an unique integer k with $0 \leq k < mn$ that has remainder k_m when divided by m and remainder k_n when divided by n.*

[†]Maybe I should mention the general case, since the reader may be interested...

reverse of "unique" is "none or many," right? (you address this below, but it needs to be made clearer here)

Proof ~~of CRT~~. Suppose, for contradiction, that there exists anonther integer k',

why can we assume this?

greater than k, with $0 \leq k' < mn$, which also has remainders k_m, k_n when divided by m and n, respectively. Then $m \mid k' - k$ and $n \mid k' - k$ are both true. Noting that m and n are coprime and $k' - k > 0$ we may write

elaborate

elaborate

$$mn \mid k' - k \implies mn \leq k' - k < mn,$$

contradiction. This shows that each pair of remainders k_m, k_n corresponds to at most one number k with $0 \leq k < mn$. Since there precisely mn pairs of remainders, and also precisely mn integers between 0 and mn, we conclude that such a k must always exist, so the proof is complete. □

exists?

elaborate

This theorem shows that every number k from the set in (4.1) corresponds to a~~n~~ unique pair of remainders when divided by n and m, and every pair (k_m, k_n) from the set in (4.3) corresponds to an unique number between 0 and mn. This will be very useful if we verify that the remainders of k when divided m and n are coprime to m and n respectively, if and only if k is coprime to mn.

which is?

By the Euclidean algorithm, we may write

$$\gcd(m, k) = \gcd(m, k - m) = \gcd(m, k - 2m) = ... = \gcd(m, k_m).$$

why?

Repeating this for n and combining, we get that $\gcd(mn, k) = \gcd(m, k)\gcd(n, k) = \gcd(m, k_m)\gcd(n, k_n)$, which equals one if and only if both $\gcd(m, k_m) = \gcd(n, k_n) = 1$.

This means the Chinese remainder theorem actually finds a one-to-one correspondence between these sets which we want to show to have equal size, which finally shows that $\varphi(mn) = \varphi(m)\varphi(n)$. □

not clear, elaborate

you have a theorem...proof within "theorem...proof;" clearly separate the two

Luckily, the proof that $\sigma(mn) = \sigma(m)\sigma(n)$ is not nearly as difficult. We will use the following fact in the proof:

Lemma 3. *Given that m and n are coprime, each divisor $d \mid mn$ can be written uniquely as the product $d_m d_n$ of divisors $d_m \mid m$ and $d_n \mid n$.*

Proof of Lemma 3. Observe the prime factorization of $d = \prod_{i=1}^{k} p_i^{\alpha_i}$. For each term factor?

because... (also, perhaps combine the pair of statements using just "and" or something along those lines)

$p_i^{\alpha_i}$ in this product holds either $\begin{cases} p_i \nmid n \\ p_i^{\alpha_i} \mid m \end{cases}$ or $\begin{cases} p_i \nmid m \\ p_i^{\alpha_i} \mid n \end{cases}$. We ~~simply~~ multiply the terms which divide m to obtain d_m, and the terms which divide n to obtain d_n. This process is unique since prime factorization is unique (**Theorem 1**), so the proof is complete. □

but divisors include 1, while prime factorization does not; right?

Now the multiplicativity of σ can be proven in one go, by using **Lemma 3** to get from (4.4) to (4.5).

$$\begin{aligned}
\sigma(mn) &= \sum_{d \mid mn} d && (4.4)\\
&= \sum_{d_m \mid m,\ d_n \mid n} d_m d_n && (4.5)\\
&= \sum_{d_m \mid m} \sum_{d_n \mid n} d_m d_n && (4.6)\\
&= \sum_{d_m \mid m} d_m \left(\sum_{d_n \mid n} d_n \right) && (4.7)\\
&= \left(\sum_{d_m \mid m} d_m \right) \left(\sum_{d_n \mid n} d_n \right) = \sigma(m)\sigma(n). && (4.8)
\end{aligned}$$

elaborate steps

□

again, proof embedded within a proof, separate

\begin{ROUGH DRAFT TERRITORY}

See how sigma and phi behave for prime powers to show lemma 1, by induction over the terms in the prime factorization, kinda.

Then show that the basel problem ∞ sum is the ∞ product: Page 230 (pdf: 242) of

[1]. Then solve **Problem.** by writing $\dfrac{\varphi(n)\sigma(n)}{n^2}$ in terms of primes:

$$\frac{\varphi(n)\sigma(n)}{n^2} = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i^{\alpha_i+1}}$$

(we get this by plugging Lemma 1)

To finish the problem, see the maximum of each term in this product, and the minimum of each term. Each term is at least $\dfrac{p_k^2 - 1}{p_k^2}$ and at most arbitrarily close to 1 but less than 1.

Then the global maximum is strictly below 1 (we only consider $n \geq 2^{\ddagger}$) Then the global minimum is strictly above $\prod_{p \in \mathbb{P}} \dfrac{p^2 - 1}{p^2} = \dfrac{6}{\pi^2}$ (Basel Problem)

So **Problem.** is done.

5 Questions

Next answer Question 1. Also rewrite it not using the interval because it's IB AA HL, not Topology 101.

The answer is: no, the bounds can't be uniformly improved. For upper bound this is trivial (n = 2^k for very big k)

For lower bound it's what convergence means. Now that I think about it, Question 1 isn't really worth doing since it's so fricking easy.

Question 2. The topmost 'dense line' is primes, (or prime powers? That would make the discussion more complete but also longer and not that much more impressive)

‡maybe consider $n = 1$ too but that's a matter of my taste & preference

Definition 2 (‘dense line’ should I come up with a better name :(). We define a subsequence b_n of the sequence a_n to be a ‘dense line’ if it

- increases
- approaches some value t
- $b_n = a_g(n)$ and $g(n) < cx \ln x$[§]

Actually, we get a dense line $b_n = a_{cp_n}$ for any positive integer d, and the line is denser the smaller c is (obviously). here p_n is the nth smallest prime. Though the issue is that if I define ‘dense lines’ like this, it’s hard to verify every dense line is generated by this description. In fact, I don’t even think that’s true... Maybe it is? But how do I define it better? This is the only formal definition I could come up with!

But that’s enough Knowledge about the dense lines that we can graph them! Is this a satisfying enough answer to Question 2?

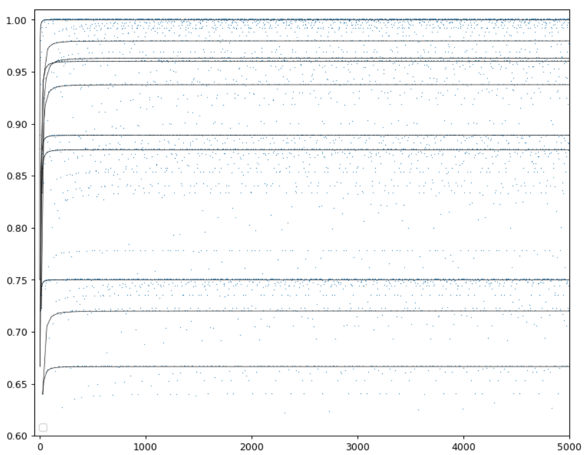

Figure 3: First 10 ‘dense lines’

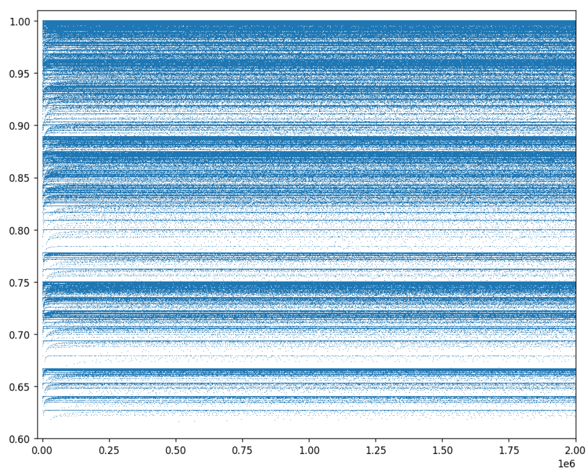

Figure 4: max from 5000 to 2,000,000

Now question 3. Look at figure 4. We see more blue ‘dense lines’ than previously, but also white regions with very few points. We are motivated to ask: Do dense lines

[§]Here $x \ln x$ is the asymptotic growth of primes (also the asymptotic growth of prime powers, cool! This condition says ‘b_n must be dense enough.’)

cover everything eventually? Will there be some strip with no points whatsoever? The answer is no, and we present an algorithm which shows this. It's really cool and relies on Nagura's bound:

Theorem 4. *For all $n \geq 25$ there is always a prime p with $n < p < 1.2n$*

And a bit of computer brute-force for $3 < n < 25$

and a tiny bit of brute-force by hand for $n = 2, n = 3$

After that though Nagura + induction can get the remaining cases. I think this is my coolest result, and I only recently come up with it

6 Concluding Remarks

I don't know what to say here. It's been fun investigating this, but seeing that the lines were just primes and constant multiples of primes is kinda anticlimactic for me; I like very hard problems and felt much more thrilled solving **Question 3.**

Also I need to write an introduction...

References

[1] Tom M. Apostol. *Introduction to Analytic Number Theory*. ~~Accessed: 2025-03-10.~~ New York: Springer-Verlag, 1976.

[2] Johan Wästlund. *Summing inverse squares by euclidean geometry*. https://www.math.chalmers.se/~wastlund/Cosmic.pdf. Accessed: 2025-03-10. Gothenburg, Sweden, 2010.