Subject: Mathematics

Investigating patterns in the product of Euler's totient function φ and the sum of divisors function σ

Research Question: What gives rise to the patterns in the graph of $f(n) = \sqrt{\varphi(n) \cdot \sigma(x)}$, and how does this relate to the Riemann zeta function?

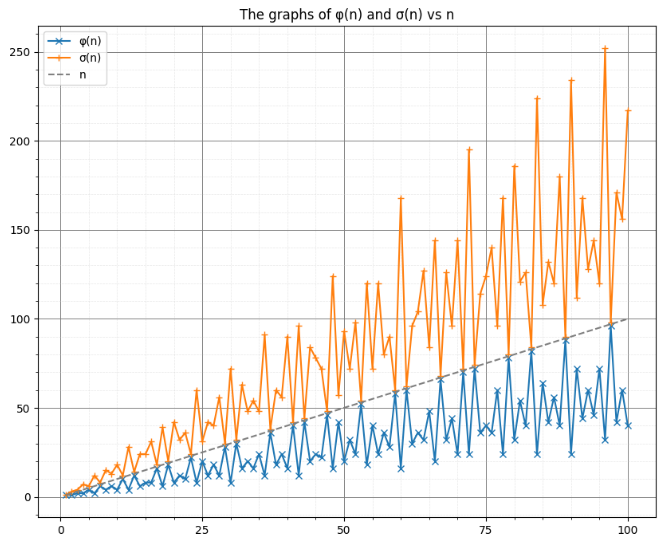
April 16, 2025

Word count: 1729

1 Presentation to the Class

$\varphi(n)$ is the "number of positive integers $\leq n$ and **coprime** to n." $\sigma(n)$ is the "sum of the positive divisors of n."

n	$\varphi(n)$	numbers	$\sigma(n)$	numbers
1	1	1	1 =	1
2	1	1	3 =	1+2
3	2	1,2	4 =	1+3
4	2	1,3	7 =	1+2+4
5	4	1,2,3,4	6 =	1+5
6	2	1,5	12 =	1+2+3+6
7	6	1,2,3,4,5,6	8 =	1+7
8	4	1,3,5,7	15 =	1+2+4+8
9	6	1,2,4,5,7,8	13 =	1+3+9
10	4	1,3,7,9	18 =	1+2+5+10



We call functions like φ and σ *"arithmetic functions"*, i.e. functions that express some property of the number.

Lemma 1.1 (Multiplicativity of φ and σ) *For any two* ***coprime*** *integers m, n, we have $\varphi(mn) = \varphi(m)\varphi(n)$ and $\sigma(mn) = \sigma(m)\sigma(n)$.*

This allows us to write closed formulas of $\varphi(n)$ and $\sigma(n)$ in terms of the prime factorization of n.

Corollary 1.1 *$f(x) = \varphi(x) \cdot \sigma(x)$ is multiplicative.*

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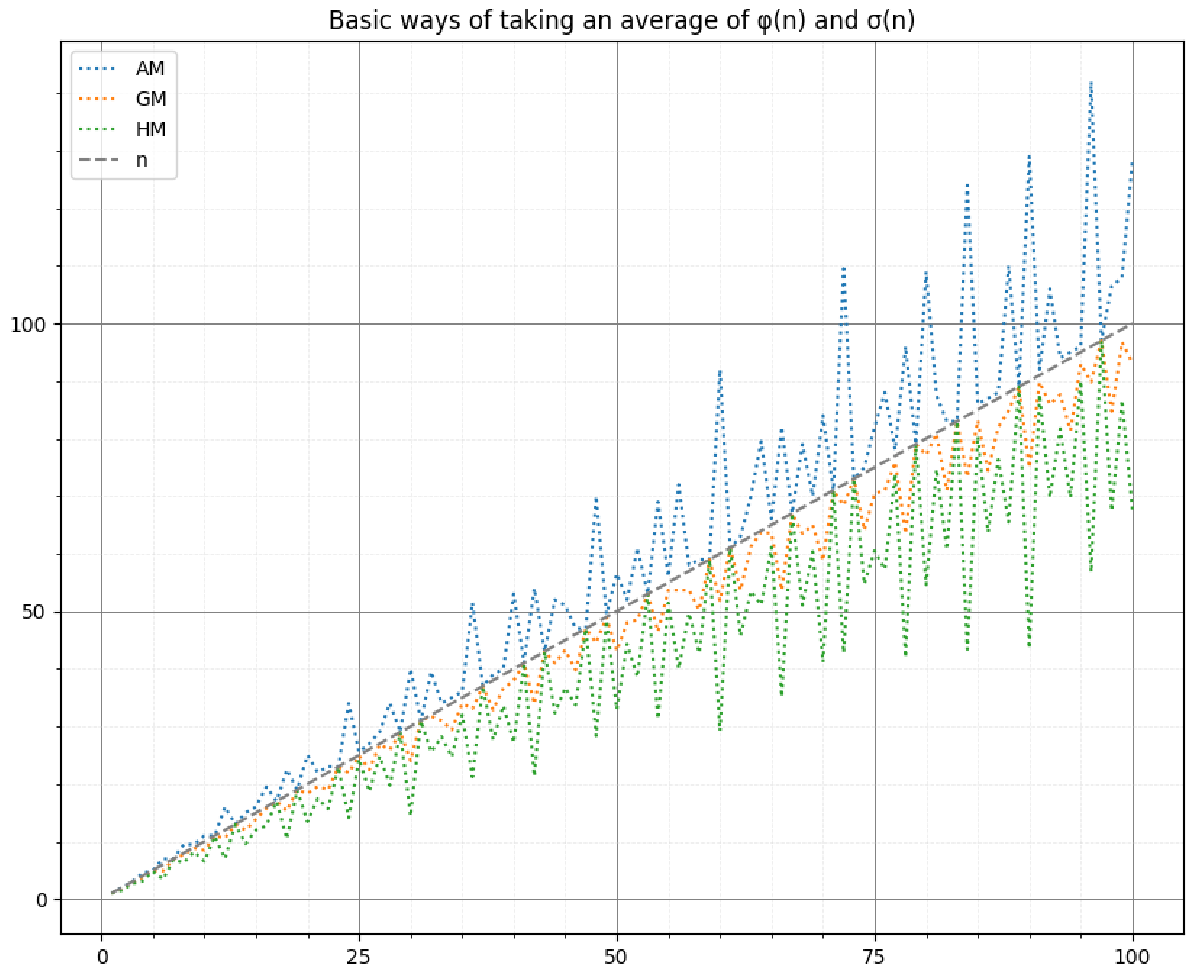
We wish to find some way of taking an average that sends the pairs of points to the line $y = n$, i.e. discover a function that reduces the amount of chaos and erraticity in this graph.

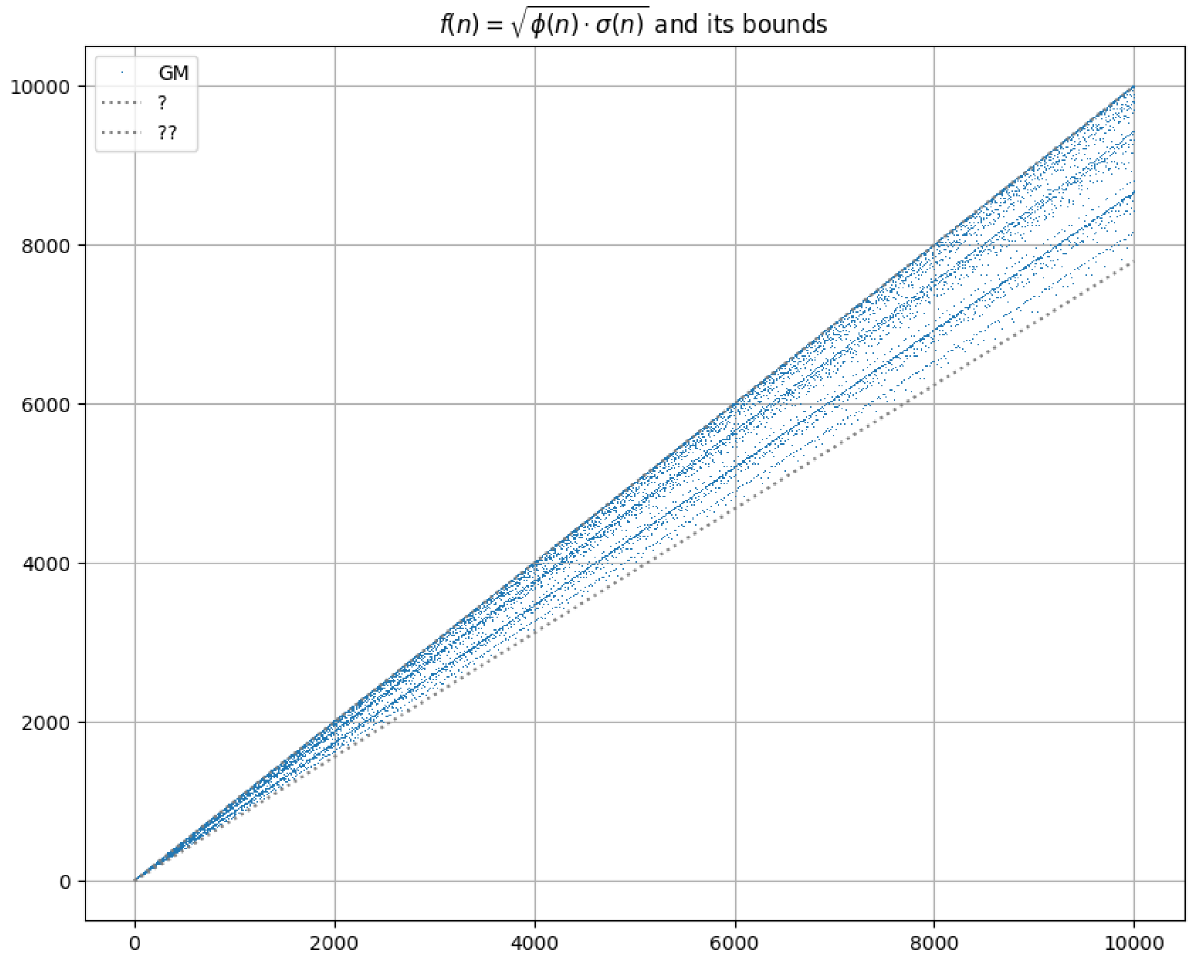
Initial guesses for ways to take an average:

$$\textbf{AM}: \ \frac{a+b}{2} \qquad \textbf{GM}: \ \sqrt{ab} \qquad \textbf{HM}: \ \frac{2}{\frac{1}{a}+\frac{1}{b}}$$



It's also possible to take some weighted version of these, to make the result closer to the target. Investigating that may be a part of my Research, if the word limit allows for that.

I hope the conclusion is that the geometric mean is the best. Then I will proceed to graph the function $f(x) = \sqrt{\varphi(n) \cdot \sigma(n)}$ in more detail and notice some lines with points packed more densely.



Since it looks linear, we are motivated to graph the function $g(n) = \dfrac{\sqrt{\varphi(n) \cdot \sigma(n)}}{n}$ in more detail, to get a clearer view of what the lines are.

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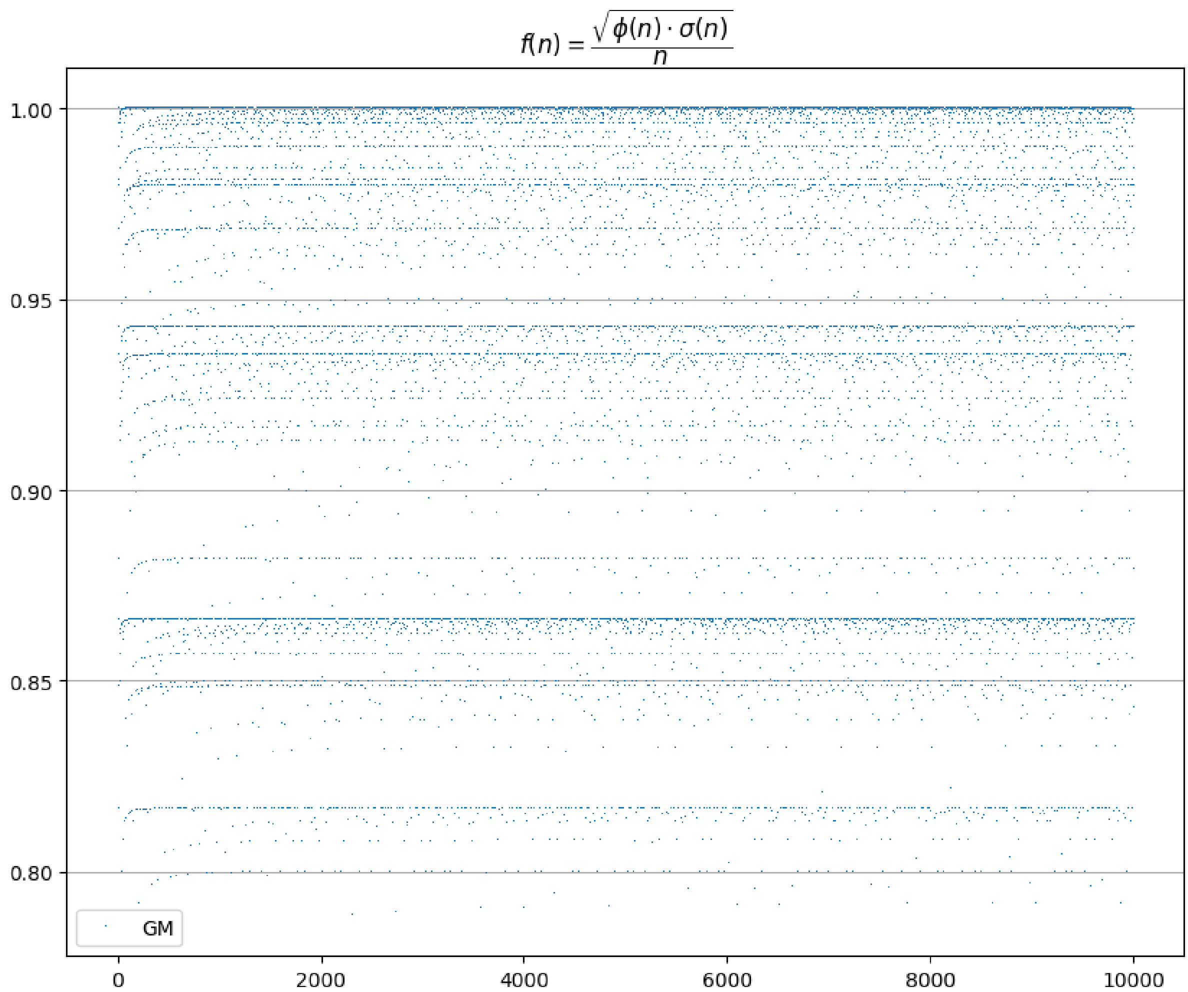

In the rest of the paper, I will compute the lower and upper bounds, show that they're asymptotically strict, and lastly explain why the patterns (denser lines) occur.

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2 Abstract

What gives rise to the patterns in the graph of $f(n) = \sqrt{\varphi(n) \cdot \sigma(x)}$, and how does this relate to the Riemann zeta function?

Contents

What gives rise to the patterns in the graph of $f(n) = \sqrt{\varphi(n) \cdot \sigma(x)}$, and how does this relate to the Riemann zeta function?

3 Introduction

Broadly speaking, number theory is the branch of mathematics concerned with the properties of integers. In a school setting, this often includes topics such as divisibility rules, prime factorization, and, most importantly, the **Fundamental Theorem of Arithmetic**.

Theorem 3.1 (Fundamental Theorem of Arithmetic) *Every integer $n > 1$ can be represented as a product of prime factors in precisely one way, up to the order of the factors.*

This theorem asserts that for any integer n, there exists a unique set of distinct prime numbers $p_1, p_2, ..., p_k$, along with their respective exponents $\alpha_1, \alpha_2, ..., \alpha_k$, such that

$$n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}.$$

This form of writing numbers as a product of primes is the key to the proofs delivered in this paper. For example, we can construct all the factors (positive divisors) of a number when given its prime factorization. Namely, they are precisely the numbers of the form

$$d = p_1^{\beta_1} p_2^{\beta_2} ... p_k^{\beta_k},$$

where $\beta_i \leq \alpha_i$ for all $1 \leq i \leq k$.

4 The zeta function

hello [1] hello!!

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5 An Explanation for the Patterns

6 Appendices

What gives rise to the patterns in the graph of $f(n) = \sqrt{\varphi(n) \cdot \sigma(x)}$, and how does this relate to the Riemann zeta function?

References

[1] Johan Wästlund. *Summing inverse squares by euclidean geometry*. https://www.math.chalmers.se/~wastlund/Cosmic.pdf. Accessed: 2025-03-10. Gothenburg, Sweden, 2010.

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