Subject: Mathematics

Investigating patterns in the product of the sum of divisors function σ and Euler's totient function φ

Research Question: What gives rise to the patterns in the graph of

$$f(n) = n^{-2}\sigma(n)\varphi(n)?$$

Word count: 1729

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1 Introduction

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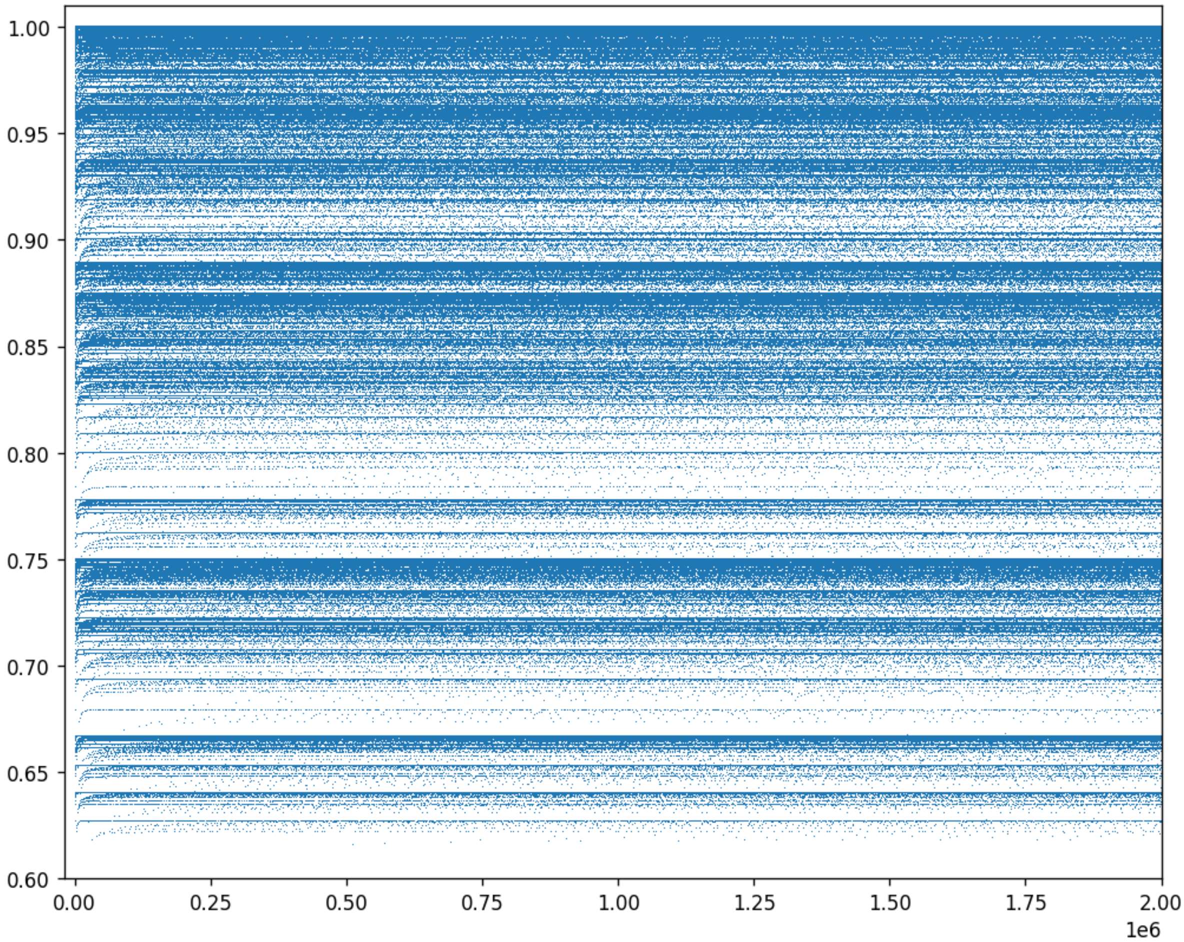

Figure 1: what?

2 Prerequisite Knowledge, Notation, and Definitions

The inarguably most important theorem in number theory is the following:

Theorem 1 (Fundamental Theorem of Arithmetic)**.** *Every positive integer has a unique prime factorization.*

This theorem is crucial in our ensuing discussion, because it allows us to express any integer $n \geq 2$ uniquely in terms of its k prime divisors $p_1, p_2, ..., p_k$, along with the positive integers $\alpha_1, \alpha_2, ..., \alpha_k$ which indicate the number of times that n is divisible by the corresponding prime factor. With this, we obtain the familiar form

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k} = \prod_{i=1}^{k} p_i^{\alpha_i} \tag{2.1}$$

In the special case $n = 1$, there are no prime factors, and by convention

$$\textit{the empty product equals 1.} \tag{2.2}$$

The reader should be comfortable with Π-notation for sequential products, which is similar to Σ but for multiplication.

2.1 Introduction to Sets

A *set* is an *unordered* collection of elements. The discussion in this paper only involves sets of numbers, such as $\mathbb{Z}$, used to denote the set of integers, and $\mathbb{R}$, used to denote the set of real numbers.

Here *unordered* means that for example, the set $A = \{1, 2\}$ is the same as the set $B = \{2, 1\}$. When two sets A, B are the same, we write $A = B$.

Notation. Let A, B, and S be sets.

- The notation $x \in S$ means that x is an element of the set S.
- For a finite set S, the symbol $|S|$ denotes the number of elements contained in S.

2.2 Set-builder Notation

Set-builder notation is a way for mathematicians to define sets symbolically rather than verbosely. Basic set-builder notation is used frequently throughout this paper.

Notation. The set S of elements in the set A that fulfill a certain condition is defined, i.e. 'built', with the following notation:

$$S = \{a \in A \ : \ \text{conditions for } a\}$$

In this case, S is a subset of A, because every element of S is an element of A. This is denoted by $S \subseteq A$. The reader should understand the following examples before proceeding:

Example 1. Build the set of positive integers, denoted $\mathbb{Z}^+$, by choosing the integers which are positive:

$$\mathbb{Z}^+ = \{x \in \mathbb{Z} \ : \ x > 0\} \tag{2.3}$$

Example 2. Build the set of positive divisors of an integer $n \neq 0$, denoted $\mathcal{D}_n$, by choosing the positive integers that divide n:

$$\mathcal{D}_n = \{x \in \mathbb{Z}^+ \ : \ x \mid n\} \tag{2.4}$$

The notation $a \mid b$ means a divides b.

Example 3. Recall that prime numbers are positive integers greater than 1 whose only positive divisors are 1 and itself. Build the set of primes, denoted $\mathbb{P}$, using the set of positive divisors defined in (2.4):

$$\mathbb{P} = \left\{x \in \mathbb{Z}^+ \ : \ x > 1,\ \mathcal{D}_x = \{1,\ x\}\right\} \tag{2.5}$$

In this case, multiple conditions on x are necessary, and notationally we separate them with a comma.

More complicated sequential sums and products also require set-builder-like notation in the subscripts, used to impose restrictions on what to take the sum or product over. (2.6) in **Definition 1** below serves as an example of this.

2.3 Key Definitions

The following functions, defined for all positive integers n, are the central focus of this study.

Definition 1 (Sum of Divisors Function σ)**.** Define $\sigma(n)$ as the sum of all positive divisors of n.

Mimicking set-builder notation, we can write

$$\sigma(n) = \sum_{d \in \mathcal{D}_n} d \tag{2.6}$$

Example 4. Compute $\sigma(12)$: Since the positive divisors of 12 are $\mathcal{D}_{12} = \{1, 2, 3, 4, 6, 12\}$, taking their sum yields $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$

Definition 2 (Euler's Totient Function φ)**.** Define $\varphi(n)$ as the number of positive integers less than or equal to n and *coprime* to n. Formally, define

$$\varphi(n) = \left|\{x \in \mathbb{Z}^+ \ : \ x \leq n, \ \gcd(x, n) = 1\}\right| \tag{2.7}$$

Terminology. We say that 'a is coprime to b', or 'a and b are coprime', if $\gcd(a, b) = 1$.

Example 5. Compute $\varphi(12)$:

The positive integers less than or equal to 12 and coprime to 12 are 1, 5, 7, 11. Since there are 4 such numbers, $\varphi(12) = 4$.

3 The Problem of Study

This investigation was inspired by the following problem from the undergraduate textbook *Introduction to Analytic Number Theory* [2].

Problem 1. Show that $\frac{6}{\pi^2} < \sigma(n)\varphi(n)n^{-2} < 1$ for all integers n greater than 1.

For simplicity, $f(n)$ will be used to denote $\sigma(n)\varphi(n)n^{-2}$ from this point onwards.

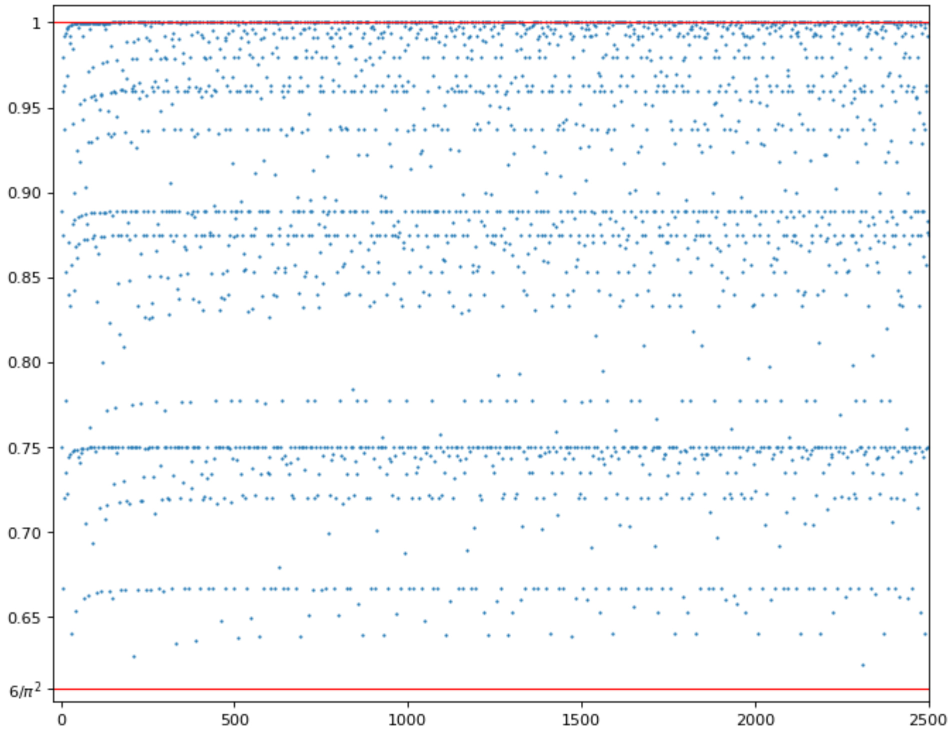

Figure 2: Graph for **Problem 1**

Figure 2 visualizes the scenario by plotting points $(n, f(n))$ (in blue) and the lower– and upper bounds (in red) on the Cartesian plane. Then, the task in **Problem 1** is to show that all blue points fall in between the red lines.

The research question of this investigation, regarding 'patterns' in the graph can now be put concretely. In addition to solving **Problem 1**, this paper will develop the necessary tools to answer the following questions.

Question 1. *Why are 'dense lines' formed?*

This question regards the horizontal lines that many of the points seem to fall in, at $y = 1,\ y = 0.75$, and $y = 0.89$ for example. This is one of the first questions arising upon seeing **Figure 2**, since the orderly patterns strike as unexpected in such a scattered and seemingly random distribution. After addressing **??**, the next natural question is,

Question 2. *Where can 'dense lines' form?*

This is also a natural thing to ask; we see many dense lines between 0.95 and 1, and much fewer below 0.70.

4 Background for Problem 1

To approach **Problem 1**, we must improve our understanding on the functions σ and φ. The following product forms for these functions in terms of the prime factorization of n shall be derived.

Lemma 1. *For any positive integer n with prime factorization $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ holds*

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1} \tag{4.1}$$

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i-1}(p_i - 1) \tag{4.2}$$

4.1 Deriving the Product Form of σ

To do this, we first show that σ is *multiplicative*, which means

$$\sigma(mn) = \sigma(m)\sigma(n) \tag{4.3}$$

holds for all coprime positive integers m, n. The following lemma is used to show *multiplicativity.*

Lemma 2. *For coprime positive integers m, n, each element d in the set $\mathcal{D}_{mn}$ is a product of a unique pair of elements in $\mathcal{D}_m$ and $\mathcal{D}_n$.*

Conversely, each pair of elements in $\mathcal{D}_m$ and $\mathcal{D}_n$ has a product which is a unique element in $\mathcal{D}_{mn}$.

Example 6. Let $m = 9,\ n = 14$. The divisors of 9 are $\mathcal{D}_9 = \{1, 3, 9\}$ and of 14 are $\mathcal{D}_{14} = \{1, 2, 7, 14\}$. Every divisor of $mn = 126$ is a product of one divisor of 9 and one divisor of 14:

$$\begin{aligned} 1 &= 1 \cdot 1, & 2 &= 1 \cdot 2, & 7 &= 1 \cdot 7, & 14 &= 1 \cdot 14, \\ 3 &= 3 \cdot 1, & 6 &= 3 \cdot 2, & 21 &= 3 \cdot 7, & 42 &= 3 \cdot 14, \\ 9 &= 9 \cdot 1, & 18 &= 9 \cdot 2, & 63 &= 9 \cdot 7, & 126 &= 9 \cdot 14. \end{aligned}$$

Proof of Lemma 2

Let d be a positive divisor of mn, and write the prime factorization of d:

$$d = \prod_{i=1}^{k} p_i^{\alpha_i} \tag{4.4}$$

In this form, each term $p_i^{\alpha_i}$ either

1. divides m and is coprime to n, or
2. divides n and is coprime to m,

because d is a divisor of mn and m, n are coprime. This means we can write

$$d = d_m d_n \tag{4.5}$$

where d_m is the product of the terms which correspond to case 1, and d_n is the product of the terms which correspond to case 2. In this way, we guarantee that $d_m \mid m$ and $d_n \mid n$, or equivalently $d_m \in \mathcal{D}_m$ and $d_n \in \mathcal{D}_n$.

This shows that each element in $\mathcal{D}_{mn}$ is a product of some pair of elements $d_m \in \mathcal{D}_m$ and $d_n \in \mathcal{D}_n$. The process of separating the terms $p_i^{\alpha_i}$ between case 1 and case 2 does not involve choices, so the corresponding pair (d_m, d_n) is unique.

Conversely, every pair $d_m \in \mathcal{D}_m$, $d_n \in \mathcal{D}_n$ determines exactly one divisor of mn, namely their product $d_m d_n$. $\square$

Proving multiplicativity of σ

Using **Lemma 2**, *multiplicativity* can be established using **Definition 1**.

$$\begin{aligned}
\sigma(mn) &= \sum_{d \in \mathcal{D}_{mn}} d && (4.6)\\
&= \sum_{d_m \in \mathcal{D}_m} \sum_{d_n \in \mathcal{D}_n} d_m d_n && (4.7)\\
&= \sum_{d_m \in \mathcal{D}_m} d_m \left(\sum_{d_n \in \mathcal{D}_n} d_n \right) && (4.8)\\
&= \left(\sum_{d_m \in \mathcal{D}_m} d_m \right) \left(\sum_{d_n \in \mathcal{D}_n} d_n \right) = \sigma(m)\sigma(n). && (4.9)
\end{aligned}$$

(4.6) This is **Definition 1**.

(4.7) By **Lemma 2**, divisors of mn correspond uniquely to products $d_m d_n$ with $d_m \mid m$, $d_n \mid n$, so the sum may be rewritten as a double sum over such pairs.

(4.8) The inner and outer sums are independent, allowing the factors d_m and d_n to be separated.

(4.9) Since the inner sum depends only on n, it can be factored out of the outer sum.

Finishing the product form of σ

Multiplicativity lets us express $\sigma(n)$ in terms of the prime factorization $n = \prod_{i=1}^{k} p_i^{\alpha_i}$. Since $p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots, p_k^{\alpha_k}$ are all coprime, we may separate the terms $p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots, p_k^{\alpha_k}$ out one by one.

$$\begin{aligned}
\sigma(n) =& \sigma(p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}) && (4.10)\\
=& \sigma(p_1^{\alpha_1}) \cdot \sigma(p_2^{\alpha_2} \ldots p_k^{\alpha_k}) && (4.11)\\
=& \sigma(p_1^{\alpha_1}) \cdot \sigma(p_2^{\alpha_2}) \cdot \sigma(p_3^{\alpha_3} \ldots p_k^{\alpha_k}) && (4.12)\\
=& \ldots && (4.13)\\
=& \sigma(p_1^{\alpha_1}) \cdot \sigma(p_2^{\alpha_2}) \cdot \cdots \cdot \sigma(p_k^{\alpha_k}) && (4.14)\\
=& \prod_{i=1}^{k} \sigma(p_i^{\alpha_i}) && (4.15)
\end{aligned}$$

To obtain the desired form in (4.1), we only need to compute $\sigma(p^\alpha)$ when p is prime and α is any positive integer. This is easy since

$$\mathcal{D}_{p^\alpha} = \{1, p, p^2, \ldots, p^\alpha\} \tag{4.16}$$

which implies $\sigma(p^\alpha)$ is simply a geometric series

$$\sigma(p^\alpha) = 1 + p + p^2 + \cdots + p^\alpha = \frac{p^{\alpha+1} - 1}{p - 1} \tag{4.17}$$

Finally, substitute this into (4.15) to finish:

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1} \tag{4.1}$$

□

Example 7. Compute $\sigma(2025)$.

$$\begin{aligned} \sigma(2025) =& \sigma(3^4 \cdot 5^2) \\ =& \frac{3^5 - 1}{3 - 1} \frac{5^3 - 1}{5 - 1} \\ =& 121 \cdot 31 = 3751 \end{aligned}$$

4.2 Deriving the Product Form of φ

To do this, we first show that φ is *multiplicative*, which means

$$\varphi(mn) = \varphi(m)\varphi(n) \tag{4.18}$$

holds for all coprime positive integers m, n. The following lemma, which is a special case of the **Chinese Remainder Theorem**, is used to show *multiplicativity.*

Lemma 3. *For coprime positive integers m, n, each integer x with $0 \leq x < mn$ has a unique pair of remainders when divided by m and by n.*

Conversely, each pair of remainders is given by a unique integer within this range.

Example 8. Let $m = 3,\ n = 4$. Then $mn = 12$, so we list all integers x with $0 \leq x < 12$

together with their remainders when divided by 3 and by 4:

$$\begin{aligned}
&0 \to (0,0), \quad 3 \to (0,3), \quad 6 \to (0,2), \quad 9 \to (0,1),\\
&1 \to (1,1), \quad 4 \to (1,0), \quad 7 \to (1,3), \quad 10 \to (1,2),\\
&2 \to (2,2), \quad 5 \to (2,1), \quad 8 \to (2,0), \quad 11 \to (2,3).
\end{aligned}$$

Proof of Lemma 3

We first show, by contradiction, that each pair of remainders when dividing by m and by n is produced by at most one integer x with $0 \leq x < mn$. Let the remainders of x when divided by m and by n be x_m and x_n, respectively. Assume, for contradiction, that there exists another integer x', with $0 \leq x' < mn$, which also gives remainders x_m, x_n. Without loss of generality, assume $x' > x$.*

Since x and x' have the same remainders when divided by m, we have $m \mid x' - x$. Similarly, $n \mid x' - x$. Recall that m and n are coprime, these can be combined into $mn \mid x' - x$.

But $x' > x$ by assumption, so $x' - x$ is a positive multiple of mn, which implies

$$x' - x \geq mn \tag{4.19}$$

$$x' \geq mn + x \geq mn \tag{4.20}$$

contradicting with the assumption that $0 \leq x' \leq mn$. Therefore, each pair of remainders is be given by at most one integer x with $0 \leq x < mn$.

It is known that there are m possible remainders when dividing by m, and n possible

*This is allowed because the other case is symmetrical remainders when dividing by n. Therefore, there are mn total pairs of remainders. This is equal to the number of integers x that satisfy $0 \leq x < mn$.

Since each such x gives exactly one pair of remainders, and each pair of remainders is given by at most one such x, it is necessarily true that each pair of remainders is given by some such x. $\square$

But $\varphi(n)$ only concerns integers that are coprime to n. The following lemma is necessary.

Lemma 4. *Given coprime positive integers m, n and integer x satisfying $0 \leq x < mn$, let the remainders of x be x_m and x_n when divided by m and by n, respectively.*

Then x is coprime to mn if and only if x_m is coprime to m and x_n is coprime to n. Symbolically, the claim is

$$\gcd(x, mn) = 1 \iff \gcd(x_m, m) = \gcd(x_n, n) = 1 \tag{4.21}$$

Example 9. Look back at **Example 8**: The integers $1, 5, 7, 11$ are coprime to 12. In the corresponding pairs of remainders, the left remainder is coprime to 3, and the right remainder is coprime to 4.

Conversely, for the remaining integers $0, 2, 3, 4, 6, 8, 9, 10$, either the left remainder shares a prime factor with 3, or the right remainder shares a prime factor with 4.

Proof of Lemma 4

Consider the integer division of x by m and by n:

$$x = q_m m + x_m \tag{4.22}$$

$$x = q_n n + x_n \tag{4.23}$$

Here q_m and q_n are the quotients of division by m and by n, respectively.

If $\gcd(x, mn) > 1$, then there exists some prime p which divides x and divides either m or n. Without loss of generality, assume $p \mid m$. Then rearrange (4.22):

$$x - q_m m = x_m \tag{4.24}$$

Both terms in the left hand side are divisible by p, so x_m must be divisible by p as well. Hence p is a common divisor of x_m and m, implying that $\gcd(x_m, m) > 1$.

Conversely, again without loss of generality, assume that $\gcd(x_m, m) > 1$. Then there exists some prime p which divides both x_m and m. Since mn is a multiple of m, it must be divisible by p. Both terms in the right hand side of (4.22) are divisible by p, so x must be divisible by p as well. Hence p is a common divisor of x and mn, implying that $\gcd(x, mn) > 1$.

We have shown that $\gcd(x, mn) > 1$ holds exactly when at least one of $\gcd(x_m, m)$ and $\gcd(x_n, n)$ is greater than 1. Equivalently, $\gcd(x, mn) = 1$ holds if and only if $\gcd(x_m, m) = \gcd(x_n, n) = 1$. $\square$

Proving multiplicativity of φ

Recall that **Definition 2** defines $\varphi(n)$ as the number of positive integers $x \leq n$ which are coprime to n, while **Lemma 3** and **Lemma 4** are concerned with nonnegative integers $x < n$. To address this disparity, we simply note that for any integer $n > 1$, neither n nor 0 are coprime to n, so $\varphi(n)$ is equal to the number of nonnegative $x < n$ which are coprime to n.

Since *multiplicativity* is trivial for $m = 1$ or $n = 1$, it is only necessary to consider coprime integers m, n greater than 1.

Combining **Lemma 3** and **Lemma 4** shows that the total number of pairs of remainders (x_m, x_n) where $\gcd(x_m, m) = \gcd(x_n, n) = 1$ is equal to the number of integers x satisfying $0 \leq x < mn$ and $\gcd(x, mn) = 1$. Note that this is precisely the amended definition of $\varphi(mn)$.

Moreover, since the possible remainders when dividing by m are simply $0, 1, \ldots, m - 1$ and only those coprime to m are counted, the number of possible values for x_m is $\varphi(m)$, again using the amended definition. Similarly, the number of possible values for x_n is $\varphi(n)$.

This means the number of pairs (x_m, x_n) is $\varphi(m)\varphi(n)$. Therefore, $\varphi(mn) = \varphi(m)\varphi(n)$.

Finishing the product form of σ

Now, exactly like (4.10) – (4.15) in the derivation of the product form of σ, we deduce that for $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ holds

$$\varphi(n) = \prod_{i=1}^{k} \varphi(p_i^{\alpha_i}) \tag{4.25}$$

For any prime p and any positive integer α, we may compute $\varphi(p^\alpha)$ by counting the numbers from 1 to p^α and removing all $p^{\alpha-1}$ multiples of p:

$$\varphi(p^\alpha) = p^\alpha - p^{\alpha-1} = p^{\alpha-1}(p-1) \tag{4.26}$$

Finally, substitute this into (4.25) to finish:

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1) \tag{4.2}$$

$\square$

Example 10. Compute $\varphi(2025)$.

$$\begin{aligned}\varphi(2025) =& \varphi(3^4 \cdot 5^2)\\ =& 3^(4-1)(3-1) \cdot 5^(2-1)(5-1)\\ =& 54 \cdot 20 = 1080\end{aligned}$$

5 Solving Problem 1

'Why is pi here? And why is it squared?' – Grant Sanderson

The task is to prove $\frac{6}{\pi^2} < f(n) < 1$ for all integers $n > 1$. Now that both parts of **Lemma 1** have been derived, we can derive the following formula for $f(n) = \sigma(n)\varphi(n)n^{-2}$ in terms of the prime factorization of n.

Lemma 5. *For any positive integer n with prime factorization $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ holds*

$$f(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i^{\alpha_i+1}} \tag{5.1}$$

Proof of Lemma 5

Suppose that the integer $n > 1$ has prime factorization $n = \prod_{i=1}^{k} p_i^{\alpha_i}$. By (4.1) and (4.2),

$$\sigma(n)\varphi(n) = \left(\prod_{i=1}^{k} \frac{p_i^{\alpha_i+1}-1}{p_i-1}\right)\left(\prod_{i=1}^{k} p_i^{\alpha_i-1}(p_i-1)\right) \tag{5.2}$$

$$= \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1}-1}{\cancel{p_i-1}} p_i^{\alpha_i-1}(\cancel{p_i-1}) \tag{5.3}$$

$$= \prod_{i=1}^{k} (p_i^{\alpha_i+1}-1)p_i^{\alpha_i-1} \tag{5.4}$$

Additionally,

$$n^{-2} = \left(\prod_{i=1}^{k} p_i^{\alpha_i}\right)^{-2} \tag{5.5}$$

$$= \prod_{i=1}^{k} (p_i^{\alpha_i})^{-2} \tag{5.6}$$

$$= \prod_{i=1}^{k} p_i^{-2\alpha_i} \tag{5.7}$$

Combine (5.4) and (5.7) and simplify:

$$f(n) = \sigma(n)\varphi(n)n^{-2} = \left(\prod_{i=1}^{k} (p_i^{\alpha_i+1}-1)p_i^{\alpha_i-1}\right)\left(\prod_{i=1}^{k} p_i^{-2\alpha_i}\right) \tag{5.8}$$

$$= \prod_{i=1}^{k} (p_i^{\alpha_i+1}-1)p_i^{\alpha_i-1}p_i^{-2\alpha_i} \tag{5.9}$$

$$= \prod_{i=1}^{k} (p_i^{\alpha_i+1}-1)p_i^{\alpha_i-1-2\alpha_i} \tag{5.10}$$

$$= \prod_{i=1}^{k} (p_i^{\alpha_i+1}-1)p_i^{-\alpha_i-1} \tag{5.11}$$

$$= \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1}-1}{p_i^{\alpha_i+1}} \tag{5.12}$$

□

Example 11. Compute $f(2025)$.

Since $2025 = 3^4 \cdot 5^2$, we use (5.1) in **Lemma 5**

$$f(2025) = \frac{(3^{4+1}-1)}{3^{4+1}}\frac{(5^{2+1}-1)}{5^{2+1}} = \frac{242}{243} \cdot \frac{124}{125} = \frac{30008}{30375}$$

Verifying using the answers of **7** and **10**:

$$f(2025) = \sigma(2025)\varphi(2025)2025^{-2} = 3751 \cdot 1080 \cdot 2025^{-}2 = \frac{4051080}{4100625} = \frac{30008}{30375}$$

The value lies between $\frac{\pi^2}{6}$ and 1, as expected.

5.1 Upper bound

Actually, **Lemma 5** looks very promising in regards to solving **Problem 1**. In fact, the upper bound $\sigma(n)\varphi(n)n^{-2} < 1$ is now obvious: Every term in the product is strictly less than 1, so the whole product must be less than 1 whenever the product contains at least one term. That is, whenever $n > 1$.

5.2 Lower bound

The result $f(n) > \frac{6}{\pi^2}$ can be derived using the following lemmas:

Lemma 6. *For any positive integer n with prime factorization $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, we have*

$$f(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1}-1}{p_i^{\alpha_i+1}} \geq \prod_{i=1}^{k} \frac{p_i^2-1}{p_i^2} \tag{5.13}$$

with equality when $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 1$.

Lemma 7 (Basel Problem)**.** *The infinite product over all primes* $\displaystyle\prod_{p\ prime} \frac{p^2}{p^2-1}$ *is equal to the infinite sum* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ *and evaluates to* $\dfrac{\pi^2}{6}$

Proof of Lemma 6

We compare the left product and the right product of (5.13) term by term. For any i holds

$$\frac{p_i^{\alpha_i+1}-1}{p_i^{\alpha_i+1}} = 1 - \frac{1}{p_i^{\alpha_i+1}} \geq 1 - \frac{1}{p_i^{1+1}} = 1 - \frac{1}{p_i^2} = \frac{p_i^2-1}{p_i^2} \tag{5.14}$$

because α_i is a positive integer. Moreover, there is equality in (5.14) when $\alpha_i = 1$.

This means every term in the left product is greater or equal to the corresponding term in the right product, so the whole left product must be greater than or equal to the whole right product. Equality holds when all terms are equal, so $\alpha_i = 1$ for all i. $\square$

Example 12. Bound $f(2025)$ from below.

The prime factorization is $2025 = 3^4 5^2$. Therefore, by **Lemma 6**,

$$f(2025) \geq \frac{3^2-1}{3^2} \cdot \frac{5^2-1}{5^2} = \frac{8}{9} \cdot \frac{24}{25} = \frac{64}{75}$$

Proof of Lemma 7

The author will take for granted that the infinite sum $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, and a proof will be omitted. For the interested reader, there is a purely elementary geometrical proof[4] popularised by the internet mathematician 3Blue1Brown[3]. A much shorter proof involving integration[1] by the author of *Introduction to Analytic Number Theory* is also recommended.

This paper will only show that the infinite product over all primes and the infinite sum over positive integers are equal.

$$
\begin{aligned}
\prod_{p \text{ prime}} \frac{p^2}{p^2-1} &= \prod_{p \text{ prime}} \frac{1}{1-\frac{1}{p^2}} && (5.15)\\
&= \prod_{p \text{ prime}} \left(\sum_{i=0}^{\infty} \left(\frac{1}{p^2}\right)^i \right) && (5.16)\\
&= \prod_{p \text{ prime}} \left(1 + \frac{1}{p^2} + \frac{1}{p^4} + \dots\right) && (5.17)\\
&= \left(1 + \frac{1}{2^2} + \frac{1}{2^4} + \dots\right)\left(1 + \frac{1}{3^2} + \frac{1}{3^4} + \dots\right)\dots && (5.18)\\
&= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} \dots = \sum_{n=1}^{\infty} \frac{1}{n^2} && (5.19)
\end{aligned}
$$

(5.15) Divide the numerator and denominator by p^2.

(5.16) Expand $\frac{1}{1-1/p^2}$ into a geometric series.

(5.17) Write without Σ-notation.

(5.18) Write without Π-notation.

(5.19) Explained below

The step between (5.18) and (5.19) feels like a jump. But each term in the infinite sum in (5.19) is produced exactly once when the infinite product in (5.19) is expanded. For example, the term $\frac{1}{4^2}$ arises from selecting $\frac{1}{2^4}$ in the factor corresponding to $p = 2$ and 1 in all the remaining factors. Similarly, the term $\frac{1}{6^2}$ arises from selecting $\frac{1}{2^2}$ in the factor for $p = 2$, $\frac{1}{3^2}$ in the factor for $p = 3$, and 1 in all others. In general, each term $\frac{1}{n^2}$ is produced exactly once upon expanding the product, and it corresponds to the unique prime factorisation of n, by the Fundamental Theorem of Arithmetic.

Therefore,

$$\prod_{p \text{ prime}} \frac{p^2}{p^2 - 1} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{5.20}$$

Finishing the lower bound

By **Lemma 6**, for $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ holds

$$f(n) \geq \prod_{i=1}^{k} \frac{p_i^2 - 1}{p_i^2} \tag{5.21}$$

Each factor $\dfrac{p_i^2 - 1}{p_i^2}$ is less than 1, so adding more such factors (i.e. extending the product to more primes) strictly decreases the value. Hence

$$\prod_{i=1}^{k} \frac{p_i^2 - 1}{p_i^2} > \prod_{p \text{ prime}} \frac{p^2 - 1}{p^2} = \left(\prod_{p \text{ prime}} \frac{p^2}{p^2 - 1} \right)^{-1} \tag{5.22}$$

The equality is strict because the left product has finitely many terms, By **Lemma 7** the product on the right equals $(\frac{\pi^2}{6})^{-1} = \frac{6}{\pi^2}$. Therefore,

$$f(n) > \frac{6}{\pi^2}. \tag{5.23}$$

$\square$

6 Answering Question 1.

Next we turn our attention from the bounds of $f(n)$ to the patterns formed in its graph.

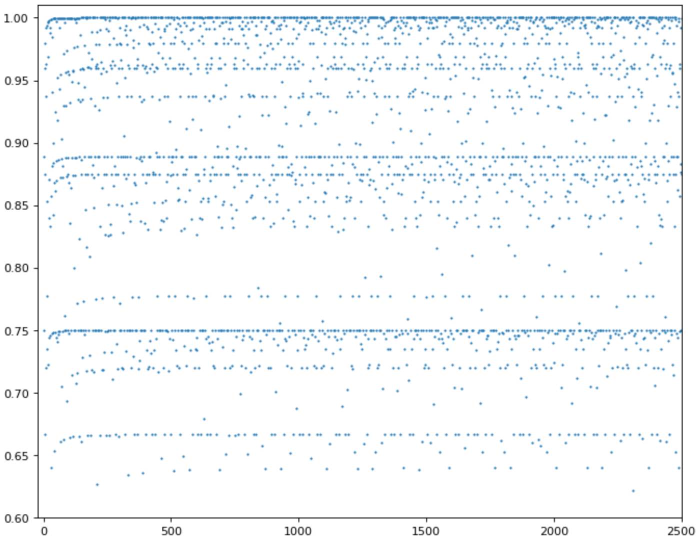

Figure 3: Clean graph of $(n, f(n))$

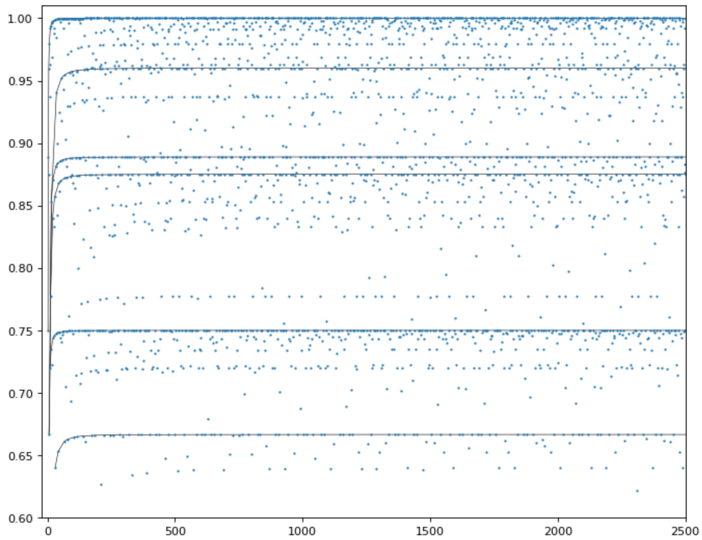

Figure 4: First six 'dense lines'

Draw **Figure 2** again, without the red lines indicating the bounds (**Figure 3**) and highlight the six densest lines (**Figure 4**).

From **Lemma 5** it can be seen that if p is prime, then

$$f(p) = \frac{p^2 - 1}{p^2} \tag{6.1}$$

Since p^2 is generally much larger than 1, $f(p)$ will often be very close to 1. We might guess that the points which form the 'dense line' at $y = 1$ are points of the form $(p, f(p))$, where p is prime. In **Figure 5** below, points of this form are highlighted; this seems to confirm our guess on why 'dense lines' are formed. Rather, seeing this graph lets us define 'dense lines' in a useful way, something that must be done before further discussion.

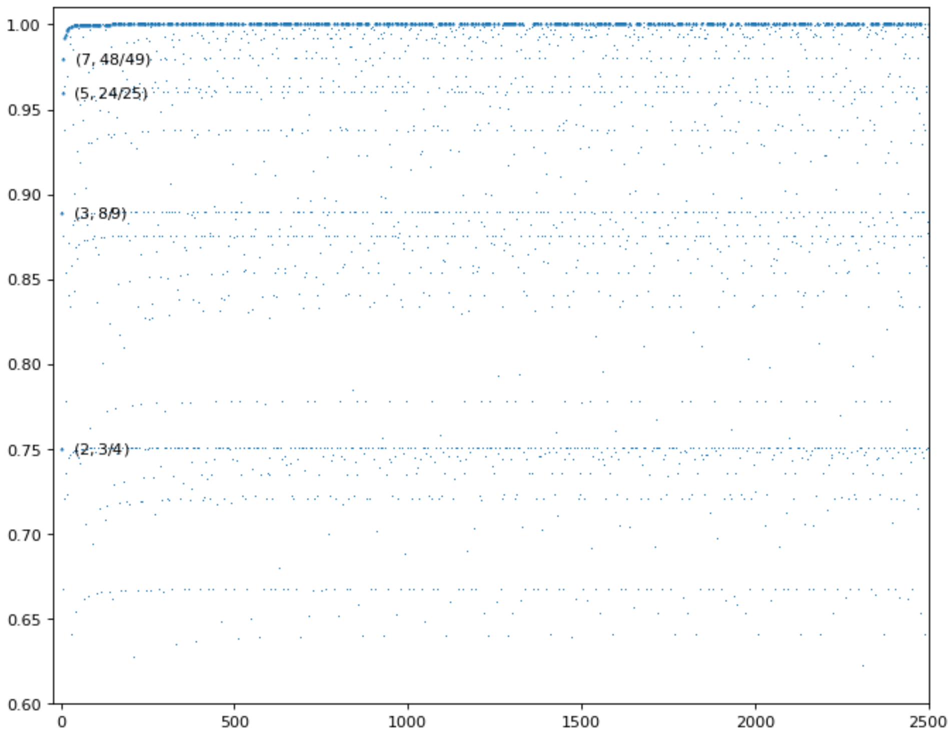

Figure 5: Highlight the points $(p, f(p))$

Definition 3. Let $p_1, p_2, \ldots$ be the sequence of prime numbers in ascending order. We say that an infinite sequence of numbers $a_1, a_2, \ldots$ form a 'dense line' if there exists positive real constants M, c, k such that for all terms a_i of the sequence holds the following:

$$\frac{a_i}{p_i} \leq M \tag{6.2}$$

$$|f(a_i) - k| \leq \frac{c}{a_i^2} \tag{6.3}$$

Explaining the Definition of Dense Lines

Condition (6.2) requires that the sequence (a_i) does not grow significantly faster than the sequence of prime numbers (p_i). In particular, it ensures that the ratio $\frac{a_i}{p_i}$ remains bounded by some constant M. If (a_i) grows too rapidly, no such M can exist, and the sequence cannot be considered dense with respect to the primes.

Example 13. Let $a_i = p_i^2$, where p_i denotes the i-th prime number. Then

$$\frac{a_i}{p_i} = p_i$$

which is unbounded, since the primes are unbounded. Hence, the sequence (a_i) fails to satisfy (6.2) and is not a dense line.

The minimal constant M satisfying (6.2) for a given sequence (a_i) is a meausre of its density relative to the primes. A smaller M corresponds to a denser sequence. †

Condition (6.3) ensures that the points $(a_i, f(a_i))$ asymptotically lie close to a horizontal line. Because the bound $\frac{c}{a_i^2}$ decreases rapidly to zero with increasing a_i, this condition guarantees that the deviation of $f(a_i)$ from k diminishes along the sequence, so that $(f(a_i))$ converges rapidly toward k. The constant c indicates 'how quickly the points fall onto the line.'

Example 14 (The primes form a dense line at $t = 1$)**.** Let $a_i := p_i$ be the i-th prime. We claim that $(p_i)_{i \geq 1}$ is a dense line at $y = 1$.

Proof. **(Condition 1.)** For each $x > 0$ the set $\{a_i : a_i \leq x\}$ is exactly the set of primes $\{p_i : p_i \leq x\}$. Hence

$$\frac{\left|\{a_i : a_i \leq x\}\right|}{\left|\{p_i : p_i \leq x\}\right|} = \frac{\left|\{p_i : p_i \leq x\}\right|}{\left|\{p_i : p_i \leq x\}\right|} = 1.$$

The limit as $x \to \infty$ therefore exists and equals $1 > 0$, so condition (1) is satisfied.

(Condition 2.) Recall the definition of f used in this paper:

$$f(n) = \frac{\sigma(n)\,\varphi(n)}{n^2}.$$

† $M = \limsup_{i\to\infty} \frac{a_i}{p_i}$ would be better, for the reader who knows lim sup.

If p is prime then $\sigma(p) = 1 + p$ and $\varphi(p) = p - 1$. Consequently

$$f(p) = \frac{(1+p)(p-1)}{p^2} = \frac{p^2 - 1}{p^2} = 1 - \frac{1}{p^2}.$$

As p increases, $1/p^2$ decreases, so the sequence

$$f(p_1),\ f(p_2),\ f(p_3), \ldots$$

is strictly increasing. Moreover

$$\lim_{i\to\infty} f(p_i) = \lim_{p\to\infty} \left(1 - \frac{1}{p^2}\right) = 1.$$

Hence condition (2) is satisfied with $t = 1$. □

Motivation and further remarks.

- Why require monotonicity of $f(a_i)$? The monotonicity requirement simplifies the picture: it guarantees that the subsequence approaches the limit t from one side and avoids oscillatory behaviour that might make the asymptotic interpretation less transparent. In many natural arithmetic subsequences (for example, values taken at primes, or values taken on an increasing sequence of integers with strictly increasing structural complexity) the corresponding f-values are monotone or eventually monotone, so the condition is not overly restrictive for intended applications.

- The choice of comparing counts to the primes (rather than to x itself) is deliberate because many sets of arithmetic interest are as sparse as the primes (or sparser), so a normalization by the primes is a natural way to detect nontrivial "largeness" among sparse sets. If one preferred to measure density among all integers, one could replace the denominator in (1) by x; the present definition instead measures the relative frequency among primes.

- Examples. The prime example above shows the maximal possible limit $t = 1$ can be attained. Other subsequences can produce intermediate limits $t \in \left(\frac{6}{\pi^2}, 1\right)$; for instance, any subsequence of primes whose relative density among primes tends to a positive constant $c \in (0, 1]$ will still have f-limit 1 (because $f(p) \to 1$), while more elaborate constructions (choosing composite a_i whose arithmetic structure forces $f(a_i)$ to cluster near a prescribed value t) can realize other t-values in the allowed interval.

7 Questions

Next answer Question 1. Also rewrite it not using the interval because it's IB AA HL, not Topology 101.

The answer is: no, the bounds can't be uniformly improved. For upper bound this is trivial (n $= 2^k$ for very big k)

For lower bound it's what convergence means. Now that I think about it, Question 1 isn't really worth doing since it's trivial.

Question 2. The topmost 'dense line' is primes, (or prime powers? That would make the discussion more complete but also longer and not that much more impressive)

Definition 4 ('dense line' should I come up with a better name :()**.** We define a subsequence b_n of the sequence a_n to be a 'dense line' if it

- increases
- approaches some value t
- $b_n = a_g(n)$ and $g(n) < cx \ln x$[‡]

Actually, we get a dense line $b_n = a_{dp_n}$ for any positive integer d, and the line is denser the smaller d is (obviously). here p_n is the nth smallest prime. Though the issue is that if I define 'dense lines' like this, it's hard to verify every dense line is generated by this description. In fact, I don't even think that's true... Maybe it is? But how do I define it better? This is the only formal definition I could come up with!

[‡]Here $x \ln x$ is the asymptotic growth of primes (also the asymptotic growth of prime powers, cool! This condition says 'b_n must be dense enough.')

But that's enough knowledge about the dense lines that we can graph them! Is this a satisfying enough answer to Question 2?

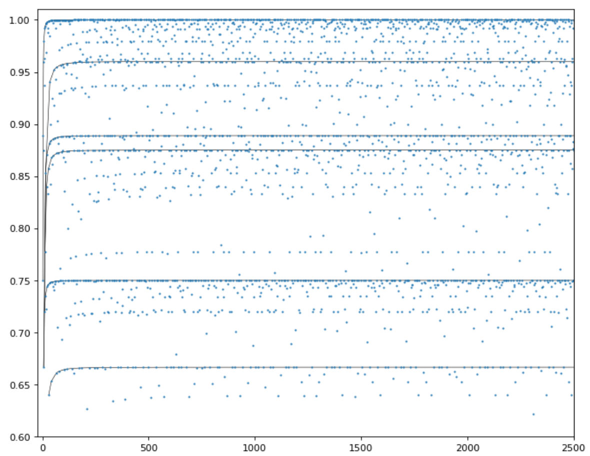

Figure 6: First 6 'dense lines'

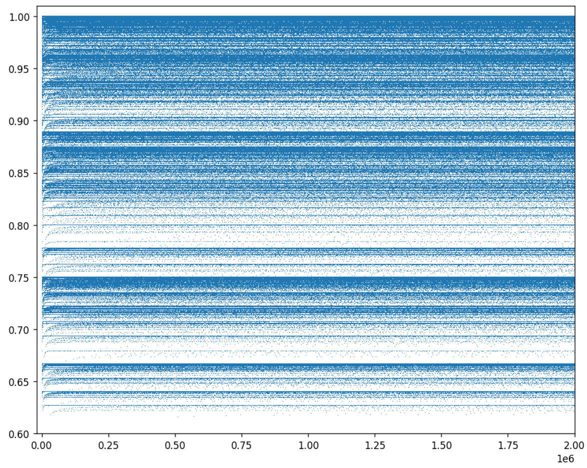

Figure 7: plotting up to $n = 2,000,000$

Now question 3. Look at figure 4. We see more blue 'dense lines' than previously, but also white regions with very few points. We are motivated to ask: Do dense lines cover everything eventually? Will there be some strip with no points whatsoever? The answer is no, and we present an algorithm which shows this. It's really cool and relies on Nagura's bound:

Theorem 2. *For all $n \geq 25$ there is always a prime p with $n < p < 1.2n$*

And a bit of computer brute-force for $3 < n < 25$

and a tiny bit of brute-force by hand for $n = 2, n = 3$

After that though Nagura + induction can get the remaining cases. I think this is my coolest result, and I only recently come up with it

8 Concluding Remarks

I don't know what to say here. It's been fun investigating this, but seeing that the lines were just primes and constant multiples of primes is kinda anticlimactic for me; I like very hard problems and felt much more thrilled solving **Question 3.**

Also I need to write an introduction...

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