Subject: Mathematics

Investigating patterns in the product of Euler's totient function φ and the sum of divisors function σ

Research Question: What gives rise to the patterns in the graph of $f(n) = \varphi(n)\sigma(n) \cdot n^{-2}$, and how does this relate to the Riemann zeta function?

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Contents

1 Introduction

Write this when most of the essay is finished

2 Prerequisite Knowledge, Notation, and Definitions

The concepts investigated in this paper are strictly elementary number theory, which means we're mostly working with positive integers. Thus, for simplicity, you can assume all variables mentioned in this paper are positive integers, unless otherwise stated.

Theorem 2.1 (Fundamental Theorem of Arithmetic)**.** *Every positive integer has an unique prime factorization.*

This theorem is crucial in our ensuing discussion, because it allows us to express any number n uniquely in terms of its k prime divisors $p_1, p_2, ..., p_k$, along with their respective exponents $\alpha_1, \alpha_2, ..., \alpha_k$, as the product $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k} = n$.[1]

Some more complicated sums and products are discussed in this paper, which require set-builder-like[2] notation to impose restrictions on the numbers we're summing over. This notation will allow us to cleanly write formulas for the two functions that will be extensively discussed:

Definition 2.1 (Sum of Divisors Function σ and Euler's Totient Function φ)**.**

$$\sigma(n) = \sum_{d \in \mathbb{Z}^+,\ d|n} d$$

$$\varphi(n) = \sum_{\substack{m \in \mathbb{Z}^+, m \leq n \\ \gcd(m,n)=1}} 1$$

The comma inside the subscripts simply means 'and', so verbosely we would say that $\sigma(n)$ takes the sum of all numbers d which divide n, and $\varphi(n)$ counts the number of numbers $m \leq n$ which are coprime to n.

[1] for $n = 1$ simply set $k = 0$

[2] see https://en.wikipedia.org/wiki/Set-builder_notation

3 The Problem of Study

The

*Prove these statements through multiplicativity and primes.

*Whether it's before or after proving the lemma, I shall shortly hereafter motivate the product of φ and σ: the $p_i - 1$'s cancel out and we get

$$\varphi(n) * \sigma(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i^{\alpha_i + 1} - 1)$$

4 The zeta function

hello [**basel**] hello!!

5 An Explanation for the Patterns

6 Appendix