Session 1 Difficult Problems - Solutions

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January 12, 2025

1. Suppose we have a system of three linear equations in three variables such that a times the first equation plus b times the second equation equals the third equation, where and are a and b are nonzero constants.

   a) Must there be constants c and d such that c times the second equation plus d times the third equation equals the first equation?

   b) If so, find and in terms of a and b.

   Solution 1. Yes, such c and d exist. This will be proven as follows:
   Let E_1, E_2, and E_3 denote the first, second, and third equations, respectively. Then from the given information, we have $E_3 = aE_1 + bE_2$. Solving for E_1, we get

   $$E_1 = \frac{E_3 - bE_2}{a} = \frac{-b}{a}E_2 + \frac{1}{a}E_3$$

   Then, clearly, $c = \dfrac{-b}{a}$ and $d = \dfrac{1}{a}$ satisfy the condition in part (a).

2. Find the value of $a^3b^7c^{14}$ given that $a^3b^2c = 108$ and $a^2b^3c^5 = 240$.

   Solution 2. We will be finished if we find values for constants x and y such that $(a^3b^2c)^x(a^2b^3c^5)^y = a^3b^7c^{14}$, since that would equal 108^x240^y.

   Expanding the brackets and simplifying, we get

   $$(a^3b^2c)^x(a^2b^3c^5)^y = a^{3x+2y}b^{2x+3y}c^{x+5y} = a^3b^7c^{14},$$

   which corresponds to the following linear system of equations, where each equation represents the relation of the exponent of each variable a, b, c.

   $$\begin{cases} 3x + 2y = 3 \\ 2x + 3y = 3 \\ x + 5y = 14 \end{cases}$$

   Solving this, we easily obtain $(x, y) = (-1, 3)$, so $a^3b^7c^{14} = 108^{-1}240^3 = \dfrac{240^3}{108} = 12800$.

3. Find the value of $a + b + c$ given that

   $$2a - b + 5c = 13$$
   $$2a + 3b + c = 75$$

Solution 3. We want to find x and y such that x multiplied by the left-hand side of the first equation plus y multiplied by the left-hand side of the second equation is equal to $a+b+c$. Formally,

$$\begin{aligned} a+b+c &= x(2a-b+5c)+y(2a+3b+c) \\ &= a(2x+2y)+b(-x+3y)+c(5x+y) \end{aligned}$$

Using the same technique as **Solution 2**, we get the following system of equations:

$$\begin{cases} 2x+2y=1 \\ -x+3y=1 \\ 5x+y=1, \end{cases}$$

which yields $(x,y)=(\frac{1}{8},\frac{3}{8})$.
Hence, $a+b+c=\frac{1}{8}(2a-b+5c)+\frac{3}{8}(2a+3b+c)=\frac{1}{8}\cdot 13+\frac{3}{8}\cdot 75=\frac{119}{4}$.

4. Find all solutions to the system of equations $a-2b=-4, a^2-2b^2=-14$.

Solution 4. Substituting $a=-4+2b$ from the first equation into the second one, we get

$$\begin{aligned} a^2-2b^2 &= -14 \\ (-4+2b)^2-2b^2=4b^2-16b+16-2b^2=2b^2-16b+16 &= -14, \text{ so} \\ 2b^2-16b+30 &= 0 \\ b^2-8b+15 &= 0 \end{aligned}$$

Here you can use the quadratic formula, but the upcoming lesson we will teach you how to factor quadratics, i.e. write

$$b^2-8b+15=(b-3)(b-5)=0 \implies b=3 \text{ or } b=5.$$

Substituting these values of b to the first equation yields $a=2$ and $a=6$, so the solutions for this system of equations are $(a,b)=(2,3)$ and $(6,5)$.

5. A tennis player computes her "win ratio" by dividing the number of matches she has won by the total number of matches she has played. At the start of a weekend, her win ratio is exactly 0.500. During the weekend she plays four matches, winning three and losing one. At the end of the weekend her win ratio is greater than 0.503. What is the largest number of matches that she could have won before the weekend began?

Solution 5. Let n be the number of games that the tennis player has won at the start of the weekend. Since her win ratio is exactly 0.5, she has played exactly $2n$ games. During the weekend, she won three games and lost one game, so her win ratio is now

$$\begin{aligned}
\frac{n+3}{2n+4} &> 0.503. \text{ Solving for n, we get} \\
n+3 &> (2n+4)0.503 = 1.006n + 2.012 \\
0.988 &> 0.006n \\
n &< \frac{0.988}{0.006} = \frac{988}{6} = 164 + \frac{2}{3}.
\end{aligned}$$

Since n is an integer, the largest possible value for n is 164.

6. A right triangle has both a perimeter and an area of 30. Find the side lengths of the triangle.

Solution 6. Let the legs be of lengths a and b. The given information produces the system of equations

$$\begin{cases} \frac{ab}{2} = 30 \\ a + b + \sqrt{a^2+b^2} = 30. \end{cases}$$

Manipulating the second equation,

$$\begin{aligned}
a + b + \sqrt{a^2+b^2} &= 30 \\
\sqrt{a^2+b^2} &= 30 - (a+b) \\
a^2 + b^2 &= 30^2 - 2\cdot 30(a+b) + (a+b)^2 = 900 - 60(a+b) + a^2 + 2ab + b^2 \\
60(a+b) &= 900 + 2ab
\end{aligned}$$

Since the first equation gives us $\frac{ab}{2} = 30 \implies 2ab = 120$, we can substitute that to receive $60(a+b) = 900 + 120 = 1020 \implies a + b = 17$. Substituting $b = 17 - a$ from this into the first equation, we get

$$\begin{aligned}
\frac{a(17-a)}{2} &= 30 \\
a(17-a) &= 60 \\
a^2 - 17a + 60 &= 0 \\
(a-5)(a-12) &= 0,
\end{aligned}$$

so $a = 5$ and $a = 12$ are solutions, which correspond to $b = 12$ and $b = 5$ respectively. Thus the hypotenuse is $\sqrt{5^2+12^2} = \sqrt{25+144} = \sqrt{169} = 13$, so our triangle's side lengths are $5, 12$ and 13.

7. Find all x such that $-4 < \dfrac{1}{x} < 3$

Solution 7. Clearly, $x \neq 0$. We cannot multiply the whole inequality by x, because the direction of the inequality must be reversed for negative x. Thus we will consider the chain of inequalities as two separate inequalities,

$$\begin{cases} -4 < \dfrac{1}{x} \\ \dfrac{1}{x} < 3. \end{cases}$$

When x is negative, the second equation is always true. Then we may multiply both sides of the first equation by x and reverse the direction, to get

$$-4x > 1 \implies x < -\frac{1}{4}.$$

When x is positive, the first equation is always true, and the second one yields

$$1 < 3x \implies x > \frac{1}{3}.$$

Thus, all values of x to satisfy the original inequality are all values such that either $x > \frac{1}{3}$ or $x < -\frac{1}{4}$.

8. If $\dfrac{x^2y}{z} = 24$ and $\dfrac{y^4z}{x} = 30$, find the value of $\dfrac{x^8}{(yz)^5}$.

Solution 8. The technique is the same as in **Solution 2**, after writing $\dfrac{x^2y}{z} = x^2y^1z^{-1}$ and $\dfrac{y^4z}{x} = x^{-1}y^4z^1$.

The answer should be equal to $\dfrac{384}{25}$.

9. ...

Solution 9. Sorry for not including the figure. Just pretend problem 9 doesn't exist.

10. Consider the following system of linear equations. Characterize the values of k such that this system has no solution, infinitely many solutions, and precisely one solution.

$$\begin{cases} kx + y + z = k \\ x + ky + z = k \\ x + y + kz = k \end{cases}$$

Solution 10. Adding the three equations yields $(k+2)x + (k+2)y + (k+2)z = 3k$, so $x + y + z = \frac{3k}{k+2}$.
Subtracting this from the first equation, we get

$$kx - x = k - \frac{3k}{k+2}(k-1)x \qquad = \frac{k^2 + 2k - 3k}{k+2}x(k-1) = \frac{k^2 - k}{(k+2)} = \frac{k(k-1)}{(k+2)}.$$

If $k = 1$, $k - 1 = 0$, both sides will be equal to 0, so there will be infinitely many solutions.
Else $k - 1$ is nonzero, so we may divide both sides by it to obtain

$$x = \frac{k}{k+2}.$$

With the same steps (by symmetry), we will arrive to $y = \dfrac{k}{k+2}$ and $z = \dfrac{k}{k+2}$.
If $k = -2$, $k+2 = 0$, we will be dividing by zero, impossible, which means there are no solutions.
For all other values of k, there will be precisely one solution, namely $x = y = z = \dfrac{k}{k+2}$.

11. Let a, b, c be nonzero constants. Solve the system

$$\begin{cases} ay + bx = c \\ az + cx = b \\ bz + cy = a \end{cases}$$

for (x, y, z) in terms of a, b, and c.

Solution 11. This should be mechanically easy: In the first equation, solve for x in terms of y, a, b, c and substitute the result into the second one, where you can solve for z in terms in terms of y, a, b, c.

Then substitute that result into the third equation, to solve for y in terms of a, b, and c.
Since we have expressions for x and z in terms of y, a, b, c, we can just substitute the previous result in the place of y in these expressions, and after simplifying we may express x and z in terms of a, b, and c, so we are done.

The result should be:

$$(x, y, z) = \left(\frac{b^2 + c^2 - a^2}{2bc}, \frac{a^2 + c^2 - b^2}{2ac}, \frac{a^2 + b^2 - c^2}{2ab} \right).$$

12. The binomial coefficients can be arranged in rows to form Pascal's Triangle (where row n is $\binom{n}{0}, \binom{n}{1}, ..., \binom{n}{n}$). In which row of Pascal's Triangle do three consecutive entries occur that are in the ratio $3:4:5$?

Solution 12. Our goal is to find a positive integer n such that for some integer k, $0 \leq k \leq n-2$, we have

$$\binom{n}{k} : \binom{n}{k+1} : \binom{n}{k+2} = 3:4:5.$$

We can obtain linear equations in n and k by taking the ratios of the binomial coefficients as follows:

$$\frac{3}{4} = \frac{\binom{n}{k}}{\binom{n}{k+1}} = \frac{\frac{n!}{k!(n-k)!}}{\frac{n!}{(k+1)!(n-k-1)!}} = \frac{(k+1)!(n-k-1)!}{k!(n-k)!} = \frac{k+1}{n-k},$$

and

$$\frac{4}{5} = \frac{\binom{n}{k+1}}{\binom{n}{k+2}} = \frac{\frac{n!}{(k+1)!(n-k-1)!}}{\frac{n!}{(k+2)!(n-k-2)!}} = \frac{(k+2)!(n-k-2)!}{(k+1)!(n-k-1)!} = \frac{k+2}{n-k-1}.$$

From $\frac{3}{4} = \frac{k+1}{n-k}$, we have $4k+4 = 3n-3k$, and from $\frac{4}{5} = \frac{k+2}{n-k-1}$, we have $5k+10 = 4n-4k-4$. Thus, we obtain the system of equations

$$\begin{cases} 7k = 3n+4 \\ 9k = 4n-14. \end{cases}$$

Solving for n yields $n = 62$, so the desired row is row 62.