Subject: Mathematics

Investigating patterns in the product of Euler's totient function φ and the sum of divisors function σ

Research Question: What gives rise to the patterns in the graph of $f(n) = \sqrt{\varphi(n) \cdot \sigma(x)}$, and how does this relate to the Riemann zeta function?

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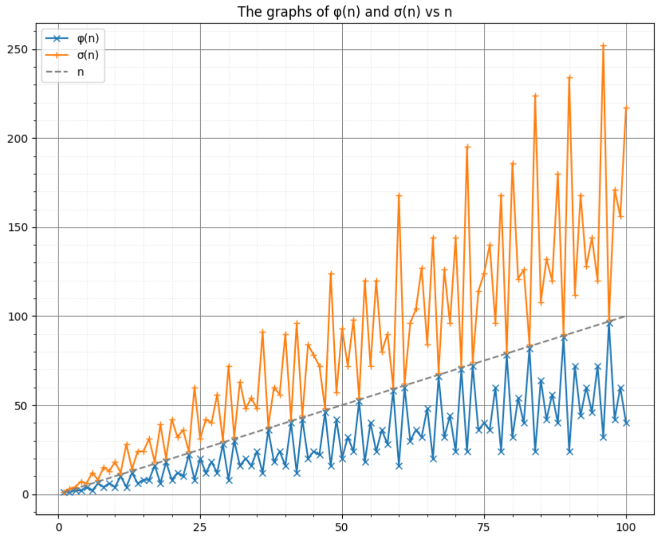
Word count: 1729

Presentation to the Class

$\varphi(n)$ is the "number of positive integers $\leq n$ and **coprime** to n." $\sigma(n)$ is the "sum of the positive divisors of n."

n	$\varphi(n)$	numbers	$\sigma(n)$	numbers
1	1	1	1 =	1
2	1	1	3 =	1+2
3	2	1,2	4 =	1+3
4	2	1,3	7 =	1+2+4
5	4	1,2,3,4	6 =	1+5
6	2	1,5	12 =	1+2+3+6
7	6	1,2,3,4,5,6	8 =	1+7
8	4	1,3,5,7	15 =	1+2+4+8
9	6	1,2,4,5,7,8	13 =	1+3+9
10	4	1,3,7,9	18 =	1+2+5+10



We call functions like φ and σ *"arithmetic functions"*, i.e. functions that express some property of the number.

Lemma 0.1 (Multiplicativity of φ and σ) *For any two **coprime** integers m, n, we have $\varphi(mn) = \varphi(m)\varphi(n)$ and $\sigma(mn) = \sigma(m)\sigma(n)$.*

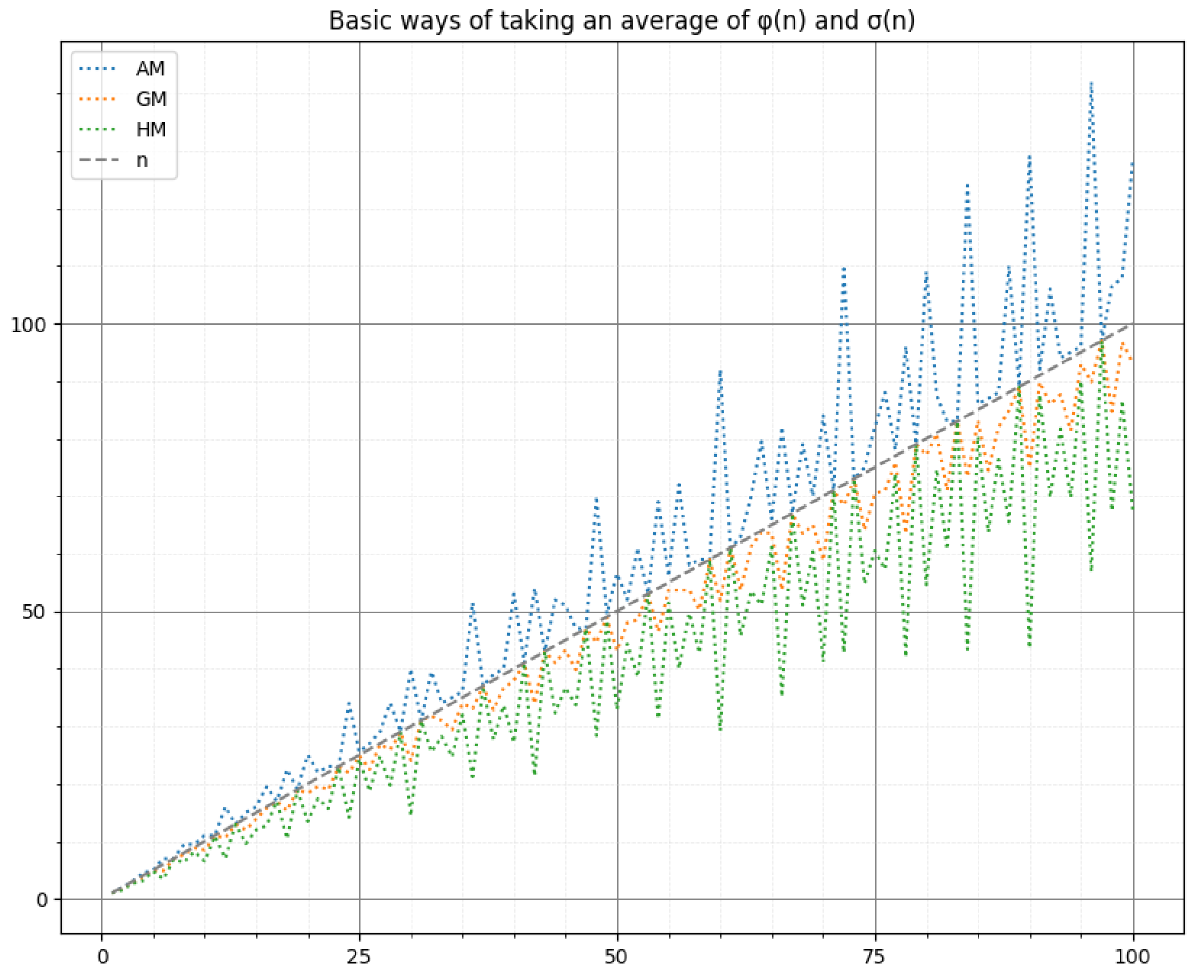
This allows us to write closed formulas of $\varphi(n)$ and $\sigma(n)$ in terms of the prime factorization of n.

Corollary 0.1 *$f(x) = \varphi(x) \cdot \sigma(x)$ is multiplicative.*

We wish to find some way of taking an average that sends the pairs of points to the line $y = n$, i.e. discover a function that reduces the amount of chaos and erraticity in this graph.

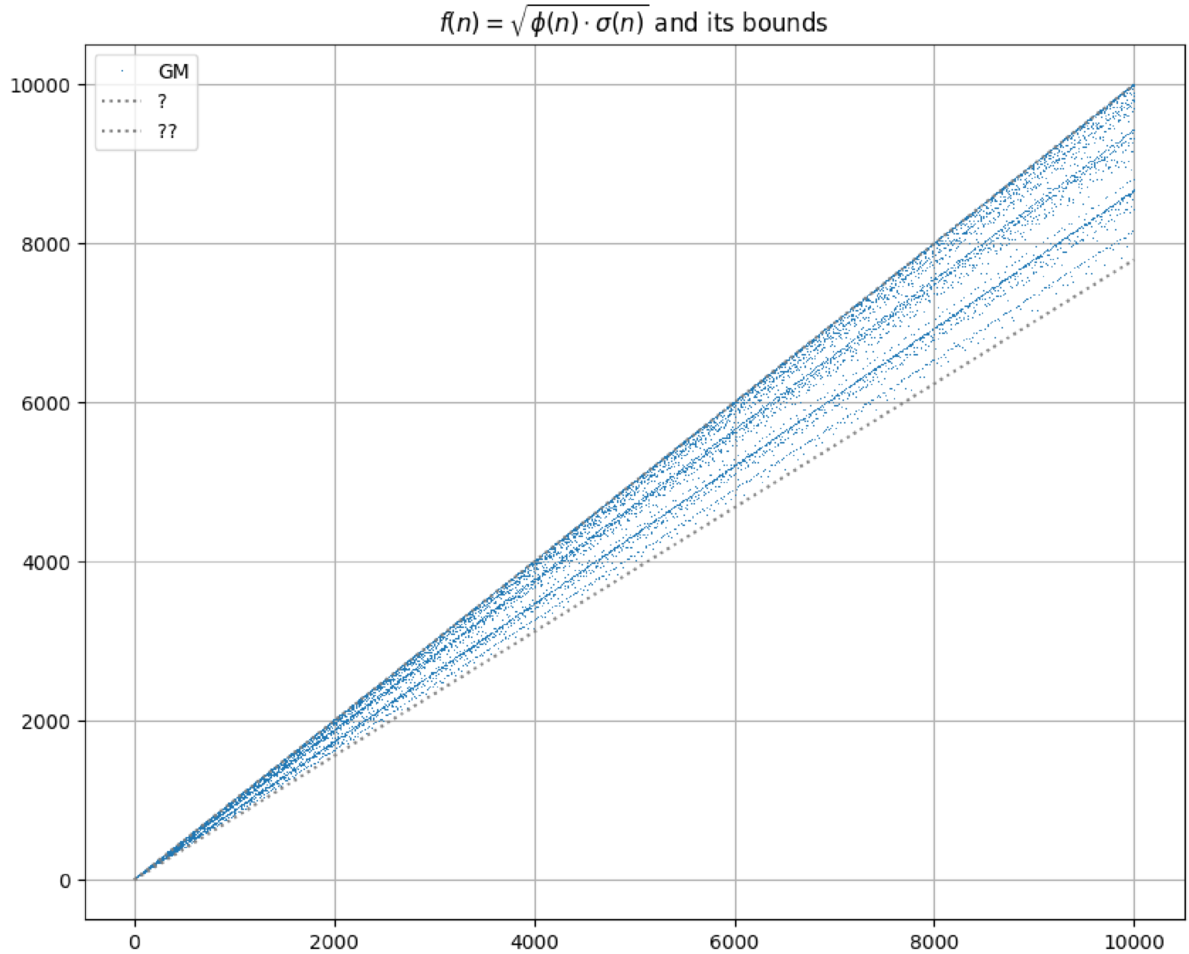
Initial guesses for ways to take an average:

$$\textbf{AM}: \ \frac{a+b}{2} \qquad \textbf{GM}: \ \sqrt{ab} \qquad \textbf{HM}: \ \frac{2}{\frac{1}{a}+\frac{1}{b}}$$

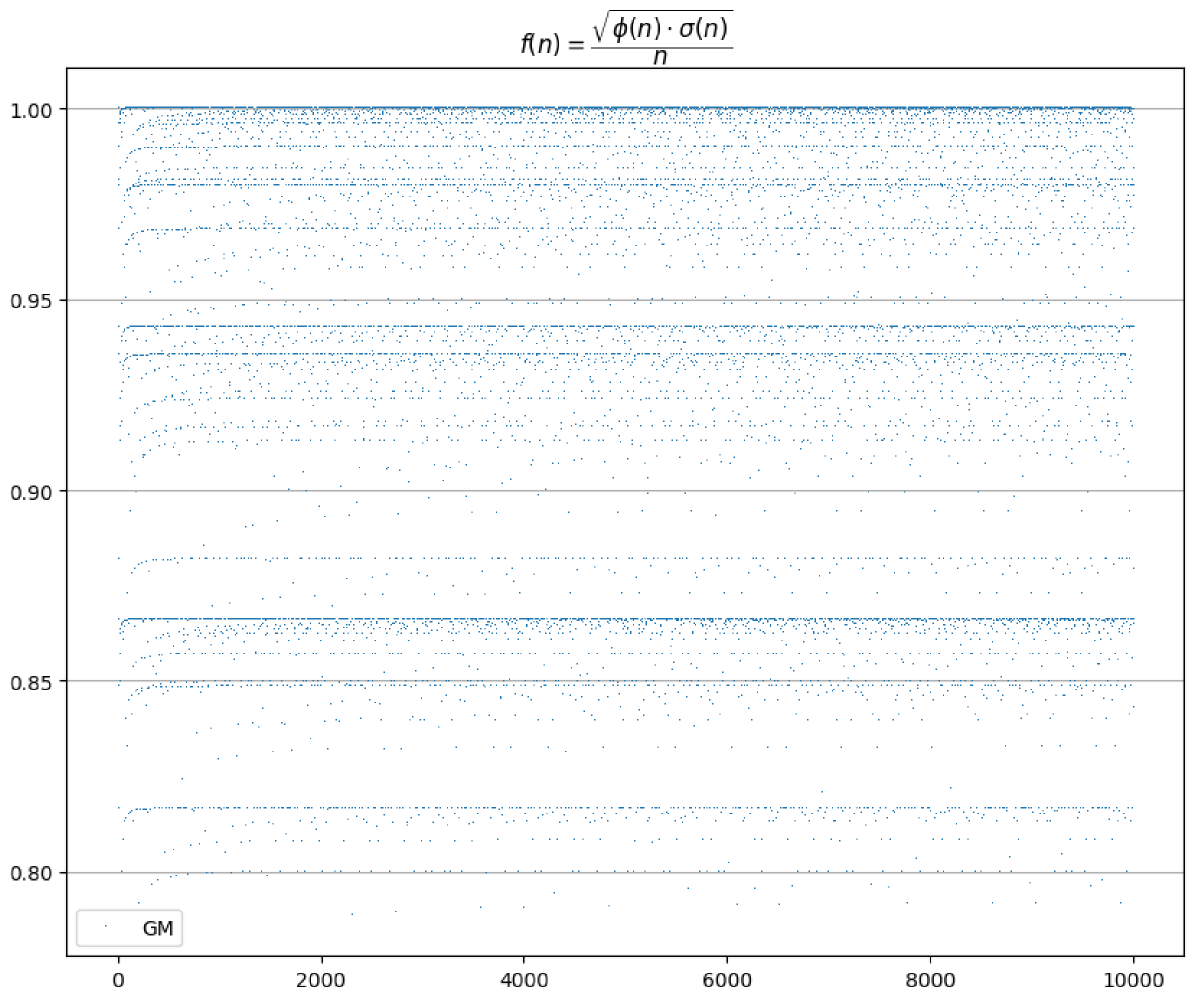


It's also possible to take some weighted version of these, to make the result closer to the target. Investigating that may be a part of my Research, if the word limit allows for that.

I hope the conclusion is that the geometric mean is the best. Then I will proceed to graph the function $f(x) = \sqrt{\varphi(n) \cdot \sigma(n)}$ in more detail and notice some lines with points packed more densely.



Since it looks linear, we are motivated to graph the function $g(n) = \dfrac{\sqrt{\varphi(n) \cdot \sigma(n)}}{n}$ in more detail, to get a clearer view of what the lines are.



In the rest of the paper, I will compute the lower and upper bounds, show that they're asymptotically strict, and lastly explain why the patterns (denser lines) occur.

Abstract

I'll write this later, when most other things are decided.

Contents

1 Prerequisite Knowledge

The reader should be familiar with mathematical notation and basic definitions like 'co-prime' and 'divisor'. Additionally, understanding of *Fundamental Theorem of Arithmetic* is required:

Theorem 1.1 (Fundamental Theorem of Arithmetic) *Every positive integer has an unique prime factorization.*

In other words, for any n, there exists a unique set of distinct prime numbers $p_1, p_2, ..., p_k$, along with their respective exponents $\alpha_1, \alpha_2, ..., \alpha_k$, such that

$$n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}.$$

Representing a number in this form will play an exceptionally important role in this paper.

2 An Introduction to Arithmetic Functions

A special class of functions called *arithmetic functions* play a vital role in the study of number theory, which is the study of positive integers and their properties. The two arithmetic functions concerned in this paper are the following.

Definition 2.1 (Sum of Divisors Function) *Denoted by $\sigma(n)$, it returns the sum of all positive divisors of n.*

Definition 2.2 (Euler's Totient Function) *Denoted by $\varphi(n)$, it returns the number of positive integers $\leq n$ and coprime to n.*

Mathematics relies on abstraction, so properties like these are encoded using functions. For example, the property "n is a perfect number if it is equal to the sum of its proper divisors" can be expressed symbolically as:

$$S = \{n \in \mathbb{N} : \sigma(n) = 2n\}$$

Euler's totient function is best known as a part of *Euler's Theorem*, upon which RSA encryption relies. See

For further references see Something Linky or go to the next url: http://www.overleaf.com or open the next file File.txt

This form of writing numbers as a product of primes is the key to the proofs delivered in this paper. For example, we can construct all the factors (positive divisors) of a number when given its prime factorization. Namely, they are precisely the numbers of the form

$$d = p_1^{\beta_1} p_2^{\beta_2} ... p_k^{\beta_k},$$

where $\beta_i \leq \alpha_i$ for all $1 \leq i \leq k$.

3 The zeta function

hello [**basel**] hello!!

4 An Explanation for the Patterns

5 Appendix