On the integration of systems of equations in total differentials [†]

The problem of the existence of integrals of a given system of s equations in total differentials of r variables, when the system is not completely integrable, was not the subject of extensive research until the article of Biermann *Ueber n simultane Differentialgleichungen der Form* $\sum X_\mu dx_\mu = 0$ published in 1885 in volume 30 of *Schlöm. Zeitschrift.* There he investigated only the maximum number of independent variables that we should to take for there to exist a family of integral varieties filling up space. He then found that this maximum number, *when these coefficients are generic*, is equal to the integer quotient of the total number r of variables by the sum of the number of equations s and one. Moreover, the remainder of this division indicates the number of functions in the independent variables that we can take arbitrarily without the problem ceasing to be solvable. After this article, not much was done on this subject except presenting the proofs of these results under other forms, without ever arriving at perfect rigour, and almost nothing is done on the case where the coefficients of the differential system are not generic.

We can arrive at precise and general results by taking into consideration the bilinear covariants of the left hand sides of the equations of the given system; the bilinear covariants, introduced by Frobenius and Darboux, has already been very fecund in the study of a single Pfaffian equation. In short, in considering the given equations, we express, using a geometric language, that each tangent at a given point A to an integral variety M passing through this point is contained in a certain $r - s$ dimensional flat variety (P) associated with this point. But if we introduce the bilinear covariants, we find not only that every flat variety of dimensions 1, 2, ... tangent to an integral variety is contained in (P), but, furthermore, any two straight lines in this flat variety (T) satisfy certain bilinear relations with respect to their direction parameters. Or, if we represent a straight line based on A by a point in a $r - 1$ dimensional space R, *the image of a tangent to M must be contained in a flat variety (H) of R, and furthermore the straight line joining the images of two tangents to the same integral variety M must belong to a certain number of linear complexes associated to A.*

[†] *Annales École Normale*, 3rd series, 18, 1901, p. 241–311.

In summary, we associate to each point A of the space not only a flat variety (H), but also a set of linear complexes in this flat variety. It is clear that the nature of these linear complexes must influence the existence and the degree of indeterminacy of the integral varieties.

By calling the set of a point A and a p dimensional variety passing through this point the name E_p, and by agreeing to say that E_p is *integral* whenever its image in R is completely contained in (H) and, furthermore, contains only the straight lines belonging to the linear complexes corresponding to A, we see that the necessary and sufficient condition for a variety to be integral is that all of its elements are integral.

If we then try to make a m dimensional integral variety pass through a known $m-1$ dimensional integral variety, we find that this is possible whenever through any integral element E_{m-1} there passes an integral element E_m. The solution is given by a Kowalewski system, and it is unique if through an arbitrary E_{m-1} there passes only one E_m.

Granted this, we are led to define an integer n in the following manner:

Through an arbitrary point A, there passes at least one integral element E_1;

Through an arbitrary integral element E_1, there passes at least one integral element E_2, etc.;

Through an arbitrary integral element E_{n-1}, there pases at least one integral element E_n;

Finally, through an arbitrary integral element E_n, there does not pass any integral element E_{n+1}.

The integer n thus defined is called the *genre* of the system.

From this we can draw precise conclusions on the existence of integrals of the given system. For this, suppose that, in a general manner, the integral elements E_{i+1} passing through an arbitrary integral element E_i depend on r_{i+1} parameters (if this element is unique, we agree to give r_{i+1} the value zero). Then we have here a system of geometrical conditions determining the n dimensional integrals completely:

Given an arbitrary point μ_0, an arbitrary variety μ_{r-r_1} passing through this point, an arbitrary variety μ_{r-r_2} passing through μ_{r-r_1}, etc., an arbitrary variety μ_{r-r_n} passing through $\mu_{r-r_{n-1}}$, there exists one and only one integral variety M_n passing through μ_0 having in common with μ_{r-r_1} a 1 dimensional variety, ..., with μ_{r-r_i} a i dimensional variety and contained in μ_{r-r_n}.

By translating this statement into analytic language and specialising it to the case where we can obtain all the integral varieties once and only once, we can show that the general n dimensional integral is determined, in a unique manner, by a system of

s_n	arbitrary functions of	n	arguments,
s_{n-1}	arbitrary functions of	$n-1$	arguments,
...		...	
s_1	arbitrary functions of	1	arguments,

and

$$s \qquad \text{arbitrary constants,}$$

by setting

$$\begin{aligned} s_n &= r_n, \\ s_{n-1} &= r_{n-1} - r_n - 1, \\ &\dots \\ s_1 &= r_1 \quad - r_2 - 1, \\ s &= r \quad - r_1 - 1. \end{aligned}$$

Furthermore, these integers s are all non-negative and *they are in increasing order, or at least they cannot decrease from s_n to s.*

We can also give the word *arbitrary* that appears in these statements a precise definition.

We hence see the important role played by these integers s and the simple manner in which they depend on the flat variety (H) and the system of linear complexes which we talke about above.

In particular, if the coefficients of the given equations are not subject to any specialisation, which is the case studied by Biermann, the genre n is the integer quotient of r by $s+1$, and if we denote the remainder by σ, we have

$$s_n = \sigma, \qquad s_{n-1} = s_{n-2} = \cdots = s_1 = s,$$

such that the general integral depends on σ arbitrary functions of n arguments, s arbitrary functions of $n-1$ arguments, etc., s arbitrary constants. This is evidently the result found by Biermann with a lot more precision.

The differential systems for which the integers s_n are zero enjoy particularly more interesting properties. We call them *systems of the first kind.*

In a general manner, the integration can be simplified if several of the numbers s are zero. If s_ν is the smallest index that this happens, we have

$$s_\nu = s_{\nu+1} = \cdots = s_n = 0.$$

For these systems, through an arbitrary integral element $E_{\nu-1}$ there passes one and only one integral element E_n. Similarly, it suffices to specify the varieties μ_0, μ_{r-r_1}, $\dots$, $\mu_{r-r_{\nu-1}}$ which we have talked about above to *determine* the integral M_n and *the search of this integral amounts to that of a system of genre ν.* It suffices to make an arbitrary but determined variety $\mu_{r-r_\nu-1}$ pass through $\mu_{r-r_{\nu-1}}$, and through this variety passes a family of varieties μ_{r-r_ν} depending on $r_\nu = n-\nu$ parameters and filling all of the space. To each of there corresponds an integral variety M_ν. The locus of these varieties M_ν, when we vary the $n-\nu$ parameters they depend on, is the variety M_n we search for. In short, we are lead to a system of $r-r_\nu$ variables of genre ν, but whose coefficients depend on $n-\nu$ arbitrary constants. In the case where ν is equal to 1, this is the Lie-Mayer method for integrating completely integrable systems. We can call ν the *true genre* of the system.

In another train of ideas, there is another case where integration can be simplified, i.e., the case where there passes through each point A a *characteristic* element, by which we mean an integral element E_p such that all elements formed with E_p and a linear integral element are integral, or, as we can say, E_p is associated to any linear integral element. We can then show that the system of equations in total differentials defining the characteristics is *completely integrable.* In other words, there exists a family of p dimensional varieties admitting at each of their points a corresponding characteristic element E_p. These varieties, which we call *characteristics*, depend on $r-p$ parameters, and there passes one and only one of them through each point of the space. For the differential system of genre n where there exists characteristic elements E_p, every non-singular integral variety M_n is generated by characteristic varieties depending on $n-p$ parameters, and if two integral varieties M_n and M'_n have a point in common, they have in common the characteristic variety based on the point.

Finally if we take the $r-p$ parameters that the characteristics depend on and any other generic p functions as the new variables, they system can be made into a form that it contains only the first $r-p$ variables. Furthermore, the search for the $r-p$ variables, in other words the integration of the characteristic system, can in general be simplified by taking into account the properties of the linear complexes associated with the given system.

In particular, if we have a system of genre n of first kind for which s_1 is equal to 1, which is the case of one single equation, there is always characteristic varieties of $n-\nu+1$ dimensions, ν denoting the true genre of the system. Once these characteristics are found by operations whose order decreases by two each time, we only have to integrate a system in $r-n+\nu-1$ variables of genre $\nu-1$.

There also exists simple theorems in the case where s_1 is equal to 2, but the study of these theorems would lead us to the theory of the *classification* of systems in total differentials.

It is hardly necessary to remark the links among all these theories and the theory of systems of partial differential equations. We will content ourselves to indicate the agreement of the results found by the general degree of indeterminacy of a Pfaffian system and those found by Delassus [†] on the degree of indeterminacy of a general integral of an involutive system of partial differential equations. However, although Delassus has made the system under a particular form, by further differentiating the variables depending on unknown functions completely, we see that in this case there is no difference between the two kinds of variables, and the origin of the numbers s, s_1, …, s_n shows their invariance under all change of variables of the dependent and independent variables.

The first two sections of this article introduce integral elements, with linear complexes

[†] *Extension du théorème de Cauchy aux systèmes les plus généraux d'équations aux dérivées partielles* [*Ann. de l'Éc. Norm.* (3), vol. XIII, p. 421–467].

that we have already talked about. Section III treats the problem consisting of making an integral variety M_{m+1} pass through an integral variety M_m. Sections IV and V give some theorems, arithmetic in character, on the genre n and the numbers r_i and s_i. Section VI contains the exposition of the Cauchy problem and the degree of indeterminacy of the general integral of a system of genre n. Section VII is denoted to systems of the first kind and the generalised Lie-Mayer method. Finally section VIII is concerned with systems admitting characteristics in the sense of the word given above, and gives some indications on the search for these characteristics.

These researches can be extended in various directions, and the problem of the *classification* of the differential systems can already, as we see, be tackled in the form of a preliminary problem, which is *the search for system of linear complexes of genre n.* Another very important question is the study of *singular* integral varieties. It is not difficult to define them, but what is interesting is the study of the new differential systems defining them. As for the original classification problem, we can without much difficulty show a certain number of interesting results, but I will not pursue this point.

I.

Consider a system of equations in total differentials of r variables $x_1, x_2, \dots, x_r$,

$$(1)\qquad \begin{cases} \omega \equiv a_1 dx_1 + a_2 dx_2 + \cdots + a_r dx_r = 0, \\ \varpi \equiv b_1 dx_1 + b_2 dx_2 + \cdots + b_r dx_r = 0, \\ \quad \cdots \\ \chi \equiv l_1 dx_1 + l_2 dx_2 + \cdots + l_r dx_r = 0, \end{cases}$$

the coefficients $a, b, \dots, l$ being functions of the variables x. We can regard a certain number n of the variables x as independent and the other $r-n$ as functions of them. Then the system (1) which establishes linear relations among the total differentials of the $r-n$ functions and the n independent variables is equivalent as a whole to a system of (linear) equations in partial derivatives which the $r-n$ functions must satisfy [†]. Employing a geometrical language, we can say that the equations that define the $r-n$ dependent variables as functions of the n independent variables represent a *n-dimensional variety* M_n in a r dimensional space, and the system (1) can be regarded as establishing the conditions which must be satisfied by the differentials of the coordinates $x_1, \dots, x_r$ at a point of the variety under an arbitrary displacement on this variety. But if we observe that these differentials (or their ratios, which are the only things coming into play) are none other than the *direction parameters* of the *tangent* to the variety under the displacement considered, we can say that the system (1) expresses that *tangents to a variety M_n at any point of the space passing by the point satisfy certain conditions*

[†] Furthermore, we know that every system of equations in partial derivatives can be transformed into a system of equations in total differentials by regarding some of the partial derivatives of the unknown functions as new dependent variables, as necessary.

depending only on the point considered, and the form of the equations (1) shows that these tangents must lie in a certain flat variety [(†)] determined by the point.

To integrate the system (1), where we suppose the number of independent variables is equal to n, is therefore to resolve the following problem:

To each point of the space we attach a flat variety [(‡)] *passing by the point. Determine an n-dimensional variety M_n such that at each of its points all tangents to this variety lie within the flat variety corresponding to this point.*

Every variety M_n satisfying this condition will be called an *integral* variety. This condition, thus stated, which the integral manifolds satisfy, is *independent* of the dimension n of the varieties.

Let us call the set of a point and a straight line passing through this point a *linear element*, and furthermore, let us say that the set of a point of a variety and a tangent at this point to this variety constitutes a linear element of this variety. Let us also call linear elements satisfying equations (1) (where dx_1, dx_2, $\ldots$, dx_r are regarded as direction parameters of the straight line of the element) *linear integral elements.* We can then state the following proposition:

For a variety to be integral, it is necessary and sufficient that all of its linear elements are integral.

II.

As well as the linear elements, we are going to consider what we will call 2, 3, ... *dimensional elements.* In general, we call *the set of a point and a p-dimensional flat variety passing through the point* a *p-dimensional element* and we denote such an element by the general symbol E_p. We say that the element E_p contains the element E_q $(p > q)$ if the two elements are at the same point and if the flat variety of the first element contains the flat variety of the second entirely. In particular, a linear element will be denoted by the symbol E_1.

We call p-dimensional elements whose linear elements belong to a variety M, or more simply the elements formed by the linear elements of M, the *elements E_p of a variety M.* If the variety M is n dimensional, it admits elements of 2, 3, $\ldots$, n dimensions, but it does not admit $n+1$ dimensional element. At each point it admits only one n dimensional element, which is formed by the set of linear elements at this point.

Every element E_p of a *integral* variety clearly enjoys the property that it contains only *integral* linear elements, *but it also satisfies other conditions which can be established independently of any particular integral variety.*

[(†)] A flat variety is, as we know, defined by linear equations. A straight line is a *one* dimensional flat variety.

[(‡)] The dimension of this plane variety is, naturally, the same for all points of the space. It is equal to the difference between r and the number of equations (1).

To derive these conditions, imagine that the coordinates of a point of a integral variety M_n are expressed by means of n parameters u, v, ..., and consider two displacements on this variety, the first obtained by varying only the parameter u and keeping all the others constant, and the second by varying only the parameter v. Let us denote by the symbols d and δ the differentials relative to these two displacements. We evidently have, according to (1),

$$\omega_d \equiv a_1 dx_1 + a_2 dx_2 + \cdots + a_r dx_r = 0,$$
$$\omega_\delta \equiv a_1 \delta x_1 + a_2 \delta x_2 + \cdots + a_r \delta x_r = 0,$$

and then

$$\omega' \equiv \delta\omega_d - d\omega_\delta = 0.$$

Forming this expression and remarking that the symbols d and δ *commute* ($d\delta = \delta d$), and finally do the same for all the equations of the system (1), we arrive at the following system:

$$(2) \qquad \begin{cases} \omega' \equiv \sum\limits_{i,k} \left(\dfrac{\partial a_i}{\partial x_k} - \dfrac{\partial a_k}{\partial x_i} \right) (dx_i \delta x_k - dx_k \delta x_i) = 0, \\ \ldots \\ \chi' \equiv \sum\limits_{i,k} \left(\dfrac{\partial l_i}{\partial x_k} - \dfrac{\partial l_k}{\partial x_i} \right) (dx_i \delta x_k - dx_k \delta x_i) = 0. \end{cases}$$

The system (2) *is satisfied by every pair of displacements on the integral variety*, or by the set of a point on the variety and two tangents at this point, or generically, *by two linear integral elements at the same point belonging to the same integral variety.*

Let us call an element *formed by linear integral elements such that any two linear elements satisfy the system* (2) the 2, 3, ... dimensional *integral* elements. We then have the following proposition:

Every 1, 2, ... *dimensional element of any integral variety is integral.*

To simplify the language, we will agree to say that two linear integral elements at the same point satisfying the system (2) are *associated* (†). Then an *integral* element of dimension 2, 3, ... is *an element formed by linear integral elements whose every two elements are associated.* According to the linear form of the equations (2), for an element E_p to be integral, it *suffices* that p independent linear elements (‡) of E_p are integral and any two of them are associated. (Furthermore every element E_p can be defined by p independent linear elements defined at the same point.)

(†) Two linear elements associated to a third one are not necessarily associated themselves.

(‡) We say that p linear elements are *independent* when they do not belong to the same $p-1$ dimensional element.

The expressions ω', ϖ', $\ldots$, χ', which are the left hand sides of the equations (2), are called the *bilinear covariants* [†] of the Pfaffian expressions ω, ϖ, $\ldots$, χ. According to the manner that they are obtained and conforming to their names, we see that they are relative covariants under any change of variables.

We can give the system (2) a geometric interpretation. Let us consider the different linear integral elements defined on a given point A of the space and let us project them unto a $r-1$ dimensional flat variety (P) not passing through A, the point of view we take being from the point A itself. Then each element is defined by the *mark* of its straight line on the flat variety projected unto, that is to say by a point on this variety (P). With our notations, the quantities dx_1, dx_2, $\ldots$, dx_r are the homogeneous coordinates of this point in (P). To say that a linear element is integral is to say that the coordinates of its projection satisfy the equations (1), i.e., are contained in a certain flat variety (Q) within (P). If we now take two associated linear integral element and their projections on (P), the quantities $dx_i\delta x_k - dx_k\delta x_i$ are precisely the Plücker coordinates of the straight line joining these two projections. The first of the equations (2) expresses a linear and homogeneous relation between these coordinates, i.e., expresses that this line belongs to a certain linear complex, and this is true for all other equations of (2).

In summary, *to say that two linear elements defined at the same point A are integral and associated is to say that by projecting them onto a $p-1$ dimensional flat variety (P), the line joining the marks of these two elements are entirely contained in a certain flat variety (Q) and, moreover, belongs to a certain number of linear complexes as well.*

And that *to say that an element E_p defined on A is integral is to say that the flat variety, that is, the mark of this element on (P), is situated completely within (Q) and, furthermore, each of the straight lines of this variety belongs to a certain number of linear complexes.*

To summarise, at each point A the given system we attach, in an arbitrarily chosen $\sigma - 1$ dimensional flat variety (P) not passing through A, a flat variety (Q) and a set of linear complexes in this variety (Q).

If we apply a change of variables, the elements defined on a point A are linked homographically to the corresponding elements defined on the corresponding point A' and *the set of linear complexes corresponding to A also undergoes a simple homographic transformation* [‡].

From this very simple remark it already follows an important consequence that, if two systems of equations in total differentials (of the same number of variables) do not correspond to points of flat varieties (Q) and linear complexes that are deducible from each other by a homographic transformation, then it is imposible to reduce one of the given systems to the other by a change of variables. More precisely, if we denote the

[†]Their introduction into the Pfaffian problem is due to Frobenius [*Ueber das Pfaff'sche Problem*, *J. de Crelle*, 1. LXXXII; 1877] and Darboux [*Sur le problème de Pfaff* (*Bull. Sc. Math.*, 2nd series, vol. VI; 1882)].

[‡]It is obvious that, if we simply change the flat variety projected unto, we obtain two systems of complexes equivalent under a homographic transformation, since they are the *projections* of each other. If we replace the equations (1) by others that form an equivalent system, it is also obvious that neither (Q) nor the set of linear complexes in (Q) is changed.

variables of the second system in total differentials by $y_1, y_2, \ldots, y_r$ and the systems analogous to (1) and (2) by $(1)'$ and $(2)'$, we aim to find the conditions under which we can pass from the system [(1), (2)] to the system [$(1)'$, $(2)'$] by a linear transformation acting on $dx_1, \ldots, dx_r$ as well as $\delta x_1, \ldots, \delta x_r$.

Three cases can arise. If this is not possible for any systems of values of x and y, then no change of variables can transform one of the given systems to the other. If this is possible subject to the condition that certain algebraic relations linking x and y hold, then every change of variables effecting the transformations we look for, *if it exists*, must respect these relations. Finally, if this is possible regardless of the values of x and y, then we can say no more about the required change of variables, *if it exists*.

We also notice, without needing to emphasise, that the classification of systems of equations in total differentials requires the preliminary classifications of all systems of linear complexes, where we do not regard two systems of complexes deducible from each other by a homographic transformation as distinct. This is, in other words, the search for the *types* of the systems of linear complexes.

To apply the preceding to an example, let us consider the system

$$\begin{cases} \omega = dz - p\,dx - q\,dy = 0, \\ \varpi = dp - u\,dq - a\,dx - b\,dy = 0, \end{cases} \tag{3}$$

where the variables are x, y, z, p, q, u and a and b denote two given functions of these six variables. The integration of this system, considered as of two independent variables x and y becomes the integration of a second order partial differential equations admitting a system of first order characteristics and, with the usual notations, this equation is obtain by eliminating u from the two relations

$$r - us - a = 0,$$
$$s - ut - b = 0.$$

Here the flat variety (Q) is three dimensional, since the homogeneous coordinates of one of its points are defined when we specify dx, dy, dq, du. We can therefore regard (Q) as ordinary space. Here there are *two* linear complexes. In space, a system of two linear complexes is always reducible to one of the following three by a homographic transformation:

$$p_{12} = p_{34} = 0, \tag{α}$$
$$p_{12} = p_{13} + p_{24} = 0, \tag{β}$$
$$p_{12} = p_{13} = 0, \tag{γ}$$

p_{ik} being the Plücker coordinates of the straight line. The case (α) gives the set of straight lines meeting two straight lines not within the same plane. The case (β) gives the set of tangents to a quadric fixed with respect to different points of a fixed generatrix of this quadric. The case (γ) gives the set of straight lines situated in a fixed plane and the lines all originate from a fixed point of the plane.

To each of these cases there corresponds a type of second order equation of the indicated form. To the case (α) corresponds the equations whose two second order characteristic systems are distinct. To the case (β) correspond the equations with coinciding characteristics, obtained by expressing that the equation

$$r + 2us + u^2t + 2\varphi(u, x, y, z, p, q) = 0$$

admits a double root for u, the function φ being *generic*. Finally to the case (γ) correspond those equations that in the last equation the function φ satisfy a certain equation in second order derivatives, which was the subject of research of Goursat. Their interest lies in the fact that we can integrate them by systems of ordinary differential equations, as we will see in section VIII.

III.

Having settled with these preliminary notions, we are now going to be concerned with what can be called the *preliminary Cauchy problem.* The problem that we name thus is the following:

Given a p dimensional integral variety M_p of a system of equations in total differentials, extend M_p into a $p+1$ dimensional integral variety M_{p+1}.

An obvious observation is that whenever the problem is solvable, through each element E_p of M_p there passes at least one *integral* element E_{p+1}, namely the element E_{p+1} of M_{p+1}. We therefore immediately arrive at our first necessary condition.

For the Cauchy problem to be solvable, it is necessary that through each element E_p of the given integral variety M_p there passes at least one integral element E_{p+1}.

Without investigating if this condition is sufficient, which it is not, we are going to concern ourselves with a particular case, but a case that nonetheless presents a great generality. *We will suppose, in the following, that the given system is such that through each integral element E_p in the space there passes at least one integral element E_{p+1}.* In other words, the property that elements E_p of M_p satisfy is supposed to hold for all integral elements E_p of the space.

With this hypothesis, *the Cauchy problem is always solvable.* However, before carrying out the proof of this proposition, it will be useful for us to present several geometrical remarks on the integral elements E_{p+1} containing a given integral element E_p. If we define the element E_p by means of p independent linear elements $\varepsilon^{(1)}$, $\varepsilon^{(2)}$, …, $\varepsilon^{(p)}$, we can define an element E_{p+1} containing E_p by means of a new linear element ε independent of the previous p elements. We will have the elements E_{p+1} we look for by expressing that ε is an *integral* linear element and it is *associated* to each of the linear elements $\varepsilon^{(1)}$, $\varepsilon^{(2)}$, …, $\varepsilon^{(p)}$. It follows from this that *the locus of the integral elements E_{p+1} containing an integral element E_p is a flat element* (not necessarily integral), since if ε and ε' provide two distinct solutions E_{p+1} and E'_{p+1}, the $p+2$ linear elements $\varepsilon^{(1)}$, $\varepsilon^{(2)}$, …, $\varepsilon^{(p)}$, ε, ε' determine an element E_{p+2} and every linear element of E_{p+2} is integral and associated with $\varepsilon^{(1)}$, $\varepsilon^{(2)}$, ..., $\varepsilon^{(p)}$. In other words, all the elements E_{p+1} contained in E_{p+2} and containing E_p are integral.

Analytically, the elements E_{p+1} containing E_p depend on $r-p$ homogeneous parameters [†]. The equations expressing that E_{p+1} is integral are *linear* with respect to these parameters.

Suppose that, for an *arbitrary* element E_p, these equations reduce to $r-p-s-1$ independent ones, s being zero or positive. Then through each arbitrary integral element E_p there will pass at least an integral element E_{p+1}. There will pass one and only one if s is zero, and an infinite number of elements depending on s arbitrary constants if s is positive. We say in these two cases that there passes ∞^s elements. The locus of all these elements is an element E_{p+s+1}.

It is possible that for a particular integral element E_p there is yet greater indetermination: we say in this case that the integral element E_p is *singular.* An integral variety M_p whose elements E_p are all singular will be called a *singular* integral variety.

We now arrive at the solution of the Cauchy problem. We are going to prove the following theorem:

Given a non-singular integral variety M_p, there always passes through this variety at least one integral variety M_{p+1}. There passes one and only one if each non-singular element E_p belongs to one and only one integral element E_{p+1}. There passes an infinite number of varieties depending on s arbitrary functions of $p+1$ arguments if each non-singular integral element E_p belongs to ∞^s integral elements E_{p+1}.

To make it more precise, take a particular non-singular element E_p^0 of M_p. Let $(x_1^0, x_2^0, \dots, x_r^0)$ be the coordinates of the point where this element is based at. Suppose that the variety M_p is *analytic*, that is to say in a neighbourhood of the point (x_i^0), $r-p$ of the coordinates x, which we take to be x_{p+1}, ..., x_r, are expressed as holomorphic functions in $x_1 - x_1^0$, $x_2 - x_2^0$, ..., $x_p - x_p^0$. Then the $r-p$ equations of the element E_p^0 can be solved with respect to dx_{p+1}, ..., dx_r. Take a particular integral element E_{p+1}^0 passing through E_p^0. The $r-p-1$ linear equations defining it can be solved with respect to $r-p-1$ of the differentials $dx_{p+1}, \dots, dx_r$, which we take to be dx_{p+2}, ..., dx_r. If an integral variety M_{p+1} admits the element E_{p+1}^0, this signifies that x_{p+2}, ..., x_r are expressed as holomorphic functions in x_1, ..., x_{p+1} in a neighbourhood of the point considered. For convenience of the following exposition, we are going to change the notations by continuing to use x_1, x_2, ..., x_p but replacing x_{p+1} by x and the other variables x_{p+2}, ..., x_r by z_1, z_2, ..., z_m $(m = r-p-1)$.

[†] For example, if the equations of E_p are

$$P_1 = P_2 = \cdots = P_{r-p} = 0,$$

where the Ps are linear forms in dx_1, ..., dx_r, the equations of E_{p+1} are

$$\frac{P_1}{\lambda_1} = \frac{P_2}{\lambda_2} = \cdots = \frac{P_{r-p}}{\lambda_{r-p}}.$$

With these notations the equations of the variety M_p are

$$\tag{4} \begin{cases} x = \varphi(x_1, x_2, \dots, x_p), \\ z_1 = \varphi_1(x_1, x_2, \dots, x_p), \\ \quad \dots \\ z_m = \varphi_m(x_1, x_2, \dots, x_p), \end{cases}$$

and the variety M_{p+1} we look for can be defined by specifying z_1, z_2, $\dots$, z_m as holomorphic functions of x, x_1, $\dots$, x_p in a neighbourhood of x_0, x_1^0, $\dots$, x_p^0.

Now let us effect a change of variables conserving the variables x_1, x_2, $\dots$, x_p; z_1, $\dots$, z_m and taking the new variable x to be the quantity $x - \varphi$, which obvious does not change any conventions that we have made previously. This implies in the formulae (4) we suppose $\varphi \equiv 0$ and $x^0 = 0$.

To complete the statement of these preliminary conventions, we suppose that the coefficients a, b, $\dots$, l of system (1) are holomorphic in a neighbourhood of $(x^0, x_1^0, \dots, z_m^0)$.

The variety M_{p+1} we search for is defined by m functions z_1, z_2, $\dots$, z_m of the $p+1$ variables x, x_1, $\dots$, x_p, holomorphic in the neighbourhood of $(0, x_1^0, \dots, x_p^0)$ and subject to be reduced to m previously given functions φ_1, φ_2, $\dots$, φ_m in x_1, x_2, $\dots$, x_p for $x = 0$.

The equations determining these functions reduce to the equations (1) by replacing dz_1, $\dots$, dz_m by their values and identifying the relevant quantities, but *we are going to substitute the system thus obtained by another system containing a bigger number of equations expressing simply that the elements E_{p+1} of the variety M_{p+1} are integral.*

For this, observe that each element E_{p+1} of M_{p+1} can be defined by $p+1$ independent linear elements, namely those that we obtain by varying only one of the independent variables x, x_1, $\dots$, x_p. These $p+1$ elements that we will call $\varepsilon^{(1)}$, $\dots$, $\varepsilon^{(p)}$ are defined by

$$\tag{ε} \frac{dx}{1} = \frac{dx_1}{0} = \cdots = \frac{dx_p}{0} = \frac{dz_1}{\frac{\partial z_1}{\partial x}} = \cdots = \frac{dz_m}{\frac{\partial z_m}{\partial x}},$$

$$\tag{$\varepsilon^{(1)}$} \frac{dx}{0} = \frac{dx_1}{1} = \cdots = \frac{dx_p}{0} = \frac{dz_1}{\frac{\partial z_1}{\partial x_1}} = \cdots = \frac{dz_m}{\frac{\partial z_m}{\partial x_1}},$$

$$\tag{$\varepsilon^{(p)}$} \frac{dx}{0} = \frac{dx_1}{0} = \cdots = \frac{dx_p}{1} = \frac{dz_1}{\frac{\partial z_1}{\partial x_p}} = \cdots = \frac{dz_m}{\frac{\partial z_m}{\partial x_p}}.$$

We will divide the equations expressing that E_{p+1} is integral into two groups. In the first group we express that the element E_p defined by $\varepsilon^{(1)}$, $\varepsilon^{(2)}$, $\dots$, $\varepsilon^{(p)}$ is integral, and in the second group we express that ε is integral and associated with $\varepsilon^{(1)}$, $\varepsilon^{(2)}$, $\dots$, $\varepsilon^{(p)}$.

If one of the equations of the system is

$$\omega \equiv a\,dx + a_1 dx_1 + \cdots + a_p dx_p + b_1 dz_1 + \cdots + b_m dz_m = 0,$$

we will set

$$\Omega \equiv a + b_1 \frac{\partial z_1}{\partial x} + \cdots + b_m \frac{\partial z_m}{\partial x},$$
$$\Omega_i \equiv a_i + b_1 \frac{\partial z_1}{\partial x_i} + \cdots + b_m \frac{\partial z_m}{\partial x_i}, \qquad (i = 1, 2, \ldots, p).$$

With these notations, the equations of the first group are, as it is easy to see,

$$\text{(I)} \qquad \begin{cases} \Omega_i = 0, \qquad \dfrac{\partial \Omega_i}{\partial x_j} - \dfrac{\partial \Omega_j}{\partial x_i} = 0 \qquad (i, j = 1, 2, \ldots, p), \\ \ldots \end{cases}$$

and those of the second group are, for example,

$$\text{(II)} \qquad \begin{cases} \Omega = 0, \qquad \dfrac{\partial \Omega}{\partial x_i} - \dfrac{\partial \Omega_i}{\partial x} = 0 \qquad (i = 1, 2, \ldots, p), \\ \ldots \end{cases}$$

the dots are from the other equations $\varpi = 0, \ldots, \chi = 0$ of the given system. The symbol $\frac{\partial f}{\partial x_i}$ denotes the derivation with respect to x_i, regarding z_1, z_2, $\ldots$, z_m as functions of x_i.

The equations (I) do not contain $\frac{\partial z_1}{\partial x}, \ldots, \frac{\partial z_m}{\partial x}$ and those in the second group are *linear* with respect to these quantities. We can simplify them further by using the equations (I).

Let us now consider the hypotheses we already made. When the element E_p defined by $\varepsilon^{(1)}$, $\varepsilon^{(2)}$, $\ldots$, $\varepsilon^{(p)}$ is integral, the equations φ must satisfy for E_{p+1} to be integral are compatible. This signifies that, *when the equations are satisfied, the equations* (II)*, considered as equations linear in* $\frac{\partial z_1}{\partial x}, \ldots, \frac{\partial z_m}{\partial x}$ *are algebraically compatible.* More precisely, they reduce to $m - s$ linearly independent equations. In particular, by hypothesis, this is true for the system of values $(0, x_1^0, \ldots, x_m^0)$. To fix ideas, we suppose that these $m - s$ equations are, for the initial values, solvable with respect to

$$\frac{\partial z_1}{\partial x}, \qquad \frac{\partial z_2}{\partial x}, \qquad \ldots \qquad \frac{\partial z_{m-s}}{\partial x},$$

and let us write them as

$$\text{(II}'\text{)} \qquad \begin{cases} \dfrac{\partial z_1}{\partial x} = \Phi_1 \left(x, x_i, z_k, \dfrac{\partial z_k}{\partial x_j}, \dfrac{\partial z_{m-s+1}}{\partial x}, \ldots, \dfrac{\partial z_m}{\partial x} \right), \\ \qquad \ldots, \\ \dfrac{\partial z_{m-s}}{\partial x} = \Phi_{m-s} \left(x, x_i, z_k, \ldots, \dfrac{\partial z_m}{\partial x} \right), \end{cases}$$

the Φs being holomorphic in their arguments in a neighbourhood of their initial values (and linear with respect to $\frac{\partial z_{m-s+1}}{\partial x}, \ldots, \frac{\partial z_m}{\partial x}$).

Granted this, instead of satisfying the set of equations (I) and (II), we satisfy only the equations (II′), recalling that the equations (I) and (II′) imply the equations (II) algebraically.

We are now going to find a solution of the equations (II′) satisfying the following conditions: $z_1, \ldots, z_m$ are holomorphic functions of $x, x_1, \ldots, x_p$ in a neighbourhood of $(0, x_1^0, \ldots, x_p^0)$ and on $x = 0$ reduce to the m given functions $\varphi_1, \ldots, \varphi_m$ in $x_1, \ldots, x_p$.

The system (II′) is a Kowalewski system. According to the work on these systems, there exists one and only one solution, holomorphic in a neighbourhood of $(0, x_1^0, \ldots, x_p^0)$, such that $z_{m-s+1}, \ldots, z_m$ are arbitrarily given (holomorphic) functions and that $z_1, \ldots,$ z_s reduce to s given functions of $x_1, \ldots, x_p$ on $x = 0$.

Granted this, we take

$$z_{m-s+1} = f_{m-s+1}(x, x_1, \ldots, x_p),$$
$$\ldots$$
$$z_m = f_m(x, x_1, \ldots, x_p)$$

the s functions f being subject to the single condition that they reduce to the s given functions $\varphi_{m-s+1}, \ldots, \varphi_m$ on $x = 0$. These s functions chosen, the system (II′) *admite a single solution satisfying the stated conditions.*

Furthermore, we see that we can always arrange them in a manner that for $x = 0$, $x_i = x_i^0$, the s quantities $\frac{\partial z_{m-s+1}}{\partial x}, \ldots, \frac{\partial z_m}{\partial x}$ take arbitrarily fixed values, that is to say *for the variety M_{p+1} thus determined to admit at will the integral elements E_{p+1} passing through E_p^0.*

The original problem is not yet solved, since even if it is clear that the integral varieties we search for can only be found among the varieties just determined by Kowalewski's theorem, it does not follow that these varieties are really *integral.* In other words, we still have to prove that these varieties satisfy the equations (I) and (II). To this end, *we are going to show that, if a variety M_{p+1} determined as above satisfy the equations* (I) *and* (II) *for a certain value of x, it also satisfy them for an infinitesimally near value $x + \delta x$.*

If this is proved, as for $x = 0$ the equations (I) express that the variety M_p which M_{p+1} reduces to is integral, which is none other than our hypothesis, and that the equations (II′) are assumed to be verified for the variety M_{p+1}, and hence the equations (II) as well, it follows that the equations (I) and (II) are satisfied for all values of x.

Now suppose for a certain value of x the equations (I) and (II) are satisfied. We have then, in particular, for this value of x,

$$\Omega = 0, \qquad \Omega_i = 0, \qquad \frac{\partial \Omega_i}{\partial x} - \frac{\partial \Omega}{\partial x_i} = 0.$$

But if Ω is zero, the same is true for its derivative $\frac{\partial \Omega}{\partial x_i}$ taken with respect to the variable x_i *independent of* x. Therefore $\frac{\partial \Omega_i}{\partial x}$ is zero for the value of x considered. To say that Ω_i and $\frac{\partial \Omega_i}{\partial x}$ vanish for the value x is the same as saying that Ω_i vanishes for the infinitesimally close value $x + \delta x$. Similarly, this is true of $\frac{\partial \Omega_i}{\partial x_j}$ and analogous quantities on $x + \delta x$. Therefore, *on* $x + \delta x$*, the equations* (I) *are satisfied.* By hypothesis, the equations (II′) are satisfied as well, and as a consequence the equations (II) which are equivalent to (II′) if we take into considerations (I) are satisfied. Therefore, *on* $x + \delta x$*, all of the equations* (I) *and* (II) *are satisfied.*

The theorem is hence demonstrated. We will give it the name *Cauchy's theorem*, by analogy with a well known theorem in the theory of first order partial differential equations, which is a particular case of this.

As an application we apply this theorem to the system (3). We see that each linear integral element at an arbitrarily given point can be represented by a point in ordinary space and an integral element E_2 is represented by a straight line which, in the general case, is subject to meet two fixed lines not lying in the same plane. It follows from this in an obvious way that, through every integral element, there passes one and only one two dimensional integral element (through a point in ordinary space there passes one and only one straight line meeting two fixed lines.) Therefore through every non singular integral variety M_1 there passes one and only one integral variety M_2. The singular linear elements are here those that are represented by the different points on two fixed straight lines. The singular integral varieties M_1 can hence be divided into two distinct series, and they are none other than what we call the *characteristics* in the theory of second order equations.

Let us return to the general case. An integral variety M_1 of the system (3) is obtained, by example, by taking x, y, z, p, q to be five functions of the same variable parameter subject to the equation

$$dz - p\,dx - q\,dy = 0,$$

and by determining u by the equation

$$p' - uq' - ax' - by' = 0.$$

In geometrical language, we thus obtain in the space (x, y, z) the set of a curve and a developable surface circumscribing this curve, and Cauchy's theorem shows that *second order partial differential equations equivalent to the system* (3) *always admit one and only one integral surface in the space* (x, y, z) *passing through an arbitrarily given curve and circumscribing along this curve an arbitrarily given developable surface.*

IV.

Cauchy's theorem makes the importance of the following property of the system (1) evident, which is that each integral element E_p belongs to at least one integral element E_{p+1}. This justifies the following definition:

We say that a system of total differential equations is of GENRE n *if the integral elements with respect to the system satisfy the following conditions:*

Through an arbitrary point it passes at least one integral element E_1*, through an arbitrary integral element* E_1 *there passes at least one integral element* E_2*, etc.;*

Through an arbitrary integral element E_{n-1} there passes at least one integral element E_n; But through an arbitrary integral element E_n there does not pass any integral element E_{n+1}.

More precisely, we assume that there passes

through an arbitrary point	∞^{r_1}	integral elements E_1,
through an arbitrary integral element E_1	∞^{r_2}	integral elements E_2,
....		
through an arbitrary integral element E_{n-1}	∞^{r_n}	integral elements E_n,

some of the numbers r_1, r_2, $\ldots$, r_n may be zero, and we continue to denote by r the number of variables, which is to say that there are ∞^r points.

We also sometimes say that the system, *considered as of $i \leq n$ independent variables*, is *in involution*.

A system of genre *zero* necessarily entails that

$$dx_1 = dx_2 = \cdots = dx_r = 0,$$

and we can leave such systems aside.

According to the preceding and Cauchy's theorem, we see immediately the following property of a system of genre n:

A system of genre n always admits at least one integral variety M_1 passing through an arbitrary point, an integral variety M_2 passing through an arbitrary integral variety M_1, etc., an integral variety M_n passing through an arbitrary integral variety M_{n-1}.

We will agree to say that an integral element E_n is *singular* if it belongs to at least one integral element E_{n+1}, an integral element E_{n-1} is singular if it belongs to more than ∞^{r_n} integral elements E_n, or if the ∞^{r_n} integral elements that it belongs to are all singular, etc., and finally a point is singular if it belongs to more than ∞^{r_1} integral elements E_1, or if the ∞^{r_1} linear elements that based on it are all singular.

The conditions a singular integral element must satisfy are *equality* conditions, and hence we can see clearly that we can always, in an infinite number of ways, find a series of integral elements

$$E_0, \qquad E_1, \qquad E_2, \qquad \ldots, \qquad E_n,$$

where E_0 denotes a point belonging to all the following elements in the series and none of the following elements are singular. Then we can claim the existence of an integral variety M_1 passing through the point E_0 and admitting the element E_1, of an integral variety M_2 passing through M_1 and admitting the element E_2, $\ldots$, of an integral variety passing through M_{n-1} and admitting the element E_n, *but finally we can claim that through M_n there passes no integral variety M_{n+1}*, since the element E_n does not belong to any integral element E_{n+1}.

Therefore, *a system of genre n does not admit integral varieties M_{n+1} passing through an ordinary integral variety.*

These propositions make the importance of the *genre* of a system of equations in total differentials evident.

V.

The numbers r, r_1, r_2, $\ldots$, r_n play a big role in the study of the indeterminacy of the most general n dimensional general integral variety. Before beginning this study, let us prove some remarkable properties of these numbers.

We are first going to prove the following theorem:

In the series

$$r, \qquad r_1, \qquad r_2, \qquad \ldots, \qquad r_n,$$

each number is greater than the following by at least one.

Indeed, first the linear elements defined on a point in the space depend on $r-1$ parameters. But these are not necessarily all integral, therefore

$$r - 1 \geq r_1.$$

In a generic manner, take a non-singular integral element E_{p-1}. By hypothesis, this element belongs to ∞^{r_p} integral elements E_p, where at least one of them is non-singular. Each of them can be defined by a linear (integral) element independent of E_{p-1}, which gives us $r_p + 1$ linear elements

$$\varepsilon, \qquad \varepsilon_1, \qquad \varepsilon_2, \qquad \ldots, \qquad \varepsilon_{r_p}$$

independent among themselves and of E_{p-1}. We can suppose, for example, that the integral element (E_{p-1}, ε) is not singular. This element, in turn, belongs to $\infty^{r_{p+1}}$ integral elements E_{p+1}, each of which is defined by means of a linear element independent of (E_{p-1}, ε), but which is necessarily expressed in terms of E_{p-1}, ε, ε_1, $\ldots$, ε_{r_p}. It is therefore necessary that we can find $r_{p+1} + 1$ such independent elements. Therefore we have

$$r_p \geq r_{p+1} + 1.$$

Q.E.D.

It follows from this that each of the numbers

$$r, \quad r_1 + 1, \quad r_2 + 2, \quad \ldots, \quad r_i + i, \quad \ldots, \quad r_{n-1} + n - 1, \quad r_n + n$$

is at least equal to n, since these numbers cannot increase and the last one is at least equal to n.

Here is a second proposition:

Through every non-singular element E_{p-1} there passes $\infty^{r_p + r_{p+1} - 1}$ integral elements E_{p+1} $(p \leq n-1)$.

Indeed, take a non-singular integral element E_{p-1}. Let

$$\varepsilon, \qquad \varepsilon_1, \qquad \varepsilon_2, \qquad \ldots, \qquad \varepsilon_{r_p}$$

be r_p+1 linear elements independent among themselves and of E_{p-1} which define r_p+1 independent integral elements E_p. Suppose, to fix ideas, that the element (E_{p-1}, ε_1) that we will denote by E_p^0 is not singular. Also suppose, as always possible, that one of the integral elements E_{p+1} passing through E_p^0 is $(E_{p-1}, \varepsilon, \varepsilon_1)$. Let E_{p+1}^0 be this element. Every integral element E_{p+1} passing through E_{p-1} is obtained by adjoining two linear elements ε', ε'' depending on ε, ε_1, $\ldots$, ε_{r_p}. In general, there will exist only one combination element linear in ε' and ε'' and depending on $(\varepsilon_1, \ldots, \varepsilon_{r_p})$ (since this is so for the particular element E_{p+1}^0). Therefore we see that every integral element E_{p+1} passing through E_{p-1} can be obtained in only one manner by taking a linear element ε' depending on ε_1, ε_2, $\ldots$, ε_{r_p}, and by making the element E_{p+1} to pass through the element E_p thus obtained in the most general manner. But ε' depends on r_p-1 parameters, and hence it is the same for E_p. Moreover, through E_p passes exactly $\infty^{r_{p+1}}$ integral elements E_{p+1} (since for the particular non singular element E_p^0, this is the case). Therefore E_{p+1} depends on

$$r_p - 1 + r_{p+1}$$

parameters.

The proof proceeds in the same manner for $p=1$.

We are going to prove in the same way that if $p \leq n-2$, *through a non-singular integral element E_{p-1} there passes $\infty^{r_p-2+r_{p+1}-1+r_{p+2}}$ integral elements E_{p+2}.*

Let us continue to use the same notations. Denote by E_p^0 a non-singular integral element passing through E_{p-1} and let it be (E_{p-1}, ε_2), by E_{p+1}^0 a non-singular integral element passing through E_p^0 and let it be $(E_{p-1}, \varepsilon_1, \varepsilon_2)$, and by E_{p+2}^0 an integral element passing through E_{p+1}^0 and let it be $(E_{p-1}, \varepsilon, \varepsilon_1, \varepsilon_2)$. Then every integral element E_{p+2} can be obtained *in only one manner* by adjoining to E_{p-1} an linear element ε' depending on $(\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{r_p})$ and making an integral element E_{p+2} to pass the element E_p thus obtained: the particular integral element E_{p+1}^0 indeed contains only one integral element E_p satisfying this condition, namely E_p^0. But the element ε' depends on r_p-2 parameters, and hence this is true for E_p as well. Moreover, through E_p which is not singular (since, in particular, E_p^0 is not) there passes $\infty^{r_{p+1}+r_{p+2}-1}$ integral elements E_{p+2}. Therefore the number of parameters E_{p+2} depends on is equal to

$$(r_p-2) + (r_{p+1}-1) + r_{p+2}.$$

Q.E.D.

We see how the theorem generalises step by step. Generally, *if $p \leq n-i$, through a non-singular integral element E_{p-1} there passes integral elements E_{p+i} depending on*

$$(r_p - i) + (r_{p+1} - (i-1)) + \cdots + (r_{p+i-1} - 1) + r_{p+i} = r_p + \cdots + r_{p+i} - \frac{i(i+1)}{2}$$

arbitrary constants.

Of course, the locus of all these elements is not, in general, a flat element, except when i is zero.

In particular, through a non-singular point of space there passes an infinite number of integral elements E_n depending on

$$r_1 + r_2 + \cdots + r_n - \frac{n(n-1)}{2}$$

arbitrary constants. If $n = r$, then $r_1 = n-1, \ldots, r_n = 0$, and there is only one integral element E_n.

Finally, here is the last very important theorem on the series of numbers r:

In the series of positive or zero integers

$$r - r_1 - 1, \qquad r_1 - r_2 - 1, \qquad \ldots, \qquad r_{n-1} - r_n - 1,$$

each number is greater than or equal to the following number.

The fact that the numbers considered are positive or zero results from the first theorem proved on the series

$$r, \qquad r_1, \qquad \ldots, \qquad r_n.$$

To prove the theorem stated, consider a non-singular integral element E_{p-1}. It is possible to find a non-singular integral element E_p^0 passing through E_{p-1}, and in turn a non-singular integral element E_{p+1}^0 passing through E_p^0, and finally an integral element E_{p+2}^0 through E_{p+1}^0. (We suppose that $p \leq n-2$). Let ε, ε_1, ε_2 be three linear elements independent of E_{p-1} which define E_{p+2}^0. These three elements are therefore integral, associated with E_{p-1} and associated among themselves. But there exists $r_p + 1$ independent linear integral elements associated with E_{p-1}. We can therefore denote them by

$$\varepsilon, \qquad \varepsilon_1, \qquad \varepsilon_2, \qquad \ldots, \qquad \varepsilon_{r_p}.$$

Let us find all the integral elements E_{p+2} containing E_{p-1}. Each of them will contain at least a linear element ε'' which is linear in

$$\varepsilon_2, \qquad \varepsilon_3, \qquad \ldots, \qquad \varepsilon_{r_p},$$

and in general, it will contain only one (as E_{p+2}^0). Similarly it will contain one, and in general only one linear element ε' linear in

$$\varepsilon_1, \qquad \varepsilon_3, \qquad \ldots, \qquad \varepsilon_{r_p},$$

and finally one and only one ε''' linear in

$$\varepsilon, \qquad \varepsilon_3, \qquad \ldots, \qquad \varepsilon_{r_p}.$$

We see that, in general, an element E_{p+2} we search for will be *defined* by three linear elements ε', ε'', ε'''. Each of them depends on $r_p - 2$ parameters, which gives a total of

$$3(r_p - 2)$$

parameters. For the element to be integral, it is necessary and sufficient that these three elements are associated among themselves. But an arbitrary element ε linear in ε, ε_1, $\ldots$, ε_{r_p}, is associated to $r_{p+1}+1$ other independent elements of the same form. In other words, to express that an arbitrary element linear in ε, $\ldots$, ε_{r_p} depending on r_p parameters is associated to a particular element of the same form, it is necessary that these r_p parameters satisfy $r_p - r_{p+1} - 1$ relations. Coming back to our three elements ε', ε'', ε''', we therefore see that, for every two of them to be associated, there must be *at most* $r_p - r_{p+1} - 1$ relations among these parameters, which gives in total *at most*

$$3(r_p - r_{p+1} - 1)$$

relations. As there are

$$3(r_p - 2)$$

parameters, we see that *the integral elements E_{p+2} passing through a non-singular integral element E_{p-1} depend on* AT LEAST

$$3(r_p - 2) - 3(r_p - r_{p+1} - 1) = 3r_{p+1} - 3$$

parameters.

Or, according to a preceding theorem, this number of parameters is equal to

$$r_p + r_{p+1} + r_{p+2} - 3,$$

we therefore have

$$r_p + r_{p+1} + r_{p+2} - 3 \geq 3r_{p+1} - 3,$$

i.e.,

$$r_p - r_{p+1} \geq r_{p+1} - r_{p+2}.$$

Q.E.D.

This proof also holds if p is equal to 1.

We can complete this theorem by the following remark:

If n is the genre of the system, we have

$$r_{n-1} - r_n - 1 \geq r_n.$$

Indeed, let E_{n-2} be a non-singular integral element. Let us denote by (E_{n-2}, ε) or E^0_{n-1} a non-singular integral element passing through E_{n-2}, and by $(E_{n-2}, \varepsilon, \varepsilon_1)$ or E^0_n a non-singular integral element passing through E^0_{n-1}. We can find $r_{n-1}+1$ independent linear integral elements associated to E_{n-2}, and as ε and ε_1 are already two among them, we can denote them by

$$\varepsilon, \qquad \varepsilon_1, \qquad \varepsilon_2, \qquad \ldots, \qquad \varepsilon_{r_{n-1}}.$$

Through E^0_{n-1} there passes exactly ∞^{r_n} integral elements E_n. We can therefore suppose that they reduce to

$$(E_{n-2}, \varepsilon, \varepsilon_1), \qquad (E_{n-2}, \varepsilon, \varepsilon_2), \qquad \ldots, \qquad (E_{n-2}, \varepsilon, \varepsilon_{r_n+1}).$$

Now take the integral element (E_{n-2}, ε_1). It also belongs to (at least) ∞^{r_n} integral elements E_n. We can obtain each of them by means of a linear element formed with

$$\varepsilon, \qquad \varepsilon_1, \qquad \varepsilon_2, \qquad \dots, \qquad \varepsilon_{r_{n-1}},$$

which is associated to ε_1. But if we take those which are formed with

$$\varepsilon, \qquad \varepsilon_1, \qquad \dots, \qquad \varepsilon_{r_n+1},$$

there is only ε, otherwise, for example if there is ε_2, the element

$$(E_{n-2}, \varepsilon, \varepsilon_1, \varepsilon_2)$$

will be integral, which contradicts the hypothesis since it passes through the *non-singular* element E_n^0. Therefore there exists at least r_n independent linear elements that can be formed with

$$\varepsilon_{r_n+2}, \qquad \dots, \qquad \varepsilon_{r_{n-1}},$$

and we necessarily have [†]

$$r_{n-1} - r_n - 1 \geq r_n.$$

Q.E.D.

From these several theorems it follows the series of inequalities

$$r - r_1 - 1 \geq r_1 - r_2 - 1 \geq \dots \geq r_{n-1} - r_n - 1 \geq r_n. \tag{5}$$

The numbers in this series play a very big role. We denote them by

$$s, \qquad s_1, \qquad s_2, \qquad \dots, \qquad s_n$$

by setting

$$\begin{cases} s &= r - r_1 - 1, \\ s_1 &= r_1 - r_2 - 1, \\ & \dots \\ s_{n-1} &= r_{n-1} - r_n - 1, \\ s_n &= r_n. \end{cases} \tag{6}$$

A particularly interesting case is that in the series of s there is a term that is zero. Suppose that s_ν $(\nu < n)$ is the first one with this property. Then we necessarily have, according to the inequality (5),

$$s_\nu = s_{\nu+1} = \dots = s_n = 0.$$

The following considerations allow us to derive this result using another method and at the same time lead us to new and important properties of these systems.

[†] The proof does not hold if $n = 1$. But the theorem still holds and it is trivial to give the proof.

Consider a non-singular integral element $E_{\nu-1}$. Let $(E_{\nu-1}, \varepsilon)$ be a non-singular integral element containing $E_{\nu-1}$, ε denoting a linear integral element independent of $E_{\nu-1}$ and associated with $E_{\nu-1}$. Through this element $(E_{\nu-1}, \varepsilon)$ there passes $\infty^{r_{\nu+1}}$ integral elements of $\nu+1$ dimensions, that is to say, as $s_\nu = 0$ and $r_{\nu+1}$ is equal to $r_\nu - 1$, we can find r_ν and only r_ν linear integral elements independent among themselves and with (E_ν, ε) and associated with $E_{\nu-1}$ and ε. Let them be

$$\varepsilon_1, \qquad \varepsilon_2, \qquad \ldots, \qquad \varepsilon_{r_\nu}.$$

But we cannot find more than $r_\nu + 1$ linear intergal elements that are independent among themselves and of $E_{\nu-1}$ and associated to $E_{\nu-1}$. Therefore *every integral element associated to $E_{\nu-1}$ is formed linearly with*

$$E_{\nu-1}, \qquad \varepsilon, \qquad \varepsilon_1, \qquad \ldots, \qquad \varepsilon_{r_\nu}.$$

It follows from this that any two of these elements are associated, for example, ε_1 and ε_2, since the integral element $(E_{\nu-1}, \varepsilon_1)$ belonging to at least $\infty^{r_\nu+1} = \infty^{r_{\nu-1}}$ integral elements of $\nu+1$ dimensions is associated with *at least* r_ν linear integral elements independent among themselves and of $(E_{\nu-1}, \varepsilon_1)$, and as there are *at most* r_ν having this property, namely

$$\varepsilon, \qquad \varepsilon_2, \qquad \ldots, \qquad \varepsilon_{r_\nu}.$$

we see that in particular $(E_{\nu-1}, \varepsilon_1)$ is associated with ε_2. We see that, moreover, the element $(E_{\nu-1}, \varepsilon_1)$ belongs to *exactly* $\infty^{r_{\nu+1}}$ integral elements of $\nu+1$ dimensions.

Now take a $\nu+1$ dimensional integral element passing through $E_{\nu-1}$ and let it be $(E_{\nu-1}, \varepsilon_1, \varepsilon_2)$. We see that, just as before, it belongs to *exactly* $\infty^{r_\nu - 2}$ integral elements of $\nu+2$ dimensions, we therefore have

$$r_{\nu+2} = r_\nu - 2,$$

and so on: a $\nu - 1 + r_\nu$ dimensional element passing through $E_{\nu-1}$, which we denote by $(E_{\nu-1}, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{r_\nu})$, belongs to exactly $1 = \infty^{r_\nu - r_\nu}$ integral elements of ν dimensions, and hence there passes through $E_{\nu-1}$ one and only one integral elements of $\nu + r_\nu$ dimensions, namely $(E_{\nu-1}, \varepsilon, \varepsilon_1, \ldots, \varepsilon_{r_\nu})$, and through this element there does not pass any $\nu + r_\nu + 1$ dimensional integral element. It follows from this that all integral elements passing through $E_{\nu-1}$ are non-singular elements.

In summary, *if we have*

$$s_\nu = r_\nu - r_{\nu+1} - 1 = 0,$$

then the genre of the system is

$$n = \nu + r_\nu$$

and there passes through a non-singular integral element $E_{\nu-1}$ one and only one integral element E_n. The locus of the integral elements passing through $E_{\nu-1}$ is the element E_n, and moreover we have the equalities

$$r_\nu = r_{\nu+1} + 1 = r_{\nu+2} + 2 = \cdots = n - \nu,$$

which entails

$$s_\nu = s_{\nu+1} = \cdots = s_{n-1} = s_n = 0.$$

In particular, if ν is equal to 1, any two linear integral elements based on a non-singular point are associated. An integral element is simply an element formed with the linear integral elements.

To end this section, we are going to determine the numbers r, r_1, $\ldots$, r_n for a system of h total differential equations in r variables, *by supposing that the coefficients are not subject to any specialisation.*

First, we have, manifestly,

$$r_1 = r - (h+1).$$

Suppose in a generic manner that the genre n is greater than p and we know r_p. If E_{p-1} denote an arbitrary integral element, every linear integral element associated with E_{p-1} can be formed linearly with E_{p-1} and the $r_p + 1$ other linear elements

$$\varepsilon, \qquad \varepsilon_1, \qquad \varepsilon_2, \qquad \ldots, \qquad \varepsilon_{r_p}.$$

Let us find how many integral elements E_{p+1} pass through the integral element (E_{p-1}, ε). For this, we must adjoin to ε a linear element ε' that can be formed with

$$\varepsilon_1, \qquad \varepsilon_2, \qquad \ldots, \qquad \varepsilon_{r_p},$$

and which is associated to ε. But this element ε' depends on $r_p - 1$ parameters, and we need h equations to express that this element is associated with ε. We therefore have, if $r_p - 1 \geq h$,

$$r_{p+1} = r_p - 1 - h,$$

and if $r_p - 1 < h$, there is no $p+1$ dimensional integral element. Therefore we see that *we go from a number r to the following number by subtracting $h+1$*, and this we do this as many times as possible

$$r_1 = r - (h+1),$$
$$r_2 = r - 2(h+1),$$
$$\cdots$$

and, the genre n is the integer quotient of r divided by $h+1$ and r_n is equal to the remainder k from this division,

$$r_n = r - n(h+1) = k.$$

The genre of a system whose coefficients are not specialised is therefore equal to the integer quotient of the number of variables by the numbers equations plus one.

The series of numbers s is in this case

$$s = s_1 = \cdots = s_{n-1} = h, \qquad s_n = k.$$

In particular, if there is only one equation, the genre is half of the number of variables: it is n if there are $2n$ or $2n+1$ variables. In the first case, an integral variety M_{n-1} belongs to one and only one integral variety M_n. This is a well known result.

VI.

We are now going to find a system of conditions *determining* all integral varieties M_n subject to these conditions, where n denote the *genre* of the system of equations in total differentials (1).

Let us first make the obvious remark that all the results proved until now continue to hold if we adjoin to the equations (1) a certain number of *algebraic* equations

$$(1') \qquad \begin{cases} f_1(x_1, x_2, \dots, x_r) = 0, \\ \qquad\qquad \dots \\ f_h(x_1, x_2, \dots, x_r) = 0. \end{cases}$$

Indeed, it suffices to adjoin to the equations (1) those that are obtained by taking the total differentials of the equations $(1')$ and in the new system consider only the points of the space that satisfy the equations $(1')$, which we name *integral points.*

Let us find what changes the genre and the integers r_i undergo when we thus adjoin h *arbitrary* algebraic equations. We obtain, in sum, a new system of equations in total differentials whose integral varieties are those of the integral varieties of the old system subject to be completely contained in the arbitrary variety μ represented by the equations $(1')$. It is first obvious that the number r is reduced by h units. In other words there are no more than ∞^{r-h} points to consider. *We assume that these points are not all singular* (with respect to the old system), otherwise the variety μ will be said to be *non-arbitrary.*

Take now a *non-singular* point E_0 of μ. There passes through this point ∞^{r_1} integral elements E_1, that is to say we can find $r_1 + 1$ independent linear elements

$$\varepsilon, \qquad \varepsilon_1, \qquad \varepsilon_2, \qquad \dots, \qquad \varepsilon_{r_1}.$$

On the other hand, the element e_{r-h} which are the locus of the linear elements of μ also contains $r - h$ independent linear elements. If we have

$$r_1 + 1 + r - h \leq r,$$

we will suppose that e_{r-h} does not contain any linear integral element, which is the general case. Then the second system is of genre zero,

$$r' = r - h, \qquad r_1 - h < 0.$$

If, on the contrary,

$$r_1 + 1 + r - h > r,$$

then e_{r-h} contains *at least* $r_1 + 1 - h$ linear integral elements. We will suppose that, which is obviously the general case, e_{r-h} contains exactly $r_1 + 1 - h$ linear integral elements. We then have

$$r' = r - h, \qquad r'_1 = r_1 - h.$$

We will furthermore suppose that the $\infty^{r_1 - h}$ linear integral elements of e_{r-h} *are not all singular.*

Let ε be a non-singular linear integral element of e_{r-h}. Through ε there passes ∞^{r_2} integral elements E_2 of the system (1), that is to say there exists r_2+1 independent linear integral elements associated with ε, which we will call

$$\varepsilon_1, \qquad \varepsilon_2, \qquad \ldots, \qquad \varepsilon_{r_2+1}.$$

On the other hand, e_{r-h} contains, besides ε, $r-h-1$ linear independent elements. If we have

$$(r_2+2)+(r-h-1) \leq r,$$

i.e.,

$$r_2 < h,$$

then e_{r-h} does not in general contain integral element E_2, which we assume. In this case, we therefore have

$$r_2 < h, \qquad n' = 1, \qquad r' = r-h, \qquad r_1' = r_1 - h.$$

But if $r_2 \geq h$, e_{r-h} contains *at least* r_2+1-h independent linear integral elements associated to ε. We suppose that, which is obviously the general case, that e_{r-h} contains *exactly* r_2+1-h of them, that is to say through ε there passes ∞^{r_2-h} integral elements contained in e_{r-h}. We suppose that, moreover, at least one of them is non-singular. We then have

$$r' = r-h, \qquad r_1' = r_1 - h, \qquad r_2' = r_2 - h, \qquad n \geq 2.$$

We see how we can continue this and what are the properties that we must assume the variety μ to have for it to be justified the name *arbitrary. In this case, if r_m denote the last number r larger or equal than h, the genre becomes equal to m and we have*

$$r' = r-h, \qquad r_1' = r_1 - h, \qquad \ldots, \qquad r_m' = r_m - h.$$

It is clear that the conditions a variety must satisfy for it to fail to be arbitrary are *equality* conditions. In particular, we can find on an arbitrary variety a non-singular point E_0, an integral element E_1^0 based at E_0, a non-singular integral element E_2^0 containing E_1^0, $\ldots$, a non-singular integral element E_m^0 containing E_{m-1}^0, but through E_m^0 there does not pass any integral element E_{m+1} belonging to the variety and the number of integral elements E_i belonging to μ passing through the integral element E_{i-1}^0 must be exactly ∞^{r_i-h}.

This being established, we are going to consider a non-singular point μ_0. Let us make an arbitrary $r-r_1$ dimensional variety μ_{r-r_1} pass through this point, an arbitrary $r-r_2$ dimensional variety pass through μ_{r-r_1} , etc., an arbitrary $r-r_n$ dimensional variety pass through $\mu_{r-r_{n-1}}$ [†]. To each of these varieties corresponds a certain system of

[†]This is always possible. Indeed, consider a non-singular integral element E_1^0 based at E_0, a non-singular integral element (E_1^0, ε_1), or E_2^0, containing $E_1^0, \ldots$, a non-singular integral element $(E_{n-1}^0, \varepsilon_{n-1})$ or E_n^0 containing E_{n-1}^0. Let us then denote by e_{r-r_1} an element formed with E_1^0 and the $(r-r_1-1)$ other non-integral linear elements, by e_{r-r_2} an element formed with e_{r-r_1}, ε_1 and the r_1-r_2-1 other linear elements that are non-integral or not associated to $E_1^0, \ldots$, by e_{r-r_n} an element formed with $e_{r-r_{n-1}}$, ε_{n-1} and the $r_{n-r}-r_n-1$ other linear elements that are non-integral or not associated with E_{n-1}^0. It suffices to take μ_{r-r_1} to be a variety admitting the element ε_{r-r_1}, take μ_{r-r_2} a variety admitting the element e_{r-r_2}, etc.

equations in total differentials. For the variety μ_{r-r_1}, we have $h = r_1$, such that

$$n' = 1, \qquad r' = r - r_1, \qquad r'_1 = 0,$$

for μ_{r-r_2}, we have $h = r_2$ and then

$$n'' = 2, \qquad r'' = r - r_2, \qquad r''_1 = r_1 - r_2, \qquad r''_2 = 0,$$

and so on.

From this it follows that the given system admits one and only one integral variety M_1 passing through μ_0 and contained in μ_{r-r_1} (since the system that gives the integral varieties contained in μ_{r-r_1} is of genre 1 and r'_1 is zero). Moreover this variety is not singular, since it admits (*see* note) a non-singular linear element.

Similarly the integral varieties contained in μ_{r-r_2}, being given by a system of genre 2 with $r'_2 = 0$ and M_1 being a non-singular integral variety of the system, it follows that, according to Cauchy's theorem, there exists one and only one integral variety M_2 passing through M_1 and contained in μ_{r-r_2}. Moreover this variety is not singular.

We can continue step by step until an integral variety M_{n-1} contained in $\mu_{r-r_{n-1}}$. Then there exists one and only one integral variety M_n contained in μ_{r-r_n}, and this variety is not singular. Then, finally, there does not exist any integral variety M_{n+1} passing through M_n.

In summary, applying Cauchy's theorem multiple times, we arrive at the following result:

Given

an arbitrary point	μ_0,		
an arbitrary variety	μ_{r-r_1}	*passing through*	μ_0,
an arbitrary variety	μ_{r-r_2}	*passing through*	μ_{r-r_1},
	$\dots$		$\dots$
an arbitrary variety	μ_{r-r_n}	*passing through*	$\mu_{r-r_{n-1}}$,

there exists one and only one integral variety M_n passing through μ_0,

having in common with	μ_{r-r_1}	*a variety* M_1,
having in common with	μ_{r-r_2}	*a variety* M_2,
	$\dots$	$\dots$
having in common with	$\mu_{r-r_{n-1}}$	*a variety* M_{n-1},
and contained entirely in	μ_{r-r_n}.	

Moreover, through this variety M_n there does not pass any integral variety M_{n+1} [†].

[†] The statement continue to hold if the genre is *greater* than n, but then the last part, according to which there does not pass through M_n any integral variety M_{+1}, must be removed.

The problem consisting of finding M_n according to the conditions stated will be called the *Cauchy problem.* The *general integral* will be the set of the integral varieties M_n that can be obtained by the preceding procedure.

We will now try to formulate the Cauchy problem in an *analytic way*, or rather, by conforming to the preceding statement of the problem, we are going to determine the general integral M_n *by a set of analytic conditions which makes its degree of indeterminacy evident.* For this, we start from a non-singular point E_0 and we denote by ε_1 a non-singular integral element based on this point, by $(\varepsilon_1, \varepsilon_2)$ a non-singular integral element E_2 passing through ε_1, …, by (E_{n-1}, ε_n) a non-singular integral element E_n passing through E_{n-1}, such that

$$\varepsilon_1, \qquad \varepsilon_2, \qquad \dots, \qquad \varepsilon_n$$

are n independent linear integral elements that are associated among themselves.

The element E_n can be defined by a system (Σ) of $r-n$ independent linear equations in dx_1, dx_2, …, dx_r. We assume that the indices are chosen in a manner that these equations are solvable with respect to dx_{n+1}, …, dx_r. The element E_{n-1} will be in turn defined by the system (Σ) adjoined with a linear equation in dx_1, dx_2, …, dx_n. Let us suppose that it is solvable with respect to dx_n. We write it as

$$(E_{n-1}) \qquad dx_n = \alpha_{n,1} dx_1 + \dots + \alpha_{n,n-2} dx_{n-2} + \alpha_{n,n-1} dx_{n-1}.$$

Similarly, we have E_{n-2} by adjoining to the preceding equations an equation linear in dx_1, …, dx_{n-1} solvable, for example, with respect to dx_{n-1}. We write it as

$$(E_{n-2}) \qquad dx_{n-1} = \alpha_{n-1,1} dx_1 + \dots + \alpha_{n-1,n-2} dx_{n-2},$$

and so on, until we arrive at the element E_1 that we obtain by adjoining to the equations defining E_2 a linear equation in dx_2, dx_1, solvable, for example, with respect to dx_2. We write it as

$$(E_1) \qquad dx_2 = \alpha_{2,1} dx_1.$$

Let us now denote by (P_0) the flat variety that is the locus of the linear integral elements passing by the point E_0. It obviously contains E_n and it is (r_1+1) dimensional. It is therefore defined by $r - r_1 - 1 = s$ linear equations solvable with respect to s of the differentials dx_{n+1}, …, dx_r. We denote these s differentials by

$$dz_1, \qquad dz_2, \qquad \dots, \qquad dz_s.$$

Note that these s equations are non other than the given equations (1) themselves. Let us denote by (P_1) the flat variety that is the locus of linear elements associated to E_1. It is evidently contained in (P_0) and it contains E_n. Moreover it is $r_2 + 2$ dimensional. It is therefore defined by $r - r_2 - 2 = s + s_1$ equations among we find the s equations of (P_0). We therefore obtain them by adjoining to these s equations s_1 other that are solvable to s_1 of the differentials other than $dz_1, \ldots, dz_s$; $dx_1, \ldots, dx_n$. By changing our notation, let us write these differentials as

$$dz_1^{(1)}, \qquad dz_2^{(1)}, \qquad \ldots, \qquad dz_{s_1}^{(1)}.$$

Similarly the flat variety (P_2) which is the locus of the linear integral elements associated with E_2 is obtained by adjoining to the $s+s_1$ equations of (P_1) s_2 other equations solvable with respect to

$$dz_1^{(2)}, \qquad dz_2^{(2)}, \qquad \ldots, \qquad dz_{s_2}^{(2)},$$

the $z^{(2)}$ being s_2 variables other than $x_1, \ldots, x_n$, z and $z^{(1)}$. And so on. The flat variety (P_{n-1}) which is the locus of linear integral elements associated with E_{n-1} introduces s_{n-1} variables

$$z_1^{(n-1)}, \qquad \ldots, \qquad z_{s_{n-1}}^{(n-1)},$$

and finally, the element E_n will be defined by adjoining to the equations defining (P_{n-1}) $r - s - s_1 - \cdots - s_{n-1} = r_n = s_n$ new equations solvable with respect to s_n variables other than $x_1, \ldots, x_n, z, z^{(1)}, \ldots, z^{(n-1)}$ which we denote by

$$z_1^{(n)}, \qquad z_2^{(n)}, \qquad \ldots, \qquad z_{s_n}^{(n)}.$$

We can summarise the equations defining (P_0), (P_1), $\ldots$, (P_{n-1}), E_n, E_{n-1}, $\ldots$, E_1 in the following table:

$$
(E_1)\left\{\begin{array}{l}
(E_{n-2})\left\{\begin{array}{l}
(E_{n-1})\left\{\begin{array}{l}
(E_n)\left\{\begin{array}{l}
(P_{n-1})\left\{\begin{array}{l}
(P_1)\left\{\begin{array}{l}
(P_0)\left\{\begin{array}{ll} dz & = [dz^{(1)}, dz^{(2)}, \ldots, dz^{(n)}, dx], \end{array}\right. \\
\begin{array}{ll} dz^{(1)} & = [dz^{(2)}, \ldots, dz^{(n)}, dx], \end{array} \\
\qquad \cdots \\
\begin{array}{ll} dz^{(n-1)} & = [dz^{(n)}, dx], \end{array}
\end{array}\right. \\
dz^{(n)} = [dx],
\end{array}\right. \\
dx_n = \alpha_{n,1}dx_1 + \alpha_{n,2}dx_2 + \cdots + \alpha_{n,n-1}dx_{n-1},
\end{array}\right. \\
dx_{n-1} = \alpha_{n-1,1}dx_1 + \alpha_{n,2}dx_2 + \cdots + \alpha_{n-1,n-2}dx_{n-2},
\end{array}\right. \\
\qquad \cdots
\end{array}\right. \\
dx_2 = \alpha_{2,1}dx_1.
\end{array}\right.
$$

The first line expresses that each of the differentials $dz_1, dz_2, \ldots, dz_s$ is expressed in linear combinations of the differentials $dz_1^{(1)}, \ldots, dx_n$.

With these conventions, we will make the following coordinate transformation. Without changing the variables $z, z^{(1)}, \ldots, z^{(n)}$, we take, as new variables,

$$
\begin{aligned}
x'_1 &= x_1, \\
x'_2 &= x_2 - \alpha_{21}x_1, \\
x'_3 &= x_3 - \alpha_{31}x_1 - \alpha_{32}x_2, \\
&\cdots \\
x'_n &= x_n - \alpha_{n1}x_1 - \alpha_{n2}x_2 - \cdots - \alpha_{n,n-1}x_{n-1}.
\end{aligned}
$$

In other words we assume the coefficients α_{ij} to be all zero.

We denote by (after this coordinate transformation has been applied)

$$a_1, \quad \dots, \quad a_n; \quad c_1, \quad \dots, \quad c_s; \quad c_1^{(1)}, \quad \dots, \quad c_{s_1}^{(1)}, \quad \dots, \quad c_1^{(n)}, \quad \dots, \quad c_{s_n}^{(n)}$$

the coordinates of the point E_0.

Finally note that every integral variety M_n admitting the element E_n can be defined by $r-n$ equations solvable respect to z, $z^{(1)}$, ..., $z^{(n)}$ (according to the form of equations of E_n itself). This is also true for all the integral varieties M_n admitting an element sufficient close to E_n. We can therefore take x_1, x_2, ..., x_n as independent variables for these varieties.

This granted, to be sure that we obtain *arbitrary* varieties μ_{r-r_1}, μ_{r-r_2}, ..., let us find an element e_{r-r_1} admitting only a single linear integral element E_1 and passing through E_0, that is to say *cutting the element* (P_0) *following* E_1, an element e_{r-r_2} admitting only a single two dimensional element containing E_1 which we call E_2, and passing through e_{r-r_1}, that is to say *cutting the element* (P_1) *following* E_2, etc., an element e_{r-r_n} admitting only one integral element containing E_{n-1} which we call E_n and passing through $e_{r-r_{n-1}}$, that is to say *cutting the element* (P_{n-1}) *following* E_n. Every variety μ_{r-r_i} admitting the element e_{r-r_i} *or a sufficiently close element* will obviously satisfy the conditions imposed on *arbitrary* varieties. We just need to find the elements e_{r-r_n}, $e_{r-r_{n-1}}$, ..., e_{r-r_1} enjoying these properties stated just now. It suffices to take e_{r-r_n} to be the system

$$dz^{(n)} = [dx],$$

$e_{r-r_{n-1}}$ to be the system obtained by adjoining to the preceding equations the following

$$\begin{aligned} dz^{(n-1)} &= [dz^{(n)}, dx], \\ dx_n \quad &= 0, \end{aligned}$$

$e_{r-r_{n-2}}$ the system obtained by adjoining to the preceding the equation

$$\begin{aligned} dz^{(n-2)} &= [dz^{n-1}, dz^{(n)}, dx], \\ dx_{n-1} \; &= 0, \end{aligned}$$

and so on. The square brackets on the right hand sides denote the same linear combinations as in the equations defining (P_0), (P_1), ..., (P_{n-1}), E_n.

After this, we are justified to define μ_{r-r_n} by the equations

$$(A_n) \qquad \begin{cases} z_1^{(n)} = \varphi_1^{(n)}(x_1, x_2, \dots, x_n), \\ \quad \dots \\ z_{s_n}^{(n)} = \varphi_{s_n}^{(n)}(x_1, x_2, \dots, x_n), \end{cases}$$

to define $\mu_{r-r_{n-1}}$ by the preceding equations and the following

$$(A_{n-1}) \qquad \begin{cases} z_1^{(n-1)} = \varphi_1^{(n-1)}(x_1, x_2, \dots, x_{n-1}), \\ \quad \dots \\ z_{s_{n-1}}^{(n-1)} = \varphi_{s_{n-1}}^{(n-1)}(x_1, x_2, \dots, x_{n-1}), \end{cases}$$

$$(B_n) \qquad x_n = a_n,$$

and so on, and define μ_{r-r_1} by the equations already written and

$$(A_1)\qquad \begin{cases} z_1^{(1)} = \varphi_1^{(1)}(x_1), \\ \quad\dots \\ z_{s_1}^{(1)} = \varphi_{s_1}^{(1)}(x_1), \end{cases}$$

$$(B_2)\qquad x_2 = a_2,$$

and finally the point μ by all the equations already written and

$$(A_0)\qquad \begin{cases} z_1 = \varphi_1, \\ z_2 = \varphi_2, \\ \quad\dots \\ z_s = \varphi_s, \end{cases}$$

$$(B_1)\qquad x_1 = a_1.$$

In these formulae, *the quantities $\varphi_1,\ \varphi_2,\ \dots,\ \varphi_s$ are arbitrary constants sufficiently close to $c_1,\ c_2,\ \dots,\ c_s$. As for the functions $\varphi^{(1)},\ \varphi^{(2)},\ \dots,\ \varphi^{(n)}$, they are arbitrary holomorphic functions in a neighbourhood of*

$$x_1 = a_1, \qquad x_2 = a_2, \qquad \dots, \qquad x_n = a_n$$

such that for this system of values the functions and their first order partial derivatives take sufficiently close values of certain fixed values.

With these hypotheses there exists one and only one integral variety M_n passing through μ_0 and having in common with μ_{r-r_1} a one dimensional variety, with μ_{r-r_2} a two dimensional variety, etc., with $\mu_{r-r_{n-1}}$ a $n-1$ dimensional variety, and finally contained entirely in μ_{r-r_n}. This variety is on the other hand defined by

$$r - n = s_n + s_{n-1} + \dots + s$$

functions $z^{(n)}$, $z^{(n-1)}$, …, z of the independent variables x_1, x_2, …, x_n. To say that M_n is contained in μ_{r-r_n} is to say that the first s_n functions $z_1^{(n)}$, …, $z_{s_n}^{(n)}$ are equal to the given functions $\varphi_1^{(n)}$, …, $\varphi_{s_n}^{(n)}$. If, on the other hand, M_n has in common with $\mu_{r-r_{n-1}}$ a $n-1$ dimensional variety, this variety can only be obtained by making $x_n = a_n$ in the expressions of the functions z, $z^{(1)}$, …. It is therefore necessary that for $x_n = a_n$, the s_{n-1} functions $z_1^{(n-1)}$, …, $z_{s_{n-1}}^{(n-1)}$ reduce to given functions $\varphi_1^{(n-1)}$, …, $\varphi_{s_{n-1}}^{(n-1)}$. And so on.

It follows from this that *under the conditions indicated, the system* (1)*, considered as defining $z_1,\ \dots,\ z_{s_n}^{(n)}$ as functions of $x_1,\ \dots,\ x_n$, admits one and only one solution for which the unknown functions are holomorphic in a neighbourhood of $x_1 = a_1,\ \dots,\ x_n = a_n$ such that the s_n functions $z^{(n)}$ are identical to various functions:*

$$\begin{array}{ll} z_1^{(n)} \text{ to the arbitrary function } & \varphi_1^{(n)}(x_1, \dots, x_n), \\ \dots & \dots \\ z_{s_n}^{(n)} \text{ to the arbitrary function } & \varphi_{s_n}^{(n)}(x_1, \dots, x_n), \end{array}$$

the s_{n-1} *functions* $z^{(n-1)}$ *reduce on* $x_n = a_n$ *to*

$$\begin{array}{ll} z_1^{(n-1)} \text{ to the arbitrary function } & \varphi_1^{(n-1)}(x_1, \ldots, x_{n-1}), \\ \ldots & \ldots \\ z_{s_{n-1}}^{(n-1)} \text{ to the arbitrary function } & \varphi_{s_{n-1}}^{(n-1)}(x_1, \ldots, x_{n-1}), \end{array}$$

and so on. The s_1 *functions* $z^{(1)}$ *reduce on* $x_2 = a_2, \ldots, x_n = a_n$

$$\begin{array}{ll} z_1^{(1)} \text{ to the arbitrary function } & \varphi_1^{(1)}(x_1), \\ \ldots & \ldots \\ z_{s_1}^{(1)} \text{ to the arbitrary function } & \varphi_{s_1}^{(1)}(x_1), \end{array}$$

and finally the s *functions* z *reduce for* $x_1 = a_1, \ldots, x_n = a_n$

$$\begin{array}{ll} z_1 \text{ to the arbitrary constant } & \varphi_1, \\ \ldots & \ldots \\ z_s \text{ to the arbitrary constant } & \varphi_s. \end{array}$$

It is clear, on the other hand, that *every integral variety* M_n *admitting an element near the previously defined particular element* E_n, *or* $(\varepsilon_1, \ldots, \varepsilon_n)$, *can be obtained by the preceding procedure, the functions and the constants* φ *being well-determined in a unique manner.*

We can therefore say that *every integral variety* M_n *admitting a* n *dimensional integral element sufficiently close to a given non-singular integral element is completely defined by the set of*

$$\begin{array}{llll} s_n & \text{arbitrary functions of } n & \text{arguments} & x_1, x_2, \ldots, x_n, \\ s_{n-1} & \text{arbitrary functions of } n-1 & \text{arguments} & x_1, x_2, \ldots, x_{n-1}, \\ \ldots & \qquad\qquad\qquad \ldots & & \ldots \\ s_1 & \text{arbitrary functions of } 1 & \text{argument} & x_1, \\ s & \text{arbitrary constants}, & & \end{array}$$

with the condition that for certain given values of the independent variables, the arbitrary elements and their first order derivatives take values sufficiently close to certain fixed constants.

It is in this sense that we say an integral variety M_n depends on s arbitrary constants, s_1 arbitrary functions of one argument, etc., s_n arbitrary functions of n arguments.

We can say that *the numbers in the series*

$$\text{(S)} \qquad s, \quad s_1, \quad s_2, \quad \ldots, \quad s_n$$

measure the indeterminacy of the integral variety M_n. The geometric origin of these numbers shows that *the measure of indeterminacy does not change if we apply any* *change of variables*, since this amounts to simply apply a homographic transformation on the integral elements based on a point, which obviously does not change the values of the numbers r and hence the numbers s.

Let us recall the property of the series (S) expressed by the inequalities

$$s \geq s_1 \geq s_2 \geq \cdots \geq s_{n-1} \geq s_n,$$

and also the values of r in terms of s:

$$\begin{aligned}
r_n &= s_n, \\
r_{n-1} &= s_n + s_{n-1} + 1, \\
r_{n-2} &= s_n + s_{n-1} + s_{n-2} + 2, \\
&\dots \\
r_1 &= s_n + s_{n-1} + \cdots + s_1 + n - 1, \\
r &= s_n + s_{n-1} + \cdots + s + n,
\end{aligned}$$

As a particular case, if we take a system of h equations in total differentials in r variables with generic coefficients, we have seen that the genre n is equal to the integer quotient of r by $h+1$, and by denoting the remainder by k, we have

$$s = s_1 = \cdots = s_{n-1} = h, \qquad s_n = k.$$

We therefore have the following theorem:

The general integral M_n of a system of h equations in total differentials whose coefficients are arbitrary functions where n denote the integer quotient of r by $h+1$ and k the remainder depends on

$$\begin{array}{lll}
k & \textit{arbitrary functions of } n & \textit{arguments}\ , \\
h & \textit{arbitrary functions of } n-1 & \textit{arguments}\ , \\
h & \textit{arbitrary functions of } n-2 & \textit{arguments}\ , \\
\dots & & \dots \\
h & \textit{arbitrary functions of } 1 & \textit{arguments}\ ,
\end{array}$$

and on h arbitrary constants.

This is the result found by Biermann with much more precision. *We can add that there are in general no $n+1$ dimensional integral.*

If h is equal to 1 and r is even, which is consequently equal to $2n$, there is no arbitrary function of n arguments. If r is odd, and hence equal to $2n+1$, there is one arbitrary function of n arguments.

Let us return to the general case. The results stated continue to hold, *even if the genre is greater than n*, if we take r_n to the be value of s_n and $r_i - r_{i+1} - 1$ to be the values of the other s_i. *It suffices that the given system, considered as in n independent* *variables, to be in involution.* But if the genre is greater than n, s_n may be *greater* than s_{n-1}.

The preceding results are simple if s_n is zero. Then the general integral depends only on arbitrary functions of at most $n-1$ arguments.

The analytic search for the integral M_n amounts to the integration of n successive Kowalewski systems. The first gives the s functions of x_1 which z_1, z_2, …, z_s reduce to when we set

$$x_2 = a_2, \qquad \dots, \qquad x_n = a_n,$$

this is a system of ordinary differential equations that we obtain by replacing in the given system of equations $z^{(1)}$ by $\varphi^{(1)}(x_1)$, $z^{(2)}$ by $\varphi^{(2)}(x_1, a_2)$, …, $z^{(n)}$ by $\varphi^{(n)}(x_1, a_2, \dots, a_n)$.

The second Kowalewski system gives the $s + s_1$ functions of x_1, x_2 which z_1, …, z_s, $z_1^{(1)}$, …, $z_{s_1}^{(1)}$ reduce to when we set

$$x_3 = a_3, \qquad \dots, \qquad x_n = a_n,$$

these functions reduce to the known functions of x_1 on $x_2 = a_2$. And so on. The last system gives the $s + s_1 + \cdots + s_{n-1}$ functions of x_1, x_2, …, x_{n-1} which z_1, …, $z_{s_{n-1}}^{(n-1)}$ reduce to when we set

$$x_n = a_n,$$

these functions reduce to known functions of x_1, …, x_{n-2} when we set $x_{n-1} = a_{n-1}$.

To clarify all the preceding results by a very simple example, take the system formed by the single equation

$$dz - p\,dx - q\,dy = 0, \tag{7}$$

where x, y, z, p, q are five variables. Here there is one equation expressing that two linear integral elements are associated, which is

$$dx\,\delta p - dp\,\delta x + dy\,\delta q - dq\,\delta y = 0. \tag{8}$$

Here $r = 5$ and $r_1 = 3$. As for r_2, the equations defining the linear integral elements associated to a given linear element $(\delta x, \delta y, p\,\delta x + q\,\delta y, \delta p, \delta q)$ contain two independent ones, namely

$$dz - p\,dx - q\,dy = 0,$$
$$\delta p\,dx + \delta q\,dy - \delta x\,dp - \delta y\,dq = 0,$$

then $r_2 = 1$. We therefore have

$$s = 1, \qquad s_1 = 1, \qquad s_2 = 1.$$

A non-singular point E_0 is for example

$$x = y = z = p = q = 0.$$

An integral element E_2 passing through this point is for example

$$(E_2) \qquad dz = dp = dq = 0,$$

and a non-singular integral element E_1 contained in E_2 is for example

$$(E_1) \qquad dz = dp = dq = dy = 0.$$

Here the element (P_0) is given by (7) where we set $p = q = 0$,

$$(P_0) \qquad dz = 0,$$

the element (P_1) is given, according to (8), by

$$(P_1) \qquad dz = dp = 0.$$

Therefore there exists one and only one integral formed by three functions z, p, q in x and y, holomorphic in a neighbourhood of $x = y = 0$ such that

$$\begin{array}{lll} q \text{ is identical to } f(x,y), & & \\ p \text{ reduces to} & \varphi(x) & \text{for } y = 0, \\ z \text{ reduces to} & c & \text{for } x = y = 0, \end{array}$$

where c is a rather small constant, f and φ are arbitrary functions holomorphic in a neighbourhood of $x = 0$, $y = 0$ and together with their first order derivatives taking on $x = y = 0$ rather small values.

Here there are two Kowalewski systems. The first gives the function z of x which reduces to c on $x = 0$, when $p = \varphi(x)$ and $q = f(x, 0)$. It is obviously given by

$$\frac{dz}{dx} = p = \varphi(x),$$

from which

$$z = c + \int_0^r \varphi(x)dx.$$

The second Kowalewski system gives the functions p and z in x and y which reduce to respectively $\varphi(x)$ and $c + \int_0^x \varphi(x)dx$ on $y = 0$ when we set $q = f(x, y)$. This system is [*see* the formulae (II) of section IV]

$$\frac{\partial z}{\partial y} - f(x,y) = 0,$$
$$\frac{\partial p}{\partial y} - \frac{\partial f}{\partial x} = 0,$$

and gives

$$\begin{aligned} &z = c + \int_0^x \varphi(x)dx + \int_0^y f(x,y)dy, \\ &p = \varphi(x) + \int_0^y \frac{\partial f}{\partial x}dy, \\ &q = f(x,y). \end{aligned}$$

We will finish this section by giving some definitions. In the series

$$s, \quad s_1, \quad \dots, \quad s_n$$

that measures the indeterminacy of the general integral M_n of the system (1) of genre n, the first number s is none other than the number of independent equations in dx_1, $\dots$, dx_r of the system (1), that is to say, using the notations of §I, it is the degree of the principal minor of the matrix

$$(\Delta) \qquad \begin{Vmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \\ \dots & \dots & \dots & \dots \end{Vmatrix}.$$

We give the number s_1 a particular name: we call it the *character* [(†)] of the system. Observe that $s + s_1$ is none other than the number of independent equations expressing that a linear element $(dx_1, \dots, dx_r)$ is integral and associated to an arbitrary linear integral $(\delta x_1, \dots, \delta x_r)$. We set

$$a_{ik} = \frac{\partial a}{\partial x_k} - \frac{\partial a_k}{\partial x_i}, \qquad \dots, \qquad l_{ik} = \frac{\partial l_i}{\partial x_k} - \frac{\partial l_k}{\partial x_i},$$

theses equations are

$$(1) \qquad \begin{cases} a_1 dx_1 + a_2 dx_2 + \dots + a_r dx_r = 0, \\ \qquad\qquad\qquad\qquad\qquad \dots \\ l_1 dx_1 + l_2 dx_2 + \dots + l_r\, dx_r = 0, \end{cases}$$

$$(2) \qquad \begin{cases} \sum_i a_{1i}\delta x_i dx_1 + \ldots + \sum_i a_{ri}\delta x_i dx_r = 0, \\ \qquad\qquad\qquad\qquad\qquad \dots \\ \sum_i l_{1i}\delta x_i dx_1 + \ldots + \sum_i l_{ri}\delta x_i dx_r = 0, \end{cases}$$

Therefore if we consider the matrix

$$(\Delta_1) \qquad \begin{Vmatrix} a_1 & a_2 & \dots & a_r \\ \dots & \dots & \dots & \dots \\ l_1 & l_2 & \dots & l_r \\ \sum a_{1i}\delta x_i & \sum a_{2i}\delta x_i & \dots & \sum a_{ri}\delta x_i \\ \dots & \dots & \dots & \dots \\ \sum l_{1i}\delta x_i & \sum l_{2i}\delta x_i & \dots & \sum l_{ri}\delta x_i \end{Vmatrix}$$

where δx_1, $\dots$, δx_r are arbitrary elements subject to the equations

$$\begin{gathered} a_1\delta x_1 + \ldots + a_r\delta x_r = 0, \\ \dots \\ l_1\delta x_1 + \ldots + l_r\delta x_r = 0, \end{gathered}$$

(†) This designation is due to, I think, H. von Webber, *Zur Invariantentheorie der Systeme Pfaff'scher Gleichungen* (*Peipz. Ber.*, p. 207–229; 1898).

the character s_1 of the system is the difference between the degree of the principal minor of the matrix (Δ_1) and the degree of the principal minor of the matrix (Δ).

We can give the other numbers s_2, s_3, ..., the names of second, third, ... *characters* of the system (1). The are calculated by the degrees of principal minors just as s and s_1 are. But *instead of saying that a system of genre n has its n-th character s_n, we say that the system is of the (s_n+1)-th kind.* A system of the first kind is therefore a system for which $s_n = 0$, and it enjoys the property that through an integral variety M_{n-1} there passes one *and only one* integral variety M_n.

VII.

In this paragraph we are going to be concerned with systems of the first kind for which the $(n-1)$-th character s_{n-1} is zero. Suppose in a general manner that s_ν is the first number that is zero in the series

$$s_1, \quad s_2, \quad \ldots, \quad s_n,$$

ν being smaller than n. In §V we have seen some properties of these systems, which we now recall:

Through a non-singular intergal element $E_{\nu-1}$ there passes one and only one integral element E_n. This element E_n is the locus of the integral elements passing through $E_{\nu-1}$, and none of these elements are singular.

We have furthermore

$$r_n = 0, \quad r_{n-1} = 1, \quad r_{n-2} = 2, \quad \ldots, \quad r_\nu = n-\nu, \quad r_{\nu-1} \leq n-\nu-2.$$

As a corollary to the property of integral elements passing through a non-singular integral element $E_{\nu-1}$, we are going to prove the following theorem:

Through a non-singular integral variety $M_{\nu-1}$ there passes one and only one integral variety M_n.

For the proof, let us make an arbitrary variety μ_{r-r_ν} pass through $M_{\nu-1}$, which is always possible, and we can ensure that the integral variety $M_{\nu-1}$ is not singular. If, in particular, $E_{\nu-1}$ is a non-singular integral element of $M_{\nu-1}$, the variety μ_{r-r_ν} will admit one and only one integral element E_ν passing through $E_{\nu-1}$. Granted this, let M_n be any integral variety passing through $M_{\nu-1}$. It naturally admits the *unique* integral element E_n passing through $E_{\nu-1}$. On the other hand, the sum of the dimensions of M_n and μ_{r-r_ν} is

$$r + n - r_\nu = r + \nu,$$

then these two varieties have in common an variety of dimension *at least* ν, and this variety is necessarily integral. But μ_{r-r_ν} does not contain $\nu+1$ dimensional integral elements passing through $E_{\nu-1}$, and hence this integral variety common to M_n and μ_{r-r_ν} is of *exactly* ν dimensions. Let us denote it by M_ν.

Granted this, we know that through a non-singular integral variety $M_{\nu-1}$ there passes one and *only one* ν dimensional integral variety contained in the arbitrary variety μ_{r-r_ν}: *therefore the variety M_ν is determined in a unique manner when we specify μ_{r-r_ν}*. In other words, *if through $M_{\nu-1}$ there passes two n dimensional integral varieties M_n and M'_n, these two varieties cut μ_{r-r_ν} following the same variety M_ν regardless of the particular μ_{r-r_ν} passing through $M_{\nu-1}$*.

From this it follows that the two varieties M_n and M'_n are identical. Since if A is any point on the first variety, we can always make a variety μ_{r-r_ν} pass through A and $M_{\nu-1}$. To this variety corresponds an integral variety M_ν situated inside M_n and passing through A, but it is also situated inside M'_n, and therefore the point A belongs to M'_n and the two varieties are coincidental.

In a more precise and rigorous manner, let us make an arbitrary variety $\mu_{r-r_\nu-1}$ pass through $M_{\nu-1}$, that is to say $\mu_{r-r_\nu-1}$ does not admit integral elements passing through $E_{\nu-1}$ other than $E_{\nu-1}$ itself, which can always be arranged. Let us then make a family of varieties μ_{r-r_ν} depending on $r_\nu = n-\nu$ parameters and *filling all of the space* pass through this determined variety $\mu_{r-r_\nu-1}$ [(†)]. These varieties are all *arbitrary*, since they obviously contain only one integral element E_ν passing through $E_{\nu-1}$, and we know that every integral element passing through $E_{\nu-1}$ is non-singular. Each of them therefore contains one and only one integral variety M_ν passing through $M_{\nu-1}$ and all the varieties M_ν belong to some integral variety M_n passing through $M_{\nu-1}$. We can add that *M_n is the locus of these varieties M_ν when the $n-\nu$ parameters they depend on vary*, since each of them is contained in M_n and, on the other hand, through any point of M_n there passes one of the varieties μ_{r-r_ν} (which fills up the space) and hence, the corresponding variety M_ν. Hence M_n is determined in a unique manner.

We summarise the results that we just obtained in the following manner:

Through a non-singular integral variety $M_{\nu-1}$ there passes one and only one integral variety M_n. To obtain it we make an arbitrary variety $\mu_{r-r_\nu-1}$ pass through $M_{\nu-1}$ and through this latter variety a family of varieties μ_{r-r_ν} depending on $r_\nu = n-\nu$ parameters and filling up the space. For each of these varieties μ_{r-r_ν} we determine the integral variety M_ν passing through $M_{\nu-1}$ which is contained entirely in μ_{r-r_ν}. The geometrical locus of these varieties M_ν, when we vary the $n-\nu$ parameters they depend on, is the integral variety M_n we search for.

Furthermore, *the variety M_ν is the integral of a system of equations in total differentials in $r-r_\nu$ variables of genre ν, but whose coefficients depend on $n-\nu$ parameters.*

[(†)] If

$$f_1 = f_2 = \cdots = f_{r_\nu+1} = 0$$

are the equations of $\mu_{r-r_\nu-1}$, it obviously suffices to take

$$f_1 - t_1 f_{r_\nu+1} = f_2 - t_2 f_{r_\nu+1} = \cdots = f_{r_\nu} - t_{r_\nu} f_{r_\nu+1} = 0.$$

From this we deduce the following theorem which refers to the Cauchy problem itself:

Consider a system of equations in total differentials of genre n whose character s_ν is zero ($\nu < n$). Given an arbitrary point μ_0 and an arbitrary variety μ_{r-r_1} passing through this point, etc., an arbitrary variety $\mu_{r-r_{\nu-1}}$ passing through $\mu_{r-r_{\nu-2}}$, there exists one and only one integral variety M_n passing through μ_0 having in common with μ_{r-r_1} a 1 dimensional variety, etc., with $\mu_{r-r_{\nu-1}}$ a $\nu-1$ dimensional variety. To obtain it we choose an arbitrary variety $\mu_{r-r_\nu-1}$ pass through $\mu_{r-r_{\nu-1}}$ and through this latter variety a family of varieties μ_{r-r_ν} depending on $r_\nu = n-\nu$ parameters and filling up the space. Each of these varieties contains one and only one integral variety M_ν passing through μ_0, having in common with μ_{r-r_1} a 1 dimensional variety, etc., with $\mu_{r-r_{\nu-1}}$ a $\nu-1$ dimensional variety. The geometrical locus of these varieties M_ν, when we vary the $n-\nu$ parameters that they depend on, is the integral variety M_n we search for.

Indeed, it suffices to observe that the $\nu-1$ dimensional integral variety situated inside $\mu_{r-r_{\nu-1}}$ is the same for M_n and all the M_ν. Then we only have to apply the preceding theorem to this $\nu-1$ dimensional integral variety.

The last theorem shows that *the Cauchy problem for the given system of genre n amounts to a Cauchy problem for a new system of genre ν, but the coefficients of the new system depend on $n-\nu$ parameters.* The numbers $s, s_1, s_2, \ldots, s_\nu$ have, furthermore, the same values for the two systems.

We say that the integer ν is the *true genre* of the system.

We are now going to translate the preceding results analytically. Let us use the notations of §VI. Here a simplification arises due to the fact that $s_n, s_{n-1}, \ldots, s_\nu$ are zero and hence there are no variables $z^{(n)}, z^{(n-1)}, \ldots, z^{(\nu)}$.

The variety $\mu_{r-r_{\nu-1}}$ is defined by

$$(A_{\nu-1}) \qquad \begin{cases} z_1^{(\nu-1)} = \varphi_1^{(\nu-1)}(x_1, x_2, \ldots, x_{\nu-1}), \\ \quad \ldots \\ z_{s_{\nu-1}}^{(\nu-1)} = \varphi_{s_{\nu-1}}^{(\nu-1)}(x_1, x_2, \ldots, x_{\nu-1}), \end{cases}$$

$$(B_\nu) \qquad x_n = a_n, \quad x_{n-1} = a_{n-1}, \quad \ldots, \quad x_\nu = a_\nu.$$

The variety $\mu_{r-r_{\nu-2}}$ is defined by the preceding equations and also

$$(A_{\nu-2}) \qquad \begin{cases} z_2^{(\nu-1)} = \varphi_2^{(\nu-1)}(x_1, x_2, \ldots, x_{\nu-2}), \\ \quad \ldots \\ z_{s_{\nu-2}}^{(\nu-2)} = \varphi_{s_{\nu-2}}^{(\nu-2)}(x_1, x_2, \ldots, x_{\nu-2}), \end{cases}$$

$$(B_{\nu-1}) \qquad x_{\nu-1} = a_{\nu-1},$$

and so on, as in the general case.

We can now take the variety defined by the $n-\nu+1 = r_\nu+1$ equations (B_ν)

$$x_n = a_n, \quad x_{n-1} = a_{n-1}, \quad \ldots, \quad x_\nu = a_\nu.$$

as the variety μ_{r-r_n-1}, and take the varieties defined by

$$(\mu_{r-r_\nu})\qquad \begin{cases} x_{\nu+1}-a_{\nu+1}=t_1(x_\nu-a_\nu),\\ x_{\nu+2}-a_{\nu+2}=t_2(x_\nu-a_\nu),\\ \qquad\dots\\ \quad x_n-a_n=t_{n-\nu}(x_\nu-a_\nu). \end{cases}$$

as the varieties μ_{r-r_ν}.

The preceding results stated in a geometrical manner can be now expressed in the following manner:

The given system admits one and only one integral for which $z_1, \dots, z^{(\nu-1)}_{s_{\nu-1}}$ are holomorphic functions in $x_1, x_2, \dots, x_n$ in a neighbourhood of

$$x_1=a_1,\quad x_2=a_2,\quad \dots,\quad x_n=a_n,$$

and respectively reduce

$$\left.\begin{array}{ll} z_1^{(\nu-1)} \text{ to the arbitrary function } & \varphi_1^{(\nu-1)}(x_1,x_2,\dots,x_{\nu-1})\\ \dots & \dots\\ z_{s_{\nu-1}}^{(\nu-1)} \text{ to the arbitrary function } & \varphi_{s_{\nu-1}}^{(\nu-1)}(x_1,x_2,\dots,x_{\nu-1}) \end{array}\right\}\text{ for }\begin{cases} x_\nu=a_\nu,\\ x_{\nu+1}=a_{\nu+1},\\ \dots\\ x_n=a_n\end{cases}$$

$$\left.\begin{array}{ll} z_1^{(\nu-2)} \text{ to the arbitrary function } & \varphi_1^{(\nu-2)}(x_1,x_2,\dots,x_{\nu-2})\\ \dots & \dots\\ z_{s_{\nu-2}}^{(\nu-2)} \text{ to the arbitrary function } & \varphi_{s_{\nu-2}}^{(\nu-2)}(x_1,x_2,\dots,x_{\nu-2}) \end{array}\right\}\text{ for }\begin{cases} x_{\nu-1}=a_{\nu-1},\\ \dots\\ x_n=a_n\end{cases}$$

$\dots$

$$\left.\begin{array}{ll} z_1^{(1)} \text{ to the arbitrary function } & \varphi_1^{(1)}(x_1)\\ \dots & \dots\\ z_{s_1}^{(1)} \text{ to the arbitrary function } & \varphi_{s_1}^{(1)}(x_1) \end{array}\right\}\text{ for }\begin{cases} x_2=a_2,\\ \dots\\ x_n=a_n\end{cases}$$

$$\left.\begin{array}{ll} z_1 \text{ to the arbitrary constant } & \varphi_1\\ \dots & \dots\\ z_s \text{ to the arbitrary constant } & \varphi_s \end{array}\right\}\text{ for }\begin{cases} x_1=a_1,\\ \dots\\ x_n=a_n\end{cases}$$

To obtain these functions, we replace in the given equations of the differential system

$$\begin{array}{lll} x_{\nu+1} & \text{by } a_{\nu+1}+ & t_1(x_\nu-a_\nu),\\ x_{\nu+2} & \text{by } a_{\nu+2}+ & t_2(x_\nu-a_\nu),\\ \dots & \qquad\dots & \\ x_n & \text{by } a_n & +\,t_{n-\nu}(x_\nu-a_\nu), \end{array}$$

where we regard t as constants. The new system obtained then admits one and only one integral for which $z_1, \ldots, z^{\nu-1}_{s_{\nu-1}}$ are holomorphic functions in $x_1, x_2, \ldots, x_\nu$ in a neighbourhood of

$$x_1 = a_1, \quad x_2 = a_2, \quad \ldots, \quad x_\nu = a_\nu,$$

and respectively reduce

$$\begin{array}{lll}
z_i^{(\nu-1)} & \text{to } \varphi_i^{(\nu-1)}(x_1, x_2, \ldots, x_{\nu-1}) & \text{for } x_\nu = a_\nu, \\
z_j^{(\nu-2)} & \text{to } \varphi_j^{(\nu-2)}(x_1, x_2, \ldots, x_{\nu-2}) & \text{for } x_{\nu-1} = a_{\nu-1}, x_\nu = a_\nu \\
\ldots & \ldots & \ldots \\
z_k^{(1)} & \text{to } \varphi_k^{(1)}(x_1) & \text{for } x_2 = a_2, \ldots, x_\nu = a_\nu, \\
z_h & \text{to } \varphi_h & \text{for } x_1 = a_1, \ldots, x_\nu = a_\nu,
\end{array}$$

$$(i = 1, 2, \ldots, s_{\nu-1}; j = 1, 2, \ldots, s_{\nu-2}; k = 1, 2, \ldots, s_1; h = 1, 2, \ldots, s).$$

If in the functions thus found we replace the $n - \nu$ parameters t they the depend on by respectively

$$t_1 = \frac{x_{\nu+1} - a_{\nu+1}}{x_\nu - a_\nu},$$
$$\ldots$$
$$t_{n-\nu} = \frac{x_n - a_n}{x_\nu - a_\nu},$$

we obtain the integral of the primitive system that we search for.

In summary, we see that, *if the integral variety M_n of a system in involution (that is to say of genre greater or equal than n) depends on arbitrary functions of $\nu - 1$ arguments but not on arbitrary functions of ν arguments, we can carry out the search for this integral variety as the search for the integral M_ν of a system of genre ν and consequently to a problem in ν independent variables, but the coefficients of the new system will depend on $n - \nu$ parameters.*

Indeed, it suffices to observe that as M_n does not depend on arbitrary functions of n arguments, we necessarily have $r_n = 0$ and then the genre of the system is exactly n.

In particular, if the general integral M_n of a system in involution depends only on *arbitrary constants*, we have $\nu = 1$, the *character* s_1 of the system is hence zero and the system is completely integrable. We can make the search of its integral to that of a system of genre 1, that is to say a system of ordinary differential equations. *The method reduces to that of Lie-Mayer for integration completely integrable systems.* If the equations of the system are solvable with respect to $dz_1, dz_2, \ldots, dz_s$, the other differentials being $dx_1, dx_2, \ldots, dx_n$, we replace

$$\begin{array}{ll}
x_2 & \text{by } a_2 + t_1(x_1 - a_1), \\
\ldots & \ldots \\
x_n & \text{by } a_n + t_{n-1}(x_1 - a_1),
\end{array}$$

and we look for the integral of the new system such that z_1, z_2, $\ldots$, z_s reduce on $x_1 = a_1$ to given arbitrary constants φ_1, φ_2, $\ldots$, φ_s. We then replace in the functions obtained t_i by $\frac{x_{i+1}-a_{i+1}}{x_1-a_1}$.

As an example, we take the simplest case possible where we do not have a completely integrale system. We choose

$$\nu = 2, \quad n = 3, \quad s_1 = 1, \quad s = 1.$$

Then

$$r_3 = 0, \quad r_2 = 1, \quad r_1 = 3, \quad r = 5.$$

The following equation corresponds to this case

$$x_4 dx_1 - x_4 x_5 dx_2 - (x_2 x_4 + x_3 + x_1 x_5) dx_5 = 0.$$

We will not verify the above and we will simply apply the generalised Lie-Mayer method to find the general integral of this equation. We see easily that the point

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 1, \quad x_5 = 0$$

is not singular, that the linear element E_1 based on this point

$$dx_1 = dx_2 = dx_3 = dx_4 = 0$$

is integral. Here (P_0) has equation

$$(P_0) \qquad dx_1 = 0,$$

as for (P_1), we easily find that

$$(P_1) \qquad dx_1 = dx_3 = 0$$

and we can check that this element E_3 is integral. Moreover, E_1 is not singular.

We can therefore take the variables x_1, x_3, x_4, x_2, x_5 to be the variables denoted by z, $z^{(1)}$, x_3, x_2, x_1 in the general theory, respectively.

There is therefore one and only one integral, such that

$$\begin{aligned} &x_3 \text{ reduce to } f(x_5) && \text{for } x_2 = 0, x_4 = 1, \\ &x_1 \text{ reduce to } c && \text{for } x_2 = 0, x_4 = 1, x_5 = 0. \end{aligned}$$

To obtain it, it suffices to replaces in the equation

$$x_4 - 1 \quad \text{by} \quad t x_2,$$

which gives

$$(t x_2 + 1) dx_1 - (t x_2 + 1) x_5 dx_2 - (t x_2^2 + x_2 + x_3 + x_1 x_5) dx_5 = 0.$$

We must first search the function x_1 in x_5 which reduces to c for $x_5 = 0$ when we set $x_3 = f(x_5)$, $x_2 = 0$. This function is given by

$$\frac{dx_1}{dx_5} = f(x_5) + x_1 x_5,$$

from which we deduce

$$x_1 = ce^{x_5^2} + e^{x_5^2/2} \int_0^{x_5} e^{-x_5^2/2} f(x_5) dx_5 = \varphi(x_5).$$

We must then search for the two functions x_3 and x_1 in x_2, x_5 which reduce to $f(x_5)$ and $\varphi(x_5)$ for $x_2 = 0$. They are given by

$$\begin{aligned} \frac{\partial x_1}{\partial x_2} &= x_5, \\ 1 &= 1 + \frac{\partial}{\partial x_2} \frac{x_3 + x_1 x_5}{t x_2 + 1}. \end{aligned}$$

The second one gives

$$x_3 + x_1 x_5 = (t x_2 + 1) \left[f(x_5) + c x_5 e^{x_5^2/2} + x_5 e^{x_5^2/2} \int_0^{x^5} e^{-x_5^2/2} f(x_5) \right],$$

and the first one

$$x_1 = x_2 x_5 + c e^{x_5^2/2} + e^{x_5^2/2} \int_0^{x_5} e^{-x_5^2/2} f(x_5) dx_5.$$

By replacing in the first formula $t x_2 + 1$ by x_4, we obtain the general integral that can be written as

$$\begin{aligned} x_1 - x_2 x_5 &= F(x_5), \\ x_3 + x_1 x_5 &= x_4 F'(x_5) \end{aligned}$$

by setting

$$F(x_5) = c e^{x_5^2} + e^{x_5^2/2} \int_0^{x_5} e^{-x_5^2/2} f(x_5) dx_5.$$

VIII.

In this last section we are going to be concerned with certain systems for which the Kowalewski system determining the integral variety M_{p+1} passing through a given integral variety presents certain simple properties that make this integration easy. Such a system, by using the notations of §III, is, if we restrict ourselves to the case of $s_{p+1} = 0$, solvable with respect to

$$\frac{\partial z_1}{\partial x}, \quad \frac{\partial z_2}{\partial x}, \quad \ldots, \quad \frac{\partial z_m}{\partial x},$$

the right hand sides depending on the variables and the first order derivatives of the unknown functions z with respect to the independent variables $x_1, x_2, \ldots, x_p$ other than x.

If we solve the Cauchy problem for a first order partial differential equation in one unknown function, by a change of independent variables we are led precisely to a Kowalewski system *where the right hand sides do not depend on the derivatives* $\frac{\partial z_i}{\partial x_k}$. Then we are actually led to a system of ordinary differential equations.

Let us find in what cases the same situation arises. As we have done in §III, we denote by ε, $\varepsilon^{(1)}$, $\varepsilon^{(2)}$, $\ldots$, $\varepsilon^{(p)}$ the $p+1$ linear elements

$$(\varepsilon) \qquad \frac{dx}{1} = \frac{dx_1}{0} = \cdots = \frac{dx_p}{0} = \frac{dz_1}{\frac{\partial z_1}{\partial x}} = \cdots = \frac{dz_m}{\frac{\partial z_m}{\partial x}},$$

$$([\varepsilon^{(1)}]) \qquad \frac{dx}{0} = \frac{dx_1}{1} = \cdots = \frac{dx_p}{0} = \frac{dz_1}{\frac{\partial z_1}{\partial x_1}} = \cdots = \frac{dz_m}{\frac{\partial z_m}{\partial x_1}},$$

$$\ldots$$

$$([\varepsilon^{(p)}]) \qquad \frac{dx}{0} = \frac{dx_1}{0} = \cdots = \frac{dx_p}{1} = \frac{dz_1}{\frac{\partial z_1}{\partial x_p}} = \cdots = \frac{dz_m}{\frac{\partial z_m}{\partial x_p}},$$

the Kowalewski equations express that the element ε is integral and associated to the element E_p $[\varepsilon^{(1)}, \varepsilon^{(2)}, \ldots, \varepsilon^{(p)}]$, *subject to the condition that the element E_p is integral.*

Granted this, suppose that the Kowalewski equations do not depend on $\frac{\partial z_i}{\partial x_k}$, i.e., a sum of $\varepsilon^{(1)}, \ldots, \varepsilon^{(p)}$. Then the values of $\frac{\partial z_1}{\partial x}, \ldots, \frac{\partial z_m}{\partial x}$ that they determine furnish a linear integral element ε, depending only on the point where it is based at, *which is associated to all the linear elements E_p passing through the point*, since an arbitrary element E_p can always be formed linearly by ε and an element of the form $[\varepsilon^{(1)}, \varepsilon^{(2)}, \ldots, \varepsilon^{(p)}]$.

In summary, *there passes through each point of the space a linear integral element enjoying the property that it is associated to any linear integral element based on the same point.*

This linear element is necessarily *singular*, since it belongs to ∞^{r_1-1} integral elements E_2. We say that it is *characteristic.*

In general, *an integral element E_p based on a non-singular point of the space is called* characteristic *if it is associated to any linear integral element based on the same point.*

All linear elements contained in a characteristic element E_h $(h > 1)$ are themselves characteristic and the locus of the characteristic linear elements is necessarily a flat characteristic element, which is the largest characteristic element based on the point.

To obtain the linear characteristic elements based on a given non-singular point analytically, let us denote by

$$\delta x_1, \quad \delta x_2, \quad \ldots, \quad \delta x_r$$

the coordinates of such an element and by

$$dx_1, \quad dx_2, \quad \ldots, \quad dx_r$$

the coordinates of a variable integral element based on the same point. We have, to determine δx_i, the equations

$$\begin{cases} a_1\delta x_1 + \cdots + a_r\delta x_r = 0, \\ \cdots \\ l_1\delta x_1 + \cdots + l_r\delta x_r = 0, \\ \sum a_{1i}dx_i\delta x_1 + \cdots + \sum a_{ri}dx_i\delta x_r = 0, \\ \cdots \\ \sum l_{1i}dx_i\delta x_1 + \cdots + \sum l_{ri}dx_i\delta x_r = 0, \end{cases}$$

where the notations are the same as in §I [†]. Moreover, these equations must hold whatever the values of

$$dx_1, \quad dx_2, \quad \ldots, \quad dx_r,$$

subject to the conditions that these quantities satisfy the s equations (1):

$$(1) \qquad \begin{cases} a_1dx_1 + a_2dx_2 + \cdots + a_rdx_r = 0, \\ \cdots \\ l_1dx_1 + l_2dx_2 + \cdots + l_rdx_r = 0. \end{cases}$$

Then, the equation in $dx_1, \ldots, dx_r$

$$\sum a_{1i}\delta x_i dx_1 + \cdots + \sum a_{ri}\delta x_i dx_r = 0$$

must be a consequence of the equations (1). In other words, *all the minors formed with $s+1$ columns of the matrix*

$$(A) \qquad \left\| \begin{matrix} a_1 & a_2 & \ldots & a_r \\ \cdots & \cdots & \cdots & \cdots \\ l_1 & l_2 & \ldots & l_r \\ \sum a_{1i}\delta x_i & \sum a_{2i}\delta x_i & \ldots & \sum a_{ri}\delta x_i \end{matrix} \right\|$$

must be zero. The same holds if we replace in this matrix the last line by the $s-1$ analogous lines deduced from the last $s-1$ equations of (1), which gives the matrices (B), $\ldots$, (L).

Ultimately, *the equations determining the characteristic linear elements are of two sorts: first the s equations*

$$(1') \qquad \begin{cases} a_1\delta x_1 + a_2\delta x_2 + \cdots + a_r\delta x_r = 0, \\ \cdots \\ l_1\delta x_1 + l_2\delta x_2 + \cdots + l_r\delta x_r = 0. \end{cases}$$

[†] We have set simply

$$a_{ik} = \frac{\partial a_i}{\partial x_k} - \frac{\partial a_k}{\partial x_i}, \qquad \ldots, \qquad l_{ik} = \frac{\partial l_i}{\partial x_k} - \frac{\partial l_k}{\partial x_i}.$$

which expresses that this element is integral, and then the equations obtained by annihilating all the minors formed with $s+1$ columns of the s matrices (A), …, (L),

$$(A)\qquad \begin{Vmatrix} a_1 & \dots & a_r \\ \dots & \dots & \dots \\ l_1 & \dots & l_r \\ \sum a_{1i}\delta x_i & \dots & \sum a_{ri}\delta x_i \end{Vmatrix},$$

$$\dots,$$

$$(L)\qquad \begin{Vmatrix} a_1 & \dots & a_r \\ \dots & \dots & \dots \\ l_1 & \dots & l_r \\ \sum l_{1i}\delta x_i & \dots & \sum l_{ri}\delta x_i \end{Vmatrix}.$$

If among these equations there are less than r independent ones, there exists characteristic linear elements and these equations determine their locus, that is to say the largest characteristic element based on the point.

If the given system is completely integral, any two linear integral elements are associated, and then, the equations of the characteristic elements reduce to the equations $(1')$. The principal minors of the matrices (A), …, (L) are of degree s by taking $(1')$ into account.

Here are now some simple fundamental properties of the characteristic elements:

Given a characteristic element E_p, every non-singular *integral element E_n contains E_p.* Indeed, otherwise the smallest element containing E_n and E_p will be at least $n+1$ dimensional and it will necessarily be integral since E_n and E_p are associated. The integral element E_n belonging to an integral element E_{n+1} will therefore be singular. Obviously here n denotes the genre of the given system.

If through every non-singular point of the space there passes a characteristic element, the given differential system is of the first kind. Since if we consider a characteristic element ε, then every non-singular integral element E_n contains ε, and there certainly exists integral elements E_{n-1} that does not contain ε and naturally, among these integral elements there are those that are not singular [†]. Let E_{n-1} be one of them. Through E_{n-1} there passes ∞^{r_n} integral elements E_n and at least one of them is non-singular, that is to say contains ε. If r_n is equal to or greater than 1, there will be at least one integral element E_n other than (E_{n-1}, ε). Let us call it (E_{n-1}, ε'). But then the element $(E_{n-1}, \varepsilon, \varepsilon')$ will be integral and the non-singular element (E_{n-1}, ε) will belong to another integral element of $n+1$ dimensions, which is impossible. Therefore r_n *must be zero*, that is to say the given differential system is of the first kind. *Therefore the systems of the first kind are the only ones that there can have characteristic elements.*

Similarly, we see that *if there exists a characteristic integral element E_p, the true genre of the system is at most $n-p+1$.* Since there certainly exists a non-singular integral element E_{n-p} having no common element with E_p and every non-singular integral element E_n passing through E_{n-p} must contain E_p, therefore it is determined in a

[†] Otherwise every non-singular integral element E_{n-1} will be subject to an *equality* condition: that which contains ε.

unique manner and we can denote it by (E_{n-p}, E_p). If E_{n-p} belongs to another integral element E_n, the element (E_n, E_p) will be integral and be at least $n+1$ dimensional. On the other hand it contains (E_{n-p}, E_p) which will then be singular. Hence E_{n-p} belongs to only one integral element E_n. Then, finally, the true genre of the system is at most $n-p+1$.

We can add that *if there exists a non-singular integral element E_{n-1} containing E_p, the true genre is at most $n-p$.* Since there always exists an integral element E_{n-p-1} contained in E_{n-1} and having no common element with E_p. If an n dimensional integral element passing through E_{n-p-1} contains E_p, it will also contain E_{n-1}, and then is completely and uniquely determined, since the non-singular integral element E_{n-1} belongs to one n dimensional integral element E_n which is itself non-singular. If now there passes through E_{n-p-1} another integral element E'_n, the element (E'_n, E_p) will be at least $n+1$ dimensional and integral. On the other hand, it will contain (E_{n-p-1}, E_p), that is to say E_{n-1}, which is impossible, since there does not pass through E_{n-1} any integral element of more than n dimensions. Therefore, through E_{n-p-1} there passes a single n dimensional integral element. Therefore, finally, the true genre of the system is at most $n-p$.

From these properties we can extract the following, for which only the statement is necessary since the proof is evident:

If for a differential system of genre n there passes through each non-singular point of the space a p dimensional characteristic element, then all the non-singular integral varieties M_n passing through a non-singular point have a p dimensional element based on the point in common, and vice versa.

Let us now see what we can draw from the existence of characteristic elements for the determination of n dimensional non-singular integral varieties.

First suppose that there exists a *linear* characteristic element. Then this linear element associates every arbitrary point to a certain straight line D passing through the point. As we know, there exists a family of curves (one dimensional varieties) such that at each of their points they are tangent to the straight line D corresponding to this point. These curves depend on $r-1$ parameters and through each non-singular point of the space there passes one and only one such curve. We call them the *characteristic curves* and they are obviously integral curves.

Granted this, consider a non-singular variety M_n. On each of its non-singular points it admits a non-singular integral element which then contains the characteristic element ε based on the point. In other words, at each of its points the variety M_n has among its tangents the straight line D corresponding to this point. On M_n there therefore exists a family of curves tangent at each of their points to the corresponding straight line D. These curves depend on $n-1$ parameters and through each non-singular point of M_n there passes one and only one such curve. But it is obvious that these curves are *characteristic curves*. Therefore, we arrive at the following result:

Every non-singular integral variety M_n is generated by a family of characteristic *curves depending on $n-1$ parameters. Through each non-singular point of M_n there passes one and only one of these curves. If two non-singular integral varieties M_n have a non-singular point in common, they have all the characteristics based on the point in common.*

It follows from this that *given a non-singular integral variety M_{n-1} not generated by characteristic curves, we will have the integral variety M_n passing through M_{n-1} by making the characteristic curves passing through the point.*

We therefore have the solution of the Cauchy problem when M_{n-1} is not generated by characteristic curves.

We are now going to show these results analytically, which will allow us to see clearly what the problem of integration reduces to when we know the characteristic curves.

In our present case, the characteristic curves are given by a system of $r-1$ equations in total differentials. They are the equations found previously that determine the characteristic element based on each point of the space. Let

$$y_1 = C_1, \quad y_2 = C_2, \quad \dots, \quad y_{r-1} = C_{r-1}$$

be $r-1$ independent first integrals of these equations. They determine the characteristic curves. Let us make a change of variables by taking y_1, y_2, $\dots$, y_{r-1} and a quantity y_r independent of the previous $r-1$ as the new variables. With these new variables, the system of linear integral elements and the associated linear elements do not change: then, the system of equations in total differentials determining the characteristics remain the same. They are therefore

$$dy_1 = dy_2 = \cdots = dy_{r-1} = 0.$$

The equations of the transformed system therefore must first be satisfied for $dy_1 = \cdots = dy_{r-1} = 0$. Then, we can make this system under the form

$$(1_1) \qquad \begin{cases} dy_1 + b_{1,s+1}dy_{s+1} + \cdots + b_{1,r-1}dy_{r-1} = 0, \\ \qquad \cdots \\ dy_s + b_{s,s+1}dy_{s+1} + \cdots + b_{s,r-1}dy_{r-1} = 0, \end{cases}$$

b depending on y_1, y_2, $\dots$, y_r. Now express that the integral element

$$\frac{dy_1}{0} = \cdots = \frac{dy_{r-1}}{0} = \frac{dy_r}{1}$$

is associated to every other integral element $(dy_1, \dots, dy_r)$. We first have

$$\frac{\partial b_{1,s+1}}{\partial y_r} dy_{s+1} + \cdots + \frac{\partial b_{1,r-1}}{\partial y_r} dy_{r-1} = 0,$$

under the condition that dy satisfy (1_1), that is to say, we have

$$\frac{\partial b_{1,s+1}}{\partial y_r} = \cdots = \frac{\partial b_{1,r-1}}{\partial y_r} = 0.$$

In other words, *all the coefficients b are independent of y_r.*

The transformed system can therefore be made under a form such that in the coefficients and differentials there are only the $r-1$ variables

$$y_1, \quad y_2, \quad \dots, \quad y_{r-1}.$$

We therefore see that the number of variables is reduced by one and, to find the varieties M_n of the original system, it suffices to find the integral varieties M_{n-1} of the new system. *The genre of the new system is reduced by one, but the degree of indeterminacy does not change.* Only that the new system can no longer be of the first kind.

Thus, *when we have integrated the differential equations of the characteristics, we are lead to a new differential system with one less variable, the genre having been reduced by one.* We have

$$s' = s, \quad s'_1 = s_1, \quad \dots, \quad s'_{n-1} = s_{n-1},$$
$$n' = n - 1,$$
$$r' = r - 1, \quad r'_1 = r_1 - 1, \quad \dots, \quad r'_{n-1} = r_{n-1} - 1.$$

Let us now pass to the case where *there passes through each point of the space at least one characteristic element E_p of dimensions at least two.* There are then at least $r - p$ independent linear equations in $dx_1, \dots, dx_r$ determining E_p. We may think that these equations do not in general determine a completely integrable differential system, but *this is not true. This differential system, which we will call the* characteristic *differential system, is always completely integrable.*

To see this, it suffices to choose in each E_p a particular linear element ε, that is to say it suffices to adjoint to the differential system any $p-1$ determined linear equations. We then have a system of $r-1$ independent equations, which are then completely integrable and we denote by

$$y_1, \quad y_2, \quad \dots, \quad y_{r-1}$$

its system of $r-1$ independent first integrals. By a change of variables, the equations of the system, as we have just seen, depend only on $y_1, \dots, y_{r-1}$. The characteristic differential system therefore changes into a system of $r - p$ equations *in $r - 1$ variables.* We reason in this case as we have done above until we have reduced the variables to be no more than $r - p$ in number, which we call

$$z_1, \quad z_2, \quad \dots, \quad z_{r-p}.$$

Then it is clear that the characteristic differential system is none other than

$$dz_1 = dz_2 = \dots = dz_{r-p} = 0.$$

Therefore, *the characteristic differential system is completely integrable and we can, by a change of variables, make the given system under a form such that its coefficients and differentials only depend on the $r-p$ first integrals of the characteristic system.*

We also see that *there exists a family of p dimensional integral varieties admitting at each of their points the characteristic element E_p. We call them* characteristic varieties. *They depend on $r-p$ parameters and, at each non-singular point of the space, there passes one and only one such variety.*

Every non-singular integral variety M_n is generated by a family of characteristic varieties depending on $n-p$ parameters. There passes one and only one of these varieties through every non-singular point of M_n. If two non-singular n-dimensional integral varieties have a non-singular point in common, they have the same characteristic variety at the point in common.

If a non-singular integral variety M_{n-p} does not have any curve in common at each of its points with the characteristic variety based on the point, to have the unique integral variety M_n passing through M_{n-p} it suffices to take the characteristic variety based on the points of M_{n-p} pass through each point of M_{n-p}.

Finally, *the general determination of the integral M_n amounts to the integration of a new differential system whose genre as well as the number of variables are reduced by p, but who has the same degree of indeterminacy as the given system.*

To see this last point, it suffices to recall that the true genre of the given system is at most $n-p+1$. Then we have

$$s_n = s_{n-1} = \cdots = s_{n-p+1} = 0.$$

We then have

$$\begin{gathered} n' = n-p, \\ s'_{n-p} = s_{n-p}, \quad \ldots, \quad s'_1 = s_1, \quad s' = s, \\ r'_{n-p} = s_{n-p} = r_{n-p} - p, \quad \ldots, \quad r'_1 = r_1 - p, \quad r' = r - p. \end{gathered}$$

But we must not forget that the reduction to this new system presupposes the preliminary determination of the characteristic varieties. The generalised Lie-Mayer method allows us to arrive at a system of genre $n-p+1$ (instead of $n-p$) *without prior integration*, but this system depends on the particular Cauchy problem that we want to solve.

Observe that if the number of variables of the given differential system can be reduced by p units by a suitable change of variables, the characteristic differential system is necessarily formed, at most, by $r-p$ independent equations. We therefore have the following theorem, originally stated under a slightly different form by von Weber [†], which is itself just a generalisation of a theorem due to Frobenius for the case of a single equation:

The minimal number of variables that by a change of variables we can make the coefficients and differentials of a given system to depend on is equal to the number of linearly independent equations of its characteristic system. The integration of this characteristic system furnishes the variables.

[†] *Loc. cit.*

Finally, to finish this subject, we are going to prove the existence of characteristic elements *in differential systems of first kind whose character is equal to one.*

Take a differential system of genre n for which we have $s_1 = 1$. Then the numbers s_2, s_3, ... cannot surpass s_1, that is to say 1, and we have, to fix ideas,

$$s_1 = s_2 = \cdots = s_{\nu-1} = 1, \qquad s_\nu = \cdots = s_n = 0,$$

ν being the true genre (which can be equal to n).

Granted this, consider a non-singular point E_0 and the set of linear integral elements based on the point. They form an element E_{r_1+1}. In what follows we will only talk about the elements situated inside E_{r_1+1}, that is to say the elements formed with the linear integral elements. (We furthermore have $r_1 + 1 = n + \nu - 1$.)

Take an integral element E_n and a linear element ε not contained in E_n. The locus of the (integral) linear elements associated with ε is a $r_1 + 1 - s_1 = r_1$ dimensional element. Therefore this element cut E_n following an element H_{n-1} (of $n + r_1 - (r_1 + 1) = n - 1$ dimensions). All the linear elements contained in H_{n-1} are then associated with E_n and ε, that is to say with the element E_{n+1}: (E_n, ε).

Now take a linear element ε' not contained in E_{n+1}. The locus of the linear elements associated to ε' is again a r_1 dimensional element cutting H_{n-1} following at least a $n-2$ dimensional element H_{n-2} and all the linear elements of H_{n-2} are associated with E_{n+1} and ε', that is to say the element E_{n+2}: (E_{n+1}, ε'). We can continue this step by step: we will have an element H_{n-3} whose linear elements are associated with an element E_{n+3} and so on, until we finally arrive at an element $H_{n-\nu+1}$ whose elements are associated to an element $E_{n+\nu+1}$, that is to say E_{r_1+1}. In other words, there exists an element $H_{n-\nu+1}$ whose linear elements are integral and associated with *any* linear integral element. This element $H_{n-\nu+1}$ is therefore *characteristic*.

It follows from this that *the given differential system of genre n, true genre ν and character* 1 *admits characteristic varieties of $n - \nu + 1$ dimensions. It then leads to, according to the determination of these characteristics, a system of genre $\nu - 1$.*

This result applies to a single Pfaffian equation (provided that it is of first kind). We hence rediscover the characteristic varieties of systems of first order partial differential equations of a single unknown function.

In particular, *if the general integral of a differential system depends on a single arbitrary function of a single argument (and of arbitrary constants), the integration leads to that of the characteristic system, which is completely integrable, and to that of a system of ordinary differential equations* (†).

If the general integral of a system of first kind depends on one arbitrary function of 1, 2, ..., $\nu - 1$ arguments (and on arbitrary constants), $\nu - 1$ being at least equal to 2, we can prove (‡) that the system can, without integration, be made under the following

(†)Beudon has proved this result for a system of partial differential equations in one unknown function. He is, in a series of notes and articles, concerned with partial differential equations with the property that they admit characteristic varieties in the sense given to this word in the text. *See*, in particular, *Sur les systèmes d'équations aux dérivées partielles dont les caractéristiques dépendent d'un nombre fini de constantes arbitraires* (*Annales de l'École Normale*, vol. XIII, supplement, p. 3–51; 1896)

(‡)*See*, in particular, von Weber, *loc. cit.*

form: first, a system of $s-1$ completely integrable equations, then, a s-th equation which can be made under the form

$$dz - p_1 dx_1 - \cdots - p_{\nu-1} dx_{\nu-1} = 0,$$

which by a suitable integration leads to the characteristic differential system.

The problem of integrating the characteristic differential system is hence not an arbitrary integration problem of a completely integrable system in total differentials. To see this, imagine that we have found a first integral y_1 and let us focus on the variety $y_1 = C$, where C is an arbitrary constant.

Now consider, at an arbitrary point A in this variety, the element E_{r-1}: $dy_1 = 0$. The characteristic element E_p on A is necessarily contained in the element E_{r-1}, but if we find the linear integral elements of E_{r-1} which are characteristics *with respect to only the elements of* E_{r-1}, we can, in certain cases, find those elements that are not contained in E_p, such that we obtain an characteristic element E_q containing E_p $(q > p)$, but which is characteristic only when we do not go out of the element E_{r-1}. In other words, the characteristic differential system of the given system where we set $y_1 = C$, $dy_1 = 0$ can contain at least one less equation than the original characteristic system. We find a first integral y_2 of this system, and so on. We will arrive at a certain number of first integrals $y_1, y_2, \ldots, y_h$, such that when we set $y_1 = C_1, \ldots, y_h = C_h$, the differential system obtained has all of its characteristic equations satisfied.

It is then clear that the equations of the given system can all be put under the form

$$\alpha_1 dy_1 + \cdots + \alpha_h dy_h = 0$$

and we arrange that, by a suitable choice of s linearly independent equations defining the system, the coefficients α that are independent among themselves and with y define the different integrals of y of the characteristic system.

This is how we precede in the case of a single Pfaffian equation. Take, for fixing the ideas, an equation in five variables with unspecified coefficients. If we represent a linear element by a point in three dimensional space R_3, the linear integral elements are represented by the points in a certain plane (P) of this space, and the images of two associated linear integral elements are such that the straight line joining them belongs to a certain linear complex. But in ordinar space, the straight lines of a linear complex situated in a plane (P) all pass through a fixed point A of the plane. The point A is therefore the image of a linear characteristic element. The characteristic differential system therefore admits three independent first integrals. We find from them a y_1, which determines in the space R_3 a plane (Q). The linear integral elements satisfying $dy_1 = 0$ also have images in R_3 that are points belonging at the same time to (P) and (Q), that is to say the straight line (D) of the intersection of these two planes. But now *two arbitrary points of this straight line are associated*, such that we have a second characteristic differential system formed with a single equation [the equation of the straight line (D) in the plane (Q)]. Let y_2 be its first integral. Then

$$dy_1 = dy_2 = 0$$

are, if we would like, the equations of the straight line (D). The equation of the plane (P), which is none other than the given Pfaffian equation, is then of the form

$$dy_2 - y_3 dy_1 = 0,$$

and y_3 is the third first integral we search for, since it is obvious that the characteristic system of the equation put into its new form can be none other than

$$dy_1 = dy_2 = dy_3 = 0.$$

To take another example, consider the case of two equations in six variables. The genre of the system is, in the general case, equal to $2 = \frac{6}{2+1}$. We can represent a linear element by a point in a space R_5 of five variables. The images of linear integral elements are then situated in a space R_3 of three dimensions, and in this space the straight lines joining two associated points belong to two linear complexes. We have seen, in §II, that three cases can arise. Take the last, the one where the straight lines of the complex are the straight lines passing through a fixed point A of R_3 and, moreover, the straight lines are situated inside a certain plane (P) passing through A. Here therefore there is a linear characteristic element whose image is A.

The characteristic differential system will admit five independent first integrals. We will find first a y_1. By replacing y_1 by C, we will have in R_5 a space R_4 cutting R_3 following the plane (Q). In R_4, the images of the linear integral elements are situated in the plane (Q), which naturally passes through A and, in the plane (Q), the lines joining two associated points are the lines based on A. Therefore there is here again a single linear characteristic element. The new characteristic system is formed with five equations defining the point A in R_4. Let y_2 be a first integral of this new system. It defines in R_4 a space R'_3 cutting (Q) following a straight line (D) passing through A, but then all the points of (D) are associated among themselves. The new characteristic system is therefore formed with *two* equations defining (D) in R'_3. We only have to search the two independent first integrals y_3 and y_4 of this system.

We have hence four integrals to search for, by operations of order respectively 5, 4, 2 and 1.

In reality, we can again simplify this integration after the first operation and restrict ourselves to three integrals given by the operations of orders 5, 3 and 1. But, for this, we need to take into consideration certain covariant equations, which are beyond the scope of this article.

There is also a case where the integration simplifies: the one where the first integral y_1 gives a space R_4 containing the plane (P), that is to say the case *where the three equations defining* (P) *admit an integrable combination.* In this case, the images of the linear integral elements of the new system are the points of (P) and *these points are all associated with each other.* The new characteristic system is formed by the *two* equations defining (P) in R_4. By integrating them, we will have two first integrals y_2 and y_3, and the equations of (P) in R_5 are then

$$dy_1 = dy_2 = dy_3 = 0.$$

The space R_2, which is the locus of the images of the linear integral elements, passes through (P), and hence the two equations defining it, that is to say the given equations, are of the form

$$dy_2 - y_4 dy_1 = 0,$$
$$dy_3 - y_5 dy_1 = 0,$$

y_4 and y_5 being two first integrals other than y_1, y_2 and y_3. Here we have, by the same occasion, a *canonical form* of the system.

The operations to apply, in this particular case, is of order 3, 2 and 1, since it suffices, in sum, to integrate the three equations defining the plane (P), these three equations forming a completely integrable system.

Differential systems in involution (†)

1 Given a Pfaffian system in n variables

$$\omega_d = a_1 dx_1 + a_2 dx_2 + \cdots + a_n dx_n, \tag{1}$$

we know that we can adjoin to it a covariant expression that is bilinear with respect to two systems of differentials denoted by the symbols d and δ:

$$\begin{aligned} \omega_{d\delta} = d\omega_\delta - \delta\omega_d &= da_1\delta x_1 - \delta a_1 dx_1 + \cdots + da_n\delta x_n - \delta a_n dx_n \\ &= \sum_{(i,k)} \left(\frac{\partial a_k}{\partial x_i} - \frac{\partial a_i}{\partial x_k}\right)(dx_i\delta x_k - dx_k\delta x_i). \end{aligned} \tag{2}$$

The coefficients of the bilinear covariant of a Pfaffian system are not arbitrary functions of variables. They satisfy the relations given by the following considerations:

If we consider any alternating bilinear differential expression:

$$\Omega_{d\delta} = \sum_{(i,k)} a_{ik}(dx_i\delta x_k - dx_k\delta x_i) \qquad (a_{ik} = -a_{ki}, a_{ii=0}), \tag{3}$$

we can adjoin to it a covariant expression that is trilinear with respect to three systems of differentials denoted by the symbols d, δ and D,

$$\Omega_{d\delta D} = d\Omega_{\delta D} + \delta\Omega_{Dd} + D\Omega_{d\delta} = \sum a_{ijk} \begin{pmatrix} dx_i & dx_j & dx_k \\ \delta x_i & \delta x_j & \delta x_k \\ Dx_i & Dx_j & Dx_k \end{pmatrix}, \tag{4}$$

where we have

$$a_{ijk} = \frac{\partial a_{ij}}{\partial x_k} + \frac{\partial a_{jk}}{\partial x_i} + \frac{\partial a_{ki}}{\partial x_j}.$$

If $\Omega_{d\delta}$ is the bilinear covariant of a Pfaffian expression ω, we can verify without difficulty that $\Omega_{d\delta D}$ is identically zero, and conversely, if the trilinear covariant of $\Delta_{d\delta}$

(†) Chapter 1 of *Sur la structure des groupes infinis de transformations* (Annales École Normale, 3rd series, 21, 1904).

is identically zero, we can show that there exists a Pfaffian expression (defined uniquely up to an exact differential of a function) whose bilinear covariant is $\Omega_{d\delta}$.

The identity expressed by the preceding theorem is the analogue, or rather the dual, of Jacobi's identity in the theory of complete systems. We will use it frequently in the following and we will give it the name *fundamental identity.*

2 We will use symbolic notations devised to simply calculations. If ω and ϖ are two Pfaffian expressions, we denote, symbolically,

$$\omega\varpi = \omega_d\varpi_\delta - \omega_\delta\varpi_d. \tag{5}$$

Similarly, if ω, ϖ and χ are three Pfaffian expressions, we set

$$\omega\varpi\chi = \begin{vmatrix} \omega_d & \varpi_d & \chi_d \\ \omega_\delta & \varpi_\delta & \chi_\delta \\ \omega_D & \varpi_D & \chi_D \end{vmatrix}. \tag{6}$$

With these notations, we have

$$\omega\varpi = -\varpi\omega, \qquad \omega\omega = 0,$$
$$\omega\varpi\chi = \varpi\chi\omega = \chi\omega\varpi = -\varpi\omega\chi = -\chi\varpi\omega = -\omega\chi\varpi.$$

We denote the bilinear covariant of a Pfaffian expression ω simply by ω', such that we have

$$\omega' = \sum \left(\frac{\partial a_k}{\partial x_i} - \frac{\partial a_i}{\partial x_k}\right) dx_i dx_k.$$

If A denotes a function in x, the bilinear covariant of $A\omega$ is

$$dA\omega + A\omega'$$

and the trilinear covariant of the bilinear expression $A\omega\varpi$ is

$$dA\omega\varpi + A\omega'\varpi - a\omega\varpi'. \tag{7}$$

Finally, if we consider n independent Pfaffian expressions in dx_1, dx_2, $\dots$, dx_n, which we write as

$$\omega_1, \quad \omega_2, \quad \dots \quad \omega_n,$$

then every Pfaffian expression can be written under the form

$$a_1\omega_1 + a_2\omega_2 + \cdots + a_n\omega_n,$$

every bilinear expression under the form

$$\sum_{(i,k)} a_{ik}\omega_i\omega_k,$$

and every trilinear under the form

$$\sum_{ijk} a_{ijk}\omega_i\omega_j\omega_k.$$

3 Given a system of Pfaffian equations, the theory on the existence and the degree of indeterminacy of the *general* integral varieties of the system can be summarised as follows:

Let us call the set of a point $(x_1, x_2, \ldots, x_n)$ and a straight line (of direction parameters dx_1, dx_2, ..., dx_n) passing through the point a *linear element*, and similar the set of a point and a p dimensional plane passing through this point a p *dimensional element*. A linear element is said to be *integral* when its coordinates satisfy the equations of the Pfaffian system. Two linear integral elements, based on the same point, are said to be *in involution* when their coordinates

$$dx_1, \quad \ldots \quad dx_n \qquad \text{and} \qquad \delta x_1, \quad \ldots \quad \delta x_n$$

annihilate all the bilinear covariants of the left hand sides of the equations of the system. Finally a p-th order element is said to be *integral* when all its linear elements are integral and any two of them are in involution.

Granted this, let s be the number of linearly independent equations expressing that a linear element, based on an *arbitrary* point, is integral (this is the number of independent equations of the system). We obviously have

$$s \leq n.$$

If s is equal to n, there are no linear integral element and the system does not admit any *general* integral variety. If s is less than n, through each point there passes at least one linear integral element.

Let E be an arbitrary linear integral element and let $s + s_1$ be the number of linearly independent equations expressing that a linear element is integral and in involution with E. We obviously have

$$s + s_1 \leq n - 1.$$

If $s + s_1$ is equal to $n - 1$, through E there does not pass any integral element of order 2 and at most the system can admit only one dimensional integral varieties. If $s + s_1$ is less than $n - 1$, through E there passes at least one second order integral element.

Let E' be an arbitrary second order integral element and let $s + s_1 + s_2$ be the number of linearly independent equations expressing that a linear element is integral and in involution with E'. We obviously have

$$s + s_1 + s_2 \leq n - 2$$

and so on.

Finally we arrive at a certain integer m which indicates the maximum number of dimensions of the general integral varieties of the system, and we have the $m+1$ integers

$$s, \quad s_1, \quad s_2, \quad \ldots \quad s_m$$

which cannot increase and which indicate the degree of indeterminacy of the general m dimensional integral variety. It depends on

s_m	arbitrary functions in	m	arguments,
s_{m-1}	arbitrary functions in	$m-1$	arguments,
...	...	...	...
s_1	arbitrary functions in	1	arguments,
s	arbitrary constants.		

Finally, we have

$$s + s_1 + \cdots + s_m = n - m.$$

If p is any integer not greater than m, we say that the system, considered as of p independent variables, is in involution. We have

$$s + s_1 + \cdots + s_p \leq n - p \tag{8}$$

and the non-singular p dimensional integral elements based on an arbitrary point depend on

$$Q = p(n-p) - ps - (p-1)s_1 - (p-2)s_2 - \cdots - 2s_{p-2} - s_{p-1}$$

parameters.

Observe that, if we suppose the equations of the system to be solved with respect to the differentials of s of the variables, the number of the other *dependent* variables is precisely $n - p - s$. Denoting this number by q, the number Q is then

$$Q = pq - (p-1)s_1 - (p-2)s_2 - \cdots - s_{p-1}. \tag{9}$$

4 The theory we just summarised must be further developed to resolve the following question whose practical importance is evident.

Determine the integral varieties of a given dimension p of a given Pfaffian system, under the condition that these integral varieties must not imply any algebraic relations among p specified variables of the given variables, or more generally, they must not imply any linear relations among p Pfaffian forms specified in advance

$$\omega_1, \quad \omega_2, \quad \ldots \quad \omega_p$$

(independent among themselves and with the left hand sides of the equations of the system).

In particular, it is necessary to have all these varieties as general integral varieties of a new Pfaffian system *in involution*.

The left hand sides

$$\theta_1, \quad \theta_2, \quad \ldots \quad \theta_s$$

of the equations of the system and the p given expressions

$$\omega_1, \quad \omega_2, \quad \ldots \quad \omega_p$$

form $s+p$ linearly independent expressions. We can adjoin to them $q = n - s - p$ other expressions, independent among themselves and with respect to the old ones, which we write as

$$\varpi_1, \quad \varpi_2, \quad \dots \quad \varpi_q.$$

Then, *using the equations of the systems*, the bilinear covariants of θ are the bilinear covariants with respect to ω and ϖ. We are first going to show that we can always assume it to be of the form

$$\theta'_k = \sum_{ij} c_{ijk}\omega_i\omega_j + \sum_{i,\rho} a_{i\rho k}\omega_i\varpi_\rho \qquad (k = 1, 2, \dots, s). \tag{10}$$

Indeed, every p dimensional integral varieties satisfies not only the given equations, but also equations of the form

$$\varpi_k = l_{k1}\omega_1 + \dots + l_{kp}\omega_p \qquad (k = 1, 2, \dots, q), \tag{11}$$

l_{ki} being functions of the variables such that all the θ'_k are zero if we replace in them ϖ by the preceding values. It may arise that the compatibility of these equations that l must satisfy requires particular relations among the given variables. If these relations do not leave the ωs independent, the problem is unsolvable. Otherwise they effectively reduce the number q. We then need to start again with the new system obtained, until we arrive at either a stage where the problem is unsolvable, or at a system of compatible equations for l. Then we can express all of the coefficients of l as functions of the given variables and a certain number of auxiliary variables y_1, y_2, ... The bilinear covariants of the Pfaffian system obtained by adjoining to the old system the equations (11) will then be, using the equations of the system, of the form

$$\sum \gamma_{ij}\omega_i\omega_j + \sum a_{ik}\omega_i dy_k,$$

which proves the proposition [†].

We henceforth assume that θ'_k are of the form (10).

5 We are first going to search for the necessary and sufficient conditions for *a Pfaffian system, considered as of p independent variables, to be in involution, such that arbitrary integral elements of order less than or equal to p do not introduce any linear relations among the ω*. If this is the case, the varieties we look for will be just the *general* integral varieties of the given system.

To this end, first, it is obviously necessary that there are integral elements of order p defined by equations of the form (11). If this is the case, we can, by adjoining to ϖ certain linear combinations of ω, ensure that

$$\varpi_1 = \varpi_2 = \dots = \varpi_q = 0$$

[†]Indeed, the covariants θ' all vanish if we use (11). As for the equations (11) themselves, they give rise to the covariants such that

$$\varpi'_k - l_{k1}\omega'_1 - \dots - l_{kp}\omega'_p + \omega_1 dl_{k1} + \dots + \omega_p dl_{kp},$$

and it suffices to replace, after expanding ω' and ϖ', ϖ by their values (11).

defines an integral element. Consequently, *we can suppose the coefficients c_{ijk} to be all zero.*

Granted this, consider the integers s_1, s_2, ..., s_p defined in §**3**. Since every linear integral element is defined by a system of equations

$$\frac{\omega_1}{u_1} = \frac{\omega_2}{u_2} = \cdots = \frac{\omega_p}{u_p} = \frac{\varpi_1}{v_1} = \cdots = \frac{\varpi_1}{v_q}, \tag{12}$$

the number s_1 indicates the number of independent equations of the system

$$\sum_{i,\rho} a_{i\rho k} u_i \varpi_\rho - \sum_{i,\rho} a_{i\rho k} v_\rho \omega_i = 0, \qquad (k = 1, 2, \ldots, s), \tag{13}$$

where u_i and v_ρ are arbitrary constants and ω_i, ϖ_ρ are the variables. *As these equations must not entail any relations among* ω, the number s_1 is the rank (degree of the principal determinant) of the matrix of s rows

$$\left\|\begin{matrix} \sum a_{i11} u_i & \sum a_{i21} u_i & \ldots & \sum a_{iq1} u_i \\ \cdots & \cdots & \cdots & \cdots \\ \sum a_{i1s} u_i & \sum a_{i2s} u_i & \ldots & \sum a_{iqs} u_i \end{matrix}\right\|.$$

For s_2, ..., s_p, consider the matrix of ps rows

$$\left\|\begin{matrix} \sum a_{i11} u_i & \sum a_{i21} u_i & \ldots & \sum a_{iq1} u_i \\ \cdots & \cdots & \cdots & \cdots \\ \sum a_{i1s} u_i & \sum a_{i2s} u_i & \ldots & \sum a_{iqs} u_i \\ \sum a_{i11} u'_i & \sum a_{i21} u'_i & \ldots & \sum a_{iq1} u'_i \\ \cdots & \cdots & \cdots & \cdots \\ \sum a_{i1s} u'_i & \sum a_{i2s} u'_i & \ldots & \sum a_{iqs} u'_i \\ \cdots & \cdots & \cdots & \cdots \\ \sum a_{i1s} u_i^{(p-1)} & \sum a_{i2s} u_i^{(p-1)} & \ldots & \sum a_{iqs} u_i^{(p-1)} \end{matrix}\right\| \tag{14}$$

where u_i, u'_i, ..., $u_i^{(p-1)}$ are p^2 arbitrary values. The number $s_1 + s_2$ is the rank of the matrix by taking the first $2s$ rows, $s_1 + s_2 + s_3$ the rank of the matrix obtained by taking the first $3s$ rows, and so on.

Granted this, the formulae (8) and (9) show that we have

$$s_1 + s_2 + \cdots + s_p \leq q$$

and the number of arbitrary parameters that the most general p-th order integral element of the form (11) depends on is

$$pq - (p-1)s_1 - (p-2)s_2 - \cdots - s_{p-1}.$$

6 Conversely, *given a Pfaffian system whose covariants* θ'_k *has the form* (10), *let us form, by means of the coefficients* $a_{i\rho k}$, *the matrix* (14) *and consider the integers*

$$\sigma_1, \quad \sigma_2, \quad \dots \quad \sigma_p$$

which are the ranks of the matrices obtained by taking in (14) *successively the first*

$$s, \quad 2s, \quad \dots \quad ps$$

rows. The number of arbitrary parameters that the most general p-th order integral element which does not introduce any linear relations among the ω *depends on can never exceed the integer*

$$pq - (p-1)\sigma_1 - (p-2)\sigma_2 - \cdots - \sigma_{p-1}.$$

If this value is attained, the system is in involution and its general p *dimensional integral varieties do not introduce any linear relations among* ω.

First, since according to the statement itself, assume that there exists p-th order integral elements of the form (11), and we can suppose all c_{ijk} in the formulae (10) to be zero.

Furthermore, we can, by means of a suitable linear transformation on ω, suppose that the rank of the p matrices considered do not decrease if we take

$$u^{(i)}_{i+1} = 1, \qquad u^{(i)}_j = 0 \quad \text{for} \quad j \neq i+1.$$

Finally we can always set

$$\theta'_\rho = \omega_1 \varpi_{\rho 1} + \omega_2 \varpi_{\rho 2} + \cdots + \omega_p \varpi_{\rho p}, \qquad (\rho = 1, 2, \dots, s), \tag{15}$$

$\varpi_{\rho i}$ being linear combinations of ϖ.

From the hypothesis it then follows that among $\varpi_{\rho 1}$ there are exactly σ_1 independent ones, which we write as

$$\varpi_{11}, \quad \varpi_{21}, \quad \dots \quad \varpi_{\sigma_1 1},$$

the $\varpi_{\rho 1}$ for which ρ is greater than σ_1 depend on the preceding ones. Similarly among $\varpi_{\rho 2}$ $(\rho \leq \sigma_1)$ there are σ_2 independent ones, which we write as

$$\varpi_{12}, \quad \varpi_{22}, \quad \dots \quad \varpi_{\sigma_2 2},$$

the $\varpi_{\rho i}$ for which ρ is greater than σ_2 depend on the $\sigma_1 + \sigma_2$ preceding ones, and so on.

In other words, the sp combinations $\varpi_{\rho i}$ are independent when we have

$$\rho \leq \sigma_i.$$

These are said to be *principal.* There are in total

$$\sigma_1 + \sigma_2 + \cdots + \sigma_p$$

of them, and the other $\varpi_{\rho i}$ are expressed linearly in terms of the principal ones for which the second index does not exceed i. Finally, it may happen that among ϖ_k there are some that cannot be expressed in the principal $\varpi_{\rho i}$, and there are in total

$$q - (\sigma_1 + \sigma_2 + \cdots + \sigma_p)$$

of them.

Granted this, we can first, in every case, express the last ones in arbitrary ways in terms of ω, which already gives

$$p(q - \sigma_1 - \sigma_2 - \cdots - \sigma_p) \tag{16}$$

arbitrary parameters. As for the expressions $\varpi_{\rho i}$, if we set

$$\varpi_{\rho i} = l_{\rho i 1}\omega_1 + l_{\rho i 2}\omega_2 + \cdots + l_{\rho i p}\omega_p,$$

the condition that θ'_ρ vanishes gives

$$l_{\rho ij} = l_{\rho ji},$$

such that $l_{\rho ij}$ satisfy the two following conditions:

1. *If we give the last index j any specified value, every linear relation among $\varpi_{\rho i}$ also holds among the corresponding $l_{\rho ij}$;*
2. *We have*

$$l_{\rho ij} = l_{\rho ji}. \tag{17}$$

It follows that the only coefficients $l_{\rho ij}$ that can be taken arbitrarily are *at most* those that have

$$\rho \leq \sigma_i, \qquad \rho \leq \sigma_j. \tag{18}$$

We call these the *principal* coefficients l. They have

$$\sigma_1 + 2\sigma_2 + \cdots + p\sigma_p \tag{19}$$

in number. By adding to it the number (16) we obtain the maximum number of parameters that a p-th order integral element depends on

$$pq - (p-1)\sigma_1 - (p-2)\sigma_2 - \cdots - \sigma_{p-1}.$$

The first part of the theorem is hence proved.

Now suppose that this number is attained, that is to say we can satisfy the two conditions stated above by taking the *principal* coefficients $l_{\rho ij}$ arbitrarily. We are going to show that the system is in involution and the corresponding numbers s_i are equal to the numbers σ_i.

Consider an *arbitrary* linear integral element E_1. We can always, by means of a linear transformation on ω, suppose it to be defined by the relations

$$\frac{\omega_1}{1} = \frac{\omega_2}{0} = \cdots = \frac{\omega_p}{0} = \frac{\varpi_{\rho i}}{v_{\rho i1}},$$

the $v_{\rho i1}$ being arbitrary quantities that satisfy the same relations as the corresponding $\varpi_{\rho i}$, in other words they are all expressible in terms of those that have

$$\rho \leq \sigma_i.$$

The equations expressing that a linear integral element is in involution with E_1 is then

$$\varpi_{\rho 1} - v_{\rho 11}\omega_1 - v_{\rho 21}\omega_2 - \cdots - v_{\rho p1}\omega_p = 0. \tag{20}$$

These equations do not entail any relations among ω, otherwise, there will be a certain index i such that there is among $\varpi_{\rho 1}$ a relation that cannot be satisfied by the corresponding $v_{\rho i1}$. Then it would be impossible to find a system of quantities $v_{\rho ij}$, that the given $v_{\rho i1}$ is part of such that the two conditions stated above are satisfied without there being any relations between the *principal* $v_{\rho i1}$, since there would be among the $v_{\rho 1i}$ a relation that cannot be satisfied by the quantities $v_{\rho i1}$.

Hence we see that s_1 is equal to σ_1 and then the integer m is equal or greater than 2.

Similarly an *arbitrary* second order integral element E_2 can always be assumed to be defined by the two elements

$$(1, 0, \ldots, 0; v_{\rho i1}) \quad \text{and} \quad (0, 1, \ldots, 0; v_{\rho i2}).$$

As the involution condition for these two elements gives

$$v_{\rho 12} = v_{\rho 21},$$

$v_{\rho i1}$ as well as $v_{\rho i2}$ being, from now on, subject to the condition that they satisfy among themselves the same relations as $\varpi_{\rho i}$, we can arbitrarily take $v_{\rho 12} = v_{\rho 21}$ for which ρ does not exceed σ_2, and take the other $v_{\rho i1}$ and $v_{\rho i2}$ for which ρ does not exceed σ_i arbitrarily. The system expressing that a linear integral element is in involution with E_2 is then

$$\begin{cases} \varpi_{\rho 1} - v_{\rho 11}\omega_1 - v_{\rho 21}\omega_2 - \cdots - v_{\rho p1}\omega_p = 0, \\ \varpi_{\rho 2} - v_{\rho 12}\omega_1 - v_{\rho 22}\omega_2 - \cdots - v_{\rho p2}\omega_p = 0, \end{cases} \qquad (\rho = 1, 2, \ldots, s). \tag{21}$$

It contains $\sigma_1 + \sigma_2$ independent equations and does not introduce any relations among ω, otherwise there will be a certain index i such that there is among $\varpi_{\rho 2}$ and $\varpi_{\rho 1}$ a relation that cannot be satisfied by the corresponding $v_{\rho i2}$ and $v_{\rho i1}$. Then, it would be impossible to find a system of quantities $v_{\rho ij}$, which the given $v_{\rho i1}$ and $v_{\rho i2}$ are a part of such that the conditions stated above are satisfied without there being any relations among the *principal* $v_{\rho i1}$ and $v_{\rho i2}$, since there would be among $v_{\rho 1i}$ and $v_{\rho 2i}$ a relation (the same one as among $\varpi_{\rho 1}$ and $\varpi_{\rho 2}$) that cannot be satisfied by the quantities $v_{\rho i1}$ and

$v_{\rho i2}$. We hence see that s_2 is equal to σ_2 and that, if p is greater than 2, there passes through each arbitrary integral element E_2 at least one integral element E_3.

We can continue in this way and we find that

$$s_1 = \sigma_1, \quad s_2 = \sigma_2, \quad \dots \quad s_p = \sigma_p,$$

which proves the theorem.

7 Whenever we have a system of pqr quantities

$$a_{i\rho k} \qquad (i = 1, 2, \dots, p; \rho = 1, 2, \dots, q; k = 1, 2, \dots, s)$$

satisfying the conditions of the theorem in §**6**, we say that it constitute an *involutive* system. It is useful, for the following, to study some remarkable properties of these systems.

Let us continue to use the notations of the preceding section

$$\sum_{i,k} a_{ik\rho}\omega_i\varpi_k = \omega_1\varpi_{\rho 1} + \omega_2\varpi_{\rho 2} + \dots + \omega_p\varpi_{\rho p}, \qquad (\rho = 1, 2, \dots, s), \tag{22}$$

$\varpi_{\rho i}$ being linear combinations of ϖ that can all be expressed by means of

$$\sigma_1 + \sigma_2 + \dots + \sigma_p$$

variables among them, which we have called *principal*, and which satisfy

$$\rho \leq \sigma_i.$$

It is possible to find a system of quantities $l_{\rho ij}$ having the following three properties:

1. Every relation among the $\varpi_{\rho i}$ holds for the corresponding $l_{\rho ij}$, j denoting any given index;

2. We have, whatever the values ρ, i and j are,

$$l_{\rho ij} = l_{\rho ji};$$

3. The *principal* quantities $l_{\rho ij}$, i.e., those that have

$$\rho \leq \sigma_i, \qquad \rho \leq \sigma_j$$

can be chosen *arbitrarily.*

Note that, according to this, if we denote by

$$X_{\rho i} = 0, \qquad (\rho > \sigma_i)$$

the equation determining the non-principal expression $\varpi_{\rho i}$ in terms of the *principal* ϖ whose second index is less than or equal to i and denote by

$$X^j_{\rho i} = 0, \qquad (j \leq i, \rho > \sigma_i)$$

the equation that we deduce from it by replacing all the $\varpi_{\rho' i'}$ by the corresponding $l_{\rho' i' j}$, *we then obtain a system of equations defining the non-principal* $l_{\rho ij}$ *in terms of the principal* $l_{\rho ij}$ *completely.* It suffices to take, step by step, the equations

$$X^1_{\rho 1} = 0, \quad X^1_{\rho 2} = 0, \quad X^2_{\rho 2} = 0, \quad X^1_{\rho 3} = 0, \quad X^2_{\rho 3} = 0, \quad X^3_{\rho 3} = 0, \quad \dots$$

Granted this, we can *prolong* the involutive system by *defining*, using $\sigma_1 + 2\sigma_2 + \cdots + p\sigma_p$ new variables, the bilinear expressions

$$(23) \qquad \omega_1 \varpi_{\rho i1} + \omega_2 \varpi_{\rho i2} + \cdots + \omega_p \varpi_{\rho ip}, \qquad (\rho = 1, 2, \dots, s; i = 1, 2, \dots, p),$$

which here play the same role as the expressions (22) for the given system and $\varpi_{\rho ij} = \varpi_{\rho ji}$ are linked by exactly the same relations as $l_{\rho ij}$. For this new system, we have

$$\begin{aligned}
\sigma'_1 &= \sigma_1 + \sigma_2 + \cdots + \sigma_p, \\
\sigma'_2 &= \phantom{\sigma_1 + {}} \sigma_2 + \cdots + \sigma_p, \\
&\cdots \\
\sigma'_p &= \phantom{\sigma_1 + \sigma_2 + \cdots + {}} \sigma_p,
\end{aligned}$$

we claim that *it is also involutive.*

Indeed, we are going to determine in the following manner a system of $\frac{sp(p+1)(p+2)}{6}$ quantities

$$l_{\rho ijk} = l_{\rho jik} = l_{\rho ikj} = l_{\rho kij} = l_{\rho jki} = l_{\rho kji}.$$

We take the following quantities arbitrarily

$$l_{\rho ijk}, \qquad (i \geq j \geq k, \rho \leq \sigma_i)$$

which we name *principal.* As for the rest, they are completely determined in terms of the preceding ones by the equations

$$X^{jk}_{\rho i} = 0, \qquad (k \leq j \leq i, \rho > \sigma_i)$$

obtained by replacing in $X_{\rho i}$ the $\varpi_{\rho' i'}$ by the corresponding $l_{\rho' i' jk}$. We arrange these equations in the order of increasing i, and those corresponding to the same i are arranged in increasing order of j (from 1 to i) and those with the same values of i and j are arranged in increasing order of k (from 1 to j).

Granted this, we can show without difficulty that *every relation among* $\varpi_{\rho i}$ *holds for the corresponding* $l_{\rho i\alpha\beta}$, *regardless of the indices* α *and* β. Similarly, every relation among $\varpi_{\rho ij}$ holds for the corresponding $l_{\rho ij\alpha}$, regardless of the index α.

The $l_{\rho ijk}$ therefore play the same role for $\varpi_{\rho ij}$ as $l_{\rho ij}$ for $\varpi_{\rho i}$. But the number of these quantities that are arbitrary is precisely

$$\sigma'_1 + 2\sigma'_2 + \cdots + p\sigma'_p,$$

and then, the system considered is really involutive.

8 The involutive system of quantities $a_{ik\rho}$ also enjoys another property which will be useful for us. *If we try to determine the most general bilinear expression* $\Pi_{\rho i}$ *annihilating the following trilinear expressions identically*

$$\omega_1\Pi_{\rho 1} + \omega_2\Pi_{\rho 2} + \cdots + \omega_p\Pi_{\rho p}, \qquad (\rho = 1, 2, \ldots, s), \tag{24}$$

and satisfying the same relations as $\varpi_{\rho i}$ *do, we find*

$$\Pi_{\rho i} = \omega_1\chi_{\rho i1} + \omega_2\chi_{\rho i2} + \cdots + \omega_p\chi_{\rho ip}, \tag{25}$$

where $\chi_{\rho ij} = \chi_{\rho ji}$ *are arbitrary Pfaffian expressions satisfying the same relations as* $l_{\rho ij}$.

In the statement, we suppose that $\Pi_{\rho i}$ are bilinear expressions with respect to ω and a number of other independent Pfaffian expressions.

First, it is evident that, if we take $\Pi_{\rho i}$ to be the expressions of the form (25), they satisfy the question. It is the converse that we need to prove.

As it is easy to see, the $\Pi_{\rho i}$, which are annihilated by multiplying with ω, can always be made into the form (25), the $\chi_{\rho ij}$ being for the moment arbitrary. Furthermore, we can suppose that every relation among $\Pi_{\rho i}$ holds also for $\chi_{\rho i\alpha}$ regardless of the index α. Granted this, suppose we have already shown that there is an integer $h < p$ such that we have

$$\chi_{\rho\alpha\beta} = \chi_{\rho\beta\alpha}, \qquad (\alpha, \beta = 1, 2, \ldots, h).$$

We claim that this property also holds for $h + 1$. Omitting the terms in

$$\omega_{h+2}, \quad \omega_{h+3}, \quad \ldots \quad \omega_p,$$

the expression (24) takes the form

$$\omega_1\omega_{h+1}(\chi_{\rho,1,h+1} - \chi_{\rho,h+1,1}) + \cdots + \omega_h\omega_{h+1}(\chi_{\rho,h,h+1} - \chi_{\rho,h+1,h}).$$

It follows that we have, *by also omitting* ω_{h+1},

$$\begin{cases} \chi_{\rho,1,h+1} = \chi_{\rho,h+1,1} + \lambda_{\rho 11}\omega_1 + \cdots + \lambda_{\rho 1h}\omega_h, \\ \qquad\cdots \\ \chi_{\rho,h,h+1} = \chi_{\rho,h+1,h} + \lambda_{\rho h1}\omega_1 + \cdots + \lambda_{\rho hh}\omega_h, \end{cases} \tag{26}$$

$\lambda_{\rho ij}$ being finite quantities satisfying

$$\lambda_{\rho ij} = \lambda_{\rho ji}, \qquad (i, j = 1, 2, \ldots, h).$$

Then, we can set, by omitting $\omega_{h+2}, \ldots, \omega_p$

$$\Pi_{\rho 1} = \omega_1\chi_{\rho 11} + \cdots + \omega_h\chi_{\rho 1h} + \omega_{h+1}\chi_{\rho,h+1,1} - \lambda_{\rho 11}\omega_1\omega_{h+1} - \cdots - \lambda_{\rho 1h}\omega_h\omega_{h+1},$$
$$\cdots$$
$$\Pi_{\rho h} = \omega_1\chi_{\rho h1} + \cdots + \omega_h\chi_{\rho hh} + \omega_{h+1}\chi_{\rho,h+1,h} - \lambda_{\rho h1}\omega_1\omega_{h+1} - \cdots - \lambda_{\rho hh}\omega_h\omega_{h+1}.$$

By adding to the *principal* $\chi_{\rho ij}$ expressions of the form $\mu_{\rho ij}\omega_{h+1}$ as necessary, we can manifestly make them such that we have

$$\lambda_{\rho ij} = 0, \qquad (i, j = 1, 2, \dots, h; \rho \leq \sigma_i, \rho \leq \sigma_j),$$

this modification necessarily entails analogous modifications for the non-principal $\chi_{\rho ij}$ as well as for $\chi_{\rho,h+1,i}$, but as for the latter they always enter the expressions of $\Pi_{\rho 1}, \dots, \Pi_{\rho h}$ multiplied by ω_{h+1}, this has no importance for them. We therefore finally see that $\Pi_{\rho i}$ are now presented in the form desired, up to bilinear expressions such as

$$\Psi_{\rho i} = \lambda_{\rho i1}\omega_1\omega_{h+1} + \cdots + \lambda_{\rho ih}\omega_h\omega_{h+1}, \qquad (i = 1, 2, \dots, p)$$

with

$$\begin{aligned} &\lambda_{\rho ij} = \lambda_{\rho ji}, && (i, j = 1, 2, \dots, h), \\ &\lambda_{\rho ij} = 0, && (\rho \leq \sigma_i, \rho \leq \sigma_j; i = 1, 2, \dots, p; j = 1, 2, \dots, h). \end{aligned}$$

By expressing these necessary relations for $\Pi_{\rho i}$, we see that these relations must also hold for $\Psi_{\rho i}$. Then, $\lambda_{\rho ij}$ satisfy the fundamental property of $l_{\rho ij}$, and as *the principal ones among them are zero*, the rest are also zero. The theorem is hence proved.

9 With the two preceding theorems, we can, *by using only the fundamental identity*, show that if we prolong a system in involution we obtain a system in involution again, a property that we can easily deduce *a priori* for the existence and indeterminacy of integral varieties of the given system.

If to the equations of the system we adjoin the new equations

$$\bar{\varpi}_{\rho i} = \varpi_{\rho i} - l_{\rho i1}\omega_1 - l_{\rho i2}\omega_2 - \cdots - l_{\rho ip}\omega_p = 0, \qquad \begin{pmatrix} \rho = 1, 2, \dots, s \\ i = 1, 2, \dots, p \end{pmatrix}, \tag{27}$$

the covariants of the left hand sides of the new equations of the system are manifestly, after using those equations themselves and the old equations, under the form

$$\bar{\varpi}'_{\rho i} = \omega_1\chi_{\rho i1} + \omega_2\chi_{\rho i2} + \cdots + \omega_p\chi_{\rho ip} + \cdots, \tag{28}$$

the terms not written out depend only on ω, and $\chi_{\rho ij}$ being are just the differentials $dl_{\rho ij}$ up to terms in ω_i. By applying the fundamental identity to the covariants

$$\theta'_\rho = \omega_1\bar{\varpi}_{\rho 1} + \omega_2\bar{\varpi}_{\rho 2} + \cdots + \omega_p\bar{\varpi}_{\rho p} \pmod{\theta_1, \dots, \theta_s},$$

we obviously obtain

$$0 = \omega_1\bar{\varpi}'_{\rho 1} + \omega_2\bar{\varpi}'_{\rho 2} + \cdots + \omega_p\bar{\varpi}'_{\rho p} \pmod{\theta_1, \dots, \theta_s; \bar{\varpi}_{\sigma 1}, \dots, \bar{\varpi}_{\sigma p}},$$

and then, according to the theorem of §**8**, we have

$$\bar{\varpi}'_{\rho i} = \omega_1\chi_{\rho i1} + \cdots + \omega_p\chi_{\rho ip} \pmod{\theta_1, \dots, \theta_s; \bar{\varpi}_{\sigma 1}, \dots, \bar{\omega}_{\sigma p}}, \tag{29}$$

$\chi_{\rho ij} = \chi_{\rho ji}$ being linked by the same relations as $l_{\rho ij}$.

Since according to the formula (28) the principal $\chi_{\rho ij}$ are necessarily independent among themselves and with respect to ω, we see that, by applying the theorem of §**7**, the prolonged system is in involution.

Remark. *This theorem may fail to hold if we only apply a partial prolongation of the given system, that is to say if we adjoin to this system only some of the equations* (27). As an example, this happens if we prolong the system in involution

$$\theta_1' = \omega_1 \varpi_1,$$
$$\theta_2' = \omega_2 \varpi_2$$

by adjoining the single equation

$$\bar{\varpi} = \varpi_1 + \varpi_2 - u\omega_1 - v\omega_2 = 0.$$

10 After studying systems in involution, we are going to consider a Pfaffian system for which the bilinear covariants have been reduced to the form (10) and we assume that the number of arbitrary parameters that the most general p-th order integral element depends on is *less* than the integer

$$\sigma_1 + 2\sigma_2 + \cdots + p\sigma_p + p(q - \sigma_1 - \sigma_2 - \cdots - \sigma_p),$$

where σ are numbers defined by means of the matrix (14). We are going to show that the Pfaffian system can be *prolonged* in a way to satisfy the conditions of the theorem in §**6**.

We say that the bilinear covariants of the system is made into *normal* form when, using the equations of the system and writing only the terms containing ϖ, we can divide the covariants into a certain number of groups having the following property. To each group is associated an integer N such that it contains as many covariants $\Pi_{\alpha_1,\alpha_2,\ldots,\alpha_p}$ as there are combinations of positive or zero integers α satisfying

$$\alpha_1 + \alpha_2 + \cdots + \alpha_p = N$$

with

$$\Pi_{\alpha_1,\alpha_2,\ldots,\alpha_p} = \omega_1 \varpi_{\alpha_1+1,\alpha_2,\ldots,\alpha_p} + \omega_2 \varpi_{\alpha_1,\alpha_2+1,\ldots,\alpha_p} + \cdots + \omega_p \varpi_\alpha \varpi_{\alpha_1,\alpha_2,\ldots,\alpha_p+1}.$$

We also assume that we can associate to each of these groups another integer h between 0 and p such that if

$$h, \quad h', \quad h'', \quad \ldots$$

are the integers associated with the first, second, third, ... groups, the ϖ of the first group which have the last $p-h$ indices are zero, the ϖ' of the second group which have the last $p-h'$ indices zero, etc., are independent among themselves and moreover all the other ϖ, ϖ', ... depend on the preceding. We call these last expressions the *principal expressions* ϖ.

If we apply an *arbitrary* linear transformation on ω, the covariants will not cease to be under the normal form, and the ϖ in the same group undergo a linear transformation.

If we establish a relation among the variables, this will translates into a linear relation in ϖ, ϖ', ..., and ω. Suppose that it effectively contains the principal expressions ϖ of the first group and the integer $h^{(i)}$ associated to the groups whose principal expressions enter wholly or partly into the relations considered are all greater than or equal to h. Then we can, by a linear transformation on $\omega_1, \ldots, \omega_h$ make them under a form such that the relation effectively contains

$$\varpi_{00\ldots N+1,0\ldots 0} \qquad (\alpha_h = N+1).$$

If this is the case, all of ϖ, ϖ', ... are expressed in terms of the principal expressions of the second, third, ... groups *and the coefficients of* $\omega_1, \ldots, \omega_{h-1}$ *in the first group.* We can then replace the first group by a certain number of other groups *for which the integer* h *is diminished by one.*

Finally, observe that if we can prolong the system by the procedure of §**4**, the covariants of the new system are again under a canonical form, with the same number of groups, the same integers h, h', ..., and the integers N are increased by one and the necessary linear relations hold among the new principal expressions. Indeed, by omitting combinations of $\omega_1, \ldots, \omega_p$ whose coefficients depend only on the old variables, if necessary, we have formulae of the following form

$$\varpi_{\alpha_1,\alpha_2,\ldots,\alpha_p} = t_{\alpha_1+1,\alpha_2,\ldots,\alpha_p}\omega_1 + t_{\alpha_1,\alpha_2+1,\ldots,\alpha_p}\omega_2 + \cdots + t_{\alpha_1,\alpha_2,\ldots,\alpha_p+1}\omega_p,$$

and t, t', ... depend only on those of the first group for which the $p-h$ last indices are zero, etc. It suffices to take as principal expressions of the prolonged system the differentials of these coefficients t, only that, if these t are not really independent, the new principal expressions must be linked by the relations, each of which dissolves a group of covariants by decreasing the associated integer h by one unit, as we have just seen.

11 Granted this, let us denote by

$$v_0, \quad v_1, \quad \ldots \quad v_p$$

the total number of the groups of covariants of the old system for which the integer h is equal to, respectively

$$0, \quad 1, \quad \ldots \quad p.$$

Suppose that we never arrive at a prolongation leaving the integers v invariant. Then the integer v_p cannot be increased, and we necessarily arrive at a stage where it no longer changes. We can then reason successively for all the integers v and prove that after a certain prolongation none of these integers can change.

To say that none of the integers v changes is the same as saying that in the prolongation all the t for which the $p-h$ last indices are zero, all the t' for which the last $p-h'$ indices are zero, etc., are arbitrary. If we use

τ_1 to denote the number of principal expressions for which the first index is at least equal to 1;

τ_2 to denote the number of principal expressions for which the first index is zero without the second being zero as well;

τ_3 to denote the number of principal expressions for which the first two indices are zero without the third being zero, etc.;

the numbers of arbitrary t, t', ... is, as it is easy to see, equal to

$$\tau_1 + 2\tau_2 + 3\tau_3 + \cdots + p\tau_p, \tag{30}$$

and moreover the number of independent ϖ, vp', ... is

$$\tau_1 + \tau_2 + \cdots + \tau_p.$$

For the system considered, we manifestly have

$$\begin{cases} \sigma_1 \geq \tau_1, \\ \sigma_1 + \sigma_2 \geq \tau_1 + \tau_2, \\ \cdots \\ \sigma_1 + \sigma_2 + \cdots + \sigma_{p-1} \geq \tau_1 + \tau_2 + \cdots + \tau_{p-1}, \\ \sigma_1 + \sigma_2 + \cdots + \sigma_{p-1} + \sigma_p = \tau_1 + \tau_2 + \cdots + \tau_{p-1} + \tau_p. \end{cases} \tag{31}$$

On the last line we have equality, each of the two sides being at least equal to the number of independent ϖ, ϖ', ... From the inequalities (21) we deduce the following:

$$\begin{cases} \sigma_p \leq \tau_p, \\ \sigma_{p-1} + \sigma_p \leq \tau_{p-1} + \tau_p, \\ \cdots \\ \sigma_2 + \cdots + \sigma_{p-1} + \sigma_p \leq \tau_2 + \cdots + \tau_{p-1} + \tau_p, \\ \sigma_1 + \sigma_2 + \cdots + \sigma_{p-1} + \sigma_p = \tau_1 + \tau_2 + \cdots + \tau_{p-1} + \tau_p, \end{cases} \tag{32}$$

and also, by adding them together,

$$\sigma_1 + 2\sigma_2 + \cdots + p\sigma_p \leq \tau_1 + 2\tau_2 + \cdots + p\tau_p. \tag{33}$$

The number of arbitrary parameters the most general p-th order integral element depends on is therefore, according to (33) *at least* equal to the integer

$$\sigma_1 + 2\sigma_2 + \cdots + p\sigma_p.$$

On the other hand, according to our theorem (§**6**), it is *at most* equal to it. Therefore they are equal and the system is in involution. Furthermore the indeterminacy is defined by the numbers

$$s_1 = \tau_1, \quad s_2 = \tau_2, \quad \ldots \quad s_p = \tau_p.$$

12 As we can see, the preceding proof is analogous to several known proofs on the possibility of making a system of partial differential equations into a system in involution. Despite of this, we would like to emphasise the practical importance of the theorem of §**6**: to make a given Pfaffian system into one in involution, we calculate, after each prolongation is effected, the values of the integers σ_1, σ_2, ..., σ_p and we stop when the number of new variables the prolongation provides attains the integer value $\sigma_1 + 2\sigma_2 + \cdots + p\sigma_p$. Unce this system is in involution, one more prolongation would give the new values

$$\begin{cases} \sigma_1' = \sigma_1 + \sigma_2 + \cdots + \sigma_p, \\ \sigma_2' = \qquad \sigma_2 + \cdots + \sigma_p, \\ \quad \cdots \\ \sigma_p' = \qquad\qquad\qquad \sigma_p. \end{cases} \tag{34}$$

Also observe that if a system is in involution with a value of the integer σ_p equal to at least 1, an *arbitrary* relation among the variables will keep the system in involution, and σ_p decreases simply by one. Instead of one relation we can take any number, less or equal to σ_p, of relations, then σ_p will decrease by this number. Without wanting to elaborate on this point, this observation can nonetheless show that an *arbitrary* system of m equations in partial differential equations (not necessarily of the same order) of m unknown functions is always in involution when we have derived from the equations of the system of order less than the maximum order the system with all the orders maximum.