Theories of finite-dimensional connected groups and differential geometry treated by the method of moving frames

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La théorie des groupes finis et continus
et la géométrie différentielle
traitées par la méthode du repère mobile

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Preface

The role played by the notion of group in all branches of mathematics is well-known. This notion is also in the foundation of elementary geometry, as H. Poincaré has shown, and the modern evolution of geometry has never ceased to show its increasing importance in geometrical theories such as Riemannian geometry and its generalisations which have for a long time appeared to be independent questions. The present work, which is a reproduction with certain modifications of a course given at the Sorbonne during the winter semester of 1931–1932, studies simultaneously the fundamental theorems of the theory of finite dimensional connected groups of S. Lie and disciplines for which the method of moving frames rest on. Indeed there exists between these two mathematical disciplines a very close relation which the following remark can help one's understanding: the classical equations of Darboux, obtained by every movement depending on several parameters of a moving trihedral, are none other than the *structural equations* of the Euclidean displacement group, by giving to this expression "structural equation" the sense that it has in the theorem that I have developed in 1904–1905 in the structure of connected groups of Sophus Lie, finite or infinite dimensional. Darboux's equations for this reason contain in them all of Euclidean geometry. When we pass from elementary geometry to a geometry founded on an arbitrary group G, Darboux's equations are replaced by equations established in 1888 by Maurer, the importance of which has long been ignored: Maurer's equations are in the foundation of all differential geometry founded on the group G, and one will necessarily encounter them when one tries to generalise the method of moving frame in one of these geometries.

This work is divided to three parts. The first part is for familiarising the reason with the method of moving frame in Euclidean geometry, especially in the case, left aside by Darboux, where the choice of moving frame to use is not immediate and hence requires solving a preliminary problem. The solution of this problem follows a general principle whose first application we see here (the theory of minimal curves and ruled surfaces with isotropic generatrices). The second part introduces the frames attached to an arbitrary group and expound the first notions of the theory of finite dimensional connected groups and the principles of the method of moving frame. Finally, the third part introduces the structural equations of Maurer-Cartan, and shows their use in the theory of moving frames and their role in the third fundamental theorem of S. Lie. The last chapter is devoted to the study of the structure of finite dimensional groups, considered from the classical point of view of S. Lie.

The editing of this work is due almost entirely to Mr Jean Leray, whose beautiful works in other domains of mathematics are well-known. I must highlight that Mr Leray did not confine himself to a simple editing of notes taken in the course: the material has been elaborated further, improved on several important points, and presented in a more rational order by him. Here I express all my gratitude to Mr Leray for his precious collaboration, which I am sure contributes greatly to making the theories expounded in this work more accessible.

I would also like to thank Mr Julia for having accepted again this work in his collection. My thanks goes equally to the publisher Gauthier-Villars who, as usual, has taken great care in the typesetting of this work.

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Part I

The method of moving trihedrals in Euclidean geometry

Introduction

1 Ribaucour the First studied the differential properties of surfaces with help of a moving rectangular trihedral. Darboux made an extremely fruitful use of moving trihedrals in his *Leçon sur la théorie générale des surfaces* (*Lessons on the general theory of surfaces*) [1][†]. This method is, at roots, a use of the properties of the displacement group of space, and in the following we will emphasise this point. As Poincaré has commented, the notion of displacement group always plays a fundamental role in geometry: the equality of two figures is defined by the possibility of superimposing the two: i.e., the transformation that transforms one into the other [‡].

Affine geometry, conformal geometry, projective geometry, etc., can all be developed in the same way using moving frames, and the relation between the properties of a moving frame and the properties of the conformal group, affine group, etc., proves to be fundamental, as we will show in detail.

But there are even more to it: the use of moving frames permits us to study the structure of any finite-dimensional continuous group itself, and to establish *inter alia* the three fundamental theorems of Sophus Lie, while never losing sight of the geometrical interpretation of our reasoning.

This first part of the work is exclusively devoted to familiarising the reader with the use of the method of moving frames.

(†)The numbers between the square brackets refer to entries in the Bibliography Index at the end of the book.

(‡)Let us enunciate three axioms of geometry: a figure F is equal to itself; if F is equal to F', then F' is equal to F; if F is equal to F' and F' to F'', then F is equal to F''. These three axioms are equivalent to the following: the identical transformation (which leaves every point in the space fixed) is a displacement; the inverse transformation of a displacement is a displacement; the transformation obtained by applying successively two displacements is a displacement. The three properties of displacements that we have just stated are what defines a set of transformations which form a group. This example shows that how we can treat the axioms relative to the equality of figures and the axioms relative to their displacements as essentially the same.

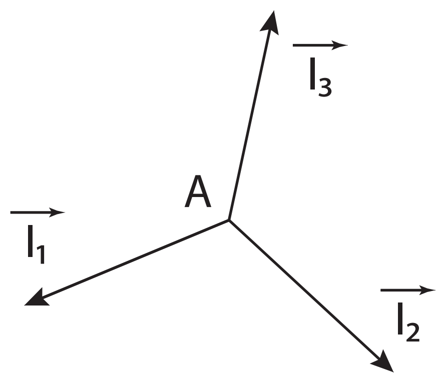
Chapter 1

Moving rectangular trihedrals. Real space curves

I. Infinitesimal displacement of a rectangular trihedral [†]

2 Review of some kinematic notions. Consider a moving solid body. It is convenient for studying its geometry to invariantly attach a right rectangular trihedral **T** to it. Let **A** be the vertex and $\vec{\mathbf{I}}_1, \vec{\mathbf{I}}_2, \vec{\mathbf{I}}_3$ be three vectors of length 1 with **A** as the origin which are along the edges of **T**. If **M** is a point on the solid body, the components x, y, z of the vector $\overrightarrow{\mathbf{AM}}$ with respect to this trihedral are constants.

Figure 1



The derivative of the vector relation

$$\overrightarrow{\mathbf{AM}} = x\vec{\mathbf{I}}_1 + y\vec{\mathbf{I}}_2 + z\vec{\mathbf{I}}_3$$

[†] *See* G. Darboux [1], vol. 1, chapters 1, 5 and 7.

shows that the velocity of $\mathbf{M}$ is [†]

$$\frac{d\vec{\mathbf{M}}}{dt} = \frac{d\vec{\mathbf{A}}}{dt} + x\frac{d\vec{\mathbf{I}}_1}{dt} + y\frac{d\vec{\mathbf{I}}_2}{dt} + z\frac{d\vec{\mathbf{I}}_3}{dt}.$$

Knowledge of the velocity of each point of the solid body therefore can be deduced from the knowledge of the twelve components of the vectors

$$\frac{d\vec{\mathbf{A}}}{dt}, \qquad \frac{d\vec{\mathbf{I}}_1}{dt}, \qquad \frac{d\vec{\mathbf{I}}_2}{dt}, \qquad \frac{d\vec{\mathbf{I}}_3}{dt}.$$

It is inappropriate to measure these vectors with respect to a fixed trihedral $\mathbf{T}_0$: the values of these components will then depend on both the choice of $\mathbf{T}$ and of $\mathbf{T}_0$. We will measure these components with respect to $\mathbf{T}$ itself, which we take to depend on the minimal number of arbitrary parameters. We therefore write

$$(1) \qquad \begin{cases} \dfrac{d\vec{\mathbf{A}}}{dt} = \xi_1\vec{\mathbf{I}}_1 + \xi_2\vec{\mathbf{I}}_2 + \xi_3\vec{\mathbf{I}}_3, \\ \dfrac{d\vec{\mathbf{I}}_1}{dt} = p_{11}\vec{\mathbf{I}}_1 + p_{12}\vec{\mathbf{I}}_2 + p_{13}\vec{\mathbf{I}}_3, \\ \dfrac{d\vec{\mathbf{I}}_2}{dt} = p_{21}\vec{\mathbf{I}}_1 + p_{22}\vec{\mathbf{I}}_2 + p_{23}\vec{\mathbf{I}}_3, \\ \dfrac{d\vec{\mathbf{I}}_3}{dt} = p_{31}\vec{\mathbf{I}}_1 + p_{32}\vec{\mathbf{I}}_2 + p_{33}\vec{\mathbf{I}}_3. \end{cases}$$

But the cross products $\vec{\mathbf{I}}_k \times \vec{\mathbf{I}}_l$ are constants, from which

$$\vec{\mathbf{I}}_k \times d\vec{\mathbf{I}}_l + \vec{\mathbf{I}}_l \times d\vec{\mathbf{I}}_k = 0,$$

i.e.,

$$p_{kl} + p_{ik} = 0.$$

Then the formula (1) becomes

$$(2) \qquad \begin{cases} \dfrac{d\vec{\mathbf{A}}}{dt} = \xi_1\vec{\mathbf{I}}_1 + \xi_2\vec{\mathbf{I}}_2 + \xi_3\vec{\mathbf{I}}_3, \\ \dfrac{d\vec{\mathbf{I}}_1}{dt} = p_{12}\vec{\mathbf{I}}_2 + p_{13}\vec{\mathbf{I}}_3, \\ \dfrac{d\vec{\mathbf{I}}_2}{dt} = -p_{12}\vec{\mathbf{I}}_1 + p_{23}\vec{\mathbf{I}}_3, \\ \dfrac{d\vec{\mathbf{I}}_3}{dt} = -p_{13}\vec{\mathbf{I}}_1 - p_{23}\vec{\mathbf{I}}_2. \end{cases}$$

Therefore six functions $\xi_1, \xi_2, \xi_3, p_{12}, p_{13}, p_{23}$ are sufficient to determine the velocity of every point on the solid body at each instant t. Furthermore, we know that these velocities are obtained when we simultaneously apply the translation defined by the vector (ξ_1, ξ_2, ξ_3) and the rotation defined by the vector $(p_{23}, -p_{13}, p_{12})$.

[†] $\dfrac{d\vec{\mathbf{A}}}{dt}$ is the vector whose components are the derivatives of the coordinates of $\vec{\mathbf{A}}$, and $\dfrac{d\vec{\mathbf{I}}_1}{dt}$ is the vector whose components are the derivatives of the components of $\vec{\mathbf{I}}_1$.

3 Movements depending on several parameters. The consideration in §**2** has applications in many diverse questions in geometry. It is useful to generalise it so as to extend its field of applications. Therefore, let us consider a trihedral **T** depending [†] on ρ parameters $u_1, \dots, u_\rho$. Let **M** be a fixed point associated with this trihedral. We have

$$\overrightarrow{\mathbf{AM}} = x\vec{\mathbf{I}}_1 + y\vec{\mathbf{I}}_2 + z\vec{\mathbf{I}}_3,$$

x, y, z being three constants. There exists ρ systems of relations such as the following

$$(3)\qquad \begin{cases} \dfrac{\partial\vec{\mathbf{M}}}{\partial u_q} = \dfrac{\partial\vec{\mathbf{A}}}{\partial u_q} + x\dfrac{\partial\vec{\mathbf{I}}_1}{\partial u_q} + y\dfrac{\partial\vec{\mathbf{I}}_2}{\partial u_q} + z\dfrac{\partial\vec{\mathbf{I}}_3}{\partial u_q}, \\ \dfrac{\partial A}{\partial u_q} = \xi_{1q}\vec{\mathbf{I}}_1 + \xi_{2q}\vec{\mathbf{I}}_2 + \xi_{3q}\vec{\mathbf{I}}_3, \\ \dfrac{\partial\vec{\mathbf{I}}_1}{\partial u_q} = p_{12q}\vec{\mathbf{I}}_2 + p_{13q}\vec{\mathbf{I}}_3, \\ \dfrac{\partial\vec{\mathbf{I}}_2}{\partial u_q} = -p_{12q}\vec{\mathbf{I}}_1 + p_{23q}\vec{\mathbf{I}}_3, \\ \dfrac{\partial\vec{\mathbf{I}}_3}{\partial u_q} = -p_{13q}\vec{\mathbf{I}}_1 - p_{23q}\vec{\mathbf{I}}_2. \end{cases}$$

The use of differential notation permits us to condense these ρ systems into one. Set

$$d\vec{\mathbf{M}} = \frac{\partial\vec{\mathbf{M}}}{\partial u_1}du_1 + \frac{\partial\vec{\mathbf{M}}}{\partial u_2}du_2 + \cdots + \frac{\partial\vec{\mathbf{M}}}{\partial u_\rho}du_\rho,$$
$$\dots$$
$$d\vec{\mathbf{I}}_3 = \frac{\partial\vec{\mathbf{I}}_3}{\partial u_1}du_1 + \frac{\partial\vec{\mathbf{I}}_3}{\partial u_2}du_2 + \cdots + \frac{\partial\vec{\mathbf{I}}_3}{\partial u_\rho}du_\rho.$$

The system becomes

$$(4)\qquad \begin{cases} d\vec{\mathbf{M}} = d\vec{\mathbf{A}} + x\,d\vec{\mathbf{I}}_1 + y\,d\vec{\mathbf{I}}_2 + z\,d\vec{\mathbf{I}}_3, \\ d\vec{\mathbf{A}} = \omega_1\vec{\mathbf{I}}_1 + \omega_2\vec{\mathbf{I}}_2 + \omega_3\vec{\mathbf{I}}_3, \\ d\vec{\mathbf{I}}_1 = \omega_{12}\vec{\mathbf{I}}_2 + \omega_{13}\vec{\mathbf{I}}_3, \\ d\vec{\mathbf{I}}_2 = -\omega_{12}\vec{\mathbf{I}}_1 + \omega_{23}\vec{\mathbf{I}}_3, \\ d\vec{\mathbf{I}}_3 = -\omega_{13}\vec{\mathbf{I}}_1 - \omega_{23}\vec{\mathbf{I}}_2. \end{cases}$$

The ω are linear homogeneous forms in du_q, whose coefficients are functions of u_q. These forms are called *Pfaffian forms.* We call the preceding equations *the relative components of an infinitesimal displacement of a trihedral.*

[†] We always restrict ourselves to the case where the set of values of the parameters constitute a domain and the coordinates of different points of this trihedral are functions of continuous parameters and their first and second derivatives.

Think of du_q as infinitesimally small for a moment and suppose all of these forms are zero except ω_1. The displacement of $\mathbf{M}$ is the translation parallel to $\vec{\mathbf{I}}_1$ of magnitude ω_1. Now suppose that all of the forms are zero except ω_{12}. The displacement of M is then, up to an infinitesimal quantity of second order, the rotation around the axis $\vec{\mathbf{I}}_3$ of angle ω_{12}. The formulae (4) therefore show how the most general infinitesimal displacement is obtained by superimposing three infinitesimal translations parallel to the three edges of T and three infinitesimal rotations around these edges.

4 The relative components have a geometrical meaning. Consider a family of trihedrals depending on one or several parameters $u_1, \ldots, u_\rho$. The Pfaffian forms $\omega_1, \ldots, \omega_{23}$ are completely determined by the data of the trihedrals corresponding to the values (u_q) and $(u_q + du_q)$ of the parameters. They depend only on the relative position of these two trihedrals and their values remain the same when we apply a displacement to the family of trihedrals together or when we change our choice of parameters.

The functions ξ_{iq} and p_{ijq} depend on the relative positions of these trihedrals and on the chosen parameters. Then, in every equation where none of these parameters has been specified, their use is less convenient than the forms ω. We will therefore use the latter systematically. This geometrical meaning of the components ω has *an important consequence*: consider two equal families of trihedrals and consider two trihedrals $\mathbf{T}_1$ and $\mathbf{T}_2$ in the first family corresponding to the values (u_q) and $(u_q + du_q)$ of the parameters. Let $\mathbf{T}_1^\star$ and $\mathbf{T}_2^\star$ be the corresponding trihedrals in the second family and (v_q) and (v_q+dv_q) their parameters. The figure $(\mathbf{T}_1, \mathbf{T}_2)$ is equal to the figure $(\mathbf{T}_1^\star, \mathbf{T}_2^\star)$, and $\mathbf{T}_2$ is placed with respect to $\mathbf{T}_1$ as $\mathbf{T}_2^\star$ is with respect to $\mathbf{T}_1^\star$. From this, it follows the equality

$$(5) \qquad \begin{cases} \omega_i(u; du) = \omega_i(v; dv), \\ \omega_{ji}(u; du) = \omega_{ji}(v; dv). \end{cases}$$

5 An integration problem. Now we are going to find if the converse of this proposition holds and to tackle the following integration problem: finding all families of trihedrals corresponding to the forms ω_i, ω_{ij} given *a priori*. We will see at the beginning of the third part of this course that the forms ω are subject to certain compatibility conditions when the family of trihedrals depend on more than one parameters. We say that the *structure* of these forms are not arbitrary and we name these conditions the structure equations.

We first consider the families depending one parameter t: hence suppose that we are given the forms [†]

$$\omega_i = \xi_i(t)dt, \qquad \omega_{ji} = p_{ji}(t)dt, \qquad (i = 1, 2, 3; j < i).$$

We claim that to these there correspond families of trihedrals and these families are all equal. According to the previous paragraph this proposition is equivalent to the

[†] We assume that the functions $\xi_i(t)$ and $p_{ji}(t)$ are defined for all real values of t and they admit continuous first order derivatives.

following: there exists one and only one family of trihedrals **T** admitting these forms as the components of their infinitesimal displacement and whose element corresponding to $t = 0$ is an *a priori* given trihedral. Let us choose this trihedral as the reference trihedral and adopt the following notation

$$\begin{array}{rll} x, y, z & \text{are the coordinates of the point } \mathbf{A}\text{, the apex of the trihedral } \mathbf{T}, & \\ \alpha, \beta, \gamma & \text{are the components of the first vector,} & \vec{\mathbf{I}}_1, \\ \alpha', \beta', \gamma' & \text{are the components of the second vector,} & \vec{\mathbf{I}}_2, \\ \alpha'', \beta'', \gamma'' & \text{are the components of the third vector,} & \vec{\mathbf{I}}_3. \end{array}$$

We obtain three systems of relations linking these unknowns by considering successively the components along the three axes of the coordinates of the vectors appearing in the system (2). The first of these systems is

$$(2_1) \qquad \begin{cases} \dfrac{dx}{dt} = \xi_1\alpha + \xi_2\alpha' + \xi_3\alpha'', \\ \dfrac{d\alpha}{dt} = p_{12}\alpha' + p_{13}\alpha'', \\ \dfrac{d\alpha'}{dt} = -p_{12}\alpha + p_{23}\alpha'', \\ \dfrac{d\alpha''}{dt} = -p_{13}\alpha - p_{23}\alpha'. \end{cases}$$

This system (2_1) admits only one solution such that we have for $t = 0$

$$x = 0, \quad \alpha = 1, \quad \alpha' = 0, \quad \alpha'' = 0.$$

By doing the same for the other two systems, we see that there exists only one trihedral depending on t that satisfies (2) and coincides with the reference trihedral on $t = 0$. It remains to verify that this variable trihedral is rectangular and that the vectors $\vec{\mathbf{I}}_1$, $\vec{\mathbf{I}}_2$, $\vec{\mathbf{I}}_3$ are of unit length. We deduce from (2) the six relations (where $p_{ij} = -p_{ji}$)

$$\frac{d(\vec{\mathbf{I}}_i \times \vec{\mathbf{I}}_j)}{dt} = \sum_{k=1}^{3} p_{ik}\vec{\mathbf{I}}_i \times \vec{\mathbf{I}}_k + \sum_{k=1}^{3} p_{jk}\vec{\mathbf{I}}_j \times \vec{\mathbf{I}}_k.$$

These relations constitute a linear system with respect to the six quantities $\vec{\mathbf{I}}_i \times \vec{\mathbf{I}}_j$. The only solution satisfying the given initial values imposed is obvious and it is

$$\vec{\mathbf{I}}_1^2 = \vec{\mathbf{I}}_2^2 = \vec{\mathbf{I}}_3^2 = 1, \qquad \vec{\mathbf{I}}_1 \times \vec{\mathbf{I}}_2 = \vec{\mathbf{I}}_2 \times \vec{\mathbf{I}}_3 = \vec{\mathbf{I}}_3 \times \vec{\mathbf{I}}_1 = 0.$$

The proposition is then established.

6 Equality of two ρ-parameter families having the same relative components. Let us finish proving the converse of **§4** [formulae (5)]: consider two families of trihedrals **T**

and $\mathbf{T}^\star$ depending on ρ parameters $u_1, u_2, \dots, u_\rho$ and $v_1, v_2, \dots, v_\rho$ respectively and suppose that a continuous and bijective correspondance established between $\mathbf{T}$ and $\mathbf{T}^\star$ realises the equalities

$$\left\{\begin{aligned} \omega_i(u;du) &= \omega_i^\star(v;dv),\\ \omega_{ji}(u;du) &= \omega_{ji}^\star(v;dv). \end{aligned}\right. \tag{5}$$

Let us choose two trihedrals $\mathbf{T}_1$ and $\mathbf{T}_2$ in the first family and let $\mathbf{T}_1^\star$ and $\mathbf{T}_2^\star$ be their analogues in the second family. We can find [(†)] a family of trihedrals depending on one parameter $\mathbf{T}_3$ belonging to the first family such that $\mathbf{T}_3$ coincides with $\mathbf{T}_1$ at $t=0$ and with $\mathbf{T}_2$ at $t=1$. Let $\mathbf{T}_3^\star$ be the analogous trihedral of $\mathbf{T}_3$. The components $\omega^\star$ of the infinitesimal displacement of $\mathbf{T}_3^\star$ are equal to the components ω of the infinitesimal displacement of $\mathbf{T}_3$. The preceding paragraph affirms that under these conditions the displacement which makes $\mathbf{T}_1$ and $\mathbf{T}_1^\star$ superimpose the corresponding $\mathbf{T}_3$ and $\mathbf{T}_3^\star$, and hence in particular $\mathbf{T}_2$ and $\mathbf{T}_2^\star$. But as $\mathbf{T}_2$ and $\mathbf{T}_2^\star$ are two arbitrary elements of the families $\mathbf{T}$ and $\mathbf{T}^\star$, this displacement superimpose all corresponding elements of the two families $\mathbf{T}$ and $\mathbf{T}^\star$. Q.E.D.

We state the conclusions of §**4** and §**6**

7 The fundamental condition of equality. *Consider a bijective correspondance established between two ρ parameter families of trihedrals* $\mathbf{T}$ *and* $\mathbf{T}^\star$. *For a single displacement to suffice to superimpose all the corresponding trihedrals, it is necessary and sufficient that the correspondance in question makes the components ω of infinitesimal displacement of* $\mathbf{T}$ *equal to the components $\omega^\star$ of the infinitesimal displacement of the analogous* $\mathbf{T}^\star$.

We note on the other hand the conclusion of §**5**.

Structure theorem. *The components of the infinitesimal displacement of a moving rectangular trihedral depending on one parameter are not subject to any structure conditions* (in other words: these components are the formes in one completely arbitrary variable).

8 Remark concerning §5. In the course of this paragraph we have characterised the position of an arbitrary trihedral in space with the help of 12 parameters linked by 6 relations: the parameters are x, y, z, α, β, γ, α', β', γ', α'', β'', γ'' and the relations are $\alpha^2+\beta^2+\gamma^2=1$, $\alpha\alpha'+\beta\beta'+\gamma\gamma'=0, \dots$

It follows from these complications that are not conducive to generalisations, as we have seen. Therefore let us utilise six independent parameters to define a trihedral. These are, for example, the three coordinates of the apex and the three Euler angles (φ,θ,ψ) and let us prove a new conclusion of §**5**.

"We can choose in one and only one way the parameters x, y, z, φ, θ, ψ as functions of one parameter t such that the trihedral has at $t=0$ a fixed position and the components of its infinitesimal displacement ω_i, ω_{ji} $(j<i)$ are equal to given forms $\xi_i(t)dt$, $p_{ji}(t)dt$."

[(†)] Since the families are assumed to be connected [note in §**3**].

Suppose that ω_i and ω_{ji} $(j < i)$ are independent forms in the six differentials dx, dy, dz, $d\varphi$, $d\theta$, $d\psi$. The system

$$\omega_i = \xi_i(t)dt, \qquad \omega_{ji} = p_{ji}(t)dt$$

is equivalent to a system of six differential equations

$$\begin{aligned} \frac{dx}{dt} &= f_1(x, y, z, \varphi, \theta, \psi, t), & \frac{d\varphi}{dt} &= f_4(x, y, z, \varphi, \theta, \psi, t), \\ \frac{dy}{dt} &= f_2(x, y, z, \varphi, \theta, \psi, t), & \frac{d\theta}{dt} &= f_5(x, y, z, \varphi, \theta, \psi, t), \\ \frac{dz}{dt} &= f_3(x, y, z, \varphi, \theta, \psi, t), & \frac{d\psi}{dt} &= f_6(x, y, z, \varphi, \theta, \psi, t). \end{aligned}$$

This system admits one and only one solution corresponding to the values of the unknowns given at $t = 0$. The proposition is hence established.

But the hypothesis that the six forms ω_i, ω_{ji} are dependent is absurd. We may then actually find a differential system

$$\frac{dx}{g_1(x, y, z, \varphi, \theta, \psi)} = \frac{dy}{g_2} = \frac{dz}{g_3} = \frac{d\varphi}{g_4} = \frac{d\theta}{g_5} = \frac{d\psi}{g_6},$$

whose integrals annihilate these six forms: there exists moving trihedrals whose components of infinitesimal displacement are zero, and each point of it is then fixed. This is absurd.

The conclusion of §**5** can also be established by a second, less complicated and better reasoning. It is interesting to note that we just used the components of infinitesimal displacement of a most general trihedral in the question relative to the families of trihedrals depending on a single parameter.

II. Application of the theory of one-parameter families of trihedrals to the study of space curves

9 The various geometrical elements attached to a curve. We will now rapidly review the classical theory of space curves. Consider a real space curve C. We start by define at each of its point, with the help of various infinitesimal geometrical constructions, the *tangent*, the differential of the *arc* ds, and to choose the sign of ds it is necessary to *orient* the curve. We then define the *osculating plane*, the *principal normal* and the *binormal*.

Observe that the tangent is a first order geometrical element, that the osculating plane, the principal normal and the binormal are second order geometrical elements [†]. This simply signifies that two curves having a first order contact have the same tangent, and they have the same osculating plane, principal normal, binormal when they have a

[†] In our study we exclude the inflection points of C.

second order contact. Recall that the set of curves having a P-th order contact between any two of them at a given point constitute a *P-th order contact element*, and this element is sait to belong to each of these curves. Suppose that the curve considered are defined by the functions $y(x)$ and $z(x)$ and their tangents are not parallel to the yz plane. A P-th order contact element can be characterised analytically by the data of one value of x and the corresponding values taken by the functions

$$y(x), \quad z(x), \quad \frac{dy}{dx}, \quad \frac{dz}{dx}, \quad \ldots, \quad \frac{d^P y}{dx^P}, \quad \frac{d^P z}{dx^P}.$$

The quantity ds is an *first order differential element*: this signifies that it is expressed by means of dx and the coordinates of the first order contact element.

10 The relative components of the displacement of Frenet trihedral. Consider the right rectangular trihedrals $\mathbf{A}\vec{\mathbf{I}}_1\,\vec{\mathbf{I}}_2\,\vec{\mathbf{I}}_3$ at each point $\mathbf{A}$ of the curve C having the following properties: the orthogonal vectors $\vec{\mathbf{I}}_1$, $\vec{\mathbf{I}}_2$, $\vec{\mathbf{I}}_3$ are of length 1, $\vec{\mathbf{I}}_1$ is along the positive tangent, $\vec{\mathbf{I}}_2$ is along the principal normal and $\vec{\mathbf{I}}_3$ is along the binormal. At each point of $\mathbf{M}$ we then find two trihedrals attached and we go from one of them to the other by changing $\vec{\mathbf{I}}_2$ to $-\vec{\mathbf{I}}_2$ and $\vec{\mathbf{I}}_3$ to $-\vec{\mathbf{I}}_3$.

When $\mathbf{M}$ traces the curve C, each of these trihedrals generate a connected family of trihedrals. These trihedrals are second order geometrical elements.

Let us introduce the relative components ω of the infinitesimal displacement of these trihedrals. The definition of the tangent is expressed by the relations $\omega_2 = 0$, $\omega_3 = 0$, that of ds by the relation $\omega_1 = ds$, that of the principal normal by the relation $\omega_{13} = 0$, and ω_{12}, ω_{13} are of the form $\omega_{12} = \rho\, ds$, $\omega_{23} = \tau\, ds$, ρ and τ being functions in s. Under these conditions the formulae (4) furnish the *Frenet formulae*

$$(6) \qquad \begin{cases} d\vec{\mathbf{A}} = ds\,\vec{\mathbf{I}}_1, \\ d\vec{\mathbf{I}}_1 = \qquad\qquad \rho\, ds\,\vec{\mathbf{I}}_2, \\ d\vec{\mathbf{I}}_2 = -\rho\, ds\,\vec{\mathbf{I}}_1 \qquad\qquad + \tau\, ds\,\vec{\mathbf{I}}_3, \\ d\vec{\mathbf{I}}_3 = \qquad\qquad -\tau\, ds\,\vec{\mathbf{I}}_2. \end{cases}$$

If we use the other trihedral $\mathbf{M}\,\vec{\mathbf{I}}_1\,\vec{\mathbf{I}}_2\,\vec{\mathbf{I}}_3$, ρ will be changed to $-\rho$ whereas τ retains the same value. For one of these trihedrals ρ is therefore positive, and we will name this one the *Frenet trihedral*, and its determination requires only the knowledge of second order contact elements, and hence we will also call the trihedral *second order trihedral.* We will apply the Frenet formulae (6) to this trihedral. The functions ρ and τ, which have intrinsic meaning, are the curvature and torsion. We will call them the second order and third order invariants.

Frenet formulae allows us to readily calculate ρ and τ by derivation when we know the explicit equations of C. They show that if we change the orientation of C, ρ and τ will retain the same values, whereas ds will be multiplied by -1. They provide the geometrical interpretation of ρ and τ. Finally, note that they permit us to calculate, at a point $\mathbf{A}$,

$$\begin{aligned}
&\frac{d\vec{\mathbf{A}}}{ds} \text{ when we know on the point: } \vec{\mathbf{I}}_1,\\
&\frac{d^2\vec{\mathbf{A}}}{ds^2} \text{ when we know on the point: } \vec{\mathbf{I}}_1, \rho,\\
&\frac{d^3\vec{\mathbf{A}}}{ds^3} \text{ when we know on the point: } \vec{\mathbf{I}}_1, \vec{\mathbf{I}}_2, \vec{\mathbf{I}}_3, \rho, \frac{d\rho}{ds}, \tau,\\
&\frac{d^P\vec{\mathbf{A}}}{ds^P} \text{ when we know on the point: } \vec{\mathbf{I}}_1, \vec{\mathbf{I}}_2, \vec{\mathbf{I}}_3, \rho, \dots, \frac{d^{P-2}\rho}{ds^{P-2}}, \tau, \dots, \frac{d^{P-3}\rho}{ds^{P-3}}.
\end{aligned}$$

11 The preceding results are sufficient to resolve two questions that arise naturally:

Contact problem. *Given an integer P, two real curves C and $C^\star$ and two points on these curves $\mathbf{A}_0$ and $\mathbf{A}_0^\star$, find all displacements transforming $\mathbf{A}_0^\star$ to $\mathbf{A}_0$ and $C^\star$ into a curve having at $\mathbf{A}_0$ with the curve C a contact of order at least equal to P.*

Equality problem. *Find all the displacements that superimpose two given real curves C and $C^\star$.*

It will be *convenient* for us to say that two *oriented* curves have an order P contact at a point only in the following case: they have at this point such a contact (here we abstract away their orientations), and moreover their positive tangents coincide at this point. Similarly a displacement will be said to superimpose two oriented curves only if it superimposes the two curves as well as their orientation.

The solutions of the contact problem consists of the solutions of two contact problems relative to the curve C oriented arbitrarily and the curve $C^\star$ oriented in one or the other sense. Similarly the equality problem decomposes into two equality problems relative to the oriented curves. It is the problems relative to oriented curves that we are going to learn to solve.

12 Contact problem of two oriented curves. First suppose $P \geq 2$. We exclude the case where one of the points $\mathbf{A}_0$, $\mathbf{A}_0^\star$ is an infection point. Let us choose these points as the origins of the curvilinear abscissas. If the problem is solvable, the displacement we are searching for superimpose the two Frenet trihedrals with apexes $\mathbf{A}_0$ and $\mathbf{A}_0^\star$, and moreover the elements of order up to P:

$$\rho, \quad \dots \quad \frac{d^{P-2}\rho}{ds^{P-2}},$$

and, if $P > 2$,

$$\tau, \quad \dots \quad \frac{d^{P-3}\tau}{ds^{P-3}},$$

each take the same value on the two points. Conversely suppose that this displacement is applied and these conditions are satisfied. Let $\mathbf{A}$ and $\mathbf{A}^\star$ be the two points on C and the displaced curve $C^\star$, whose curvilinear abscissas are both equal to s. According to the last line of §**10**, we have on $s = 0$,

$$\mathbf{A} = \mathbf{A}^\star, \quad \frac{d\vec{\mathbf{A}}}{ds} = \frac{d\vec{\mathbf{A}}^\star}{ds}, \quad \dots \quad \frac{d^P\vec{\mathbf{A}}}{ds^P} = \frac{d^P\vec{\mathbf{A}}^\star}{ds^P}.$$

Taylor's formula permits us to deduce from these equations that when s tends to zero, $\dfrac{\overline{\mathbf{A}\mathbf{A}^\star}}{s^{P+1}}$ remains bounded: the two curves have a contact of order at least equal to P.

Hence for the problem to be solvable, it is necessary and sufficient that each of the elements ρ, $\dfrac{d\rho}{ds}$, $\dots$; τ, $\dfrac{d\tau}{ds}$, $\dots$, whose order is at least equal to P, to have the same value at $\mathbf{A}_0$ and $\mathbf{A}_0^\star$. If the problem is solvable, the displacement we are looking for is unique: it is the one that superimpose the two second order trihedrals at $\mathbf{A}_0$ and $\mathbf{A}_0^\star$.

Case where $P = 1$. The problem is always solvable. For enunciating the solutions as in the case where $P \geq 2$, let us use the following definition: a right rectangular trihedral $\mathbf{A}\vec{\mathbf{I}}_1\,\vec{\mathbf{I}}_2\,\vec{\mathbf{I}}_3$ will be called a *first order trihedral* attached at a point $\mathbf{A}$ of a (oriented) curve when $\vec{\mathbf{I}}_1$ is along the (positive) tangent of the point $\mathbf{A}$. The displacements we search for are those that superimpose a particular first order trihedral attached at $\mathbf{A}_0^\star$ unto the most general first order trihedral attached at $\mathbf{A}_0$.

13 Equality problem. When C and $C^\star$ are analytic the equality problem can be resolved by an intermediate contact problem: the necessary and sufficient condition for a displacement which makes the points $\mathbf{A}_0$ and $\mathbf{A}_0^\star$ superimposes the curves C and $C^\star$ is that which realises at these points a contact of order larger than every integer P. Using the conclusion of the preceding paragraph and noting that ρ and τ are analytic functions of s, we obtain the following solvability condition: the curvature ρ and the torsion τ must each be the same function of the curvilinear abscissas s along the two curves.

However, it is preferable to tackle the problem differently. We now only suppose that the coordinates of a point along the curve are functions of a parameter admitting fourth order derivatives. The problem of superimposing two oriented curves C and $C^\star$ is equivalent to the problem of superimposing the families of Frenet trihedrals attached to these curves.

The fundamental condition for equality (§**7**, p. 18) applies: this makes the problem into the search for all correspondances between the curvilinear abscissas s and $s^\star$ of the two curves realising the three equalities

$$ds = ds^\star, \qquad \rho(s)ds = \rho^\star(s^\star)ds^\star, \qquad \tau(s)ds = \tau^\star(s^\star)ds^\star.$$

The search for these correspondances proceed as the following:

Two cases can be distinguished:

1. *ρ and τ are constants.* It it then necessary that $\rho^\star$ and $\tau^\star$ have the same constant values. This is also sufficient, and there exists an infinite number of correspondances, namely $s = s^\star + C$ where C denote an arbitrary constant. Each of the two curves can slide along themselves by a continuous displacement.

2. *ρ and τ are not both constants.* Suppose for example the curvature to be non-constant. Let $\dfrac{d\rho}{ds} = F(\rho)$, $\tau = \Phi(\rho)$. It is then necessary to have $\dfrac{d\rho^\star}{ds^\star} = F(\rho^\star)$, $\tau^\star = \Phi(\rho^\star)$ with the same functions F and Φ. This condition is sufficient: if indeed we establish between s and $s^\star$ the correspondance given by $\rho(s) = \rho^\star(s^\star)$ and

$$ds = \frac{d\rho}{F(\rho)} = \frac{d\rho^\star}{F(\rho^\star)} = ds^\star.$$

14 Finally, let us use the structure theorem stated in §**7**. We have determined that, whatever the functions $\rho(s)$ and $\tau(s)$, there corresponds to them two families of trihedrals, equal between the two families, depending on the parameter s and satisfying the equations (6). Let C be the position of the apex $\mathbf{A}$ of these trihedrals. The equations (6) prove that the trihedrals are the Frenet trihedrals at C. Then:

Structure theorem. *The curvature and the torsion of a space curve are arbitrary functions of the arc length.*

III. Study of space curves based on the theory of multi-parameter families of trihedrals

15 Introduction. In the course of the preceding paragraph, the two problems of contact and of equality have not been analysed: they are found to be solved by a series of doubtlessly ingenious considerations, but which are also difficult to generalise. We are going to take up the study of them again by a more systematic method, which will prove easy to apply to numerous problems of the same nature but which are more complicated than our treatment in the preceding.

We are first concerned with the contact problem. We will learn to solve them recurrently order by order. This solution will lead us to define successively the various order elements attached at a point on a real curve. With the help of these elements, we will state the solvability conditions and we will define their solutions. After the contact problem has been completely treated, we will find ourselves to have sufficient knowledge to solve the equality problem.

16 Principle of recurrent solution of the contact problem. The most general order P contact element depends on $2P+3$ parameters $[x, y, z, \ldots, y^{(P)}, z^{(P)}]$. We can therefore envisage it as a point in a $2P+3$ dimensional space. There then correspond to a order P contact element of a curve C a curve Γ in a $2P+3$ dimensional space. For two curves C and $C^\star$ of the three dimensional space to have a contact of at least order P, it is necessary and sufficient that the corresponding curves Γ and $G^\star$ have a point in common: for the order of the contact to be greater than P it is necessary and sufficient that Γ and $\Gamma^\star$ are tangent at this point. Indeed, for the last two curves to be tangent, it is necessary and sufficient that at the common point of abscissa x, the other coordinates

y, z, y', z', $\ldots$, $y^{(P)}$, $z^{(P)}$, considered as functions in x, are equal as well as their first derivatives, but this signifies the one-to-one equality of the quantities y, z, y', z', $\ldots$, $y^{(P)}$, $z^{(P)}$, $y^{(P+1)}$, $z^{(P+1)}$ and $y^\star$, $z^\star$, $y^{\star\prime}$, $\ldots$, $y^{\star(P)}$, $z^{\star(P)}$, $y^{\star(P+1)}$, $z^{\star(P+1)}$, or in other words, that the two given curves C and $C^\star$ have a order $P+1$ contact.

We can now make the contact condition of two curves Γ and $\Gamma^\star$ under another form by expressing that there exists two points α, $\alpha^\star$ on the two curves infinitesimally close to the common point α_0 and the distance between them is zero up to an infinitesimally small error as measured against the distance between α_0 and α. We deduce from this the following contact condition for the curve C.

Contact condition. *The necessary and sufficient condition for two curves C and $C^\star$ having a point $\mathbf{A}_0$ in order P order contact to have a order $\geq P+1$ contact at the point is the following: there must exist on C a point $\mathbf{A}$ and on $C^\star$ a point $\mathbf{A}^\star$ which are infinitesimally close to $\mathbf{A}_0$ for which the relations expressing that C and $C^\star$ have a contact of order $\geq P$ at $\mathbf{A}$ and $\mathbf{A}^\star$ are realised up to infinitesimally small errors compared with the distance between $\mathbf{A}_0$ and $\mathbf{A}$.*

The contact problem therefore will be treated by a recurrent procedure: to solve the contact problem of order $P+1$ we search among the solutions of the order P problem those solutions that satisfy the condition that we just stated.

For brevity, *assume that we have already solved the first order contact problem* and defined for this purpose the tangent and first order trihedrals (*see* the end of **§12**). We also assume the orientation to be already defined and we only consider the problems relative to oriented curves. A oriented contact element is characterised by the associated connected family of first order trihedrals and it is completely determined by one of these trihedrals. The first order trihedrals of a curve depend on two parameters: we utilise one parameter t characterising the position of the apex of the trihedral on the curve, which we name the *principal parameter*, and one *secondary parameter* θ which is the angle that we need to turn the trihedral $(t,0)$ around the positive tangent to obtain the trihedral (t,θ). The sufficient condition for first order contact of two curves is that a particular first order trihedral of the first curve is also a first order trihedral of the second curve.

17 Application of the contact condition to the problem of second order contact. Consider a displacement resolving the first order problem: it superimposes an arbitrary first order trihedral $\mathbf{T}_0^\star$ attached to $\mathbf{A}_0^\star$ to a certain first order trihedral $\mathbf{T}_0$ attached at $\mathbf{A}_0$. Call the parameters of the trihedrals $(t_0^\star,\theta_0^\star)$ and (t_0,θ_0). For the displacement to solve the second order contact problem, it is necessary and sufficient that given a first order trihedral $\mathbf{T}^\star$ of $C^\star$ infinitesimally close to $\mathbf{T}_0^\star$ whose apex is a point $\mathbf{A}^\star$ infinitesimally close to $\mathbf{A}_0^\star$, we can find a first order trihedral $\mathbf{T}$ on C that the displacement considered superimposes it unto $\mathbf{T}^\star$ up to infinitesimally small quantities compared to $\overline{\mathbf{A}_0^\star\mathbf{A}^\star}$. This condition is analytically expressed as follows:

Let $\omega_i(t,\theta,dt,d\theta)$, $\omega_{ij}(t,\theta,dt,d\theta)$; $\omega_i^\star(t^\star,\theta^\star,dt^\star,d\theta^\star)$, $\omega_{ij}^\star(t^\star,\theta^\star,dt^\star,d\theta^\star)$ be the components of infinitesimal displacement of a first order trihedral of C and $C^\star$. The system

$$
(c) \qquad \begin{cases} \omega_i(t_0, \theta_0, dt, d\theta) = \omega_i(t_0, \theta_0, dt, d\theta), & (i = 1, 2, 3), \\ \omega_{ji}(t_0, \theta_0, dt, d\theta) = \omega_{ji}(t_0, \theta_0, dt, d\theta), & (j < i), \end{cases}
$$

where $dt^\star$ and $d\theta^\star$ are given, must admit a solution in dt and $d\theta$.

The contact problem to solve is therefore equivalent to the following problem: associate to a value $\theta_0^\star$ chosen according to our convenient all the values of θ_0 realising the condition (c).

18 Details of the nature of the forms ω_i, ω_{ji}. The movement of the trihedral of parameters (t, θ) is completely determined by the knowledge of θ and the movement of the trihedral $(t, 0)$. It is therefore natural to learn to calculate the forms $\omega(t, \theta, dt, d\theta)$ as functions of the forms $\omega(t, 0, dt, 0)$, θ and $d\theta$. This is what we are going to do. For this, observe that we have, according to the formula (4) on page 15 and the definition of first order trihedral,

$$
(7) \qquad \begin{cases} \omega_1(t, \theta, dt, d\theta) = \vec{\mathbf{I}}_1 \times d\vec{\mathbf{A}}, \qquad \omega_2(t, \theta, dt, d\theta) = \omega_3(t, \theta, dt, d\theta) = 0, \\ \omega_{ji}(t, \theta, dt, d\theta) = \vec{\mathbf{I}}_i \times d\vec{\mathbf{I}}_j. \end{cases}
$$

Denote the trihedral corresponding to $\theta = 0$ by $\mathbf{A}\vec{\mathbf{J}}_1\vec{\mathbf{J}}_2\vec{\mathbf{J}}_3$. We have

$$
(8) \qquad \vec{\mathbf{I}}_1 = \vec{\mathbf{J}}_1, \qquad \vec{\mathbf{I}}_2 = \vec{\mathbf{J}}_2 \cos\theta + \vec{\mathbf{J}}_3 \sin\theta, \qquad \vec{\mathbf{I}}_3 = -\vec{\mathbf{J}}_2 \sin\theta + \vec{\mathbf{J}}_3 \cos\theta.
$$

We also have analogous formulae to (7):

$$
(9) \qquad \begin{cases} \omega_1(t, \theta, dt, d\theta) = \vec{\mathbf{J}}_1 \times d\vec{\mathbf{A}}, \qquad \omega_2(t, \theta, dt, d\theta) = \omega_3(t, \theta, dt, d\theta) = 0, \\ \omega_{ji}(t, \theta, dt, d\theta) = \vec{\mathbf{J}}_i \times d\vec{\mathbf{J}}_j. \end{cases}
$$

Let us replace the quantities $\vec{\mathbf{I}}_1$, $\vec{\mathbf{I}}_i$ and $d\vec{\mathbf{I}}_j$ by their expressions from (8). In the formulae obtained let us use the relations (9), and they become

$$
(10) \qquad \begin{cases} \omega_1(t, \theta, dt, d\theta) = \omega_1(t, 0, dt, 0), \\ \omega_2(t, \theta, dt, d\theta) = \omega_3(t, \theta, dt, \theta) = 0, \\ \omega_{12}(t, \theta, dt, d\theta) = \omega_{12}(t, 0, dt, 0) \cos\theta + \omega_{13}(t, 0, dt, 0) \sin\theta, \\ \omega_{13}(t, \theta, dt, d\theta) = -\omega_{12}(t, 0, dt, 0) \sin\theta + \omega_{13}(t, 0, dt, 0) \cos\theta, \\ \omega_{23}(t, \theta, dt, d\theta) = d\theta + \omega_{23}(t, 0, dt, 0). \end{cases}
$$

The differential $d\theta$ does not appear in the forms ω_1, ω_2, ω_3, ω_{12}, ω_{13}: these components only depend on the choice of the trihedral (t, θ) and the position of the apex of the trihedral $(t + dt, \theta + d\theta)$. We will agree in all analogous cases to call these components the *principal components*.

There exists certain first order trihedrals whose matrix of principal components is particularly simple. The form ω_{13} is zero for two values of θ. Let us consider [(†)] the values which make the quotient $\dfrac{\omega_{12}(t,\theta,dt)}{\omega_1(t,\theta,dt)}$ positive, which is independent of dt. We call the corresponding trihedral "the second order trihedral" or "the Frenet trihedral" and the value ρ corresponding to $\dfrac{\omega_{12}}{\omega_1}$ "the curvature".

On the other hand, observe that according to (10), the form ω_1 is the same for every first order trihedral. It constitutes a *first order differential invariant.* We call it "the differential of the arc element" and denote it by ds. Henceforth we assume that $t = s$.

The Frenet formulae [(6), p. 20] show that these definitions correspond well to the usual definitions of the arc element and the Frenet trihedral which have been reviewed in the preceding paragraph.

19 Return to the second order contact problem (§17). Consider the condition (c) again. Let us choose $\theta_0^\star$ in a way that the corresponding trihedral $\mathbf{T}_0^\star$ is the second order trihedral of the curve $C^\star$ at $\mathbf{A}_0^\star$. The condition (c) requires θ_0 to be such that at the point A_0 we have $\omega_{13} = 0$, $\dfrac{\omega_{12}}{\omega_1} > 0$. The trihedral $\mathbf{T}_0$ is therefore necessarily the second order trihedral of the curve C at $\mathbf{A}_0$. The system which appear in the condition (c) can now be written as

$$\begin{aligned} ds &= ds^\star, \\ \rho\, ds &= \rho^\star ds^\star, \\ d\theta + \omega_{23}(s,0,ds,0) &= d\theta^\star + \omega_{23}^\star(s^\star,0,ds^\star,0). \end{aligned}$$

The condition (c) is equivalent to the condition $\rho = \rho^\star$. The equality of two curvatures is therefore the only solvability condition of the problem. This condition, once satisfied, the problem admits a unique solution: it is the displacement which superimposes the Frenet trihedral $\mathbf{T}_0^\star$ with the Frenet trihedral $\mathbf{T}_0$.

20 Third order contact problem. Let us now apply the contact condition (§**16**), which is very easy since the second order problem admits at most one solution. This solution, when it exists, is the displacement which superimposes the Frenet trihedral $\mathbf{A}_0$ and $\mathbf{A}_0^\star$. Let us assume this. The condition for it to realise a order ≥ 3 contact is the following: there must exist a point $\mathbf{A}$ on C and a point $\mathbf{A}^\star$ on C, infinitesimally close to $\mathbf{A}_0$, such that the Frenet trihedrals and the relative curvatures at these points coincide up to infinitesimally small quantities compared to $\overline{\mathbf{A}_0\mathbf{A}}$. This condition is equivalent to the following, where the forms ω and $\omega^\star$ represent the infinitesimal displacement components

(†) We suppose the curve to be oriented in the sense of increasing parameters and we exclude from consideration the points on which $\omega_{12}(t,0,dt,0)$ and $\omega_{13}(t,0,dt,0)$ are both zero.

of the Frenet trihedrals of the two curves: the system

$$\begin{gathered}\rho + \frac{d\rho}{ds}ds = \rho^\star + \frac{d\rho^\star}{ds^\star}ds^\star,\\ \omega_i(s, ds) = \omega_i^\star(s^\star, ds^\star),\\ \omega_{ji}(s, ds) = \omega_{ji}^\star(s^\star, ds^\star),\end{gathered}$$

must admit a solution for which neither ds nor $ds^\star$ is zero.

The solvability conditions of the problem is therefore

$$\rho = \rho^\star, \qquad \frac{d\rho}{ds} = \frac{d\rho^\star}{ds^\star}, \qquad \tau = \tau^\star.$$

If these conditions are satisfied, the solution is unique.

From the above it is easy to deduce successively the solutions *of order* 4, …, P *contact problems* [c.f. §**12**, p. 21].

21 Equality problem. The family of second order trihedrals attached at an oriented curve C is such that if we displace C this family will transform by the same displacement. Let us recall the reasons: the family of first order trihedrals possesses this property, and moreover, a displacement does not alter the relative components ω of the infinitesimal displacement of a trihedral. The second order trihedrals are defined as first order trihedrals whose (principal) components ω satisfy certain relations.

On the other hand, the family of second order trihedrals depend on a single parameter.

Through this, the equality problem of two oriented curves then becomes the equality of two families of one-parameter trihedrals, and this problem is solved easily by the fundamental condition for equality in paragraph §**7** (p. 18) (c.f. §**13**, p. 22).

Also, the *structure theorem* of paragraph §**14** (p. 23) results immediately from the *structure theorem* of paragraph §**7** (p. 18).

IV. The method of reduced equations

21 *bis.* We can obtain the Frenet equations by another method than the one we just developed, the method of *reduced equations*. Given a oriented space curve C, the method consists of attaching at each of its points **A** a rectangular trihedral such that the equations of the curve, which we assume to be analytic, when expanded in a neighbourhood of **A** have their first coefficients taking the simplest form possible. The x-axis is naturally the positive tangent here, the y-axis is the principal normal and the z-axis is the binormal. The equations of the curve is of the form

$$(11) \qquad \begin{cases} y = \dfrac{1}{2}ax^2 + \dfrac{1}{6}cx^3 + \cdots, \\ z = \dfrac{1}{6}bx^3 + \cdots. \end{cases}$$

The abscissa x of a point $\mathbf{A}'$ infinitesimally close to $\mathbf{A}$ is infinitesimally small which has an intrinsic significance and which is just the arc element ds. The coefficients a and b are invariants attached at the point A, the first is manifestly of second order and the second third order.

The Frenet formulae give the infinitesimal displacement components ω_i, ω_{ij} of the trihedral attached at the moving point $\mathbf{A}$ on the curve [formula (4), p. 15]

$$\begin{aligned} d\mathbf{A} &= \quad \omega_1\vec{\mathbf{I}}_1 + \omega_2\vec{\mathbf{I}}_2 + \omega_3\vec{\mathbf{I}}_3, \\ d\vec{\mathbf{I}}_1 &= \qquad\qquad \omega_{12}\vec{\mathbf{I}}_2 + \omega_{13}\vec{\mathbf{I}}_3, \\ d\vec{\mathbf{I}}_2 &= -\omega_{12}\vec{\mathbf{I}}_1 \qquad\qquad + \omega_{23}\vec{\mathbf{I}}_3, \\ d\vec{\mathbf{I}}_3 &= -\omega_{13}\vec{\mathbf{I}}_3 - \omega_{23}\vec{\mathbf{I}}_2. \end{aligned}$$

Now take a fixed point $\mathbf{B}$ on the curve and let x, y, z be its coordinates with respect to a trihedral $\mathbf{T}$ of the origin $\mathbf{A}$. They satisfy equation (11) where the coefficients a, b, ..., dependent on $\mathbf{A}$, are for example functions of the curvilinear abscissa s of $\mathbf{A}$. But by expressing that the point of relative coordinates x, y, z remains fixed when the trihedral $\mathbf{T}$ varies, we obtain the relations

$$\begin{aligned} dx + \omega_1 - y\omega_{12} - z\omega_{13} &= 0, \\ dy + \omega_2 + x\omega_{12} - z\omega_{23} &= 0, \\ dz + \omega_3 + x\omega_{13} + y\omega_{23} &= 0. \end{aligned}$$

Following (11), let us express

$$\begin{aligned} dy &= \qquad (ax + \cdots)dx + \frac{1}{2}da\,x^2 + \cdots, \\ dz &= \left(\frac{1}{2}bx^2 + \cdots\right)dx + \frac{1}{6}db\,x^3 + \cdots, \end{aligned}$$

the relations become

$$(12)\quad \begin{cases} \omega_2 + x\omega_{12} - \left(\frac{1}{6}bx^3 + \cdots\right)\omega_{23} \\ = (ax + \cdots)\left[\omega_1 - \left(\frac{1}{6}ax^2 + \cdots\right)\omega_{12} - \left(\frac{1}{6}bx^3 + \cdots\right)\omega_{13}\right] - \frac{1}{2}da\,x^2 + \cdots, \\ \omega_3 + x\omega_{13} + \left(\frac{1}{2}ax^2 + \cdots\right)\omega_{23} \\ = \left(\frac{1}{2}bx^2 + \cdots\right)\left[\omega_1 - \left(\frac{1}{6}ax^2 + \cdots\right)\omega_{12} - \left(\frac{1}{6}bx^3 + \cdots\right)\omega_{13}\right] - \frac{1}{6}db\,x^2 + \cdots \end{cases}$$

These relations must be satisfied regardless of the position of the point $\mathbf{A}$ and *regardless of the fixed point* $\mathbf{B}$, and therefore they are identities in x by equating the terms independent of x and the terms in x, we first obtain

$$\omega_2 = 0, \qquad \omega_3 = 0,$$

then

$$\omega_{12} = a\omega_1, \qquad \omega_{13} = 0,$$

the terms in x^2 in the second equation then gives

$$a\omega_{23} = b\omega_1.$$

The infinitesimally small quantity ω_1 is manifestly equal to ds, since ω_1 is the abscissa of $\mathbf{A} + d\mathbf{A}$, and we deduce from it

$$\begin{aligned}
\frac{d\mathbf{A}}{ds} &= \vec{\mathbf{I}}_1, \\
\frac{d\vec{\mathbf{I}}_1}{ds} &= a\vec{\mathbf{I}}_2, \\
\frac{d\vec{\mathbf{I}}_2}{ds} &= -a\vec{\mathbf{I}}_1 + \frac{b}{a}\vec{\mathbf{I}}_3, \\
\frac{d\vec{\mathbf{I}}_3}{ds} &= -\frac{b}{a}\vec{\mathbf{I}}_2.
\end{aligned}$$

We then observe two things:

1. The knowledge of the reduced form of the equations of the curve permits us to rederive the Frenet formulae,

2. We have

$$a = \rho, \qquad \frac{b}{a} = \tau \quad \text{or} \quad b = \rho\tau.$$

By going further in our identification of terms in equations (12) we calculate step by step the coefficients of the right hand sides of equations (11) as functions of ρ, τ, $\frac{d\rho}{ds}$, $\frac{d\tau}{ds}$, etc., which proves in another way that a space curve is completely determined, up to a displacement, by the knowledge of ρ and τ as functions of s. The identification of the terms in x^2 in the first equation of (12) gives for example

$$c = \frac{da}{ds} = \frac{d\rho}{ds}.$$

Chapter 2

Theory of minimal curves

22 Introduction. The theory of the preceding chapter can be easily extended to the case of an arbitrary complex curve [†], except if the curve is minimal. Indeed, consider a curve $x(t)$, $y(t)$, $z(t)$ which is minimal, i.e., its tangent is isotropic:

$$x'^2 + y'^2 + z'^2 = 0, \tag{1}$$

$$x'x'' + y'y'' + z'z'' = 0. \tag{2}$$

These two relations show that the normal plane and the osculating plane are coincident. On the other hand the tangent can no longer be taken as a vector $\vec{\mathbf{I}}_1$ of length 1. Hence the Frenet trihedral loses its significance.

The purpose of this chapter is to solve the contact and equality problem of two minimal curves [‡]. We will imitate the last section of the previous chapter. Here the use of tri-rectangular trihedrals is less convenient, and hence we will devote the first section to the properties of another species of trihedrals. We then attach different arbitrary families of such trihedrals that we will call first order trihedrals, second order trihedrals, etc., to the minimal curve. In the third section we will solve the equality and contact problems by using these definitions, and the names that we have given, i.e., first order trihedrals, etc., will be justified. We finish by some complements, various formulae and some geometrical interpretations.

I. Cyclic trihedrals

23 Definition. Given a right rectangular trihedral $\mathbf{O}xyz$, let us consider the trihedral constituted by the point $\mathbf{O}$ and the three vectors $\vec{\mathbf{I}}$, $\vec{\mathbf{I}}''$, $\vec{\mathbf{I}}'''$ having $\mathbf{O}$ as the origin whose

[†] Each time when we study a problem where complex quantities appear, every function that we consider will be supposed to be analytic.

[‡] The theory of minimal curves was founded by S. Lie [4], p. 694–709, whereas the notion of a *pseudo-arc* is due to E. Vessiot (*Comptes rendu*, 140, 1905, p. 1381).

components are

$$\begin{matrix} \frac{1}{2}, & \frac{i}{2}, & 0, \\ 0, & 0, & 1, \\ 1, & -i, & 0. \end{matrix}$$

No displacement leaves this trihedral fixed, since one such displacement would leave the trihedral $\mathbf{O}xyz$ fixed. This trihedral, as well as all trihedrals equal to it, are called right cyclic trihedrals, and their symmetric partners are called left cyclic trihedrals. There exists one and only one displacement which transforms a given cyclic trihedral to another given cyclic trihedral of the same handedness. No displacement can transform a right cyclic trihedral into a left cyclic trihedral, otherwise the displacement can transform the trihedral $\mathbf{O}xyz$ into its mirror image. The right and left cyclic trihedrals constitute two families without common elements.

If $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2\vec{\mathbf{I}}_3$ is a cyclic trihedral, the matrix of quantities

$$\begin{pmatrix} \vec{\mathbf{I}}_1{}^2 & \vec{\mathbf{I}}_1 \times \vec{\mathbf{I}}_2 & \vec{\mathbf{I}}_1 \times \vec{\mathbf{I}}_3 \\ \vec{\mathbf{I}}_2 \times \vec{\mathbf{I}}_1 & \vec{\mathbf{I}}_2{}^2 & \vec{\mathbf{I}}_2 \times \vec{\mathbf{I}}_3 \\ \vec{\mathbf{I}}_3 \times \vec{\mathbf{I}}_1 & \vec{\mathbf{I}}_3 \times \vec{\mathbf{I}}_2 & \vec{\mathbf{I}}_3{}^2 \end{pmatrix} \quad \text{is} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{3}$$

If moreover the trihedral is a right one, we have

$$\vec{\mathbf{I}}_1 \wedge \vec{\mathbf{I}}_2 = i\vec{\mathbf{I}}_1, \qquad \vec{\mathbf{I}}_2 \wedge \vec{\mathbf{I}}_3 = i\vec{\mathbf{I}}_3, \qquad \vec{\mathbf{I}}_3 \wedge \vec{\mathbf{I}}_1 = i\vec{\mathbf{I}}_2 \tag{4}$$

and

$$(\vec{\mathbf{I}}_1 \wedge \vec{\mathbf{I}}_2) \times \vec{\mathbf{I}}_3 = i. \tag{5}$$

The relations (3), (4) and (5) are not independent. We do not need to make this point precise, but only need to know the following *theorem*:

Every trihedral $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2\vec{\mathbf{I}}_3$ *satisfying* (3) *and* (5) *is a right cyclic trihedral.*

Indeed, a displacement will make the planes $\vec{\mathbf{I}}_1\mathbf{O}\vec{\mathbf{I}}_3$ and $x\mathbf{O}y$ coincident in a way that $\vec{\mathbf{I}}_1$ and $\vec{\mathbf{I}}$ are along the same isotropic line. $\vec{\mathbf{I}}_3$ and $\vec{\mathbf{I}}''$ are then also along the same isotropic line. A rotation around $\mathbf{O}z$ then allows us to have the equality $\vec{\mathbf{I}}_1 = \vec{\mathbf{I}}$. Then the relation

$$\vec{\mathbf{I}}_1 \times \vec{\mathbf{I}}_3 = \vec{\mathbf{I}} \times \vec{\mathbf{I}}''$$

entails $\vec{\mathbf{I}}_3 = \vec{\mathbf{I}}''$. The unit vectors $\vec{\mathbf{I}}_2$ and $\vec{\mathbf{I}}'$, which are perpendicular to $\vec{\mathbf{I}}_1$ and $\vec{\mathbf{I}}_3$ are equal or opposite. The relation (5) shows that we cannot have $\vec{\mathbf{I}}_2 = -\vec{\mathbf{I}}'$, so necessarily $\vec{\mathbf{I}}_2 = \vec{\mathbf{I}}'$. And the theorem is proved.

24 Components of an infinitesimal displacement. The most general displacement of a cyclic trihedral depends on *six* parameters. We will not make the choice precise. Let us vary the trihedrals. Consider a point $\mathbf{M}$ fixed respect to a trihedral. We have

$$\overrightarrow{\mathbf{AM}} = x\vec{\mathbf{I}}_1 + y\vec{\mathbf{I}}_2 + z\vec{\mathbf{I}}_3,$$

x, y, z being constants. Then

$$\overrightarrow{d\mathbf{M}} = \overrightarrow{d\mathbf{A}} + x\,d\vec{\mathbf{I}}_1 + y\,d\vec{\mathbf{I}}_2 + z\,d\vec{\mathbf{I}}_3.$$

To know the infinitesimal displacement of the trihedral, it therefore suffices to know the *twelve* Pfaffian forms ω defined by the relations

$$\left\{\begin{aligned} \overrightarrow{d\mathbf{A}} &= \omega_1\ \vec{\mathbf{I}}_1 + \omega_2\ \vec{\mathbf{I}}_2 + \omega_3\ \vec{\mathbf{I}}_3,\\ d\vec{\mathbf{I}}_1 &= \omega_{11}\vec{\mathbf{I}}_1 + \omega_{12}\vec{\mathbf{I}}_2 + \omega_{13}\vec{\mathbf{I}}_3,\\ d\vec{\mathbf{I}}_2 &= \omega_{21}\vec{\mathbf{I}}_1 + \omega_{22}\vec{\mathbf{I}}_2 + \omega_{23}\vec{\mathbf{I}}_3,\\ d\vec{\mathbf{I}}_3 &= \omega_{31}\vec{\mathbf{I}}_1 + \omega_{32}\vec{\mathbf{I}}_2 + \omega_{33}\vec{\mathbf{I}}_3. \end{aligned}\right. \tag{6}$$

At most six of these forms are linearly independent forms in the differentials of the six parameters.

It is easy to make this more precise. All the quantities $\vec{\mathbf{I}}_i \times \vec{\mathbf{I}}_j$ are constants, hence

$$\vec{\mathbf{I}}_i \times d\vec{\mathbf{I}}_j + \vec{\mathbf{I}}_j \times d\vec{\mathbf{I}}_i = 0.$$

Let us replace the differentials in this relation by their expressions in (6) and use the relations (3). It becomes

$$\omega_{13} = 0, \quad \omega_{22} = 0, \quad \omega_{31} = 0, \quad \omega_{12} + \omega_{23} = 0, \quad \omega_{11} + \omega_{33} = 0, \quad \omega_{21} + \omega_{32} = 0.$$

The six quantities

$$\omega_1, \qquad \omega_2, \qquad \omega_3, \qquad \omega_{11}, \qquad \omega_{12}, \qquad \omega_{21}$$

therefore suffice to define the infinitesimal displacement of the trihedral. We will call them the relative components of the displacement, and we henceforth will write (6) under one or the other forms of the following

$$\left\{\begin{aligned} \overrightarrow{d\mathbf{A}} &= \quad \omega_1\ \vec{\mathbf{I}}_1 + \omega_2\ \vec{\mathbf{I}}_2 + \omega_3\vec{\mathbf{I}}_3,\\ d\vec{\mathbf{I}}_1 &= \quad \omega_{11}\vec{\mathbf{I}}_1 + \omega_{12}\vec{\mathbf{I}}_2,\\ d\vec{\mathbf{I}}_2 &= \quad \omega_{21}\vec{\mathbf{I}}_1 - \omega_{12}\vec{\mathbf{I}}_3,\\ d\vec{\mathbf{I}}_3 &= -\omega_{21}\vec{\mathbf{I}}_2 - \omega_{11}\vec{\mathbf{I}}_3; \end{aligned}\right. \tag{7}$$

$$\left\{\begin{aligned} \omega_1 &= \vec{\mathbf{I}}_3 \times \overrightarrow{d\mathbf{A}}, & \omega_{11} &= \vec{\mathbf{I}}_3 \times d\vec{\mathbf{I}}_1 = -\vec{\mathbf{I}}_1 \times d\vec{\mathbf{I}}_3,\\ \omega_2 &= \vec{\mathbf{I}}_2 \times \overrightarrow{d\mathbf{A}}, & \omega_{12} &= \vec{\mathbf{I}}_2 \times d\vec{\mathbf{I}}_1 = -\vec{\mathbf{I}}_1 \times d\vec{\mathbf{I}}_2,\\ \omega_3 &= \vec{\mathbf{I}}_1 \times \overrightarrow{d\mathbf{A}}, & \omega_{21} &= \vec{\mathbf{I}}_3 \times d\vec{\mathbf{I}}_2 = -\vec{\mathbf{I}}_2 \times d\vec{\mathbf{I}}_3. \end{aligned}\right. \tag{8}$$

25 Properties of the relative components ω. We claim that *these six components are independent forms of the differentials of the six parameters.* Otherwise the differential system

$$\omega_1 = \omega_2 = \omega_3 = \omega_{11} = \omega_{12} = \omega_{21} = 0$$

admit solutions other than those that makes all the parameters constant, and there would exist moving trihedrals whose infinitesimal displacement components are zero and each of its points remains fixed, which is absurd [c.f. §**8**, p. 18].

Let us now pose the following *integration problem*: let us specify the following forms arbitrarily

$$\begin{cases} \omega_1 = \xi_1(t)dt, & \omega_2 = \xi_2(t)dt, & \omega_3 = \xi_3(t)dt, \\ \omega_{11} = \xi_{11}(t)dt, & \omega_{12} = \xi_{12}(t)dt, & \omega_{21} = \xi_{21}(t)dt. \end{cases} \tag{9}$$

We look for all families of cyclic trihedrals depending on the parameter p which admit these forms as their infinitesimal displacement [c.f. §**5**, p. 16].

Let $u_1, \ldots, u_6$ be the parameters of the most general cyclic trihedral. Then the components ω constitute six forms in du_q which are linearly independent, and the equations (9) is equivalent to a system of six differential equations

$$\frac{du_q}{dt} = f_q(u_1, u_2, \ldots, u_6, t).$$

There therefore exists one and only one family in the space considered whose trihedral of parameter $t = 0$ is an arbitrarily given trihedral.

It follows from this in particular the STRUCTURE THEOREM:

The infinitesimal displacement components of a moving cyclic trihedral depending on one parameter are not subject to any structure conditions [c.f. §**7**, p. 18].

26 Equality condition. If we subject all trihedrals in a family to the same displacement, the relative components ω of their infinitesimal displacements remain the same, since they define the *relative* positions of two infinitesimally near trihedrals and have an intrinsic meaning.

In particular every displacement transforms one solution of the integration problem stated in paragraph §**25** to another solution of the same problem.

All solutions of the problem are therefore equivalent. Then the following theorem is proved for the case $\rho = 1$:

FUNDAMENTAL CONDITION FOR EQUALITY. *Consider a bijective correspondence established between two ρ parameter families of cyclic trihedrals* $\mathbf{T}$ *and* $\mathbf{T}^\star$*. For the same displacement to superimpose the corresponding trihedrals, it is necessary and sufficient that the would-be correspondence make the infinitesimal displacement components ω of* $\mathbf{T}$ *equal to the components $\omega^\star$ of the infinitesimal displacement of the corresponding trihedral* $\mathbf{T}^\star$ [c.f. §**7**, p. 18].

The necessity of the condition stated is also established regardless of ρ (≤ 6).

To prove sufficiency and completely establish the theorem, we proceed as in paragraph §**6** (p. 17). Suppose this condition is satisfied. The theorem holds for $\rho = 1$, and every one parameter family of trihedrals in the family $\mathbf{T}$ is equal to the corresponding family of trihedrals in $\mathbf{T}^\star$. Hence the equality of the two families $\mathbf{T}$ and $\mathbf{T}^\star$ themselves. Q.E.D.

II. Definition of the elements of various orders attached to a minimal curve

27 Definition of first order trihedrals. We define them to be the right cyclic trihedrals whose apex belongs to the given minimal curve and whose first vector $\vec{\mathbf{I}}_1$ is tangent to the curve at $\mathbf{A}$. The second vector $\vec{\mathbf{I}}_2$ therefore belongs to the normal and osculating plane [c.f. the introduction to chapter 2, §**22**].

Relations characterising a first order trihedral. The family of first order trihedrals are distinguished among the families of cyclic trihedrals by having its apex $\mathbf{A}$ on the curve and by having $\overrightarrow{d\mathbf{A}}$ parallel to $\vec{\mathbf{I}}_1$. According to formulae (7), this translates into the two equations

$$\omega_2 = 0, \qquad \omega_3 = 0. \tag{10}$$

Comparison of two first order trihedrals. Let $\mathbf{A}\vec{\mathbf{J}}_1\vec{\mathbf{J}}_2\vec{\mathbf{J}}_3$ be a particular first order trihedral. Let us find the most general first order trihedral $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2\vec{\mathbf{I}}_3$ having the same apex. According to the definition of first order trihedral, we have

$$\begin{aligned}
\vec{\mathbf{I}}_1 &= \alpha\vec{\mathbf{J}}_1,\\
\vec{\mathbf{I}}_2 &= \beta\vec{\mathbf{J}}_2 + \lambda\vec{\mathbf{J}}_1,\\
\vec{\mathbf{I}}_3 &= \gamma[\vec{\mathbf{J}}_3 + \mu\vec{\mathbf{J}}_2 + \rho\vec{\mathbf{J}}_1].
\end{aligned}$$

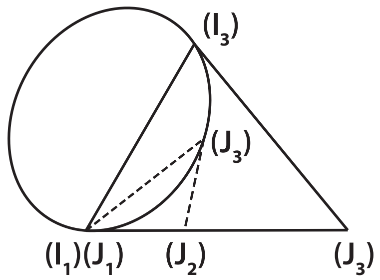


Figure 1: *Schemata of the plane at infinity*
where the ombilical point and the points at infinity $(\mathbf{I}_1)$, $(\mathbf{J}_1)$, ..., and the lines along the vectors $\vec{\mathbf{I}}_1$, $\vec{\mathbf{J}}_1$, ... are shown.

To express that the trihedral $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2\vec{\mathbf{I}}_3$ is a right trihedral we use the equations (3) and (5) [theorem of §**23**, p. 31]. We obtain

$$\beta^2 = 1, \qquad \alpha\gamma = 1, \qquad \beta\mu + \lambda = 0, \qquad \mu^2 + 2\rho = 0, \qquad \alpha\beta\gamma = 1.$$

Then

$$\begin{cases} \vec{\mathbf{I}}_1 = \alpha\vec{\mathbf{J}}_1, \\ \vec{\mathbf{I}}_2 = \vec{\mathbf{J}}_2 + \lambda\vec{\mathbf{J}}_1, \\ \vec{\mathbf{I}}_3 = \dfrac{1}{\alpha}\left[\vec{\mathbf{J}}_3 - \lambda\vec{\mathbf{J}}_2 - \dfrac{\lambda^2}{2}\vec{\mathbf{J}}_1\right]. \end{cases} \tag{11}$$

The family of first order trihedrals therefore depends on the *principal parameter* t, which characterises the position f the point $\mathbf{A}$ on the curve, as well as two secondary parameters $\alpha \neq 0$ and λ. The family of first order trihedrals is connected.

Comparison of the infinitesimal displacement components of two first order trihedrals. Suppose that the two trihedrals $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2\vec{\mathbf{I}}_3$ and $\mathbf{A}\vec{\mathbf{J}}_1\vec{\mathbf{J}}_2\vec{\mathbf{J}}_3$ varies while conserving the same apex. Let ω be the infinitesimal displacement components of the first trihedral, and $\bar{\omega}$ those of the second. We will calculate the components ω as functions of the components $\bar{\omega}$, λ, α, $d\lambda$ and $d\alpha$. For this we use the formulae (3), (7), (8) and (11):

$$\begin{aligned} \omega_1 &= \vec{\mathbf{I}}_3 \times \overrightarrow{d\mathbf{A}} = \frac{1}{\alpha}\left[\vec{\mathbf{J}}_3 - \lambda\vec{\mathbf{J}}_2 - \frac{\lambda^2}{2}\vec{\mathbf{J}}_1\right] \times \overrightarrow{d\mathbf{A}} = \frac{1}{\alpha}\bar{\omega}_1, \\ \omega_{11} &= \vec{\mathbf{I}}_3 \times d\vec{\mathbf{I}}_1 = \frac{1}{\alpha}\left[\vec{\mathbf{J}}_3 - \lambda\vec{\mathbf{J}}_2 - \frac{\lambda^2}{2}\vec{\mathbf{J}}_1\right][\alpha\, d\vec{\mathbf{J}}_1 + \vec{\mathbf{J}}_1 d\alpha] = \bar{\omega}_{11} - \lambda\bar{\omega}_{12} + \frac{d\alpha}{\alpha}, \\ \omega_{12} &= \vec{\mathbf{I}}_2 \times d\vec{\mathbf{I}}_1 = [\vec{\mathbf{J}}_2 + \lambda\vec{\mathbf{J}}_1][\alpha\, d\vec{\mathbf{J}}_1 + \vec{\mathbf{J}}_1 d\alpha] = \alpha\bar{\omega}_{12}, \\ \omega_{21} &= \vec{\mathbf{I}}_3 \times d\vec{\mathbf{I}}_2 = \frac{1}{\alpha}\left[\vec{\mathbf{J}}_3 - \lambda\vec{\mathbf{J}}_2 - \frac{\lambda^2}{2}\vec{\mathbf{J}}_1\right][d\vec{\mathbf{J}}_2 + \lambda\, d\vec{\mathbf{J}}_1 + \vec{\mathbf{J}}_1 d\lambda] \\ &= \frac{1}{\alpha}\left[\bar{\omega}_{21} - \frac{\lambda^2}{2}\bar{\omega}_{12} + \lambda\bar{\omega}_{11} + d\lambda\right]. \end{aligned}$$

In sum,

$$\begin{cases} \omega_1 = \dfrac{1}{\alpha}\bar{\omega}_1, \qquad \omega_{11} = \bar{\omega}_{11} - \lambda\bar{\omega}_{12} + \dfrac{d\alpha}{\alpha}, \\ \omega_{12} = \alpha\bar{\omega}_{12}, \qquad \omega_{21} = \dfrac{1}{\alpha}\left[\bar{\omega}_{21} - \dfrac{\lambda^2}{2}\bar{\omega}_{12} + \lambda\bar{\omega}_{11} + d\lambda\right]. \end{cases} \tag{12}$$

Let us now focus on the linear combinations of the components of ω in which the differentials of the secondary parameters $d\alpha$, $d\lambda$ do not appear: these are linear combinations in ω_1, ω_2, ω_3 and ω_{12}, and their values depend uniquely on the choice of the trihedral $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2\vec{\mathbf{I}}_3$ and the infinitesimal displacement of its apex. We call the following the *principal components of order* 1:

$$\boxed{\omega_1, \quad \omega_2 \text{ (which vanishes)}, \quad \omega_3 \text{ (which vanishes)}, \quad \omega_{12}.}$$

Remark. We could have seen these are the principal components: their linear combinations are linear combinations of the forms ω which vanishes when dt vanishes. The hypothesis $t = 0$ implies that $\overrightarrow{d\mathbf{A}} = 0$ and $d\vec{\mathbf{I}}_i$ is parallel to $\vec{\mathbf{I}}_i$, from which $\omega_1 = \omega_2 = \omega_3 = 0$, $\omega_{12} = 0$. The components ω_1, ω_2, ω_3, ω_{12} are therefore the principal components. There are no others, since the number of principal components is the number of components of ω minus the number of secondary parameters, $6 - 2 = 4$.

28 Second order trihedrals. According to the formulae (12) every point **A** on the curve possesses a first order trihedral such that

$$\omega_1 = \omega_{12}. \tag{13}$$

We will call these the second order trihedrals.

This definitions fails when $\bar{\omega}_{12} = 0$, in which case the component ω_{12} is zero for all first order trihedrals. Then it becomes impossible to distinguish among the first order trihedrals with the help of the relations relative to the principal components. We will examine this exceptional case in more details in paragraph §**30** (p. 39).

Comparisons of two second order trihedrals. Now denote a particular second order trihedral by $\mathbf{A}\vec{\mathbf{J}}_1\vec{\mathbf{J}}_2\vec{\mathbf{J}}_3$ and $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2\vec{\mathbf{I}}_3$ the most general second order trihedral. The formulae (12) tell us that in the equations (11) λ is arbitrary, but

$$\alpha = \pm 1. \tag{14}$$

The families of second order trihedrals therefore decompose into two sub-families containing no common element.

By an arbitrary procedure we distinguish one of these two sub-families. This operation will be called *orientating* of the curve, analogous to the case of real curves: the family of first order trihedrals attached at a non-oriented real curve is disconnected and decomposes into two connected sub-families, and choosing one of these sub-families amounts to orientating the curve. Similarly, we call only the distinguished sub-family of the second order trihedrals attached to our minimal curve oriented.

According to (11) and (12), if $\mathbf{A}\vec{\mathbf{J}}_1\vec{\mathbf{J}}_2\vec{\mathbf{J}}_3$ and $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2\vec{\mathbf{I}}_3$ are two second order trihedrals of an oriented curve,

$$
\begin{cases}
\vec{\mathbf{I}}_1 = \vec{\mathbf{J}}_1, \\
\vec{\mathbf{I}}_2 = \vec{\mathbf{J}}_2 + \lambda \vec{\mathbf{J}}_1, \\
\vec{\mathbf{I}}_3 = \vec{\mathbf{J}}_3 - \lambda \vec{\mathbf{J}}_2 - \dfrac{\lambda^2}{2} \vec{\mathbf{J}}_1;
\end{cases}
\tag{15}
$$

$$
\begin{cases}
\omega_1 = \bar{\omega}_1, \\
\omega_{11} = \bar{\omega}_{11} - \lambda \bar{\omega}_1, \\
\omega_{21} = \bar{\omega}_{21} - \dfrac{\lambda^2}{2} \bar{\omega}_1 + \lambda \bar{\omega}_{11} + d\lambda,
\end{cases}
\tag{16}
$$

λ being arbitrary.

The relations (11) and (12) also shows that to every second order $\mathbf{A}\vec{\mathbf{J}}_1\vec{\mathbf{J}}_2\vec{\mathbf{J}}_3$ on a oriented curve there corresponds a second order trihedral $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2\vec{\mathbf{I}}_3$ on the curve oriented in the opposite sense such that we have

$$
\begin{cases}
\vec{\mathbf{I}}_1 = -\vec{\mathbf{J}}_1, \\
\vec{\mathbf{I}}_2 = \vec{\mathbf{J}}_2, \\
\vec{\mathbf{I}}_3 = -\vec{\mathbf{J}}_3;
\end{cases}
\tag{17}
$$

$$
\begin{cases}
\omega_1 = -\bar{\omega}_1, \\
\omega_{11} = \bar{\omega}_{11}, \\
\omega_{21} = -\bar{\omega}_{21}.
\end{cases}
\tag{18}
$$

According to (16) the component ω_1 is the same for all second order trihedrals of an oriented curve, and this is a Pfaffian form depending only on the principal parameter and its constitutes a *second order differential invariant*. We will call it the differential of the *pseudo-arc* and we denote it by $d\sigma$ [c.f. the end of §**18**, p. 25]. The first formula (18) shows that $d\sigma$ transforms into its opposite when we change the orientation of the curve.

The linear combinations of an second order frame where $d\lambda$ does not appear are the linear combinations of the following *principal components*:

Order 1				Order 2
$\omega_1,$	$\omega_2(=0),$	$\omega_3(=0),$	$\omega_{12}(=\omega_1)$	ω_{11}

We could have seen this: the second order frame depend on *one* less secondary parameter than the first order frames. The linear combinations of ω where the differentials of the second order parameters do not appear therefore must be the linear combinations of *one* principal component at most. The hypothesis $dt = 0$ entails, according to (15), $d\vec{\mathbf{I}}_1 = 0$ and therefore, according to (7), $\omega_{11} = 0$. The second order principal component is therefore ω_{11}.

29 Third order trihedrals. We distinguish in the family of second trihedrals those special trihedrals which we will call third order trihedrals, those which are distinguished by a condition for the third order principal component, which is

$$\omega_{11} = 0. \tag{19}$$

Every oriented curve has a unique third order trihedral at each of its point, $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2\vec{\mathbf{I}}_3$. According to (18) its mirror image with respect to $\vec{\mathbf{I}}_2$, $\mathbf{A}\vec{\mathbf{J}}_1\vec{\mathbf{J}}_2\vec{\mathbf{J}}_3$ defined by (17) is a third order trihedral of the curve oriented in the opposite sense.

We set $\omega_{21} = k\,d\sigma$. We call the quantity k curvature, or fourth order invariant (†).

According to (18) the value of the curvature is independent of the orientation. We obtain formulae analogous to the *Frenet formulae* (§**10**, p. 20):

$$\begin{cases} \overrightarrow{d\mathbf{A}} = d\sigma\,\vec{\mathbf{I}}_1, & d\vec{\mathbf{I}}_2 = d\sigma(k\vec{\mathbf{I}}_1 - \vec{\mathbf{I}}_3), \\ d\vec{\mathbf{I}}_1 = d\sigma\,\vec{\mathbf{I}}_2, & d\vec{\mathbf{I}}_3 = -k\,d\sigma\,\vec{\mathbf{I}}_2. \end{cases} \tag{20}$$

Structure theorem. Let us specify a function $k(\sigma)$ arbitrarily. According to the structure theorem stated in paragraph §**25** (p. 34), there exists a cyclic trihedral $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2\vec{\mathbf{I}}_3$ depending on the parameter σ such that the equations (20) hold. The first of these equations shows that the apex $\mathbf{A}$ describes a minimal curve C. Moreover, $\omega_2 = \omega_3 = 0$, therefore the trihedral is a first order trihedral of C; $\omega_1 = \omega_{12}$, and hence the trihedral is second order; $\omega_{11} = 0$, and hence the trihedral is a third order trihedral for C, suitably oriented, and the curvature of C is $k(\sigma)$ at the point with the pseudo-arc σ. Hence the structure theorem: *Along a minimal curve the curvature is an arbitrary function of the pseudo-arc.*

30 Exceptional case. The preceding definitions fail for *the curves whose components* ω_{12} *of the first order trihedral are constantly zero* (‡).

For these trihedrals there then exists only one principal component ω_1, and the preceding, which consists of defining the second order trihedral by the relations among the principal components of the first order trihedral is inapplicable.

Let us consider two such curves C and $C^\star$. Let $\mathbf{T}$ be the family of first order trihedrals on C, $\mathbf{T}^\star$ the family on $C^\star$. $\mathbf{T}$ is a one-parameter family, and we can associated at the moving trihedral $\mathbf{T}$ a moving trihedral $\mathbf{T}^\star$ whose infinitesimal displacement has the same components such that to a particular trihedral $\mathbf{T}_0$ there corresponds an arbitrarily chosen trihedral $\mathbf{T}_0^\star$: for this it suffices to integrate the equations

$$\omega_1 = \omega_1^\star, \qquad \omega_{11} = \omega_{11}^\star, \qquad \omega_{21} = \omega_{21}^\star,$$

which constitute a system of three differential equations in three unknown functions, the initial values being fixed. Then the displacement superimposing the arbitrary first order

(†) The name invariant signifies that k does not vary when the curve undergoes a displacement.

(‡) Similarly the family of real curves contains the exceptional category of straight lines, for which we cannot attach Frenet trihedrals.

trihedral $\mathbf{T}_0$ and $\mathbf{T}_0^\star$ of C and $C^\star$ superimpose these two curves (see the *fundamental condition of equality*, §**26**, p. 34).

In particular a three parameter family of displacement permits us to transform the curve C into itself: every displacement transforming one of the first order trihedral of C to any other leaves the curve fixed. This shows that no first order trihedral has intrinsic special property: hence the failure of the procedure that we used to define the second order trihedrals.

On the other hand the curves under consideration are very simple: the relation $\omega_{12} = 0$ entails $d\vec{\mathbf{I}}_1 = \omega_{11}\vec{\mathbf{I}}_1$, and the tangent remains fixed along the curve parallel to a fixed direction, and hence these curves are isotropic lines [†].

III. Problems of equality and contact

31 Equality problem. The definitions that we have given in the course of the preceding section are such that if we subject our minimal curve to an arbitrary displacement, the elements k and $d\sigma$ do not change and the first, second and third order trihedrals are transformed by the same displacement: this gives the intrinsic significance of the components ω [c.f. §**26**, p. 34].

In particular, the problem of *determining the displacements which superimpose two given oriented minimal curves C and $C^\star$* is equivalent to the problem of finding the displacements which superimpose the third order trihedrals C and $C^\star$. The fundamental condition for equality (§**26**, p. 34) makes the problem into searching all bijective correspondances that can be established between C and $C^\star$ satisfying the equations

$$k(\sigma) = k^\star(\sigma^\star), \qquad d\sigma = d\sigma^\star. \tag{21}$$

If the equality problem is posed for two non-oriented curves C and $C^\star$, we must construct, other than the preceding correspondences, those that satisfy the equations

$$k(\sigma) = k^\star(\sigma^\star), \qquad d\sigma = -d\sigma^\star. \tag{22}$$

Each of the correspondences obtained will furnish one of the displacement superimposing C and $C^\star$.

Let us examine the *special case* where one of the curvature is constant. The other curvature must have the same constant, otherwise the problem of superimposing the curves is unsolvable. The correspondances (21) and (22) each depends on one arbitrary parameter, and hence we can superimpose C and $C^\star$ by making the two coincident at two of their arbitrarily given points, in the same or opposite orientation. In particular, the curve C can slide on itself (as a helix), and if C_1 is the curve obtained by changing the orientation of C, displacements would permit us to superimpose C and C_1.

[†] The family of displacement transforming a non-isotropic line into itself depends on two parameters, is disconnected and consists of two connected sub-families. However, we have just seen that the family of displacements transforming one isotropic line to itself depends on three parameters and is connected.

32 Contact problem of order 1. This problem can be solved immediately: consider two minimal curves C and $C^\star$ and two of their points $\mathbf{A}_0$ and $\mathbf{A}_0^\star$. The displacements superimposing C and $C^\star$ and realising a contact of order ≥ 1 there are those that makes an arbitrarily chosen first order trihedral $\mathbf{T}_0^\star$ at $\mathbf{A}_0^\star$ coincident with the different first order trihedrals $\mathbf{T}_0$ attached at $\mathbf{A}_0$. These displacements therefore depend on two parameters.

On the other hand, we have determined that a first order contact element is determined by the data of the associated family of first order trihedrals and *vice versa*.

We will solve *the contact problems of order greater than one* by equivalence, by applying the stated contact condition in paragraph §**16** (p. 23). This condition amounts to no longer defining the difference between two points in terms of their distance, but, for example, in terms of the greatest absolute value of the difference of their coordinates.

33 Contact problem of order 2. Consider a displacement solving the first order contact problem: it superimposes any first order trihedral $\mathbf{T}_0^\star$ attached at $\mathbf{A}_0^\star$ to a first order trihedral $\mathbf{T}_0$ attached at $\mathbf{A}_0$. For the displacement to realise a contact of order ≥ 2, it is necessary and sufficient that we can find an infinitesimal displacement of first order trihedral $\mathbf{T}_0$ and an infinitesimal displacement of the trihedral $\mathbf{T}_0^\star$ which are equal and which displace the origin of these trihedrals. This condition translates into a system of linear equations in terms of the differentials of the parameters

$$\omega_1 = \omega_1^\star, \qquad \omega_{11} = \omega_{11}^\star, \qquad \omega_{12} = \omega_{12}^\star, \qquad \omega_{21} = \omega_{21}^\star, \tag{23}$$

this system must admit a solution which does not annihilate the differentials of the principal parameters.

Suppose that we have chosen a second order trihedral $\mathbf{T}_0^\star$: $\omega_1^\star = \omega_{12}^\star$. We must have $\omega_1 = \omega_{12}$: $\mathbf{T}_0$ must equally be a second order trihedral. If this is the case then the solution we are searching for of the system (23) surely exists: since ω_1, ω_{11} and ω_{21} are linear independent forms in the differentials of the principal parameters and the two secondary parameters [c.f. §**27**, p. 40].

The the solutions of second order contact problem are the displacements that superimpose a second order trihedral of $\mathbf{A}_0^\star$ with the different second order trihedrals of $\mathbf{A}_0$.

The second order contact problem is therefore determined by the families of second order trihedrals and *vice versa*. Similarly $d\sigma^2$ is determined by the data of the second order contact element and the differential of one of the coordinates x, y and z.

Henceforth we will assume the two curves C and $C^\star$ to be oriented and we consider a second order contact to be established between the two curves only when the second order trihedral of the *oriented* curve C coincide with a second order trihedral of the *oriented* curve $C^\star$.

34 Contact problem of order 3. Consider a contact of order ≥ 2 established at a point $\mathbf{A}_0$ between two oriented minimal curves C and $C^\star$: every second order trihedral $\mathbf{T}_0^\star$ of $C^\star$ having its apex at $\mathbf{A}_0$ coincides with a second order trihedral $\mathbf{T}_0$ of C with apex

$\mathbf{A}_0$. For the contact to realise a contact of order ≥ 3, it is necessary and sufficient that we can find two equal infinitesimal displacements of these second order trihedrals. The condition easily leads to the following form: there must exist between the differentials of the principal parameters of the two curves a relation realising the equality of the principal components

$$\omega_1 = \omega_1^\star, \qquad \omega_{11} = \omega_{11}^\star.$$

Let us choose a third order trihedral $\mathbf{T}_0^\star$ of C: $\omega_{11}^\star = 0$. We must have $\omega_{11} = 0$, $\mathbf{T}_0$ must be the third order trihedral of C, and the correspondence to be established between the differentials of the principal parameters is defined by $\omega_1 = \omega_1^\star$ and it always exists.

The solution of the third order contact problem is therefore the displacements that superimpose the third order trihedrals of the given curves.

The third order contact element is determined by the third order trihedral and *vice versa.*

35 Contact problems of order > 3**.** Consider a contact of order ≥ 3 established at a point $\mathbf{A}_0$ between two oriented curves C and $C^\star$. Their third order trihedrals coincide. For the contact to be of order ≥ 4, it is necessary and sufficient that we can find two equal infinitesimal displacements of these two trihedrals: this condition is equivalent to the equality $k = k^\star$.

Suppose $k = k^\star$. For the contact to be of order ≥ 5, it is necessary and sufficient that we can find two infinitesimal displacements of these two trihedrals satisfying the relation $dk = dk^\star$. This condition is equivalent to $\dfrac{dk}{d\sigma} = \dfrac{dk^\star}{d\sigma^\star}$, etc. From this we have the following theorem:

For the solution of the *contact problem of order* $P \geq 4$ of two points $\mathbf{A}_0$ and $\mathbf{A}_0^\star$ of two oriented minimal curves C and $C^\star$, the solvability condition is that we have, at these points,

$$k = k^\star, \qquad \frac{dk}{d\sigma} = \frac{dk^\star}{d\sigma^\star}, \qquad \dots, \qquad \frac{d^{P-1}k}{d\sigma^{P-1}} = \frac{d^{P-1}k^\star}{d\sigma^{\star P-1}}.$$

When these relations are satisfied the problem admits a single solution: the displacement which superimposes the third order trihedrals of the points $\mathbf{A}_0$ and $\mathbf{A}_0^\star$.

IV. Complements

To practically solve the equality and contact problems, we need to know how to effectively determine the various order trihedrals and the invariant k. We are going to show how we can achieve this aim in two ways: by calculating them and by the construction of infinitesimal geometry.

36 Analytic determination of the trihedrals of various order, of k and of $d\sigma$. Consider a minimal curve traced by a point (x, y, z), the reference trihedral used being tri-rectangular. We have

$$dx^2 + dy^2 + dz^2 = 0.$$

Therefore there exists a parameter t such that

$$\frac{2dx}{1-t^2} = \frac{2dy}{i(1+t^2)} = \frac{dz}{t}.$$

Let $f(t)$ be the common value of these fractions. We have

$$\begin{aligned}
x(t) &= \int_{t_0}^{t} \frac{1-t^2}{2} f(t)dt + x_0,\\
y(t) &= \int_{t_0}^{t} i\frac{1+t^2}{2} f(t)dt + y_0,\\
z(t) &= \int_{t_0}^{t} tf(t)dt + z_0.
\end{aligned}$$

These formulae, called *Weierstrass's formulae*, furnish the parametric representation of the most general minimal curve.

We say that the following trihedral $\mathbf{A}\vec{\mathbf{J}}_1\vec{\mathbf{J}}_2\vec{\mathbf{J}}_3$ is a particular first order trihedral:

$$\begin{array}{lllll}
\vec{\mathbf{J}}_1 & \text{has components:} & \dfrac{1-t^2}{2}, & i\dfrac{1+t^2}{2}, & t,\\
\vec{\mathbf{J}}_2 & \text{has components:} & -t, & it, & 1,\\
\vec{\mathbf{J}}_3 & \text{has components:} & 1, & -i, & 0.
\end{array}$$

Indeed, $\vec{\mathbf{J}}_1$ is just the tangent and we have

$$\begin{gathered}
\vec{\mathbf{J}}_1{}^2 = 0, \qquad \vec{\mathbf{J}}_2{}^2 = 1, \qquad \vec{\mathbf{J}}_3^2 = 0, \qquad \vec{\mathbf{J}}_1 \times \vec{\mathbf{J}}_2 = 0,\\
\vec{\mathbf{J}}_2 \times \vec{\mathbf{J}}_3 = 0, \qquad \vec{\mathbf{J}}_1 \times \vec{\mathbf{J}}_3 = 0, \qquad (\vec{\mathbf{J}}_1 \wedge \vec{\mathbf{J}}_2) \times \vec{\mathbf{J}}_3 = i.
\end{gathered}$$

Formulae (8) permit use to calculate the components $\bar{\omega}$ of the infinitesimal displacement of this trihedral:

$$\begin{aligned}
\bar{\omega}_1 &= \vec{\mathbf{J}}_3 \times \overrightarrow{d\mathbf{A}} = f(t)dt, & \bar{\omega}_{11} &= \vec{\mathbf{J}}_3 \times d\vec{\mathbf{J}}_1 = 0,\\
\bar{\omega}_{12} &= \vec{\mathbf{J}}_2 \times d\vec{\mathbf{J}}_1 = dt, & \bar{\omega}_{21} &= \vec{\mathbf{J}}_3 \times d\vec{\mathbf{J}}_2 = 0.
\end{aligned}$$

The most general first order trihedral is defined by the formulae (11) in paragraph §**27** (p. 36), and the formulae (12) of paragraph §**27** takes the form

$$\text{(24)} \qquad \left\{\begin{array}{ll}
\omega_1 = \dfrac{1}{\alpha} f(t)dt, & \omega_{11} = -\lambda\, dt + \dfrac{d\alpha}{\alpha},\\
\omega_{12} = \alpha\, dt, & \omega_{21} = \dfrac{1}{\alpha}\left(-\dfrac{\lambda^2}{2}dt + d\lambda\right).
\end{array}\right.$$

The second order trihedrals are defined by the condition $\omega_1 = \omega_{12}$ which is equivalent to $\alpha = \sqrt{f}$. The third order trihedrals are defined by the condition $\omega_1 = \omega_{12}$, $\omega_{11} = 0$, which is equivalent to

$$\alpha = \sqrt{f}, \quad \lambda = \frac{1}{2}\frac{f'}{f}.$$

Then, for these third order trihedrals,

$$\omega_1 = d\sigma, \qquad \omega_{21} = k\, d\sigma,$$

we have

$$d\sigma = \sqrt{f}\, dt, \qquad k = \frac{1}{f}\left[\frac{1}{2}\frac{f''}{f} - \frac{5}{8}\frac{f'^2}{f^2}\right]. \tag{25}$$

Application. Let us determine the constant curvature minimal curves. According to (25) the problem consists of integrating the equation

$$\frac{1}{f}\left[\frac{1}{2}\frac{f''}{f} - \frac{5}{8}\frac{f'^2}{f^2}\right] = \text{constant}.$$

Let us search a particular solution of the form $f = at^m$.
We obtain

$$k = \frac{1}{at^{m+2}}\left[\frac{1}{2}m(m-1) - \frac{5}{8}m^2\right].$$

If $k = 0$ then we can take $m = -4$. Otherwise we need to take

$$m = -2 \qquad \text{and} \qquad a = \frac{1}{2k}.$$

We learned in paragraph §**31** (p. 40) that the set of curves with given constant curvature can be deduced by the particular curves thus obtained by applying the most general displacement.

We are going to determine by a second, more geometric procedure the curves in question.

37 Applications of the "Frenet formulae" (20) **to the search of the curves with constant curvature.** First suppose that $k = 0$. We have $d\vec{\mathbf{I}}_3 = 0$, and therefore $\vec{\mathbf{I}}_3$ is a constant vector, let us call it $\vec{\mathbf{K}}_3$. The equation

$$\vec{\mathbf{I}}_2 = -\vec{\mathbf{I}}_3 d\sigma$$

gives

$$\vec{\mathbf{I}}_2 = -\sigma\vec{\mathbf{K}}_3 + \vec{\mathbf{K}}_2,$$

$\vec{\mathbf{K}}_2$ being a new constant vector. The equation

$$d\vec{\mathbf{I}}_1 = \vec{\mathbf{I}}_2 d\sigma \qquad \text{gives} \qquad \vec{\mathbf{I}}_1 = -\frac{1}{2}\sigma^2\vec{\mathbf{K}}_3 + \sigma\vec{\mathbf{K}}_2 + \vec{\mathbf{K}}_1,$$

$\vec{\mathbf{K}}_1$ being a third constant vector. Hence, $\mathbf{A}_0$ denoting a fixed point,

$$\overrightarrow{\mathbf{A}_0\mathbf{A}} = -\frac{1}{6}\sigma^3\vec{\mathbf{K}}_3 + \frac{1}{2}\sigma^2\vec{\mathbf{K}}_2 + \sigma\vec{\mathbf{K}}_1.$$

We obtain a *space cubic* osculating to the plane at infinity at a point of the ombilical conic.

Now suppose that k is constant but non-zero. According to formulae (20), we have

$$d\left(\vec{\mathbf{I}}_1 + \frac{1}{k}\vec{\mathbf{I}}_3\right) = 0,$$

therefore

$$\vec{\mathbf{I}}_1 + \frac{1}{k}\vec{\mathbf{I}}_3$$

is a constant vector (of length $\sqrt{2/k}$). With this result, let us introduce a point $\mathbf{M}$ defined by the relation

$$\overrightarrow{\mathbf{AM}} = R\vec{\mathbf{I}}_2,$$

where R is a constant length. We have

$$\overrightarrow{d\mathbf{M}} = \overrightarrow{d\mathbf{A}} + R\,d\vec{\mathbf{I}}_2 = [(1+kR)\vec{\mathbf{I}}_1 - R\vec{\mathbf{I}}_3]d\sigma.$$

Let use take

$$R = -\frac{1}{2k},$$

it becomes

$$\overrightarrow{d\mathbf{M}} = \frac{1}{2}\left[\vec{\mathbf{I}}_1 + \frac{1}{k}\vec{\mathbf{I}}_3\right]d\sigma.$$

The vector $\dfrac{\overrightarrow{d\mathbf{M}}}{d\sigma}$ is a fixed vector: the point $\mathbf{M}$ describes a curve Δ parallel to the vector $\vec{\mathbf{I}}_1 + \dfrac{1}{k}\vec{\mathbf{I}}_3$. Δ is not an isotropic line, it is perpendicular to $\vec{\mathbf{I}}_2$, and therefore $\mathbf{A}$ remains on the cylinder with axis Δ and radius R. Then *the curves with non-zero constant curvature are the minimal circular helices.*

Given such a helix and one of its points $\mathbf{A}$, the two third order trihedrals $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2\vec{\mathbf{I}}_3$, the curvature k and the pseudo-arc σ are easy to determine. Let $\mathbf{AM}$ be the normal to the cylinder at $\mathbf{A}$, $\mathbf{M}$ being its point of contact with the axis. This line carries two unit vectors, of the two, $\vec{\mathbf{I}}_2$ is characterised by the following property: all of the right cyclic trihedrals having $\vec{\mathbf{I}}_2$ as the second vector have their first vectors along one of the isotropic curves in the tangent plane to the cylinder at $\mathbf{A}$, and these isotropic lines must be tangent to the minimal helix. We therefore set

$$\overrightarrow{\mathbf{AM}} = R\vec{\mathbf{I}}_2,$$

and we have

$$k = -\frac{1}{2R}.$$

$\vec{\mathbf{I}}_1$ and $\vec{\mathbf{I}}_3$ are determined up to multiples of ± 1 by the requirement that the vector $\vec{\mathbf{I}}_1 + \frac{1}{k}\vec{\mathbf{I}}_3$ have its support on the generatrix of the cylinder. Finally, the relation

$$\overrightarrow{d\mathbf{M}} = \frac{1}{2}\left(\vec{\mathbf{I}}_1 + \frac{1}{k}\vec{\mathbf{I}}_3\right) d\sigma$$

gives us the pseudo-arc element.

38 Geometric interpretation of k. There exists one and only one oriented minimal circular helix which has a contact of order 4 with the given oriented minimal curve at a given point on the curve: this is the helix having the same curvature and third order trihedral as the curve at this point. The helix corresponding to the opposite direction is manifestly the same helix oriented in the opposite sense. This helix is therefore attached geometrically at this point of the curve, independent of orientation (an exceptional case needs to be noted: the case where the cuve has vanishing curvature, and there no minimal helix can realise a fourth order contact with the curve, but such a contact can be obtained with a minimal cubic with zero curvature).

We then obtain a geometrical interpretation of k: it is the curvature of this helix.

Similarly, the arc element of the curve is an infinitesimal quantity equivalent to the arc element of the helix, the two third order trihedrals of the curve are also those of the helix. We are going to find for the elements of order $<$ another geometric interpretation without using this helix, which is a function of the fourth order contact element.

For this, we ar going to study the properties of the minimal curve at a neighbourhood of one of its points $\mathbf{A}$.

39 Reduced equations. We can, as we have done in paragraph §**21***bis.*, try to find the reduced equations of the curve in a neighbourhood of one of its pints $\mathbf{A}$ directly and derive from these the Frenet formulae. Let us follow the procedure in reverse here. Let

$$\begin{aligned} y &= f(x,\sigma) = P_1(x,\sigma) + P_2(x,\sigma) + \cdots + P_n(x,\sigma) + \cdots, \\ z &= \varphi(x,\sigma) = Q_1(x,\sigma) + Q_2(x,\sigma) + \cdots + Q_n(x,\sigma) + \cdots, \end{aligned}$$

be the equations of the curve with respect to the right cyclic trihedral attached at the point $\mathbf{A}$, $P_i(x,\sigma)$ and $Q_i(x,\sigma)$ denoting homogeneous polynomials of degree i in x whose coefficients depend on the curvilinear abscissa σ of the point $\mathbf{A}$. Let us take a fixed point $\mathbf{B}$ of coordinates x, y, z with respect to the cyclic trihedral of origin $\mathbf{A}$ on the curve. By expressing that the point of the *moving* coordinates (x, y, z) is *fixed*, we obtain, according to the Frenet formulae (20) (p. 39), the relations

$$\begin{aligned} \frac{dx}{d\sigma} + 1 + ky &= 0, \\ \frac{dy}{d\sigma} + x - kz &= 0, \\ \frac{dz}{d\sigma} - y \qquad &= 0. \end{aligned}$$

From this, it follows the relations

$$\begin{aligned}
&-\frac{\partial f}{\partial x}(1+kf)+\frac{\partial f}{\partial \sigma}+x-k\varphi=0,\\
&-\frac{\partial \varphi}{\partial x}(1+kf)+\frac{\partial \varphi}{\partial \sigma}-f \qquad\;\; =0.
\end{aligned}$$

There relations must hold regardless of the point **A** and the fixed point **B**, and therefore are identities in x and σ. By equating the independent terms of x and the coefficients of different powers of x on the left hand sides, we obtain successively

$$\begin{aligned}
&P_1=0, && Q_1=0,\\
&P_2=\frac{1}{2}x^2, && Q_2=0,\\
&P_3=0, && Q_3=-\frac{1}{6}x^3,\\
&P_4=-\frac{1}{12}kx^4, && Q_4=0,\\
&P_5=-\frac{1}{60}\frac{dk}{d\sigma}x^5, && Q_5=\frac{1}{15}kx^5.
\end{aligned}$$

The reduced equations of the curve are therefore

$$\tag{26}
\begin{cases}
y=\ \ \dfrac{1}{2}x^2-\dfrac{1}{12}kx^4-\dfrac{1}{60}\dfrac{dk}{d\sigma}x^5+\cdots,\\
z=-\dfrac{1}{6}x^3+\dfrac{1}{15}kx^5+\cdots
\end{cases}$$

We can restrict our calculation to y only, since we have

$$dy^2+2dx\,dz=0,$$

from which

$$\frac{dz}{dx}=-\frac{1}{2}\left(\frac{dy}{dx}\right)^2.$$

40 Geometric constructions of pseudo-arc. *a.* The abscissa x of a point $\mathbf{A}'$ infinitesimally close to **A** gives the pseudo-arc σ of Δ' when we take **A** as the origin. The square of the distance $\mathbf{AA}'=c$ is

$$\tag{27} c^2=y^2+2xz\sim\frac{1}{4}x^4-\frac{1}{3}x^4\sim-\frac{1}{12}x^4\sim-\frac{1}{12}d\sigma^4,$$

$d\sigma^4$ is therefore an infinitesimally small quantity equivalent to $-12c^2$.

This formula gives four values to $d\sigma$ corresponding to the two possible orientations and the two possible definitions of the right cyclic trihedrals as in paragraph §**23** (p. 31). However we want a geometrical interpretation which gives us only two values of $d\sigma$, corresponding to the two orientations of the curve.

b. For this, let us choose arbitrarily a direction $\mathbf{D}$ not lying in the normal and osculating plane $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2$, and a direction $\mathbf{D}'$ in the place which is not parallel to the tangent at $\mathbf{A}$. Let us consider the parallelepiped having the diagonal $\mathbf{AA}'$ whose edges are parallel to the tangent at $\mathbf{A}$, $\mathbf{D}'$ and $\mathbf{D}$. if we modify the choice of $\mathbf{D}$ and $\mathbf{D}'$ we obtain other parallelepipeds, but their (oriented) volumes are the same infinitesimally small quantities v. Let us calculate v by making the following particular choice: $\mathbf{D}$ is parallel to $\vec{\mathbf{I}}_3$ and $\mathbf{D}'$ is parallel to $\vec{\mathbf{I}}_2$. The formulae (26) (§**39**, p. 47) and (5) (§**23**, p. 32) give

$$v = -\frac{i}{12}x^6 + \cdots \sim -\frac{i}{12}d\sigma^6.$$

We then obtain, besides (27), the formula

$$d\sigma^6 = 12iv, \tag{28}$$

which gives a second geometric interpretation of $d\sigma$.

c. Only by dividing (27) and (28) do we obtain a rational expression of $d\sigma^2$

$$d\sigma^2 = -\frac{iv}{c^2}. \tag{29}$$

Let us show how this formula permits us to determine $d\sigma$ when we know the rectangular coordinates of the parametric equations of the curve $x(t)$, $y(t)$, $z(t)$. We have

$$\overrightarrow{\mathbf{AA}'} = \frac{\overrightarrow{d\mathbf{A}}}{dt}dt + \frac{1}{2}\frac{\overrightarrow{d^2\mathbf{A}}}{dt^2}(dt)^2 + \frac{1}{6}\frac{\overrightarrow{d^3\mathbf{A}}}{dt^3}(dt)^3 + \cdots$$

Take $\mathbf{D}'$ to be the direction of $\dfrac{\overrightarrow{d^2\mathbf{A}}}{dt^2}$, $\mathbf{D}$ the direction of $\dfrac{\overrightarrow{d^3\mathbf{A}}}{dt^3}$. It becomes

$$v = \frac{1}{12}V(dt)^6,$$

V being the volume of the parallelepiped with edges

$$\frac{\overrightarrow{d\mathbf{A}}}{dt}, \qquad \frac{\overrightarrow{d^2\mathbf{A}}}{dt^2}, \qquad \frac{\overrightarrow{d^3\mathbf{A}}}{dt^3},$$

i.e.,

$$V = \begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}.$$

On the other hand, c^2 is an infinitesimally small quantity equivalent to

$$\left(\frac{\overrightarrow{d\mathbf{A}}}{dt}dt + \frac{1}{2}\frac{\overrightarrow{d^2\mathbf{A}}}{dt^2}dt^2 + \frac{1}{6}\frac{\overrightarrow{d^3\mathbf{A}}}{dt^3}dt^3\right)^2.$$

But

$$\left(\frac{\overrightarrow{d\mathbf{A}}}{dt}\right)^2 = 0,$$

from which

$$\frac{\overrightarrow{d\mathbf{A}}}{dt} \times \frac{\overrightarrow{d^2\mathbf{A}}}{dt^2} = 0, \qquad \left(\frac{\overrightarrow{d^2\mathbf{A}}}{dt^2}\right)^2 + \frac{\overrightarrow{d\mathbf{A}}}{dt} \times \frac{\overrightarrow{d^3\mathbf{A}}}{dt^3} = 0,$$

and

$$c^4 = \frac{1}{4}\left(\frac{\overrightarrow{d^2\mathbf{A}}}{dt^2}\right)^2 dt^4 + \frac{1}{3}\frac{\overrightarrow{d\mathbf{A}}}{dt} \times \frac{\overrightarrow{d^3\mathbf{A}}}{dt^3} dt^4 + \cdots = -\frac{1}{12}\left(\frac{\overrightarrow{d^2\mathbf{A}}}{dt^2}\right)^2 dt^4 + \cdots$$

The formula (29) therefore gives

$$d\sigma^2 = +i\frac{\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}}{x''^2 + y''^2 + z''^2} dt^2.$$

41 Parametric reduced equations. We can, using the reduced equations (26) of the curve, establish the expressions in the coordinates x, y, z of a point near $\mathbf{A}$ as a function of the pseudo-arc σ of $\mathbf{M}$ calculated by taking $\mathbf{A}$ as the origin. Indeed, the Frenet equations (20) give, on successive derivations,

$$\frac{\overrightarrow{d\mathbf{A}}}{d\sigma} = \vec{\mathbf{I}}_1, \qquad \frac{\overrightarrow{d^2\mathbf{A}}}{d\sigma^2} = \vec{\mathbf{I}}_2, \qquad \frac{\overrightarrow{d^3\mathbf{A}}}{d\sigma^3} = k\vec{\mathbf{I}}_1 - \vec{\mathbf{I}}_3, \qquad \frac{\overrightarrow{d^4\mathbf{A}}}{d\sigma^4} = \frac{dk}{d\sigma}\vec{\mathbf{I}}_1 + 2k\vec{\mathbf{I}}_2, \qquad \cdots$$

We deduce from it the limit expansion

$$\text{(30)} \qquad \begin{cases} x = \sigma + \frac{1}{6}k\sigma^3 + \frac{1}{24}\frac{dk}{d\sigma}\sigma^4 + \cdots \\ y = \frac{1}{2}\sigma^2 + \frac{1}{12}k\sigma^4 + \cdots \\ z = -\frac{1}{6}\sigma^3 + \cdots \end{cases}$$

the omitted terms are of at least fifth order.

These formulae allow a *geometric construction of the Frenet frame.* First the vector $\vec{\mathbf{I}}_1$ is the limit of $\frac{\overrightarrow{\mathbf{AA}'}}{\sigma}$. To construct $\vec{\mathbf{I}}_2$, consider two points $\mathbf{A}'$ and $\mathbf{A}''$ near $\mathbf{A}$ of abscissas σ and $-\sigma$. Let $\mathbf{P}$ be right between $\mathbf{A}'\mathbf{A}''$. We have, according to (30)

$$\overrightarrow{\mathbf{AP}} = \frac{1}{2}\sigma^2\vec{\mathbf{I}}_2 + \cdots$$

Therefore $\vec{\mathbf{I}}_2$ is the limit of the vector $2\frac{\overrightarrow{\mathbf{AP}}}{\sigma^2}$.

Remark. We can also proceed in the following manner, without using σ. A sphere centered at $\mathbf{A}$ with infinitesimally small radius cuts the curve at four points infinitesimally close to $\mathbf{A}$. Of the six lines joining the point $\mathbf{A}$ to the middle of the six segments determined by the four points taken two by two, four are along the tangent at $\mathbf{A}$ to the curve and two are along the vector $\vec{\mathbf{I}}_2$. To know which one of the two vectors of length 1 along the line is $\vec{\mathbf{I}}_2$, we use the following property: every right cyclic trihedral having $\vec{\mathbf{I}}_2$ as the second vector has its first vector along one of the two isotropic lines of the plane perpendicular to $\vec{\mathbf{I}}_2$ at $\mathbf{A}$: this line must be the tangent of the curve.

42 Important remark. Thus we see how the "*Frenet formulae*" (20), by the intermediacy of the "*reduced equations*" (26) and (30), allow us to obtain the geometric constructions and the analytic expressions of the trihedrals and the various order invariants and the differential invariants attached to the curve. Let us also, in all similar questions, give a capital importance to the analogous formulae, which we continue to call "*Frenet formulae*". On the other hand, we will not concern ourselves with defining the trihedrals, the invariants and the differential invariants obtained from the explicit equations [which are here the Weierstrass formulae (p. 43)].

Let us make a last observation. To solve the contact and equality problems of curves in the complex domain, we are obliged to develop three different theories according to whether we are considering a straight line, a non-minimal curve or a minimal curve. But the possibilities have not been thus exhausted, since the plane curves situated in an isotropic plane also requires a special theory, but one which does not present any difficulty now, and hence we leave it aside from our discussion.

Chapter 3

Study of real ruled surfaces

43 The order of contact between two ruled surfaces. A ruled surface is a surface generated by a moving straight line called the generatrix G, and on this surface this line by one parameter. The set of straight lines in the space, on the other hand, depends on four parameters. Hence a straight line can be represented by a point in a four dimensional space, and a ruled surface by a curve in this representative space. If two of these curves have at a common point a contact [†] of order P (≥ 0), then the corresponding ruled surfaces have a common generatrix in common, and we say that they have a contact of order P along this generatrix. In this question, we name the set of real ruled surfaces having a contact of order P along a given generatrix the "contact elements of order P".

I. Elements of various orders

44 Zeroth order trihedrals [‡]. It is convenient to assume that the generatrix G is oriented and to introduce a certain orientation on the ruled surfaces. Henceforth a generatrix will be considered to belong simultaneously to two oriented surfaces only if they have the same orientation on the two surfaces.

We call zeroth order trihedrals attached to an oriented generatrix G the tri-rectangular right trihedrals $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2\vec{\mathbf{I}}_3$ whose first vector $\vec{\mathbf{I}}_1$ is along G and has the orientation of G.

Comparison of two zeroth order trihedrals $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2\vec{\mathbf{I}}_3$ *and* $\mathbf{B}\vec{\mathbf{J}}_1\vec{\mathbf{J}}_2\vec{\mathbf{J}}_3$. We have

$$\tag{1} \begin{cases} \mathbf{A} = \mathbf{B} + \rho\,\vec{\mathbf{J}}_1 \qquad \text{(this signifies } \overrightarrow{\mathbf{BA}} = \rho\,\vec{\mathbf{J}}_1\text{)}, \\ \vec{\mathbf{I}}_1 = \vec{\mathbf{J}}_1, \\ \vec{\mathbf{I}}_2 = \vec{\mathbf{J}}_2\cos\theta + \vec{\mathbf{J}}_3\sin\theta, \\ \vec{\mathbf{I}}_3 = -\vec{\mathbf{J}}_2\sin\theta + \vec{\mathbf{J}}_3\cos\theta, \end{cases}$$

ρ and θ being arbitrary parameters.

[†] Henceforth we say that two curves having a point in common without being tangent to each other have a contact of order 0.

[‡] The reasoning employed in this paragraph is related to the remark of paragraph §**27**, p. 35.

The family of zeroth order trihedrals of one ruled surface is therefore a function of three parameters: the principal parameter which the generatrix generating the surface depends on, and two secondary parameters. By our convection of orientation that we have adopted, this family is connected.

By replacing $\vec{\mathbf{I}}_1$ by $-\vec{\mathbf{I}}_1$ and $\vec{\mathbf{I}}_2$ by $-\vec{\mathbf{I}}_2$, we transform this family into the family of zeroth order trihedral oriented in the opposite sense.

Consider the *six* components ω_i, ω_{ji} of the instantaneous displacement of the zeroth order trihedral $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2\vec{\mathbf{I}}_3$. Let us focus our attention to their linear combinations which do not involve the differentials of the *two* secondary parameters. These linear combinations are obtained by linearly combining four of them, which we will call the *zeroth order principal components*, and they are characterised by the property that they vanish when the differential of the principal parameters are zero, i.e., when the trihedral varies on the same generatrix. Then $\overrightarrow{d\mathbf{A}}$ is parallel to $\vec{\mathbf{I}}_1$ and $d\vec{\mathbf{I}}_1$ is zero, hence

$$\omega_2 = \omega_3 = \omega_{12} = \omega_{13} = 0.$$

Hence the zeroth order principal components are

$$\boxed{\omega_2, \quad \omega_3, \quad \omega_{12}, \quad \omega_{13}.}$$

We will determine how these four principal components depend on the secondary parameters. Let $\mathbf{B}\vec{\mathbf{J}}_1\vec{\mathbf{J}}_2\vec{\mathbf{J}}_3$ be a second zeroth order moving trihedral corresponding constantly to the same generatrix as the first. We have the relations (1). Let $\bar{\omega}_i$, $\bar{\omega}_{ji}$ be the components of its infinitesimal displacements. The position of $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2\vec{\mathbf{I}}_3$ is determined by the knowledge of ρ, θ and the position of $\mathbf{B}\vec{\mathbf{J}}_1\vec{\mathbf{J}}_2\vec{\mathbf{J}}_3$, and we can calculate the quantities ω_i, ω_{ji} with the help of the quantities $\bar{\omega}_i$, $\bar{\omega}_{ji}$, ρ, θ, $d\rho$ and $d\theta$. But the principal components ω_2, ω_3, ω_{12} and ω_{13} do not manifestly depend on ρ, θ and the principal components $\bar{\omega}_2$, $\bar{\omega}_3$, $\bar{\omega}_{12}$, $\bar{\omega}_{13}$. The formulae (1) will permit us to obtain their effective expressions, and we will do the calculation by assuming that ρ and θ are constants and hence have vanishing differentials. We have

$$\begin{aligned}
\omega_2 = \vec{\mathbf{I}}_2 \times \overrightarrow{d\mathbf{A}} &= (\ \ \vec{\mathbf{J}}_2 \cos\theta + \vec{\mathbf{J}}_3 \sin\theta) \times (\overrightarrow{d\mathbf{B}} + \rho\, d\vec{\mathbf{J}}_1) \\
&= (\ \ \bar{\omega}_2 \cos\theta + \bar{\omega}_3 \sin\theta) + \rho(\ \ \bar{\omega}_{12} \cos\theta + \bar{\omega}_{13} \sin\theta), \\
\omega_3 = \vec{\mathbf{I}}_3 \times \overrightarrow{d\mathbf{A}} &= (-\vec{\mathbf{J}}_2 \sin\theta + \vec{\mathbf{J}}_3 \cos\theta) \times (\overrightarrow{d\mathbf{B}} + \rho\, d\vec{\mathbf{J}}_1) \\
&= (-\bar{\omega}_2 \sin\theta + \bar{\omega}_3 \cos\theta) + \rho(-\bar{\omega}_{12} \sin\theta + \bar{\omega}_{13} \cos\theta), \\
\omega_{12} = \vec{\mathbf{I}}_2 \times d\vec{\mathbf{I}}_1 &= (\ \ \vec{\mathbf{J}}_2 \cos\theta + \vec{\mathbf{J}}_3 \sin\theta) \times d\vec{\mathbf{J}}_1 = \ \ \bar{\omega}_{12} \cos\theta + \bar{\omega}_{13} \sin\theta, \\
\omega_{13} = \vec{\mathbf{I}}_3 \times d\vec{\mathbf{I}}_1 &= (-\vec{\mathbf{J}}_2 \sin\theta + \vec{\mathbf{J}}_3 \cos\theta) \times d\vec{\mathbf{J}}_1 = -\bar{\omega}_{12} \sin\theta + \bar{\omega}_{13} \cos\theta.
\end{aligned}$$

In summary

$$\text{(2)} \qquad \begin{cases}
\ \ \omega_2 = (\ \ \bar{\omega}_2 \cos\theta + \bar{\omega}_3 \sin\theta) + \rho(\ \ \bar{\omega}_{12} \cos\theta + \bar{\omega}_{13} \sin\theta), \\
\ \ \omega_3 = (-\bar{\omega}_2 \sin\theta + \bar{\omega}_3 \cos\theta) + \rho(-\bar{\omega}_{12} \sin\theta + \bar{\omega}_{13} \cos\theta), \\
\omega_{12} = \ \ \bar{\omega}_{12} \cos\theta + \bar{\omega}_{13} \sin\theta, \\
\omega_{13} = -\bar{\omega}_{12} \sin\theta + \bar{\omega}_{13} \cos\theta.
\end{cases}$$

45 First order trihedrals. These formulae show that among the zeroth order trihedrals attached at a generatrix G there are two trihedrals [(†)] for which ω_{13} and ω_2 are zero and they are symmetric with respect to G. Each of these two trihedrals gives a connected family and these two families do not have any common elements. We will distinguish one of these families, which will subject our ruled surface to *a second operation of orientation*. The trihedrals of this family will be called first order trihedrals, and each generatrix possesses one and only one such trihedral.

Of the four zeroth order principal components, two remain different from zero: ω_3 and ω_{12}. We set $\omega_{12} = d\sigma$, $\omega_3 = k\, d\sigma$ and we call k a first order invariant and $d\sigma$ a first order differential invariant.

We also set $\omega_1 = \alpha\, d\sigma$ and $\omega_{23} = \beta\, d\sigma$. We say that α and β are the two second order invariants.

The displacement of first order trihedral is subject to the "*Frenet formulae*"

$$\left\{\begin{array}{ll} \overrightarrow{d\mathbf{A}} = d\sigma(\alpha\vec{\mathbf{I}}_1 + k\vec{\mathbf{I}}_3), & d\vec{\mathbf{I}}_2 = d\sigma(-\vec{\mathbf{I}}_1 + \beta\vec{\mathbf{I}}_3), \\ d\vec{\mathbf{I}}_1 = d\sigma\, \vec{\mathbf{I}}_2, & d\vec{\mathbf{I}}_3 = d\sigma(-\beta\vec{\mathbf{I}}_2). \end{array}\right. \tag{3}$$

Remark. Four orientations of the surface is possible. When we pass from one to another the first order trihedral changes to one of its symmetric image with respect to one of its edges.

STRUCTURE THEOREM. *Let us specify the functions $k(\sigma)$, $\alpha(\sigma)$ and $\beta(\sigma)$ arbitrarily. According to the structure theorem stated in paragraph* §**7** (p. 18)*, there exists a right tri-rectangular trihedral $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2\vec{\mathbf{I}}_3$ depending on the parameter σ satisfying the equations* (3)*. This is a zeroth order trihedral of the ruled surface which $\vec{\mathbf{I}}_1$ generates. The relations $\omega_{13} = 0$, $\omega_2 = 0$ show that this is a first order trihedral of this surface. Hence the structure theorem: on a ruled surface, k, α and β are arbitrary functions of σ.*

Remark. If $k = 0$, the surface is developable and the point A describes the edge of the cusp and the trihedral $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2\vec{\mathbf{I}}_3$ is just the Frenet trihedral of this uve, but the Frenet formulae (3) where we set $k = 0$ are not the Frenet formulae in the theory of curves. The quantity $d\sigma$ in formulae (3) is equal to the angle of the elementary contingence $\rho\, ds$. We then have $\alpha = \dfrac{1}{\rho}$, $\beta = \dfrac{\tau}{\rho}$. These results should not surprise us, since there is no reason to find the same invariants, and similarly in the following when we consider the curve as a *set of points* or an *envelope of straight lines*, the contact problems are not the same, etc.

II. PROBLEMS OF EQUALITY AND CONTACT; GEOMETRICAL CONSTRUCTIONS

[(†)] An exceptional case should be noted: the case for which $\omega_{12}^2 + \omega_{13}^2 = 0$. Since the surface is real this condition can hold only if $\omega_{12} = \omega_{13} = 0$, from which $d\vec{\mathbf{I}}_1 = 0$. If this is the case for every generatrix, the vector $\vec{\mathbf{I}}_1$ is then constant and the surface is a cylinder.

46 The equality problem. Let us find all the ways to superimpose two given oriented ruled surfaces. The fundamental condition for equality (§**7**, p. 18) shows that the problem amounts to establishing every correspondence between their generatrix which are bijective and realise the four equalities

$$d\sigma = d\sigma^\star, \qquad k = k^\star, \qquad \alpha = \alpha^\star, \qquad \beta = \beta^\star.$$

In particular the only ruled surfaces that can slide onto themselves are those along which k, α and β are constants (we generate them by subjecting a straight line to a helical movement).

47 The contact problem. *Zeroth order contact problem.* The displacements superimposing two given generatrices G and $G^\star$ of two oriented surfaces are now those that superimpose a zeroth order trihedral of $G^\star$ to the most general zeroth order trihedral of G.

To tackle *the contact problem of order* P (> 0), we use a *contact condition* analogous to that in paragraph §**16** (p. 23).

The necessary and sufficient condition for two oriented ruled surfaces to have a contact of order $\geq P+1$ along a generatrix G_0 is the following: there must exist a generatrix of the first surface and a generatrix of the second surface infinitesimally close to G_0 such that the relations expressing that the two surfaces have a contact of order $\geq P$ along the generatrices G and $G^\star$ are realised up to infinitesimally small quantities of order higher than the difference between the parameters of G_0 and G.

By applying this contact condition we obtain step by step the following conclusion: the only displacement that can realise a contact of order $P \geq 1$ along two generatrices of two oriented ruled surfaces is those that superimpose the first order trihedrals of these generatrices, and for it to realise such a contact, it is necessary and sufficient that the following relations hold:

$$\begin{aligned}
&\text{for } P = 1: \quad && k = k^\star, \\
&\text{for } P = 2: \quad && k = k^\star, \qquad \alpha = \alpha^\star, \qquad \beta = \beta^\star, \qquad \frac{dk}{d\sigma} = \frac{dk^\star}{d\sigma^\star}, \\
&\text{for } P > 2: \quad && k = k^\star, \qquad \alpha = \alpha^\star, \qquad \beta = \beta^\star,
\end{aligned}$$

$$\frac{d^l k}{d\sigma^l} = \frac{d^l k^\star}{d\sigma^{\star l}}, \qquad \frac{d^n \alpha}{d\sigma^n} = \frac{d^n \alpha^\star}{d\sigma^{\star n}}, \qquad \frac{d^n \beta}{d\sigma^n} = \frac{d^n \beta^\star}{d\sigma^{\star n}}, \qquad (l \leq P-1; n \leq P-2).$$

48 Geometric constructions. Let us determine the tangent plane to the surface at a point $\mathbf{C}$ on the generatrix. Set $\overrightarrow{\mathbf{AC}} = \rho\vec{\mathbf{I}}_1$. We have

$$\overrightarrow{d\mathbf{C}} = (\alpha\vec{\mathbf{I}}_1 + k\vec{\mathbf{I}}_3)d\sigma + \vec{\mathbf{I}}_1 d\rho + \rho\vec{\mathbf{I}}_2 d\sigma.$$

The tangent plane at $\mathbf{C}$ therefore contains the vector $\vec{\mathbf{I}}_1$ and the perpendicular vector $\rho\vec{\mathbf{I}}_2 + k\vec{\mathbf{I}}_3$.

We then see that the plane $(\vec{\mathbf{I}}_1, \vec{\mathbf{I}}_2)$ is none other than the asymptotic plane, the plane $(\vec{\mathbf{I}}_1, \vec{\mathbf{I}}_3)$ is the central plane and $\mathbf{A}$ is the central point, i.e., the point where the tangent plane is perpendicular to the asymptote. Finally $\dfrac{k}{\rho}$ is the tangent of the angle which we must turn around the axis $\vec{\mathbf{I}}_1$ to pass from the asymptotic plane to the tangent plane at $\mathbf{C}$, and k is therefore the parameter of this situation.

The formula $d\mathbf{A} = (\alpha\vec{\mathbf{I}}_1 + k\vec{\mathbf{I}}_3)d\sigma$ shows that $\dfrac{k}{\alpha}$ is the tangent of the angle that the striction line (i.e., the locus of the central point) cuts the generatrix.

Finally for a fixed point $\mathbf{O}$, let us draw a vector $\overrightarrow{\mathbf{OP}}$ equal to $\vec{\mathbf{I}}_1$. $\mathbf{P}$ describes a spherical curve, the formula $d\overrightarrow{\mathbf{P}} = \vec{\mathbf{I}}_2 d\sigma$ shows that σ is the arc of this spherical curve, and the formale $d\vec{\mathbf{I}}_2 = (-\vec{\mathbf{I}}_1 + \beta\vec{\mathbf{I}}_3)d\sigma$ shows that β is the geodesic curvature.

Chapter 4

Study of ruled isotropic surfaces

I. Elements of order 1; contacts of order 1

49 Zeroth order trihedrals. By isotropic ruled surfaces we mean ruled surfaces whose generatrices are isotropic. We continue to define contact as we have done in paragraph §**43** (p. 51).

The zeroth order trihedrals attached at a generatrix G are the right cyclic trihedrals $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2\vec{\mathbf{I}}_3$ whose first vector is along G.

For two isotropic ruled surfaces to have a *zeroth order contact*, it is necessary and sufficient that their zeroth order trihedrals coincide.

Comparison of two zeroth order trihedrals, $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2\vec{\mathbf{I}}_3$ and $\mathbf{B}\vec{\mathbf{J}}_1\vec{\mathbf{J}}_2\vec{\mathbf{J}}_3$ corresponding to the same generatrix. We have [c.f. formulae (11), §**27**, p. 35]

$$(1)\qquad \begin{cases} \mathbf{A} = \mathbf{B} + \rho\,\vec{\mathbf{J}}_1 \qquad \text{(this signifies } \overrightarrow{\mathbf{BA}} = \rho\vec{\mathbf{J}}_1\text{)}, \\ \vec{\mathbf{I}}_1 = \mu\vec{\mathbf{J}}_1, \\ \vec{\mathbf{I}}_2 = \vec{\mathbf{J}}_2 + \lambda\vec{\mathbf{J}}_1, \\ \vec{\mathbf{I}}_3 = \dfrac{1}{\mu}\left[\vec{\mathbf{J}}_3 - \lambda\vec{\mathbf{J}}_2 - \dfrac{\lambda^2}{2}\vec{\mathbf{J}}_1\right], \end{cases}$$

λ, μ and ρ being arbitrary parameters.

The zeroth order trihedrals of a isotropic ruled surface therefore depend on a principal parameter and three secondary parameters.

Consider the six components of instantaneous displacement: ω_1, ω_2, ω_{11}, ω_{12}, ω_{21}. Let us consider linear combinations of these components that do not involve differentials of the three secondary parameters. These are obtained as linear combinations of three of them which we will call the *zeroth order principal components*, and they are characterised by the property that they are zero whenever the differential of the principal parameter is zero. When this differential is zero, $d\vec{\mathbf{I}}_1$ and $\overrightarrow{d\mathbf{A}}$ are parallel to $\vec{\mathbf{I}}_1$, and therefore $\omega_{12} = \omega_2 = \omega_3 = 0$. Hence the zeroth order components are

$$\boxed{\omega_{12}, \quad \omega_2, \quad \omega_3.}$$

Let us find how these three components ω_2, ω_3 and ω_{12} depend on the secondary parameters. Consider again the two zeroth order trihedrals of the same generatrix, $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2\vec{\mathbf{I}}_3$ and $\mathbf{B}\vec{\mathbf{J}}_1\vec{\mathbf{J}}_2\vec{\mathbf{J}}_3$ and denote by ω and $\bar{\omega}$ the respective components of their infinitesimal displacement. The formulae (1), together with the formulae (7) and (8) of paragraph §**24** (p. 33) allow use to express ω_2, ω_3 and ω_{12} as functions of $\bar{\omega}_2$, $\bar{\omega}_3$, $\bar{\omega}_{12}$, λ, μ and ρ. The calculation can be done as if $d\lambda$, $d\mu$ and $d\rho$ are identically zero. We obtain

$$\begin{aligned}
\omega_2 &= \vec{\mathbf{I}}_2 \times \overrightarrow{d\mathbf{A}} = (\vec{\mathbf{J}}_2 + \lambda\vec{\mathbf{J}}_1) \times (\overrightarrow{d\mathbf{B}} + \rho\, d\vec{\mathbf{J}}_1) = \bar{\omega}_2 + \lambda\bar{\omega}_3 + \rho\bar{\omega}_{12},\\
\omega_3 &= \vec{\mathbf{I}}_1 \times \overrightarrow{d\mathbf{A}} = \qquad \mu\vec{\mathbf{J}}_1 \times (\overrightarrow{d\mathbf{B}} + \rho\, d\vec{\mathbf{J}}_1) = \mu\bar{\omega}_3,\\
\omega_{13} &= \vec{\mathbf{I}}_2 \times d\vec{\mathbf{I}}_1 = (\vec{\mathbf{J}}_2 + \lambda\vec{\mathbf{J}}_1) \times \mu\, d\vec{\mathbf{J}}_1 \qquad = \mu\bar{\omega}_{12}.
\end{aligned}$$

in sum:

$$(2) \qquad \begin{cases} \omega_2 = \bar{\omega}_2 + \lambda\bar{\omega}_3 + \rho\bar{\omega}_{12},\\ \omega_3 = \mu\bar{\omega}_3,\\ \omega_{12} = \mu\bar{\omega}_{12}. \end{cases}$$

50 First order trihedrals. The formulae (2) reveal that the expression $\dfrac{\omega_3}{\omega_{12}}$ has the same value [†] for all zeroth order trihedrals, and we name it the *first order invariant k*.

And we have

$$\omega_2 = \bar{\omega}_2 + (\rho + \lambda k)\bar{\omega}_{12},$$

there therefore exists among the zeroth order trihedrals attached at a generatrix certain trihedrals whose ω_2 is zero. We call them *first order trihedrals*.

Henceforth we will set

$$(3) \qquad \omega_2 = 0, \qquad \omega_3 = k\omega_{12}.$$

The first order contact problem can be solved easily by using the contact condition stated in paragraph §**47**: for the problem to be solvable, it is necessary and sufficient that the two first order invariants of the generatrices which we want to make coincide are equal. When this condition is satisfied, the displacements realising the contact are those that superimpose the most general first order trihedral of the first generatrix to a particular first order trihedral of the second generatrix.

Comparison of two first order trihedrals corresponding to the same generatrix: $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2\vec{\mathbf{I}}_3$ and $\mathbf{B}\vec{\mathbf{J}}_1\vec{\mathbf{J}}_2\vec{\mathbf{J}}_3$.

[†] We assume that ω_{12} is not zero. If ω_{12} is identically zero, $d\vec{\mathbf{I}}_1$ would be constantly parallel to $\vec{\mathbf{I}}_1$ and $\vec{\mathbf{I}}_1$ would be a fixed direction. The surface would then be a cylinder, the case that we exclude from our consideration.

The formulae (1) still apply provided we set

$$\rho + \lambda k = 0.$$

As for the other terms, we have

$$(4) \qquad \begin{cases} \mathbf{A} = \mathbf{B} - \lambda k \vec{\mathbf{J}}_1, \\ \vec{\mathbf{I}}_1 = \mu \vec{\mathbf{J}}_1, \\ \vec{\mathbf{I}}_2 = \vec{\mathbf{J}}_2 + \lambda \vec{\mathbf{J}}_1, \\ \vec{\mathbf{I}}_3 = \dfrac{1}{\mu}\left[\vec{\mathbf{J}}_3 - \lambda \vec{\mathbf{J}}_2 - \dfrac{\lambda^2}{2}\vec{\mathbf{J}}_1\right]. \end{cases}$$

The first order trihedrals of an isotropic ruled surface therefore depend on a principal parameter and *two* secondary parameters.

The first order trihedrals depend on *one* less secondary parameter than the zeroth order trihedrals. They are hence linear combinations of their components where the differentials of the secondary parameters must be linear combinations of at most *one* principal component. It will be called the first order principal component. When the trihedral $\mathbf{B}\vec{\mathbf{J}}_1\vec{\mathbf{J}}_2\vec{\mathbf{J}}_3$ remains fixed and λ and μ vary, we have

$$\overrightarrow{d\mathbf{A}} = -k\, d\lambda\, \vec{\mathbf{J}}_1, \qquad d\vec{\mathbf{I}}_2 = d\lambda\, \vec{\mathbf{J}}_1,$$

and then

$$\overrightarrow{d\mathbf{A}} + k\, d\vec{\mathbf{I}}_2 = 0, \qquad \omega_1 + k\omega_{21} = 0,$$

$\omega_1 + k\omega_{21}$ is therefore the first order principal component. In summary, the first order principal components are

Order 0	Order 1
$\omega_{12}, \quad \omega_2(=0), \quad \omega_3(=k\omega_{12})$	$\omega_1 + k\omega_{21}$

To know how $\omega_1 + k\omega_{21}$ depends on the secondary parameters, let us use the formulae (4) and the formulae (7) and (8) of paragraph §**24** (p. 33). We effect the calculation by replacing $d\lambda$ and $d\mu$ by 0 (here we are not justified to replace dk by o):

$$\begin{aligned} \omega_1 + k\omega_{21} = \vec{\mathbf{I}}_3 \times [\overrightarrow{d\mathbf{A}} + k\, d\vec{\mathbf{I}}_2] &= \frac{1}{\mu}\left[\vec{\mathbf{J}}_3 - \lambda\vec{\mathbf{J}}_2 - \frac{\lambda^2}{2}\vec{\mathbf{J}}_1\right] \times [\overrightarrow{d\mathbf{B}} + k\, d\vec{\mathbf{J}}_2 - \lambda\, dk\, \vec{\mathbf{J}}_1] \\ &= \frac{1}{\mu}(\bar{\omega}_1 + k\bar{\omega}_{21}) - \frac{\lambda}{\mu} dk. \end{aligned}$$

The formulae which indicate how the principal components of order ≤ 2 depend on the secondary parameters are therefore

$$(5) \qquad \begin{cases} \omega_{12} = \mu\bar{\omega}_{12}, \\ \omega_1 + k\omega_{21} = \dfrac{1}{\mu}(\bar{\omega}_1 + k\bar{\omega}_{21}) - \dfrac{\lambda}{\mu} dk. \end{cases}$$

An peculiarity arises here which we have not encountered before: the invariant k appears before we have constructed the Frenet trihedral. It will then require, in the course of the third section of this chapter (§**55**), a small modification of our usual procedure. We will in the second section first study a special case: the cases where we consider the isotropic ruled surfaces whose k is equal to a given constant. But before that we are going to study a *very exceptional* case.

51 Surfaces whose k is constant and whose $\omega_1 + k\omega_{21}$ vanishes identically. The formulae (5) show that $\omega_1 + k\omega_{21}$ is identically zero regardless of which first order trihedral we choose. They do not allow us to distinguish the special trihedrals in the family of first order trihedrals. We are in a case analogous to the one encountered in paragraph §**30** (p. 39).

Consider two such surfaces for which k has the same value, and let $\mathbf{T}$ and $\mathbf{T}^\star$ be their first order trihedrals. Let us choose a one-parameter family of trihedrals $\mathbf{T}$. We can make them correspond to a moving trihedral $\mathbf{T}^\star$ whose instantaneous displacement has the same infinitesimal components such that two arbitrarily chosen trihedrals $\mathbf{T}_0$ and $\mathbf{T}_0^\star$ correspond to each other: it suffices to integrate the equations

$$\omega_1 = \omega_1^\star, \qquad \omega_{12} = \omega_{12}^\star, \qquad \omega_{11} = \omega_{11}^\star,$$

which constitute a system of three differential equations in three unknown functions.

The fundamental condition for equality (§**26**, p. 34) allow us to deduce from this the following consequence: the displacement superimposing arbitrary first order trihedrals $\mathbf{T}_0$ and $\mathbf{T}_0^\star$ superimposes the two surfaces.

In particular a three parameter family of displacements allow us to transform each of these surfaces to themselves, and we see that no first order trihedral can possess special intrinsic property.

The reader could suspect that these surfaces are spheres. It is easy to verify this: consider the point

$$\mathbf{P} = \mathbf{A} + k\vec{\mathbf{I}}_2. \tag{6}$$

The formulae above allow us to establish

$$\overrightarrow{d\mathbf{P}} = 0.$$

$\mathbf{P}$ is therefore fixed. $\mathbf{A}$ describes a sphere centred at $\mathbf{P}$ and $\vec{\mathbf{I}}_2$ is along the radius and the length of this radius is $\pm k$ (the sign of k changes with the system of generatrices with which we generate the sphere).

52 Geometrical construction of the invariant k. Consider a isotropic ruled surface and one of its generatrices. There exists one and only one sphere having a first order contact with the surface along this generatrix (indeed this sphere is defined by the data of two of its generatrices which are infinitesimally close). Its radius is $\pm k$. Its centre is according to (6) the point $\mathbf{P} = \mathbf{A} + k\vec{\mathbf{I}}_2$ (the formulae (4) allow us to verify that the point is independent of the choice of first order trihedral: $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2\vec{\mathbf{I}}_3$).

Observe that the centre of a sphere along to the isotropic plane (†) of each of its generatrices: $\mathbf{P}$ is therefore situated on the characteristic line of the isotropic plane passing by the generatrix of our ruled surface, and this characteristic line is also isotropic. It is the locus of the points

$$\mathbf{A} + k\vec{\mathbf{I}}_2 + x\vec{\mathbf{I}}_1, \tag{7}$$

x being a variable parameter. On the other hand we easily verify that

$$d(\mathbf{A} + k\vec{\mathbf{I}}_2 + x\vec{\mathbf{I}}_1)$$

is a vector parallel to the isotropic plane $(\vec{\mathbf{I}}_1, \vec{\mathbf{I}}_2)$.

The quantity k is the distance (‡) from the generatrix to the characteristic line. More precisely, let us choose any right cyclic trihedral $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2\vec{\mathbf{I}}_3$ such that $\mathbf{A}$ and $\vec{\mathbf{I}}_1$ are along the generatrix: the value of k is such that the characteristic line is the locus of the points (7). We thus obtain a second geometrical interpretation of k.

This characteristic line envelops a minimal curve and generates a developable surface whose tangent planes are the isotropic planes of the generatrices of our ruled surface: to characterise the rule surface it therefore suffices to specify this minimal curve and the length k attached at each of its points. The geometrical interpretation of these new elements that we attache at our isotropic rule surface can therefore be realised by the intermediate elements of this minimal curve.

II. Surfaces with constant k

53 Second order trihedrals. When k has a constant value, the formulae (5) are written

$$\omega_{12} = \mu\bar{\omega}_{12},$$
$$\omega_1 + k\omega_{21} = \frac{1}{\mu}(\bar{\omega}_1 + k\bar{\omega}_{21}).$$

We suppose that (§) ω_{12} is $\omega_1 + k\omega_{21}$ which is different from zero. Each generatrix then possesses a first order trihedral such that

$$\omega_{12} = \omega_1 + k\omega_{21}. \tag{8}$$

(†) An isotropic plane is a plane tangent to a umbilical point and it contains only one direction of isotropic lines.

(‡) Every secant of two isotropic straight lines in the same isotropic plane is a common perpendicular line to them, and they have the same length. On the contrary, in a non-isotropic plane two parallel isotropic lines have no common perpendicular, and it is impossible to define the distance between these two parallels lines.

(§) The surfaces whose ω_{12} is identically zero are cylinders (c.f. the note on p. 58). Those for which $\omega_1 + k\omega_{21}$ vanish identically are spheres (c.f. §**51**, p. 60).

These trihedrals constitute two distinct, connected families which can be deduced from each other by changing $\vec{\mathbf{I}}_1$ into $-\vec{\mathbf{I}}_1$ and $\vec{\mathbf{I}}_3$ into $-\vec{\mathbf{I}}_3$.

We choose one of these families arbitrarily, which constitutes an operation of *orientation.* The trihedrals of this family will be called second order trihedrals of the oriented isotropic ruled surface.

Consider two isotropic ruled surface whose invariants k have the same value. *The second order contact problem* is always solvable, and their solutions are the displacements superimposing the second order trihedrals of the two surfaces.

Comparison of two second order trihedrals corresponding to the same generatrix $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2\vec{\mathbf{I}}_3$ and $\mathbf{B}\vec{\mathbf{J}}_1\vec{\mathbf{J}}_2\vec{\mathbf{J}}_3$. We obtain, by replacing μ by 1 in (4),

$$(9)\qquad \begin{cases} \mathbf{A} = \mathbf{B} - \lambda k\vec{\mathbf{J}}_1, \\ \vec{\mathbf{I}}_1 = \vec{\mathbf{J}}_1, \\ \vec{\mathbf{I}}_2 = \vec{\mathbf{J}}_2 + \lambda\vec{\mathbf{J}}_1, \\ \vec{\mathbf{I}}_3 = \vec{\mathbf{J}}_3 - \lambda\vec{\mathbf{J}}_2 - \dfrac{\lambda^2}{2}\vec{\mathbf{J}}_1. \end{cases}$$

We also have

$$\omega_{12} = \bar{\omega}_{12},$$

hence ω_{12} does not depend on the principal parameter and its differential, and it is *a second order differential invariant.* We denote it by $d\sigma$.

The second order trihedrals depend on *one* parameter less than the first order trihedrals. The linear combinations of their components involving no differentials of the secondary parameters must therefore be linear combinations of *one* principal component at most. We will call it the second order principal component. When the trihedral $\mathbf{B}\vec{\mathbf{J}}_1\vec{\mathbf{J}}_2\vec{\mathbf{J}}_3$ remains fixed and λ varies, $d\vec{\mathbf{I}}_1 = 0$, hence $\omega_{11} = 0$. ω_{11} is therefore the second order principal component. In summary the second order principal components are

Order 0	Order 1	Order 2
$\omega_{12}, \quad \omega_2(=0), \quad \omega_3 = (k\omega_{12})$	$\omega_1 + k\omega_{21}(=\omega_{12})$	ω_{11}

The formulae (9) and the formulae (7) and (8) of paragraph §**24** give

$$\omega_{11} = \vec{\mathbf{I}}_3 \times d\vec{\mathbf{I}}_1 = \left(\vec{\mathbf{J}}_3 - \lambda\vec{\mathbf{J}}_2 - \frac{\lambda^2}{2}\vec{\mathbf{J}}_1\right) \times d\vec{\mathbf{J}}_1 = \bar{\omega}_{11} - \lambda\bar{\omega}_{12}.$$

From which we get the formula indicating how the second order principal component depends on the secondary parameter:

$$(10)\qquad \omega_{11} = \bar{\omega}_{11} - \lambda\, d\sigma.$$

54 Third order trihedrals. The formula (10) shows that each generatrix of our oriented surface possesses one and only one second order trihedral such that

$$\omega_{11} = 0.$$

We call this the third order trihedral. A change of orientation transforms this trihedral to its symmetric image with respect to $\vec{\mathbf{I}}_2$.

We also set $\omega_{21} = \alpha\, d\sigma$ and we call α the *fourth order invariant.*

With these conditions the displacement of the third order trihedral is regulated by the "*Frenet formulae*"

$$\begin{cases} \overrightarrow{d\mathbf{A}} = d\sigma(1 - k\alpha)\vec{\mathbf{I}}_1 + d\sigma\, k\vec{\mathbf{I}}_3, \\ d\vec{\mathbf{I}}_1 = d\sigma\, \vec{\mathbf{I}}_2, \\ d\vec{\mathbf{I}}_2 = d\sigma(\alpha\vec{\mathbf{I}}_1 - \vec{\mathbf{I}}_3), \\ d\vec{\mathbf{I}}_3 = -\alpha\, d\sigma\, \vec{\mathbf{I}}_2. \end{cases} \tag{11}$$

55 Complements. *Third order contact problem.* Consider two oriented surfaces for which k has the same constant value. The contact problem of order $P \geq 3$ admits at most one solution: the displacement superimposing two third order trihedrals.

The problem is always solvable for $P = 3$.

It is solvable for $P > 3$ only if α, $\dots$, $\dfrac{d^{P-1}\alpha}{d\sigma^{P-1}}$ have the same values for the two generatrices that we consider.

The equality problem. This is solvable only if the constant k is the same for the two surfaces considered: the fundamental condition of paragraph §**26** (p. 34 makes it into searching all correspondances between the generatrices of two surfaces realising the equalities

$$\alpha(\sigma) = \alpha^\star(\sigma^\star), \qquad d\sigma = d\sigma^\star.$$

Structure theorem. Reasoning analogous to those in paragraph §**29** allows us to deduce from the structure theorem stated in paragraph §**25** (p. 34) the following result:

k and $\alpha(\sigma)$ can be a constant and an arbitrary function of σ respectively.

Geometrical constructions. Consider again the point

$$\mathbf{P} = \mathbf{A} + k\vec{\mathbf{I}}_2,$$

the centre of the sphere having a first order contact with the surface whose generatrix is along $\vec{\mathbf{I}}_1$. We have, according to the Frenet formulae (11),

$$\overrightarrow{d\mathbf{P}} = d\sigma\, \vec{\mathbf{I}}_1, \qquad d\vec{\mathbf{I}}_2 = d\sigma(\alpha\vec{\mathbf{I}}_1 - \vec{\mathbf{I}}_3),$$
$$d\vec{\mathbf{I}}_1 = d\sigma\, \vec{\mathbf{I}}_2, \qquad d\vec{\mathbf{I}}_3 = -\alpha\, d\sigma\, \vec{\mathbf{I}}_2.$$

These formulae are the Frenet formulae of minimal curves up to notational changes [paragraph §**29**, formulae (20), p. 39]. Then the point $\mathbf{P}$ is the characteristic point of the isotropic plane passing through the generatrix of our ruled surface, $d\sigma$ is [†] the pseudo-arc and α is the curvature of the minimal curve $\mathbf{P}$ generates. $\mathbf{P}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2\vec{\mathbf{I}}_3$ is the "Frenet trihedral".

If $k = 0$, $\mathbf{A}$ is coincident with $\mathbf{P}$ and we recover the Frenet formulae for minimal curves.

III. Surfaces with variable k

56 Second order trihedrals. The first order invariant k, when it is not a constant along the curve, constitutes a natural principal parameter.

The formulae (5) (p. 59) show that each generatrix possesses one and only one first order trihedral such that

$$\omega_{12} = dk, \qquad \omega_1 + k\omega_{21} = 0. \tag{12}$$

We will call it the *second order trihedral.*

Second order contact problem. *Consider two generatrices G_0 and $G_0^\star$ of two surfaces and a displacement realising a contact of order ≥ 1 along these two generatrices: it superimposes one of the first order trihedrals $\mathbf{T}_0^\star$ attached at $G_0^\star$ to a first order trihedral $\mathbf{T}_0$ attached at G_0. For the order of contact to be greater than* 1, *it is necessary and sufficient that the condition* (*c*) *in paragraph* §**17** (p. 25) *is satisfied.*

This condition is expressed as follows: there exists equal infinitesimal displacements of the first order trihedral $\mathbf{T}_0$ and $\mathbf{T}_0^\star$ for which

$$dk = dk^\star \neq 0.$$

In other words the system

$$dk = dk^\star, \qquad \omega_1 = \omega_1^\star, \qquad \omega_{12} = \omega_{12}^\star, \qquad \omega_{21} = \omega_{21}^\star, \qquad \omega_{11} = \omega_{11}^\star$$

must admit a solution such that $dk \neq 0$.

Finding such a solution evidently becomes finding a solution for the following system, where the non-principal components have been excluded

$$dk = dk^\star, \qquad \omega_{12} = \omega_{12}^\star, \qquad \omega_1 + k\omega_{21} = \omega_1^\star + k^\star\omega_{21}^\star.$$

Suppose that $\mathbf{T}_0^\star$ is a second order trihedral of $G_0^\star$, $\mathbf{T}_0$ then must be a second order trihedral of G_0, and if $\mathbf{T}_0$ is such a trihedral then the system above is solvable. We hence arrive at the following conclusion:

If the first order contact problem is solvable, i.e., if the parameters k and $k^\star$ of two given generatrices are equal, then the second order contact problem admits one and only one solution: the displacement superimposing the two second order trihedrals.

[†] This is true for an orientation concordant with the ruled surface and the minimal curve.

57 The Frenet formulae. Let us consider the instantaneous displacement of a second order trihedral. Set

$$\omega_{21} = \alpha\, dk, \qquad \omega_{11} = \beta\, dk,$$

α and β we will call the *third order invariants.* We have the *Frenet formulae*

$$\left\{\begin{aligned} \overrightarrow{d\mathbf{A}} &= -\alpha k\, dk\, \vec{\mathbf{I}}_1 + k\, dk\, \vec{\mathbf{I}}_3, \\ d\vec{\mathbf{I}}_1 &= \quad \beta\, dk\, \vec{\mathbf{I}}_1 + \quad dk\, \vec{\mathbf{I}}_2, \\ d\vec{\mathbf{I}}_2 &= \quad \alpha\, dk\, \vec{\mathbf{I}}_1 - \quad dk\, \vec{\mathbf{I}}_3, \\ d\vec{\mathbf{I}}_3 &= - \ \alpha\, dk\, \vec{\mathbf{I}}_2 + \beta dk\, \vec{\mathbf{I}}_3, \end{aligned}\right. \tag{13}$$

The contact problems of order $P \geq 3$ admit at most one solution: the displacement superimposing the second order trihedrals. For this solution to exist, it is necessary and sufficient that the following conditions are realised:

For $P = 3$,

$$k = k^\star, \qquad \alpha = \alpha^\star, \qquad \beta = \beta^\star.$$

For $P > 3$,

$$k = k^\star, \qquad \alpha = \alpha^\star, \qquad \beta = \beta^\star,$$

$$\frac{d\alpha}{dk} = \frac{d\alpha^\star}{dk^\star}, \quad \ldots, \quad \frac{d^{P-3}\alpha}{dk^{P-3}} = \frac{d^{P-3}\alpha^\star}{dk^{\star P-3}}, \quad \ldots, \quad \frac{d\beta}{dk} = \frac{d\beta^\star}{dk^\star}, \quad \frac{d^{P-3}\beta}{dk^{P-3}} = \frac{d^{P-3}\beta^\star}{dk^{\star P-3}}.$$

The equality problem. *The necessary and sufficient condition for two surfaces to be equal is that the functions $\alpha(k)$ and $\beta(k)$ are identically the same on the two surfaces.*

Geometrical constructions. The point $\mathbf{P} = \mathbf{A} + k\vec{\mathbf{I}}_2$ of paragraph §**52** is no longer the characteristic point of the isotropic plane of the generatrix. Indeed, we have

$$\frac{\overrightarrow{d\mathbf{P}}}{dk} = \vec{\mathbf{I}}_2,$$

and the characteristic point $\mathbf{Q}$ of this isotropic plane is distinguished by the following property: $\overrightarrow{d\mathbf{Q}}$ is parallel to $\vec{\mathbf{I}}_1$. We easily deduce

$$\mathbf{Q} = \mathbf{P} - \vec{\mathbf{I}}_1.$$

This is the point $\mathbf{Q}$ generating the minimal curve introduced in the last part of paragraph §**51** and using it we can obtain the geometrical constructions. On the other hand the relations

$$\vec{\mathbf{I}}_1 = \overrightarrow{\mathbf{QP}}, \qquad \vec{\mathbf{I}}_2 = \frac{\overrightarrow{d\mathbf{P}}}{dk}, \qquad \overrightarrow{\mathbf{AP}} = k\vec{\mathbf{I}}_2$$

suffice to define the second order trihedral geometrically.

Part II

Fundamental notions in the theory of finite-dimensional continuous groups

Chapter 5

The moving frame of a finite-dimensional continuous group

I. Transformations; groups; moving frames

58 Transformations, their inverses and products. Let $\mathbf{M}$ be a point in a n-dimensional space. This point is defined by a system of n coordinates $x_1, \ldots, x_n$. Let D be a domain in this space which may be the entire space. By the *transformations* of D unto itself we mean all correspondances that attach to each point $\mathbf{M}$ in D a unique point $\mathbf{M}'$ also belonging to D. Such a transformation is denoted by a letter, for example T. We write

$$\mathbf{M}' = T\mathbf{M}.$$

Example. A displacement is constituted by a transformation of Euclidean space unto itself.

Suppose that when the point $\mathbf{M}$ successively moves to through points in D once and once only, the corresponding point $\mathbf{M}'$ also moves successively through all the points of D once and only once. The transformation then attaches to each point $\mathbf{M}'$ of D a unique corresponding point $\mathbf{M}$, which we will call the *inverse transformation* of T and we denote by the symbol T^{-1}. We write

$$T^{-1}\mathbf{M}' = \mathbf{M}.$$

Example. Every displacement admites an inverse transformation which is again a displacement.

In the following we will only consider transformations possessing inverses.

Consider two transformations T and T' operating on the same domain D. Consider the point $\mathbf{M}' = T\mathbf{M}$, then the point $\mathbf{M}'' = T'\mathbf{M}'$, and finally the transformation that moves $\mathbf{M}''$ to $\mathbf{M}$: $\mathbf{M}'' = T'(T\mathbf{M})$. We call this transformation the *product of the transformation T with the transformation T'*. We denote it by the symbol $T'T$.

In general the two products TT' and $T'T$ constitute different transformations. We say that the transformations T and T' *commute* when $TT' = T'T$.

Example. The product of two displacements is a displacement. The product of two translations T and T' is a translation, $TT' = T'T$: any two translations commute. The product of two rotations in general is not a rotation. In general two rotations do not commute.

We have by definition

$$T''(T'T)\mathbf{M} = T''[T'(T\mathbf{M})] \qquad \text{and} \qquad (T''T')T\mathbf{M} = T''[T'(T\mathbf{M})].$$

Therefore

$$T''(T'T) = (T''T')T.$$

We express this by saying that the product of transformations is *associative*. The product of any number of transformations is perfectly determined by these transformations and the order in which we apply their product.

Inverse of a product. Consider two transformations T and T' operating on the same domain D and each admitting an inverse transformation. We claim that $(TT')^{-1}$ exists and that

$$(TT')^{-1} = T'^{-1}T^{-1}.$$

Indeed the relation

$$TT'\mathbf{M} = \mathbf{M}'$$

is equivalent to the relation

$$T'\mathbf{M} = T^{-1}\mathbf{M}',$$

therefore we get the relation

$$\mathbf{M} = T'^{-1}T^{-1}\mathbf{M}'.$$

Similarly the inverse of product of any number of transformations possessing inverses is the product of the inverse transformations multiplied in the opposite order:

$$(T_1T_2\ldots T_{\alpha-1}T_\alpha)^{-1} = T_\alpha^{-1}T_{\alpha-1}^{-1}\ldots T_2^{-1}T_1^{-1}.$$

The identity transformation is the one that associate each point of D to the point itself. We represent it by the symbol 1. We evidently have

$$1T = T, \qquad T1 = T, \qquad TT^{-1} = 1, \qquad T^{-1}T = 1.$$

Therefore all transformations commute with their inverses and with the identity transformation.

59 Groups. *Definition:* a transformation group is a set of transformations acting on the same domain which possess the two following properties:

1. the inverse of each transformation in the group exists and belongs to the group;

2. the product of any two transformations in the group belongs to the group.

Remark. Let S be one transformation of the group. This group contains the inverse S^{-1} and hence the product $SS^{-1} = 1$. Therefore every group contains the identical transformation.

Examples. The set of rotations of the space does not form a group. On the other hand the set of translations form a group. Similarly all displacements form a group.

By the *subgroup* of a given group we mean every subset of transformations of this group that is itself a group.

Let us indicate several subgroups of the displacement group: the displacements which leave a given point fixed, those that leave two given points separately invariant, those that leave the set of two points invariant, those that leave a given straight line invariant, those that leave an oriented line invariant, the displacements that leave a point on a curve fixed and transforms this curve to a new curve having a p-th order contact with the old curve: these are the displacements transforming a given p-th order trihedral of a point considered into all the other similar trihedrals.

After we state the first fundamental theorem in the theory of groups (§**78**, p. 90), the families of transformations that we will consider will be groups exclusively.

60 Parameter space. Consider a family of transformations. We can always establish a continuous and bijective correspondence between the transformations in this family and the points of a suitable chosen representative space. This representative space is called the *parameter space*.

When this space is connected, the family is said to be *connected*.

Examples. The displacements of a three dimensional space leaving a given line fixed is a group that is not connected. The displacements and the products of the displacements by the reflections constitute a disconnected group.

When the parameter space is connected and is of finite dimensions r, the family is called *connected and finite-dimensional*.

Example. The displacement of three dimensional space constitute a connected and finite-dimensional group for which $r = 6$.

In this case the coordinates $a_1, \ldots, a_r$ of a point a in the space of parameters are called the *parameters* of the transformation S_a corresponding to the point, and r is "the number of parameters" or the "order" of the group. We use the parameters $0, \ldots, 0$ for the identity transformation, when it belongs to the family.

The transformations of a finite-dimensional and connected family are therefore given by the formulae

$$\left\{\begin{aligned} x'_1 &= \varphi_1(x_1, \ldots, x_n; a_1, \ldots, a_r), \\ &\ldots \\ x'_n &= \varphi_n(x_1, \ldots, x_n; a_1, \ldots, a_r). \end{aligned}\right. \tag{1}$$

The values of the variables x_i and the parameters a_k can be real or complex. We always suppose that the functions φ_i are analytic with respect to their $n + r$ arguments.

Let us point out that in general,

$$\frac{D(\varphi_1, \ldots, \varphi_n)}{D(x_1, \ldots, x_n)} \neq 0, \tag{2}$$

when the transformations in the family possess inverses: indeed it must then be possible to solve the system (1) with respect to $x_1, \ldots, x_n$ when we specify $x'_1, \ldots, x'_n, a_1, \ldots, a_r$.

61 Transformed system of reference. The formulae (1) presuppose that a system of reference $\mathbf{R}_0$ is defined on the domain D which is the set of all the points $\mathbf{M}$ of coordinates $x_1, \ldots, x_n$. To define a system of reference $\mathbf{R}$ on D is to attribute at every point of D a system of coordinates such that at two different points of D we have two different system of coordinates.

Let T be a transformation of the domain D into itself which admits an inverse T^{-1}. By definition, *to transform the reference system* $\mathbf{R}$ *by the transformation* T is to attach at the point $T\mathbf{M}$ the coordinates that the point $\mathbf{M}$ has in the system $\mathbf{R}$. Two different points of D are transformed into two different points of D by T, and hence we attache to them two different system of coordinates. Then we have obtained a new system of reference, which will be denoted by $T\mathbf{R}$.

Remark. The coordinates of a point $\mathbf{M}'$ with respect to $T\mathbf{R}$ are the coordinates of the point $T^{-1}\mathbf{M}'$ with respect to $\mathbf{R}$.

Example. To transform a system of Cartesian coordinates by a displacement is to subject the reference trihedral to the displacement in question.

From the given definition we immediately deduce the following consequences:

a. Let two systems of references be defined on the same domain D. There exists at most one transformation which transforms the first system into the second.

b. Consider two systems of references $\mathbf{R}$ and $\mathbf{R}'$. If $\mathbf{R}' = T\mathbf{R}$, then $\mathbf{R} = T^{-1}\mathbf{R}'$.

c. We have $T'(T\mathbf{R}) = (T'T)\mathbf{R}$.

d. Consider two figures F and F' between the points of which there exist a bijective correspondence. Suppose this correspondence is such that every point of F has with respect to a certain system of reference $\mathbf{R}$ the same coordinates as its corresponding point in F' with respect to a second system of reference $\mathbf{R}'$. If $\mathbf{R}' = T\mathbf{R}$, then $F' = TF$.

62 Equations of a transformation in relative coordinates. Consider a domain D on which a reference system $\mathbf{R}_0$ is defined. Let S and T be two transformations of D into itself, defined by the following equations:

$$\begin{aligned} &\text{for } S: \qquad x_i' = \varphi(x_1, \ldots, x_n), \\ &\text{for } T: \qquad x_i' = \psi(x_1, \ldots, x_n). \end{aligned}$$

We assume that S possesses an inverse. We say that $\mathbf{R}_0$ is a "*absolute reference system*" and $S\mathbf{R}_0$ is a "*relative reference system*". We will determine the equations of the transformation T with respect to the relative reference system.

Consider a point $\mathbf{M}$ in D and its transformed point $T\mathbf{M}$. The relative coordinates of $\mathbf{M}$ are the absolute coordinates of $S^{-1}\mathbf{M}$, and the relative coordinates of $T\mathbf{M}$ are the absolute coordinates of $S^{-1}T\mathbf{M} = (S^{-1}TS)S^{-1}\mathbf{M}$.

To express this result more clearly, we will call the *analytic transformation* T the transformation T considered acting not on a geometric point but on n coordinates. By contrast we call the *geometrical transformation* T the point transformation defined by the equations of T when x_i and x_i' represent absolute coordinates there. The conclusion obtained above is the following:

The geometrical transformation T, with respect to the reference system $S\mathbf{R}_0$, is represented by the analytic transformation $S^{-1}TS$.

This transformation $S^{-1}TS$ is the one which transforms x_i into x_i' defined by the n equations

$$\varphi_i(x_1', \ldots, x_n') = \psi_i[\varphi_1(x_1, \ldots, x_n), \ldots, \varphi(x_1, \ldots, x_n)]. \tag{3}$$

Transforming a transformation by another. Let S and T be two transformations operating on the same domain and let S have an inverse. STS^{-1} is called "T transformed by S".

This definition allows us to say the following: when we transform a frame $\mathbf{R}_0$ by a transformation S, the analytic transformation T corresponding to a given geometrical transformation is transformed by S^{-1}.

Now we state the following properties of the transformed transformation:

a. The product of the transformed transformations T_1 and T_2 by S is the product of T_1 and T_2 transformed by S:

$$(ST_1S^{-1})(ST_2S^{-1}) = S(T_1T_2)S^{-1}.$$

b. T transformed by S_1 and then by S_2 is T transformed by S_1S_2:

$$S_1(S_2TS_2^{-1})S_1^{-1} = (S_1S_2)T(S_1S_2)^{-1}.$$

63 The moving frame of a family of transformations. During the first part of this work it has been useful for us to employ a system of moving reference deduced from a system of absolute reference by an arbitrary displacement. Now for the study of a connected and finite dimensional family of transformations S_a whose inverses exist and operate on the same domain D, let $\mathbf{R}_0$ be an absolute frame of D. We again introduce the family of reference systems

$$\mathbf{R}_a = S_a \mathbf{R}_0.$$

In geometry we usually represent a system of Cartesian references by a trihedral with numbered edges, and in the preceding chapters we have represented them by three vectors with a common origin. We could also have characterised each Cartesian system by the position of an ellipsoid whose axes have lengths 1, 2, 3, etc.

In a similar way, let us trace a figure F_0 in D and associate the system of reference $\mathbf{R}_a = S_a\mathbf{R}_0$ to the figure S_aF_0. We make sure that two distinct systems of reference $\mathbf{R}_a = S_a\mathbf{R}_0$ and $\mathbf{R}_b = S_b\mathbf{R}_0$ correspond to two distinct figures S_aF_0 and S_bF_0: F_0 must be different from all the figures $S_a^{-1}S_bF_0$.

We can, for example, construct F_0 with a finite number of points $\mathbf{A}_1$, $\mathbf{A}_2$, …, $\mathbf{A}_p$: $\mathbf{A}_1$ will be a point that not all transformations $S_a^{-1}S_b$ leave fixed. Let Σ_1 be those transformations of $S_a^{-1}S_b$ that leaves $\mathbf{A}_1$ fixed, then $\mathbf{A}_2$ will be a point that is not left fixed by all transformations Σ_1. Let Σ_2 be those transformations of $S_a^{-1}S_b$ leaving $\mathbf{A}_1$ and $\mathbf{A}_2$ fixed, then $\mathbf{A}_3$ will be a point that is not left fixed by all the transformations in Σ_2, …. The last point $\mathbf{A}_p$ will be such that Σ_{p+1} is reduced to the identity transformation.

Having chosen such a figure F_0A which is distinct from all the transformed figures $S_a^{-1}S_bF_0$, we will call the figure S_aF_0 the *frame* attached at $\mathbf{R}_a$. Instead of speaking of a system of reference, we will henceforth speak of the corresponding frame: we shall find it advantageous to deal with this more concrete concept.

The moving frame of a group.

Suppose that the transformations S_a constitute a group.

The condition that F_0 must be different from all the figures $S_a^{-1}S_bF_0$ then reduces to the following: F_0 must not be invariant under any transformation other than the identity.

As $\mathbf{R}_a = S_aS_b^{-1}\mathbf{R}_b$, all the positions of the moving frame are deduced from any one of them by transforming with all the transformations of the group.

The group with respect to $\mathbf{R}_0$ is the set of analytic transformations S_a, and with respect to $\mathbf{R}_b$ it is hence the set of analytic transformations $S_b^{-1}S_aS_b$. But we can choose S_a in a way such that $S_b^{-1}S_aS_b$ is any arbitrary transformation S_α of the group: we take $S_a = S_bS_\alpha S_b^{-1}$. Then the set of transformations of a group is defined by the same equations in absolute coordinates as in relative coordinates.

The absolute frame $\mathbf{R}_0$ is a position of the moving frame that cannot be distinguished from any other.

We are going to indicate the frames that are commonly used in the study of the most elementary groups.

64 The moving frame of a linear transformation group. The linear transformation

$$x' = ax + b \qquad (a > 0)$$

constitute a continuous group on one variable with two parameters.

The identity transformation ($a = 1, b = 0$) is the only one that leaves two arbitrarily chosen points invariant, for example the points with abscissae 0 and 1. These are the two numbered points, or more simply the vector having the first as the origin and the second and its tip, which we chose as a frame $\mathbf{R}_0$. Transforming $\mathbf{R}_0$ by the transformation $x' = ax + b$, we obtain a vector $\vec{\mathbf{I}}$ having a point $\mathbf{A}$ with abscissa b as origin and the point of abscissa $a+b$ as the tip with length a. The set of our frames is therefore the set of vectors oriented in the same direction as the x-axis. Let $\mathbf{M}$ be a point on this axis. What is its abscissa with respect to the frame $\vec{\mathbf{I}}$? We need to consider the transformation of the group which transforms this frame into the frame $\mathbf{R}_0$. It multiplies the length of all segments by $\dfrac{1}{a}$ and transforms $\mathbf{A}$ to $\mathbf{O}$, and $\mathbf{M}$ to a point $\mathbf{P}$ (*fig.* 1). We have $\mathbf{X} = \overrightarrow{\mathbf{OP}}$, $\overrightarrow{\mathbf{OP}} = \dfrac{1}{a}\overrightarrow{\mathbf{AM}}$, and therefore $\mathbf{X}$ is defined by the relation $\overrightarrow{\mathbf{AM}} = \mathbf{X} \cdot \vec{\mathbf{I}}$.

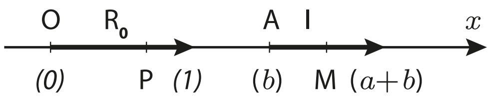


Figure 1

65 The moving frame of the affine group. A point in the plane is represented by two Cartesian coordinates x and y. The affine group is the connected, finite dimensional group defined by the relations

$$\begin{cases} x' = ax + by + c, \\ y' = a'x + b'y + c', \end{cases} \qquad \begin{vmatrix} a & b \\ a' & b' \end{vmatrix} > 0. \tag{4}$$

The transformation (4) associate collinear points to collinear points, parallel lines to parallel lines, and vectors $\vec{\mathbf{V}}_i$ satisfying a linear homogeneous relation

$$\sum_{(i)} x_i \vec{\mathbf{V}}_i = 0$$

to vectors $\vec{\mathbf{V}}'_i$ satisfying the same relation

$$\sum_{(i)} x_i \vec{\mathbf{V}}'_i = 0.$$

Only the identity transformation leaves the origin $\mathbf{O}$, the point $(0, 1)$ and the point

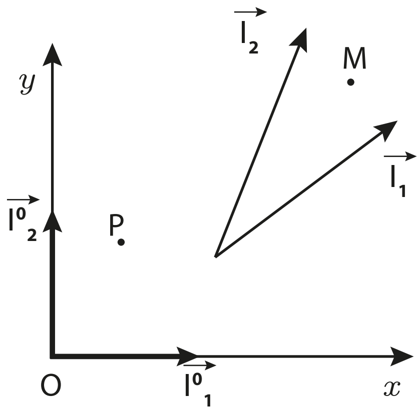


Figure 2

$(1,0)$ separately invariant. Therefore we can choose the figure formed by two vectors $\vec{\mathbf{I}}_1{}^0$ and $\vec{\mathbf{I}}_2{}^0$ having $\mathbf{O}$ as the common origin and having the points $(0,1)$ and $(1,0)$ as the tips. These vectors, transformed by (4), becomes two vectors $\vec{\mathbf{I}}_1$ and $\vec{\mathbf{I}}_2$ having the point $\mathbf{A}$ (c, c') as the origin and the components (a, a') and (b, b') as the components (*fig.* 2). Let us determine the relative coordinates X and Y of a point $\mathbf{M}$ in the place. The transformation transforming $\vec{\mathbf{I}}_1$ and $\vec{\mathbf{I}}_2$ into $\vec{\mathbf{I}}_1{}^0$ and $\vec{\mathbf{I}}_2{}^0$ transforms $\mathbf{M}$ into a point $\mathbf{P}$ whose absolute coordinates are X and Y. These coordinates are defined by the relation

$$\overrightarrow{\mathbf{OP}} = X\vec{\mathbf{I}}_1{}^0 + Y\vec{\mathbf{I}}_2{}^0.$$

Therefore X and Y are two numbers defined by the relation

$$\overrightarrow{\mathbf{AM}} = X\vec{\mathbf{I}}_1 + Y\vec{\mathbf{I}}_2.$$

Analogous considerations apply to the affine group in n dimensions.

66 The moving frame of the projective group (first definition). If we represent a point in a plane as its three homogeneous coordinates, then the projective group is the group defined by the relation

$$(5) \qquad \begin{cases} x' = ax + by + cz, \\ y' = a'x + b'y + c'z, \\ z' = a''x + b''y + c''z, \end{cases} \qquad \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} \neq 0.$$

This group is connected and finite dimensional. It actually acts on two variables and depends on 8 parameters. It transforms collinear points into collinear points, straight lines into straight lines, and it preserves the anharmonic ratio. Let us investigate what transformations leave the points $\alpha_0(1,0,0)$, $\beta_0(0,1,0)$, $\gamma_0(0,0,1)$, $\delta_0(1,1,1)$ invariant: we find the transformations $x' = ax$, $y' = ay$, $z' = az$, which are all equivalent to the identity transformation. We therefore can take the frame $\mathbf{R}_0$ to be the figure formed by the four preceding points numbered in the order indicated. $\mathbf{R}_0$ is transformed by (5) into a figure formed by the points $\alpha(a, a', a'')$, $\beta(b, b', b'')$, $\gamma(c, c', c'')$, $\delta(a+b+c, a'+b'+c', a''+b''+c'')$, i.e., into the figure formed by four arbitrary points where any three of its points are not collinear.

Let us determine the relative homogeneous coordinates X, Y, Z of a point $\mathbf{M}$ in the plane. The transformation that moves $\mathbf{R}$ to $\mathbf{R}_0$ transforms $\mathbf{M}$ into a point $\mathbf{P}$. This point has absolute coordinates [†]

$$
\begin{aligned}
&X = (\beta_0\alpha_0, \beta_0\gamma_0, \beta_0\delta_0, \beta_0\mathbf{P}),\\
&X = (\alpha_0\beta_0, \alpha_0\gamma_0, \alpha_0\delta_0, \alpha_0\mathbf{P}),\\
&Z = 1.
\end{aligned}
$$

The relative coordinates of $\mathbf{M}$ are therefore

$$X = (\beta\alpha, \beta\gamma, \beta\delta, \beta\mathbf{M}); \qquad Y = (\alpha\beta, \alpha\gamma, \alpha\delta, \alpha\mathbf{M}); \qquad Z = 1.$$

Analogous considerations apply for projective groups in n dimensions.

67 The moving frame of the projective group (second definition). The transformations of the projective group are no longer defined by the formulae (5), but the following, which is also a group of transformations,

$$
\text{(6)} \qquad \begin{cases} x' = ax + by + cz, \\ y' = a'x + b'y + c'z, \\ z' = a''x + b''y + c''z, \end{cases} \qquad \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} = 0.
$$

To each given geometrical transformation there now corresponds the well-determined coefficients a, b, c, $\ldots$, c''. Granted this, the data of three numbers (x, y, z) will be said to define *an analytic point* coincident with the *geometrical point* with the same coordinates. The transformed image of an analytic point (x, y, z) by the operatrion T of the group the formulae (6) defines will be the analytic point $\mathbf{M}'(x', y', z')$. Note that we again have $T(T'\mathbf{M}) = (TT')\mathbf{M}$. Consider p analytic points

$$\mathbf{M}_1(x_1, y_1, z_1), \qquad \mathbf{M}_2(x_2, y_2, z_2), \qquad \ldots,$$

and suppose that their coordinates satisfy three relations

$$
\begin{aligned}
a_1x_1 + a_2x_2 + \cdots + a_px_p = 0,\\
a_1y_1 + a_2y_2 + \cdots + a_py_p = 0,\\
a_1z_1 + a_2z_2 + \cdots + a_pz_p = 0,
\end{aligned}
$$

which we will agree to write as

$$a_1\mathbf{M}_1 + a_2\mathbf{M}_2 + \cdots + a_p\mathbf{M}_p = 0.$$

[†] We represent, as used here, the anharmonic ration by of four converging lines D_1, D_2, D_3, D_4 by (D_1, D_2, D_3, D_4).

Let $\mathbf{M}'_1$, ..., $\mathbf{M}'_p$ be the transformed images of these points by the same operation in the group, we have

$$a_1\mathbf{M}'_1 + a_2\mathbf{M}'_2 + \cdots + a_p\mathbf{M}'_p = 0.$$

We denote the value of the following determinant by the symbol $[\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3]$

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

and using the rules of multiplication, we obtain the relation

$$[\mathbf{M}'_1, \mathbf{M}'_2, \mathbf{M}'_3] = [\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3].$$

Only the identity transformation leaves each of the following analytic points invariant

$$\mathbf{A}^0(0,0,1), \qquad \mathbf{A}_1^0(1,0,0), \qquad \mathbf{A}_2^0(0,1,0).$$

The frame $\mathbf{R}_0$ that we will use in the following will be formed by these three analytic points. The transformation (6) transforms $\mathbf{R}_0$ into the frame $\mathbf{R}$ formed by the three analytic points

$$\mathbf{A}(c, c', c''), \qquad \mathbf{A}_1(a, a', a''), \qquad \mathbf{A}_2(b, b', b'').$$

Conversely, three analytic points $\mathbf{A}$, $\mathbf{A}_1$, $\mathbf{A}_2$ such that

$$[\mathbf{A}, \mathbf{A}_1, \mathbf{A}_2] = 1$$

constitute a frame $\mathbf{R}$.

Consider a point in the plane. Let us find its coordinates X, Y, Z relative to $\mathbf{R}$ (they are determined up to a factor). They are the coordinates of an analytic point $\mathbf{M}$ coincident with the given geometrical point. The transformation that transforms $\mathbf{R}$ into $\mathbf{R}_0$ transforms $\mathbf{M}$ into a point $\mathbf{P}$ of absolute coordinates X, Y, Z. These coordinates are defined by the relation

$$\mathbf{P} = X\mathbf{A}_1^0 + Y\mathbf{A}_2^0 + Z\mathbf{A}_3^0.$$

Therefore the relative coordinates of $\mathbf{M}$ are defined by

$$\mathbf{M} = X\mathbf{A}_1 + Y\mathbf{A}_2 + Z\mathbf{A}_3.$$

II. Infinitesimal displacement components of a moving frame

Differential calculus can be applied to the theory of groups by the intermediacy of two notions that we are going to introduce: infinitesimal transformations and the components of infinitesimal movements of a moving frame.

68 Infinitesimal transformations. Consider a transformation T_ε depending on an infinitesimally small parameter ε which is very close to the identity. Assume that T_ε is defined by the equations

$$x_i' = x_i + \varepsilon\eta_i(x_1, \ldots, x_n) + \cdots \tag{7}$$

where the dots denote the higher order terms in the infinitesimal ε.

The equation (7) signifies that, when we move from a point to its transformed image, the coordinates x_i undergoes a variation whose principal part is

$$\delta x_i = \varepsilon\eta_i(x_1, \ldots, x_n).$$

An arbitrary function $f(x_1, \ldots, x_n)$ under the same conditions undergoes a variation $f(x_1', \ldots, x_n') - f(x_1, \ldots, x_n)$ whose principal part is

$$\delta f = \varepsilon \sum_{i=1}^{n} \eta_i(x_1, \ldots, x_n) \frac{\partial f(x_1, \ldots, x_n)}{\partial x_i}. \tag{8}$$

This formula (8) includes the n formulae (7). The consideration of δf has the advantage of no longer explicitly referring to the choice of coordinates.

By an *infinitesimal transformation* we mean an operator attaching to a function $f(x_1, \ldots, x_n)$ a function

$$Xf = \sum_{i=1}^{n} \xi_i(x_1, \ldots, x_n) \frac{\partial f(x_1, \ldots, x_n)}{\partial x_i}.$$

An infinitesimal transformation is therefore a linear and homogeneous operator. Every linear and homogeneous combination of infinitesimal transformations is an infinitesimal transformation.

By definition we associate the *infinitesimal transformation* defined by (8) with the *infinitesimally small transformation* T_ε defined by (7). The expression δf will be called the *symbol* of this infinitesimally small transformation and of this infinitesimal transformation.

Consider an infinitesimally small displacement obtained by applying three successive translations parallel to the axes of coordinates, of respective length ε_1, ε_2 and ε_3, and then three rotations around the axes of coordinates, of respective angles ε_{23}, ε_{31}, ε_{12}. This infinitesimally small displacement subjects the coordinates x, y, z to variations whose principal parts are

$$\begin{aligned}
\delta x &= \varepsilon_1 \qquad\qquad - \varepsilon_{12} y + \varepsilon_{31} z,\\
\delta y &= \varepsilon_2 + \varepsilon_{12} x \qquad\qquad - \varepsilon_{23} z,\\
\delta z &= \varepsilon_3 - \varepsilon_{31} x + \varepsilon_{23} y.
\end{aligned}$$

Therefore there corresponds to this infinitesimally small displacement the infinitesimal transformation

$$\begin{aligned}
Xf &= \varepsilon_1 \frac{\partial f}{\partial x} + \varepsilon_2 \frac{\partial f}{\partial y} + \varepsilon_3 \frac{\partial f}{\partial z}\\
&\quad + \varepsilon_{23}\left(y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y}\right) + \varepsilon_{31}\left(z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}\right) + \varepsilon_{12}\left(x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}\right).
\end{aligned}$$

The transformations whose symbols are

$$X_1 f = \frac{\partial f}{\partial x}, \qquad X_2 f = \frac{\partial f}{\partial y}, \qquad X_3 f = \frac{\partial f}{\partial z}$$

are called the unit *infinitesimal translations* parallel to the axes $\mathbf{O}x$, $\mathbf{O}y$, $\mathbf{O}z$; the transformations whose symboles are

$$X_{23} f = y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y}, \qquad X_{31} f = z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}, \qquad X_{12} f = x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x},$$

are called the unit *infinitesimal rotations* around the axes $\mathbf{O}x$, $\mathbf{O}y$, $\mathbf{O}z$.

69 Equations of an infinitesimal transformation in relative coordinates. We distinguished geometrical transformations from analytic transformations (§**62**, p. 73). An infinitesimally small transformation and its corresponding infinitesimal transformation are simultaneously called geometric and analytic.

Consider a domain D on which a system of reference $\mathbf{R}_0$ is defined, and a transformation S possessing an inverse acting on D:

$$x'_i = \varphi_i(x_1, \ldots, x_n). \tag{S}$$

Consider an infinitesimal geometric transformation which, with respect to $\mathbf{R}_0$, has the symbole

$$Xf = \sum_{i=1}^{n} \eta_i(x_1, \ldots, x_n)\frac{\partial f}{\partial x_i}. \tag{T}$$

Let us find what its symbol is with respect to the reference system $S\mathbf{R}_0$.

To this end consider η_i as infinitesimally small quantities. To the infinitesimally small transformation $S^{-1}TS$ there corresponds an infinitesimal transformation whose symbole

$$Yf = \sum_{k=1}^{n} \zeta_k(x_1, \ldots, x_n)\frac{\partial f}{\partial x_k} \tag{9}$$

is the symbol we search for by virtue of paragraph §**62** (p. 73).

To determine the functions ζ_k it suffices to replace, in the formula (3) of paragraph §**62**, x'_k by $x_k + \zeta_k$ and $\psi_k(y_1, \ldots, y_n)$ by

$$y_k + \eta_k(y_1, \ldots, y_n),$$

then neglect infinitesimally small quantities of orders higher than 1. It becomes

$$\sum_{k=1}^{n} \frac{\partial \varphi_i}{\partial x_k}\zeta_k = \eta_i[\varphi_1(x_1, \ldots, x_n), \ldots, \varphi_n(x_1, \ldots, x_n)]. \tag{10}$$

The determinant of the quantities $\dfrac{\partial \varphi_i}{\partial x_k}$ is non-zero, according to the inequality (2) (p. 72). We solve the system (10) with respect to ζ_k and substitute the values obtained in (9), and we will then obtain the symbol of the infinitesimally transformation $S^{-1}TS$ we look for, which is called the *transformed image* of T by S^{-1}.

Example. If T is the infinitesimal translation

$$Xf = \varepsilon_1 \frac{\partial f}{\partial x} + \varepsilon_2 \frac{\partial f}{\partial y},$$

and S is the rotation

$$\begin{aligned} x' &= x\cos c - y\sin c,\\ y' &= x\sin c + y\cos c, \end{aligned}$$

then the transformed image of T by S^{-1} is the infinitesimal translation

$$Yf = (\varepsilon_1\cos c + \varepsilon_2 \sin c)\frac{\partial f}{\partial x} + (-\varepsilon_1 \sin c + \varepsilon_2\cos c)\frac{\partial f}{\partial y}.$$

70 Relative components of a moving frame. Consider the moving frame of a family of transformations and let $\mathbf{R}_a$ and $\mathbf{R}_{a+da}$ be two infinitesimally close positions in this space. The geometrical transformation transforming $\mathbf{R}_a$ into $\mathbf{R}_{a+da}$ is the geometrical transformation $S_{a+da}S_a^{-1}$. With respect to the frame $\mathbf{R}_a$, it constitutes, according to paragraph §**62**, the analytic transformation [†] $S_a^{-1}(S_{a+da}S_a^{-1})S_a = S_a^{-1}S_{a+da}$.

Let us recall some circumstances that arise when S_a is the group of displacements and $\mathbf{R}_a$ is a moving rectangular trihedral. The infinitesimal transformation $S_a^{-1}S_{a+da}$ is an infinitesimal displacement. It symbol is therefore of type

$$\begin{aligned} &\omega_1(a,da)X_1f + \omega_2(a,da)X_2f + \omega_3(a,da)X_3f +\\ &\quad \omega_{23}(a,da)X_{23}f + \omega_{31}(a,da)X_{31}f + \omega_{12}(a,da)X_2f, \end{aligned} \tag{11}$$

X_1f, X_2f, X_3f being unit infinitesimal translations parallel to the coordinates, $X_{23}f$, $X_{31}f$, $X_{12}f$ being unit infinitesimal rotations around the coordinate axes [c.f. §**68**, example p. 79]. ω_1, ω_2, ω_3, ω_{23}, ω_{31}, ω_{12} are six Pfaffian forms in the six parameters a_1, $\dots$, a_6, they are the six relative components of the infinitesimal displacement of the trihedral. Recall that they are linearly independent.

Definition. We say that a moving frame $\mathbf{R}_0$ *possesses relative components when the symbol of its infinitesimal transformation* $S_a^{-1}S_{a+da}$ *is of type*

$$\omega_1(a,da)X_1f + \cdots + \omega_r(a+da)X_rf, \tag{12}$$

the following circumstances arise: the infinitesimal transformations X_1f, $\dots$, X_rf *depending on neither the parameters* a_1, $\dots$, a_r *nor their differentials* $\omega_1(a,da)$, $\dots$,

[†] We can also reason as the following: the geometric transformation transforming $\mathbf{R}_a$ into $\mathbf{R}_{a+da}$, with respect to $\mathbf{R}_a$, has the same equations as the transformation transforming $\mathbf{R}_0$ into $S_a^{-1}\mathbf{R}_{a+da} = S_a^{-1}S_{a+da}\mathbf{R}_0$ and hence it constitutes the analytic transformation $S_a^{-1}S_{a+da}$.

$\omega_r(a, da)$ *are* r *Pfaffian forms in* r *parameters that* $\mathbf{R}_a$ *depends on. At a point in the parameter space, these forms are in general independent.*

We name $\omega_1, \ldots, \omega_r$ *the "relative components" of* $\mathbf{R}_a$.

Remark concerning the independence of the forms $\omega_1, \ldots, \omega_r$. Suppose that the symbol of $S_a^{-1}S_{a+da}$ is (12) but the forms $\omega_1(a, da), \ldots, \omega_r(a, da)$ are not independent for any value of the parameters $a_1, \ldots, a_r$. We can apply the reasoning already employed in the course of paragraph §**8** and §**25** (p. 18 and 34) to this situation: we can find a system of differential equations

$$\frac{da_1}{\chi_1(a_1, \ldots, a_r)} = \frac{da_2}{\chi_2(a_1, \ldots, a_r)} = \cdots = \frac{da_r}{\chi_r(a_1, \ldots, a_r)},$$

such that all the forms $\omega_s(a, da)$ are zero when the quantities $a_1, \ldots, a_r$ varies while continuing to satisfy these differential equations. $S_a^{-1}S_{a+da}$ is then always the identity and the transformation S_a remains fixed. Hence every transformation in the family is indistinguishable from an infinite connected family of other transformations of the family: the transformations S_a actually depends on a number of parameters less than r.

It is therefore unnecessary to check that the forms $\omega_1, \ldots, \omega_r$ are linearly independent in general when we know that the frame considered cannot be considered as depending on less than r parameters.

71 Rules for calculating the relative components. Consider a family of transformations S_a possessing inverses

$$x_i' = \varphi_i(x_1, \ldots, x_n; a_1, \ldots, a_r). \tag{1}$$

We are going to investigate if its moving frame possesses relative components and we are going to determine these components.

Let $\mathbf{M}$ be a point with absolute coordinates $x_1, \ldots, x_n$ and let

$$x_1 + \delta x_1, \qquad \ldots, \qquad x_n + \delta x_n$$

be the absolute coordinates of the point $\mathbf{P} = S_a^{-a}S_{a+da}\mathbf{M}$. We have

$$S_a\mathbf{P} = S_{a+da}\mathbf{M},$$

then

$$\varphi_i(x_1 + \delta x_1, \ldots, x_n + \delta x_n; a_1, \ldots, a_r) = \varphi_i(x_1, \ldots, x_n; a_1 + da_1, \ldots, a_r + da_r),$$

so

$$\sum_{k=1}^{n} \frac{\partial\varphi_i(x_1, \ldots, x_n; a_1, \ldots, a_r)}{\partial x_k}\delta x_k = \sum_{p=1}^{r} \frac{\partial\varphi_i(x_1, \ldots, x_n; a_1, \ldots, a_r)}{\partial a_p}da_p. \tag{13}$$

The inequality (2) (p. 72) ensures that these equations can be solved with respect to δx_k. The symbol of the infinitesimal transformation $S_a^{-1}S_{a+da}$ is obtained by substituting the value of δx_k thus obtained into the relation

$$\delta f = \sum_{k=1}^{n} \delta x_k \frac{\partial f}{\partial x_k}. \tag{14}$$

For the relative components to exist, it is necessary and sufficient that δf is of type (12).

Example. The group of linear transformations

$$x' = ax + b, \qquad (a > 0).$$

The relations (13) and (14) take the form

$$a\,\delta x = x\,da + db, \qquad \delta f = \frac{da}{a} x \frac{\partial f}{\partial x} + \frac{db}{a}\frac{\partial f}{\partial x}.$$

The frame of this group hence possesses the relative components

$$\omega_1 = \frac{da}{a}, \qquad \omega_2 = \frac{db}{a}.$$

These formulae can also be established by a geometric reasoning: consider the point $\mathbf{M}$ whose abscissa is x with respect to $\mathbf{R}_a$ and the point $\mathbf{P}$ whose abscissa is x with respect to $\mathbf{R}_{a+da}$. $x + \delta x$ is the abscissa of $\mathbf{P}$ with respect to $\mathbf{R}_a$. We therefore have (c.f. **§64**, p. 75)

$$\delta x = \frac{\overrightarrow{\mathbf{MP}}}{a}.$$

But, according to paragraph **§64**,

$$\overrightarrow{\mathbf{MP}} = \overrightarrow{\mathbf{AM}}\frac{da}{a} + db.$$

On the other hand

$$x = \frac{\overrightarrow{\mathbf{AM}}}{a}$$

from which

$$\delta x = x\frac{da}{a} + \frac{db}{a}.$$

Relative components of the frame of a group. Consider a finite dimensional connected group S_a depending on r parameters and the number r cannot be further reduced. An infinitesimally small transformation S_{da} of this group has a symbol of type

$$da_1 X_1 f + \cdots + da_r X_r f,$$

or, more generally, of type

$$\varepsilon_1 X_1 f + \cdots + \varepsilon_r X_r f,$$

$\varepsilon_1, \ldots, \varepsilon_r$ being homogeneous linear combinations with constant coefficients of $da_1, \ldots, da_r$.

We say that $X_1 f, \ldots, X_r f$ are the r *bases of the infinitesimal transformations of the group*, and we name $\varepsilon_1, \ldots, \varepsilon_r$ the components of the transformation S_{da}. All infinitesimal transformations of the group are homogeneous linear combinations with constant coefficients of $X_1 f, \ldots, X_r f$.

$S_a^{-1} S_{a+da}$ is an infinitesimally small transformation of the group. Let $\omega_1(a, da), \ldots, \omega_r(a, da)$ be its components. The symbol of $S_a^{-1} S_{a+da}$ is

$$\omega_1(a, da) X_1 f + \cdots + \omega_r(a, da) X_r f.$$

Therefore *the moving frame of a group always possesses relative components.* [Example: the moving trihedral.]

The infinitesimal transformations of a group are linearly independent: otherwise we can assume that $X_1 = 0$ and replace ω_1 by 0 in the symbol of $S_a^{-1} S_{a+da}$, and then the forms $\omega_1, \ldots, \omega_r$ are not independent, which contradicts the hypothesis that the number of parameters of the group is exactly r (c.f. §**70**, remark concerning ..., p. 81).

The relative components $\omega_1(a, da), \ldots, \omega_r(a, da)$ of a group is not only independent in general, but also independent at every point in the parameter space. We can then give them arbitrary values $db_1, \ldots, db_r$ by choosing $S_{a+da} = S_a S_{db}$.

72 Relative components of an affine frame. Consider the affine group and the frames defined in paragraph §**65** (p. 75).

In the first place let us indicated what the components of an infinitesimal transformation S_{da} are (we suppose that the parameters are chosen in a way such that the identity transformation corresponds to vanishing values of the parameters). The frame $\mathbf{R}_{da}$ is constituted by a point $\mathbf{A}$ and two vectors $\vec{\mathbf{I}}_1$ and $\vec{\mathbf{I}}_2$. To characterise it, it suffices to specify with respect to the frame $\mathbf{R}_0$ the coordinates $\omega_1(0, da)$, $\omega_2(0, da)$ of $\mathbf{A}$, the components $1 + \omega_{11}(0, da)$, $\omega_{12}(0, da)$ of $\vec{\mathbf{I}}_1$ and the components $\omega_{21}(0, da)$, $1 + \omega_{22}(0, da)$ of $\vec{\mathbf{I}}_2$. The linearly independent forms $\omega_1(0, da)$, $\omega_2(0, da)$, $\omega_{11}(0, da)$, $\omega_{12}(0, da)$, $\omega_{21}(0, da)$ and $\omega_{22}(0, da)$, the number of which is equal to the number of parameters, are chosen as the components of the infinitesimal transformation S_{da}.

Now consider a frame $\mathbf{R}_a$ formed by $\mathbf{A}$, $\vec{\mathbf{I}}_1$, $\vec{\mathbf{I}}_2$ and a frame $\mathbf{R}_{a+da}$ formed by $\mathbf{A} + \overrightarrow{d\mathbf{A}}$, $\vec{\mathbf{I}}_1 + d\vec{\mathbf{I}}_1$, $\vec{\mathbf{I}}_2 + d\vec{\mathbf{I}}_2$. Let use transform the figure formed by these two frames by the transformation S_a^{-1}. We will obtain the frame $\mathbf{R}_0$ and a frame $\mathbf{R}'$ formed by a point $\mathbf{A}'$ and two vectors $\vec{\mathbf{I}}'_1$, $\vec{\mathbf{I}}'_2$. By definition

$$\begin{aligned} \overrightarrow{\mathbf{OA}'} &= \omega_1\ (a, da)\vec{\mathbf{I}}_1{}^0 + \omega_2\ (a, da)\vec{\mathbf{I}}_2{}^0, \\ \vec{\mathbf{I}}'_1 - \vec{\mathbf{I}}_1{}^0 &= \omega_{11}(a, da)\vec{\mathbf{I}}_1{}^0 + \omega_{12}(a, da)\vec{\mathbf{I}}_2{}^0, \\ \vec{\mathbf{I}}'_2 - \vec{\mathbf{I}}_2{}^0 &= \omega_{21}(a, da)\vec{\mathbf{I}}_1{}^0 + \omega_{22}(a, da)\vec{\mathbf{I}}_2{}^0. \end{aligned}$$

According to the properties of the affine group that we have already used, it follows that

$$
(15) \qquad \begin{cases} \overrightarrow{d\mathbf{A}} = \omega_1\ (a,da)\vec{\mathbf{I}}_1 + \omega_2\ (a,da)\vec{\mathbf{I}}_2, \\ d\vec{\mathbf{I}}_1 = \omega_{11}(a,da)\vec{\mathbf{I}}_1 + \omega_{12}(a,da)\vec{\mathbf{I}}_2, \\ d\vec{\mathbf{I}}_2 = \omega_{21}(a,da)\vec{\mathbf{I}}_1 + \omega_{22}(a,da)\vec{\mathbf{I}}_2. \end{cases}
$$

The relations (15) allow us to very simply determine the relative components $\omega_1(a,da)$, $\omega_2(a,da)$, $\omega_{11}(a,da)$, $\omega_{12}(a,da)$, $\omega_{21}(a,da)$, $\omega_{22}(a,da)$ when we know the relative position of two frames $\mathbf{R}_a$ and $\mathbf{R}_{a+da}$.

73 Relative components of a projective frame. Consider the projective group and the frames defined in paragraph §**67** (p. 77). To characterise the frame $\mathbf{R}_{da}$, it suffices to specify the coordinates of the analytic points determining it:

$$
\begin{vmatrix} 1+\omega_{00}(0,da) & \omega_{01}(0,da) & \omega_{02}(0,da) \\ \omega_{10}(0,da) & 1+\omega_{11}(0,da) & \omega_{12}(0,da) \\ \omega_{20}(0,da) & \omega_{21}(0,da) & 1+\omega_{22}(0,da) \end{vmatrix}.
$$

Furthermore the determinant formed by these 9 coordinates must have the value 1 up to second order quantities, in other words $\omega_{00}+\omega_{11}+\omega_{22}=0$. The linearly independent forms $\omega_{00}(0,da)$, $\omega_{01}(0,da)$, $\omega_{02}(0,da)$, $\omega_{10}(0,da)$, $\omega_{11}(0,da)$, $\omega_{12}(0,da)$, $\omega_{20}(0,da)$, $\omega_{21}(0,da)$, the total number of which is that of the parameters of the group, are chosen as the components of the infinitesimal transformation S_{da}.

Now consider a frame $\mathbf{R}_a$ formed by three analytic points $\mathbf{A}$, $\mathbf{A}_1$ and $\mathbf{A}_2$ and a frame $\mathbf{R}_{a+da}$ formed by $\mathbf{A}+d\mathbf{A}$, $\mathbf{A}_1+d\mathbf{A}_1$, $\mathbf{A}_2+d\mathbf{A}_2$. Let us transform the figure formed by these frames by S_a^{-1}, and we will obtain the frame $\mathbf{R}_0$ and a frame $\mathbf{R}'$ formed by $\mathbf{A}'$, $\mathbf{A}'_1$ and $\mathbf{A}'_2$. By definition

$$
\begin{aligned} \mathbf{A}' - \mathbf{A}^0 &= \omega_{00}(a,da)\mathbf{A}^0 + \omega_{01}(a,da)\mathbf{A}^0_1 + \omega_{02}(a,da)\mathbf{A}^0_2, \\ \mathbf{A}'_1 - \mathbf{A}^0_1 &= \omega_{10}(a,da)\mathbf{A}^0 + \omega_{11}(a,da)\mathbf{A}^0_1 + \omega_{12}(a,da)\mathbf{A}^0_2, \\ \mathbf{A}'_2 - \mathbf{A}^0_2 &= \omega_{20}(a,da)\mathbf{A}^0 + \omega_{21}(a,da)\mathbf{A}^0_1 + \omega_{22}(a,da)\mathbf{A}^0_2. \end{aligned}
$$

According to the properties of the projective group that we have already used, it follows that

$$
(16) \qquad \begin{cases} d\mathbf{A}\ = \omega_{00}(a,da)\mathbf{A} + \omega_{01}(a,da)\mathbf{A}_1 + \omega_{02}(a,da)\mathbf{A}_2, \\ d\mathbf{A}_1 = \omega_{10}(a,da)\mathbf{A} + \omega_{11}(a,da)\mathbf{A}_1 + \omega_{12}(a,da)\mathbf{A}_2, \\ d\mathbf{A}_2 = \omega_{20}(a,da)\mathbf{A} + \omega_{21}(a,da)\mathbf{A}_1 + \omega_{22}(a,da)\mathbf{A}_2. \end{cases}
$$

The relations (16) allow us to very simply determine the relative components $\omega_{ij}(a,da)$ when we know the two frames $\mathbf{R}_a$ and $\mathbf{R}_{a+da}$. We also, furthermore

$$
(17) \qquad \omega_{00}(a,da) + \omega_{11}(a,da) + \omega_{22}(a,da) = 0.
$$

74 Absolute components of a moving frame. Consider the moving frame $\mathbf{R}_a$ of a family of transformations S_a.

The transformation transforming $\mathbf{R}_a$ to $\mathbf{R}_{a+da}$ is, with respect to $\mathbf{R}_0$, the analytic transformation $S_{a+da}S_a^{-1}$.

Definition. We say that $\mathbf{R}_a$ possesses absolute components when the symbol of transformation $S_{a+da}S_a^{-1}$ is of the type

$$\varpi_1(a, da)Y_1 f + \cdots + \varpi_r(a, da)Y_r f, \tag{18}$$

the infinitesimal transformations $Y_1 f$, …, $Y_r f$ being independent of a and da and ϖ_1, …, ϖ_r being r Pfaffian forms that are in general at an arbitrary point of the space independent.

This independence of ϖ_1, …, ϖ_r on the other hand necessarily holds when r is the smallest number of parameters that $\mathbf{R}_a$ is a function of.

The rule for calculating the absolute components. We assume that the transformation S_a is defined by the equations

$$x_i' = \varphi_i(x_1, \ldots, x_n; a_1, \ldots, a_r). \tag{1}$$

Consider a point $\mathbf{M}$ with absolute coordinates x_1, …, x_n, let $x_1 + \delta x_1$, …, $x_n + \delta x_n$ be the absolute coordinates of the point $\mathbf{P} = S_{a+da}S_a^{-1}\mathbf{M}$ and let y_1, …, y_n be the absolute coordinates of the point $\mathbf{Q} = S_a^{-1}\mathbf{M}$. We have $\mathbf{M} = S_a\mathbf{Q}$, $\mathbf{P} = S_{a+da}\mathbf{Q}$, and then

$$x_i' = \varphi_i(x_1, \ldots, x_n; a_1, \ldots, a_r), \qquad x_i + \delta x_i = \varphi_i(y_1, \ldots, y_n; a_1 + da_1, \ldots, a_r + da_r).$$

Hence

$$\delta x_i = \sum_{p=1}^{r} \frac{\partial \varphi_i(y_1, \ldots, y_n; a_a, \ldots, a_r)}{\partial a_p} da_p.$$

The symbol of the infinitesimal transformation $S_{a+da}S_a^{-1}$ is then

$$\delta f = \sum_{i,p} \frac{\partial \varphi_i(y_1, \ldots, y_n; a_1, \ldots, a_r)}{\partial a_p} \frac{\partial f(x_1, \ldots, x_n)}{\partial x_i} da_p, \tag{19}$$

y_1, …, y_n being determined by the system

$$x_i = \varphi_i(y_1, \ldots, y_n; a_1, \ldots, a_r). \tag{20}$$

For the absolute components to exist it is necessary and sufficient that δf is of type (18).

Example. Consider the family of real transformations possessing inverses,

$$x' = ax^3 + b \qquad (a > 0).$$

The equations (19) and (20) are written

$$\delta f = (y^3 da + db)\frac{\partial f}{\partial x}, \qquad x = ay^3 + b.$$

From which

$$\delta f = \frac{da}{a} x \frac{\partial f}{\partial x} + \left(db - b\frac{da}{a}\right)\frac{\partial f}{\partial x}.$$

The moving frame of this family therefore possesses the absolute components

$$\varpi_1 = \frac{da}{a}, \qquad \varpi_2 = db - b\frac{da}{a}.$$

Example. The group of linear transformations

$$x' = ax + b \qquad (a > 0).$$

The equations (19) and (20) are written

$$\delta f = (y\,da + db)\frac{\partial f}{\partial x}, \qquad x = ay + b.$$

From which

$$\delta f = \frac{da}{a} x \frac{\partial f}{\partial x} + \left(db - b\frac{da}{a}\right)\frac{\partial f}{\partial x}.$$

We find the same formula as in the preceding example.

This formula can be established by geometrical considerations. Consider two points **M** and **P** such that the abscissa of **M** with respect to $\mathbf{R}_a$ is equal to the abscissa of **P** with respect to $\mathbf{R}_{a+da}$. Let x and $x + \delta x$ be their absolute abscissae. **P** is obtained by subjecting **M** to a homothetic transformation having the origin of $\mathbf{R}_a$ as the centre and $\dfrac{a+da}{a}$ as the power, and then a translation of magnitude db (c.f. §**64**, p.75). Then

$$\delta x = \frac{da}{a}(x - b) + db.$$

The symbol of $S_{a+da}S_a^{-1}$ is therefore

$$\delta f = \frac{da}{a} x \frac{\partial f}{\partial x} + \left(db - b\frac{da}{a}\right)\frac{\partial f}{\partial x}.$$

Absolute components of the moving frame of a group. Consider a finite dimensional connected group S_a in r parameters. Let $\varpi_1(a, da)$, …, $\varpi_r(a, da)$ be the components of the infinitesimally small transformation $S_{a+da}S_a^{-1}$. The symbol of $S_{a+da}S_a^{-1}$ is

$$\varpi_1(a + da)X_1 f + \cdots + \varpi_r(a + da)X_r f.$$

The moving frame of a group therefore always possesses absolute components.

These absolute components are independent at all points in the parameter space: we can arbitrarily specify their values db_1, …, db_r by choosing $S_{a+da} = S_{db}S_a$.

75 Relations between the absolute components ϖ_p and the relative components ω_p. Consider a finite dimensional and connected family of transformations S_a possessing inverses. The transformation $\Sigma_a = S_a^{-1}$ form a family of the same nature. We have

$$S_a^{-1} S_{a+da} = \Sigma_a \Sigma_{a+da}^{-1}.$$

But the infinitesimally small transformation $\Sigma_a \Sigma_{a+da}^{-1}$ and the transformation $\Sigma_{a-da} \Sigma_a^{-1}$, which can be deduced from the other by replacing a by $a - da$, differ by only infinitesimally small quantities of second order. The corresponding infinitesimal transformations are therefore identical. In other words

$$S_a^{-1} S_{a+da} = \Sigma_{a-da} \Sigma_a^{-1}, \tag{21}$$

and similarly

$$S_{a+da} S_a^{-1} = \Sigma_a^{-1} \Sigma_{a-da}. \tag{21$'$}$$

The relative components of the moving frame of one of these families considered therefore exist whenever the absolute components of the moving frame of the other family exist, and when these components exist, they have opposite values.

This proposition allows us to transform the study of absolute components into the study of relative components: this is what will be useful to us in paragraph **§82**.

Suppose that the transformations S_a constitute a group. We can set $\Sigma_a = S_a^{01} = S_a$. The relation (21), which is now written

$$S_a^{-1} S_{a+da} = S_{\alpha-d\alpha} S_\alpha^{-1}$$

gives us

$$\omega_p(a, da) = -\varpi_p(\alpha, d\alpha). \tag{22}$$

This relation shows that the *relative components* becomes the *absolute components with a sign change* when we apply a change of parameters consisting of attributing to one transformation the *parameters* that the inverse transformation has.

Example. Consider the group of linear transformations. The inverse of $x' = ax + b$ is $x' = \alpha x + \beta$, where

$$\alpha = \frac{1}{\alpha}, \qquad \beta = -\frac{b}{a}.$$

According to paragraph **§71** (p. 82)

$$\omega_1(a, b, da, db) = \frac{da}{a}, \qquad \omega_2(a, b, da, db) = \frac{db}{a}.$$

The formulae (22) give us

$$\varpi_1(\alpha, \beta, d\alpha, d\beta) = -\frac{da}{a} = \frac{d\alpha}{\alpha},$$
$$\varpi_2(\alpha, \beta, d\alpha, d\beta) = -\frac{db}{a} = \alpha\, d\left(\frac{\beta}{\alpha}\right) = d\beta - \frac{\beta}{\alpha} d\alpha.$$

This result conforms well to the calculation in paragraph **§74**.

III. Three theorems concerning infinitesimal displacement components of a moving frame

76 Fundamental condition of equality [†]**.** *Consider a moving frame* $\mathbf{R}_a$ *and two families of frames* $\mathbf{R}_u$ *and* $\mathbf{R}_v$ *generated by this moving frame. Suppose that a bijective correspondance has been established between the frames* $\mathbf{R}_u$ *and* $\mathbf{R}_v$*. For the same transformation* T *to make the corresponding frames in the two families coincide, it is necessary and sufficient that the correspondance searched for makes the relative components* $\omega_p(u, du)$ *of the frame* $\mathbf{R}_u$ *equal to the relative components* $\omega_p(v, dv)$ *of the frame* $\mathbf{R}_v$*.*

[Phrased another way: if the frames $\mathbf{R}_u$ and $\mathbf{R}_v$ always have the same relative position, then the relative components of their infinitesimal displacements are the same, and *vice versa.*]

Proof. Let $\mathbf{R}_0$ be the absolute frame and S_u, S_v the transformations transforming $\mathbf{R}_0$ to $\mathbf{R}_u$ and $\mathbf{R}_v$ respectively.

The hypothesis

$$\mathbf{R}_u = T\mathbf{R}_v$$

entails

$$S_u = TS_v,$$

from which, by derivation,

$$S_{u+du} = TS_{v+dv}.$$

The last two relations results in

$$S_u^{-1}S_{u+du} = S_v^{-1}S_{v+dv},$$

hence

$$\omega_p(u, du) = \omega_p(v, dv).$$

Conversely, the identity

$$\omega_p(u, du) = \omega_p(v, dv).$$

entails the identity

$$S_u^{-1}S_{u+du} = S_v^{-1}S_{v+dv},$$

which can be written as

$$S_{u+du}S_{v+dv}^{-1} = S_uS_v^{-1},$$

this last relation shows that $S_uS_v^{-1}$ is a fixed transformation T, and we have

$$\mathbf{R}_u = T\mathbf{R}_v.$$

[†] This theorem generalises the equality conditions stated in paragraph §**7** and §**26** (p. 18 and p. 34).

77 Structure theorem. We are going to generalise the structure theorems in paragraphs §**7** and §**26** (p. 18 and p. 34) by considering the following problem:

Consider a moving frame $\mathbf{R}_a$ possessing relative components. Can we apply a displacement of this moving frame, which is a function of one variable t, such that its position for $t = 0$ is a given frame $\mathbf{R}_1$ and its relative components are given forms $\lambda_1(t)dt, \ldots, \lambda_r(t)dt$?

The point $a(t)$ in the parameter space corresponding to the frame searched for is the point that coincides with the image $a(0)$ of $\mathbf{R}_1$ for $t = 0$ and whose coordinates satisfy the r differential equations

$$\omega_1(a, da) = \lambda_1(t)dt, \qquad \ldots, \qquad \omega_r(a, da) = \lambda_r(t)dt. \tag{23}$$

As $\omega_1, \ldots, \omega_r$ are linearly independent, the equations (23) express $\dfrac{da_1}{dt}, \ldots, \dfrac{da_r}{dt}$ as functions of $a_1, \ldots, a_r$ and t. The integration of this differential system however presents some difficulties, and here is an example: suppose that $\mathbf{R}_a$ is the family of right rectangular trihedrals whose origin is at a distance less than 1 from the origin of the absolute trihedral. This family of frames possesses relative components, but the integration of system (23) is not always possible, since they may lead to any right tri-rectangular trihedral. The integration of system (23) is guaranteed to be possible only in the cases where we enlarge the given family to all right tri-rectangular trihedrals.

Similarly, in general, for the integration of the system (23) to be always possible, it is necessary that we enlarge the range of definition of the given frame. This *enlargement* consists of adjoining to the set of given positions of the moving frame all positions that we obtain by transforming these positions a number of times by the transformations leading one of the frames to another. Analogous considerations to those at the beginning of the following paragraph allow us to show that the moving frame thus obtained depends on the same number of parameters and possesses relative components as well. These proofs are long and not very instructive: we will not pursue them.

Finally the following theorem holds for the moving frames whose *range of definition has been thus enlarged*:

Structure theorem. *The components of infinitesimal displacements of a moving frame are not subject to any structure conditions when we vary this moving frame as a function of one parameter.*

N.B. This theorem applies not only to the relative components, but to the absolute components as well: according to paragraph §**75** [formula (22), p. 88], a change of parameters actually allows us to consider the forms $-\varpi_p$ as the relative components of a frame.

78 Relation between groups and the moving frames possessing the relative components. Consider a connected and finite dimensional family of transformations S_a whose moving frame $\mathbf{R}_a$ possesses relative components. Let us enlarge the range of definition of $\mathbf{R}_a$ such that the structure theorem above holds.

Let us specify three positions $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{R}_3$ of $\mathbf{R}_a$. Consider a frame $\mathbf{R}(t)$ belonging to the family $\mathbf{R}_a$ depending on the parameter t and coinciding with $\mathbf{R}_1$ for $t = 0$ and with $\mathbf{R}_2$ for $t = 1$. The structure theorem affirms the existence of a frame $\mathbf{R}^\star(t)$ belonging to the family $\mathbf{R}_a$ coinciding with $\mathbf{R}_a$ for $t = 0$ and whose relative components are equal to those of $\mathbf{R}(t)$. Let $\mathbf{R}_4$ be the position $\mathbf{R}^\star(t)$ occupies for $t = 1$. The fundamental condition of equality affirms that the transformation that transforms $\mathbf{R}_1$ to $\mathbf{R}_3$ transforms $\mathbf{R}(t)$ to $\mathbf{R}^\star(t)$, and therefore in particular $\mathbf{R}_2$ into $\mathbf{R}_4$. This shows that the transformation transforming any frame $\mathbf{R}_1$ of the family $\mathbf{R}_a$ to another $\mathbf{R}_3$ transforms every frame $\mathbf{R}_2$ of this family into a frame $\mathbf{R}_4$ belonging to the same family.

The set of transformations transforming any one of the frames $\mathbf{R}_a$ to any other therefore constitutes a group. This group consists of the transformations T_a transforming one particular position $\mathbf{R}_1$ of $\mathbf{R}_a$ to the set of all other positions of the frame $\mathbf{R}_a$:

$$\mathbf{R}_a = T_a \mathbf{R}_1. \tag{24}$$

If we choose $\mathbf{R}_1$ as an absolute frame, the moving frame considered becomes the moving frame of the group T_a. This moving frame therefore possesses absolute components as well (c.f. §**74**, last part). According to paragraph §**75** a change of parameters swaps the role of the absolute components and the relative components. We can therefore affirm, conversely, that the existence of the absolute components entails the existence of the relative components. Hence *the relative components and the absolute components exist simultaneously.*

The formula (24) can be written

$$T_a + S_a S_1^{-1}.$$

Hence the following proposition: consider a family of transformations S_a whose moving frame $\mathbf{R}_a$ possesses the components and let S_1 be an arbitrarily chosen particular transformation of S_a. The transformations $S_a S_1^{-1}$ constitute a group, at least if the structure theorem applies to the frame $\mathbf{R}_a$ without needing to enlarge its range of definition. The transformations $S_a S_1^{-1}$ belong to a connected and finite dimensional group and their images in the parameter space of the group constitute a part of this space having the same number of dimensions as the space itself: such a set of transformations will be called the *kernel of the group*. This set, containing the identity transformation for $S_a = S_1$, constitute the a certain neighbourhood of the identity transformation inside the group.

Example. The moving frame of the real transformations

$$x' = ax^3 + b \qquad (a > 0; -1 < b < 1),$$

possesses relative and absolute components (c.f. §**71** and §**74**, examples). Let us choose the transformation $x' = x^2$ for S_1. $S_a S_1^{-1}$ is the transformation

$$x' = ax + b \qquad (a > 0; -1 < b < 1).$$

These transformations constitute a kernel of the group

$$x' = ax + b \qquad (a > 0).$$

Let us restrict to the case where the family S_a contains the identity transformation [†]: set $S_1 = 1$. The formula (24) reduces to $T_a = S_a$, and our conclusions take the following form:

First fundamental theorem in group theory (Sophus Lie). *Consider a connected and finite dimensional family of transformations possessing inverses for all elements. Suppose that this family contains the identity transformation. For this family to be a kernel of a group it is necessary and sufficient that the infinitesimal displacement of its moving frame possess relative components [or absolute components].*

This theorem allows us to know practically if a family of transformations constitutes a kernel of a group, since we have practical methods to know if the relative [or absolute] components exist (c.f. §**71** and §**74**).

Henceforth we will not consider any family of transformations that is not a group or any frame that is not a particular position of a moving frame of a group.

IV. Parameter groups

79 Definition of parameter groups. Consider a transformation group S_a and its moving frame $\mathbf{R}_a$. Each transformation S_a transforms every frame $\mathbf{R}_\xi$ into another frame $\mathbf{R}_{\xi'}$. We will use Θ_a to denote the transformation acting on the parameter space that associates the point ξ' to the point ξ. S_a are the transformations transforming the frames $\mathbf{R}_\xi$ into each other. It follows that Θ_a constitute a group, which will be called the *parameter group.*

In other words the most general transformation Θ_a of the parameter group is defined as the following:

We write

$$\xi' = \Theta_a \xi \qquad \text{when} \qquad S_{\xi'} = S_a S_\xi. \tag{25}$$

The relation $S_c = S_a S_b$ entails $\Theta_c = \Theta_a \Theta_b$: the correspondance existing between the operations S_a and Θ_a of a group and its associated parameter group therefore associates products of transformations with products of associated transformations, multiplied in the same order. We express this by saying that they respect *the law of composition of these groups.* We say that two groups are *holoedrically isomorphic* [‡] if we can establish a bijective correspondance between their transformations respecting their laws of transformation. *Therefore every group is holoedrically isomorphic to its parameter group.*

[†] Recall that a group necessarily contains the identity transformation.

[‡] An “holoedric isomorphism” is simply an isomorphism in modern terminology. —Translator.

On the other hand the transformation transforming a frame to another is unique. Therefore there exists one and only one transformation in the group Θ_a that allow us to transform a given point ξ to another given point ξ'. We express this by saying that *the parameter group is simply transitive.*

80 Pfaffian forms left invariant by parameter groups. Consider a variable point ξ in the parameter space and a fixed transformation Θ_a of the parameter group and the point $\xi' = \Theta_a \xi$. The fundamental condition for equality (p. 89) tells us that

$$\omega_p(\xi', d\xi') = \omega_p(\xi, d\xi).$$

We express this condition by saying that *the parameter group leaves the r Pfaffian forms $\omega_p(\xi, d\xi)$ invariant.*

Let us determine all differential forms $\Omega(\xi, d\xi)$ left invariant by the parameter group. The forms $\omega_p(\xi, d\xi)$ are linearly independent (c.f. §**70**, p. 93). Every differential form $\Omega(\xi, d\xi)$ can therefore be written

$$\Omega(\xi, d\xi) = a_1(\xi)\omega_1(\xi, d\xi) + \cdots + a_r(\xi)\omega_r(\xi, d\xi).$$

Requiring that they are invariant under the transformation Θ_a transforming ξ into ξ':

$$a_1(\xi)\omega_1(\xi, d\xi) + \cdots + a_r(\xi)\omega_r(\xi, d\xi) = a_1(\xi')\omega_1(\xi', d\xi') + \cdots + a_r(\xi')\omega_r(\xi', d\xi').$$

But we know that $\omega_p(\xi, d\xi) = \omega_p(\xi', d\xi')$, hence

$$[a_1(\xi) - a_1(\xi')]\omega_1(\xi, d\xi) + \cdots + [a_r(\xi) - a_r(\xi')]\omega_r(\xi, d\xi) = 0.$$

Then $a_p(\xi) = a_p(\xi')$. On the other hand the points ξ and ξ' can be arbitrarily chosen, therefore each of the functions a_p reduces to a constant.

Henceforth we give the name *relative [or absolute] components of an infinitesimal displacement of a moving frame $\mathbf{R}_\xi$ not long to the forms $\omega_p(\xi, d\xi)$ [or $\varpi_p(\xi, d\xi)$], but also to their linear combinations of constant coefficients.*

The conclusion of this paragraph can be stated as the following: *the only forms $\Omega(\xi, d\xi)$ left invariant by the parameter group are the relative components of the infinitesimal displacement of the frame $\mathbf{R}_\xi$.*

The converse of this proposition will be established in paragraph §**83**, p. 96.

81 Infinitesimal transformations of parameter groups. Consider a transformation Θ_ε of infinitesimal parameters ε_i. It transforms the point ξ into the point $\xi + \delta\xi$ defined by the relation $\mathbf{R}_{\xi+\delta\xi} = S_\varepsilon \mathbf{R}_\xi$. This relation can be written

$$S_{\xi+\delta\xi} = S_\varepsilon S_\varepsilon \qquad \text{or rather} \qquad S_{\xi+\delta\xi} S_\xi^{-1} S_\varepsilon.$$

It is therefore equivalent to the r relations

$$\varpi_p(\xi, \delta\xi) = \varepsilon_p. \tag{26}$$

Then the symbol of the infinitesimal transformation Θ_ε can be obtained by eliminating $\delta\xi_p$ from the equations (26) and the equation

$$\delta f(\xi_1,\ldots,\xi_r) = \frac{\partial f(\xi_1,\ldots,\xi_r)}{\partial \xi_1}\delta\xi_1 + \cdots + \frac{\partial f(\xi_1,\ldots,\xi_r)}{\partial \xi_r}\delta\xi_r.$$

Remark. The forms $\varpi_p(\xi, d\xi)$ being linearly independent, the differential df can be expressed in one and only one way in the form

$$df = \varpi_1(\xi, d\xi)A_1 f + \cdots + \varpi_r(\xi, d\xi)A_r f, \tag{27}$$

the symbols $A_p f$ representing the independent infinitesimal transformations of the differentials $d\xi_p$. When we have put df under this form, the symbol of the most general infinitesimal transformation of the first parameter group can be obtained immediately: it is

$$\delta f = \varepsilon_1 A_1 f + \cdots + \varepsilon_r A_r f. \tag{28}$$

Example. Consider the group of linear transformations

$$x' = ax + b \qquad (a > 0).$$

Recall that (c.f. p. 82 and p. 86)

$$\omega_1(a, b, da, db) = \frac{da}{a}, \qquad \omega_2 = \frac{db}{a}, \qquad \varpi_1 = \frac{da}{a}, \qquad \varpi_2 = db - b\frac{da}{a}.$$

Consider the transformation S_a

$$x' = ax + b$$

and the transformation S_ξ

$$x' = \xi x + \eta.$$

The transformation $S_a S_\xi$ has the equation

$$x' = a\xi x + a\eta + b.$$

The transformation Θ_a of the first parameter group therefore transforms the point (ξ, η) into a point with coordinates

$$\xi' = a\xi, \qquad \eta' = a\eta + b. \tag{29}$$

We verify without difficulty that

$$\omega_p(a\xi, a\eta + b; a\,d\xi, a\,d\eta) = \omega_p(\xi, \eta; d\xi, d\eta).$$

The most general infinitesimal transformation of the first parameter group is obtained by setting

$$a = 1 + \varepsilon_1, \qquad b = \varepsilon_2.$$

According to the formula (29) it transforms the point (ξ, η) into a point $(\xi + \delta\xi, \eta + \delta\eta)$ defined by the two equations

$$\delta\xi = \varepsilon_1 \xi, \qquad \delta\eta = \varepsilon_1 \eta + \varepsilon_2,$$

the symbol of this infinitesimal transformation is therefore

$$\delta f = \varepsilon_1 \left(\xi \frac{\partial f}{\partial \xi} + \eta \frac{\partial f}{\partial \eta} \right) + \varepsilon_2 \frac{\partial f}{\partial \eta}. \tag{30}$$

Let us verify that the formulae (27) and (28) give us the same result. (27) is written

$$df = \frac{d\xi}{\xi} \left(\xi \frac{\partial f}{\partial \xi} + \eta \frac{\partial f}{\partial \eta} \right) + \left(d\eta - \eta \frac{d\xi}{\xi} \right) \frac{\partial f}{\partial \eta},$$

this formula reduces to (30) when we replace df by δf, $\varpi_1 = \dfrac{d\xi}{\xi}$ by ε_1 and $\varpi_2 = d\eta - \eta \dfrac{d\xi}{\xi}$ by ε_2.

82 The second parameter groups. Let us use T to denote the transformation of the parameter space associating the point b to every point a such that $S_b = S_a^{-1}$. T is identical to its inverse T^{-1}.

We call the transformed image of the first parameter group by T the *second parameter group*, i.e., the set of transformations $T\Theta_a T$.

Let us transform the reference system chosen in the parameter space by T. The relative and absolute components are, under this transformation, the Pfaffian forms that are originally called $-\varpi_p$ and $-\omega_p$ [c.f. formula (22), p. 88]. The parameter group is, under this transformation, the group that was originalled called the second parameter group. From which we have the following three propositions:

1. A group is holoedrically isomorphic to its second parameter group.

2. The forms left invariant by the second parameter group are the absolute components of the moving frame.

3. Let f be an arbitrary function in $\xi_1, \ldots, \xi_r$ and let us express the differential df under the form

$$df = \omega_1(\xi, d\xi) B_1 f + \cdots + \omega_r(\xi, d\xi) B_r f,$$

the most general infinitesimal transformation of the second parameter group has its symbol

$$\delta f = \varepsilon_1 B_1 f + \cdots + \varepsilon_r B_r f.$$

We denote the most general transformation of the second parameter group by $\Psi_a = T\Theta_a^{-1} T$. The relation $\xi'' = \Psi_a \xi$ is equivalent to the following:

$$\xi'' = T\Theta_a^{-1} T\xi, \qquad T\xi'' = \Theta_a^{-1} T\xi, \qquad S_{\xi''}^{-1} = S_a^{-1} S_\xi^{-1}, \qquad S_{\xi''} = S_\xi S_a.$$

Ψ_a is therefore defined as the following:

We write

$$\xi'' = \Psi_a \xi \qquad \text{when} S_{\xi''} = S_\xi S_a. \tag{31}$$

Let us consider a transformation Θ_a of the parameter group and a transformation Ψ_b of the second parameter group and let ξ' and ξ'' the respective transformed images of ξ by $\Theta_a \Psi_b$ and by $\Psi_b \Theta_a$. We have, according to (25) and (31),

$$\begin{aligned} S_{\xi'} &= S_a S_{\eta'}, \quad \text{where} \quad S_{\eta'} = S_\xi S_b, \\ S_{\xi''} &= S_{\eta''} S_b, \quad \text{where} \quad S_{\eta''} = S_a S_\xi. \end{aligned}$$

Therefore

$$S_{\xi'} = S_a S_\xi S_b, \qquad S_{\xi''} = S_a S_\xi S_b,$$

ξ' and ξ'' are the same, $\Theta_a \Psi_b = \Psi_b \Theta_a$: *each of the transformation of the parameter group is exchangeable with each of the transformation of the second parameter group.*

V. Some problems of integration

83 Construction of parameter groups from relative components. Consider a connected, finite dimensional group S_a. Its parameter group Θ_a leaves the r components $\omega_1(\xi, d\xi)$, …, $\omega_r(\xi, d\xi)$ invariants. We are going to prove that this property characterises this group.

Theorem. *The transformations of the parameter group are the only ones leaving the r forms $\omega_p(\xi, d\xi)$ invariant.*

To establish this proposition we are going to prove the following: given any r linearly independent [(†)] Pfaffian forms $\omega_p(\xi, d\xi)$, there exists at most one transformation of the parameter space into itself leaving them invariant and transforming a given point ξ_0 to another given point ξ'_0.

Let ξ_1 be an arbitrary point in the parameter space and let ξ be a variable point depending on a parameter t and coinciding with ξ_0 for $t = 0$ and with ξ_1 for $t = 1$. Under these conditions the forms $\omega_p(\xi, d\xi)$ reduce to certain forms in t and dt: $\lambda_p(t)dt$.

If the transformation we look for exists, the coordinates of the transformed image ξ' of ξ are functions of t and equal to the coordinates of ξ_0 for $t = 0$ and satisfy the differential system

$$\omega_p(\xi', d\xi') = \lambda_p(t)dt.$$

This system can be solved with respect to the r differentials $d\xi'$ [(†)]. Their integrals that take the given initial values are therefore well defined, and hence the position of the transformed image ξ'_1 of ξ_1 cannot be indeterminate.

Remark. We are not stating that given any r Pfaffian forms $\pi_p(\xi, d\xi)$ there always exists a group of transformations depending on r parameters which leaves them invariant. This proposition is false. Let us find, for example, the transformations that leaves the two forms $\pi_1 = \dfrac{d\xi}{\eta}$ and $\pi_2 = d\eta$ invariant.

The relation $\dfrac{d\xi'}{\eta'} = \dfrac{d\xi}{\eta}$ implies that ξ' must be a function of ξ along. We therefore have $\xi' = f(\xi)$ with $\eta f'(\xi) = \eta'$.

The relation $d\eta' = d\eta$ shows that $f'(\xi)$ cannot only be 1. The transformation we search therefore reduce to the following: $\xi' = \xi + a$, $\eta' = \eta$. There is only a single parameter a.

[(†)] We have established that the forms $\omega_p(\xi, d\xi)$ are linearly independent (§**71**, p. 82).

The components ω_p of the infinitesimal displacement of a moving frame satisfy certain structure conditions. The study of these conditions is the subject of study in the third part of this work.

84 Complements. The preceding theorem can be thought of as a corollary of the following [†].

THEOREM. *Consider a R dimensional space of coordinates $X_1, \ldots, X_R$ and N ($< R$) independent Pfaffian forms $\pi_1(X, dX), \ldots, \pi_N(X, dX)$.* We will call the *integral varieties* of the Pfaffian system

$$\pi_1(X, dX) = 0, \qquad \ldots, \qquad \pi_N(X, dX) = 0$$

all varieties of the space that annihilate these forms when the point $(X_1, \ldots, X_R)$ describes it. *We claim that for a given point $\mathbf{M}_0$ there passes at most one $R-N$ dimensional integral variety.*

Indeed, let us trace through $\mathbf{M}_0$ any $N+1$ dimensional variety

$$G_1(X_1, \ldots, X_R) = 0, \qquad \ldots, \qquad G_{R-N-1}(X_1, \ldots, X_R) = 0,$$

on every point of which the forms

$$\pi_1(X, dX), \qquad \ldots, \qquad \pi_N(X, dX), \qquad dG_1, \qquad \ldots, \qquad dG_{R-N-1}$$

are linearly independent. The curve that is the intersection of this $N+1$ dimensional variety and the variety we are looking for cannot be indeterminate, since it is the integral through $\mathbf{M}_0$ of the differential system

$$\pi_1(X, dX) = 0, \qquad \ldots, \qquad \pi_N(X, dX) = 0, \qquad dG_1 = 0, \qquad \ldots, \qquad dG_{R-N-1} = 0.$$

This suffices to establish the stated theorem.

Remark I. Observe that determining the integral varieties of a Pfaffian system becomes the integration of differential systems.

Remark II. Of course, given arbitrarily N Pfaffian forms $\pi_1(X, dX), \ldots, \pi_N(X, dX)$, there does not necessarily pass through each point of $\mathbf{M}_0$ of the space $(X_1, \ldots, X_R)$ a $R-N$ dimensional integral variety of the system

$$\pi_1 = 0, \qquad \ldots, \qquad \pi_N = 0.$$

For example the particular system

$$A(X, Y, Z)dX + B(X, Y, Z)dY + C(X, Y, Z)dZ = 0$$

does not have such an integral variety unless the vector (A, B, C) is always perpendicular to its rotation. Systems possessing this property are called *completely integrable.*

[†]It suffices to set $R = 2r$, $N = r$, $X_1 = \xi_1, \ldots, X_r = \xi_r$, $X_{r+1} = \xi'_1, \ldots, X_R = \xi'_r$, $\pi_{r+1} = \omega_1(\xi, d\xi) - \omega_1(\xi', d\xi'), \ldots, \pi_N = \omega_r(\xi, d\xi) - \omega_r(\xi', d\xi')$.

85 Construction of a group from its infinitesimal transformations (first procedure). Consider a group of transformations S_a operating on a domain D. Let $X_1, \ldots, X_r$ be the infinitesimal transformations of this group. Consider a point $\mathbf{M}_0$ of D whose coordinates are $(x_1^0, \ldots, x_n^0)$ and its transformed image $\mathbf{M}_1(x_1^1, \ldots, x_n^1)$ by S_a. Let us introduce a variable transformation $S(t)$ belong to the group which depends on the parameter t and which coincides with the identity for $t = 0$ and with S_1 for $t = 1$. Let $\mathbf{M}(x_1, \ldots, x_n)$ be the transformed image of $\mathbf{M}_0$ by $S(t)$.

When t undergoes the increment dt, $\mathbf{M}$ undergoes the infinitesimal transformation $S(d + dt) \cdot S(t)^{-1}$ and $x_1, \ldots, x_n$ hence undergoes the increment

$$\begin{aligned}
dx_1 &= \varpi_1(t, dt) X_1 x_1 + \cdots + \varpi_r(t, dt) X_r x_1, \\
&\ldots \\
dx_n &= \varpi_1(t, dt) X_n x_1 + \cdots + \varpi_r(t, dt) X_r x_n.
\end{aligned}$$

If we think the $\varpi_p(t, dt)$ as given, these equations constitute a differential system whose integral $x_1(t), \ldots, x_n(t)$ corresponding to the initial values $x_1^0, \ldots, x_n^0$ is unique. The coordinates $x_1^1, \ldots, x_n^1$ are the values that the functions $x_1(t), \ldots, x_n(t)$ take for $t = 1$. But the structure theorem of paragraph §**77** (p. 90) tells us that $\varpi_1(t, dt)$, $\ldots$, $\varpi_r(t, dt)$ are arbitrary Pfaffian forms in t: $\lambda_1(t)dt, \ldots, \lambda_r(t)dt$. We can therefore construct the group from its infinitesimal transformations, by the following procedure:

Choose r bounded functions $\lambda_1(t), \ldots, \lambda_r(t)$ arbitrarily. We search for the general solution $x_i(t)$ of the differential system

$$\frac{dx_i}{dt} = \lambda_1(t) X_1 x_i + \cdots + \lambda_r(t) X_r x_i, \qquad (i = 1, \ldots, n), \tag{32}$$

and we correspond to a point $\mathbf{M}_0$ of coordinates $x_i(0)$ the point $\mathbf{M}_i$ of coordinates $x_i(1)$. We then obtain the most general transformation of the group.

Under a more intuitive form: the most general transformation of the group is the product of an infinity of infinitesimally small transformations of components $\lambda_s(t)dt$. To remember this we usually say that *the infinitesimal transformations of the group generate the group.*

N.B. We do not claim that we can find a group possessing arbitrarily chosen infinitesimal transformations. This proposition is false. The infinitesimal transformations of a group satisfy structure equations that we will state in the third part of this work (complement to the second fundamental theorem, p. 234).

86 Construction of a group from its infinitesimal transformations (second procedure). Let us apply the following construction:

Let us specify r arbitrary constants $a_1, \ldots, a_r$ and search the general solution $x_i(t)$ of the differential system

$$\frac{dx_i}{dt} = a_1 X_1 x_i + \cdots + a_r X_r x_i, \qquad (i = 1, \ldots, n), \tag{33}$$

and let us correspond to the point $\mathbf{M}_0$ *of coordinates* $x_i(0)$ *the point* $\mathbf{M}_1$ *of coordinates* $x_i(1)$.

The transformations thus obtained belong to the group and they depend on the r parameters $a_1, \ldots, a_r$ and not on any smaller number of parameters, since $X_1, \ldots, X_r$ are linearly independent. They therefore constitue *a kernel of the group* (†) that we aim to construct. This kernel contains the identity transformation.

We claim that *we obtain the most general transformation of the group by multiplying the transformation belonging to the kernel of the group by a finite number* (‡). Indeed, given any transformation S of the group, we can link it to the identity transformation by a series of transformations

$$1, \qquad S_1, \qquad \ldots, \qquad S_n, \qquad S,$$

such that all the transformations

$$S_1, \qquad S_1^{-1}S_2, \qquad S_2^{-1}S_3, \qquad \ldots, \qquad S_{n-1}^{-1}S_n, \qquad S_n^{-1}S$$

are close enough to the identity to belong to the kernel of the group [15]. But

$$S = S_1 \cdot S_1^{-1}S_2 \cdot S_2^{-1}S_3 \ldots S_{n-1}^{-1}S_n \cdot S_n^{-1}S.$$

87 Construction of the set of transformed images of a point by the operations of a group for which we know the infinitesimal transformations. Consider a group operating on a domain D whose infinitesimal transformations are $X_1, \ldots, X_r$. We say that two points of D are *homologous* when a transformation in the group allow us to transform one of the points into the other. The homologous points of a point $\mathbf{M}_0$ of coordinates $x_1^0, \ldots, x_n^0$ constitute the set of points $\mathbf{M}$ whose coordinates can be obtained by integrating the system (32), where we take $\lambda_1(t), \ldots, \lambda_r(t)$ to be any functions possible, and the initial values are

$$x_1(0) = x_1^0, \qquad \ldots, \qquad x_n(0) = x_n^0.$$

In other words, for two points of D to be homologous, it is necessary and sufficient that one moving point in D can go from one to the other with velocity within a linear combination with bounded coefficients of the r vectors

$$(X_1x_1, \ldots, X_1x_n), \qquad (X_2x_1, \ldots, X_2x_n), \qquad (X_rx_1, \ldots, X_rx_n). \tag{34}$$

For example, if $r = n$, two points of D are necessarily homologous when we can join them by a path along which the determinant $|X_px_i|$ does not vanish.

If $r \geq n$, two points of D are necessarily homologous when we can join them by a path along which the matrix $\|X_px_i\|$ is of constant rank n.

(†) In paragraph **§201** we will show that, for example in the case of the group of linear transformations of two variables, that this kernel of the group cannot be used to recover the whole group. [*See* p. 219 the discussion of the transformations obtained by integrating the system (33)].

(‡) We remember this fact by saying that the kernel of the group *generates* the group.

If $r < n$, the homologous points of a given point constitute a r dimensional variety whose tangent hyperplane contains the r vectors (34). The functions $f(x_1, \ldots, x_n)$ which are constants on this variety are therefore those that satisfy the system

$$X_1 f = 0, \qquad \ldots, \qquad X_r f = 0. \tag{35}$$

Then the left hand sides of the equations

$$f_1(x_1, \ldots, x_n) = c_1, \qquad \ldots, \qquad f_{n-r}(x_1, \ldots, x_n) = c_{n-r}$$

of this variety have $n - r$ independent integrals of the system (35).

Chapter 6

On various relations that may exist between two groups

I. Similar groups

88 Consider a transformation S operating on a n dimensional domain D:

$$x_i' = \varphi_i(x_1, \ldots, x_n). \tag{1}$$

Let us use, instead of the old coordinate system $\mathbf{R}_0$, a new coordinate system $\mathbf{R}_1$. The same geometrical operation then translates into a new analytic transformation S_i

$$X_i' = \Phi_i(X_1, \ldots, X_n). \tag{2}$$

We say that the transformations (1) and (2) are *similar*.

Example. Suppose that $\mathbf{R}_1$ is the transformed image of a system of coordinates $\mathbf{R}_0$ by a transformation T transforming D into itself: $\mathbf{R}_1 = T\mathbf{R}_0$. S_1 is then the transformed image [(†)] of S by T^{-1}, $S_1 = T^{-1}ST$. A transformation is therefore similar to all of its transformed images.

Similarly, given two *groups* G and G' operating on two domains D and D', we say that they are *similar* when, by means of a suitable change of coordinates of the domain D, the transformations of G have the same analytic expressions as those of the group G'. In other words there exists a bijective correspondence C between the points of D' and D that identifies the transformations of G' with those of G. We say that G' is the *transformed image* of G by C.

Two similar groups can be considered geometrically identical: to study one of them is to study the other.

Example. The group of rotations around a point $\mathbf{A}$ is similar to the group of rotations around another point $\mathbf{A}'$. The change of coordinates that proves this similitude is the transformation $\overrightarrow{\mathbf{AA}'}$. Also, this translation and its rotations belong to the displacement group.

[(†)] C.f. *Remarks* (§**61**, p. 72).

Now consider in place of the displacement group an arbitrary group G, in place of rotations around $\mathbf{A}$ a subgroup g of G, and in place of the translation $\overrightarrow{\mathbf{AA}'}$ a transformation S of G not in g and in place of the rotations around $\mathbf{A}'$ the transformations SgS^{-1} which are the transformed images of g by S and constitute a subgroup of G, namely the transformed image of the subgroup g by S. This subgroup is similar to g.

The preceding examples show the utility of the following definition: two transformations of a group G are said to be *homologous* when one is the transformed image of the other by a transformation of G.

Similarly two subgroups of a group G are said to be homologous when one of them is the transformed image of the other by a transformation in G. For two transformations or two subgroups to be homologous, it is hence necessary and sufficient that we can find, in the family of transformed images of the absolute frame of G, two frames relative to which they have the same equations.

Finally, note that two transformations or two homologous subgroups homologous to a third are homologous themselves. The relations

$$S_1 = S_a^{-1} S S_a \qquad \text{and} \qquad S = S_b^{-1} S_2 S_b$$

entails that

$$S_1 = (S_b S_a)^{-1} S_2 (S_b S_a).$$

II. Notion of isomorphism

89 Introduction. The various aspects presented by the displacement group. Consider the displacement group of space. It has equations that are essentially different according to whether coordinates are the point coordinates or the tangent coordinates. More precisely, we can consider the group not only by its points, but also by the planes: we obtain two groups G and G' which are not similar. Let us generalise: let each of the "objects" consist of a set of points (for example planes, lines, spheres, ellipsoids). We know what transforming one of these objects to another is, and let us consider a class of such objects containing the transformed images of each such object in the class by all displacements. The displacement group operates on the objects in the class. It takes different forms according to the choice of the class: the group that we obtain are not similar if we consider the class of planes, of lines, of ellipsoids, of ellipsoids whose axes are of length 1, 2 and 3, of ellipsoids of revolution, of spheres, or of spheres of radius one. The study of the group G is therefore linked to the study of a considerable number of other groups.

It is important to note that we are led to similar groups by the consideration of the class of points and by those of the spheres of radius 1. Indeed we can establish between these two classes a bijective correspondance such that two homologous objects remain homologous after undergoing the same displacement. Two such classes are called *equivalent*.

However, on the other hand, the elementary geometry defines the displacement of an oriented line and of a direction of lines: we can speak of the classe of oriented lines and of the class of directions of lines. We establish easily that each of these classes are not equivalent to the class of objects each of which is constituted by a set of points. Here we are led to the search of the more general "objects" than those defined in the last paragraph. We will manage to achieve this by a recurrence method: having defined certain classes of objects, we will consider the new objects that are sets whose elements are the objects already defined. Displacing one of these sets by a displacement is displacing each of its elements by this displacement.

Hence let us consider the objects ω_1 constituted by three aligned points $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ such that the translation $\overrightarrow{\mathbf{AB}}$ repeated three times transforms $\mathbf{A}$ into $\mathbf{C}$: we have a class of objects equivalent to those of the vectors $\overrightarrow{\mathbf{AC}}$. Let us then introduce the objects ω_2 each of the objects of which is constituted by a set of objects $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ corresponding to the vectors $\overrightarrow{\mathbf{AC}}$ along the same line and having the same direction: the class of objects ω_2 is equivalent to those of oriented lines. Let us now introduce the class of objects ω_3 each object of which is constituted by the set of objects ω_2 corresponding to parallel oriented lines: we have a class of objects equivalent to those of the directions of oriented lines.

Remark. Every translation is equivalent to the identical transformation when we act them on the objects ω_3.

90 Definition of isomorphism. Given a class ω of objects constructed by the recurrence procedure indicated in the preceding paragraph, the displacements define a group Γ of transformations operating on this class. And the following property connects the group Γ to the displacement group G:

There exists a correspondence between the transformations of G and of Γ. To every transformation of Γ there corresponds at least one transformation of G and to every transformation of G there corresponds only one transformation of Γ. To the product of two transformations of G always corresponds the product, multiplied in the same order, or the homologous transformations of Γ.

We express the fact that two groups G and Γ possessing these properties by saying that Γ *is homomorphic or meriedrically isomorphic to G.* This notion of meriedric isomorphism which we hinted at in paragraph §**89** is of the greatest importance.

If the correspondence existing between the transformations of G and the transformations of Γ is bijective, then the two groups are said to be *holoedrically isomorphic.* This is the case for a group and its parameter group (c.f. §**79**, p. 92). [†]

Examples and remarks. The displacement group operating on the directions of oriented straight lines is meriedrically isomorphic to the displacement group operating on the points, but it is not holoedrically isomorphic to it: to the identical transformation of the first correspond

[†] A "meriedric isomorphism" is a surjective homomorphism in modern terms, where as a "holoedric isomorphism" is simply an isomorphism. —Translator.

all translations in the second. If a first group is meriedrically isomorphic to a second and if this second is itself meriedrically isomorphic to a third, then the first group is meriedrically isomorphic to the third. Similarly two groups holoedrically isomorphic to the same group are holoedrically isomorphic themselves.

Consider a group G and a group Γ meriedrically isomorphic to G. Γ operate on a set of points or objects Δ. We consider a transformation of G which also operate on Δ, its action being by definition that of the homologous transformation of Γ: *this convention* is practical, since it does not alter the way the transformations of G are composed of and it is in accordance with the conventions in paragraph §**89** relative to the displacement group.

Example. Suppose that Δ is the set of frames of G and Γ is the parameter group. G and Γ are isomorphic. The convention that we have just given coincides with the definition previously stated of the transformed image of a frame by an operation of G.

91 The classes of objects a given group transforms. We will generalise the considerations of paragraph §**89** in this paragraph. Consider, instead of the displacement group, any group G. Consider a set of objects ω whose transformed images are defined by G and suppose that this set contains the transformed images of each of its elements by all the transformation in G: we say that it is *a class.* The group of transformations G of the objects of this class will be denoted by the symbol $G(\omega)$. $G(\omega)$ is necessarily homomorphic to G.

Consider such a class ω and its set Ω of objects ω. Let us define the transformed image of one such set Ω by an operation of G as the set of transformed images of its elements. Suppose that the set of Ω contain the transformed images of all its elements. Then the group $G(\Omega)$ of transformations of Ω by the operations of G is meriedrically isomorphic to G. Ω contains one class of objects which G operate on: this remark has furnished for us a recurrence procedure allowing us to construct the classes of objects more and more extensive and new groups meriedrically isomorphic to G.

Remark that we have already used this practical method to make the affine group operate on the class of objects constituted by the vectors (§**65**, p. 75), and the projective group on the class constituted by the analytic points (§**67**, p. 77).

Two groups $G(\Omega_1)$ and $G(\Omega_2)$ can be similar: it is necessary and sufficient that there exists between the objects Ω_1 and Ω_2 a bijective correspondence such that two homologous objects remain homologous after being subjected to the same transformation of G: two classes possessing this property will be said to be *equivalent classes.*

In this chapter we will solve the two following questions, about which it is useful to note that they are entirely equivalent: given a group G, construct the groups which are meriedrically isomorphic to it, such that every group meriedrically isomorphic to G are similar to one of the groups that we have constructed.

Given a group G, construct the classes of objects on which it operates, such that every class of objects transformed by G are equivalent to one of the classes of objectes we have constructed.

III. Identifying objects in a given class of objects

92 Introduction. During the first part of this work, we have always dealt with geometrical objects in a manner already a little abstract: the contact elements of various orders, and we have learnt to attach to each of these contact elements a frame or a family of frame [†] and furthermore, eventually scalar quantities, such that we can always give the following answer to the contact problems:

For two contact elements to be equal, it is necessary and sufficient that the associated system of scalar quantities are identically the same [‡].

The displacements superimposing two equal contact elements are those that transform a frame attached at the first in a frame attached at the second.

In other words, each contact element is characterised by *a family of frames*, relative to which it occupies the same position, and by a system of numbers, *the invariants*, which concerns "its form".

We will show how this procedure, by which we have identified the contact elements of various orders, is of very broad reach.

For this let us distinguish several cases.

Definition. A group G is said to *operate transitively* on a class of objects ω when there always exists a transformation of this group transforming one arbitrary chosen object ω to another arbitrarily chosen object ω.

When this transformation is always unique, we say that the group $G(\omega)$ is *simply transitive.*

Example. Every parameter group is simply transitive.

A group $G(\omega)$ which is not transitive is called *intransitive.*

Example. The displacement group operates transitively on the points, intransitively on the spheres.

93 Identification of a classe of objects ω transformed transitively by a group G. The identification of these objects does not necessitate the use of invariants and is very easy. Let us choose arbitrarily one of these objects ω_1 and a frame $\mathbf{R}_1$. Attach at each object ω the family [§] of frames $\mathbf{R}$ relative to which it occupies the position of ω_1 with respect to $\mathbf{R}_1$. In other words: let g_1 be the subgroup of transformations of G leaving ω_1 fixed

[†] Which are tri-rectangular or cyclic frames.

[‡] This identity is in particular realised when no scalar is attached to the contact elements considered.

[§] This family is obtained by transforming $\mathbf{R}_1$ by the set of transformations of the group which transform ω_1 into ω.

and let S be one of the transformations transforming ω_1 into ω. The frames that we attach at ω are the frames $Sg_1\mathbf{R}_1$. The family of frames thus attached at each object ω characterise it completely: for two objects ω to be identical, it is necessary and sufficient that their family of frames have a comment element. And then they have all elements in common. The transformations transforming one into the other of the two objects ω are those that makes a frame attached at one of them into a frame attached at the other.

Example. During the first part of this work, we have, in various times, *practically* identified a straight line by the family of trihedrals whose first edge is along this line.

Generalities of the operation called orientation. Let us point out a very common circumstance which arises when for example a group G is the group of real displacements and ω is the class of real lines: the transformations of the subgroup g_1 can constitute several distinct connected components. Then let g_1' be the one component containing the identical transformation. The transformations of g_1' are characterised by the property that it can be reduced continuously to the identity in g_1. They constitute the largest connected subgroup of g_1. It is useful to consider the object ω_1' which is constituted by the family of frames $g_1'\mathbf{R}_1$, and the class of its transformed images ω' by the set of transformations of G: the family of frames attached to an object ω decomposes into connected sub-families, each of which is a family of frames attached to an object ω'. The ω' are called "the oriented objects ω", and every object ω will be considered as the union of a finite number or a set of disconnected oriented objects ω'.

94 Identification of a classe of objects ω transformed intransitively by a group G. This class decomposes into subclasses δ each of which is constituted by the set of transformed images of the same object ω. No two subclasses of δ have any common element, and the group G operates transitively on each of them [†].

Let us indicate the example of the displacement group, which operates transitively on the second order contact elements of real curves. In particular it transforms in a special fashion those whose curvature is zero.

We begin by identifying the subclasses δ with the help of a system of numbers, called the invariants. Then the elements of each of the subclasses on which the group G operates transitively are represented as we have represented them in the preceding paragraph. An object ω is then characterised by the following date: the invariants of its subclass.

This procedure of identification of one class of objects is of the greatest practical importance. We have already said that it is the base of the method of moving frames. We are going to use it in paragraphs **§95** and **§96** to solve the problem of constructing all groups isomorphic to a given group.

We will make the construction of subclasses of the class ω precise and make the choice of invariants precise in the following case which is particularly interesting: the objects ω are the

[†] In other words it attaches to each subclass δ a group G_δ transforming the objects constituting δ transitively.

points of a domain D and the group G is defined by the analytic equations

$$x'_i = \varphi(x_1, \dots, x_n; a_1, \dots, a_r). \tag{1}$$

The subclasses δ are the varieties. Let us find those that pass through a given point $\mathbf{A}_0(x_1^0, \dots, x_n^0)$. It is composed of the set of transformed points of $\mathbf{A}_0$ by all the operations of G. Its equations are therefore obtained by eliminating the r parameters a from the n equations

$$x_i = \varphi_i(x_1^0, \dots, x_n^0; a_1, \dots, a_r).$$

Furthermore eliminating these parameters from the following equations have the same effect

$$x_i^0 = \varphi_i(x_1, \dots, x_n; a_1, \dots, a_r).$$

Suppose for example that we can choose the coordinates such that each point $\mathbf{A}$ of D has one and only one transformed image whose first r coordinates are zero (†). Each variety δ has one and only one point $\mathbf{A}_1$ whose first r coordinates are zero. We can choose the other $n-r$ coordinates $x_{r+1}^1, \dots, x_n^1$ of this point as parameters. The equations of this variety is obtained then by resolving with respect to a the r equations

$$0 = \varphi_i(x_1, \dots, x_n; a_1, \dots, a_r) \qquad (i = 1, 2, \dots, r),$$

by substituting the values obtained in the $n-r$ functions

$$\varphi_i(x_1, \dots, x_n; a_1, \dots, a_r) \qquad (i = r+1, \dots, n),$$

by equating the functions in x thus constructed

$$y_1(x_1, \dots, x_n), \qquad \dots, \qquad y_{n-r}(x_1, \dots, x_n)$$

to $n-r$ arbitrary constants $x_{r+1}^i, \dots, x_n^i$ (‡).

A more general procedure furnishing the varieties δ is the following: given a point $\mathbf{A}$ of D we each a transformed image $\mathbf{A}_0$ of this point satisfying certain conditions that determine it uniquely which are independent of $\mathbf{A}$: the parameters the points $\mathbf{A}_0$ depend on are functions of the coordinates of $\mathbf{A}$ which are *invariants by every transformation of the group* (§), and by equating them to arbitrary constants we obtain the equations of the varieties δ.

IV. Groups isomorphic to a given group

95 Finding transitive groups meriedrically isometric to G. Consider a class of objects ω transformed transitively by G. Let g_1 be the subgroup of the transformations of G leaving the particular object ω_1 fixed. During paragraph **§93** we have considered the family of frames $g_1\mathbf{R}_1$ and the class of the transformed images of this family by all transformations of G. From these considerations it follows that this class is equivalent to the class ω.

(†) This is always possible in a sufficiently small neighbourhood of a *generic* point of the domain D.

(‡) This procedure is due to Sophus Lie.

(§) This expression signifies that the values of the functions in $x_1, \dots, x_n$ remain unaltered when we substitue the quantities $x_1, \dots, x_n$ by their transformed images using any operation of the group.

On this subject, let us define the following: g being a subgroup of G, let us consider the set of points representing it in the parameter space and let c be the class of the transformed images of this set by the operations of G: this class c will be called "*the body defined by* g" [(†)].

According to the preceding, *a class* ω *for which* G *operates transitively is equivalent to a body* c_1 *defined by the subgroup* g_1 *leaving an object* ω_1 *arbitrarily chosen from the class* ω *invariant.*

Let g_2 be the subgroup leaving the object $\omega_2 = S\omega_1$ invariant. $g_2 = Sg_1S^{-1}$ is the transformed image of the subgroup g_1 by S. Hence *two homologous bodies* (which we understand as two bodies defined by two homologous subgroups) *are equivalent.*

Conversely the transformations of G leaving two homologous objects of two equivalent bodies invariant constitute the same subgroup. This subgroup, as we have just seen, is homologous to those defining these two bodies: *two equivalent bodies are therefore homologous.*

We are now able to indicate the procedure for resolving two equivalent problems posed at the end of paragraph **§91** (p. 104), provided that we limit ourselves to the search of transitive groups:

We construct a set of subgroups of g, whose homologues will constitute all the subgroups of G. We consider the body c defined by them. Every class of objects transformed transitively by G is equivalent to one of these bodies. The transitive groups meriedrically isomorphic to G are the groups similar to the groups $G(c)$.

96 Finding intransitive groups meriedrically isometric to G. Paragraph **§94** leads to the following considerations. Consider a set of bodies c_s where the same body may be represented several times in it. Let C be the union of all the objects of these bodies. Two elements of C are considered identical only in the case where they belong to the same c_s and are identical in this body. Every class transformed *intransitively* by G is equivalent to *one such union* C *of bodies* c_s, and every intransitive group meriedrically isomorphic to G is similar to one of the groups $G(C)$.

Remark. A union C of bodies c_s is transformed into an equivalent union when we substitue one or several bodies c_s with their homologous bodies.

97 Distinction between holoedric and meriedric isomorphisms. $G(\omega)$ can fail to be holoedrically isomorphic to G: this is the case when the transformations γ_ω common to the subgroups of G leaving the various elements objects ω fixed do not reduce to the identity. These transformations γ_ω obviously constitute a subgroup of G. For two transformations S' and S'' of G to operate on the class ω in the same way, it is necessary and sufficient that $S'^{-1}S''$ is a transformation γ_ω.

Example. The complete study of the group G of displacement entails the study of every group meriedrically isomorphic to it, therefore in particular the study of the way that G operate on

[(†)] Every body is transformed transitively by G.

the points of infinite ω, i.e., the study of the projective transformations of the plane leaving a conic fixed. This group $G(\omega)$ is not holoedrically isomorphic to G: the transformations γ_ω are translations. The study of projection transformations of the plane leaving a conic fixed does not entail the study of the group of displacements of the space.

Let us indicate an important property of the subgroup γ_ω: whatever the object ω, the transformation γ_ω and the transformation S of G are, we must have $\gamma_\omega S\omega = S\omega$, therefore $S^{-1}\gamma_\omega S\omega = \omega$. Then the transformed images of the transformations γ_ω by the various operations of G also belong to the subgroup γ_ω. A subgroup possessing this property of being identical to all of its homologues is called an *invariant subgroup.*

If an invariant subgroup of G is contained in another subgroup of G, it is manifestly contained in all the transformed images of this last group. From this follows the following proposition. Suppose that $G(\omega)$ is transitive and let ω_1 be one of the objects ω and g_1 be the transformations leaving ω_1 fixed. Then γ_ω can be defined as largest invariant subgroup of G contained in g_1.

98 Important remarks. The study of groups isomorphic to a given grou G thus amounts to the determination of all the subgroups of G. We will learn in chapter 9 (§**115**, p. 133) to what problems of differential and integral calculus we can reduce this determination to. This determination does not further necessitate the knowledge of the group G itself, but only the knowledge of the way its operations compose, its "law of composition" [(†)], that is to say its parameter groups.

We say that to give such a law is to give *an abstract group.* The preceding teaches us how to realise all the groups whose law of composition is that of the abstract group, *to construct all concrete realisations of a given abstract group*, when we know all the subgroups of this abstract group. The law of composition of a group, that is to say *its structure*, or its parameter group, is therefore what is fundamental about the group.

On this subject let us prove the following theorem:

For two groups to be holoedrically isomorphic, it is necessary and sufficient that their parameter groups are similar. Indeed, consider two isomorphic groups G and G'. By definition we can establish between the transformations of G and G' a bijective corres-

[(†)] Such a law possesses the following characteristics:

It operates on a set of abstract elements. One of these elements is called the identity and is denoted by 1. It associates to two elements given in a certain order, a and b, a third element, called the product and denoted by ab. This product is associative

$$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$

We have

$$a \cdot 1 = 1 \cdot a.$$

To every element a corresponds to an element called the inverse and denoted by a^{-1}, such that

$$a \cdot a^{-1} = a^{-1} \cdot a = 1.$$

pondance associating a product of the transformations of G to the project, multiplied in the same order, of the two homologous transformations of G'. Consider a point ξ of the parameter space of G and its homologue ξ', and a transformation T of the first parameter group of G and its homologue T'. The point $T\xi$ has the homologue $T'\xi'$. The parameter groups of G and G' are therefore similar.

Conversely, consider two groups whose parameter groups are similar. This similitude establishes a correspondence between the points ξ and ξ' of the two parameter groups. We deduce from this with out difficulty an isomorphic between these two groups. Q.E.D.

COROLLARY. *Given a group, if we modify the choice of its parameters, the new parameter group is similar to the original parameter group.*

This paragraph leads naturally to the search for the relations existing between a group and its parameter group. This is what we will study in the following chapter. The conclusions of this chapter will be the basis for chapter 8.

Chapter 7

Relations between a group and its parameter group

I. Simply transitive group

99 Consider a simply transitive group G. Let us choose the coordinates of the origin $\mathbf{A}$ as the frame $\mathbf{R}_0$ [(†)]. The frame $\mathbf{R} = S\mathbf{R}_0$ will be, under these conditions, the point $S\mathbf{A}$. Let us adopt the coordinates of this point as the parameters of the transformation S and of the frame $\mathbf{R}$. The group is then identical to its parameter group.

The corollary and the theorem of paragraph §**98** allow us to deduce from this result the following:

Every simply transitive group is similar to its parameter group.
Two simply transitive groups cannot be isomorphic without being similar.

II. Transitive group

100 Generalities. Now consider a transitive group G which operates on the points of a domain D. Suppose a system of absolute coordinates, $\mathbf{R}_0$, whose origin is a point $\mathbf{A}$ inside the domain D is defined. For any other frame $\mathbf{R} = S\mathbf{R}_0$, we will call the transformed point $\mathbf{M} = S\mathbf{A}$ of $\mathbf{A}$ by the operation S transforming $\mathbf{R}_0$ into $\mathbf{R}$ the origin of this frame. For such a frame $\mathbf{R}$ and the transformation S let us choose the coordinates $x_1, x_2, \ldots, x_n$ of the origin $\mathbf{M}$ of $\mathbf{R}$ and $r - n$ other suitable quantities [(‡)] $u_1, u_2, \ldots, u_{r-n}$ as parameters.

Granted this convention, if a transformation of parameters $a_1, \ldots, a_r$ transforms the point $(x_1, \ldots, x_n)$ into the point $(x'_1, \ldots, x'_n)$ defined by the formulae

$$x'_i = \varphi_i(x_1, \ldots, x_n; a_1, \ldots, a_r), \tag{1}$$

[(†)] It is suppose to be inside the domain where the group acts.

[(‡)] We must assume the subgroup of the transformations of G leaving a point of D fixed to be connected. In other words we assume that the points of D are oriented objects (c.f. §**93**, p. 105).

then it transforms the frame $(x_1, \ldots, x_n; u_1, \ldots, u_{r-n})$ into a frame of origin $(x'_1, \ldots, x'_n)$, i.e., into a frame of parameters $x'_1, \ldots, x'_n; u'_1, \ldots, u'_{r-n}$ defined by the formulae

$$(2) \qquad \begin{cases} x'_i = \varphi_i(x_1, \ldots, x_n; a_1, \ldots, a_r), & (i = 1, \ldots, n), \\ u'_j = \psi_j(u_1, \ldots, u_{r-n}; x_1, \ldots, x_n; a_1, \ldots, a_r), & (j = 1, \ldots, r-n). \end{cases}$$

Given a group G operating on n variables $x_1, x_2, \ldots, x_n$, we agree to say that a group G' operating on these n variables x_i and the p other variables $y_1, y_2, \ldots, y_p$ is a *holoedric prolongation* of G

1. If it transforms the variables x in the same manner as the group G;

2. If to a transformation T of G there corresponds only one transformation T' of G' operating on x_i in the same manner as T.

The parameter group of a given transitive group G can hence be regarded as a holoedric prolongation of G, and we have thus established the following result:

Every transitive group G admits a simply transitive holoedric prolongation. It is obvious that such a prolongation is *necessarily similar to the parameter group.*

Remarks. This result allows us to state under a new form the condition for two transitive groups to be holoedrically isomorphic: it is necessary and sufficient that they admit similar holoedric prolongations.

It also allows us to give a new statement to the problem consisting of searching for all transitive groups isomorphic to a given group. We can suppose that it is about the parameter group in r variables. The problem is hence searching for all choices of the variables such that n $(< r)$ of these variables $x_1, \ldots, x_n$ are transformed into themselves by a formula analogous to (1), the transformed images of the other variables being furnished by formulae analogous to (2) such that two transformations transforming the first n variables in the same manner also transform the $r - n$ others in the same manner.

101 Practical construction of the simply transitive holoedric prolongation of a give group. We are now going to indicate the precise procedure we will use to choose the parameters conforming to the prescriptions of paragraph §**100**.

Attach at each point $\mathbf{M}$ of D a particular transformation $S_{\mathbf{M}}$ of G transforming $\mathbf{A}$ into $\mathbf{M}$. Granted this, let S be any transformation of G and $\mathbf{M}$ the transformed image of $\mathbf{A}$ by this transformation. The transformation $S_{\mathbf{M}}^{-1}S$ changes $\mathbf{A}$ into $\mathbf{M}$, then $\mathbf{M}$ into $\mathbf{A}$. It hence belongs to the subgroup g of the operations of G leaving the point $\mathbf{A}$ fixed. The operation S therefore can be made in a single way under the form $S = S_{\mathbf{M}}g$.

This established, to conform to paragraph §**100**, it suffices to choose the coordinates $x_1, \ldots, x_n$ of $\mathbf{M}$ and the parameters $u_1, \ldots, u_{r-n}$ defining the different transformations of the subgroup g leaving the point $\mathbf{A}$ fixed as the parameter.

102 Examples. *The group of planar displacements.* Let us choose the translation $\overrightarrow{\mathbf{AM}}$ as the transformation $S_{\mathbf{M}}$, and choose as the parameters of the rotation g leaving $\mathbf{A}$ fixed, the angle c of this rotation. The equations of $S_{\mathbf{M}}$ are therefore

$$x' = x + x_0, \qquad y' = y + y_0,$$

and those of g are

$$\begin{aligned} x' &= x\cos c - y\sin c, \\ y' &= x\sin c + y\cos c. \end{aligned}$$

The most general transformation S of the group of parameters $(x_0, y_0; c)$ has equations

$$\begin{cases} x' = x\cos c - y\sin c + x_0, \\ y' = x\sin c + y\cos c + y_0. \end{cases} \tag{3}$$

It transforms the frame $(x, y; u)$ into the frame $(x', y'; u')$ defined by the formulae

$$\begin{cases} x' = x\cos c - y\sin c + x_0, \\ y' = x\sin c + y\cos c + y_0, \\ u' = u + c. \end{cases} \tag{4}$$

Group of linear transformations: $x' = ax + b$ $(b > 0)$. Let us choose the translation $\overrightarrow{\mathbf{AM}}$ as the transformation $S_{\mathbf{M}}$, and choose as the parameters of the homothetic transformations leaving $\mathbf{A}$ fixed the ratio of these homothetic transformations.

The equation of $S_{\mathbf{M}}$ is hence $x' = x + x_0$ and that of g is $x' = ax$.

From this, the equation of the transformation S of the group, whose parameters are x_0; a, is

$$x' = ax + x_0. \tag{5}$$

It transforms the frame $(x; u)$ into the frame $(x'; u')$ defined by

$$\begin{cases} x' = ax + x_0, \\ u' = au. \end{cases} \tag{6}$$

Group of homographic transformation: $x' = \dfrac{ax + b}{a'x + b'}$, $ab' - ba' > 0$.

Let us choose as the equation of the transformation $S\mathbf{M}$

$$x' = x + x_0$$

and as the equation of the transformations of the subgroup g

$$x' = \frac{x}{\beta x + \alpha}, \qquad (\alpha > 0).$$

The transformation of the group whose parameters are x_0, α, β is therefore the transformation

$$x' = \frac{x}{\beta x + \alpha} + x_0. \tag{7}$$

It transforms the frame $(x; u, v)$ into the frame $(x'; u', v')$ defined by the formulae

$$(8) \qquad \begin{cases} x' = \dfrac{x}{\beta x + \alpha} + x_0, \\ u' = \dfrac{u}{\alpha}(\beta x + \alpha)^2, \\ v' = \dfrac{v}{\alpha}(\beta x + \alpha)^2 + \dfrac{\beta}{\alpha}(\beta x + \alpha). \end{cases}$$

III. Intransitive group

103 Now consider an intransitive group G with p invariants operating on a domain D whose defining equations

$$(1) \qquad x'_i = \varphi_i(x_1, \ldots, x_n; a_1, \ldots, a_r)$$

are analytic. Let δ be the analytic varieties each constituted by the set of transformed images of a point of D, which we already know how to determine in paragraph **§94** (p. 106). Let ζ_1, ζ_2, …, ζ_p be the parameters parametrizing the varieties δ.

Let us trace an analytic variety λ_0 having one and only one point in common with each of the varieties δ (†). Let λ be the transformed images of λ_0.

Example. If G is the rotation group around the origin, the varieties δ are the spheres centred at the origin. We can take λ to be the half-line originating at the origin.

Through every point of D there passes at least one variety λ. The group $G(\lambda)$ is transitive and is holoedrically isomorphic to the group G. We denote by ξ_1, ξ_2, … the parameters parametrizing the varieties λ (‡). Let

$$(9) \qquad \begin{cases} \xi'_1 = \psi_1(\xi_1, \xi_2, \ldots; a_1, \ldots, a_r), \\ \quad \ldots \end{cases}$$

be the equations defining the transformations of the group $G(\lambda)$.

Let us then introduce the new objects μ, each of which is constituted by one of the varieties λ and a point on this variety. We call the point the origin of μ. The intransitive group $G(\mu)$ will allow us to compare the groups G and $G(\lambda)$. To define an object μ it suffices to give the parameters ξ_1, ξ_2, … of the variety λ which it is a part of, and the parameters ζ_1, ζ_2, … of the variety δ which the origin of μ belongs to.

(†) It will be necessary in certain cases to restrict to one part of the domain D, inside which the existence of the variety λ is guaranteed. It only is under this restriction that the considerations of this section are rigorous.

(‡) When we want to construct a frame for the group G, it is not useful to form it with a finite number of points belonging to arbitrarily chosen varieties δ, which we have already played a special role. It is preferable to form a frame with a finite number of lines λ.

The equations defining the group $G(\mu)$ are

$$(10)\qquad \begin{cases} \xi_1' = \psi_1(\xi_1, \xi_2, \dots; a_1, \dots, a_r), \\ \quad \dots, \\ \zeta_1' = \zeta_1, \\ \quad \dots \end{cases}$$

The group $G(\mu)$ therefore appears as the group $G(\lambda)$ prolonged by the addition of variables transformed identically.

On the other hand, we can choose the coordinates $x_1, \dots, x_n$ of the origin and a certain number of other suitable parameters $u_1, u_2, \dots$ as the parameters of an object μ. If we compare the old choice of parameters $\xi_1, \dots, \zeta_1, \dots$ to the new one, we see immediately that ζ_i are functions in $x_1, \dots, x_n$. The equations defining $G(\mu)$ now appear under the following form:

$$(11)\qquad \begin{cases} x_1' = \varphi_1(x_1, \dots, x_m; a_1, \dots, a_r), \\ \quad \dots \\ u_1' = \theta_1(x_1, \dots, x_m; u_1, \dots; a_1, \dots, a_r), \\ \quad \dots \end{cases}$$

and $G(\mu)$ appears as the holoedric prolongation of G. These two aspects, (9) and (10), of $G(\mu)$ proves that the intransitive group G can be holoedrically prolonged into a group similar to a transitive group $G(\lambda)$, which is itself prolonged by the addition of variables transformed identically. But according to the preceding paragraph, $G(\lambda)$ possesses a holoedric prolongation similar to its parameter group, which is that of G. We hence obtain the following conclusion:

Every intransitive group G with p invariants can be prolonged holoedrically such that to obtain a group similar to the parameter group prolonged by the addition of p variables that are transformed identically.

N.B. These p new variabels can be considered as p functions of the variables transformed by G.

104 Examples. Consider the intransitive group G in two parameters acting on two variables:

$$(1')\qquad \begin{cases} x' = x, \\ y' = y + ax + b. \end{cases}$$

The lines δ are the lines $x = \text{const}$. We choose the line $y = 0$ as λ_0. Its transformed image by $(1')$ is the line

$$y' = ax' + b.$$

λ can therefore be any line of the plane that is not parallel to the y axis.

Consider such a line $y = \xi x + \eta$. Its transformed image by $(1')$ is the line

$$y' = \xi \cdot x' + \eta + ax' + b.$$

The group $G(\lambda)$ hence has equations

$$(9') \qquad \begin{cases} \xi' = \xi + a, \\ \eta' = \eta + b. \end{cases}$$

G is hence similar to the group of planar translations and μ can be parameterised with the tangential coordinates (ξ, η) of the line λ and the abscissa ζ of its origin. Then $G(\mu)$ has equations

$$(10') \qquad \begin{cases} \xi' = \xi + a, \\ \eta' = \eta + b, \\ \zeta' = \zeta. \end{cases}$$

Hence $G(\mu)$ appears as the group of planer translations prolonged by the addition of one variable that is transformed identically. On the other hand, we can choose, as the parameters of μ, the coordinates (x, y) of the origin and the angular coefficient ξ of the line that forms a part of it. The equations $(1')$ and the first equation of $(10')$ gives us the equations of $G(\mu)$

$$(11') \qquad \begin{cases} x' = x, \\ y' = y + ax + b, \\ \xi' = \xi + a, \end{cases}$$

$G(\mu)$ is a holoedric prolongation of G.

The formulae allow us to go from $(10')$ to $(11')$ are

$$(12') \qquad x = \xi, \qquad y = \xi\zeta + \eta.$$

In this case $G(\lambda)$ is simply transitive, therefore similar to the parameter group. The formulae $(9')$ also show that $G(\lambda)$ is identical to the parameter group when we choose ξ and η as the parameters of λ and a and β as the parameters of the transformations of the group.

Chapter 8

Equations defining the operations of a finite-dimensional connected group

Introduction

105 Let us recall the results stated in 93 and 96: the parameter group leaves the relative components $\omega_p(\xi, d\xi)$ invariant, and this property characterises its transformations. The aim of the present chapter is to combine this result with those of the preceding chapter. We will then obtain a procedure [(†)] allowing us to construct the partial differential equations characterising the operations of a given finite dimensional and connected group.

We assume that the most general transformation of a group we will consider is defined by n analytic functions depending on n variables and r parameters,

$$x'_i = \varphi_i(x_1, \ldots, x_n; a_1, \ldots, a_r), \qquad (i = 1, 2, \ldots, n). \tag{1}$$

I. The case of simply transitive groups

106 Theorem. *A simply transitive group can be characterised by the property that it leaves n $(= r)$ Pfaffian forms $\omega_i(x, dx)$ invariant.*

Indeed, it can be identified with its parameter group (c.f. §**99**, p. 111).

[(†)] This procedure hence allows us to apply the same operation as the procedure called "the method of elimination of constants". But we will make use of all the simplifications that are possible due to the fact that the transformations in question constitute a group.

Application. Consider a one-parameter transitive group. Its transformations are defined by a differential equation

$$\alpha(x')dx' = \alpha(x)dx.$$

From which, denoting a primitive of the function $\alpha(x)$ by $\varphi(x)$ and the parameter by a,

$$\varphi(x') = \varphi(x) + a.$$

A change of coordinates which is effected by setting $y = \varphi(x)$ hence allows us to reduce the defining equation of the group to the form $y' = y + a$. We express this fact by saying that *all one-parameter simply transitive group is similar to the group of translations on the line.*

Complement. Consider an n dimensional domain $x_1,\ x_2,\ \dots,\ x_n$ and n Pfaffian forms $\omega_1(x, dx),\ \dots,\ \omega_n(x, dx)$. The set of bijective transformations of this domain leaving these n forms invariant manifestly constitute a group: every transformation in this set admits an inverse belonging to the set and the product of two transformations of the set is a transformation in the set, since the equations

$$\omega_i(x, dx) = \omega_i(x', dx'), \qquad \omega_i(x', dx') = \omega_i(x'', dx'')$$

entails

$$\omega_i(x, dx) = \omega_i(x'', dx'').$$

But this group may depend on a number r of parameters that is less than n.

We have already given an example on page 96: the group leaving the forms $\omega_1 = \dfrac{dx_1}{x_2}$, $\omega_2 = dx_2$ invariant is the one-parameter group

$$x_1' = x_1 + a, \qquad x_2' = x_2.$$

107 Determination of the invariant Pfaffian forms. We will deduce the Pfaffian forms that a simply transitive group left invariant from the defining equations (1) of the group. We have in these equations $n = r$. Let us apply the general method of forming differential equations that we call the method of elimination of constants: consider any two points $(x_1, \dots, x_n)$ and $(X_1, \dots, X_n)$ and let S_a be the transformation that transforms the first into the second:

$$X_i = \varphi_i(x_1, \dots, x_n; a_1, \dots, a_n), \qquad (i = 1, 2, \dots, n). \tag{2}$$

Keeping the a_i fixed, let us vary the x_i infinitesimally. The variations of the right hand sides of (2) are

$$dX_i = \frac{\partial\varphi_i(x_1, \dots, x_n; a_1, \dots, a_n)}{\partial x_1} dx_1 + \cdots + \frac{\partial\varphi_i(x_1, \dots, x_n; a_1, \dots, a_n)}{\partial x_n} dx_n. \tag{3}$$

The group being simply transitive, the parameters a_i can be expressed in terms of the quantities x_i and X_i. Let us substitute them into (3). It becomes

$$(4) \qquad dX_i = a_{i1}(X_1, \dots, X_n; x_1, \dots, x_n)dx_1 + \alpha_{in}(X_1, \dots, X_n; x_1, \dots, x_n)dx_n.$$

This system [†] can be made under the form

$$\omega_i(X, dX) = \omega_i(x, dx).$$

This fact is not obvious. This should not surprise us: we have not used the hypothesis that the transformations (2) form a group.

Consider a third point $(x'_1, x'_2, \dots, x'_n)$ and, as previously done, let $(x'_1 + dx'_1, \dots, x'_n + dx'_n)$ be the coordinates of the transformed image of the point $(X_i + dX_i)$ by the transformation transforming (X_i) to the point (x'_i). We have

$$(5) \qquad dX_i = \alpha_{i1}(X_1, \dots, X_n; x'_1, \dots, x'_n)dx'_1 + \cdots + \alpha_{in}(X_1, \dots, X_n; x'_1, \dots, x'_n)dx'_n.$$

The relations (4) and (5) gives us

$$\begin{aligned} &\alpha_{i1}(X_1, \dots, X_n; x_1, \dots, x_n)dx_1 + \cdots + \alpha_{in}(X_1, \dots, X_n; x_1, \dots, x_n)dx_n \\ &= \alpha_{i1}(X_1, \dots, X_n; x'_1, \dots, x'_n)dx'_1 + \cdots + \alpha_{in}(X_1, \dots, X_n; x'_1, \dots, x'_n)dx'_n. \end{aligned}$$

The point (X_i) being chosen arbitrarily once and for all, the r linearly independent forms

$$\alpha_{i1}(X_1, \dots, X_n; x_1, \dots, x_n)dx_1 + \cdots + \alpha_{in}(X_1, \dots, X_n; x_1, \dots, x_n)dx_n$$

are invariant under the transformation transforming the point (x_i) into the point (x'_i), that is to say by every transformation of the group.

We have not only learnt to calculate these invariant forms but also given a second proof of their existence, and this proof is very apt to be generalised.

Remark. The rule of calculating these forms results equally from the use of moving frames and the considerations using which paragraph **§106** begins. We are going to show it by a reasoning that does not differ essentially from the previous:

Assume that, as in paragraph **§99** (p. 111), the frame $\mathbf{R}_0$ is a point $\mathbf{A}$. Let the coordinates of this point be (X_i). The components $\omega_i(x, dx)$ are none other than the coordinates of the point $(x_i + dx_i)$ relative to the frame that the point (x_i) constitutes. Let a_i be the parameters of the transformation S_a transforming the frame (x_i) into the frame $\mathbf{R}_0 \equiv \mathbf{A}$. Let $(X_i + dX_i)$ be the transformed image of the point $(x_i + dx_i)$ by S_a. We have

$$dX_i = \omega_i(x, dx),$$

[†] Which is equivalent to the partial differential equations

$$\frac{\partial X_i}{\partial x_k} = \alpha_{ik}(X_1, \dots, X_n; x_1, \dots, x_n).$$

but

$$X_i = \varphi_i(x_1, \dots, x_n; a_1, \dots, a_n),$$

$$dX_i = \frac{\partial \varphi_i(x_1, \dots, x_n; a_1, \dots, a_n)}{\partial x_1} dx_1 + \dots + \frac{\partial \varphi_i(x_1, \dots, x_n; a_1, \dots, a_n)}{\partial x_n} dx_n.$$

The forms $\omega_i(x, dx)$, invariant under the transformations of the group, are hence obtained by solving the equations (2) with respect to a_i and by substituting the values obtained into the right hand sides of (3).

Example: Consider the group of transformations

$$x' = ax, \qquad (a > 0),$$

(2) is now written $X = ax$; (3) is now written $dX = a\,dx$.

The elimination of a gives us $dX = X\dfrac{dx}{x}$. Let us choose $X = 1$. We obtain the invariant form $\dfrac{dx}{x}$.

From this it is evident that the differential equation $\dfrac{dx'}{x'} = \dfrac{dx}{x}$ and the existence of a relation $x' = ax$ are equivalent.

II. The case of transitive groups

108 Preliminaries. Suppose the parameters are chosen as indicated in paragraph **§100** (p. 111) of the preceding chapter, namely $x_1, x_2, \dots, x_n; u_1, \dots, u_{r-n}$. The parameter grou p is a holoedric prolongation of the given group.

The transformations of the parameter group are characterised by the property that they leave the relative components of the instantaneous displacement of the moving frame invariant. They are r Pfaffian forms

$$\omega_p(x, u, dx, du).$$

Recall that we calculate these forms from the relations

$$(6) \qquad \begin{cases} \displaystyle\sum_{p=1}^{r} \omega_p(a, da) X_p f = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \delta x_j; \\ \displaystyle\sum_{p=1}^{r} \frac{\partial \varphi_i(x_1, \dots, x_n; a_1, \dots, a_r)}{\partial a_p} da_p = \sum_{j=1}^{n} \frac{\partial \varphi_i(x_1, \dots, x_n; a_1, \dots, a_r)}{\partial x_j} \delta x_j \\ (i = 1, \dots, n). \end{cases}$$

Let us examine several particular cases.

Group of planar displacements [Equations (3) of the preceding chapter, p. 113]. We obtain

$$(7) \qquad \begin{cases} \omega_1(x, u, dx, du) = \cos u\, dx + \sin u\, dy, \\ \omega_2(x, u, dx, du) = -\sin u\, dx + \cos u\, dy, \\ \omega_3(x, u, dx, du) = du. \end{cases}$$

Let us verify by a direct calculation that the equations

$$\omega_i(x, u, dx, du) = \omega_i(x', u', dx', du')$$

defines a transformation of the parameter group [equation (4) of the preceding chapter].

The invariance of the third Pfaffian equation gives us $du = du'$, therefore $u' = u + c$, c being a constant.

The invariance of the forms ω_1 and ω_2 give us

$$\begin{aligned} dx' &= \cos c\, dx - \sin c\, dy, \\ dy' &= \sin c\, dx + \cos c\, dy, \end{aligned}$$

from which

$$\begin{aligned} x' &= x\cos c - y\sin c + x_0, \\ y' &= x\sin c + y\cos c + y_0. \end{aligned}$$

Q.E.D.

Group of linear transformations [equation (5) of the preceding chapter, p. 113]. We have

$$\text{(8)} \qquad \begin{cases} \omega_1(x, u, dx, du) = \dfrac{dx}{u}, \\ \omega_2(x, u, dx, du) = \dfrac{du}{u}. \end{cases}$$

Conversely the equations

$$\frac{dx'}{u'} = \frac{dx}{u}, \qquad \frac{du'}{u'} = \frac{du}{u}$$

entail

$$u' = au, \qquad x' = ax + x_0,$$

a and x_0 being constants: we find the equations of the parameter group [equations (6) of the preceding chapter].

Group of homographic transformations [equations (7) of the preceding chapter, p. 113]. We have

$$\text{(9)} \qquad \omega_1 = u\, dx, \qquad \omega_2 = 2v\, dx - \frac{du}{u}, \qquad \omega_3 = \frac{v^2}{u} dx - \frac{dv}{u}.$$

Let us now show by a direction calculation that the general integral of the system

$$u' dx' = u\, dx, \qquad 2v' dx' - \frac{du'}{u'} = 2v\, dx - \frac{du}{u}, \qquad \frac{v'^2}{u'} dx' - \frac{dv'}{u'} = \frac{v^2}{u} dx - \frac{dv}{u},$$

is given by the formulae (8) of the preceding chapter.

From the relation

$$u' dx' = u\, dx$$

it follows that x' is a function of x:

$$x' = f(x),$$

and we have

$$u' = \frac{u}{f'(x)},$$

then ω_3 gives us an equation where only f and its derivatives appear:

$$\frac{f'''}{f'} - \frac{3}{2}\frac{f''^2}{f'^2} = 0. \tag{10}$$

The function

$$f(x) = \frac{x}{\beta x + \alpha} + x_0$$

satisfies this equation for all constants x_0, α and β, and it is also a general integral, see the way the constants of integration enter in the general integral. Then the equations (8) of the preceding chapter is established again.

We determine for example how the differential equations of the transformations of the group can follow from the forms ω_i. At the end of this section we will know that this fact is general.

It is remarkable that in all the preceding examples we obtain n forms ω_p containing only the differentials dx_i. We are going to show that this result holds in all cases.

109 Lemma. *We can always form n distinct linear combinations of ω_p with constant coefficients in which the differentials du_i do not appear.*

Indeed, consider the absolute frame $\mathbf{R}_0$, and suppose that its parameters are 0, …, 0. We can form n linear combinations of the forms $\omega_p(0,0,dx,du)$ that contains only the differentials dx_i. Suppose that these are precisely ω_1, ω_2, …, ω_n. Specifying the values of these first n forms then becomes specifying the infinitesimal displacement of the origin of the frame $\mathbf{R}_0$.

The forms $\omega_1(x,u,dx,du)$, …, $\omega_n(x,u,dx,du)$ can therefore be considered as the relative components of the infinitesimal displacement of the moving frame $\mathbf{R}$. This displacement is measured with respect to this frame. These forms are zero at the same time as when this displacement, i.e., dx_1, …, dx_n are zero. They therefore depend only on the differentials [†] dx_1, …, dx_n. Q.E.D.

Let us give a second proof which is more analytic in its reasoning. Let S_a be the operation of the group transforming $\mathbf{R}_0$ into $\mathbf{R}$. Consider an arbitrary frame infinitesimally close to $\mathbf{R}_0$ and let dx_i^0, …, du_{r-n}^0 be its parameters. S_0 transforms it into a frame of parameters $x_1 + dx_1$, …, $u_{r-n} + du_{r-n}$. By virtue of the invariance of the forms ω_i, we have

$$\omega_p(0,0,dx^0,du^0) = \omega_p(x,u,dx,du).$$

It follows that the system

$$dx_1^0 = 0, \qquad \ldots, \qquad dx_n^0 = 0,$$

which is by hypothesis equivalent to the system

$$\omega_1(0,0,dx^0,du^0) = 0, \qquad \ldots, \qquad \omega_n(0,0,dx^0,du^0) = 0,$$

[†] Henceforth we will suppress the expression du in the symbol representing these forms.

is also equivalent to the system

$$\omega_1(0,0,dx,du)=0, \qquad \dots, \qquad \omega_n(0,0,dx,du)=0.$$

On the other hand it follows from (1) that

$$dx_i=\sum_{k=1}^{n}\frac{\partial\varphi_i(0,\dots,0;a_1,\dots,a_r)}{\partial x_k}dx_k^0.$$

The system

$$dx_1^0=0, \qquad \dots, \qquad dx_n^0=0,$$

is hence equivalent to the system

$$dx_1=0, \qquad \dots, \qquad dx_n=0.$$

In summary the two systems

$$dx_1=0, \qquad \dots, \qquad dx_n=0$$

and

$$\omega_1(x,u,dx,du)=0, \qquad \dots, \qquad \omega_n(x,u,dx,du)=0$$

are equivalent. This proves that the forms

$$\omega_1(x,u,dx,du), \qquad \dots, \qquad \omega_n(x,u,dx,du)$$

contain only the differentials dx_i.

110 Lemma. *If the differentials u_j do not appear in the first n forms*

$$\omega_1(x,u,dx), \qquad \dots, \qquad \omega_n(x,u,dx),$$

then either at least one of these forms effectively depend on the variables u_i, or the group considered is simply transitive.

Indeed, suppose the forms

$$\omega_1(x,u,dx), \qquad \dots, \qquad \omega_n(x,u,du)$$

are independent of the parameters u_j. Every operation of the group transforms the quantities x_i into the quantities x_i' such that

$$\omega_i(x,dx)=\omega_i(x',dx') \qquad (i=1,2,\dots,n),$$

and the n forms $\omega_i(x,dx)$ are linearly independent. But we have proved in paragraph §**83** (p. 96) that there exists in a n dimensional space at most one transformation leaving the n linearly independent forms invariant and transforming a given point into another given point. Q.E.D.

111 Digression. We are going to show how the choice of parameters indicated in paragraph **§101** (chapter 7, p. 112) has as a consequence other properties of the forms ω_i. These new properties are on the other hand not very important.

Suppose, as in the preceding chapter, that the forms

$$\omega_1(x, u, dx, du), \qquad \dots, \qquad \omega_n(x, u, dx, du)$$

characterise the infinitesimal relative transformation undergone by the origin of the moving frame $\mathbf{R}$. Let

$$\bar{\omega}_1(x, dx), \qquad \dots, \qquad \bar{\omega}_r(x, dx)$$

be the relative components of the infinitesimal transformation undergone by the frame $\mathbf{R}_{\mathbf{M}}$ when the parameters x_i vary by dx_i: they are the parameters of the infinitesimal transformation $\mathbf{R}_{\mathbf{M}}^{-1}\mathbf{R}_{\mathbf{M}+d\mathbf{M}}$.

Granted this, let $S_{\mathbf{M}}g_u$ be the operation transforming the frame $\mathbf{R}_0$ into the frame of parameter $(x; u)$, and let $S_{\mathbf{M}+d\mathbf{M}}g_{u+du}$ the operation transforming the frame $\mathbf{R}_0$ into the frame $(x + dx; u + du)$. The quantities $\omega_i(x, u, dx, du)$ are the parameters of the infinitesimal transformation

$$(S_{\mathbf{M}}g_u)^{-1}S_{\mathbf{M}+d\mathbf{M}}g_u, \qquad g_u^{-1}S_{\mathbf{M}}^{-1}S_{\mathbf{M}+d\mathbf{M}}g_{u+du}.$$

This transformation is the product of the two following

$$g_u^{-1}S_{\mathbf{M}}^{-1}S_{\mathbf{M}+d\mathbf{M}}g_u, \qquad g_u^{-1}g_{u+du}.$$

Their components are therefore the sum of components with the same indices of these two transformations. But the first is the transformed image of the infinitesimal transformation $S_{\mathbf{M}}^{-1}S_{\mathbf{M}+d\mathbf{M}}$ of parameter $\bar{\omega}_i(x, dx)$ by g_u^{-1}. Its parameters are therefore of the form

$$\bar{\omega}_i(x, u, dx).$$

As for the second, its parameters are of the form $\omega_i^\star(u, du)$ and they exist only for $i > n$.

We therefore have, finally,

$$\text{(11)} \qquad \begin{cases} \omega_i(x, u, dx) = \bar{\omega}_i(x, u, dx) & \text{for } i \leq n, \\ \omega_i(x, u, dx, du) = \omega_i^\star(u, du) + \bar{\omega}_i(x, u, dx) & \text{for } i > n. \end{cases}$$

Thus we have just shown that we can choose the parameters such that x_i do not appear in the coefficients of du_j. This fact is of little importance. It is more interesting to note that the preceding allows us to deduce the forms $\omega_i(x, u, dx, du)$ from the knowledge of the forms $\bar{\omega}_i(x, dx)$ and the forms $\omega_i^\star(u, du)$ relative to the subgroup g.

Group of planar displacements. The group g is the group

$$x' = x\cos u - y\sin u,$$
$$y' = x\sin u + y\cos u.$$

We deduce from it

$$\omega_3^\star(u, du) = du.$$

We have, according to the conventions made in paragraph **§102** (p. 113),

$$\bar{\omega}_1(x, dx) = dx, \qquad \bar{\omega}_2(x, dx) = dy, \qquad \bar{\omega}_3(x, dx) = 0,$$

from which (see the last of the examples of paragraph §**69**, 80)

$$\begin{aligned}\bar{\omega}_1(x,u,dx) &= \cos u\,dx + \sin u\,dy,\\ \bar{\omega}_2(x,u,dx) &= -\sin u\,dx + \cos u\,dy,\\ \bar{\omega}_3(x,u,dx) &= 0.\end{aligned}$$

We obtain the formulae (7) of page 120 by substituting these equations into (11).

Group of linear transformations. g is the subgroup

$$x' = ux.$$

We have

$$\omega_2^\star(u,du) = \frac{du}{u}.$$

We have, according to the convention made in paragraph §**102**,

$$\bar{\omega}_1(x,dx) = dx, \qquad \bar{\omega}_2(x,dx) = 0.$$

From which

$$\bar{\omega}_1(x,u,dx) = \frac{dx}{u}, \qquad \bar{\omega}_2(x,u,du) = 0,$$

(11) now gives us the formulae (8) on page 121.

Group of homographic transformations. g is the subgroup

$$x' = \frac{x}{vx+u}.$$

Let us write this equation under the form

$$\frac{1}{x'} = u\frac{1}{x} + v.$$

We see that g is similar to the group of linear substitutions. We can therefore choose

$$\omega_2^\star(u,du) = \frac{du}{u}, \qquad \omega_3^\star(u,du) = \frac{dv}{u}.$$

On the other hand

$$\bar{\omega}_1(x,dx) = dx, \qquad \bar{\omega}_2(x,dx) = 0, \qquad \bar{\omega}_3(x,dx) = 0.$$

We easily deduce from this a result equivalent to (9) (p. 121).

112 Consequences of the lemmas established in §109 and §110. Consider a connected, finite-dimensional and transitive group defined by the analytic equations

$$x'_i = \varphi_i(x_1,\ldots,x_n;a_a,\ldots,a_r). \tag{1}$$

Let us make a choice of parameters such that the n forms $\omega_1, \ldots, \omega_n$ depend only on the differentials $dx_1, \ldots, dx_n$, which we write as

$$\omega_i(x,u,dx) = \sum_{k=1}^{n} \alpha_{ik}(x,u)dx_k, \qquad (i = 1,\ldots,n).$$

Suppose that the group is not simply transitive. Some of the coefficients α_{ik} depend on the parameters u. We modify, if necessary, the definition of the parameters u such that those that α_{ik} do not depend on are $u_{p+1}, \ldots, u_{r-n}$ and that two different systems of values of $u_1, \ldots, u_p$ always correspond to two different systems of values α_{ik}.

Let us find how the group operate on the p quantities $u_1, \ldots, u_p$. It is not necessary to go back to the parameter group and the relations defining $u_1, \ldots, u_p$ in terms of the old parameters. We have, for each transformation S_a,

$$\sum_{k=1}^{n} \alpha_{ik}(x,u)dx_k = \sum_{l=1}^{n} \alpha_{il}(x',u')dx'_l,$$

from which

$$\alpha_{ik}(x,u) = \sum_{l=1}^{n} \alpha_{il}(x',u')\frac{\partial \varphi_l(x_1,\ldots,x_n;a_1,\ldots,a_r)}{\partial x_k}.$$

These equations can be solved with respect to $u_1, \ldots, u_p$. Indeed, the given data of $\alpha_{ik}(x,u)$ determines by hypothesis the values of $u_1, \ldots, u_p$. We obtain

$$u_h = \Psi_h(x'_1,\ldots,x'_n;u'_1,\ldots,u'_p;a_1,\ldots,a_r), \qquad (h=1,\ldots,p). \tag{12}$$

The formulae (12) define the transformed quantities $u_1, \ldots, u_p$ of the variables $x'_1, \ldots, x'_n$, $u'_1, \ldots, u'_p$ by S_a^{-1}.

Example. *Group of planar displacement.* We have

$$\begin{aligned}
\omega_1(x,u,dx) &= \cos u\,dx + \sin u\,dy,\\
\omega_2(x,u,dx) &= -\sin u\,dx + \cos u\,dy.
\end{aligned}$$

Let x', y' be the transformed quantities of x, y by the transformation S_a:

$$\begin{aligned}
x' &= x\cos c - y\sin c + a,\\
y' &= x\sin c + y\cos c + b.
\end{aligned}$$

We have

$$\begin{aligned}
\cos u\,dx + \sin u\,dy &= \cos u'dx' + \sin u'dy',\\
-\sin u\,dx + \cos u\,dy &= -\sin u'dx' + \cos u'dy'.
\end{aligned}$$

From which

$$\begin{aligned}
\cos u &= \cos u'\frac{\partial x'}{\partial x} + \sin u'\frac{\partial y'}{\partial x}, &\qquad \sin u &= \cos u'\frac{\partial x'}{\partial y} + \sin u'\frac{\partial y'}{\partial y},\\
\sin u &= \sin u'\frac{\partial x'}{\partial x} - \cos u'\frac{\partial y'}{\partial x}, &\qquad \cos u &= -\sin u'\frac{\partial x'}{\partial y} + \cos u'\frac{\partial y'}{\partial y},
\end{aligned}$$

that is to say

$$u = u' - c.$$

Group of homographic transformations. We have

$$\omega_1(x,u,dx) = \frac{dx}{u}.$$

Let x' be the transformed quantity of x by the transformation S_a:

$$x' = \frac{x}{\beta x + \alpha} + x_0.$$

We have

$$\frac{dx'}{u'} = \frac{dx}{u},$$

from which

$$u = \frac{u'}{\alpha}(\beta x + \alpha)^2.$$

The formulae (12) thus calculated, let us consider the transitive group of transformations defined by (12). This is a holoedric prolongation of the given group and it acts on $n+p$ variables

$$x_1, \quad \ldots, \quad x_n; \quad u_1, \quad \ldots, \quad u_p.$$

For this group, let us make the choice of parameters defined in paragraph §**100**. Let $x_1, \ldots, x_n, u_1, \ldots, u_p, v_1, \ldots, v_{r-n-p}$ be the parameters thus chosen. There exists $n+p$ forms linear in $dx_1, \ldots, dx_n, du_1, \ldots, du_p$, with coefficients functions of x_i, u_h, v_l, which are left invariant by the prolonged group. Among these forms we find $\omega_1(x,u,dx), \ldots, \omega_n(x,u,dx)$. Let $\omega_1(x,u,v,dx,du), \ldots, \omega_p(x,u,v,dx,du)$ be the p others. Two alternative cases can arise: either the last p forms are independent of v_l, and according to paragraph §**110** the prolonged group is simply transitive, or we can modify the definition of the parameters v_l such that the following circumstances are realised: set

$$\omega_k(x,u,v,dx,du) = \sum_{i=1}^{n} \beta_{ki}(x,u,v)dx_i + \sum_{h=1}^{p} \gamma_{kh}(x,u,v)du_h,$$

the coefficients β_{ki} and γ_{kh} depending only on q of the parameters v, which are written $v_1, \ldots, v_q$, and two different systems of values of $v_1, \ldots, v_q$ always correspond two different systems of values of these coefficients. By using (1) and (12) and the invariance of the forms $\omega_k(x,u,v,dx,du)$, let us calculate the transformed quantities $v_1, \ldots, v_q$ in $x'_1, \ldots, x'_n, u'_1, \ldots, u'_p, v'_1, \ldots, v'_q$ by the transformation S_a^{-1} of the group. We obtain formulae analogous to (12):

$$(13) \qquad v_i = \theta_i(x'_1, \ldots, x'_n; u'_1, \ldots, u'_p; v'_1, \ldots, v'_q; a_1, \ldots, a_r), \qquad (i = 1, \ldots, q).$$

The transitive group defined by (1), (12), (13) is a holoedric prolongation of the two preceding groups. It operates on $n+p+q$ variables $x_1, \ldots, x_n, u_1, \ldots, u_p, v_1, \ldots, v_q$. Either it is similar to the parameter group, or the procedure applied twice allows us to prolong it into a group operating on a greater number of variables. But the number of these variables cannot exceed r. Therefore after a finte number of prolongation of such nature we obtain a group similar to the parameter group.

We have just learnt to prolong holoedrically a given transitive group G operating on n variables $x_1, \ldots, x_n$ into a simply transitive group operating on r variables

$$x_1, \quad \ldots, \quad x_n; \quad u_1, \quad \ldots, \quad u_p; \quad v_1, \quad \ldots, \quad v_q; \quad \ldots; \quad w_1, \quad \ldots, \quad w_s,$$

such that the prolonged group leaves r independent forms invariant, namely

$$\text{the } n \text{ forms} \qquad \omega_1(x,u,dx), \quad \ldots, \quad \omega_n(x,u,dx),$$

which effectively modifies every variation of one of the parameters u,

$$\text{the } p \text{ forms} \qquad \omega_1(x,u,v,dx,du), \quad \ldots, \quad \omega_p(x,u,v,dx,du),$$

which effectively modifies every variation of one of the parameters v,

$$\text{the } s \text{ forms} \qquad \begin{cases} \omega_1(x,u,v,\ldots,w,dx,du,dv,\ldots,dw), \\ \qquad\qquad \ldots \\ \omega_s(x,u,v,\ldots,w,dx,du,dv,\ldots,dw). \end{cases}$$

This simply transitive group can be chosen as the parameter group of G. Its operations, as well as transforming the variables among themselves, transform the variables x_i and u_j among themselves, the variables x_i, u_j, v_k, $\ldots$ among themselves. The operations of this simply transitive group are characterised by the property that they leave the r forms

$$\omega_1(x,u,dx), \quad \ldots, \quad \omega_n(x,u,dv), \quad \omega_1(x,u,v,dx,du), \quad \ldots, \\ \omega_1(x,u,v,dx,du,dv), \quad \ldots$$

Remark. According to paragraph §**83** (p. 96) the integration of a certain differential equation allows us to construct the equations (1), (12), (13), $\ldots$ from the preceding r forms. *A priori* it seems that we cannot construct the equations (1) of the group without constructing the equations (12), (13), $\ldots$, i.e., without effectively constructing the equations of the parameter group. But this is not necessary: the following paragraph will show that the functions $\varphi_1, \ldots, \varphi_n$ in $x_1, \ldots, x_n$ appearing on the right hand sides of (1) are defined by a system of partial differential equations in which the quantities u, v, $\ldots$, w are absent.

113 Partial differential equations defining the transformations of a group. Consider a variable frame depending on only the parameters $x_1, \ldots, x_n$. Suppose for example that all the other parameters are constant and, to simply the notation, are zero. The transformed image of this variable frame by any fixed operation S_a of G is another variable frame of parameters

$$x'_1, \quad \ldots, \quad x'_n, \quad u'_1, \quad \ldots, \quad u'_p, \quad \ldots, \quad w'_1, \quad \ldots, \quad w'_s.$$

We have

$$
(14)\qquad \begin{cases}
\omega_i(x', u', dx') = \omega_i(x, 0, dx), \\
\omega_h(x', u', v', dx', du') = \omega_h(x, 0, 0, dx, 0), \\
\cdots \\
\omega_m(x', u', \dots, w', dx', du', \dots, dw') = \omega_m(x, 0, \dots, 0, dx, 0, \dots, 0) \\
(i = 1, \dots, n; h = 1, \dots, p; m = 1, \dots, s).
\end{cases}
$$

According to paragraph §**84** (p. 97), there exists at most one system of functions

$$
x'_1, \quad \dots, \quad x'_n, \quad u'_1, \quad \dots, \quad u'_p, \quad \dots, \quad w'_1, \quad \dots, \quad w'_s
$$

of the variables x_1, …, x_n satisfying the equations (14) and are equal to a system of r given numbers when x_i are all zero.

Then for the functions $x'_1(x_1, \dots, x_n)$, …, $x'_n(x_1, \dots, x_n)$ to define a transformation of G, it is necessary and sufficient that we can find the functions $u'_1(x_1, \dots, x_n)$, …, $w'_s(x_1, \dots, x_n)$ such that the relations (14) are satisfied. A change of notations allow us to state this result as the following: for the functions $x'_1(x_1, \dots, x_n)$, …, $x'_n(x_1, \dots, x_n)$ to define a transformation of G, it is necessary and sufficient that we can find $r-n$ other functions $u_1(x_1, \dots, x_n)$, …, $w_s(x_1, \dots, x_n)$ such that we have

$$
(15_1)\qquad \omega_i(x, u, dx) = \omega_i(x', 0, dx'),
$$

$$
(15_2)\qquad \omega_h(x, u, v, dx, du) = \omega_h(x', 0, 0, dx', 0),
$$

$$
\cdots
$$

$$
(15_3)\qquad \omega_m(x, u, \dots, w, dx, du, \dots, dw) = \omega_m(x', 0, \dots, 0, dx', 0, \dots, 0).
$$

The n relations (15_1) is equivalent to the system

$$
(16_1)\qquad \alpha_{ij}(x, u) = \sum_{k=1}^{n} \alpha_{ik}(x', 0) \frac{\partial x'_k}{\partial x_j},
$$

the p relations (15_2) is equivalent to the system

$$
(16_2)\qquad \beta_{hi}(x, u, v) + \sum_{k=1}^{p} \gamma_{hk}(x, u, v) \frac{\partial u_k}{\partial x_i} = \sum_{j=1}^{n} \beta_{hj}(x', 0, 0) \frac{\partial x'_j}{\partial x_i}.
$$

$$
\cdots
$$

By hypothesis, the equations (16_1) can be solved with respect to u_1, …, u_p. It becomes

$$
(17_1)\qquad u_h = F_h \left(x_1, \dots, x_n, x'_1, \dots, x'_n, \frac{\partial x'_1}{\partial x_1}, \frac{\partial x'_1}{\partial x_2}, \dots, \frac{\partial x'_n}{\partial x_n} \right).
$$

Similarly, by solving the equations (16_2) with respect to v_1, …, v_q, we see that v_i is equal to a known function of x_i, x'_i, $\dfrac{\partial x'_i}{\partial x_j}$, $\frac{\partial u_h}{\partial x_j}$. Using equation (17_1), it becomes

$$
(17_2)\qquad v_l = G_l \left(x_1, \dots, x_n, x'_1, \dots, x'_n, \frac{\partial x'_1}{\partial x_1}, \dots, \frac{\partial x'_n}{\partial x_n}, \frac{\partial^2 x'_1}{\partial x_1^2}, \frac{\partial^2 x'_1}{\partial x_1 \partial x_2}, \dots, \frac{\partial^2 x'_n}{\partial x_n^2} \right).
$$

$$\dots$$

etc.

Let us replace in the relations (15_3) the quantities u_h, v_l, ..., w_m by their expressions (17). We obtain a system of equations where only the functions $x'_i(x_1, \dots, x_n)$ and their derivatives appear. This system of equations constitute the necessary and sufficient condition for the functions $x'_i(x_1, \dots, x_n)$ to define a transformation of the group G. They are *the partial differential equations of the transformations of the group* G. Following S. Lie, we call them *the defining equations* of the group G.

EXAMPLES. *Group of planar displacements.* The formulae (7) (p. 120) show that the parameters x, y, u used in these formulae fullfil the condition stated at the end of paragraph §**112**. The formulae (15) are written

$$\begin{aligned} \cos u\, dx + \sin u\, dy &= dx', \\ -\sin u\, dx + \cos u\, dy &= dy', \\ du &= 0, \end{aligned}$$

and the formulae (16)

$$\cos u = \frac{\partial x'}{\partial x}, \qquad \sin u = \frac{\partial x'}{\partial y}, \qquad \sin u = -\frac{\partial y'}{\partial x},$$
$$\cos u = \frac{\partial y'}{\partial y}, \qquad \frac{\partial u}{\partial x} = 0, \qquad \frac{\partial u}{\partial y} = 0.$$

The partial differential equations defining the displacements are therefore

$$\frac{\partial y'}{\partial x} = -\frac{\partial x'}{\partial y}, \qquad \frac{\partial y'}{\partial y} = \frac{\partial x'}{\partial x}, \qquad \left(\frac{\partial x'}{\partial y}\right)^2 + \left(\frac{\partial x'}{\partial x}\right)^2 = 1, \qquad \frac{\partial^2 x'}{\partial x^2} = 0, \qquad \frac{\partial^2 x'}{\partial x\, \partial y} = 0.$$

Group of linear transformations. The formulae (8) (p. 121) show that the parameters x, u used in these formulae fullfil the conditions stated at the end of paragraph §**112**. The formulae (15) are written, with the condition $u' = 1$ instead of $u' = 0$,

$$\frac{dx}{u} = dx', \qquad \frac{du}{u} = 0,$$

and the formulae (16) becomes

$$\frac{1}{u} = \frac{dx'}{dx}.$$

The differential equation defining linear transformations is therefore

$$\frac{d^2 x'}{dx^2} = 0.$$

Group of homographic transformations. The formulae (9) (121) show that the parameters x, u, v fullfil the conditions sated at the end of paragraph §**112**. The formulae (15) are written, with the condition $u' = 1$, $v' = 0$,

$$u\, dx = dx', \qquad 2v\, dx - \frac{du}{u} = 0, \qquad \frac{v^2}{u} dx - \frac{dv}{u} = 0,$$

and the formulae (16) give

$$u = \frac{dx'}{dx}, \qquad v = \frac{1}{2u}\frac{du}{dx}.$$

The differential equation defining the homographic transformations is therefore

$$\frac{d}{dx}\left[\frac{\left(\dfrac{d^2x'}{dx^2}\right)}{\left(\dfrac{dx'}{dx}\right)}\right] = \frac{1}{2}\frac{\left(\dfrac{d^2x'}{dx^2}\right)^2}{\left(\dfrac{dx'}{dx}\right)^2}$$

The method using which we thus form the defining equations of a given group shows that *the order* (†) *of the system of partial differential equations is equal to the number of classes* (‡) *of the variables* u_i, v_l, …, w_m *introduced in paragraph* **§112**: this order is 1 in the case of a simply transitive group, 2 in the case of the group of planar displacements and the group of linear transformations, 3 in the case of homographic transformations.

III. The case of intransitive groups

114 Let us recall the result proved in paragraph **§103** (p. 114).

Every intransitive group G with p invariants can be prolonged holoedrically in a way to obtain a group similar to the parameter group, which is itself prolonged by adding p variables transformed identically, and these p new variables can be considered as p functions of the variables x_1, …, x_n transformed by G. The group of parameters is characterised by the property that it leaves r Pfaffian forms invariant.

Then an *intransitive group* of n variables, r parameters and p invariants can be *characterised* by the property of *leaving invariant*:

p functions, y_1, …, y_p of the variables x_1, …, x_n;

r Pfaffian forms $\omega_1(x, u, dx, du)$, …, $\omega_r(x, u, dx, du)$ which involve the n variables x_1, …, x_n and $r + p - n$ auxiliary variables u_1, …, u_{r+p-n}.

Remark. The differentials of the p invariants constitute p new invariant Pfaffian forms which involve x_1, …, x_n, which gives $r + p$ linearly independent Pfaffian forms left invariant by the group. We can substitute the r forms $\omega_i(x, u, dx, du)$ with any system of r forms of the type

$$\sum_{j=1,\dots,n} a_{ij}(y)\omega_j(x, u, dx, du) + \sum_{l=1,\dots,p} b_{il}(y)dy_l,$$

where $a_{ij}(y)$ and $b_{il}(y)$ are arbitrary functions of y_1, …, y_p provided the determinant of $a_{ij}(y)$ is non-zero. These circumstances differ considerably from those that arise in the case of transitive groups.

(†) Let us recall that this order is, by definition, the highest order of derivatives that occurs.

(‡) u_1, …, u_p constitute the first class, v_1, …, v_q the second class, etc.

We will not pursue further the examination of these differential conditions defining the group, since the study of an intransitive group amounts to that of the transitive group (*see* **§94**, p. 106), and we would only be repeating for each of the variaties δ the content of the preceding paragraphs.

Example. Consider again the intransitive group studied in paragraph **§104** (p. 115):

$$\begin{cases} x' = x, \\ y' = y + ax + b. \end{cases} \tag{1'}$$

We know that we can prolong it holoedrically into the group

$$\begin{cases} x' = x, \\ y' = y + ax + b, \\ \xi' = \xi + a, \end{cases} \tag{11'}$$

and that the change of variables

$$x = \zeta, \qquad \eta = y - \xi x \tag{12'}$$

transforms this group into the similar group

$$\begin{cases} \xi' = \xi + a, \\ \eta' = \eta + b, \\ \zeta' = \zeta. \end{cases} \tag{10'}$$

This last group is a parameter group prolonged by an identity transformation. The Pfaffian equations left invariant by the parameter group are

$$\omega_1 = d\xi, \qquad \omega_2 = d\eta.$$

The group $(1')$ is therefore characterised by the property of leaving ζ, $d\xi$, $d\eta$ invariant.

This is a system of conditions of the type that we discovered at the beginning of this paragraph. Let us write them by using the variables x, y, ξ:

Invariant: x;

Invariant forms: $\omega_1 = d\xi$; $\omega_2 = dy - x\,d\xi - \xi\,dx$.

Remark. We can substitute ω_2 with the simpler form

$$dy - \xi\,dx.$$

It is easy to verify by direct calculation that the equations $(1')$ result from this system of conditions.

Chapter 9

Realisations of a given abstract group. Subgroups of a group

115 Introduction. The preceding chapter, which is related to the definition of a group by a system of differential equations, leads us naturally to transform the determination of all groups realising a given abstract group into problems of differential and integral calculus, to which we have already devoted section iv. of chapter 6 (p. 107).

The problem of differential calculus that we will meet will be to know when a system of n Pfaffian equations in n variables is completely integrable: its solution, which is the basis of the third part of this course, will be given only at the beginning of the third part (§**166**, p. 180).

The problem of integral calculus will be to determine the integral varieties of a completely integral system: we know that it leads to the integration of differential systems (c.f. §**84**, *Remark I*, p. 97)

I. Transitive group realising a given abstract group

116 Construction of transitive groups realising a given abstract group. Consider an abstract group, i.e., a parameter group G. We assume it to be defined in a domain D of a r dimensional space by r analytic equations

$$\xi'_p = \Phi_p(\xi_1, \ldots, \xi_r; a_1, \ldots, a_r), \qquad (p = 1, \ldots, r). \tag{1}$$

It is characterised by the property of leaving invariant r forms

$$\omega_p(\xi, d\xi).$$

Consider a transitive group isomorphic to this abstract group:

(P) $\left\{\begin{array}{c} \text{We suppose that this transitive group operates on a domain } D \\ \text{of a } n \text{ dimensional space } (n < r)\text{, that it is defined by } n \text{ analytic equations} \\ x'_i = \varphi_i(x_1, \dots, x_n; a_1, \dots, a_r), \qquad (i = 1, \dots, n) \\ \text{and the subgroup leaving a point of } D \text{ fixed is connected } ^{(\dagger)}. \end{array}\right.$

We know that this transitive group can be prolonged holoedrically, by introducing new variables $u_1, \dots, u_{r-n}$, and a group similar to the parameter group G (§**100**, 111). If we substitute the new variables $(x_1, \dots, x_n; u_1, \dots, u_{r-n})$ for the variables $(\xi_1, \dots, \xi_r)$, we can always find n distinct linear combinations of ω_p with constant coefficients such that the differentials du_k do not appear (§**109**, p. 122). If we set these n linear combinations to zero, they constitute manifestly a completely integrable system, whose integral varieties are the surfaces $x_i =$ constant.

Therefore $x_1, \dots, x_n$ can be considered as parameters serving to distinguish the integral varieties of a completely integral system obtained by equating n independent linear combinations of the forms $\omega_p(\xi, d\xi)$ with constant coefficients to zero.

This theorem gives us an analytic procedure allowing us to construct all transitive groups possessing the three properties (P) realising a given abstract group.

117 Discussion. Conversely, consider such a completely integrable system obtained by equating to zero certain linear combinations of the forms $\omega_p(\xi, d\xi)$ with constant coefficients. The group G leaves the forms $\omega_p(\xi, d\xi)$ invariant, and therefore it transforms the integral varieties V of this completely integral system among themselves, and G operates transitively on the class of objects constituted by these varieties. But two problems arise: is the group of $G(V)$ transforming the varieties V among themselves defined by the analytic equations, and is it holoedrically isomorphic or meriedrically isomorphic to the given abstract group G?

Example I. Consider the displacement group. Its parameter group G leaves the six well known forms invariant: ω_1, ω_2, ω_3, ω_{23}, ω_{31}, ω_{12}.

Let us show the system

$$\omega_{23} = 0, \qquad \omega_{31} = 0, \qquad \omega_{12} = 0$$

is completely integrable. Finding in the parameter space the varieties on which these three equations are satisfied is equivalent to finding the family of trihedrals whose instantaneous rotation is constantly zero: these families are the trihedrals with parallel edges. Then the variety V is constituted by all trihedrals parallel to a given trihedral depends on three parameters. This proves that the system

$$\omega_{23} = 0, \qquad \omega_{31} = 0, \qquad \omega_{12} = 0$$

(†) In other words we suppose that the points of D are oriented objects (c.f. §**93**, p. 105).

is completely integrable and its integral varieties are the various varieties V.

$G(V)$ is the displacement group of these varieties left fixed by all transformations. $G(V)$ is therefore not holoedrically, but meriedrically isomorphic to G.

Example II. Suppose that the domain D on which the parameter group G is defined is the surface of a torus. The coordinates (ξ, η) of every point on this torus are defined up to $\pm 2k\pi$, the coordinate lines being the parallels and the meridians. Suppose that the equations of G are

$$\xi' = \xi + a, \qquad \eta' = \eta + b.$$

The forms are $\omega_p\ d\xi$ and $d\eta$. Every linear combination with constant coefficient $\alpha\, d\xi + \beta\, d\eta$ equated to zero is completely integrable and the corresponding varieties V have the equation $\alpha\xi + \beta\eta =$ constant.

The transformations $\xi' = \xi - t\beta$, $\eta' = \eta + t\alpha$ depending on one parameter t leave each of the varieties V invariant: the group $G(V)$ hence can only be meriedrically isomorphic to the group G. On the other hand, when $\dfrac{\beta}{\alpha}$ is irrational, each of the varieties V pass arbitrarily close to each of the point of the torus and the group $G(V)$ hence cannot by defined by analytic equations.

118 Distinction between holoedric and meriedric isomorphism. We suppose that the second difficulty indicated in the preceding paragraph does not arise: a variety V does not pass through the neighbourhood of any point of D an infinite number of times. Then these varieties V, when we construct them by integrating analytic differential equations [(†)], are analytic and the equations of $G(V)$ are also analytic.

We are going to indicate how we can determine if the isomorphic existing between G and $G(V)$ is holoedric or meriedric. Let us substitute the parameters $(\xi_1, \ldots, \xi_r)$ with the parameters $(x_1, \ldots, x_n; u_1, \ldots, u_{r-n})$ such that the varieties V have equations $x_1 =$ constant, $\ldots$, $x_n =$ constant. Suppose on the other hand that ω_1, $\ldots$, ω_n are forms that, when equated to zero, define these varieties V. The differentials du_j are absent from these n forms.

A first case which can arise is the following: the variables u_j are also absent from the n forms.

Given two varieties $V^0(x_i^0)$ and $V^1(x_i^1)$, there exists a transformation of G transforming a given fixed point (x_i^0, u_k^0) of V^0 into any point (x_i^1, u_k^1) of V^1, the transformations of G transforming V^0 into V^1 hence depend on $r - n$ parameters. But every transformation of $G(V)$ leave the n linearly independent forms $\omega_i(x, dx)$ invariant. Hence there exists at most one transformation transforming V^0 into V^1 (§**83**, p. 96). The group $G(V)$ is therefore simply transitive and hence meriedrically isomorphic to G.

If the preceding case does not arise, then a certain number of the forms ω_1, $\ldots$, ω_n

[(†)]C.f. §**84**, *Remark I*, p. 97.

effectively depend on some of the parameters $u_1, \ldots, u_{r-n}$. We have

$$\omega_i = \sum_{k=1}^{n} \alpha_{ik}(x,u)dx_k, \qquad (i = 1, \ldots, n).$$

If necessary, we modify the definition of the parameters u_j such that those that α_{ik} do not depend on are $u_{p+1}, \ldots, u_{r-n}$ and to two different systems of values of $u_1, \ldots, u_p$ always correspond two differential values of α_{ik}. Let us find how the group operate on the p quantities $u_1, \ldots, u_p$. It suffices to use the equations

$$x'_i = \varphi_i(x_1, \ldots, x_n; a_1, \ldots, a_r), \tag{2}$$

which indicates how G transforms the variables x_i. Indeed, we have

$$\sum_{k=1}^{n} \alpha_{ik}(x,u)dx_k = \sum_{l=1}^{n} \alpha_{il}(x',u')dx'_l,$$

hence

$$x_{ik}(x,u) = \sum_{l=1}^{n} \alpha_{il}(x',u') \frac{\partial \varphi_l(x_1, \ldots, x_n; a_1, \ldots, a_r)}{\partial x_k},$$

and from these relations follow the equations showing how the variables $u_1, \ldots, u_p$ are transformed:

$$u_h = \Psi_h(x'_1, \ldots, x'_n; u'_1, \ldots, u'_p; a_1, \ldots, a_r), \qquad (h = 1, \ldots, p). \tag{3}$$

We thus see that the group G transforms the variables $x_1, \ldots, x_n$; $u_1, \ldots, u_p$ among themselves and every operation of G transforming $x_1, \ldots, x_n$ identically transforms $u_1, \ldots, u_p$ identically.

Let Γ_1 be the transformation group of the variables $x_1, \ldots, x_n, u_1, \ldots, u_p$. The problem of finding if $G(V)$ is isomorphic to G is transformed into a group of finding if Γ_1 [which is furthermore a holoedric prolongation of $G(V)$] is isomorphic to G. But Γ_1 operate on a larger number of variables than $G(V)$.

We will then proceed with Γ_1 as we have done with $G(V)$: either Γ_1 is simply transitive or we can prolong it holoedrically into a group Γ_2 operating on a larger number of variables: $x_1, \ldots, x_n$; $u_1, \ldots, u_p$; $v_1, \ldots, v_q$.

We apply this procedure recurrently, which is very much analogous to the one employed in paragraph **§112** (p. 125). Finally we arrive at a simply transitive group Γ_α. If it is of less than r variables, $G(V)$ is meriedrically isomorphic to G; if it is of r variables, the transformations of G leaving every point of the domain invariant operated by Γ_α constitute a disconnected subgroup of G: the isomorphism is holoedric if the subgroup reduces to the identical transformation.

II. Intransitive group realising a given group

119 Construction of intransitive groups realising a given abstract group. Consider again a abstract group G defined by analytic equations and let $\omega_1(\xi, d\xi)$, …, $\omega_r(\xi, d\xi)$ be the Pfaffian forms that it leaves invariant. Let Γ be an intransitive group with p invariants, isomorphic to G and also defined by analytic relations

$$x'_i = \varphi_i(x_1, \ldots, x_n; a_1, \ldots, a_r), \qquad (i = 1, \ldots, n). \tag{2}$$

Let us prolong G by the addition of p variables y_1, …, y_p transformed identically. We can prolong Γ holoedrically by the addition of variables u_1, …, u_{r+p-n} so as to obtain a group identical (up to a similitude) to the prolongation of group G, and furthermore y_1, …, y_p appear as functions of x_1, …, x_n.

Let D be the $r + p$ dimensional domain on which this prolongation common to the group G and Γ operates. Two systems of coordinates are defined on it, namely

$$(\xi_1, \ldots, \xi_r; y_1, \ldots, y_p) \qquad \text{and} \qquad (x_1, \ldots, x_n; u_1, \ldots, u_{r+p-n}).$$

We consider in D:

1. The $r + p - n$ dimensional varieties V having equations $x_1 =$ constant, …, $x_n =$ constant transformed among themselves by the group;
2. The r dimensional varieties δ having equations $y_1 =$ constant, …, $y_p =$ constant left invariant by the group and generated by the varieties V;
3. A p dimensional analytic variety λ_0 having one and only one point in common with each of the variety δ (c.f. §**103**, p. 114). The coordinates y_1, y_2, …, y_p suffice to characterise a point of λ_0.

We construct at each point of λ_0 n Pfaffian forms in ξ_j, y_l, $d\xi_j$, dy_l which are linearly independent and which vanish when the displacements $d\xi_j$, dy_l are along the variety V passing through the point considered. We choose p of these forms to be equal to the differentials dy_1, …, dy_p and the other $n - p$ to be of the type

$$\sum_{1 \le s \le r} \alpha_{is}(y)\omega_s(\xi, d\xi). \tag{4}$$

The operations of the group leaving these n forms invariant, and therefore they are zero along each of the varieties V. This result constitutes the generalisation to the case of intransitive group of the theorem that we have stated in paragraph §**116** (p. 133).

The varieties V constitute the integral varieties of a completely integrable Pfaffian system obtained by equating to zero the p differentials dy_l and $n - p$ Pfaffian forms of type (4).

120 Discussion. Conversely, consider such a system. The group transforms the varieties V into each other since it leaves the forms ω_p and the variables y_l invariant. Through each point of D there passes a variety V. The invariant varieties δ can be generated with the varieties V. The transformation group for the varieties V is therefore an intransitive group with p invariants and which is holoedrically or meriedrically isomorphic to G. But it is not guaranteed to be analytic. When it is analytic we can determine with the help of a procedure analogous to the one in paragraph **§118** if it is effectively isomorphic to G.

Example. Suppose that g is the translation group

$$\xi_1' = \xi_1 + a_1, \qquad \dots, \qquad \xi_r' = \xi_r + a_r. \tag{1'}$$

This group is a parameter group. The forms $\omega_s(\xi, d\xi)$ are the differentials $d\xi_s$. Every system

$$\begin{array}{ll} dy_1 = 0, & \alpha_{1,1}(y)d\xi_1 + \cdots + \alpha_{1,r}(y)d\xi_r = 0, \\ \dots & \dots \\ dy_p = 0, & \alpha_{n-p,1}(y)d\xi_1 + \cdots + \alpha_{n-p,r}(y)d\xi_r = 0 \end{array}$$

is completely integrable.

The integral varieties V have equations

$$\begin{array}{ll} y_1 = \text{const.}, & \alpha_{1,1}(y) + \cdots + \alpha_{1,r}(y)\xi_r = \text{const.}, \\ \dots & \dots \\ y_p = \text{const.}, & \alpha_{n-p,1}(y) + \cdots + \alpha_{n-p,r}(y)\xi_r = \text{const.} \end{array}$$

These n constants serve to distinguish the varieties V. We will call them $x_1, \dots, x_p, x_{p+1}, \dots, x_n$. The transformation (1′) transforms the variety V with parameters $(x_1, \dots, x_n)$ into the variety V' with parameters

$$\left\{\begin{aligned} x_1' &= x_1, \\ &\dots \\ x_p' &= x_p, \\ x_{p+1}' &= x_{p+1} + \alpha_{1,1}(x_1, \dots, x_p)a_1 + \cdots + \alpha_{1,r}(x_1, \dots, x_p)a_r, \\ &\dots \\ x_n' &= x_n + \alpha_{n-p,1}(x_1, \dots, x_p)a_1 + \cdots + \alpha_{n-p,r}(x_1, \dots, x_p)a_r. \end{aligned}\right. \tag{2'}$$

Every intransitive group with p invariants whose equations are analytic and whose parameter group is the translation group (1′) is therefore similar to a group of type (2′).

121 Remark on the analytic nature of realisations of an abstract group. We have assumed above that the functions $\alpha_{i,j}$ are analytic. But if, in the formulae (2′), we choose the absolutely arbitrarily functions for $\alpha_{i,j}$, we again obtain a realisation of the translation group (1′). We can then construct examples of intransitive groups realising an abstract analytic group without being either analytic or similar to any analytic group.

On the contrary, the study that we have made on the groups realising an abstract analytic group allows us to establish the following theorem:

Every transitive group whose defining equations are continuous and who realises an analytic abstract group and is similar to an abstract analytic group is similar to a group whose defining equations are analytic.

III. Complements

122 Preliminaries. In the course of this chapter we have established the following result (c.f. §**116**, p. 133 and §**119**, p. 137). Consider an abstract group G. Every class of oriented objects transformed by analytic operations is equivalent to the class of objects constituted by the v dimensional integral varieties of a completely integrable system formed by equating to zero $r - v$ linear combinations of Pfaffian forms $\omega_1, \ldots, \omega_r$. These combinations are of constant coefficients if G operates transitively on this class, otherwise the parameters will appear in them (the invariants $y_1, \ldots, y_p$. This result must be compared to the conclusions of paragraphs §**95** and §**96** (p. 107 and 108): this comparison suggests that the varieties that we just talked about are identical to those constituted by the objects of "bodies". We are going to establish this identity by demonstrating the converse theorems in paragraphs §**123** and §**124**.

To reason in a geometrical way and to prepare for the following chapter, let us recall that a point ξ of the parameters space is the image, not only of the transformation S_ξ, but of the frame $\mathbf{R}_\xi = S_\xi \mathbf{R}_0$ as well. A variety that constitutes an object of bodies is the image of the families of frames attached at this object. A variety along which one of the Pfaffian forms ω_p is zero is the image of a family of frames whose infinitesimal displacements inside this family annihilate the component ω_p.

With these notions, let us prove the following two theorems.

123 Theorem. *Consider the family of frames* $\mathbf{R}$ *attached to an object* [(†)]. *Suppose that this family depends analytically on v parameters. We claim that there exists exactly $r - v$ independent linear combinations of the components ω_p of the infinitesimal displacement of these frames which are zero when these displacements are inside the family.*

The components ω_p of the instantaneous displacement of a frame in the family depend on v parameters. We can therefore construct $r - v$ independent linear combinations with constant coefficients which are zero for a particular position $\mathbf{R}_1$ of $\mathbf{R}$. Suppose that these are $\omega_{v+1}, \ldots, \omega_r$. Let $\mathbf{R}_2$ be a second frame in the family: $\mathbf{R}_2 = S\mathbf{R}_1$. If S remains fixed, we can vary infinitesimally $\mathbf{R}_1$ inside the family. We can also vary infinitesimally $\mathbf{R}_2$ inside the family such that its relative infinitesimal displacement is identical to that of $\mathbf{R}_1$. Hence the components $\omega_{v+1}, \ldots, \omega_r$ are also zero for this displacement of $\mathbf{R}_2$. Q.E.D.

(†) C.f. §**93**, p. 105

124 Converse theorem. *Suppose that a family of frames depends analytically on v parameters and $r - v$ linearly independent components of the instantaneous relative displacements of the frames of this family are identically zero. Then*

1. *The system obtained by equating to zero these $r - v$ components is a completely integrable system;*
2. *This family is the family attached at one of the objects operated on by the group.*

Indeed, the transformed images of this family by the operations of the group possess the same property: their images in the parameter space constitue the v dimensional varieties along which the $r - v$ components considered are zero. There passes one of these varieties through every point: therefore the system obtained by equating to zero these $r - v$ components is completely integrable, and it is impossible [†] that two of these varieties, i.e., two of the families of frames, have a common element. From this last fact results that the transformations of the group transforming the frames in the given family into each other transform this family into itself. This property characterises the families attached to the objects transformed by the group. Q.E.D.

125 Corollary. A "body" is constituted by the image variety of a subgroup of G in the parameter space and by the transformed images of this variety (§**95**). The two preceding theorems hence have the following consequences:

THEOREM. *For a v dimensional variety of the parameter space passing through the origin* [‡] *to be the image of a subgroup, it is necessary and sufficient that the $r - v$ independent linear combinations with constant coefficients of the forms $\omega_p(\xi, d\xi)$ are constantly zero along this variety. Then the system obtained by equating to zero these $r - v$ combinations is completely integrable.*

[†] By virtue of the theorem of paragraph §**81**, p. 93.

[‡] The image of the identity.

Chapter 10

Differential geometry

I. The method of moving frames

126 Introduction. Consider a connected and finite dimensional group G operating on the points of a domain D (†). We will propound to *identify* (‡) the *contact elements* of various orders of λ dimensional varieties, V_λ, traced in D. We know that it is *theoretically possible* to effect this operation (chapter 6, section iii.). We are going to develop the "method of moving frame" which will allow us to effect it *practically*.

This method is a generalisation of the method of moving trihedral what we have applied in the first part of this work to several particular cases (§), the group G being the displacement group. The method of moving frame is founded on the study of the relative components of moving frames. It is therefore rightly linked to the properties of parameter groups, whose importance, we know, is fundamental.

Once we have identified the contact elements, the *contact problems* will then be already solved. This identification constitutes equally a first step for solving the *equality problem*: to know if there exists transformations of G superimposing two given varieties in D. Let us also remark that the *application problem* (¶) of a variety onto another will be solved immediately when we have preliminarily studied the contact elements of two varieties by the method of moving frame.

127 Statements of the properties imposed on frames and invariants of various orders. We will subject the solution of the identification problem that we have just put forward to certain conditions. This paragraph will state these conditions. At the end of paragraph §**129**, it will become evident that we have respected these conditions.

(†) More generally on the objects, or elements, or a "body". In this case what we call in this section "point" must be interpreted as an element of a body under consideration. We have already considered from this point of view (chapter 3 and 4) the theory of ruled surfaces, where they are considered as varieties of *straight lines* (in this regard they are one dimensional varieties).

(‡) C.f. paragraph §**93** and §**94**, p. 105.

(§) The reader is urged to review the most complicated of these cases: those treated in chapter 4, p. 57.

(¶) We will say something about the application problems in paragraph §**196** (p. 213).

To a point **A** of a variety V_λ we will attach an infinite number of families of frames: the frames of order 0, …, of order P, …. Each of these families will contain the next one [†].

To a point **A** we will also attache the following series of numbers: μ_0 invariants of order 0, …, $\mu_P - \mu_{P-1}$ invariants of order P, ….

*The contact element of order P of a point **A** of V_λ will be distinguished by the family of frames of order P of this point and by the set of invariants of order $\leq P$ of the point.*

In other words, every frame attached at a contact element of order $P+1$ will be part of the set of frames attached at a contact element of order P. Every invariant attached to a contact element of order P will be part of the set of invariants attached to the contact element of order $P+1$.

This way of identifying them that we have chosen therefore makes it obvious that the contact condition of order P is incorporated in the contact condition of order $P+1$.

The family of P-th order frames of a point **A** in V_λ may be disconnected: it decomposes into several connected sub-families. The system of invariants of order $\leq P$ and each of these sub-family constitute, by definition [‡], *an oriented contact element of order P*. Every contact element then decomposes into a certain number of oriented contact elements.

Let us specify an oriented contact element of order P. Its most general P-th order frame depends on v_P parameters, which we will call the *secondary parameters of order P*. According to the theorem of paragraph §**123** (p. 139), the infinitesimal displacements of this frame are characterised by the property that they annihilate $r - v_P$ relative components. More precisely, the infinitesimal displacements of the most general frame of order $P-1$ attached to the given contact element annihilate $r - v_{P-1}$ independent linear combinations of $\omega_1, \dots, \omega_r$. The infinitesimal displacement of P-th order frame annihilates these $r - v_{P-1}$ linear combinations and $v_{P-1} - v_P$ others

$$\pi_\alpha = \sum_{p=1}^{r} a_{\alpha p}\omega_p, \qquad (r - v_{P-1} < \alpha \leq r - v_P). \tag{1}$$

We will call these the *principal components of order P*. The $a_{\alpha p}$ are assumed to be constants or functions of invariants of orders $\leq P$. This hypothesis is legitimate, since the data of these invariants determines the nature of infinitesimal displacements that we investigate.

Let us consider the most general contact element of order 0, i.e., a point **A** of D. According to the conclusion of paragraph §**109** (p. 122, the differentials of the coordinates of **A** are linear combinations of the principal components of order 0 and the differentials of the invariants of order 0. Consider a variable point **A** on a variety V_λ. We call the parameters that the points of V_λ depend on the *principal parameters*. The differentials of the principal parameters are therefore linear combinations of λ suitably chosen forms

[†] Or they are identical.

[‡] C.f. paragraph §**93**, p. 105.

among the principal components of order 0 and the differentials of the invariants of order 0. We will simplify the notations by supposing that these λ forms are the first λ principal components of order zero, which we denote by

$$\pi_1, \quad \ldots, \quad \pi_\lambda.$$

Consider a varying frame which always remains a frame of order P of V_λ. It depends on λ principal parameters of V_λ and on v_P secondary parameters of order P. By definition, its principal components of order $\leq P$ are independent of the differentials of the secondary parameters. They are therefore linear combinations of $\pi_1, \ldots, \pi_\lambda$. *The principal components of orders $< P$ of the frames of order P of V_λ will be linear combinations of $\pi_1, \ldots, \pi_\lambda$ with coefficients functions of invariants of orders $\leq P$.*

The differentials of the P-th order invariants and the P-th order principal components of the P-th order frames of V_λ are also linear combinations of $\pi_1, \ldots, \pi_\lambda$:

$$\text{(2)} \qquad \begin{cases} dk_\alpha = b_{\alpha 1}\pi_1 + \cdots + b_{\alpha\lambda}\pi_\lambda & (\mu_{P-1} < \alpha \leq \mu_P), \\ \pi_\alpha = b'_{\alpha 1}\pi_1 + \cdots + b'_{\alpha\lambda}\pi_\alpha & (r - v_{P-1} < \alpha \leq r - v_P). \end{cases}$$

The coefficients $b_{\alpha\beta}$, $b'_{\alpha\beta}$ are functions of invariants of orders $\leq P$ and, in general the secondary parameters of order P. We will call them *the coefficients of order P.*

128 Use of a recurrent contact condition. By hypothesis we know how to identify the points of D, i.e., the contact elements of order zero. Often geometrical intuition will also allow us to identify first order contact elements. Here let us proceed by recursion.

Suppose that we have managed to define the invariants of orders $\leq P$ and the frames of order $\leq P$, the properties stated in the preceding paragraph being verified by these invariants, by these frames, and by the coefficients of order P.

Let us apply the following *contact condition*:

For two varieties V_λ and $V^\star_\lambda$ to have a contact of order $\geq P+1$ at a point $\mathbf{A}_0$, it is necessary and sufficient that at every point $\mathbf{A}$ in an infinitesimal neighbourhood V_λ of $\mathbf{A}_0$ there corresponds a point $\mathbf{A}^\star$ in $V^\star_\lambda$ such that the conditions expressing that V_λ and $V^\star_\lambda$ have a contact of order $\geq P$ at points $\mathbf{A}$ and $\mathbf{A}^\star$ are verified up to infinitesimally small quantities of order greater than the distance $\overline{\mathbf{A}_0\mathbf{A}}$ of the points $\mathbf{A}_0$ and $\mathbf{A}$.

This condition shows immediately that the frames of order P and the invariants of orders $\leq P$ defined by recurrence in the paragraphs below allow us to identify the contact elements of order P.

129 The mechanism of the method of moving frame [†]. Suppose that we know:

1. The number of invariants of orders 1, 2, ..., P;
2. The definition of frames of orders 1, 2, ..., P;

[†] Paragraph §**172**, p. 190 indicates a better way to apply the operations of this mechanism rapidly.

3. The definition of the principal components of orders $< P$ [c.f. (1)], their expressions as functions of $\pi_1, \ldots, \pi_\lambda$ and invariants of orders $\leq P$; the expressions of the differentials of invariants of orders $1, \ldots, P-1$ as functions of $\pi_1, \ldots, \pi_\lambda$ and invariants of order P.

We obtain the corresponding data of order $P+1$ as the following.

We *orient*, if necessary, the contact element of order P, in a manner such that the family of frames of order P is connected.

We define the *principal components of order* P.

We determine how $\pi_1, \ldots, \pi_\lambda$ and the principal components of order P depend on secondary parameters of order P. We deduce from them *how the coefficients of order* P *depend on the secondary parameters of order* P.

We arrange these secondary parameters of order P in a way to *establish among the coefficients of order* P *the largest number of relations which are as simple as possible.* To these values of secondary parameters correspond particular order P frames: they are the *frames of order* $P+1$, and the values taken by the coefficients of order P which are not numerical constants constitute the *invariants of order* $P+1$. The formulae (2) then give the *expressions of the principal components of order* P and the *expressions of the differentials of the invariants of order* P as functions of $\pi_1, \ldots, \pi_\lambda$ and of invariants of order $P+1$.

Remark. The number of principal components of order P is the difference $v_{P-1} - v_P$ of the number of secondary parameters of orders $P-1$ and P: this is the number of relations that we have established among the coefficients of order $P-1$ to define the frames of order P.

130 Important remark. Choosing at certain exceptional points the secondary parameters in a way to establish among the coefficients of order P the desired relations (§**129**) could be impossible, and also could be possible in an infinite number of ways (†). This difficulty can also arise at all points in certain kinds of varieties: the principal components may be identically zero (‡), and the P-th order invariants may be constant on the variety (§). For each of these categories of varieties that can appear we must apply the method of moving frames in a special manner.

If from order $P+1$ onwards two varieties belong to two different categories, it is then impossible to realise between them a contact of order greater than P, except maybe at certain exceptional points (¶).

(†) *Example.* At an inflection point of a real curve all first order trihedrals satisfy the condition $\omega_{13} = 0$ with which the second order trihedrals are identified in general among the family of first order trihedrals (§**18**, p. 25).

(‡) *Example.* In the study of real curves we meet the category of straight lines for which ω_{13} is identically zero.

(§) *Example.* The isotropic ruled surfaces on which k is constant (chapter 2, section ii., p. 35).

(¶) For example a real curve cannot have a contact of order greater than 1 with a straight line except at its inflection points.

131 Frenet frame; equality condition. Given a variety V_λ let us determine successively the frames and the invariants of the contact elements of various orders. We will find at the end an order Q such that the frames of order $Q+1$ are identical to those of order Q and the invariants of order $Q+1$ are functions of the invariants of order $\leq Q$: indeed, the number of secondary parameters cannot decrease by more than v_0 and the number of invariants not linked by any relations on V_λ cannot exceed the number of dimension λ of V_λ. The identification of contact elements of orders greater than $Q+1$ then proceed very easily. The frames of orders greater than Q are identical to those of frames of order Q, the invariants of order $Q+2$, of order $Q+3$, ..., are the partial derivatives of orders 1, 2, ..., of the invariants of order $Q+1$ considered as functions of a system of invariants of orders $\leq Q$ which are independent on V_λ. We will call the frames of order Q the "Frenet frames", by analogy with the case of real curves.

Suppose that the fundamental group is composed of analytic transformations and consider two analytic varieties. To superimpose them is to realise at two of their points a contact of order greater than any integer P [†]. Then

The necessary and sufficient conditions for two varieties to be equal are the following: they most belong to the same category (§**130**) *and their invariants of order* $\leq Q+1$ *must be linked by the same system of relations.*

Then the operations of group G superimposing them are those that superimpose two of their Frenet frames at two points for which the invariants of order Q are the same.

On the other hand, a suitable use of the fundamental condition of equality (§**76**, p. 89) also allows us to get this result and frees us from the hypothesis that the transformations of the group and the varieties considered are analytic.

Application. Suppose that the Frenet frames of a variety V_λ still depend on secondary parameters. The solution that we just give to the equality problem tells us that the variety V_λ is transformed into itself by the connected subgroup of G transforming the Frenet frames of the same point of V_λ among themselves.

132 Problem. *Under what conditions does a variety V_λ possess the following property:* [‡] *given any two points of V_λ there always exists an operation of G transforming V_λ into itself and transforming the two points into each other?*

We are going to prove that the answer is the following: *the invariants of various orders must all be constant on V_λ.*

It is obvious that this condition is necessary. Let us prove that it is sufficient. If it holds then the Frenet frames are the first frames such that the relations among their principal components are constant coefficients. The number of these relations with constant coefficients is $r - v_Q - \lambda$. The Frenet frames of V_λ, which depend on $v_Q + \lambda$ parameters, therefore constitute, according to the theorem of paragraph §**124** (p. 140), the frames

[†] C.f. paragraph §**13**, first indentation, p. 22.

[‡] If G is the displacement group, then the sphere possesses this property.

attached at an object transformed by G: the subgroup transforming these frames into each other leaves the family they constitute, i.e., V_λ, invariant. Our proposition is then stablished, and we observe on the other hand that we can choose the family of frames attached at the object constituted by V_λ the family of Frenet frames of this variety [(†)].

II. Unimodular affine geometry; study of real place curves

133 Introduction. We will apply the method of moving frame to the study of real plane curves, relative to the group of unimodular affine transformations G. This group is defined by the equations

$$\begin{cases} x' = ax + by + c, \\ y' = a'x + b'y + c', \end{cases} \qquad \begin{vmatrix} a & b \\ a' & b' \end{vmatrix} = 1. \tag{1}$$

It is composed of the set of affine transformations (§**65**, p. 75) conserving the area. As in the case of affine group, we represent the *absolute frame* $\mathbf{R}_0$ by the vectors $\vec{\mathbf{I}}_1^0$ and $\vec{\mathbf{I}}_2^0$ having the point $(0,0)$ as the origin and the points $(1,0)$ and $(0,1)$ as the tips. A *frame* $\mathbf{R}$ is composed of two vectors $\vec{\mathbf{I}}_1$ and $\vec{\mathbf{I}}_2$ having the same origin and satisfying the relation [(‡)]

$$\vec{\mathbf{I}}_1 \wedge \vec{\mathbf{I}}_2 = 1. \tag{2}$$

To characterise the position of a frame $\mathbf{R}_{da}$ infinitesimally close to $\mathbf{R}_0$, it suffices to specify the coordinates $\omega_1(0,da), \omega_2(0,da)$ of $\mathbf{A}$, the components $1+\omega_{11}(0,da), \omega_{12}(0,da)$ of $\vec{\mathbf{I}}_1$ and the components $\omega_{21}(0,da)$, $1+\omega_{22}(0,da)$ of $\vec{\mathbf{I}}_2$. On the other hand we have, according to (2),

$$\omega_{11}(0,da) + \omega_{22}(0,da) = 0.$$

The independent linear forms

$$\omega_1(0,da), \quad \omega_2(0,da), \quad \omega_{11}(0,da), \quad \omega_{12}(0,da), \quad \omega_{21}(0,da)$$

the total number of which is the number of parameters of the unimodular affine group, will be chosen as the components of infinitesimal displacement transforming $\mathbf{R}_0$ to $\mathbf{R}_{da}$.

From here, a reasoning analogous to the one developed in paragraph §**72** (p. 84) with respect to the affine group shows that the *relative components* of a frame $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2$ are the five forms ω defined by the formulae

$$\begin{cases} \overrightarrow{d\mathbf{A}} = \omega_1\ (a,da)\vec{\mathbf{I}}_1 + \omega_2\ (a,da)\vec{\mathbf{I}}_2, \\ d\vec{\mathbf{I}}_1 = \omega_{11}(a,da)\vec{\mathbf{I}}_1 + \omega_{12}(a,da)\vec{\mathbf{I}}_2, \\ d\vec{\mathbf{I}}_2 = \omega_{21}(a,da)\vec{\mathbf{I}}_1 - \omega_{11}(a,da)\vec{\mathbf{I}}_2. \end{cases} \tag{3}$$

(†) In paragraph §**44** (chapter 3, p. 51) we have attached at a straight line the trihedrals whose first vector is along this straight line. These trihedrals are the first order trihedrals of the curve constituted by this straight line.

(‡) This relation expresses that the parallelogram whose sides are $\vec{\mathbf{I}}_1$ and $\vec{\mathbf{I}}_2$ has its volume equal to $+1$.

134 Definition of elements of orders 0 **and** 1. A contact element of order zero is constituted by a point, a contact element of order 1 by a point and a tangent. There is hence no invariants of order 0 or 1. By definition the *frames of order* 0 of a point **A** will be the frames $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2$ with apex **A**, the *frames of order* 1 of a point **A** will be the frames $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2$ whose first vector $\vec{\mathbf{I}}_1$ is tangent to the curve. The frames of order 0 constitute a connected family. The frames of order 1 constitutes two connected families. A contact element of order 1 therefore decomposes into two oriented contact elements.

The group G depends on $r = 5$ parameters. The frames of order 0 and 1 depend respectively on $v_0 = 3$ and $v_1 = 2$ parameters. The number of principal components of order 0 is therefore $r - v_0 = 2$ and that of the principal components of order 1 is $v_0 - v_1 = 1$.

When a frame of order 0 varies while its origin remains fixed, $\overrightarrow{d\mathbf{A}} = 0$, hence $\omega_1 = \omega_2 = 0$. ω_1 and ω_2 are hence the principal components of order 0. On the other hand when a frame varies while remaining a frame of order 1, $\overrightarrow{d\mathbf{A}}$ is parallel to $\vec{\mathbf{I}}_1$, hence $\omega_2 = 0$. The principal components of order 0 of the frames of order 1 are therefore

Order 0
$\omega_1, \quad \omega_2(=0)$

135 Definition of elements of order 2. Consider two variable frames of order 1 of a curve: $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2$ and $\mathbf{A}\vec{\mathbf{J}}_1\vec{\mathbf{J}}_2$. Let $\bar{\omega}$ and ω be their relative components, b and $\bar{b}$ their coefficients of order 1. We have

$$\begin{cases} \vec{\mathbf{I}}_1 = \lambda \vec{\mathbf{J}}_1, \\ \vec{\mathbf{I}}_2 = \dfrac{1}{\lambda}(\vec{\mathbf{J}}_2 + \mu \vec{\mathbf{J}}_1). \end{cases} \tag{4}$$

According to the preceding paragraph, there exists a single principal component of order 1. According to (4), when **A** is fixed, $d\vec{\mathbf{I}}_1$ is parallel to $\vec{\mathbf{I}}_1$, hence $\omega_{12} = 0$: ω_{12} is the principal component of order 1. We define the coefficient b of order 1 by the relation $\omega_{12} = b\omega_1$.

Let us calculate ω_1 and ω_{12} as functions of $\bar{\omega}_1$ and $\bar{\omega}_{12}$ and the secondary parameters λ and μ. We have

$$\begin{cases} \omega_1 = \overrightarrow{d\mathbf{A}} \wedge \vec{\mathbf{I}}_2 = \dfrac{1}{\lambda}\overrightarrow{d\mathbf{A}} \wedge (\vec{\mathbf{J}}_2 + \mu\vec{\mathbf{J}}_1) = \dfrac{1}{\lambda}\bar{\omega}_1, \\ \omega_{12} = \vec{\mathbf{I}}_1 \wedge d\vec{\mathbf{I}}_1 = \lambda^2 \vec{\mathbf{J}}_1 \wedge d\vec{\mathbf{J}}_1 \qquad\qquad = \lambda^2 \bar{\omega}_{12}, \end{cases} \tag{5}$$

from which

$$b = \lambda^3 \bar{b}. \tag{6}$$

By definition, the frames of order 2 are the frames of order 1 such that $b = 1$. There is no invariant of order 2. The principal components of order ≤ 1 of frames of order 2 are

Order 0	Order 1
$\omega_1, \quad \omega_2(=0)$	$\omega_{12}(=\omega_1)$

An exceptional category of curves arise for which the preceding definition does not make sense: the curves along which the principal component of order 1, ω_{12}, is constantly zero. The circumstances studied in paragraph §**132** (p. 145) are then realised: the operations of G transforming the frames of order 1 of the curve into each other constitute a subgroup leaving this curve invariant. On the other hand it is evident that this category of curves is the family of straight lines.

136 Definition of elements of order 3. By replacing in (6) b and $\bar{b}$ by 1 we obtain $\lambda^3 = 1$, i.e., since λ is real, $\lambda = 1$. The formulae allowing use to compare two frames of order 2 $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2$ and $\mathbf{A}\vec{\mathbf{J}}_1\vec{\mathbf{J}}_2$ are hence obtained by replacing in (4) λ by 1:

$$\begin{cases} \vec{\mathbf{I}}_1 = \vec{\mathbf{J}}_1, \\ \vec{\mathbf{I}}_2 = \vec{\mathbf{J}}_2 + \mu\vec{\mathbf{J}}_1. \end{cases} \tag{7}$$

These formulae (7) show that the family of frames of order 2 is connected: a contact element of order 2 possesses a single orientation. It attributes a determined orientation to the tangent of the curve.

Since we have defined the frames of order 2 by imposing a condition on the coefficient of order 1, there exists *one* principal component of order 2. When $\mathbf{A}$ remains fixed, $\vec{\mathbf{J}}_1$ remains fixed, hence $\omega_{11} = 0$. ω_{11} is the principal component of order 2. The coefficient b of order 2 will be defined by the relation $\omega_{11} = b\omega_1$.

Let us calculate ω_1 and ω_{11} as functions of $\bar{\omega}_1$ and $\bar{\omega}_{11}$ and of μ. By replacing (4) λ by 1 we obtain $\omega_1 = \bar{\omega}_1$. ω_1 is hence independent of the second order parameter. It is a second order invariant form. We denote it by $d\sigma$ and we name σ the *affine arc.*

We then have

$$\omega_{11} = d\vec{\mathbf{I}}_1 \wedge \vec{\mathbf{I}}_2 = d\vec{\mathbf{J}}_1 \wedge (\vec{\mathbf{J}}_2 + \mu\vec{\mathbf{J}}_1) = \bar{\omega}_{11} - \mu\bar{\omega}_{12} = \bar{\omega}_{11} - \mu\omega_1,$$

from which

$$b = \bar{b} - \mu.$$

By definition, the frames of order 3 are the frames of order 2 such that $b = 0$. There is no third order invariant. The principal components of order ≤ 2 of the frames of order 3 are hence

Order 0	Order 1	Order 2
$\omega_1, \quad \omega_2(=0)$	$\omega_{12}(=\omega_1)$	$\omega_{11}(=0)$

The order 3 frame does not depend on any secondary parameter: it constitutes the *Frenet frame.*

137 Definition of elements of order > 3. The frames of orders > 3 coincide with the Frenet frame. There exists an invariant of order 4: $k = \dfrac{\omega_{21}}{d\sigma}$. We call it the *affine curvature*. There exists invariants of orders $P > 4$, namely $\dfrac{d^{P-4}k}{d\sigma^{P-4}}$.

The relative components of the Frenet frame are

Order 0	Order 1	Order 2	Order 3
$\omega_1, \quad \omega_2 (= 0)$	$\omega_{12} (= \omega_1)$	$\omega_{11} (= 0)$	$\omega_{21} = k\, d\sigma$

In other words, we have the *Frenet formulae*

$$\begin{cases} \overrightarrow{d\mathbf{A}} = d\sigma\, \vec{\mathbf{I}}_1, \\ d\vec{\mathbf{I}}_1 = d\sigma\, \vec{\mathbf{I}}_2, \\ d\vec{\mathbf{I}}_2 = k\, d\sigma\, \vec{\mathbf{I}}_1. \end{cases} \tag{8}$$

The order Q defined in paragraph §**131** (p. 144) is 3 if k is constant. Otherwise it is 4.

The structure theorem of paragraph §**77** (p. 90) tells us that k is an arbitrary function of σ.

Equality problem. The fundamental condition of equality stated in paragraph §**76** (p. 89) leads the search for all unimodular affine transformations superimposing two curves C and $C^\star$ to the search for all bijective correspondences between their points such that

$$k = k^\star, \qquad d\sigma = d\sigma^\star.$$

138 Analytic determination of Frenet frame, curvature and affine arc. *Generalities.* Suppose we have determined at each point of the curve $[x, y(x)]$ a particular first order frame $\mathbf{A}\vec{\mathbf{J}}_1\vec{\mathbf{J}}_2$. For example this could be the following:

$$\begin{cases} \text{coordinates of } \mathbf{A}: & x, y; \\ \text{coordinates of } \vec{\mathbf{J}}_1: & 1, y'; \\ \text{coordinates of } \vec{\mathbf{J}}_2: & 0, 1. \end{cases} \tag{9}$$

The frames $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2$ of order 1 can be deduced by the formulae (4). The frames of order 2 are those of the frames of order 1 for which we have $\omega_{12} = \omega_1$. The Frenet frame is the one for which $\omega_{12} = \omega_1$ and $\omega_{11} = 0$. Let us hence calculate the components ω of the infinitesimal displacement of the frame $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2$ with the help of the components $\bar{\omega}$ which is about the particular frame $\mathbf{A}\vec{\mathbf{J}}_1\vec{\mathbf{J}}_2$. We have, first of all, according to (5),

$$\omega_1 = \frac{1}{\lambda}\bar{\omega}_1, \qquad \omega_{12} = \lambda^2 \bar{\omega}_{12}.$$

On the other hand

$$\omega_{11} = d\vec{\mathbf{I}}_1 \wedge \vec{\mathbf{I}}_2 = \frac{1}{\lambda}(a\lambda\vec{\mathbf{J}}_1 + \lambda d\vec{\mathbf{J}}_1) \wedge (\vec{\mathbf{J}}_2 + \mu\vec{\mathbf{J}}_1) = \bar{\omega}_{11} - \mu\bar{\omega}_{12} + \frac{d\lambda}{\lambda},$$

$$\begin{aligned}\omega_{12} = d\vec{\mathbf{I}}_2 \wedge \vec{\mathbf{I}}_2 &= \frac{1}{\lambda^2}(d\vec{\mathbf{J}}_2 + \mu d\vec{\mathbf{J}}_1 + \vec{\mathbf{J}}_1 d\mu) \wedge (\vec{\mathbf{J}}_2 + \mu\vec{\mathbf{J}}_1)\\ &= \frac{1}{\lambda^2}(\bar{\omega}_{21} + 2\mu\bar{\omega}_{11} - \mu^2\bar{\omega}_{12} + d\mu).\end{aligned}$$

First example. Suppose that $\mathbf{A}\vec{\mathbf{J}}_1\vec{\mathbf{J}}_2$ is the frame (9). Then

$$\begin{aligned}&\overrightarrow{d\mathbf{A}} \text{ has components } dx,\ y'dx;\\ &d\vec{\mathbf{J}}_1 \text{ has components } 0,\quad y''dx;\\ &d\vec{\mathbf{J}}_2 \text{ has components } 0,\quad 0,\end{aligned}$$

and then we have

$$\bar{\omega}_1 = dx, \qquad \bar{\omega}_{11} = 0, \qquad \bar{\omega}_{12} = y''dx, \qquad \bar{\omega}_{21} = 0.$$

From which

$$\omega_1 = \frac{1}{\lambda}dx, \qquad \omega_{12} = \lambda^2 y''dx,$$

$$\omega_{11} = -\mu y''dx + \frac{d\lambda}{\lambda}, \qquad \omega_{21} = \frac{1}{\lambda^2} = \frac{1}{\lambda^2}(-\mu^2 y''dx + d\mu).$$

These expressions of the components of ω show that the second order frames are obtained by taking in (4) $\lambda = (y'')^{-1/3}$ and by leaving μ arbitrary. The Frenet frame is then obtained by setting in (4)

$$\lambda = (y'')^{-\frac{1}{3}}, \qquad \mu = \frac{1}{y''}\frac{1}{\lambda}\frac{d\lambda}{dx} = -\frac{1}{3}\frac{y'''}{(y'')^2}. \tag{10}$$

By definition $\omega_1 = d\sigma$, from which

$$d\sigma = (y'')^{\frac{1}{3}}dx, \tag{11}$$

k is obtained by replacing λ and μ by the values (10) in the formulae

$$k = \frac{\omega_{21}}{d\sigma} = \frac{1}{\lambda}\left(-\mu^2 y'' + \frac{d\mu}{dx}\right).$$

It becomes

$$k = -\frac{\mu}{\lambda^2}\frac{d\lambda}{dx} + \frac{1}{\lambda}\frac{d\mu}{dx} = \frac{d}{dx}\left(\frac{\mu}{\lambda}\right) = \frac{d}{dx}\left(\lambda\frac{d\lambda}{dx}\right) = \frac{1}{2}\frac{d^2(\lambda^2)}{dx^2},$$

and finally

$$k = \frac{1}{2}\left(y''^{-\frac{2}{3}}\right)''. \tag{12}$$

Second example. Now suppose that the absolute frame $\mathbf{R}_0$ is rectangular and that we taking the frame $\mathbf{A}\vec{\mathbf{J}}_1\vec{\mathbf{J}}_2$ to be the first order Euclidean frame attached to each point of the curve: $\vec{\mathbf{J}}_1$ and $\vec{\mathbf{J}}_2$ have length 1, and $\vec{\mathbf{J}}_1$ makes an angle of $\frac{\pi}{2}$ with $\vec{\mathbf{J}}_2$. Let us denote by s the Euclidean curvilinear abscissa and R the radius of the Euclidean curvature of the curve. We have

$$\bar{\omega}_1 = ds, \qquad \bar{\omega}_{11} = 0, \qquad \bar{\omega}_{12} = \frac{ds}{R}, \qquad \bar{\omega}_{21} = -\frac{ds}{R},$$

from which

$$\omega_1 = \frac{1}{\lambda}ds, \qquad \omega_{12} = \frac{\lambda^2}{R}ds,$$
$$\omega_{11} = -\frac{\mu}{R}ds + \frac{d\lambda}{\lambda}, \qquad \omega_{21} = \frac{1}{\lambda^2}\left[-(1+\mu^2)\frac{ds}{R} + d\mu\right].$$

These expressions of the components ω show that the second order frames are obtained by taking in (4) $\lambda = R^{1/3}$ and leaving μ arbitrary. The Frenet frame is obtained by setting in (4)

$$\lambda = R^{\frac{1}{3}}, \qquad \mu = \frac{R}{\lambda}\frac{d\lambda}{ds} = \frac{1}{3}\frac{dR}{ds}. \tag{13}$$

By definition $\omega_1 = d\sigma$, from which

$$d\sigma = R^{-\frac{1}{3}}ds, \tag{14}$$

k is obtained by replacing λ and μ by their values (13) in the formula

$$k = \frac{\omega_{21}}{d\sigma} = \frac{1}{\lambda}\left[-(1+\mu^2)\frac{1}{R} + \frac{d\mu}{ds}\right].$$

It becomes

$$k = -\frac{1}{\lambda R} - \frac{\mu}{\lambda^2}\frac{d\lambda}{ds} + \frac{1}{\lambda}\frac{d\mu}{ds} = -\frac{1}{\lambda R} + \frac{d}{ds}\left(\frac{\mu}{\lambda}\right) = -R^{-\frac{4}{3}} + \frac{d}{ds}\left(\frac{1}{3}R^{-\frac{1}{3}}\frac{dR}{ds}\right),$$

and finally

$$k = -R^{-\frac{4}{3}} + \frac{1}{2}\frac{d^2}{ds^2}(R^{\frac{2}{3}}). \tag{15}$$

Remark concerning the affine arc. According to the Frenet formulae (8) we have, regardless of the parameter used for representing the points of the curve,

$$\overrightarrow{d\mathbf{A}} = \vec{\mathbf{I}}_1 d\sigma \qquad \text{and} \qquad \overrightarrow{d^2\mathbf{A}} = \vec{\mathbf{I}}_2 d\sigma^2 + \vec{\mathbf{I}}_1 d^2\sigma.$$

But

$$\vec{\mathbf{I}}_1 \wedge \vec{\mathbf{I}}_2 = 1,$$

from which

$$d\sigma = (d\mathbf{A} \wedge d^2\mathbf{A})^{\frac{1}{3}}. \tag{16}$$

As in the particular cases, these formulae manifestly contains the formulae (11) and (14).

139 Reduced equation. We can recover the Frenet formulae by finding directly a reduced equation of the curve in a neighbourhood of one of its points $\mathbf{A}$. By taking a cartesian frame with origin $\mathbf{A}$ and whose first vector $\vec{\mathbf{I}}_1$ is along the tangent, the equation of the curve is of the form

$$y = \frac{1}{2}\alpha x^2 + \frac{1}{6}\beta x^3 + \cdots$$

The possible changes of coordinates are

$$x' = \lambda x + \mu y, \qquad y' = \nu y \qquad \text{with} \qquad \lambda\nu = 1.$$

We can reduce the coefficient α to unity by taking $\lambda^3 = \alpha$, and the coefficient β to zero by taking $\mu = \dfrac{\beta\lambda}{3\alpha^2}$. We then have a reduced equation of the form

$$y = \frac{1}{2}x^2 + \frac{1}{4}hx^4 + \frac{1}{5}lx^5 + \cdots$$

The abscissa x of a point $\mathbf{A}'$ infinitesimally close to $\mathbf{A}$ can be regarded as the principal part of the *affine arc element* $\mathbf{AA}'$.

Let us verify that the frame thus introduced at each point is just the Frenet frame by calculating the relative components of its infinitesimal displacement. Let

$$\begin{aligned} d\mathbf{A} &= \omega_1\ \vec{\mathbf{I}}_1 + \omega_2\ \vec{\mathbf{I}}_2, \\ d\vec{\mathbf{I}}_1 &= \omega_{11}\vec{\mathbf{I}}_1 + \omega_{12}\vec{\mathbf{I}}_2, \\ d\vec{\mathbf{I}}_2 &= \omega_{21}\vec{\mathbf{I}}_1 - \omega_{11}\vec{\mathbf{I}}_2. \end{aligned}$$

The relative coordinates x, y of a fixed point $\mathbf{B}$ of the curve with respect to the frame with origin $\mathbf{A}$ varies, with this frame, according to the formulae

$$\begin{aligned} dx + \omega_1 + x\omega_{11} + y\omega_{21} &= 0, \\ dy + \omega_2 + x\omega_{12} - y\omega_{11} &= 0. \end{aligned}$$

But we have

$$dy = (x + hx^3 + lx^4 + \ldots)dx + \frac{1}{4}dh\, x^4 + \cdots,$$

from which

$$\omega_2 + x\omega_{12} - \left(\frac{1}{2}x^2 + \frac{1}{4}hx^4 + \dots\right)\omega_{11} + \frac{1}{4}dh\, x^4 + \cdots$$
$$= (x + hx^3 + lx^4 + \dots)\left[\omega_1 + x\omega_{11} + \left(\frac{1}{2}x^2 + \frac{1}{4}hx^4 + \dots\right)\omega_{21}\right].$$

By identifying the terms in x on the two sides of this relation, we have

$$\omega_2 = 0, \qquad \omega_{12} = \omega_1, \qquad -\frac{1}{2}\omega_{11} = \omega_{11},$$

equations showing the identity of the frame considered and the Frenet frame. We then have

$$\frac{1}{2}\omega_{21} + h\omega_1 = 0, \qquad -\frac{1}{4}h\omega_{11} = h\omega_{11} + l\omega_1 - \frac{1}{4}dh, \qquad \cdots$$

we deduce

$$h = -\frac{1}{2}k, \qquad l = -\frac{1}{8}\frac{dk}{d\sigma}, \qquad \cdots$$

from which the reduced equation

$$y = \frac{1}{2}x^2 - \frac{1}{8}kx^4 - \frac{1}{40}\frac{dk}{d\sigma} + \cdots \tag{17}$$

We can also deduce from the Frenet formulae (8) differentiated with respect to σ the equation

$$\mathbf{A}' = \mathbf{A} + \sigma\vec{\mathbf{I}}_1 + \frac{1}{2}\sigma^2\vec{\mathbf{I}}_2 + \frac{1}{6}k\sigma^3\vec{\mathbf{I}}_1 + \frac{1}{24}\frac{dk}{d\sigma}\sigma^4\vec{\mathbf{I}}_1 + \frac{1}{24}k\sigma^4\vec{\mathbf{I}}_2 + \cdots$$

from which the expansions

$$\begin{cases} x = \sigma + \dfrac{1}{6}k\sigma^3 + \dfrac{1}{24}\dfrac{dk}{d\sigma}\sigma^4 + \cdots, \\ y = \dfrac{1}{2}\sigma^2 + \dfrac{1}{24}k\sigma^4 + \cdots \end{cases} \tag{18}$$

140 Geometrical determination of the affine arc, Frenet frame and affine curvature. Let Δ be a direction of a straight line not parallel to the tangent at $\mathbf{A}$, and let $\mathbf{AC}'\mathbf{A}'$ be the triangle whose side $\mathbf{AC}'$ is tangent to the curve at $\mathbf{A}$ and whose side $\mathbf{C}'\mathbf{A}'$ is parallel to Δ (fig. 1). The area of this triangle, which has its principal part independent of Δ, is, according to (17), $\dfrac{1}{4}x^3 \sim \frac{1}{4}\sigma^3$, from which

$$d\sigma \sim \sqrt[3]{4\,\text{area}\,\mathbf{AC}'\mathbf{A}} \qquad \text{and} \qquad \vec{\mathbf{I}}_1 = \lim \frac{\overrightarrow{\mathbf{AA}'}}{\sigma}. \tag{19}$$

Let us now draw a line parallel to the tangent at $\mathbf{A}$, infinitesimally close to $\mathbf{A}$ and at the side of the curve. It cuts the curve at two points infinitesimally close to $\mathbf{A}$ whose

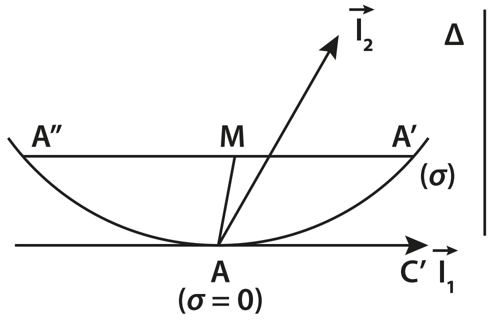


Figure 1

middle **M** (fig. 1), according to (17), has its abscissa zero. The vector $\vec{\mathbf{I}}_2$ is hence along the limit of the line **AM**, which we call the *affine normal*.

We can also say that the affine normal is the diameter of the *osculatory parabola* passing through **A** to the curve at **A**, the parabola which, according to (17), has the equation $y = \dfrac{1}{2}x^2$.

The geometrical characterisation of the affine curve will lead to the study of *curves with constant affine curvature*. They are, together with the straight lines, the only curves which are transformed by an infinite number of unimodular affine transformations into themselves (c.f. §**132**, p. 145). There exists one and only one such curves having a given k and admitting a given Frenet frame $\mathbf{O}\vec{\mathbf{K}}_1\vec{\mathbf{K}}_2$. This curve is obtained by integrating the system

$$\frac{d\overrightarrow{\mathbf{A}}}{d\sigma} = \vec{\mathbf{I}}_1, \qquad \frac{d\vec{\mathbf{I}}_1}{d\sigma} = \vec{\mathbf{I}}_2, \qquad \frac{d\vec{\mathbf{I}}_2}{d\sigma} = k\vec{\mathbf{I}}_1.$$

We choose **O** as the origin of the curvilinear abscissa and we begin with the particular case $k = 0$. In this case

$$\vec{\mathbf{I}}_2 = \vec{\mathbf{K}}_2, \qquad \vec{\mathbf{I}}_1 = \vec{\mathbf{K}}_1 + \sigma\vec{\mathbf{K}}_2, \qquad \mathbf{A} = \mathbf{O} + \sigma\vec{\mathbf{K}}_1 + \frac{1}{2}\sigma^2\vec{\mathbf{K}}_2,$$

the curve is a *parabola* with an axis parallel to $\vec{\mathbf{K}}_2$. Now suppose $k < 0$. We have

$$\begin{aligned} \vec{\mathbf{I}}_1 &= \vec{\mathbf{K}}_1 \cos(\sqrt{-k}\,\sigma) + \frac{\vec{\mathbf{K}}_2}{\sqrt{-k}} \sin(\sqrt{-k}\,\sigma), \\ \vec{\mathbf{I}}_2 &= -\sqrt{-k}\,\vec{\mathbf{K}}_1 \cos(\sqrt{-k}\,\sigma) + \vec{\mathbf{K}}_2 \sin(\sqrt{-k}\,\sigma), \end{aligned}$$

from which

$$\mathbf{A} = \left(\mathbf{O} - \frac{\vec{\mathbf{K}}_2}{k}\right) + \frac{\vec{\mathbf{K}}_1}{\sqrt{-k}} \sin(\sqrt{-k}\,\sigma) + \frac{\vec{\mathbf{K}}_2}{k} \cos(\sqrt{-k}\,\sigma).$$

The curve is hence *an ellipse*. Its centre is the point $\mathbf{O} - \frac{\vec{\mathbf{K}}_2}{k}$ and it admits two vectors equipollent to the vectors $\dfrac{\vec{\mathbf{K}}_1}{\sqrt{-k}}$ and $\dfrac{\vec{\mathbf{K}}_2}{k}$ as the two conjugate semi-diameters. Its area is hence $\pi(-k)^{-\frac{3}{2}}$. This last fact constitutes a geometrical interpretation of the affine curvature k.

Similarly the curve is a *hyperbola* if $k < 0$. It admits two vectors equipollent to $\dfrac{\vec{\mathbf{K}}_1}{\sqrt{k}}$ and $\dfrac{\vec{\mathbf{K}}_2}{k}$ as the two conjugate diameters. The area of the parallelogram whose sides are parallel to the asymptotes and whose **OA** is a diagonal is equal to $\dfrac{1}{2}k^{-\frac{3}{2}}$.

The preceding allows us to affirm that, given an arbitrary curve as well as one of its points, there exists only one conic having at this point a contact of order ≥ 4 with this curve: the constant curvature of this conic is the curvature of the curve at the point considered.

III. Projective geometry; study of real plane curves [†]

141 Introduction. We will apply the method of moving frame to the study of real planar curves relative to the group G of projective transformations. This group is defined by the equations

$$(1) \qquad \begin{cases} x' = ax + by + cz, \\ y' = a'x + b'y + c'z, \\ z' = a''x + b''y + c''z, \end{cases} \qquad \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} \neq 0,$$

where x, y, z represent the three homogeneous coordinates of a point in the plane. We use the frames constructed in paragraph §**67** (p. 77). It is composed of three analytic points $\mathbf{A}$, $\mathbf{A}_1$, $\mathbf{A}_2$ such that

$$(2) \qquad [\mathbf{A}, \mathbf{A}_1, \mathbf{A}_2] = .$$

The relative components of infinitesimal displacements of this frame [‡] are the eight quantities ω_{00}, ω_{01}, ω_{02}, ω_{10}, ω_{11}, ω_{12}, ω_{20}, ω_{21} defined by the relations

$$(3) \qquad \begin{cases} d\mathbf{A} \ = \omega_{00}\mathbf{A} + \omega_{01}\mathbf{A}_1 + \omega_{02}\mathbf{A}_2, \\ d\mathbf{A}_1 = \omega_{10}\mathbf{A} + \omega_{11}\mathbf{A}_1 + \omega_{12}\mathbf{A}_2, \\ d\mathbf{A}_2 = \omega_{20}\mathbf{A} + \omega_{21}\mathbf{A}_1 + \omega_{22}\mathbf{A}_2, \end{cases}$$

$$(3') \qquad \omega_{00} + \omega_{11} + \omega_{22} = 0.$$

142 Definition of elements of order 0 **and** 1**.** There is no invariant of order 0 or 1. The frames of order 0 of a point on a curve are the frames whose first apex coincides with the point considered. The frames of order 1 are the order 0 frames whose second apex is along the tangent of the point considered.

[†] For this section, the reader can consult E. Cartan, *Leçon sur la théorie des espaces à connexion projective* [20], chapter II, section V, p. 84–111.
[‡] C.f. paragraph §**73**, p. 85.

The group G depends on $r = 8$ parameters. The frames of orders 0 and 1 depend respectively $v_0 = 6$ and $v_1 = 5$ parameters. The number of principal components of order 0 is hence $r - v_0 = 2$ and the number of principal components of order 1 is $v_0 - v_1 = 1$.

When a frame of order 0 varies while its origin remains fixed, $d\mathbf{A}$ is proportional to $\mathbf{A}$, hence $\omega_{01} = \omega_{02} = 0$. ω_{01} and ω_{02} are the principal components of order 0. On the other hand when a frame varies while remaining a frame of order 1, $d\mathbf{A}$ is a linear combination of $\mathbf{A}$ and $\mathbf{A}_1$, hence $\omega_{02} = 0$. The principal components of order 0 of the frames of order 1 are hence

Order 0
$\omega_{01}, \quad \omega_{02}(= 0)$

143 Definition of elements of order 2. Consider two frames of order 1 at the same variable point: $\mathbf{A}\mathbf{A}_1\mathbf{A}_2$ and $\mathbf{B}\mathbf{B}_1\mathbf{B}_2$. We have

$$(4) \qquad \begin{cases} \mathbf{A} \ = \lambda\mathbf{B}, \\ \mathbf{A}_1 = \lambda_1\mathbf{B}_1 + \mu\mathbf{B}, \\ \mathbf{A}_2 = \lambda_2\mathbf{B}_2 + \rho\mathbf{B}_1 + \sigma\mathbf{B}, \end{cases} \qquad \text{where} \qquad \lambda\lambda_1\lambda_2 = 1.$$

The frames of order 1 hence constitute four connected families, characterised by the four systems of signs that can be attributed to the two parameters λ and λ_1: a contact element of order 1 decomposes into four oriented contact elements.

According to the preceding paragraph, there exists only one principal component of order 1. According to (4), when $\mathbf{B}\mathbf{B}_1\mathbf{B}_2$ remains fixed and λ, λ_1, λ_2 vary, $d\mathbf{A}_1$ is a linear combination of $\mathbf{A}$ and $\mathbf{A}_1$. Hence $\omega_{12} = 0$. ω_{12} is the principal component of order 1. We define the coefficient b of order 1 by the relation $\omega_{12} = b\omega_0$.

Let us calculate [(†)] ω_{01} and ω_{12} as functions of $\bar\omega_{01}$, $\bar\omega_{12}$, λ, λ_1, λ_2, μ, ρ, σ. We have

$$(5) \qquad \begin{cases} \omega_{01} = [\mathbf{A}, d\mathbf{A}, \mathbf{A}_2] \\ \qquad = [\lambda\mathbf{B}, \mathbf{B}\, d\lambda + \lambda(\bar\omega_{00}\mathbf{B} + \bar\omega_{01}\mathbf{B}_1), \lambda_2\mathbf{B}_2 + \rho\mathbf{B}_1 + \sigma\mathbf{B}] = \lambda^2\lambda_2\bar\omega_{01}, \\ \omega_{12} = [\mathbf{A}, \mathbf{A}_1, d\mathbf{A}_1] \\ \qquad = [\lambda\mathbf{B}, \lambda_1\mathbf{B}_1 + \mu\mathbf{B}, \mathbf{B}_1 d\lambda_1 + \mathbf{B}\, d\mu + \lambda_1 d\mathbf{B}_1 + \mu\, d\mathbf{B}] = \lambda\lambda_1^2\bar\omega_{12}. \end{cases}$$

From which

$$(6) \qquad b = \lambda_1^3\bar b.$$

By definition the frames of order 2 are the order 1 frames such that $b = 1$. There is no invariant of order 2. The principal components of order ≥ 1 of the frames of order 2 are

Order 0	Order 1
$\omega_{01}, \quad \omega_{02}(= 0)$	$\omega_{12}(= \omega_{01})$

(†) ω are the components of $\mathbf{A}\mathbf{A}_1\mathbf{A}_2$, and $\bar\omega$ are the components of $\mathbf{B}\mathbf{B}_1\mathbf{B}_2$.

A category of exceptional curves arise, for which the preceding definition does not make sense: the curves along which the displacements of first order frames satisfy the relation $\omega_{12} = 0$. The circumstances studies in paragraph §**132** are then realised. On the other hand it is obvious that the category of curves in question is the one of straight lines.

144 Definition of elements of order 3. By replacing in (6) b and $\bar{b}$ by 1 we obtain $\lambda_1^3 = 1$, i.e., $\lambda_1 = 1$. The formulae that allow us to compare two frames of order 2 at the same point, $\mathbf{A}\mathbf{A}_1\mathbf{A}_2$ and $\mathbf{B}\mathbf{B}_1\mathbf{B}_2$, are hence obtained by replacing in (4) λ_1 by 1. It becomes

$$(7) \qquad \begin{cases} \mathbf{A} \ \ = \lambda\mathbf{B}, \\ \mathbf{A}_1 = \mathbf{B}_1 + \mu\mathbf{B}, \\ \mathbf{A}_2 = \dfrac{1}{\lambda}\mathbf{B}_2 + \rho\mathbf{B}_1 + \sigma\mathbf{B}. \end{cases}$$

The frames of order 2 hence constitute two connected families, each characterised by the sign of λ: a contact element of order 2 decomposes into two oriented contact element.

There exists *one* principal component of order 2, since we have defined the frames of order 2 by imposing *one* relation on the coefficients of order 1. According to (7), when $\mathbf{B}\mathbf{B}_1\mathbf{B}_2$ remains fixed and λ, μ, ρ, σ vary, $d\mathbf{A}_1$ is proportional to $\mathbf{A}$, hence $\omega_{11} = 0$. ω_{11} is the principal component of order 1. We define the coefficient b of order 2 by the relation $\omega_{11} = b\omega_{01}$.

Let us calculate ω_{01} and ω_{11} as functions of $\bar{\omega}_{01}$, $\bar{\omega}_{11}$, λ, μ, ρ, σ. We have, according to (5), set $\lambda_1 = 1$, i.e., $\lambda\lambda_2 = 1$,

$$(8) \qquad \omega_{01} = \lambda\bar{\omega}_{01}.$$

We then have

$$\begin{aligned} \omega_{11} &= [\mathbf{A}, d\mathbf{A}_1\mathbf{A}_2] \\ &= \left[\lambda\mathbf{B}, d\mathbf{B}_1 + \mathbf{B}\,d\mu + \mu\,d\mathbf{B}, \frac{1}{\lambda}\mathbf{B}_2\rho\mathbf{B}_1 + \sigma\mathbf{B}\right] = \bar{\omega}_{11} + (\mu - \lambda\rho)\bar{\omega}_{01}. \end{aligned}$$

From which

$$(9) \qquad b = \frac{1}{\lambda}\bar{b} + \left(\frac{\mu}{\lambda} - \rho\right).$$

By definition the frames of order 3 are the order 2 frames such that $b = 0$. There is no invariant of order 3. The principal components of orders ≤ 2 of the frames of order 3 are

Order 0	Order 1	Order 2
ω_{01}, $\quad \omega_{02}(= 0)$	$\omega_{12}(= \omega_{01})$	$\omega_{11}(= 0)$

145 Definition of elements of order 4. By replacing in (9) b and $\bar{b}$ by 0, we obtain $\rho = \dfrac{\mu}{\lambda}$. The formulae allowing us to compare two frames of order 3 at the same point are hence

$$\left\{\begin{aligned} \mathbf{A} &= \lambda\mathbf{B}, \\ \mathbf{A}_1 &= \mathbf{B}_1 + \mu\mathbf{B}, \\ \mathbf{A}_2 &= \frac{1}{\lambda}(\mathbf{B}_2 + \mu\mathbf{B}_1 + \nu\mathbf{B}). \end{aligned}\right. \tag{10}$$

A contact element of order 3 hence decomposes into two oriented contact elements.

There exists one principal component of order 3. Suppose that in formulae (10) $\lambda - 1$, μ and ν are infinitesimally small. The transformation transforming the frame $\mathbf{B}\mathbf{B}_1\mathbf{B}_2$ into the frame $\mathbf{A}\mathbf{A}_1\mathbf{A}_2$ has relative components

$$\begin{pmatrix} \omega_{00} & \omega_{01} & \omega_{02} \\ \omega_{10} & \omega_{11} & \omega_{12} \\ \omega_{20} & \omega_{21} & \omega_{22} \end{pmatrix} = \begin{pmatrix} \lambda - 1 & 0 & 0 \\ \mu & 0 & 0 \\ \nu & \mu & 1 - \lambda \end{pmatrix},$$

hence $\omega_{10} - \omega_{21} = 0$. $\omega_{10} - \omega_{21}$ is the principal component of order 3. We set $\omega_{10} - \omega_{21} = b\omega_{01}$.

We have (†)

$$\begin{aligned} \omega_{10} &= [d\mathbf{A}_1, \mathbf{A}_1, \mathbf{A}_2] \\ &= \left[d\mathbf{B}_1 + \mu\, d\mathbf{B} + \mathbf{B}\, d\mu, \mathbf{B}_1 + \mu\mathbf{B}, \frac{1}{\lambda}(\mathbf{B}_2 + \mu\mathbf{B}_1 + \nu\mathbf{B})\right] \\ &= \frac{1}{\lambda}(d\mu + \mu\bar{\omega}_{00} + \bar{\omega}_{10} - \nu\bar{\omega}_{01}), \\ \omega_{21} &= [\mathbf{A}, d\mathbf{A}_2, \mathbf{A}_2] = \frac{1}{\lambda}[d\mu + \mu\bar{\omega}_{00} + \bar{\omega}_{21} + (\nu - \mu^2)\bar{\omega}_{01}], \\ \omega_{10} - \omega_{21} &= \frac{1}{\lambda}(\bar{\omega}_{10} - \bar{\omega}_{21}) + \frac{\mu^2 - 2\nu}{\lambda}\bar{\omega}_{01}. \end{aligned}$$

Using (8),

$$b = \frac{1}{\lambda^2}\bar{b} + \frac{\mu^2 - 2\nu}{\lambda^2}. \tag{11}$$

By definition the frames of order 4 are the order 3 frames such that $b = 0$. There is no invariant of order 4. The principal components of orders ≤ 3 of the frames of order 4 are

Order 0	Order 1	Order 2	Order 3
ω_{01}, $\omega_{02}(= 0)$	$\omega_{12}(= \omega_{01})$	$\omega_{11}(= 0)$	$\omega_{10} - \omega_{21}(= 0)$

(†) We can simplify the calculation by replacing $d\mu$ by 0 since the value of $\omega_{10} - \omega_{21}$ is independent of $d\mu$.

146 Definition of elements of order 5. By replacing in (11) b and $\bar{b}$ by 0 we obtain $\nu = \frac{\mu^2}{2}$. The formulae allowing us to compare the frames of order 4 at the same point are hence

$$\begin{cases} \mathbf{A} \;= \lambda \mathbf{B}, \\ \mathbf{A}_1 = \mathbf{B}_1 + \mu \mathbf{B}, \\ \mathbf{A}_2 = \dfrac{1}{\lambda}\left(\mathbf{B}_2 + \mu \mathbf{B}_1 + \dfrac{\mu^2}{2}\mathbf{B}\right). \end{cases} \tag{12}$$

A contact element of order 4 hence possesses two orientations.

There exists one principal component of order 4. Suppose that in the formulae (11) $\lambda - 1$ and μ are infinitesimally small. The transformation that transforms $\mathbf{B}\mathbf{B}_1\mathbf{B}_2$ into the frame $\mathbf{A}\mathbf{A}_1\mathbf{A}_2$ has relative components

$$\begin{pmatrix} \omega_{00} & \omega_{01} & \omega_{02} \\ \omega_{10} & \omega_{11} & \omega_{12} \\ \omega_{20} & \omega_{21} & \omega_{22} \end{pmatrix} = \begin{pmatrix} \lambda - 1 & 0 & 0 \\ \mu & 0 & 0 \\ 0 & \mu & 1 - \lambda \end{pmatrix},$$

ω_{20} is hence the principal component of order 4. We set $\omega_{20} = b\omega_{01}$.

A rather long calculation allows us to deduce from (12) that

$$\omega_{20} = [d\mathbf{A}_2 \mathbf{A}_1, \mathbf{A}_2] = \frac{\bar{\omega}_{20}}{\lambda^2}.$$

From which, using (8),

$$b = \frac{\bar{b}}{\lambda^3}. \tag{13}$$

By definition the frames of order 5 are the order 4 frames such that $b = -1$. There is no invariant of order 5. The principal components of orders ≤ 4 of the frames of order 5 are

Order 0	Order 1	Order 2	Order 3	Order 4
$\omega_{01}, \quad \omega_{02}(=0)$	$\omega_{12}(=\omega_{01})$	$\omega_{11}(=0)$	$\omega_{10} - \omega_{21}(=0)$	$\omega_{20}(=-\omega_{01})$

147 A category of exceptional curves. The preceding definition of the order 5 frames does not make sense when the displacements of the order 4 frames annihilate ω_{20}. The curves of this category present the particularities studiedin paragraph **§132** (p. 145): each of them remain invariant under the projective transformation transforming one of its order 4 frame into another.

Let us determine what these curves are. We can find a family of order 4 frames depending only on the principal parameter and for which

$$\omega_{00} = 0, \qquad \omega_{10} = 0. \tag{14}$$

Indeed, we need to determine the secondary parameters v_1 and v_2 as functions of the principal parameter u such as to satisfy the two differential equations (14). But no combinations of the forms ω_{00}, ω_{10} is a combination of the principal components. The system (14) hence can be written

$$dv_1 = f_1(u, v_1, v_2)du, \qquad dv_2 = f_2(u, v_1, v_2)du.$$

We have, under these conditions

$$d\mathbf{A} = \omega_{01}\mathbf{A}_1, \qquad d\mathbf{A}_1 = \omega_{01}\mathbf{A}_2, \qquad d\mathbf{A}_2 = 0.$$

Let us choose a new principal parameter t defined by the equation $dt = \omega_{01}$. Let $\mathbf{A}^0\mathbf{A}_1^0\mathbf{A}_2^0$ be the position of the frame $\mathbf{A}\mathbf{A}_1\mathbf{A}_2$ at $t = 0$. We have

$$\mathbf{A}_2 = \mathbf{A}_2^0, \qquad \mathbf{A}_1 = \mathbf{A}_2^0 + t\mathbf{A}_1^0, \qquad \mathbf{A} = \mathbf{A}^0 + t\mathbf{A}_1^0 + \frac{1}{2}t^2\mathbf{A}_2^0.$$

The homogeneous coordinates of the point $\mathbf{A}$ with respect to the frame $\mathbf{A}^0\mathbf{A}_1^0\mathbf{A}_2^0$ are hence

$$z = 1, \qquad x = t, \qquad y = \frac{1}{2}t^2.$$

The category of curves in question is hence the conics.

If at a point of a curve which is not a conic, the component ω_{20} is zero, the point is *sextactic*, which signifies that the osculating conic at this point has a fifth order contact with the curve.

148 Definition of elements of order 6. By replacing in (13) b and $\bar{b}$ by -1 we obtain $\lambda = 1$. The formulae allowing us to compare the frames of order 5 at the same point are hence

$$\left\{\begin{aligned} \mathbf{A} &= \mathbf{B}, \\ \mathbf{A}_1 &= \mathbf{B}_1 + \mu\mathbf{B}, \\ \mathbf{A}_2 &= \mathbf{B}_2 + \mu\mathbf{B}_1 + \frac{\mu^2}{2}\mathbf{B}. \end{aligned}\right. \tag{15}$$

A contact element of order 5 hence possesses only one orientation.

We have, according to (8), by setting $\lambda = 1$,

$$\omega_{01} = \bar{\omega}_{01},$$

ω_{01} is hence an invariant form of order 5: we denote it by $d\sigma$ and we call σ the *projective arc*.

The principal component of order 5 is obviously ω_{00}. We have

$$\omega_{00} = [d\mathbf{A}, \mathbf{A}_1, \mathbf{A}_2] = \left[d\mathbf{B}, \mathbf{B}_1 + \mu\mathbf{B}, \mathbf{B}_2 + \mu\mathbf{B}_1 + \frac{1}{2}\mu^2\mathbf{B}\right] = \bar{\omega}_{00} - \mu\omega_{01}.$$

From which, bet setting $\omega_{00} = b\omega_{01}$,

$$b = \bar{b} - \mu.$$

By definition the frames of order 6 are the order 5 frames such that $b = 0$. There is no invariant of order 6. The principal components of orders ≤ 5 of the frames of order 6 are:

Order 0	Order 1	Order 2	Order 3	Order 4	Order 5
$\omega_{01}, \quad \omega_{02}(=0)$	$\omega_{12}(=\omega_{01})$	$\omega_{11}(=0)$	$\omega_{10} - \omega_{21}(=0)$	$\omega_{20}(=-\omega_{01})$	$\omega_{00}(=0)$

The order 6 frame does not depend on secondary parameters and it constitutes the *Frenet frame*.

149 Definition of elements of order > 6. There exists an invariant of order 7: $k = -\dfrac{\omega_{10}}{d\sigma}$. We call it the *projective curvature*.

There exists an invariant of order $P > 7$: $\dfrac{d^{P-7}k}{d\sigma^{P-7}}$.

The instantaneous displacement of the Frenet frame is characterised by the *Frenet formulae*:

$$\begin{cases} d\mathbf{A} \;\, = d\sigma\,\mathbf{A}, \\ d\mathbf{A}_1 = -k\,d\sigma\,\mathbf{A} + d\sigma\,\mathbf{A}_2, \\ d\mathbf{A}_2 = -d\sigma\,\mathbf{A} - k\,d\sigma\,\mathbf{A}_1. \end{cases} \tag{16}$$

The structure theorem of paragraph §**77** (p. 90 tells us that k is an arbitrary function of σ.

The fundamental condition of equality stated in paragraph §**76** (p. 89) leads the search for all projective transformations superimposing two curves C and $C^\star$ to the search of all bijective correspondences between their points such that

$$k = k^\star, \qquad d\sigma = d\sigma^\star.$$

Remark. The Frenet frame is only defined at a point of a curve when this point is neither an inflection point nor a sextactic point, but we have to say that we have implicitly assumed that the point is not a point of cusp.

150 Curves considered as envelopes of straight lines. We can consider a curve as the envelop of its tangents. From this point of view we can introduce frames constituted by three *analytic straight lines*, each of which is defined, with respect to a fixed reference system, by three tangential coordinates u_1, u_2, u_3. The produce of an analytic straight line (u) and an analytic point (x) will be the quantity $u_1x_1 + u_2x_2 + u_3x_3$. The frame $\mathbf{A}_0\mathbf{A}_1\mathbf{A}_2$ that we have used until now can be considered as consisting of three analytic straight lines a, a_1, a_2 coinciding geometrically with the straight lines $\mathbf{A}_0\mathbf{A}_1$, $\mathbf{A}_0\mathbf{A}_2$,

$\mathbf{A}_1\mathbf{A}_2$. We completely specify them by the condition that the products $a\mathbf{A}_2$, $a_1\mathbf{A}_1$, $a_2\mathbf{A}_0$ are equal to 1. An easy calculation then gives the formulae

$$\begin{aligned} da &= -\omega_{22}a - \omega_{12}a_1 - \omega_{02}a_2, \\ da_1 &= -\omega_{21}a - \omega_{11}a_1 - \omega_{01}a_2, \\ da_2 &= -\omega_{20}a - \omega_{10}a_1 - \omega_{00}a_2. \end{aligned}$$

We will observe that if we begin with the Frenet frame considered as the set of points, we find the matrix of its relative components, when we consider it as the set of tangents

$$\begin{pmatrix} 0 & -d\sigma & 0 \\ k\,d\sigma & 0 & -d\sigma \\ d\sigma & k\,d\sigma & 0 \end{pmatrix}.$$

This differs from the matrix that we have found before simply in that $d\sigma$ is replaced by $-d\sigma$, the curvature k being the same. *The projective arc element of a curve is hence recovered with a change of sign, whereas the projective curvature remains the same.*

151 Reduced equations. We can arrive at the reduced equation by a curve in the neighbourhood of an ordinary point rapidly by beginning with the osculating conic which we can always suppose to be transformed into the equation $y = \dfrac{1}{2}x^2$. The reduced equation we find will hence be of the form

$$y = \frac{1}{2}x^2 + \alpha x^5 + \beta x^6 + \gamma x^7 + \cdots$$

The allowed changes of coordinates are those that do not change the equation of the osculating conic, namely

$$x' = \frac{\lambda(x+\mu y)}{1+\mu x + \frac{1}{2}\mu^2 y}, \qquad y' = \frac{\lambda^2 y}{1+\mu x+\frac{1}{2}\mu^2 y}.$$

We then find

$$y' - \frac{1}{2}x'^2 = \frac{\alpha\lambda^2 x^5 + \beta\lambda^2 x^6 + \cdots}{(1+\mu x+\frac{1}{2}\mu^2 y)^2},$$

from which, if we use α' and β' to denote the new coefficients of the equation,

$$\lambda^2(\alpha x^5 + \beta x^6) = \frac{\lambda^5\alpha'(x+\mu y)^5}{(1+\mu x+\frac{1}{2}\mu^2 y)^3} + \frac{\lambda^6\beta'(x+\mu y)^6}{(1+\mu x+\frac{1}{2}\mu^2 y)^4} + \cdots$$

$$\alpha' = \frac{\alpha}{\lambda^3}, \qquad \beta' = \frac{\beta + \frac{1}{2}\alpha\mu}{\lambda^4}.$$

Hence we can arrange in one and only one way such that α is a given numerical constant m and β is zero. We then have the reduced equation

$$y = \frac{1}{2}x^2 + mx^5 + \gamma x^7 + \cdots$$

The abscissa x of a point $\mathbf{A}'$ infinitesimally close to $\mathbf{A}$ gives a differential invariant which will be, up to a constant factor, the projective arc element.

To arrive at the Frenet formulae, start with the formulae

$$\begin{aligned}
d\mathbf{A} &= \omega_{00}\mathbf{A} + \omega_{01}\mathbf{A}_1 + \omega_{02}\mathbf{A}_2,\\
d\mathbf{A}_1 &= \omega_{10}\mathbf{A} + \omega_{11}\mathbf{A}_1 + \omega_{12}\mathbf{A}_2,\\
d\mathbf{A}_2 &= \omega_{20}\mathbf{A} + \omega_{21}\mathbf{A}_1 + \omega_{22}\mathbf{A}_2.
\end{aligned}$$

which give the infinitesimal displacement of the frame attached to a point $\mathbf{A}$ of the curve for which the reduced equation has the indicated form. A fixed point $\mathbf{B}$ with non-homogeneous coordinates x, y corresponds to the analytic point $\mathbf{A} + x\mathbf{A}_1 + y\mathbf{A}_2$. We express that it is fixed by writing that its differential is a multiple of the point itself. This gives the relations

$$\begin{aligned}
dx + \omega_{01} + x(\omega_{11} - \omega_{00}) + y\omega_{21} - x(x\omega_{10} + y\omega_{20}) = 0,\\
dy + \omega_{02} + x\omega_{12} + y(\omega_{22} - \omega_{00}) - y(x\omega_{10} + y\omega_{20}) = 0.
\end{aligned}$$

The method that we have already applied several times leads to the identification of the relation

$$\begin{aligned}
&\omega_{02} + x\omega_{12} + \left(\frac{1}{2}x^2 + mx^5 + hx^7 + \cdots\right)(\omega_{22} - \omega_{00})\\
&\qquad - x\left(\frac{1}{2}x^2 + mx^5 + \cdots\right)\omega_{10} - \left(\frac{1}{2}x^2 + mx^5 + \cdots\right)^2\omega_{20}\\
&\quad = (x + 5mx^4 + 7hx^6 + \cdots)\left[\omega_{01} + x(\omega_{11} - \omega_{00}) + \left(\frac{1}{2}x^2 + mx^5 + \cdots\right)\omega_{21}\right.\\
&\qquad \left. - x^2\omega_{10} - x\left(\frac{1}{2}x^2 + mx^5 + \cdots\right)\omega_{20}\right] - dh\, x^7 + \cdots
\end{aligned}$$

We deduce successively

$$\omega_{02} = 0, \qquad \omega_{12} = \omega_{01}, \qquad \omega_{11} = 0, \qquad \omega_{21} = \omega_{10}, \qquad \omega_{20} = 20m\omega_{01},$$

then

$$\omega_{00} = 0, \qquad -\frac{1}{2}m\omega_{21} + 7h\omega_{01} = 0.$$

By taking $m = -\dfrac{1}{20}$ and $h = \dfrac{1}{280}k$, we find the components of infinitesimal displacement of the Frenet frame. We then have the *reduced equation*

$$y = \frac{1}{2}x^2 - \frac{1}{20}x^5 + \frac{k}{280}x^7 + \cdots,$$

where k is the projective curvature. The y axis is the *projective normal*.

The geometric determination of the Frenet frame and the geometric characterisation of the projective arc and the projective curvature can be made by using the reduced equation. Let us restrict ourselves to indicating, following Halphen, the use that we can make of the cubic having a double point at $\mathbf{A}$ one of whose branches at this point has a sixth order contact with the curve. Its equation is

$$2xy - x^3 + \frac{4}{5}y^3 = 0.$$

We see that the second tangent at the double point is the projective normal and the straight line of the inflections is the third side $\mathbf{A}_1\mathbf{A}_2$ of the Frenet frame. The consideration of the bundle of cubics having at $\mathbf{A}$ a seventh order contact with the curve and of the *Halphen point* common to all these cubics allow us to characterise the projective curvature.

Part III

Structure constants of finite-dimensional connected groups

Chapter 11

The structural equations of E. Cartan

I. Introduction; Darboux equations

152 The relative components of the displacement of a frame depending on a single parameter are arbitrary Pfaffian forms (the structure theorem of paragraph §**77**, p. 90). This is no longer the case when the frame depends on several parameters. Darboux is the first one to have stated this, during his study of infinitesimals of surfaces by the method of moving trihedrals. Let us review his calculations [†]

Consider a moving trihedral depending on two parameters u and v. The relative components ω of its infinitesimal displacement are defined by the relations

$$(1)\qquad \begin{cases} \overrightarrow{d\mathbf{A}} = \omega_1\ \vec{\mathbf{I}}_1 + \omega_2\ \vec{\mathbf{I}}_2 + \omega_3\ \vec{\mathbf{I}}_3, \\ d\vec{\mathbf{I}}_1 = \omega_{12}\vec{\mathbf{I}}_2 - \omega_{31}\vec{\mathbf{I}}_3, \\ d\vec{\mathbf{I}}_2 = -\omega_{12}\vec{\mathbf{I}}_1 + \omega_{23}\vec{\mathbf{I}}_3, \\ d\vec{\mathbf{I}}_3 = \omega_{31}\vec{\mathbf{I}}_1 - \omega_{23}\vec{\mathbf{I}}_2 \end{cases}$$

Let us for the moment abandon the use of Pfaffian forms: set

$$(2)\qquad \begin{cases} \omega_1 = \xi\, du + \xi_1 dv, & \omega_2 = \eta\, du + \eta_1 dv, & \omega_3 = \zeta\, du + \zeta_1 dv, \\ \omega_{23} = p\, du + p_1 dv, & \omega_{31} = q\, du + q_1 dv, & \omega_{12} = r\, du + r_1 dv, \end{cases}$$

where ξ, ξ_1, η, η_1, ζ, ζ_1, p, p_1, q, q_1, r, r_1 are functions of u and v. The system (1)

[†] G. Darboux, *Leçon sur la théorie générale des surfaces*, vol. 1, book 1, chapters 5 and 7.

decomposes into two systems

$$(3)\qquad \begin{cases} \dfrac{\partial \vec{\mathbf{A}}}{\partial u} = \xi\vec{\mathbf{I}}_1 + \eta\vec{\mathbf{I}}_2 + \zeta\vec{\mathbf{I}}_3, \\ \dfrac{\partial \vec{\mathbf{I}}_1}{\partial u} = r\vec{\mathbf{I}}_2 - q\vec{\mathbf{I}}_3, \\ \dfrac{\partial \vec{\mathbf{I}}_2}{\partial u} = -r\vec{\mathbf{I}}_1 + p\vec{\mathbf{I}}_3, \\ \dfrac{\partial \vec{\mathbf{I}}_3}{\partial u} = q\vec{\mathbf{I}}_1 - p\vec{\mathbf{I}}_2 \end{cases}$$

$$(4)\qquad \begin{cases} \dfrac{\partial \vec{\mathbf{A}}}{\partial v} = \xi_1\vec{\mathbf{I}}_1 + \eta_1\vec{\mathbf{I}}_2 + \zeta_1\vec{\mathbf{I}}_3, \\ \dfrac{\partial \vec{\mathbf{I}}_1}{\partial v} = r_1\vec{\mathbf{I}}_2 - q_1\vec{\mathbf{I}}_3, \\ \dfrac{\partial \vec{\mathbf{I}}_2}{\partial v} = -r_1\vec{\mathbf{I}}_1 + p_1\vec{\mathbf{I}}_3, \\ \dfrac{\partial \vec{\mathbf{I}}_3}{\partial v} = q_1\vec{\mathbf{I}}_1 - p_1\vec{\mathbf{I}}_2 \end{cases}$$

From the first equation of (3) we deduce

$$\frac{\partial^2 \vec{\mathbf{A}}}{\partial u\,\partial v} = \frac{\partial \xi}{\partial v}\vec{\mathbf{I}}_1 + \frac{\partial \eta}{\partial v}\vec{\mathbf{I}}_2 + \frac{\partial \zeta}{\partial v}\vec{\mathbf{I}}_3 + \xi\frac{\partial \vec{\mathbf{I}}_1}{\partial v} + \eta\frac{\partial \vec{\mathbf{I}}_2}{\partial v} + \zeta\frac{\partial \vec{\mathbf{I}}_3}{\partial v},$$

from which, using the three equations (4),

$$\frac{\partial^2 \vec{\mathbf{A}}}{\partial u\,\partial v} = \left(\frac{\partial \xi}{\partial v} + q_1\zeta - r_1\eta\right)\vec{\mathbf{I}}_1 + \left(\frac{\partial \eta}{\partial v} + r_1\xi - p_1\zeta\right)\vec{\mathbf{I}}_2 + \left(\frac{\partial \zeta}{\partial v} + p_1\eta - q_1\xi\right)\vec{\mathbf{I}}_3.$$

An analogous calculation, based on the first equation of (4) and the three last equations of (3), gives us

$$\frac{\partial^2 \vec{\mathbf{A}}}{\partial v\,\partial u} = \left(\frac{\partial \xi_1}{\partial u} + q\zeta_1 - r\eta_1\right)\vec{\mathbf{I}}_1 + \left(\frac{\partial \eta_1}{\partial u} + r\xi_1 - p\zeta_1\right)\vec{\mathbf{I}}_2 + \left(\frac{\partial \zeta_1}{\partial u} + p\eta_1 - q\xi_1\right)\vec{\mathbf{I}}_3.$$

The identification of these two expressions of $\dfrac{\partial^2 \vec{\mathbf{A}}}{\partial u\,\partial v}$ gives three relations

$$(5_1)\qquad \begin{cases} \dfrac{\partial \xi}{\partial v} - \dfrac{\partial \xi_1}{\partial u} = (q\zeta_1 - q_1\zeta) - (r\eta_1 - r_1\eta), \\ \dfrac{\partial \eta}{\partial v} - \dfrac{\partial \eta_1}{\partial u} = (r\xi_1 - r_1\xi) - (p\zeta_1 - p_1\zeta), \\ \dfrac{\partial \zeta}{\partial v} - \dfrac{\partial \zeta_1}{\partial u} = (p\eta_1 - p_1\eta) - (q\xi_1 - q_1\xi). \end{cases}$$

We can also deduce from the systems (3) and (4) two expressions for each of the second derivatives $\dfrac{\partial^2\vec{\mathbf{I}}_1}{\partial u\,\partial v}$, $\dfrac{\partial^2\vec{\mathbf{I}}_2}{\partial u\,\partial v}$, $\dfrac{\partial^3\vec{\mathbf{I}}_1}{\partial u\,\partial v}$. We have, for example,

$$\frac{\partial^2\vec{\mathbf{I}}_1}{\partial u\,\partial v} = (qq_1 + rr_1)\,\vec{\mathbf{I}}_1 + \left(\frac{\partial r}{\partial v} + p_1 q\right)\vec{\mathbf{I}}_2 + \left(-\frac{\partial q}{\partial v} + rp_1\right)\vec{\mathbf{I}}_3,$$
$$\frac{\partial^2\vec{\mathbf{I}}_1}{\partial u\,\partial v} = (qq_1 + rr_1)\,\vec{\mathbf{I}}_1 + \left(\frac{\partial r_1}{\partial u} + pq_1\right)\vec{\mathbf{I}}_2 + \left(-\frac{\partial q_1}{\partial u} + r_1 p\right)\vec{\mathbf{I}}_3.$$

The identification of these second derivatives gives three new relations

$$(5_2)\qquad \begin{cases} \dfrac{\partial p}{\partial v} - \dfrac{\partial p_1}{\partial u} = qr_1 - q_1 r, \\ \dfrac{\partial q}{\partial v} - \dfrac{\partial q_1}{\partial u} = rp_1 - r_1 p, \\ \dfrac{\partial r}{\partial v} - \dfrac{\partial r_1}{\partial u} = pq_1 - p_1 q. \end{cases}$$

The components ω hence do not have arbitrary structure: they satisfy [†] *the Darboux equations* (5_1), (5_2).

The structure of the relative components of the displacement of the moving frame plays a fundamental role in all the questions that we have tackled, because the families of frames depending on several parameters involve them. The aim of this chapter is to determine this structure.

The Darboux equations (5_1), (5_2) are complicated. This complication is due to that we did not use differential notations: the particular choice of the independent variables u and v plays an apparent role here. We will avoid this inconvenience by using suitable symbols.

II. Differentials; exterior derivation

153 Differential notation. Consider n independent variables $x_1, \ldots, x_n$. We call a system of n functions whose arguments are $x_1, \ldots, x_n$ and an arbitrary number of new variables the system of differentials $dx_1, \ldots, dx_n$ of these variables. The differential of a function $f(x_1, \ldots, x_n)$ is by definition

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n.$$

Suppose that we adopt as independent variables n functions $y_1, \ldots, y_n$ of $x_1, \ldots, x_n$ in place of $x_1, \ldots, x_n$. We will agree to continue to associate with them the same differentials

$$dy_i = \sum_k \frac{\partial y_i(x_1, \ldots, x_n)}{\partial x_k} dx_k.$$

[†] Darboux has shown that these are the only equations imposed on them: this is just a corollary of the structure theorem that will be established in §**163** (p. 179).

From the rule of differentiation for the functions of functions we deduce that, thanks to our convention, df is a quantity that do not depend on the choice of independent variables.

This invariance of df under a change of variables sums up the rules indicating how a change of variables transforms the derivatives of a function.

154 Simultaneous use of two systems of differentials. It will be useful for us to attribute simultaneously to each variable x_i two differentials dx_i and δx_i. Each function $f(x_1, \ldots, x_n)$ will hence have two differentials df and δf. We will make [†] the symbol δ to act on the differentials dx_i and the symbol d on the differentials δx_i. In other words, we define the differentials δ to be of the variables other than x_i, …, x_n which dx_i depend on, and the differentials d to be of the variables other than x_1, …, x_n which δx_i depend on. We will always assume that we can make our choice such that

$$d\delta x_i = \delta dx_i. \tag{6}$$

a. Suppose for example that the differentials dx_i and δx_i are *determined*, i.e., they are the functions $\xi_i(x_1, \ldots, x_n)$, $\eta_i(x_1, \ldots, x_n)$ of the variables x_1, …, x_n. The condition (6) will then be written

$$\sum_k \left(\frac{\partial \xi_i}{\partial x_k} \eta_k - \xi_k \frac{\partial \eta_i}{\partial x_k} \right) = 0.$$

b. Suppose on the contrary that dx_1, …,dx_n, δx_1, …, δx_n are *undetermined*, i.e., constitute $2n$ new variables. To satisfy the relations (6), it then suffices to define arbitrarily $d\delta x_i = \delta dx_i$.

c. Finally suppose that the differentiation d is determined and the differentiation δ is undetermined. The expressions δdx_i have one value imposed, and the expressions $d\delta x_i$ can be defined arbitrarily. We will hence choose $d\delta x_i = \delta dx_i$.

When the relation (6) is satisfied the symbols d and δ are said to *commute*. We then have

$$\begin{aligned} df &= \sum_i \frac{\partial f}{\partial x_i} dx_i, \qquad \delta f = \sum_i \frac{\partial f}{\partial x_i} \delta x_i, \\ \delta df &= \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \delta x_j + \sum_i \frac{\partial f}{\partial x_i} \delta dx_i, \\ d\delta f &= \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} \delta x_i dx_j + \sum_i \frac{\partial f}{\partial x_i} d\delta x_i, \end{aligned}$$

hence

$$d\delta f = \delta df. \tag{7}$$

[†] On the contrary it will be absolutely useless for us to use the symbols $d^2 f$ and $\delta^2 f$.

This relation, applied to the new independent variables $y_i(x_1, \dots, x_n), \dots, y_n(x_1, \dots, x_n)$, shows that if two differentiations commute, they also commute after a change of variables.

The introduction of two commuting differentiations allows us to translate the fact that we can invert the order of derivations under a remarkably simple form: the equation (7). But this is the fact that is at the base of all of the considerations of the current chapter and it is at the base of Darboux's calculus: the equations (5_1), (5_2) express that $\dfrac{\partial^2 \vec{\mathbf{A}}}{\partial u\, \partial v} = \dfrac{\partial^2 \vec{\mathbf{A}}}{\partial v\, \partial u}$.

N.B. The relations that we will write will always be satisfied regardless of the particular choice of the symbols dx_i and δx_i, $d\delta x_i = \delta d x_i$. Nonetheless it will be useful at various moments to make such a choice. Recall for example that if dx_i are the infinitesimal increase of the variables, df is the principal part of the corresponding increase of f.

155 Exterior derivation. Consider a Pfaffian form

$$\omega(x, dx) = \sum_k a_k(x_1, \dots, x_n) dx_k. \tag{8}$$

Let us form the expression

$$\begin{aligned} d\omega(x, \delta x) - \delta\omega(x, dx) &= \sum_k (da_k \delta x_k - \delta a_k dx_k) \\ &= \sum_{i,k} \frac{\partial a_k}{\partial x_i}(dx_i \delta x_k - \delta x_i dx_k) \\ &= \sum_{(i,k)} \left(\frac{\partial a_k}{\partial x_i} - \frac{\partial a_i}{\partial x_k} \right) (dx_i \delta x_k - \delta x_i dx_k).\ ^{(\dagger)} \end{aligned} \tag{9}$$

According to (7), a necessary condition for ω to be a exact total differential is that the expression (9) is zero. We will prove that this condition is also sufficient (§**166**, third corollary of Frobenius' theorem, p. 180).

As the relation (7) plays a fundamental role, the expression (9) will occur frequently. We will denote it by an abridged symbol:

$$\omega'(x, dx, \delta x) = d\omega(x, \delta x) - \delta\omega(d, dx). \tag{10}$$

The quantity ω' remains invariant when we change the choice of independent variables. We call ω' the *exterior derivative* or the *bilinear covariant* of ω.

(†)The summation $\sum_{i,k}$ takes all the values of the indices $1 \le i \le n$, $1 \le k \le n$, whereas the summation $\sum_{(i,k)}$ takes all the values of the indices $1 \le i \le k \le n$.

156 Exterior products. ω' depends linearly on each of the series of variables $dx_1, \ldots,$ dx_n; $\delta x_1, \ldots, \delta x_n$. ω' changes sign when we permute term for term the two series of variables. A function possessing these two properties is said to be alternating bilinear.

An alternating bilinear function of the variables $dx_1, \ldots, dx_n$; $\delta x_1, \ldots, \delta x_n$. ω' depends only on the variables through the intermediacy of the determinant $dx_i\delta x_j - \delta x_i dx_j$. Each of these determinants will be denoted by the abridged notation $[dx_i dx_j]$ and will be called the exterior product of dx_i by dx_j. This exterior product changes sign when we permute the two factors. We will consider it as distributive with respect to addition, such that the exterior product of two forms

$$\omega_i = \sum_i a_i(x)dx_i, \qquad \varpi_i = \sum_i b_i(x)dx_i,$$

will be by definition

$$\begin{aligned}[\omega\varpi] &= \sum_{i,j} a_i b_j [dx_i dx_j] \\ &= \sum_{(i,j)} (a_i b_j - a_j b_i)[dx_i dx_j] = \omega(dx)\varpi(\delta x) - \omega(\delta x)\varpi(dx).\end{aligned}$$

We obviously have

$$[\omega\varpi] = -[\varpi\omega] \qquad \text{and} \qquad [\omega\omega] = 0.$$

The formulae (9) will be henceforth written as

$$\omega' = \sum_{i,j} \left(\frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j}\right)[dx_i dx_j].$$

The reader will establish easily the following formula: consider a function m and a Pfaffian form ω. The exterior derivative of $m\omega$ is

$$(m\omega)' = [dm\,\omega] + m\omega'. \tag{11}$$

N.B. The notion of bilinear covariant was introduced by Frobenius. It is the base of the theory of Pfaffian systems. The exterior multiplication was defined by H. Grassmann. A more complete treatment of these notions can be found in *Leçons sur les Invariants intégraux* of E. Cartan (Hermann, 1922, chapters 5 and 6).

III. The structural equations of E. Cartan; the second fundamental theorem of groups

157 Definition of the structure constants. Let us consider a finite dimensional connected group of transformations S_a. Let $\omega_1(a, da), \ldots, \omega_r(a, da)$ be the relative components of the infinitesimal displacement of its moving frame.

The first parameter group leaves these r forms invariant (§**80**, p. 93). In other words, consider a fixed transformation S_a, a variable transformation S_ξ and the product transformation $S_{\xi'} = S_a S_\xi$. We have

$$\omega_s(\xi', d\xi') = \omega_s(\xi, d\xi) \qquad (s = 1, \ldots, r). \tag{12}$$

This relation (12) entails

$$d\omega_s(\xi', \delta\xi') - \delta\omega_s(\xi', d\xi') = d\omega_s(\xi, \delta\xi) - \delta\omega_s(\xi, d\xi). \tag{13}$$

Hence the first parameter group also leaves the exterior derivatives ω'_s of the r components ω_s invariant.

But each of these exterior derivatives is an alternating bilinear form with respect to the differentials of the independent variables, hence also with respect to the forms ω_s: we have

$$\omega'_s(\xi, d\xi, \delta\xi) = \sum_{(p,q)} c_{pqs}(\xi)[\omega_p(\xi, d\xi)\omega_q(\xi, d\xi)], \tag{14}$$

where

$$c_{pqs}(\xi) + c_{qps}(\xi) = 0. \tag{15}$$

Similarly

$$\omega'_s(\xi', d\xi', \delta\xi') = \sum_{(p,q)} c_{pqs}(\xi')[\omega_p(\xi', d\xi')\omega_q(\xi', d\xi')]. \tag{14'}$$

The relation (13) expresses the equality of the left hand sides of (14) and (14′). For the right hand sides to be equal, it is necessary, according to (12) and (15) that

$$c_{pqs}(\xi') = c_{pqs}(\xi).$$

But we can choose ξ and ξ' as two arbitrary points in the parameter space. The quantities $c_{pqs}(\xi)$ are hence independent of the variables $\xi_1, \ldots, \xi_r$. The constants $c_{pqs} = -c_{qps}$ $(1 \le p < q \le r)$ are the $\dfrac{r^2(r-1)}{2}$ "*structure constants*" of the r parameter group considered.

The relations (14) express that the relative components ω_s of the displacement of a moving frame are subject to the relations

$$\omega'_s = \sum_{(p,q)} c_{pqs}[\omega_p\omega_q]. \tag{16}$$

The relations (22) of paragraph §**75** (p. 88) allow us to deduce from the equations (16) that the absolute components ϖ_s satisfy the relations

$$\varpi'_s = -\sum_{(p,q)} c_{pqs}[\varpi_p\varpi_q]. \tag{17}$$

The equations (16) [and (17)] was first written down by Maurer [3]. We will call them preliminarily the *Maurer equations.* But it is E. Cartan who has established that these equations reveal the structure of the components ω_s and ϖ_s (the structure theorem of paragraph §**163**, p. 179). So now we will also call the equations (16) [and (17)] the *structural equations of E. Cartan.*

Remark. A linear transformation on the forms ω_s transforms the structural constants linearly.

158 Structure equations of the linear transformation group

$$x' = ax + b \qquad (a > 0).$$

We have, according to paragraphs §**71** (p. 82) and §**74** (p. 86)

$$\omega_1 = \frac{da}{a}, \qquad \omega_2 = \frac{db}{a}, \qquad \varpi_1 = \frac{da}{a}, \qquad \varpi_2 = db - b\frac{da}{a}.$$

Hence

$$\omega_1' = 0, \qquad \omega_2' = -\frac{1}{a^2}[da\, db], \qquad \varpi_1' = 0, \qquad \varpi_2' = \frac{1}{a}[da\, db].$$

The structural equations are hence

$$\omega_1' = 0, \qquad \omega_2' = -[\omega_1\omega_2], \qquad \text{and} \qquad \varpi_1' = 0, \qquad \varpi_2' = [\varpi_1\varpi_2].$$

then

$$c_{121} = 0, \qquad c_{122} = -1.$$

159 Structure equations of the planar projective group. The eight relative components are defined geometrically by the equations (16) and (17) of paragraph §**73** (p. 85)

$$\begin{cases} d\mathbf{A} \ = \omega_{00}(d)\mathbf{A} + \omega_{01}(d)\mathbf{A}_1 + \omega_{02}(d)\mathbf{A}_2, \\ d\mathbf{A}_1 = \omega_{10}(d)\mathbf{A} + \omega_{11}(d)\mathbf{A}_1 + \omega_{12}(d)\mathbf{A}_2, \\ d\mathbf{A}_2 = \omega_{20}(d)\mathbf{A} + \omega_{21}(d)\mathbf{A}_1 + \omega_{22}(d)\mathbf{A}_2, \end{cases} \tag{18}$$

$$\omega_{00}(d) + \omega_{11}(d) + \omega_{22}(d) = 0. \tag{19}$$

Let us write out

$$d\delta\mathbf{A} = \delta d\mathbf{A} = 0,$$

we obtain

$$\omega_{00}'\mathbf{A} + \omega_{01}'\mathbf{A}_1 + \omega_{02}'\mathbf{A}_2 - [\omega_{00}\, d\mathbf{A}] - [\omega_{01}\, d\mathbf{A}_1] - [\omega_{02}\, d\mathbf{A}_2] = 0.$$

Replacing $d\mathbf{A}$, $d\mathbf{A}_1$, $d\mathbf{A}_2$ by their expressions (18), it becomes

$$\begin{aligned} &\mathbf{A}\{\omega_{00}' - [\omega_{01}\omega_{10}] - [\omega_{02}\omega_{20}]\} \\ &\quad + \mathbf{A}_1\{\omega_{01} - [\omega_{00}\omega_{01}] - [\omega_{01}\omega_{11}] - [\omega_{02}\omega_{21}]\} \\ &\quad + \mathbf{A}_2\{\omega_{02} - [\omega_{00}\omega_{02}] - [\omega_{01}\omega_{12}] - [\omega_{02}\omega_{22}]\} = 0, \end{aligned}$$

each of the braces must be zero and furnishes one structural equation. We obtain the other structural equations by writing that

$$d\delta\mathbf{A}_1 - \delta d\mathbf{A}_1 = 0, \qquad d\delta\mathbf{A}_2 - \delta d\mathbf{A}_2 = 0.$$

The structural equations are finally the following

$$\omega'_{ij} = \sum_{k=0}^{2} [\omega_{ik}\omega_{kj}] \qquad (i = 0, 1, 2; j = 0, 1, 2). \tag{20}$$

160 Structure equations of the affine group of space (c.f. §72, p. 84). The affine frame is constituted by a point $\mathbf{A}$ and three linearly independent vectors $\vec{\mathbf{I}}_1$, $\vec{\mathbf{I}}_2$, $\vec{\mathbf{I}}_3$. The twelve relative components of its displacement are defined geometrically by the formulae

$$d\mathbf{A} = \sum_{k=1}^{3} \omega_k \vec{\mathbf{I}}_k, \tag{21}$$

$$d\vec{\mathbf{I}}_i = \sum_{k=1}^{3} \omega_{ik} \vec{\mathbf{I}}_k. \tag{22}$$

By writing out $d\delta\vec{\mathbf{I}}_i - \delta d\vec{\mathbf{I}}_i = 0$ we obtain, as in the preceding paragraph, nine structural equations

$$\omega'_{ij} = \sum_{k=1}^{3} [\omega_{ik}\omega_{kj}]. \tag{23}$$

On the other hand let us write out

$$d\delta\vec{\mathbf{A}} - \delta d\vec{\mathbf{A}} = 0,$$

it becomes

$$\sum_{k=1}^{3} \omega'_k \vec{\mathbf{I}}_k - \sum_{1 \leq k \leq 3} [\omega_k d\vec{\mathbf{I}}_k] = 0,$$

from which, by replacing $d\vec{\mathbf{I}}_k$ by their expressions (22), we get three other structural equations

$$\omega'_i = \sum_{k=1}^{3} [\omega_k \omega_{ki}]. \tag{24}$$

The equations (23) and (24) constitute the set of structural equations.

The unimodular affine group, the planar affine group, the group of the displacements of the space are the subgroups of the projective group, characterised respectively (c.f. §**125**, p. 140) by the equations $\omega_{i0} = 0$ as well as

$$\omega_{11} + \omega_{22} + \omega_{33} = 0, \tag{25}$$

$$\omega_3 = 0, \qquad \omega_{13} = 0, \qquad \omega_{31} = 0, \qquad \omega_{23} = 0, \qquad \omega_{32} = 0, \qquad \omega_{33} = 0, \tag{26}$$

$$\omega_{ij} + \omega_{ji} = 0. \tag{27}$$

Their frames are the particular projective frames for which the calculations of this paragraph applies. The structural equations of these three groups are hence the equations (23) and (24), where we must naturally take into account of the respective equations (25), (26), (27).

Consider the displacement group of space and let us make explicit the components ω as indicated in the formulae (2). A brief calculation shows that the formulae (23) and (24) then reduce to the Darboux equations (5_1) and (5_2). These *Darboux equations* which play a fundamental role in the study of differentials of surfaces and Euclidean geometry are hence particular structural equations.

161 Structure equations of the homographic transformation group

$$x' = \frac{ax+b}{cx+d} \qquad (ad-bc>0).$$

According to the equations (9) of chapter 8 (§**108**, p. 120), we have

$$\omega_1 = u\,dx, \qquad \omega_2 = 2v\,dx - \frac{du}{u}, \qquad \omega_3 = \frac{v^2}{u}dx - \frac{dv}{u}.$$

From which

$$\omega_1' = [du\,dx], \qquad \omega_2' = 2[dv\,dx], \qquad \omega_3' = \frac{2v}{u}[dv\,dx] - \frac{v^2}{u^2}[du\,dx] + \frac{1}{u^2}[du\,dv].$$

But

$$[\omega_1\omega_2] = [du\,dx], \qquad [\omega_1\omega_3] = [dv\,dx],$$
$$[\omega_2\omega_3] = \frac{2v}{u}[dv\,dx] - \frac{v^2}{u^2}[dv\,dx] + \frac{1}{u^2}[du\,dv]$$

The structural equations are hence

$$\omega_1' = [\omega_1\omega_2], \qquad \omega_2' = 2[\omega_1\omega_3], \qquad \omega_3' = [\omega_2\omega_3]. \tag{28}$$

162 A theorem about Pfaffian systems. The following paragraph will then establish three essential theorems which involve the structural constants. These theorems will be immediate corollaries of the theorem of integral calculus here:

Statement. Consider r linearly independent forms in r variables $\omega_s(a, da)$ and r other forms in ρ variables $\omega_s^\star(u, du)$ ($\rho \le r$). Suppose that the two systems of forms ω_s and $\omega_s^\star$ satisfy the same structural equations

$$\omega_s' = \sum_{(p,q)} c_{pqs}[\omega_p\omega_q], \qquad \omega_s^{\star\prime} = \sum_{(p,q)} c_{pqs}[\omega_p^\star\omega_q^\star],$$

where c_{pqs} are constants. We claim that we can choose in one and only one way $a_1, \dots, a_r$ as functions of $u_1, \dots, u_\rho$ satisfying the system

$$\omega_s(a, da) = \omega_s^\star(u, du) \tag{29}$$

which associate a point a^0 to an arbitrarily given point u^0.

Proof. Consider a point $u(t)$ depending on a parameter t and coinciding with u^0 at $t = 0$. The differential system

$$\omega_s(a, da) = \omega_s^\star[u(t), du(t)] \qquad (s = 1, \dots, r) \tag{30}$$

can be solved with respect to the differentials da_s as function of $a_1, \dots, a_r, t, dt$. It possesses one and only one solution $a_1(t), \dots, a_r(t)$ such that $a_1(0), \dots, a_r(0)$ are the coordinates of the point a^0. The point $a(t)$, which has coordinates $a_1(t), \dots, a_r(t)$, is the homologue of $u(t)$ under the correspondence we search for, if it exists.

Now suppose that the point u depends not only on t, but on a second parameter θ as well. The components $\dfrac{\partial a_s}{\partial \theta}$ of the vector $\dfrac{\partial \vec{a}}{\partial \theta}$ are obtained by integrating the differentiated system (30)

$$\frac{\partial}{\partial \theta}\omega_s\left(a, \frac{\partial a}{\partial t}\right) = \frac{\partial}{\partial \theta}\omega_s^\star\left(u, \frac{\partial u}{\partial t}\right). \tag{31}$$

The equations (31) constitute, with respect to the unknowns $\dfrac{\partial a_s(t,\theta)}{\partial \theta}$, a linear differential system, and the initial values of the uknowns are

$$\frac{\partial a_s(0,\theta)}{\partial \theta} = 0.$$

Since the two differentiations $\dfrac{\partial}{\partial t}$ and $\dfrac{\partial}{\partial \theta}$ commute, we have, according to the structural equations of E. Cartan,

$$\frac{\partial}{\partial \theta}\omega_s\left(a, \frac{\partial a}{\partial t}\right) - \frac{\partial}{\partial t}\omega_s\left(a, \frac{\partial a}{\partial \theta}\right) = \sum_{p,q} c_{pqs}\omega_p\left(a, \frac{\partial a}{\partial \theta}\right)\omega_q\left(a, \frac{\partial a}{\partial t}\right),$$

$$\frac{\partial}{\partial \theta}\omega_s^\star\left(u, \frac{\partial u}{\partial t}\right) - \frac{\partial}{\partial t}\omega_s^\star\left(u, \frac{\partial u}{\partial \theta}\right) = \sum_{p,q} c_{pqs}\omega_p^\star\left(u, \frac{\partial u}{\partial \theta}\right)\omega_q^\star\left(u, \frac{\partial u}{\partial t}\right).$$

According to (30) the last factors $\omega_q\left(a, \dfrac{\partial a}{\partial t}\right)$, $\omega_q^\star\left(u, \dfrac{\partial u}{\partial t}\right)$ are equal. The equation (31) can hence he written as

$$\begin{aligned} &\frac{\partial}{\partial t}\left\{\omega_s\left(a, \frac{\partial a}{\partial \theta}\right) - \omega_s^\star\left(u, \frac{\partial u}{\partial \theta}\right)\right\} \\ &\qquad + \sum_{p,q} c_{pqs}\left\{\omega_p\left(a, \frac{\partial a}{\partial \theta}\right) - \omega_p^\star\left(u, \frac{\partial u}{\partial \theta}\right)\right\}\omega_q^\star\left(u, \frac{\partial u}{\partial t}\right) = 0. \end{aligned}$$

The braces, being zero for $t = 0$, are zero regardless of t. The solution $\dfrac{\partial a_s}{\partial \theta}$ of (31) are obtained hence by solving the system

$$\omega_s\left(a, \frac{\partial a}{\partial \theta}\right) = \omega_s^\star\left(u, \frac{\partial u}{\partial \theta}\right). \tag{32}$$

The equations (32) show that the point $a(t)$ associated with a fixed point $u(t)$ remains fixed when we modify the points $u(t')$ corresponding to values t' different from 0 and 1 continuously: the correspondance among the points $u(t)$ and $a(t)$ is hence one-to-one. On the other hand (32) expresses that this correspondance satisfies the equations (29). We have already observed that it is the only one possible.

The theorem is hence established.

N.B. The reasonings we have made require the statement of the theorem to be made precise by the following hypotheses, which are *sufficient*:

1. The space of the points u is simply connected, i.e., every closed contour can be continuously reduced to a point.
2. The space of the points a has only interior points.
3. When the integral $\int_L \sqrt{\sum_s \omega_s^2}$ over the space of a along a path L remains bounded, this path converges to a point of this space.

The necessity of the third hypothesis is made obvious by the following example in which $r = 1$:

$$\omega_1(a, da) = \frac{da}{1+a^2}, \qquad \omega_1^\star(u, du) = du,$$

the space of a and the space of u formed by two indefinite straight lines generated by the points with abscissae a and u. If we correspond to a point $u = 0$ the point $a = 0$, the equality $\omega_1(a, da) = \omega_1^\star(u, du)$ gives

$$a = \tan u.$$

a point a only corresponds to points u in the interval

$$-\frac{\pi}{2} < u < +\frac{\pi}{2},$$

and the integral

$$\int \omega_1(a, da) = \int \frac{da}{1+a^2}$$

remains bounded indeed when a is increased indefinitely, contrary to the condition 3.

If the hypothesis 3 is satisfied, the integration of the differential system (30) can be carried out step by step indefinitely. Indeed, suppose the contrary and let t_0 be the lower boud of the values of t that cannot be attained by integration. When t tends to t_0 from below, we have in the space of a a linge L for which the integral $\int \sqrt{\sum \omega_s^2}$ remains bounded since it is less than the finite integral $\int_{t=0}^{t=t_0} \sqrt{\sum_s [\omega_s^\star(u, du)]^2}$. Hence this line L tends to a point (a) corresponding to $t = t_0$ and from here we can continue the integration.

The hypothesis 3 is certainly satisfied if the Riemannian space constituted by the space of a equipped with the metric $ds^2 = \sum_s [\omega_s(a, da)]^2$ is *normal*, i.e., if all *bounded* infinite set of points admit an accumulation point [†].

163 Suppose in the theorem of the preceding paragraph the quantities $\omega_s(a, da)$ are the relative components of the infinitesimal displacement of a frame of a r parameter group, and the forms $\omega_s^\star(u, du)$ satisfy the structural equations of this group. The theorem above signifies that there exists a family of frames with parameters $a_1(u)$, ..., $a_r(u)$ whose relative infinitesimal displacement has components

$$\omega_1^\star(u, du), \qquad \ldots, \qquad \omega_r^\star(u, du).$$

This family is determined up to a choice of the point a^0, i.e., up to a transformation of the group. This is in accordance with the fundamental condition of equality (§**76**, p. 89).

The theorem above can hence be interpreted geometrically as the following (c.f. §**77**, p. 90).

STRUCTURAL THEOREM. *Given a group, the relative components of the infinitesimal displacement of its moving frame are subject only to the structural equations of E. Cartan*

$$\omega_s' = \sum_{(p,q)} c_{pqs}[\omega_p \omega_q].$$

164 Let us now consider two r parameter groups having the same structural constants and whose parameter spaces are simply connected. Let us call $\omega_s(a, da)$ and $\omega_s^\star(u, du)$ the relative components of the infinitesimal displacement of the frames of these two groups. The theorem in paragraph §**162** affirms that we can establish a bijective correspondance between the points u and the points a satisfying the equations

$$\omega_s(a, da) = \omega_s^\star(u, du).$$

The transformations of the first group and of the second group are reciprocally characterised by the property that they leave the forms ω_s and $\omega_s^\star$ invariant. The correspondance established between the points a and the points u hence establishes a correspondance between the transformations: it constitutes a holoedric isomorphism.

Conversely, two holoedrically isomorphic groups have the same parameter group and the same structural constants. Hence:

NECESSARY AND SUFFICIENT CONDITION FOR ISOMORPHISM: *Consider two groups whose parameter spaces are simply connected* [‡]*. For them to be holoedrically isomorphic, it is necessary and sufficient that they have the same structural constants.*

[†] See E. CARTAN, *Leçon sur la géométrie des espaces de Riemann*, p. 64.

[‡] Every group G is meriedrically isomorphic to a group Γ whose parameter space is simply connected. Let us construct what is called the *simply connected covering space* of the parameter space of G in topology, and Γ is the group of transformations of this space leaving invariant the Pfaffian forms left invariant by the parameter group G [15].

165 Finally suppose that in the theorem of paragraph §**162** the forms $\omega_s^\star$ are identical to the forms ω_s.

The proof of the theorem can then be completed in a way to make the hypothesis that the space of the oints a is simply connected superfluous.

The theorem considers the transformation of the space leaving the forms ω_s invariant. These transformations obviously constitute a group. The theorem consists in affirming that, in the case considered, the group operates transitively on the space of points a: it depends on r parameters, which is then a parameter group (c.f. §**83**, p. 96).

In other words:

The second fundamental theorem in the theory of groups (E. Cartan) [(†)]**.** *Consider r Pfaffian forms $\omega_s(a, da)$ in r parameters. For the transformations leaving them invariant to constitute a r parameter group, it is necessary and sufficient that the following conditions are realised:*

1. *The forms ω_s satisfy the structural equations of E. Cartan*

$$\omega_s' = \sum_{(p,q)} c_{pqs}[\omega_p \omega_q].$$

2. *The space of points a possesses only interior points. When the value of the integral $\int_L \sqrt{\sum_{s=1}^{r} |\omega_s|^2}$ is finite, the path L converges to a point of the space. The forms $\omega_s(a, da)$ are linearly independent at each point a.*

166 Complements. We are going to show how the essential passages of this chapter (§**157** and §**162**) can be deduced from a general theorem about Pfaffian forms.

Frobenius theorem. *Consider a R dimensional space of coordinates $X_1, \ldots, X_R$ and N ($< R$) linearly independent Pfaffian forms $\Pi_1(X, dX), \ldots, \Pi_N(X, dX)$. The necessary and sufficient condition for that the Pfaffian system*

$$\Pi_1(X, dX) = 0, \qquad \ldots, \qquad \Pi_N(X, dX) = 0 \tag{33}$$

to be completely integrable is the following: the exterior derivatives $\Pi_1'(X, dX, \delta X), \ldots, \Pi_N'(X, dX, \delta X)$ are zero for every choice of the differentials $dX_1, \ldots, dX_R$, $\delta X_1, \ldots, \delta X_r$ which annihilates the forms $\Pi_1, \ldots, \Pi_R$.

We can prove this theorem by reasonings not too different from those of paragraphs §**167** and §**162**. However, to vary the exposition, we are going to prove it with the a change of coordinates and the transformation of the system (33) into an equivalent system.

[(†)] When we discuss the theory of groups as done by S. Lie, the name of the "second fundamental theorem" becomes the theorem of S. Lie which will be stated in paragraph §**211** (p. 231) under the name of "complement to the second fundamental theorem".

Proof. A change of coordinates leaves the property that a system is completely integrable invariant. It also leaves the exterior derivatives invariant, and hence the Frobenius condition invariant.

Two Pfaffian systems are said to be equivalent when at each point $X_1, \ldots, X_N$ the same differentials $dX_1, \ldots, dX_n$ satisfy the two systems. They are completely integrable simultaneously.

Consider two equivalent systems, (33) and

$$\Pi_1^\star(X, dX) = 0, \qquad \ldots, \qquad \Pi_N^\star(X, dX) = 0. \tag{33*}$$

The forms $\Pi_1^\star, \ldots, \Pi_N^\star$ reduce to the forms $\Pi_1, \ldots, \Pi_N$ by a linear transformation with coefficients functions of $X_1, \ldots, X_R$:

$$\Pi_i^\star(X, dX) = A_{i1}(X)\Pi_1(X, dX) + \cdots + A_{iN}(X)\Pi_N(X, dX).$$

From which

$$\Pi_i^{\star\prime} = A_{i1}\Pi_1' + \cdots + A_{iN}\Pi_N' + [dA_{i1}\Pi_1] + \cdots + [dA_{iN}\Pi_N].$$

Suppose that the system (33) satisfies the Frobenius condition: then the brackets written above are zero when the differentials $dX_1, \ldots, dX_R, \delta X_1, \ldots, \delta X_R$ satisfy the equations (33). The Frobenius condition therefore holds simultaneously for two equivalent systems.

Granted this, let us now show that a completely integrable system (33) satisfy the Frobenius condition. We can choose, in the neighbourhood of a point, new coordinates $U_1, \ldots, U_R$ such that the equations of the integral varieties are $U_1 = \text{constant}, \ldots, U_N = \text{constant}$. The system (33) is equivalent algebraically to the following

$$dU_1 = 0, \qquad \ldots, \qquad dU_N = 0. \tag{34}$$

The exterior derivatives of $dU_1, \ldots, dU_N$ are identically zero. (34) and hence (33) therefore satisfy the Frobenius condition. Q.E.D.

Let us now show that the system (33) is completely integrable when it satisfies the Frobenius condition.

When $R = N + 1$, (33) is a system of N differential equations in N unknowns in one independent variable. Through every point passes an integral curve. (33) is hence a completely integrable system.

Then we can proceed by induction, by supposing that the proposition already proved for $R - 1$ dimensions and N equations. The system (33), when we replace in it X_R by a constant, is completely integrable, by virtue of this supposition. The integrals have equations, in a neighbourhood of the given point,

$$U_1 = \text{constant}, \qquad \ldots, \qquad U_N = \text{constant}, \qquad X_R = \text{constant},$$

by means of the introduction of a new system of coordinates $U_1, \ldots, U_{R-1}, X_R$. Necessarily, the system (33), written with these new coordinates, is equivalent to a system of the form

$$dU_1 = A_1 dX_R = 0, \qquad \ldots, \qquad dU_N - A_N dX_R = 0. \tag{35}$$

According to the hypotheses the system (35) must satisfy the Frobenius condition. Then the exterior product

$$[dA_1 dX_R] = \frac{\partial A_1}{\partial U_1}[dU_1 dX_R] + \cdots + \frac{\partial A_1}{\partial U_{R-1}}[dU_{R-1} dX_R]$$

is zero using (35). Hence

$$\frac{\partial A}{\partial U_{N+1}} = 0, \qquad \ldots, \qquad \frac{\partial A_1}{\partial U_{R-1}} = 0.$$

$A_1, \ldots, A_N$ are hence functions only of the coordinates $U_1, \ldots, U_N, X_R$: (35) is a system of N differential equations in N unknowns and one independent variable. It is therefore completely integrable. Q.E.D.

Important remark. A $R - N$ dimensional variety satisfying (35) is regular in the neighbourhood of every point on which the forms $\Pi_1, \ldots, \Pi_N$ are linearly independent with respect to their differentials. Such a variety cannot cut itself. But, considered in its totality, it can pass through the neighbourhood of the same point an infinite number of times.

First corollary of Frobenius theorem. Consider the components ω_s of the displacement of a frame of a group. The system of r equations and $2r$ variables

$$\omega_s(\xi, d\xi) = \omega_s(\xi', d\xi') \tag{36}$$

is completely integrable. But we can write

$$\omega_s'(\xi, d\xi, \delta\xi) = \sum_{p,q} c_{pqs}(\xi)\omega_p(\xi, d\xi)\omega_q(\xi, \delta\xi) \qquad [c_{pqs}(\xi) + c_{qps}(\xi) = 0].$$

According to Frobenius theorem the relations

$$\sum_{p,q} c_{pqs}(\xi)\omega_p(\xi, d\xi)\omega_q(\xi, \delta\xi) = \sum_{p,q} c_{pqs}(\xi')\omega_p(\xi', d\xi')\omega_q(\xi', \delta\xi')$$

must hold whenever $d\xi$, $d\xi'$, $\delta\xi$, $\delta\xi'$ satisfy the equations (36). Hence

$$c_{pqs}(\xi) = c_{pqs}(\xi') = \text{constant}.$$

This is the fundamental result of paragraph **§157**.

Second corollary of Frobenius theorem. The theorem of paragraph **§162** consists in affirming that the system (29) in r equations and $r+\rho$ variables is completely integrable. Proving it becomes, according to Frobenius theorem, verifying that we have

$$\omega_s' = \omega_s^{\star\prime},$$

whenever the differentials satisfy the relations (29). But this fact results immediately from the structural equations

$$\omega_s' = \sum_{(p,q)} c_{pqs}[\omega_p\omega_q], \qquad \omega_s^{\star\prime} = \sum_{(p,q)} c_{pqs}[\omega_p'\omega_q'].$$

Third corollary of Frobenius theorem. We have already said (§**155**, 171) that the necessary and sufficient condition for a form in n variables $\omega(x, dx)$ to be an exact total differential is that its exterior derivative is identically zero. We can deduce this fact from the Frobenius theorem: for $\omega(x, dx)$ to be a total exact differential, it is necessary and sufficient that the equation in $n+1$ independent variables

$$dy - \omega(x, dx) = 0$$

is completely integrable. But the Frobenius condition reduce in this case to $\omega' = 0$.

IV. Determination of groups and subgroups

The structural equations play a fundamental role in the determination of groups enjoying some given properties. We are going to indicate the simplest of these problems.

167 Search of all groups with one parameter. The translation on the straight line constitutes a one parameter group. Its parameter group has the equation

$$\xi' = \xi + a.$$

It leaves the form $\omega_1 = d\xi$ invariant. Its structural equation is

$$\omega_1' = 0. \tag{37}$$

But the structural equation of a one parameter group necessarily reduces to the equation (37). According to the isomorphism condition (§**164**, p. 179) every one parameter group is hence isomorphic to the group of translations on the straight line. Such is the case, for example, for the group of rotations on the circle, $\varphi' = \varphi + a$, where the angle φ is defined up to $2k\pi$, but the isomorphism here is only meriedric, since the space of parameters, which is a circumference, is not simply connected.

168 Search of all groups with two parameters. The transformations of the plane constitutes a two parameter group. Its parameter group has equations

$$\xi_1' = \xi_1 + a_1, \qquad \xi_2' = \xi_2 + a_2. \tag{38}$$

It leaves the forms $\omega_1 = d\xi_1$, $\omega_2 = d\xi_2$ invariant. Its structural equations are

$$\omega_1' = 0, \qquad \omega_2' = 0. \tag{39}$$

The linear transformation group $x' = ax + b$ also depends on two parameters. Its structural equations are (§**158**, p. 174)

$$\omega_1' = 0, \qquad \omega_2' = -[\omega_1\omega_2]. \tag{40}$$

Now consider any group of two parameters. Its structural equations are of type

$$\omega_1' = c_{121}[\omega_1\omega_2], \qquad \omega_2' = c_{122}[\omega_1\omega_2]. \tag{41}$$

A linear transformation with constant coefficients applied on ω_1 and ω_2 allow us to set c_{121} to zero and give c_{122} the value 0 or -1. A group in two parameters therefore have the structure (39) or (40). Hence every two parameter group is isomorphic to the group of translations of planes or the group of linear transformations in one variable.

169 Realisation of an abstract group. Determination of the subgroups of a group. Chapter 9 makes the search for transitive groups isomorphic to a given group (§**116**, p. 133) and the determination of the subgroups of a given group (§**125**, p. 140) to the following problem: *Forming the linear combinations with constant coefficients of the components $\omega_1, \ldots \omega_r$ that, when set to zero, constitute a completely integrable system.*

Frobenius theorem and the structural equations immediately give the necessary and sufficient conditions such that the system

$$\omega_1 = 0, \qquad \ldots, \qquad \omega_\rho = 0$$

is completely integrable: we must have

$$c_{pqs} = 0 \qquad \text{for} \qquad s = 1, \ldots, \rho, \quad p = s+1, \ldots, r, \quad q = s+1, \ldots, r. \tag{42}$$

Once the structural constants are known, the problem whose statement we just recalled reduces to a simple algebraic problem.

More generally chapter 9 (§**119**, p. 137) makes the search for groups, transitive or intransitive, isomorphic to a given group to the following problem: *Finding functions α_{ij} of variables $y_1, \ldots, y_p$ such that the system*

$$dy_1 = 0, \qquad \ldots, \qquad dy_p = 0, \tag{43}$$

$$\sum_{1 \le s \le r} \alpha_{is}(y)\omega_s(\xi, d\xi) = 0 \qquad (i = 1, \ldots, n-p), \tag{44}$$

is completely integrable. Frobenius theorem and the structural equations allow us to give this condition the following form: we must have

$$\sum_{s=1}^{r} \alpha_{is}(y) \sum_{(p,q)} c_{pqs}[\omega_p \omega_q] = 0, \tag{45}$$

when the forms ω_s satisfy the equations (44). The determination of the functions $\alpha_{is}(y)$ is hence simply an algebraic problem.

Chapter 12

Differential geometry (continued)

i. Implementation of the method of moving frames

170 Introduction. Consider a finite dimensional connected group G operating transitively on the points in a domain D. The problems of differential geometry concerning the varieties traced in D, studied with respect to the transformations of G, can be teated by the method of moving frame. The section i. of chapter 10 (p. 141) has described this method.

The problems of differential geometry remain the same when we replace G by a similar group. To solve them therefore it is immaterial to suppose that D and G are explicitly given. It suffices to specify what abstract group is realised by G and what bodies of objects are realised by the points of D. To characterise this abstract group, it suffices to specify the structural constants of G (at least if the parameter space of G is simply connect, c.f., *conditions of isomorphism*, §**164**, p. 179). To characterise the bodies of objects, it suffices to specify for example that it constitutes the integral varieties of the system that we obtain by equating to zero the n first relative components ω_s of the displacement of frame of G (this supposes that the points of G are oriented objects).

In sum, (up to some questions about orientation) the data of the problems of geometry solved by the method of moving frame are the structural constants of the "fundamental group G". The aim of this chapter is to show how we can apply this method by using only the structure constants. The process that we are going to describe is not only of this theoretical interest: it furnishes, *through a rapid and practical algorithm*, the number of invariants of order P and their properties.

This process does not address the questions of orientation. It cannot serve to determine practically the frames and the invariants of order P of a given variety at a given point [†]. But when we have finished applying this method, it is easy to obtain the *reduced*

[†] We have already treated the problem of their determination by the method of moving frames in the course of paragraph §**138** (p. 149).

equations [†] of the variety. These equations easily furnish geometrical interpretations, and with the help of direct calculations independent of the method of moving frame, we can deduce from them the answers to the questions about orientation still remaining (c.f. §**42**, p. 50).

171 Review of notations; applications of the structure equations of E. Cartan. Consider a λ dimensional variety V_λ and one of its oriented contact elements of order P. This contact element is specified by a family of frames depending on v_P parameters (the *frames of order* P) and a system of invariants $k_1, \ldots, k_{\mu_P}$ (the *invariants of orders* $\leq P$).

The transformations of these frames into each other constitute a subgroup of G, $\mathcal{G}_P$. We call it the *subgroup of order* P attached to the contact element that we consider.

Consider a frame that varies while remaining a frame of order P of the same point in V_λ. Among its relative components ω_i there exists $r - v_P$ linearly independent relations obtained by annihilating the last $r - v_P$ *principal components* of orders $\leq P$:

$$\text{(1)} \qquad \pi_1 = \sum_{p=1}^{r} a_{1,p}\omega_p, \qquad \ldots, \qquad \pi_{r-v} = \sum_{p=1}^{r} a_{r-v_P,p}\omega_p,$$

$a_{l,p}$ being constants or functions of the invariants of orders $\leq P$.

We can suppose, to simplify the exposition, that the forms $\omega_{r-v_P+1}, \ldots, \omega_r$ are independent linear combinations of the differentials of the v_P *secondary parameters* the frames of order P depend on, which amounts to saying that the forms $\pi_1, \ldots, \pi_{r-v_P}$ are linearly independent in $\omega_1, \ldots, \omega_{r-v_P}$.

According to the theorem of paragraph §**124** (p. 140), the system

$$\pi_1 = 0, \qquad \ldots, \qquad \pi_{r-v_P} = 0$$

is completely integrable for every system of constant values given to the principal parameters in $a_{l,p}$. Frobenius theorem (§**166**, p. 180) affirms that under these conditions the quantities

$$\sum_{p=1}^{r} a_{1,p}\omega'_p, \qquad \ldots, \qquad \sum_{p=1}^{r} a_{r-v_P,p}\omega'_p$$

is zero whenever the differentials annihilate $\pi_1, \ldots, \pi_{r-v_P}$. But the differential of $a_{l,p}$ is zero when the differentials dk_m of the invariants are. This proves that we can write

$$\text{(2)} \qquad \begin{aligned} \pi'_i = {} & \sum_{(\alpha,\beta)} C_{\alpha\beta i}[\pi_\alpha\pi_\beta] + \sum_{m,\alpha} D_{m\alpha i}[dk_m\pi_\alpha] \\ & + \sum_{\alpha,n} A_{\alpha n i}[\pi_\alpha\omega_{r-v_P+n}] + \sum_{m,n} B_{mni}[dk_m\omega_{r-v_P+n}], \end{aligned}$$

[†] We have given examples of reduced equations in paragraph §**39** [equations (26), p. 47], and in paragraph §**139** [equation (17), p. 153]. More generally we call the equations of the varieties with respect to its frames of order P the reduced equations.

the indices of summation α, β go from 1 to $r - v_P$, the index m from 1 to μ_P and the index n from 1 to v_P. The coefficients are functions of the invariants k_m which are immediately reduces to the quantities $a_{l,p}$ and *the structural constants c_{pqs} of the fundamental group G.*

Let us make a special choice of the differentials d and δ appearing in (2). The differentials d of the principal and secondary parameters are zero, the differentials δ of the secondary parameters are v_P new variables, consisting of v_P infinitesimally small increment, and the differentials $d\delta$ and δd of the principal and secondary parameters are zero. The symbols d and δ commute. The quantities $\varpi_i(\delta)$ are zero and we set

$$\omega_s(\delta) = e_s \qquad (s = r - v_P + 1, \dots, r).$$

The equations (2) are hence written

$$\delta\pi_i + \sum_{n=1}^{n=v_P} e_{r-v_P+n} \left[\sum_{\alpha=1}^{\alpha=r-v_P} A_{\alpha ni}\pi_\alpha + \sum_{m=1}^{m=\mu_P} B_{mni} dk_m \right] = 0 \tag{3}$$

These formulae indicate what increments $\pi_1, \dots, \pi_{r-v_P}$ undergo when the secondary parameters undergo the given increments: they give the infinitesimal transformations of the group $\mathcal{G}_P$, considered as operating on the principal components of orders $\leq P$.

Recall the definition of the *coefficients of order P*, $b_{\alpha\beta}$ and $b'_{\alpha\beta}$: we have, when a frame varies while remaining a frame of order P of V_λ,

$$\begin{cases} dk_\alpha = b_{\alpha 1}\pi_1 + \cdots + b_{\alpha\lambda}\pi_\lambda, & \text{where} \quad \mu_{P-1} < \alpha \leq \mu_P, \\ \pi_\alpha = b'_{\alpha 1}\pi_1 + \cdots + b'_{\alpha\lambda}\pi_\lambda, & \text{where} \quad r - v_{P-1} < \alpha \leq r - v_P. \end{cases} \tag{4}$$

When the secondary parameters undergo increments, the coefficients $b_{\alpha\beta}$ and $b'_{\alpha\beta}$ undergo increments $\delta b_{\alpha\beta}$ and $\delta b'_{\alpha\beta}$ which are obtained by differentiating the defining formulae (4). dk_α remains constant ($\delta dk_\alpha = 0$). It becomes

$$\begin{cases} 0 = \delta b_{\alpha 1}\pi_1 + \cdots + \delta b_{\alpha\lambda}\pi_\lambda + b_{\alpha 1}\delta\pi_1 + \cdots + b_{\alpha\lambda}\delta\pi_\lambda, \\ \delta\pi_\alpha = \delta b'_{\alpha 1}\pi_1 + \cdots + \delta b'_{\alpha\lambda}\pi_\lambda + b_{\alpha 1}\delta\pi_1 + \cdots + b_{\alpha\lambda}\delta\pi_\lambda. \end{cases} \tag{5}$$

Let us replace in the formulae (3) the principal components of order P, π_l, by their expressions (4) and the principal components π_l of orders $< P$ by their expressions as function of $\pi_1, \dots, \pi_\lambda$ and the invariants of orders $\leq P$. Let us substitute the values $\delta\pi_1, \dots, \delta\pi_{r-v_P}$ thus obtained into (5). Let us write that in each of the relations thus obtained the coefficients of $\pi_1, \dots, \pi_\lambda$ are zero. It becomes relations of the form

$$\begin{cases} \delta b_{\alpha\beta} = \displaystyle\sum_{n=1}^{n=v_P} f_{\alpha\beta n}(b_{\lambda\mu}, b'_{\lambda\mu}) e_{r-v_P+n}, \\ \delta b'_{\alpha\beta} = \displaystyle\sum_{n=1}^{n=v_P} \varphi_{\alpha\beta n}(b_{\lambda\mu}, b'_{\lambda\mu}) e_{r-v_P+n}. \end{cases} \tag{6}$$

The formulae (6) give the infinitesimal transformations of the group $\mathcal{G}_P$, considered as operating on the coefficients of order P.

172 The mechanism of the method of moving frames. To apply this method is to solve the following problem a certain number of times:

The structure constants of the fundamental group G are given.

We know, 1. the number of invariants of orders $\leq P$, 2. the matrix of secondary components of orders $\leq P$, 3. the expressions of the principal components of orders $< P$ of a fame of order P of V_λ and the differentials of the invariants of orders $< P$ as functions of $\pi_1, \ldots, \pi_\lambda$ and the invariants of orders $\leq P$.

We seek analogous information concerning the order $P+1$.

For this, we deduce from the matrix of secondary coefficients of orders P and $P-1$ a system of $v_{P-1} - v_P$ principal components of order P (1). We calculate their exterior derivatives (2). We define by the formulae (4) the coefficients of order P and we calculate, through the intermediate step of (5), the infinitesimal transformations (6) of $\mathcal{G}_P$, considered as operating on these coefficients. We consider the space W_P having the coefficients of order P as coordinates. We construct, following the indications in paragraph §**87** (p. 99), the set $v(b)$ of transformed images of a point b arbitrarily chosen in W_P by all the operations of $\mathcal{G}_P$. We trace in W_P in the simplest way possible a variety w_P [(†)] which meets each variety $v(b)$ at one and only one point. Every point b of W_P hence have one and only one homologue on w_P. Conforming to the prescriptions of paragraph §**129** (p. 143), we choose the frames of order P corresponding to a point b beloning to w_P the *frames of order* $P+1$. We will choose the parameters allowing us to characterise the position of this point b on w_P as the *invariants of order* $P+1$.

If for example $\mathcal{G}_P$ operate transitively on W_P, w_P is a point and there is no invariant of order $P+1$.

The formulae (4) give, when b is on w_P, the *expressions* of the principal components of order P and the differentials of the invariants of order P, as function of $\pi_1, \ldots, \pi_\lambda$ and the invariants of order $P+1$.

$\mathcal{G}_{P+1}$ is the subgroup of $\mathcal{G}_P$ leaving the point of w_P corresponding to the contact element of order $P+1$ considered fixed. The table of *secondary components of order* $P+1$ hence reduce to the table of secondary parameters of order P obtained by linking the quantities e_q using the relations (6), where we set $\delta b_{\alpha\beta} = 0$, $\delta b'_{\alpha\beta} = 0$.

Important remark. When $\lambda > 1$ it is essential to express that the principal quantities of orders 0, 1, … satisfy the structure equations of the fundamental group G. If we neglect this fact, we would be discussing the categories of varieties which do not exist (*see* for this the examples of paragraphs §**187**, §**188** and §**192**, p. 204, 205 and 208).

II. Projective geometry; study of plane curves

[(†)] If necessary w_P can be composed of several varieties.

173 Introduction. We have given a first solution to the problem stated in paragraph **§141** (p. 155), which consists of a long calculation (p. 155–161). The application of the procedure developed in the preceding paragraph not only leads more comfortably to our goal, but also lets us discover an important property of projective geometry: the existence of anharmonic of four points on a curve (c.f. §**183**, p. 198). This is what we are going to develop now.

Recall the structural equations of the group considered, which is the projective group of the plane [equations (20)

$$\omega'_{ij} = \sum_{k=0}^{2} [\omega_{ik}\omega_{kj}] \qquad (i = 0,1,2; j = 0,1,2). \tag{7}$$

174 Nature of the elements of order 0**.** The frames of order 0 attached to a point $\mathbf{A}$ will be the frames $\mathbf{A}\mathbf{A}_1\mathbf{A}_2$ having this point as the first vertex. We determine geometrically that the matrix of secondary components of order 0 is

$$\begin{pmatrix} e_{00} & e_{01} = 0 & e_{02} = 0 \\ e_{10} & e_{11} & e_{12} \\ e_{20} & e_{21} & e_{22} \end{pmatrix}, \qquad \text{where} \qquad e_{00} + e_{01} + e_{02} = 0.$$

The principal components of order 0 are therefore linear combinations of ω_{01} and ω_{02}.

There is no invariant of order 0. The coefficient of order zero $b = \dfrac{\omega_{02}}{\omega_{01}}$ is arbitrary.

175 Nature of the elements of order 1**.** The exterior derivatives of the principal components ω_{01} and ω_{02} are, according to the structural equations (7),

$$\begin{aligned} \omega'_{01} &= -[\omega_{01}\omega_{00}] + [\omega_{01}\omega_{11}] + [\omega_{02}\omega_{21}], \\ \omega'_{02} &= -[\omega_{02}\omega_{00}] + [\omega_{01}\omega_{12}] + [\omega_{02}\omega_{22}]. \end{aligned}$$

$\mathcal{G}_0$ hence acts on these principal transformations by the infinitesimal transformations [c.f. (3)]

$$\begin{aligned} \delta\omega_{01} &= \omega_{01}(e_{00} - e_{11}) - \omega_{02}e_{21}, \\ \delta\omega_{02} &= -\omega_{01}e_{12} + \omega_{02}(e_{00} - e_{22}). \end{aligned}$$

The coefficient of order 0, $b = \dfrac{\omega_{02}}{\omega_{01}}$, hence undergoes the infinitesimal transformation (†)

$$\delta b = e_{21}b^2 - (e_{22} - e_{11})b - e_{12},$$

$\mathcal{G}_0$ operates transitively on b, and there is no invariant of order 1.

We will define the frames of order 1 by the relation $b = 0$ which leads to $\omega_{02} = 0$.

(†) The expression of this infinitesimal transformation shows that $\mathcal{G}_0$ acts on b by the most general homographic transformation.

By replacing δb and b by 0 in the defining equation of δb we obtain $e_{12} = 0$, the matrix of secondary components of order 1 is hence

$$\begin{pmatrix} e_{00} & 0 & 0 \\ e_{10} & e_{11} & 0 \\ e_{20} & e_{21} & e_{22} \end{pmatrix}, \qquad \text{where} \qquad e_{00} + e_{01} + e_{02} = 0.$$

The principal components of the frames of order 1 are

Order 0	Order 1
$\omega_{01}, \quad \omega_{02}(= 0)$	ω_{12}

176 Nature of the elements of order 2. According to the structural equations (7), we have

$$\omega'_{12} = -[\omega_{02}\omega_{10}] - [\omega_{12}\omega_{11}] + [\omega_{12}\omega_{22}].$$

$\mathcal{G}_1$ hence acts on ω_{12} by the infinitesimal transformation [c.f. (3)]

$$\delta\omega_{12} = \omega_{12}(e_{11} - e_{22}).$$

The preceding expression of $\delta\omega_{01}$ reduce to

$$\delta\omega_{01} = \omega_{01}(e_{00} - e_{11}).$$

We have a first order coefficient, $b = \dfrac{\omega_{12}}{\omega_{01}}$, with

$$\delta b = b(2e_{11} - e_{22} - e_{00}).$$

$\mathcal{G}_1$ operates transitively [(†)] on b. There is no invariant of order 2. We define the frames of order 2 by the relation [(‡)] $b = 1$, which leads to $\omega_{12} = \omega_{01}$.

By replacing δb by 0 and b by 1 in the defining equations of δb, we obtain

$$2e_{11} - e_{22} - e_{00} = 0,$$

but

$$e_{00} + e_{11} + e_{22} = 0,$$

hence

$$e_{11} = 0, \qquad e_{22} = -e_{00}.$$

The matrix of secondary components of order 2 is

$$\begin{pmatrix} e_{00} & 0 & 0 \\ e_{10} & 0 & 0 \\ e_{20} & e_{21} & -e_{00} \end{pmatrix}.$$

The principal components of orders ≤ 2 of the frames of order 2 are

[(†)]The expression of δb shows that $\mathcal{G}_1$ multiplies b by an arbitrary constant.

[(‡)]The reasoning indicated is correct if the variables are complex. If the variables are real, we must reason as the following: $\mathcal{G}_1$ operates transitively on the points $b < 0$ and on the points $b > 0$. The operations of G_1 hence allow us to give b one of the values ± 1. On the other hand the transformation of $\mathbf{A}_1$ into $-\mathbf{A}_1$ and $\mathbf{A}_2$ into $-\mathbf{A}_2$ leads the case $b = -1$ to the case $b = 1$. Hence finally $b = 1$.

Order 0	Order 1	Order 2
$\omega_{01}, \quad \omega_{02}(=0)$	$\omega_{12}(=\omega_{01})$	ω_{11}

Remark. If b has the exceptional value 0, the reasoning does not go through. The straight lines constitute the exceptional category of the curves for which b is constantly zero.

177 Nature of the elements of order 3**.** According to the structural equation (7), we have

$$\omega'_{11} = -[\omega_{10}\omega_{01}] + [\omega_{12}\omega_{21}] = -[\omega_{01}\omega_{10}] + [\omega_{01}\omega_{21}],$$

$\mathcal{G}_2$ hence acts on ω_{11} by the infinitesimal transformation

$$\delta\omega_{11} = \omega_{01}(e_{10} - e_{21}).$$

According to the preceding expression of $\delta\omega_{01}$, we have

$$\delta\omega_{01} = \omega_{01}e_{00}.$$

We have a coefficient of order 2, $b = \dfrac{\omega_{11}}{\omega_{01}}$, with

$$\delta b = e_{10} - e_{21} - be_{00}.$$

$\mathcal{G}_2$ operates transitively [(†)] on b. There is no invariant of order 3. We define the frames of order 3 by the relation $b = 0$, which entails

$$\omega_{11} = 0.$$

By replacing δb and b by 0 in the defining equation of δb, we obtain $e_{10} = e_{21}$.

The matrix of secondary components of order 3 is hence

$$\begin{pmatrix} e_{00} & 0 & 0 \\ e_{10} & 0 & 0 \\ e_{20} & e_{10} & -e_{00} \end{pmatrix}.$$

The principal components of the frames of order 3 are

Order 0	Order 1	Order 2	Order 3
$\omega_{01}, \quad \omega_{02}(=0)$	$\omega_{12}(=\omega_{01})$	$\omega_{11}(=0)$	$\omega_{21} - \omega_{10}$

[(†)] $\mathcal{G}_2$ transforms b by the most general linear transformation.

178 Nature of the elements of order 4. The structural equations (7) give us

$$\begin{aligned} \omega'_{21} &= -[\omega_{01}\omega_{20}] - [\omega_{11}\omega_{21}] - [\omega_{21}\omega_{22}] = -[\omega_{01}\omega_{20}] + [\omega_{21}\omega_{00}], \\ \omega'_{10} &= \quad [\omega_{10}\omega_{00}] + [\omega_{11}\omega_{10}] + [\omega_{12}\omega_{20}] = \quad [\omega_{10}\omega_{00}] + [\omega_{01}\omega_{20}], \end{aligned}$$

from which

$$\omega'_{21} - \omega_{10} = -2[\omega_{01}\omega_{20}] + [(\omega_{21} - \omega_{10})\omega_{00}].$$

$\mathcal{G}_3$ hence acts on $\omega_{21} - \omega_{10}$ by the infinitesimal transformation

$$\delta(\omega_{21} - \omega_{10}) = 2\omega_{01}e_{20} - (\omega_{21} - \omega_{10})e_{00}.$$

On the other hand

$$\delta\omega_{01} = \omega_{01}e_{00}.$$

We have a coefficient of order 3, $b = \dfrac{\omega_{21} - \omega_{10}}{\omega_{01}}$, with

$$\delta b = 2e_{20} - 2e_{00}b.$$

$\mathcal{G}_3$ operates transitively [†] on b. There is no invariant of order 4.

We define the frames of order 4 by the relation $b = 0$ which leads to $\omega_{21} = \omega_{10}$. By setting $\delta b = b = 0$, we obtain $e_{20} = 0$. The matrix of secondary components of order 4 is hence

$$\begin{pmatrix} e_{00} & 0 & 0 \\ e_{10} & 0 & 0 \\ 0 & e_{10} & -e_{00} \end{pmatrix}.$$

The principal components of the frames of order 4 are

Order 0	Order 1	Order 2	Order 3	Order 4
$\omega_{01}, \quad \omega_{02}(=0)$	$\omega_{12}(=\omega_{01})$	$\omega_{11}(=0)$	$\omega_{21} - \omega_{10}(=0)$	ω_{20}

179 Nature of the elements of order 5. The structural equations (7) give us

$$\omega'_{20} = [\omega_{20}\omega_{00}] + [\omega_{21}\omega_{10}] - [\omega_{20}\omega_{22}] = 2[\omega_{20}\omega_{00}].$$

$\mathcal{G}_4$ hence acts on ω_{20} by the infinitesimal transformation

$$\delta\omega_{20} = -2\omega_{20}e_{00}.$$

On the other hand

$$\delta\omega_{01} = \omega_{01}e_{00}.$$

We have a coefficient of order 4, $b = \dfrac{\omega_{20}}{\omega_{01}}$, with

$$\delta b = -3e_{00}b.$$

[†] $\mathcal{G}_3$ acts on b by the most general linear transformation.

$\mathcal{G}_4$ acts transitively [†] on b. There is no invariant of order 5.

We define the frames of order 5 by the relation [‡] $b = -1$, which entails $\omega_{20} = -\omega_{01}$. By setting $\delta b = 0$, $b = -1$, we obtain $e_{00} = 0$. The matrix of secondary components of order 5 is hence

$$\begin{pmatrix} 0 & 0 & 0 \\ e_{10} & 0 & 0 \\ 0 & e_{10} & 0 \end{pmatrix}.$$

The principal components of the frames of order 5 are

Order 0	Order 1	Order 2	Order 3	Order 4	Order 5
$\omega_{01}, \quad \omega_{02}(=0)$	$\omega_{12}(=\omega_{01})$	$\omega_{11}(=0)$	$\omega_{21}-\omega_{10}(=0)$	$\omega_{20}(=-\omega_{01})$	ω_{00}

Remark. If b has the exceptional value 0, the reasoning does not go through. The curves for which b is zero, i.e., whose fourth order frames satisfy the equation $\omega_{20} = 0$, constitute an exceptional category which we will study in paragraph **§182**.

180 Nature of the elements of order 6**.** The structural equations (7) give us

$$\omega'_{00} = [\omega_{01}\omega_{10}] + [\omega_{02}\omega_{20}] = [\omega_{01}\omega_{10}].$$

$\mathcal{G}_5$ hence acts on ω_{00} by the infinitesimal transformation

$$\delta\omega_{00} = -\omega_{01}e_{10}.$$

We have, on the other hand, according to the preceding expression of $\delta\omega_{01}$,

$$\delta\omega_{01} = 0,$$

ω_{01} is hence a *fifth order invariant form.* We denote it by $d\sigma$ and call σ the projective arc.

We have a coefficient of order 5, $b = \dfrac{\omega_{00}}{\omega_{01}}$, with

$$\delta b = -e_{10}.$$

$\mathcal{G}_5$ operates transitively [§] on b. There is no invariant of order 6.

We define the frames of order 6 by the relation $b = 0$, which entails

$$\omega_{00} = 0.$$

The equation $\delta b = 0$ gives $e_{10} = 0$. All the secondary components of order 6 are zero. The frame of order 6 of a point does not depend on any parameter, hence it is the *Frenet frame.*

[†] $\mathcal{G}_4$ multiplies b by an arbitrary constant.

[‡] The reasoning indicated is correct in the field of complex numbers. In the real field $\mathcal{G}_4$ allows us to give b one of the values ± 1. The change of $\mathbf{A}_0$ into $-\mathbf{A}_0$ and $\mathbf{A}_2$ into $-\mathbf{A}_2$ allow us to transform the case where $b = -1$ into the case where $b = 1$ without altering any of the values attributed to the coefficients of orders < 4, i.e., without altering the relations $\omega_{02} = 0$, $\omega_{11} = 0$, $\omega_{12} = \omega_{01}$, $\omega_{21} - \omega_{10} = 0$.

[§] $\mathcal{G}_5$ transforms b by the addition of an arbitrary constant.

181 Nature of the elements of orders > 6. The frames of order > 6 coincide with the Frenet frame.

There exists one invariant of order 7, $k = -\dfrac{\omega_{10}}{d\sigma}$.

There exists one invariant of order P when $P > 7$: it is $\dfrac{d^{P-7}k}{d\sigma^{P-7}}$.

The matrix of the relative components of the Frenet frame is

$$\begin{pmatrix} \omega_{00} & \omega_{01} & \omega_{02} \\ \omega_{10} & \omega_{11} & \omega_{12} \\ \omega_{20} & \omega_{21} & \omega_{22} \end{pmatrix} = \begin{pmatrix} 0 & d\sigma & 0 \\ -k\,d\sigma & 0 & d\sigma \\ -d\sigma & -k\,d\sigma & 0 \end{pmatrix}.$$

In other words the Frenet formulae are the formulae already written down in paragraph §**149** [(16), p. 161].

182 Curves whose fifth order contact elements are all exceptional. We are going to say a series of remarks concerning the curves C whose frames of order 4 satisfy the relation $\omega_{20} = 0$ (c.f. §**179**, *Remark*).

The Frenet frames of the curves C are their frames of order 4. The curves C do not possess any invariant.

According to paragraph §**131** (the equality problem, p. 144) the projective transformations superimposing two curves C are those that superimpose any two of their Frenet frames.

The Frenet frame of a curve C depends on the principal parameter and two secondary parameters. The relative components consists of the matrix

$$\begin{pmatrix} \omega_{00} & \omega_{01} & 0 \\ \omega_{10} & 0 & \omega_{01} \\ 0 & \omega_{10} & -\omega_{00} \end{pmatrix}. \tag{8}$$

The projective transformations transforming C into itself depend on three parameters and leave the terms of this matrix, i.e., ω_{01}, ω_{00} and ω_{10}, invariant. According to the second fundamental theorem in *group theory* (p. 180), ω_{01}, ω_{00} and ω_{10} satisfy structural equations. These equations are obtained immediately from the structural equations (7) and the matrix (8). The are

$$\omega'_{01} = [\omega_{00}\omega_{01}], \qquad \omega'_{00} = [\omega_{01}\omega_{10}], \qquad \omega'_{10} = [\omega_{10}\omega_{00}]. \tag{9}$$

It suffices to set

$$\omega_{01} = 2\omega_1, \qquad \omega_{00} = -\omega_2, \qquad \omega_{10} = -\omega_3$$

to transform the structural equations (9) into the equations (28) of §**161** (p. 176), which are the structural equations of the homographic group. According to the *isomorphism condition* (§**164**, p. 179) the group of projective transformations transforming C into itself is hence isomorphic to the homographic transformation. We can make this more precise:

the varieties on which $\omega_1 = 0$ are those whose parameter is transformed homographically by the group leaving ω_1, ω_2 and ω_3 invariant. The group of projective transformations transforming C into itself is hence similar to the homographic group when we consider its action on the varieties $\omega_{01} = 0$. But, to annihilate ω_{01} is to maintain the principal parameter the Frenet frame depends on constant. The varieties $\omega_{01} = 0$ hence constitute a class of equivalent objects equivalent to the points of C. Then, the projective transformations transforming C into itself transform homographically the points of C, or rather transform homographically the parameter these points depend on, when this parameter is suitably chosen.

Let $\mathbf{A}\mathbf{A}_1\mathbf{A}_2$ be the most general Frenet frame of a curve C. Consider a fixed point represented by the analytic point

$$\mathbf{P} = \mathbf{A} + \lambda\mathbf{A}_1 + \mu\mathbf{A}_2.$$

We have, according to the matrix (8),

$$d\mathbf{P} = (\omega_{00} + \lambda\omega_{10})\mathbf{A} + (d\lambda + \omega_{01} + \mu\omega_{10})\mathbf{A}_1 + (d\mu + \lambda\omega_{01} - \mu\omega_{00})\mathbf{A}_2.$$

The point $\mathbf{P}$ is geometrically fixed if the right hand side is a multiple of the analytic point $\mathbf{P}$. We deduce from it

$$\left\{\begin{aligned} d\lambda + \omega_{01} - \lambda\omega_{00} + (\mu - \lambda^2)\omega_{10} &= 0,\\ d\mu + \lambda\omega_{01} - 2\mu\omega_{00} - \lambda\mu\omega_{10} &= 0. \end{aligned}\right. \tag{10}$$

Knowing the non-homogenous coordinates λ and μ of $\mathbf{P}$ with respect to a particular Frenet frame, we can deduce its coordinates with respect to the most general Frenet frame by integrating the equations (10). These equations hence constitute a completely integrable system. It will then be easy to justify this statement by applying the Frobenius theorem (p. 180).

Let us determine all the curves possessing the same equation $\mu = f(\lambda)$ with respect to all Frenet frames of C. We will have, according to (10)

$$\frac{d\mu}{d\lambda} = \frac{\lambda}{1} = \frac{2\mu}{\lambda} = \frac{\lambda\mu}{\lambda^2 - \mu},$$

which gives only one solution, namely

$$\mu = \frac{1}{2}\lambda^2. \tag{11}$$

The curve C, which evidently possesses the property considered, is hence the conic of equation (11). Its fourth order frames are those for which the equations are of the form (11).

We choose λ as the parameter of the points of C. When the frame varies, the relative parameter λ of a fixed point of the conic satisfy the first equation of (10)

$$d\lambda + \omega_{01} - \lambda\omega_{00} - \frac{1}{2}\lambda^2\omega_{10} = 0. \tag{12}$$

This equation, being a Riccati equation, the parameter λ of a fixed point $\mathbf{P}$ of the conic, relative to a frame $\mathbf{R}$, is a homographic function of the parameter Λ of the same point $\mathbf{P}$ with respect to a fixed frame $\mathbf{R}_0$. This signifies that if we apply at the point $\mathbf{P}$ the projective transformation which leads the frame $\mathbf{R}_0$ to coincide with the frame $\mathbf{R}$, the point $\mathbf{P}$ will coincide with the point $\mathbf{P}'$ of C whose parameter with respect to the frame $\mathbf{R}_0$ is Λ. In other words the projective displacements which transform the curve C to coincide with itself effect a homographic transformation on the parameters of the points of C with respect to a fixed frame, the one that transforms λ into Λ.

183 Projective development of a curve onto a conic (†)**.** The reasoning that we just applied to the conics C generalise to any curve Γ.

The fourth order frame of Γ, $\mathbf{A}\mathbf{A}_1\mathbf{A}_2$, also depend on one principal parameter and two secondary parameters. The matrix of components of these frames is

$$\begin{pmatrix} \omega_{00} & \omega_{01} & 0 \\ \omega_{10} & 0 & \omega_{01} \\ \omega_{20} & \omega_{10} & -\omega_{00} \end{pmatrix}.$$

It differs from (8) by the presence of the principal component ω_{20}. Nevertheless we can again deduce from the structural equations (7) that ω_{01}, ω_{00} and ω_{10} satisfy the structural equations (9). The transformations leaving these three forms invariant hence constitute again a three parameter group. This group transforms homographically the principal parameter the points of Γ depend on, when this parameter is suitably chosen. Given four points of Γ, the anharmonic ratio of the corresponding values of this principal parameter hence have an intrinsic sense in projective geometry. We call it the *anharmonic ratio of the four points of* Γ *considered*. Its definition involves only the contact element of order four.

We will indicate in the course of this paragraph how we can geometrically and analytically determine this anharmonic ratio.

The equation (12) is again completely integrable. We can show it easily by Frobenius theorem. We have, by deriving exteriorly its left hand sides and using the equations (12) themselves,

$$\begin{aligned} &\omega_{01}' - \lambda\omega_{00}' - \frac{1}{2}\lambda^2\omega_{10}' + \left[\left(\omega_{01} - \lambda\omega_{00} - \frac{1}{2}\lambda^2\omega_{10}\right)(\omega_{00} + \lambda\omega_{10})\right] \\ &\qquad \equiv \omega_{01}' - [\omega_{00}\omega_{01}] - \lambda\{\omega_{00}' = [\omega_{01}\omega_{10}]\} - \frac{1}{2}\lambda^2\{\omega_{10}' - [\omega_{10}\omega_{00}]\}, \end{aligned}$$

which vanishes by virtue of (9). Now observe that the osculating conic at each point of Γ has equation $\mu = \frac{1}{2}\lambda^2$ with respect to every fourth order frame attached at the point. The equation (12) hence allow us to establish a bijective correspondence between the points of different osculating conics at Γ, and this correspondence is homographic. We will get this correspondence by integrating the equation (12) and by taking the constant

(†)For more details, *see* [20], p. 51–69.

of integration Λ to have the value λ corresponding to a particular order four frame attached at a particular point $\mathbf{A}$ of Γ. Consider

$$\Lambda = \frac{\alpha\lambda + \beta}{\gamma\lambda + 1}, \tag{13}$$

α, β, γ being functions of the principal and secondary parameters the frame $\mathbf{A}\mathbf{A}_1\mathbf{A}_2$ depends on. The point with parameter Λ on the conic C_0 osculating Γ at $\mathbf{A}_0$ then corresponds to the point of parameter λ of the conic osculating Γ at $\mathbf{A}$. If the curve Γ is itself a conic, these two points will coincide: the correspondence will be the identity.

Granted this, it is natural that, when given four points $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, $\mathbf{P}_4$ of Γ, to consider on one of the osculating conics C_0, the four points $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{Q}_3$, $\mathbf{Q}_4$ corresponding to $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, $\mathbf{P}_4$, each considered as belonging to the osculating conic to Γ at the point, and to call the anharmonic ratio of the four points $\mathbf{Q}_i$ of C_0 the anharmonic ratio of the four points $\mathbf{P}_i$. If we have taken another osculating conic than C_0, we will have the same anharmonic ratio.

Analytically, according to (13), each point $\mathbf{A}$ of Γ considered as belong to the osculating conic at the point corresponds to the value $\lambda = 0$ of the relative parameter. The point corresponding to it on C_0 hence have the parameter $\Lambda = \beta$. Then, the anharmonic ratio of four points on the curve is equal to the anharmonic ratio of four corresponding values of the function β. It follows in particular that β is a function of the only principal parameter and we can choose β as the principal parameter.

We can present these in a geometrical way that is more intuitive. When we pass through a point $\mathbf{P} = \mathbf{A} + \lambda\mathbf{A}_1 + \frac{1}{2}\lambda^2\mathbf{A}_2$ of the osculating conic at $\mathbf{A}$, at the corresponding $\mathbf{P}'$ of the infinitesimally close osculating conic, we have

$$\begin{aligned} d\mathbf{P} = \left(\omega_{00} + \lambda\omega_{10} + \frac{1}{2}\lambda^2\omega_{20}\right)\mathbf{A} + \left(d\lambda + \omega_{01} + \frac{1}{2}\lambda^2\omega_{10}\right)\mathbf{A}_1 \\ + \left(\lambda\, d\lambda + \lambda\omega_{01} - \frac{1}{2}\lambda^2\omega_{00}\right)\mathbf{A}_2, \end{aligned}$$

or, by using equation (12),

$$d\mathbf{P} = (\omega_{00} + \lambda\omega_{10})\mathbf{P} + \frac{1}{2}\lambda^2\omega_{20}\mathbf{A}. \tag{14}$$

The straight line $\mathbf{PP}'$, in the limit, will pass through $\mathbf{A}$. The point correspondence between the two conics C and C' is hence extremely simple: these two conics have the point $\mathbf{A}$ in common (they have a third order contact there). We correspond to a point $\mathbf{P}$ of C the second point of intersection of $\mathbf{AP}$ with C'. It is obvious that this correspondence conserves the anharmonic ratios. Then, by following the point $\mathbf{P}$ little by little, we obtain a curve that we call the *projective expansion* of Γ which enjoys the property that at each of its points the tangent will pass by the point of contact with Γ of the conic, on which the point $\mathbf{P}$ is taken (fig. 1). Four expanding projectives obviously cut the different osculating conics of the systems of four points whose anharmonic ratio is constant, which we can call the anharmonic ratio of the four expanding lines. The anharmonic ratio of the four points of Γ is none other than the anharmonic ratio of the four expanding lines going through these points.

The expanding projectives can be regarded as guiding an application or a development of Γ onto the osculating conic C_0 at a particular point $\mathbf{A}$, each point $\mathbf{P}$ of C_0 following the expanding projective beginning from this point until reach the point where it meets Γ. The conic C_0 coincide successively with the different osculating conics C and the passage of a conic C to a infinitesimally close conic C' is none other than the projective displacement, namely the one that, with respect to one of the fourth order frames attached at the point of contact $\mathbf{A}$ of C with Γ, has all of its components zero except ω_{20}. To show this result, it suffices to observe that, according to (14), the analytic point $\mathbf{P} + d\mathbf{P}$ which, up to infinitesimally small quantities of higher orders, can be written as

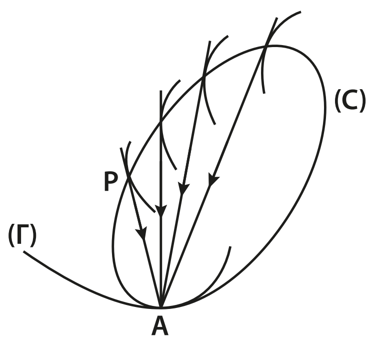
$$\mathbf{P} + d\mathbf{P} = (1 + \omega_{00} + \lambda\omega_{10}) \left(\mathbf{P} + \frac{1}{2}\lambda^2\omega_{20}\mathbf{A}\right),$$

coincides geometrically with the point $\mathbf{P} + \frac{1}{2}\lambda^2\omega_{20}\mathbf{A}$ coming from $\mathbf{P}$ by the infinitesimal projective displacement indicated. The development of C_0 onto Γ hence results from an infinity of infinitesimal projective displacements. We can conversely consider the development of Γ onto the osculating conic at one of its points $\mathbf{A}_0$: it results from an infinity of infinitesimal projective displacements each with components the elements of the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\omega_{20} & 0 & 0 \end{pmatrix},$$

The deformation of Γ considered is then analogous to that we subject a given curve to by developing it on one of its tangents in Euclidean geometry: the movement of deformation is guided by the expanding lines (in the ordinary sense) of the curve, and it is the result of an infinity number of infinitesimally small Euclidean displacements. Two expanding lines cut the different tangents of the segments whose length is constant, and similarly the four expanding projectives cut the different osculating conics of the systems of points whose anharmonic ratio is constant.

Figure 1



We have shown how the integration of equation (12) allow us to analytically determine the anharmonic ratio of four points of Γ. We can also proceed by considering a family of fourth order frames depending on a parameter t obtained by integration of the equations

$$\omega_{01} = dt, \qquad \omega_{00} = 0, \qquad \omega_{10} = 0,$$

once this integration is done, the equations in total differentials (12) becomes the ordinary differential equation

$$d\lambda + dt = 0,$$

whose general solution is

$$\Lambda = \lambda + t.$$

We hence have here $\beta = t$ and hence the anharmonic ratio of four points of Γ is equal to the anharmonic ratio of four values of t. The generating point of the expanding projective with parameter Λ is

$$\mathbf{P} = \mathbf{A} + (\Lambda - t)\mathbf{A}_1 + \frac{1}{2}(\Lambda - t)^2\mathbf{A}_2.$$

In particular the expanding line meeting at the point $\mathbf{A}_0$ ($t = 0$ of Γ is generated by the point $\mathbf{P} = \mathbf{A} - t\mathbf{A}_1 + \frac{1}{2}t^2\mathbf{A}_2$. We deduce from this easily the form of this expanding line in a neighbourhood of $\mathbf{A}_0$.

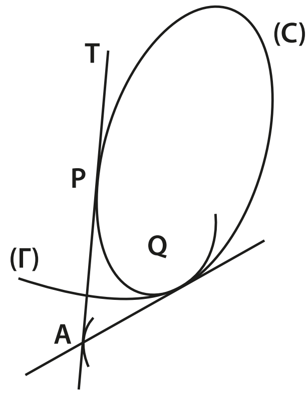
Observe that we have here $\mathbf{A}_1 = \dfrac{d\mathbf{A}}{dt}$, $\mathbf{A}_2 = \dfrac{d^2\mathbf{A}}{dt^2}$ and that the analytic point $\mathbf{A}$ satisfies a differential equation of the form

$$\frac{d^3\mathbf{A}}{dt^3} - r\mathbf{A} = 0, \tag{15}$$

by setting $\omega_{20} = r\,dt$. Conversely whenever we attach at each point of Γ an analytic point $\mathbf{A}$ satisfying, with respect to a suitably chosen parameter t, a differential equation of the form (15), the frame $\mathbf{A}\dfrac{d\mathbf{A}}{dt}\dfrac{d^2\mathbf{A}}{dt^2}$ will be a fourth order frame and the anharmonic ratio of four points of Γ will be equal to the anharmonic ratio of the four corresponding values of t. The choice of t, as we know in advance, is determined up to a homographic transformation with arbitrary constant coefficients, and the analytic point $\mathbf{A}$ depends on this choice. For one conic, the coefficient r is obviously zero.

At last let us show that it is possible to transform by duality the geometrical definition of the anharmonic ratio of four points on Γ. Consider a tangent T to C generated by the projective movement: its characteristic point $\mathbf{Q}$ always remains on the tangent at $\mathbf{A}$ to Γ (fig. 2). The anharmonic ratio of four points of Γ is the one of four such tangents, characterised by the property that the coincide successively with the tangents at four given points of Γ.

Figure 2



184 *Remark.* In the projective theory of planar curves, we are lead to introduce in an intrinsic manner two parameters, one of which, the projective arc, is defined up to an additive constant, i.e., up to a transformation of a one parameter group, and the other is defined up to a transformation of a three parameter group, the homographic group. We can ask whether there exists an intrinsic parameter defined up to the transformations of the linear group in two parameters. In projective geometry there does not exist such a thing, but in *general* (non-unimodular) affine geometry there exists. By choosing in the plane an area unit, we have seen the existence of the affine arc. By changing the unit of area, this arc undergoes a linear transformation, and hence it corresponds to the conditions desired. This is the only intrinsic parameter that exists in general affine geometry for the *parabolas*, which play in general affine geometry the same role as the conics in projective geometry. For a generic curve Γ there exists *affine expanding lines*, which we obtain by taking for each osculating parabola to Γ a point $\mathbf{P}$ such that the tangent at the point $\mathbf{P}$ is parallel to the tangent to Γ at the point where it is couched by the parabola considered. Three of these expanding lines cut the different osculating parabolas of the systems of three points whose anharmonic ratio with the point at infinity on the corresponding parabola is constant. We leave to the reader the task of sowing these different properties.

An analogous example, much simpler, is furnished by the theory of curves in the *geometry of similarity*: this is Euclidean geometry where we regard two similar figures as the same. The classic curvilinear abscissa s is a parameter defined up to a linear transformation $s \to as + b$. There is no other intrinsic parameter of the straight line, but on one curve strictly speaking the intrinsic parameter $\int \rho\, ds$ (the angle of contingency) is defined up to a constant.

III. Euclidean geometry; study of surfaces

185 The fundamental group G is the displacement group, whose structural equations are (p. 175):

$$\omega_i' = \sum_{k=1}^{3}[\omega_k\omega_{ki}], \qquad \omega_{ij}' \sum_{k=1}^{3}[\omega_{ik}\omega_{kj}] \qquad (\omega_{ij}+\omega_{ji}=0). \tag{16}$$

We will simplify the general method a bit as it is used in the study of the projective properties of plane curves. We begin with the following lemma:

Given $2n$ *linear differential forms* $\omega_1, \omega_2, \ldots, \omega_n, \varpi_1, \varpi_2, \ldots, \varpi_n$ *whose* n *parameters are independent and who satisfy the relation*

$$[\omega_1\varpi_1]+[\omega_2\varpi_2]+\cdots+[\omega_n\varpi_n]=0,$$

the forms ϖ_i *are linear combinations of the forms* ω_i*, the matrix of coefficients being symmetric.*

The proof is immediate. If there is, among the given $2n$ forms, more than n linearly independent forms, for example the $n+h$ forms $\omega_1, \ldots, \omega_n, \varpi_1, \ldots, \varpi_h$, we can express the last $n-h$ forms $\varpi_{h+1}, \ldots, \varpi_n$ as linear functions of the preceding, but then, in the given relation, the first term $[\omega_1\varpi_1$ cannot cancel with any other term. Granted this, we have

$$\varpi_i' = \sum_k a_{ik}\omega_k,$$

and by expressing that the coefficient of $[\omega_i\omega_j]$ on the left hand side of the given relation is zero, we find that $a_{ij}=a_{ji}$.

186 Nature of the elements of orders 0 and 1. It is geometrically obvious that there is no invariant of order 0 or 1. The frames of order 0 of a point $\mathbf{A}$ will be the rectangular trihedrals with the apex $\mathbf{A}$, $\mathbf{A}\vec{\mathbf{I}}_1\vec{\mathbf{I}}_2\vec{\mathbf{I}}_3$. The frames of order 1 will be those of frame 0 whose vector $\vec{\mathbf{I}}_3$ is normal to the surface.

The matrix of secondary coefficients of order 0 is

$$e_1=0, \qquad e_2=0, \qquad e_3=0, \qquad e_{13}, e_{23}, e_{12} \text{ arbitrary.}$$

The matrix of secondary coefficients of order 1 is

$$e_1=0, \qquad e_2=0, \qquad e_3=0, \qquad e_{13}=0, \qquad e_{23}=0, \qquad e_{12} \text{ arbitrary.}$$

When a frame varies while remaining a frame of order 1 of the surface, its displacement satisfies the equation

$$\omega_3 = 0. \tag{17}$$

The principal components of the frames of order 1 are hence

Order 0	Order 1
$\omega_1, \quad \omega_2, \quad \omega_3(=0)$	$\omega_{13}, \quad \omega_{23}$

187 Nature of the elements of order 2. According to the structure equations (16) and the equation (17), the exterior derivatives of the principal components ω_1, ω_2, ω_3, ω_{13} and ω_{23} are

$$\omega_1' = [\omega_2\omega_{21}], \qquad \omega_2' = [\omega_1\omega_{12}], \tag{18}$$

$$0 = \omega_3' = [\omega_1\omega_{13}] + [\omega_2\omega_{23}], \tag{19}$$

$$\omega_{13}' = -[\omega_{23}\omega_{12}], \qquad \omega_{23}' = -[\omega_{13}\omega_{21}]. \tag{20}$$

The application of the lemma of paragraph §**185** to the relation (19) allow us to set

$$\omega_{13} = a\omega_1 + b\omega_2, \qquad \omega_{23} = b\omega_1 + c\omega_2. \tag{21}$$

The exterior derivation of these equations, by using equations (18), (20) and (21), gives

$$\begin{aligned} -[(b\omega_1 + c\omega_2)\omega_{12}] &= a[\omega_2\omega_{21}] + b[\omega_1\omega_{12}] + [da\,\omega_1] + [db\,\omega_2],\\ -[(a\omega_1 + b\omega_2)\omega_{21}] &= b[\omega_2\omega_{21}] + c[\omega_1\omega_{12}] + [db\,\omega_1] + [dc\,\omega_2], \end{aligned}$$

which can also be written

$$\begin{cases} [\omega_1(da - 2b\omega_{12})] + [\omega_2(db + \overline{a-c}\,\omega_{12})] = 0,\\ [\omega_1(db + \overline{a-c}\,\omega_{12})] + [\omega_2(dc - 2b\omega_{21})] = 0. \end{cases} \tag{22}$$

The forms $da - 2b\omega_{12}$, $db + (a-c)\omega_{12}$, $dc - 2b\omega_{21}$ are, according to the lemma, linear combinations of ω_1 and ω_2. Then, if we vary only the secondary parameters, they are zero, we hence have

$$\delta a = 2e_{12}b, \qquad \delta b = e_{12}(c-a), \qquad \delta c = -2e_{12}b. \tag{23}$$

We see immediately that the group $\mathcal{G}_1$, whose infinitesimal transformation is defined by (23), leaves the two quantities $a + c$ and $ac - b^2$ invariant. We can recover this result by observing that the two differential forms

$$\omega_1^2 + \omega_2^2, \qquad \omega_1\omega_{13} + \omega_2\omega_{23} \tag{24}$$

are invariants under $\mathcal{G}_1$, and so we deduce from (18) and (20)

$$\begin{cases} \delta\omega_1 = e_{12}\omega_2, & \delta\omega_2 = -e_{12}\omega_1,\\ \delta\omega_{13} = e_{12}\omega_{23}, & \delta\omega_{23} = -e_{12}\omega_{13}. \end{cases} \tag{25}$$

Hence we can always choose the first two axes of the first order trihedral in a way such that the form $\omega_1\omega_{13} + \omega_2\omega_{23} = a\omega_1^2 + 2b\omega_1\omega_2 + c\omega_2^2$ have its second coefficient zero.

We hence define the frames of order 2 by the relation [†] $b = 0$. The corresponding values of a and c are denoted r_1 and r_2. The constitute invariants of order 2.

$\mathcal{G}_2$ is the subgroup of $\mathcal{G}_1$ satisfying the condition $\delta b = 0$ for $b = 0$. This relation entails $e_{12} = 0$, at least if we are not in the exceptional case where $a = c$ (this case is studied in paragraph §**190**). The secondary components of order 2 are hence all zero. $\mathcal{G}_2$ does not depend on any parameter, and the second order frame is the *Frenet frame*, ω_1 and ω_2 are the *invariant forms of order* 2.

[†] More rigorously by the relation $b = 0$ and the inequality $a > c$.

188 Nature of the elements of order 3. The frames of orders $P \geq 2$ are the Frenet frame.

The infinitesimal displacement of the Frenet frame has components

$$\omega_1, \quad \omega_2, \quad \omega_3 = 0, \quad \omega_{13} = r_1\omega_1, \quad \omega_{23} = r_2\omega_2, \quad \omega_{12} = \rho_1\omega_1 + \rho_2\omega_2, \tag{26}$$

ρ_1 and ρ_2 are two coefficients of order 2, which constitute two invariants of order 3.

It is convenient to use the following notation: f being a function of the principal parameters, we can write

$$df = f_{,1}\omega_1 + f_{,2}\omega_2, \tag{27}$$

the functions $f_{,1}$ and $f_{,2}$ are called the *covariant derivatives* of f of order 1. Similarly there exists covariant second derivatives, which are lined by the following relation, obtained by exteriorly differentiating (27),

$$f_{,12} - f_{,21} = \rho_1 f_{,1} + \rho_2 f_{,2}.$$

The covariant derivatives of the invariants of order 2, namely $r_{1,1}$, $r_{1,2}$, $r_{2,1}$, $r_{2,2}$ also constitute the invariants of order 3.

Let us write out that the components (26) satisfy the structural equations (16). We have already expressed that the structural equation (19) is satisfied. The structural equations concerning ω_1', ω_2', ω_{13}', ω_{23}' and ω_{12}' give us respectively

$$\begin{cases} \omega_1' = \rho_1[\omega_1\omega_2], & \omega_2' = \rho_2[\omega_1\omega_2], \\ r_{1,2} = \rho_1(r_1 - r_2), & r_{2,1} = \rho_2(r_1 - r_2), \\ \rho_{1,2} - \rho_{2,1} = r_1 r_2 + \rho_1^2 + \rho_2^2. \end{cases} \tag{28}$$

The third order invariants are hence the four covariant derivatives of r_1 and r_2.

The nature of invariants of orders > 3. Let us set aside all Weingarten surfaces [(†)] for which r_2 is a function of r_1. Then r_1 and r_2 are two independent variables whose covariant derivatives are functions: $r_{i,j} = f_{ij}(r_1, r_2)$. According to paragraph §**131** (p. 144), the invariants of orders $P > 3$ are the various partial derivatives of orders $P - 4$ of the four functions $f_{ij}(r_1, r_2)$.

189 Equality condition of two surfaces. According to paragraph §**131** (p. 144), the equality condition of two surfaces which are not Weingarten surfaces is that the four functions $f_{ij}(r_1, r_2)$ are the same for the two surfaces. If one of the surfaces is a Weingarten surface, things are a little bit more complicated. There are two cases to distinguish:

1. If $r_2 = f(r_1)$ and $r_{1,1}$ and $r_{1,2}$ are functions $\varphi(r_1)$ and $\psi(r_1)$, it is necessary and sufficient that the second surface is of the same nature, with the same functions f, φ

[(†)] These surfaces are denoted by W.

and ψ. In this case, the two surfaces can be superimposed in an infinity of ways. If, for example, $r_{1,1} \neq 0$, it suffices to solve the equations

$$r_1^\star = r_1, \qquad \omega_2^\star = \omega_2.$$

It is easy to verify that the two surfaces are surfaces of revolution or helicoids.

2. If $r_2 = f(r_1)$ and one of the covariant derivatives, for example $r_{1,1}$, is not a function of r_1, we form $r_{1,11}$ and $r_{1,12}$ which are determined functions $\varphi(r_1, r_{1,1})$ and $\psi(r_1, r_{1,1})$, and on the other hand set $r_{1,2} = \chi(r_1, r_{1,1})$. It is necessary and sufficient that there exists on the second surface analogous relations with the same functions f, φ, ψ and χ. In this case, the integer Q of paragraph §**131** is equal to 3 instead of 2 as in the preceding case.

Let us add the remark that the functions introduced are not arbitrary, as a consequence of the structural equations.

The case where r_1 and r_2 are both constants is treated later (§**191**).

Equality condition of Gauss. Gauss created the differential theory of surfaces. He used in his reasoning two quadratic forms with respect to the differentials of the parameters of the surface. Let $\mathbf{A}$ be a point of the surface and $\vec{\mathbf{I}}_3$ the normal vector of length 1 at $\mathbf{A}$, these forms are

$$F = d\vec{\mathbf{A}}^2, \qquad \Phi = -d\vec{\mathbf{A}} \times d\vec{\mathbf{I}}_3.$$

Gauss has given a necessary and sufficient condition for two surfaces S and $S^\star$ to be equal or symmetric:

"We can establish a bijective correspondence between the points of S and of $S^\star$ which identifies the form F of S with the form $F^\star$ of $S^\star$, and the form Φ of S with the form $\Phi^\star$ or $-\Phi^\star$ of $S^\star$."

It is easy to deduce this proposition from our equality condition. Consider the Frenet frame of S. We have

$$F = \omega_1^2 + \omega_2^2, \qquad \Phi = \omega_1\omega_{13} + \omega_2\omega_{23} = r_1\omega_1^2 + r_2\omega_2^2.$$

$rF - \Phi$ is not the square of a Pfaffian form unless r is equal to r_1 or r_2, in which case it is the square of $\pm\sqrt{r_1 - r_2}\,\omega_2$ or $\pm\sqrt{r_2 - r_1}\,\omega_1$. A point correspondence between S and $S^\star$ identifying F to $F^\star$ and Φ to $\pm\Phi^\star$ hence establishes the relations

$$r_1 = r_1^\star, \qquad r_2 = r_2^\star, \qquad \omega_1 = \omega_1^\star, \qquad \omega_2 = \omega_2^\star, \tag{29}$$

through perhaps transforming $S^\star$ by a symmetry and the modification of the orientation of $S^\star$.

The relations (29) entail that the covariant derivatives of various orders of r_1 and r_2 are equal to the corresponding derivatives of $r_1^\star$ and $r_2^\star$: r_1 and r_2 and their covariant derivatives are linked by the same systems of relations on S and on $S^\star$. According to our equality condition S and $S^\star$ are equal.

We can observe more simply that the equations (29) can be written

$$\omega_1 = \omega_1^\star, \qquad \omega_2 = \omega_2^\star, \qquad \omega_{13} = \omega_{13}^\star, \qquad \omega_{23} = \omega_{23}^\star,$$

We deduce from them, by means of the structural equations,

$$[\omega_2(\omega_{21} - \omega_{21}^\star)] = 0, \qquad [\omega_1(\omega_{12} - \omega_{12}^\star)] = 0,$$

the form $\omega_{12} - \omega_{12}^\star$, must be, according to the lemma of §**185**, at the same time an multiple of ω_2 and a multiple of ω_1, which means that it is identically zero. The components ω_{ij} being equal to each of the corresponding component $\omega_{ij}^\star$, the two surfaces are equal.

190 Applications of the structure theorem. THEOREM. *Consider two Pfaffian forms ω_1 and ω_2 depending on two independent variables and two functions r_1 and r_2 of these variables. If the equations* (28) *are satisfied, there exists a surface whose ω_1 and ω_2 are invariant forms of order* 2 *and whose r_1 and r_2 are invariants of order* 2.

Proof. The equations (28) express that the forms (26) satisfy the structural equations (16). There hence exists a moving frame whose relative components are the given forms. This trihedral is the Frenet trihedral of the surface generated by its apex. This surface is the one we look for.

Complements. We can choose on a surface the parameters u_1 and u_2 which remain constant on the curves along respectively the curves $\omega_1 = 0$ and $\omega_2 = 0$. We then have

$$\omega_1 = h_1(u_1 u_2) du_1, \qquad \omega_2 = h_2(u_1, u_2) du_2.$$

The covariant derivatives of a function f are

$$f_{,1} = \frac{1}{h_1}\frac{\partial f}{\partial u_1}, \qquad f_{,2} = \frac{1}{h_2}\frac{\partial f}{\partial u_2}.$$

The relations (28) can be written under these conditions

$$\rho_1 = -\frac{1}{h_2}\frac{\partial \log h_1}{\partial u_2}, \qquad \rho_2 = \frac{1}{h_1}\frac{\partial \log h_2}{\partial u_1},$$
$$\rho_1 = \frac{1}{h_2}\frac{1}{r_1 - r_2}\frac{\partial r_1}{\partial u_2}, \qquad \rho_2 = \frac{1}{h_1}\frac{1}{r_1 - r_2}\frac{\partial r_2}{\partial u_1},$$
$$\frac{1}{h_2}\frac{\partial \rho_1}{\partial u_2} - \frac{1}{h_1}\frac{\partial \rho_2}{\partial u_1} = \rho_1^2 + \rho_2^2 + r_1 r_2.$$

The theorem of this paragraph affirms that the necessary and sufficient condition for there to exist a surface corresponding to the given functions h_1, h_2, r_1 and r_2 is that we have [†]

$$\frac{\partial \log h_1}{\partial u_2} = -\frac{1}{r_1 - r_2}\frac{\partial r_1}{\partial u_2},$$
$$\frac{\partial \log h_2}{\partial u_1} = \frac{1}{r_1 - r_2}\frac{\partial r_2}{\partial u_1},$$
$$-\frac{1}{h_2}\frac{\partial}{\partial u_2}\left(\frac{1}{h_2}\frac{\partial \log h_1}{\partial u_2}\right) - \frac{1}{h_1}\frac{\partial}{\partial u_1}\left(\frac{1}{h_1}\frac{\partial \log h_2}{\partial u_1}\right)$$
$$= \frac{1}{h_2^2}\left(\frac{\partial \log h_1}{\partial u_2}\right) + \frac{1}{h_1^2}\left(\frac{\partial \log h_2}{\partial u_1}\right)^2 + r_1 r_2.$$

191 Geometrical interpretations. The lines $\omega_2 = 0$ and $\omega_1 = 0$ are characterised geometrically by the property that $d\vec{\mathbf{A}}$ and $d\vec{\mathbf{I}}_3$ remain parallel when $\mathbf{A}$ describes one of them: they are the curvature lines of the surface. The lines that annihilate $\omega_1\omega_{13}+\omega_2\omega_{23}$ are the asymptotic lines, and $f_{,1}$ and f_2 are the ratios of increments of f in the directions of the paths when we displace the respectively in the directions of the curvature lines.

The Frenet trihedral has its first two edges tangent to the curvature lines.

When $\mathbf{A}$ moves along the first curvature line,

$$d\vec{\mathbf{I}}\mathbf{A} = \omega_1\vec{\mathbf{I}}_1, \qquad d\vec{\mathbf{I}}_1 = \omega_1\rho_1\vec{\mathbf{I}}_2 + \omega_1 r_1\vec{\mathbf{I}}_3.$$

But $\dfrac{d\vec{\mathbf{I}}_1}{\omega_1}$ is along the principal normal to this curvature line. Its length is the curvature of this line, and ρ_1 and r_1 are the projections of the curvature vector on the tangent plane and normal to the surface. In other words, ρ_1 and ρ_1 are the geodesic curvature and normal curvature of the first curvature line. The principal curvatures of the surface are r_1 and r_2.

192 Surfaces whose contact element of order 2 is exceptional. We will now study the exceptional class of surfaces whose coefficients of order 1 satisfy the equations $a = c$, $b = 0$.

According to the formulae (23), $\mathcal{G}_1$ leaves a, b and c invariant. The frames of order 2 coincides with the frames of order 1. There exists one invariant of order 2: it is $r = a = c$. The principal components of order 2 are

Order 0	Order 1
$\omega_1, \quad \omega_2, \quad \omega_3(=0)$	$\omega_{13} = r\omega_1, \quad \omega_{23} = r\omega_2$

Let us write the structural equation

$$\omega'_{13} = [\omega_{12}\omega_{23}],$$

it becomes

$$r\omega'_1 + [dr\,\omega_1] = [\omega_{12}\omega_{23}],$$

[†] These formulae are, up to notational variation, the well known CODAZZI formulae in the case of a surface with respect to its curvature lines. C.f. [1], vol. 2, book 5, chapter 2.

i.e.,

$$r[\omega_2\omega_{21}] + r_{,2}[\omega_2\omega_1] = r[\omega_{12}\omega_2], \qquad \text{or} \qquad r_{,2} = 0.$$

Similarly

$$r_{,1} = 0.$$

The invariant r is hence constant on the surface. The Frenet trihedrals are the trihedrals of order 1, and the three parameter group of displacements transforming these trihedrals into one another leaves the surface invariant.

If $r = 0$, we have $d\vec{\mathbf{I}}_3 = 0$. The surface is a *plane*.

If $r \neq 0$, we have

$$d\vec{\mathbf{A}} + \frac{1}{r}d\vec{\mathbf{I}}_3 = \omega_1\vec{\mathbf{I}}_1 + \omega_2\vec{\mathbf{I}}_2 + \frac{1}{r}\omega_{31}\vec{\mathbf{I}}_1 + \frac{1}{r}\omega_{32}\vec{\mathbf{I}}_2 = 0,$$

and the vector $\frac{1}{r}\vec{\mathbf{I}}_3$ with origin $\mathbf{A}$ has its tip fixed. The surface is hence a *sphere*.

193 Surfaces invariant under a two parameter displacement group. We will now find all surfaces that are transformed into themselves under a two parameter displacement group. This group will necessarily be constituted by the displacements superimposing the Frenet trihedrals. According to paragraph §**132** (p. 145), these surfaces are the ones whose invariants are all constant, except the spheres and planes.

We have $r_{1,1} = r_{1,2} = r_{2,1} = r_{2,2} = 0$. According to the preceding paragraph, $r_1 \neq r_2$. Hence, by virtue of the equations (28), $\rho_1 = \rho_2 = 0$ and $r_1 r_2 = 0$. Suppose that, to fix ideas, $r_1 = \text{constant}$, $r_2 = 0$.

The expressions (26) give us

$$d\vec{\mathbf{A}} = \omega_1\vec{\mathbf{I}}_1 + \omega_2\vec{\mathbf{I}}_2,$$
$$d\vec{\mathbf{I}}_1 = \omega_1 r_1\vec{\mathbf{I}}_3, \qquad d\vec{\mathbf{I}}_2 = 0, \qquad d\vec{\mathbf{I}}_3 = -r_1\omega_1\vec{\mathbf{I}}_1.$$

$\vec{\mathbf{I}}_2$ is parallel to a fixed direction: the first curvature lines are parallel straight lines. The second curvature lines, being orthogonal to the first ones, are planar, and their radius of curvature r_1 is constant. They are circles.

The surfaces that we search for are hence the *cylinders of revolution.*

194 Reduced equation. Let $z = f(x, y)$ be the equation of the surface in a neighbourhood of one of its points $\mathbf{A}$ with respect to the Frenet trihedral at this point. The function f, assumed to be expanded in series in powers of x and y, will contain the coefficients which will depend on the point $\mathbf{A}$. We denote by $\overline{df}$ the differential of f in which x and y are regarded as constants. The relative coordinates x, y, z of a fixed point $\mathbf{P}$ of the surface satisfy equations expressing that the differential of the point $\mathbf{P} = \mathbf{A} + x\vec{\mathbf{I}}_1 + y\vec{\mathbf{I}}_2 + z\vec{\mathbf{I}}_3$ is zero, namely

$$\begin{aligned} &dx + \omega_1 + y\omega_{21} + z\omega_{31} = 0,\\ &dy + \omega_2 + x\omega_{12} + z\omega_{32} = 0,\\ &dz \qquad\quad + x\omega_{13} + y\omega_{23} = 0. \end{aligned}$$

We deduce, by multiplying the first equation by $-\dfrac{\partial f}{\partial x}$, the second by $-\dfrac{\partial f}{\partial y}$ and adding to the third,

$$\overline{df} + x\omega_{13} + y\omega_{23} - \frac{\partial f}{\partial x}(\omega_1 + y\omega_{21} + z\omega_{31}) - \frac{\partial f}{\partial y}(\omega_2 + x\omega_{12} + z\omega_{32}) = 0. \tag{30}$$

Let

$$f = \frac{1}{2}(ax^2 + 2bxy + cy^2) + \frac{1}{6}(\alpha x^3 + 3\beta x^2 y + 3\gamma xy^2 + \delta y^3) + \cdots$$

By equating in (30) the coefficients of different powers of x and y, and by replacing ω_{13}, ω_{23} and ω_{12} by their values (26), we obtain

$$a = r_1, \qquad b = 0, \qquad c = r_2,$$
$$dr_1 = \alpha\omega_1 + \beta\omega_2, \qquad (r_1 - r_2)\omega_{12} = \beta\omega_1 + \gamma\omega_2, \qquad dr_2 = \gamma\omega_1 + \delta\omega_2.$$

We deduce from the reduced equation

$$z = \frac{1}{2}(r_1 x^2 + r_2 y^2) + \frac{1}{6}(r_{1,1}x^3 + 3r_{1,2}x^2 y + 3r_{2,1}xy^2 + r_{2,2}y^3) + \cdots$$

and, moreover, the relations (28) already obtained

$$r_{1,2} = (r_1 - r_2)\rho_1, \qquad r_{2,1} = (r_1 - r_2)\rho_2.$$

The reduced equation makes the differential invariants of the first three orders evident.

195 Deformation of surfaces. Two surfaces are said to be applicable in the send of Gauss if we can establish a point correspondence between them which makes their first two fundamental forms identical. Let us atribute to each point **A** of one of the surfaces, S, a first order trihedral, depending consequently on two principal parameters and one secondary parameter. For each choice of this trihedral there corresponds a decomposition of the first fundamental form into a sum of two squares $\omega_1^2 = \omega_2^2$. It follows that every first order trihedral attached at S corresponds to a first order trihedral attached to $S^\star$ realising the equality

$$\omega_1 = \omega_1^\star, \qquad \omega_2 = \omega_2^\star. \tag{31}$$

It is in the possibility of relations of this form that the applicability of two surfaces resides.

The derivation of relations (31) gives, by using the structural equations (16),

$$[\omega_2(\omega_{21}^\star - \omega_{21})] = 0, \qquad [\omega_1(\omega_{12} - \omega_{12}^\star)] = 0,$$

From which we get the complementary relation

$$\omega_{12} = \omega_{12}^\star. \tag{32}$$

Another derivation gives

$$[\omega_{13}\omega_{23}] = [\omega_{13}^\star\omega_{23}^\star]$$

or rather

$$ac - b^2 = a^\star c^\star - [b^\star]^2,$$

by setting

$$\begin{aligned} \omega_{13} &= a\omega_1 + b\omega_2, & \omega_{23} &= b\omega_1 + c\omega_2, \\ \omega_{13}^\star &= a^\star\omega_1^\star + b^\star\omega_2^\star, & \omega_{23}^\star &= b^\star\omega_1^\star + c^\star\omega_2^\star. \end{aligned}$$

But we have seen that $ac - b^2$ is a second order invariant of the surface S, or more precisely it is the total curvature $r_1 r_2 = K$. From which we get Gauss' theorem, in which two applicable surfaces have the same total curvature at their corresponding points.

We are now going to find the necessary and sufficient conditions for two given surfaces to be applicable.

We can distinguish several cases.

1. *K is constant.* Every surface $S^\star$ applicable onto S must also have its total curvature constant and equal to K. This condition is sufficient, since the relations (31) and (32) are completely integrable, due to the common structural constants satisfied by the forms ω_1, ω_2, ω_{12}:

$$(33) \qquad \begin{cases} \omega_1' = [\omega_2\omega_{21}], & \omega_2' = [\omega_1\omega_{12}], \quad \omega_{12}' = -K[\omega_1\omega_2], \\ (\omega_1^\star)' = [\omega_2^\star\omega_{21}^\star], & (\omega_2^\star)' = [\omega_1^\star\omega_{12}^\star], \quad (\omega_{12}^\star)' = -K[\omega_1^\star\omega_2^\star]. \end{cases}$$

Moreover, we see that each surface itself admits an infinite applications generating a three parameter group whose equations (33) are the structural equations.

2. *K is not constant.* We are going to specialise the first order trihedral attached at each surface in a way to make the vector $\vec{\mathbf{I}}_2$ tangent to the lines of equal constant curvature. In the application of the two surfaces the trihedrals thus determined must coincide with each other. Then let

$$dK = K_1\omega_1, \qquad dK^\star = K_1^\star\omega_1^\star,$$

we must have at two corresponding points

$$K_1 = K_1^\star.$$

a. Suppose that there exists no relations between K and K_1, and let $K_{1,1} = f_1(K, K_1)$, $K_{1,2} = f_2(K, K_1)$. We must have on the surface $S^\star$ the same relations $K_{1,1}^\star = f_1(K^\star, K_1^\star)$, $K_{1,2}^\star = f_2(K^\star, K_1^\star)$. This condition is necessary and sufficient. Indeed, if we have established between the two surfaces the point correspondence defined by the equations

$$K = K^\star, \qquad K_1 = K_1^\star,$$

we have, at the same time,

$$K_{1,1} = K^\star_{1,1}, \qquad K_{1,2} = K^\star_{1,2}.$$

The relations $dK = dK^\star$, $dK_1 = dK^\star_1$ then entails

$$K_1\omega_1 = K^\star_1\omega^\star_1, \qquad K_{1,1}\omega_1 + K_{1,2}\omega_2 = K^\star_{1,1}\omega^\star_1 + K^\star_{1,2}\omega^\star_2,$$

from which

$$\omega_1 = \omega^\star_1, \qquad \omega_2 = \omega^\star_2.$$

b. Suppose at last that K_1 is a function $f(K)$. The form ω_1 is an exact differential $\dfrac{dK}{f(K)}$ and we have

$$\omega_{12} = \rho_2\omega_2.$$

Again we have two possibilities.

b_1. The function ρ_2 is independent of K. Let

$$\rho_{2,1} = \varphi_1(K, \rho_2), \qquad \rho_{2,2} = \varphi_2(K, \rho_2).$$

For the surface $S^\star$ to be applicable onto S, it is necessary and sufficient that we have

$$K^\star_1 = f(K^\star), \qquad \rho^\star_{2,1} = \varphi_1(K^\star, \rho^\star_2), \qquad \rho^\star_{2,2} = \varphi_2(K^\star, \rho^\star_2),$$

the application is realised by the point correspondence defined by

$$K^\star = K, \qquad \rho^\star_2 = \rho_2,$$

which entails

$$K^\star_1 = K_1, \qquad \rho^\star_{2,1} = \rho_{2,1}, \qquad \rho^\star_{2,2} = \rho_{2,2},$$
$$f(K^\star)\omega^\star_1 = f(K)\omega_1, \qquad \varphi_1(K^\star, \rho^\star_2) + \varphi_2(K^\star, \rho^\star_2)\omega^\star_2 = \varphi_1(K, \rho_2)\omega_1 + \varphi_2(K, \rho)\omega_2,$$

i.e.,

$$\omega^\star_1 = \omega_1, \qquad \omega^\star_2 = \omega_2.$$

b_2. The function ρ_2 depends only on K, $\rho_2 = \varphi(K)$. It then suffices that we have $K^\star_1 = f(K^\star)$, $\rho^\star_2 = \varphi(K^\star)$. The application is realised by the integration of the equations

$$K^\star = K, \qquad \omega^\star_2 = \omega_2,$$

where the second is completely integrable by using the first equation. The surface S admits ∞^1 applications onto itself: it is application unto a surface of revolution.

We will observe analogies that exist between the solution of the applicability problem and those of equality problem. This analogy points that the first problem can be made into the equality problems of the three linearly independent Pfaffian forms in three variables ω_1, ω_2, ω_{12} to the three analogous forms $\omega^\star_1$, $\omega^\star_2$, $\omega^\star_{12}$ [equations (31) and (32)].

196 Overview of the general problem of deformation. We can present Gauss' problem of deformation under the following form. Two surfaces S and $S^\star$ are said to be applicable if we can establish between these two surfaces a point correspondence such that if $\mathbf{A}$ and $\mathbf{A}^\star$ are two corresponding points, there exists a displacement transforming $\mathbf{A}^\star$ into $\mathbf{A}$ such that all points of $S^\star$ infinitesimally close to $\mathbf{A}^\star$ coincide up to second order infinitesimals with the corresponding point of S. If, indeed, T is a first order trihedral attached at $\mathbf{A}$, the trihedral $\mathbf{T}^\star$ that we must attach to $\mathbf{A}^\star$ for which the displacement considered transforms $\mathbf{T}^\star$ to coincide with $\mathbf{T}$, is characterised by the relations (31)

$$\omega_1 = \omega_1^\star, \qquad \omega_2 = \omega_2^\star.$$

The preceding statement calls for an obvious generalisation applicable to two varieties V_λ in a geometry with a given group G [12]. Two varieties V_λ and $V_\lambda^\star$ are said to be applicable to order p if we can establish between them a point correspondence such that, $\mathbf{A}$ and $\mathbf{A}^\star$ being two corresponding points, there exists a transformation of the group G which, applied to $V_\lambda^\star$, transforms $\mathbf{A}^\star$ into $\mathbf{A}$ and every point of $V_\lambda^\star$ infinitesimally close to $\mathbf{A}^\star$ to a corresponding point of V_λ up to infinitesimally small quantities of order $p = 1$. In the case of Gauss, $p = 1$. We can also add the supplementary condition that, as a result of the transformation considered, the variety $V_\lambda^\star$ have at $\mathbf{A}$ with the variety V_λ a contact of a given order $q \geq p$. The projective application of surfaces $\lambda = p = 2$ has been the subject of various researches. All problems of application can be treated by the method of moving frame [20].

Chapter 13

The third fundamental theorem in the theory of groups

I. The necessary part of the third fundamental theorem

197 Exterior derivative of a alternating bilinear differential form (†) Consider an alternating bilinear differential form

$$\Omega(\delta' x, \delta'' x) = \sum_{i,j} A_{ij}(x)(\delta' x_i \delta'' x_j - \delta'_j \delta'' x_i). \tag{1}$$

Let us introduce a third system of differentials δX_i, the symbol δ, δ', δ'' assumed to commute. By the *exterior derivative* of Ω we mean the expression

$$\Omega' = \delta\Omega(\delta', \delta'') + \delta'\Omega(\delta'', \delta) + \delta''\Omega(\delta, \delta'). \tag{2}$$

This is a trilinear form in δx_i, $\delta' x_i$, $\delta'' x_i$. The expressions $\delta\delta' x_i$, $\delta'\delta'' x_i$, $\delta''\delta x_i$ do not appear. Its value remains constant when we modify the choice of the independent variables.

To deal with such trilinear forms, it is convenient to set

$$\begin{vmatrix} \delta x_1 & \delta' x_1 & \delta'' x_1 \\ \delta x_2 & \delta' x_2 & \delta'' x_2 \\ \delta x_3 & \delta' x_3 & \delta'' x_3 \end{vmatrix} = [dx_1 dx_2 dx_3],$$

this quantity is called the *exterior product* of the three differentials dx_1, dx_2, dx_3. The exterior product is associative and distributive with respect to addition. The exterior product of forms

$$\omega_1 = \sum_i a_i dx_i, \qquad \omega_2 = \sum_i b_i dx_i, \qquad \omega_3 = \sum_i c_i dx_i,$$

(†) More generally we can define the exterior derivative of all exterior differential forms (c.f. E. Cartan, *Leçons sur les invariants intégraux*, Hermann, 1922, chapter 8, p. 65–71).

will be, by definition,

$$[\omega_1\omega_2\omega_3] = [[\omega_1\omega_2]\omega_3] = \sum_{i,j,k} a_i b_j c_k [dx_i dx_j dx_k].$$

The exterior product of three linear forms remains constant when we swap the factors by a even permutation, it changes sign when we swap the factors by an odd permutation.

The value of the exterior derivative Ω' of (1) is

$$\Omega'(\delta,\delta',\delta'') = \sum_{i,j,k} \frac{\partial A_{ij}}{\partial x_k}[dx_i dx_j dx_k] = \sum_{(i,j,k)} \left(\frac{\partial A_{ij}}{\partial k} + \frac{\partial A_{jk}}{\partial x_i} + \frac{\partial A_{ki}}{\partial x_j}\right)[dx_i dx_j dx_k].$$

Suppose that Ω is the exterior derivative of a linear form

$$\omega = \sum_i a_i(x) dx_i,$$

we have

$$A_{ij} = \frac{1}{2}\left(\frac{\partial x_j}{\partial a_i} - \frac{\partial x_i}{\partial a_j}\right)$$

and then

$$\frac{\partial A_{ij}}{\partial x_k} + \frac{\partial A_{jk}}{\partial x_i} + \frac{\partial A_{ki}}{\partial x_j} = 0.$$

All exterior derivatives hence have the value 0 *for their exterior derivatives* [†].

The reader can establish easily the following formula: consider two linear forms ω_1 and ω_2. We have

(3) $$[\omega_1\omega_2]' = [\omega_1'\omega_2] - [\omega_2'\omega_1].$$

198 Exterior derivation of the structure equations of E. Cartan

$$\omega_s' = \frac{1}{2}\sum_{p,q} c_{pqs}[\omega_p\omega_q] \qquad (1 \le s \le r,\ 1 \le p \le r,\ 1 \le q \le r).$$

The left hand side has its exterior derivative zero, since it is itself an exterior derivative. The exterior derivative of the right hand side is

$$\begin{aligned}\frac{1}{2}\sum_{p,q} c_{pqs}[\omega_p'\omega_q] &- \frac{1}{2}\sum_{p,q} c_{pqs}[\omega_q'\omega_p] \\ &= \sum_{p,q} c_{pqs}[\omega_p'\omega_q] = \frac{1}{2}\sum_{p,q,\beta} c_{pqs}c_{\alpha\beta p}[\omega_\alpha\omega_b\omega_q].\end{aligned}$$

We hence have, by replacing q by γ,

$$\sum_{\alpha,\beta,\gamma,p} c_{\alpha\beta p}c_{p\gamma s}[\omega_\alpha\omega_\beta\omega_\gamma] = 0,$$

[†] The converse of this proposition also holds, though only locally.

i.e.,

$$\sum_{(\alpha,\beta,\gamma)} \sum_{p} (c_{\alpha\beta p} c_{p\gamma s} + c_{\gamma\alpha\pi} c_{p\beta s} + c_{\beta\gamma p} c_{p\alpha s})[\omega_\alpha \omega_\beta \omega_\gamma] = 0.$$

But the quantities ω_α, ω_β, ω_γ are linearly independent when $1 \leq a < b < \gamma \leq r$. Then

$$\sum_{p} (c_{\alpha\beta p} c_{p\gamma s} + c_{\gamma\alpha\pi} c_{p\beta s} + c_{\beta\gamma p} c_{p\alpha s}) = 0.$$

199 Third fundamental theorem of the theory of groups (S. Lie). *The necessary and sufficient condition for the r^3 quantities c_{pqs} to be the structural constants of an r parameter group is that these quantities satisfy the relations*

$$\text{(4)} \qquad c_{pqs} + c_{qps} = 0 \qquad \left(\frac{r^2(r-1)}{2} \text{ relations}\right),$$

$$\text{(5)} \qquad \sum_{p=1}^{r} (c_{\alpha\beta p} c_{p\gamma s} + c_{\gamma\alpha p} c_{p\beta s} + c_{\beta\gamma p} c_{p\alpha s}) = 0 \qquad \left(\frac{r^2(r-1)(r-2)}{6} \text{ relations}\right).$$

We have just shown that they are necessary.

The proof of sufficiency is equivalent, according to the third fundamental theorem (§**165**, p. 180), to resolving the following problem: given constants c_{pqs} satisfying relations (4) and (5), construct an r dimensional space E and r Pfaffian forms ω_s of the functions of the points of E which satisfy the following conditions:

1. We have
$$\omega'_s = \sum_{(p,q)} c_{pqs}[\omega_p \omega_q];$$
2. E contains only interior points;
3. The forms ω_s are linearly independent at each point of E;
4. Whenever the integral $\int_L \sqrt{\sum_{s=1}^{r} |\omega_s|^2}$ converges, the path L converges to a point of

E.

We will impose the condition that E is simply connected [c.f. note of §**164**, p. 179].

This problem posses at most one solution, since two groups are holoedrically isomorphic when they have the same structural constants and the parameter spaces are simply connected (c.f. isomorphism condition, §**164**, p. 179).

One of the difficulties of this problems is the construction of the space E, whose topology is unknown, for which we cannot in general choose the r dimensional Euclidean space.

Before trying to establish the converse of the third fundamental theorem, we are going to define in the parameter space of a group a particularly simple coordinate system.

II. The canonical parameters of S. Lie

200 Properties of the canonical parameters. Let S be a finite dimensional connected group of transformations S_ξ and consider a variable frame $\mathbf{R}_{\xi(t)}$ depending on the time t, which coincides with $\mathbf{R}_0$ at $t = 0$ and whose movement is uniform: this means that the quotients

(6) $$\frac{\omega_1(\xi, d\xi)}{dt}, \qquad \ldots, \qquad \frac{\omega_r(\xi, d\xi)}{dt}$$

are constants $a_1, \ldots, a_r$.

When $a_1, \ldots, a_r$ are given, the point $\xi(1)$ is well determined. This point in the parameter space called by S. Lie the point of *canonical parameters* $a_1, \ldots, a_r$. The point $\xi(t)$ is of canonical parameters $a_1 t, \ldots, a_r t$.

When t varies and $a_1, \ldots, a_r$ remain fixed, the transformations $\Sigma_t = S_{\xi(t)}$ generate a one parameter group, since the infinitesimal transformation $\Sigma_t^{-1}\Sigma_{t+dt}$ has the symbol $dt(a_1 X_1 + \cdots + a_r X_r)$ and hence has the relative component dt. The parameter group is given by $dt' = dt$. We hence have

(7) $$\Sigma_{t'}\Sigma_{t''} = \Sigma_{t'+t''}.$$

This formula expresses that the product of transformations of canonical parameters $a_1 t'$, $\ldots$, $a_r t'$ and $a_1 t''$, $\ldots$, $a_r t''$ is the transformation of canonical parameters $a_1(t' + t'')$, $\ldots$, $a_r(t' + t'')$.

In other words: *in the space* $(a_1, \ldots, a_r)$ *of canonical parameters the straight lines passing through the origin represent the subgroups. Through each of the straight lines the composition law of the group reduce to the addition of coordinates of the same indices.*

For example, to the opposite values of canonical parameters, $a_1, \ldots, a_r$ and $-a_1$, $\ldots$, $-a_r$, correspond inverse transformations.

According to the formula (7), we have

(8) $$S_{\xi(t')} \cdot S_{\xi(t'')} = S_{\xi(t'')} \cdot S_{\xi(t')}.$$

On the other hand, more generally, any two transformations of a one parameter connected group commute, since, according to paragraph §**167** (p. 183), such a group is isomorphic to the group of transformations on the straight line.

The formula (8) entails

$$S^{-1}_{\xi(t)} S_{\xi(t+dt)} = S_{\xi(t+dt)} S^{-1}_{\xi(t)}.$$

The frame $\mathbf{R}_{\xi(t)}$ hence have the same absolute and relative components

(9) $$\omega_s(\xi, d\xi) = \varpi_s(\xi, d\xi) = a_s dt.$$

Consider a point that remains fixed with respect to $\mathbf{R}_{\xi(t)}$ and whose absolute coordinates are $x_1, \ldots, x_n$. The components of its velocity are

$$\frac{dx_i}{dt} = \sum_{p=1}^{r} \frac{\varpi_p}{dt} X_p x_i,$$

i.e.,

$$\frac{dx_i}{dt} = \sum_{p=1}^{r} a_p X_p x_i \qquad (i = 1, \ldots, n). \tag{10}$$

The transformation S_ξ with canonical parameters $a_1, \ldots, a_r$ is hence obtained in the following way: we integrate the system (10) and we correspond to a point $\mathbf{M}_0$ with coordinates $x_i(1)$. We have $\mathbf{M}_1 = S_\xi \mathbf{M}_0$ (c.f. §**85** and §**86**, p. 98).

201 The canonical parameters are the local coordinates in the parameter space. If we give $a_1, \ldots, a_r$ very small values $da_1, \ldots, da_r$, the corresponding point ξ in the parameter space is very close to zero and its coordinates $d\xi_i, \ldots, d\xi_r$ are defined by the relations

$$\omega_s(0, d\xi) = da_s. \tag{11}$$

These relations show that $d\xi_1, \ldots, d\xi_r$ are independent linear combinations of $da_1, \ldots, da_r$. This signifies that the functional determinant $\dfrac{D(\xi)}{D(a)}$ differs from zero at the origin of the coordinates. Then the system of canonical parameters near $(0, \ldots, 0)$ correspond bijectively to the points in the parameter space which are close to the origin.

If to a point in the parameter space always corresponds one and only one system of canonical coordinates, we can always identify the parameter space to r dimensional Euclidean space. But the example of rotational group in three dimensional Euclidean space let us see that such an identification is not always possible, even when the parameter space is simply connected.

We are going show by two very simple examples that *in general, we can use the canonical parameters as alternative coordinates of the parameter space only in a neighbourhood of the origin.*

First consider the displacement group of space. Every displacement is helicoidal. But a helicoidal displacement constituted by a translation of length l and a rotation of angle θ can be generated by an infinity of helicoidal movement whose reduced steps are $\dfrac{l}{\theta + 2k\pi}$ (k being positive, zero or negative). Hence every displacement has *an infinite number of canonical parameters.*

Consider the group of linear transformations of two real variables

$$\begin{cases} x' = \alpha x + \beta y, \\ y' = \gamma x + \delta y, \end{cases} \qquad (\alpha\delta - \beta\gamma > 0). \tag{12}$$

Its frames are pairs of vectors $\vec{\mathbf{I}}_1$ and $\vec{\mathbf{I}}_2$ such that $\vec{\mathbf{I}}_1 \wedge \vec{\mathbf{I}}_2 > 0$. We see geometrically that this family of frames is connected. The group (12) is hence also connected. The equations (10) can be written in this case

$$\begin{cases} \dfrac{dx}{dt} = a_1 x + a_2 y, \\[2mm] \dfrac{dy}{dt} = a_3 x + a_4 y. \end{cases} \tag{13}$$

The integration of the system (13) furnishes all the transformations of the group (12) possessing canonical coordinates.

We integrate (13) by finding all combinations with constant coefficients $\lambda x+\mu y$ such that

$$\frac{d}{dt}(\lambda x+\mu y)=k(\lambda x+\mu y) \qquad (k= \text{ constant}).$$

k, λ and μ are determined by the equations

$$\text{(14)} \qquad \begin{vmatrix} a_1-k & a_3 \\ a_2 & a_4-k \end{vmatrix}=0, \qquad \begin{aligned} \lambda(a_1-k)+\mu a_3&=0,\\ \lambda a_2+\mu(a_4-k)&=0. \end{aligned}$$

First suppose that the equation in k, or the characteristic equation, has two distinct roots k_1 and k_2. The finite transformation of the canonical parameters a_1, a_2, a_3 and a_4 is reducible to the form

$$\text{(15)} \qquad \lambda_1 x'+\mu_1 y'=e^{k_1}(\lambda_1 x+\mu_1 y), \qquad \lambda_2 x'+\mu_2 y'=e^{k_2}(\lambda_2 x+\mu_2 y),$$

the characteristic equation of this transformation has roots e^{k_1} and e^{k_2}: either these two roots are real and positive, or they are complex conjugates (imaginary conjugates, or real, negative and equal).

If the equation in k has a double root, necessarily real, the finite transformation of the canonical parameters a_1, a_2, a_3, a_4 is reducible to the form

$$\text{(16)} \qquad \lambda x'+\mu y'=e^{k}(\lambda x+\mu y), \qquad y'=e^k y+h(\lambda x+\mu y),$$

its characteristic equation has a double root, the real and positive root e^k.

We wee that the finite transformation

$$x'=\alpha x, \qquad y'=\beta y$$

where α and β are two distinct real numbers, does not belong to any of the categories obtained. It does not admit canonical parameters.

202 Expression of the forms ω_s as functions of canonical parameters. Consider a moving frame $\mathbf{R}_{\xi(t)}$ depending on the parameter t. Suppose it to be defined by the data of its initial position $\mathbf{R}_{\xi(0)}$ and relative components of its infinitesimal displacement

$$\omega_s[\xi(t),d\xi(t)]=\lambda_s(t)dt.$$

Suppose that these data are functions of other parameters $a_1, \ldots$. Let us subject these parameters to increments $\delta a_1, \ldots$. The functions $\lambda_s(t)$ undergo variations $\delta\lambda_s(t)$. $\mathbf{R}_{\xi(t)}$ undergoes, for each value of t, an infinitesimal displacement characterised by its relative components

$$\omega_s[\xi(t),\delta\xi(t)].$$

It is easy to determine the quantities $\omega_s[\xi(t),\delta\xi(t)]$ with the help of the quantities $\lambda_s(t)$, $\delta\lambda_s(t)$, $\omega_s[\xi(0),\delta\xi(0)]$.

The symbol δ and the symbol d of differentiation with respect to t commute. We have, according to the structural equations of E. Cartan,

$$d\omega_s[\xi(t), \delta\xi(t)] - \delta\lambda_s(t)dt = \sum_{p,q} c_{pqs}\lambda_p(t)dt\,\omega_q[\xi(t), \delta\xi(t)].$$

The forms ω_s, whose value for $t = 0$ we know, is hence obtained by integrating the linear differential system

$$\text{(17)} \qquad \frac{d\omega_s}{dt} = \delta\lambda_s + \sum_{p,q} c_{pqs}\lambda_p\omega_q.$$

In particular, suppose that the functions λ_s are constants a_s and $\xi(0) = 0$. What we just wrote down gives us the means to calculate the expression of relative components $\omega_s(a, \delta a)$ as functions of canonical parameters a_s and their differentials δa_s.

We integrate the linear differential system with constant coefficients

$$\text{(18)} \qquad \frac{d\omega_s}{dt} = \delta a_s + \sum_{p,q} c_{pqs}a_p\omega_q,$$

we choose the integrals which at $t = 0$ satisfy the initial conditions $\omega_s = 0$. This integral has value [†] $\omega_s(ta, t\,\delta a)$.

Remark. Analogous reasoning applies to the absolute components. It is convenient to substitute in place of equations (17) and (18) the following:

$$\text{(19)} \qquad \frac{d\varpi_s}{dt} = \delta\lambda_s - \sum_{p,q} c_{pqs}\lambda_p\varpi_q,$$

$$\text{(20)} \qquad \frac{d\varpi_s}{dt} = \delta a_s - \sum_{p,q} c_{pqs}a_p\varpi_q.$$

III. Proof (incomplete) of the third fundamental theorem

203 Construction of the forms ω_s verifying the structure equations of E. Cartan. Consider the constants c_{pqs} satisfying the relations

$$\text{(21)} \qquad c_{pqs} + c_{qps} = 0, \qquad (1 \le p \le r;\ 1 \le q \le r;\ 1 \le s \le r).$$

The conclusions of the preceding paragraph suggests the following operation:

[†] ta denotes the point with coordinates $ta_1, \ldots, ta_r$; $t\,\delta a$ denotes the vector with components $t\,\delta a_1, \ldots, t\,\delta a_r$.

Introduce r parameters $a_1, \ldots, a_r$ and their differentials $\delta a_1, \ldots, \delta a_r$. Write the linear differential system with constant coefficients

$$\frac{d\omega_s}{dt} = \delta a_s + \sum_{p,q} c_{pqs}\omega_q, \tag{22}$$

consider the solutions that satisfy on $t = 0$ the initial conditions $\omega_s = 0$. It depends on t, a_p, δa_p through the intermediacy of the products ta_p and $t\,\delta a_p$. It is linear and homogeneous with respect to the variables $t\,\delta a_p$. Let us call it

$$\omega_s(ta, t\,\delta a).$$

We are going to find under what conditions the Pfaffian forms $\omega_s(ta, t\,\delta a)$ satisfy the structural equations

$$\omega_s'(ta, t\,\delta a, t\,\delta' a) = \sum_{(p,q)} c_{pqs}[\omega_p(ta, t\,\delta a)\omega_q(ta, t\,\delta a)], \tag{23}$$

the forms $\omega_s(a, \delta a)$ then naturally also satisfy the equations

$$\omega_s'(a, \delta a, \delta' a) = \sum_{(p,q)} c_{pqs}[\omega_p(a, \delta a)\omega_q(a, \delta a)]. \tag{23$'$}$$

Let us calculate the exterior derivatives of the two sides of (22) with respect to the variables a_s. We obtain

$$\frac{d}{dt}\omega_s'(ta, t\,\delta a, t\,\delta' a) = \sum_{p,q} c_{pqs}[\delta a_p \omega_q(ta, t\,\delta a)] + \sum_{p,q} c_{pqs} a_p \omega_q'(ta, t\,\delta a, t\,\delta' a), \tag{24}$$

$\omega_s'(ta, t\,\delta a, t\,\delta_a')$ is the solution of this linear differential system that verifies the initial conditions $\omega_s' = 0$ at $t = 0$. The expression

$$\sum_{(p,q)} c_{pqs}[\omega_p(ta, t\,\delta a)\omega_q(ta, t\,\delta a)] \tag{25}$$

is zero when t is. For the relation (23) to hold, it is hence necessary and sufficient that the system (24) is satisfied when we replace ω_s' by the expression (23). The relation

$$\frac{d}{dt}\sum_{(p,q)} c_{pqs}[\omega_p\omega_q] = \sum_{p,q} c_{pqs}[\delta a_p \omega_q] + \sum_{p,q}\sum_{(\alpha\beta)} c_{pqs} a_p c_{\alpha\beta q}[\omega_\alpha\omega_\beta] \tag{26}$$

hence must be satisfied, when using the equation (22).

But we have, according to (22),

$$\begin{aligned}\frac{d}{dt}\sum_{(p,q)} c_{pqs}[\omega_p\omega_q] &= \sum_{(p,q)} c_{pqs}\left[\frac{d\omega_p}{dt}\omega_q\right] + \sum_{(p,q)} c_{pqs}\left[\omega_p\frac{d\omega_q}{dt}\right] \\ &= \sum_{p,q} c_{pqs}\left[\frac{d\omega_p}{dt}\omega_q\right] = \sum_{p,q} c_{pqs}[\delta a_p\omega_q] + \sum_{p,q}\sum_{\alpha,\beta} c_{pqs}c_{\alpha\beta p}a_\alpha[\omega_\beta\omega_q].\end{aligned}$$

The relation (26) is hence written

$$\sum_{p,q}\sum_{\alpha,\beta} c_{pqs}c_{\alpha\beta p}a_\alpha[\omega_\beta\omega_q] = \sum_{p,q}\sum_{(\alpha\beta)}[\omega_\alpha\omega_\beta],$$

i.e., by changing the names of the indices,

$$\sum_{\alpha,\beta,\gamma,p} a_\alpha[\omega_\beta\omega_\gamma]\left(c_{\alpha\beta p}c_{p\gamma s} - \frac{1}{2}c_{\beta\gamma p}c_{\alpha ps}\right) = 0,$$

i.e.,

$$\sum_{\alpha,p}\sum_{(\beta,\gamma)} a_\alpha[\omega_\beta\omega_\gamma](c_{\alpha\beta p}c_{p\gamma s} + c_{\gamma\alpha p}c_{p\beta s} + c_{\beta\gamma p}c_{p\alpha s}) = 0.$$

But the quantities a_α are arbitrary, the forms ω_s are linearly independent when $ta_1, \ldots,$ ta_r are sufficiently small, and the paratheses in the preceding formula must be identically zero. Our conclusion will hence be the following:

The necessary and sufficient condition for the integration of equations (22) *to furnish the forms* ω_s *verifying the structural equations* (23′) *is that the constants* c_{pqs} *satisfy the relations*

$$c_{\alpha\beta p}c_{p\gamma s} + c_{\gamma\alpha p}c_{p\beta s} + c_{\beta\gamma p}c_{p\alpha s} = 0. \tag{27}$$

204 Application of the preceding paragraph to the proof of the third fundamental theorem. Given a group, the relative components ω_s of its frame satisfy the structural equations (23′) (second fundamental theorem, §**165**, p. 180). According to the preceding paragraph the relations (27) necessarily hold. Hence we have just proved a second time the direct part of the third fundamental theorem.

Let us now deal with the converse. Given constants c_{pqs} satisfying the relations (21) and (27), the preceding paragraph tells us that we can define, in Euclidean space with coordinates $a_1, \ldots, a_r$, r Pfaffian forms $\omega_s(a, da)$ that satisfy the structural equations

$$\omega_s' = \sum_{(p,q)} c_{pqs}[\omega_p\omega_q].$$

We have $\omega_s(0, da) = da_s$. The forms $\omega_s(a, da)$ are hence independent in a neighbourhood of the point $(0, \ldots, 0)$. But currently there appears to be no reason that can allow us to affirm that they are independent at all points and a path L converges when the integral $\int_L \sqrt{\sum_{s=1}^{r}|\omega_s|^2}$ converges.

When these two conditions are satisfied [†] the transformations leaving the forms ω_s invariant constitute a group, whose parameter space is the Euclidean space constituted by points

[†] They are satisfied if the characteristic roots of the equations (22) with zero right hand sides, i.e., where we suppress terms in δa_s, are all zero (groups of rank zero).

$(a_1, \dots, a_r)$. Every transformation of this group has one and only one canonical parameter system. The examples of paragraph **§201** (p. 219) show that these circumstances are exceptional.

Thus we have not completely proved the converse of the third fundamental theorem. We have only managed to construct the solutions of the structural equations of E. Cartan. Paragraph **§218** (p. 239) will indicate what other view will allow us to establish this converse theorem completely.

Chapter 14

The structural equations of S. Lie

I. The bracket of two infinitesimal transformations

205 An infinitesimal transformation transformed by another infinitesimal transformation. Consider two infinitesimally small transformations $\mathbf{S}$ and $\mathbf{T}$, both possessing inverses and operating on the same domain D. Let us give them the symbols

$$(\mathbf{S}) \qquad \mathbf{X}f = \sum_{k=1}^{n} \xi_k(x_1, \ldots, x_n) \frac{\partial f}{\partial x_k},$$

$$(\mathbf{T}) \qquad \mathbf{Y}f = \sum_{k=1}^{n} \eta_k(x_1, \ldots, x_n) \frac{\partial f}{\partial x_k},$$

Let us find the symbol of the transformed transformation $\mathbf{S}^{-1}\mathbf{T}\mathbf{S}$ of $\mathbf{T}$ by $\mathbf{S}^{-1}$, and let it be

$$\mathbf{Z}f = \sum_{k=1}^{n} \zeta_k(x_1, \ldots, x_n) \frac{\partial f}{\partial x_k}.$$

This symbol can be deduced from (3) of paragraph §**62** (p. 73) by replacing φ_i with $x_i + \xi_i$. It becomes, by neglecting infinitesimally small quantities of order 3,

$$\zeta_i + \sum_{k=1}^{n} \zeta_k \frac{\partial \xi_i}{\partial x_k} = \eta_i + \sum_{k=1}^{n} \xi_k \frac{\partial \eta_i}{\partial x_k} + \cdots$$

The principal part of ζ_i is hence η_i. The preceding equations can hence be written

$$(1) \qquad \zeta_i = \eta_i + \sum_{k=1}^{n} \left(\xi_k \frac{\partial \eta_i}{\partial x_k} - \eta_k \frac{\partial \xi_i}{\partial x_k} \right).$$

Introduce the following *definition*: *given the two infinitesimal transformations*

$$\mathbf{X}f = \sum_{k=1}^{n} \xi_k \frac{\partial f}{\partial x_k}, \qquad \mathbf{Y}f = \sum_{k=1}^{n} \eta_k \frac{\partial f}{\partial x_k},$$

we call the following infinitesimal transformation the bracket of these two transformations, denoted by $(\mathbf{XY})$

$$(2) \qquad (\mathbf{XY}) = \sum_{k,i} \left(\xi_k \frac{\partial \eta_i}{\partial x_k} - \eta_k \frac{\partial \xi_i}{\partial x_k} \right) \frac{\partial f}{\partial x_i}.$$

The formula (1) expresses that the symbol of $\mathbf{S}^{-1}\mathbf{TS}$ is $\mathbf{Y}f + (\mathbf{XY})$. We can hence state this result as the following:

The infinitesimal transformation $\mathbf{Y}f$*, when transformed by the transformation* $\mathbf{X}f$*, is the infinitesimal transformation*

$$(3) \qquad \mathbf{Y}f + (\mathbf{YX}).$$

206 Properties of the brackets. We have

$$(4) \qquad (\mathbf{XY}) = -(\mathbf{YX}), \qquad (\mathbf{XX}) = 0.$$

We can write

$$(\mathbf{XY}) = \sum_{k,i} \left\{ \xi_k \frac{\partial}{\partial x_k} \left(\eta_i \frac{\partial f}{\partial x_i} \right) - \eta_k \frac{\partial}{\partial x_k} \left(\xi_i \frac{\partial f}{\partial x_i} \right) \right\},$$

i.e.,

$$(5) \qquad (\mathbf{XY}) = \mathbf{X}(\mathbf{Y}f) - \mathbf{Y}(\mathbf{X}f).$$

Consider two differentiations d and δ defined by the formulae

$$dx_i = \xi_i, \qquad \delta x_i = \eta_i.$$

We have

$$df = \mathbf{X}f, \qquad \delta f = \mathbf{Y}f, \qquad d\delta f - \delta df = (\mathbf{XY}).$$

These two differentiations commute when the bracket $(\mathbf{XY})$ is zero, and they commute only in this case.

Three arbitrary infinitesimal transformations $\mathbf{X}f$, $\mathbf{Y}f$, $\mathbf{Z}f$ satisfy the *Jacobi identity*

$$(6) \qquad [\mathbf{X}(\mathbf{YZ})] + [\mathbf{Y}(\mathbf{ZX})] + [\mathbf{Z}(\mathbf{XY})] = 0.$$

By expanding the left hand side, we indeed obtain

$$\begin{aligned}
&[\mathbf{X}(\mathbf{YZ})]f + [\mathbf{Y}(\mathbf{ZX})]f + [\mathbf{Z}(\mathbf{XY})]f \\
&\ = \mathbf{X}(\mathbf{YZ})f + \mathbf{Y}(\mathbf{ZX})f + \mathbf{Z}(\mathbf{XY})f - (\mathbf{YZ})\mathbf{X}f - (\mathbf{ZX})\mathbf{Y}f - (\mathbf{XY})\mathbf{Z}f \\
&\ = \mathbf{XYZ}f + \mathbf{YZX}f + \mathbf{ZXY}f - \mathbf{XZY}f - \mathbf{YXZ}f - \mathbf{ZYX}f \\
&\quad - \mathbf{YZX}f - \mathbf{ZXY}f - \mathbf{XYZ}f + \mathbf{ZYX}f + \mathbf{XZY}f + \mathbf{YXZ}f = 0.
\end{aligned}$$

207 Application to the theory of groups [†]**.** Consider a finite dimensional connected group and its infinitesimal transformations

$$\mathbf{X}_1 f, \qquad \ldots, \qquad \mathbf{X}_r f.$$

An infinitesimal transformation of the group, when transformed by another infinitesimal transformation, again belongs to the group. The conclusion of paragraph §**205** hence entails the conclusion that the braket of two infinitesimal transformations of the group is also an infinitesimal transformation of the group. We have formulae of the type

$$(7) \qquad (\mathbf{X}_p \mathbf{X}_q) = \sum_{s=1}^{r} c'_{pqs} \mathbf{X}_s f,$$

c'_{pqs} being constants.

According to (4),

$$(8) \qquad c'_{pqs} + c'_{qps} = 0.$$

The Jacobi identity entails the relations

$$(9) \qquad \sum_{p=1}^{r} (c'_{\alpha\beta p} c'_{p\gamma s} + c'_{\gamma\alpha p} c'_{p\beta s} + c'_{\beta\gamma p} c'_{p\alpha s}) = 0.$$

In paragraph §**210** we will give another proof that $(\mathbf{X}_p \mathbf{X}_q)$ is an infinitesimal transformation of the group. This second proof will have the advantage of proving that the constants c'_{pqs} are identical to the structural constants c_{pqs} of the group. The formulae (9) will henceforth constitute a new proof of the direct part of the third fundamental theorem.

II. The second fundamental theorem of S. Lie

208 Ratio between the infinitesimal transformations and the components of the moving frame. Consider a finite dimensional and connected group whose frame $\mathbf{R}_a$ varies as function of one or several parameters $a_1, \ldots$

Suppose the movement of $\mathbf{R}_a$ has been defined by the data of its *absolute components* $\varpi_s(a, da)$. Consider a fixed point with respect to $\mathbf{R}_a$. When the parameters $a_1, \ldots$ increase by $da_1, \ldots,$ this point undergoes the transformation $S_{a+da} S_a^{-1}$. Its *absolute coordinates* x_i undergoes the increase

$$(10) \qquad dx_i = \sum_{s=1}^{r} \varpi_s(a, da) \mathbf{X}_s x_i.$$

The coordinates x_i are obtained by integrating the system (10), which is hence *completely integrable.*

[†] Sophus Lie, the creator of the theory of connected groups, did not use the components ω, ϖ, but the bracket notation played a fundamental role in its reasonings.

Suppose that the movement of $\mathbf{R}_a$ is now defined by the data of its *relative components* $\omega_s(a, da)$. Consider a fixed point (with respect to $\mathbf{R}_0$) and let x_i be the *relative coordinates* (with respect to $\mathbf{R}_a$). If the quantities x_i remain fixed, the point considered would undergo a certain displacement, which we call the driving displacement. Express this driving displacement with respect to $\mathbf{R}_a$: it is obtained by transforming the point x_i by $S_a^{-1}S_{a+da}$, i.e., by giving x_i the increase $\sum_{s=1}^r \omega_s(a, da)\mathbf{X}_s x_i$. To this driving displacement let us also add the relative displacement dx_i. The absolute displacement which follows must be zero, i.e.,

$$dx_i + \sum_{s=1}^{r} \omega_s(a, da)\mathbf{X}_s x_i = 0. \tag{11}$$

The coordinates x_i are obtained by integrating this system, which is therefore *completely integrable.*

Variant. We could have reasoned as the following: the infinitesimal transformation $\mathbf{X}_s f$ and the relative components ω_s are defined, according to paragraph §**71** (p. 82), by the relations

$$\delta x_i = \sum_{s=1}^{r} \omega_s(a, da)\mathbf{X}_s x_i,$$

$$\sum_{k=1}^{n} \frac{\partial \varphi_i(x_1, \ldots, x_n; a_1, \ldots, a_r)}{\partial x_k}\delta x_k = \sum_{p=1}^{r} \frac{\partial \varphi_i(x_1, \ldots, x_n; a_1, \ldots, a_r)}{\partial a_p} da_p.$$

The system

$$dx_i + \sum_{s=1}^{r} \omega_s(a, da)\mathbf{X}_s x_i = 0 \tag{11}$$

is hence equivalent to the following

$$\sum_{k=1}^{n} \frac{\partial \varphi_i(x_1, \ldots, x_n; a_1, \ldots, a_r)}{\partial x_k} dx_k + \sum_{p=1}^{r} \frac{\partial \varphi_i(x_1, \ldots, x_n; a_1, \ldots, a_r)}{\partial a_p} da_p = 0.$$

The integrals of this system are defined by the equations

$$\varphi_i(x_1, \ldots, x_n; a_1, \ldots, a_r) = \text{ constant.}$$

The give the coordinates of the points whose transformed image under S_a are fixed. The system (11) is hence completely integrable.

Example: *Group of Euclidean displacements.* We have (c.f. §**68**, example, 79)

$$\sum_{s=1}^{r} \omega_s \mathbf{X}_s f = \omega_1 \frac{\partial f}{\partial x} + \omega_2 \frac{\partial f}{\partial y} + \omega_3 \frac{\partial f}{\partial z} + \omega_{23}\left(y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y}\right) + \omega_{31}\left(z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}\right) + \omega_{12}\left(x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}\right).$$

The system (11), whose integration furnishes the coordinates of a fixed point, is

$$\begin{aligned}
dx + \omega_1 - \omega_{12}y + \omega_{31}z &= 0,\\
dy + \omega_2 + \omega_{12}x - \omega_{23}z &= 0,\\
dz + \omega_3 - \omega_{31}x + \omega_{23}y &= 0.
\end{aligned}$$

209 Application of the Frobenius theorem. The preceding paragraph leads to the following problem:

Consider a Pfaffian system

$$dx_i + \sum_{s=1}^{r} \omega_s(a, da)\mathbf{X}_s x_i = 0, \qquad (i = 1, \dots, n); \tag{12}$$

the variables x_i does not appear in the forms ω_s and the variables a_s does not appear in the infinitesimal transformations $\mathbf{X}_s f$ (we do not assume that these forms and these infinitesimal transformations are necessarily the relative components and the infinitesimal transformations of a group). Under what condition is the system (12) completely integrable?

Let us apply Frobenius theorem (§**166**, p. 180). The exterior derivative of the left hand side of (12) must be zero:

$$\sum_{s=1}^{r} \omega_s' \mathbf{X}_s x_i + \sum_{s=1}^{r} [d\mathbf{X}_s x_i, \omega_s] = 0. \tag{13}$$

The differentials are supposed to satisfy the system (12). But it follows from (12) that, f being a function of $x_1, \dots, x_n$,

$$df = -\sum_{p=1}^{r} \omega_p \mathbf{X}_p f.$$

In particular

$$d\mathbf{X}_s x_i = -\sum_{p=1}^{r} \omega_p \mathbf{X}_p(\mathbf{X}_s x_i).$$

The condition (13) hence can be written

$$\sum_{s} \omega_s' \mathbf{X}_s x_i - \sum_{s,p} [\omega_p \omega_s] \mathbf{X}_p(\mathbf{X}_s x_i) = 0,$$

i.e.,

$$\sum_{s} \omega_s' \mathbf{X}_s x_i - \sum_{(p,q)} [\omega_p \omega_q] \{\mathbf{X}_p(\mathbf{X}_q x_i) - \mathbf{X}_q(\mathbf{X}_p x_i)\} = 0,$$

or rather

$$\sum_{s} \omega_s' \mathbf{X}_s - \sum_{(p,q)} [\omega_p \omega_q](\mathbf{X}_p \mathbf{X}_q) = 0. \tag{14}$$

This relation (14) hence expresses that the system (12) is completely integrable.

Suppose that the variables a_s are r in number and the r forms ω_s are independent. We can then set

$$\omega_s' = \sum_{(p,q)} c_{pqs} [\omega_p \omega_q], \tag{15}$$

The coefficients c_{pqs} being eventually functions of $a_1, \ldots, a_r$. (14) now takes the form

$$\sum_{(p,q)} [\omega_p \omega_q] \left\{ \sum_{s} c_{pqs} \mathbf{X}_s - (\mathbf{X}_p \mathbf{X}_q) \right\} = 0.$$

Hence we must have

$$(\mathbf{X}_p \mathbf{X}_q) = \sum c_{pqs} \mathbf{X}_s. \tag{16}$$

Suppose that the transformations $\mathbf{X}_s$ are linearly independent: according to (16) the coefficients c_{pqs} cannot depend on the variables a_s. The are constants.

By means of the hypotheses stated, the fact that the system (12) is completely integrable hence entails the existence of constants c_{pqs} such that the relations (15) and (16) hold.

Conversely, when such constants exist, the relations (14) are satisfied, the system (12) is hence completely integrable.

210 The structure equations of S. Lie. Consider again a group, its infinitesimal transformations $\mathbf{X}_s$ and its relative and absolute components of its frames, ω_s and ϖ_s. The systems (10) and (11) are completely integrable. There hence exists, according to the preceding paragraph, constants c_{pqs} such that

$$(\mathbf{X}_p \mathbf{X}_q) = \sum_{s=1}^{r} c_{pqs} \mathbf{X}_s, \tag{17}$$

$$\omega_s' = \sum_{(p,q)} c_{pqs} [\omega_p \omega_q], \tag{18}$$

$$\varpi_s' = -\sum_{(p,q)} c_{pqs} [\varpi_p \varpi_q]. \tag{19}$$

We just derived again the structural equations of E. Cartan [(18) and (19)], as well as the equations (7), which all shows that the constants c'_{pqs} appearing in them are the structural constants of the group.

The equations (17) are called the *structural equations of S. Lie.*

Example: *Displacement group.* The structural equations of E. Cartan are (c.f. §**160**, p. 175)

$$\omega_i' = \sum_k [\omega_k \omega_{ki}], \qquad \omega_{ij}' = \sum_k [\omega_{ik}\omega_{kj}]$$

(i, j, k take the values 1, 2, 3, and we have $\omega_{ij} + \omega_{ji} = 0$ as well as $\mathbf{X}_{ij} + \mathbf{X}_{ji} = 0$).

The infinitesimal transformation $\mathbf{X}_i$, $\mathbf{X}_{ij}$ must hence satisfy the structural equations of S. Lie:

$$(\mathbf{X}_i\mathbf{X}_j) = 0, \qquad (\mathbf{X}_{ik}\mathbf{X}_{kj}) = \mathbf{X}_{ij},$$
$$(\mathbf{X}_i\mathbf{X}_{ij}) = \mathbf{X}_j, \qquad (\mathbf{X}_i\mathbf{X}_{jk}) = 0, \qquad (i, j, k \text{ different}).$$

On the other hand it is easy to verify them directly, using the expressions of $\mathbf{X}_i$, $\mathbf{X}_{ij}$ given in paragraph §**68** (p. 79).

211 Group generated by infinitesimal transformations satisfying the structure equations of S. Lie. Suppose we are given r linearly independent infinitesimal transformations $\mathbf{X}_1, \ldots, \mathbf{X}_r$ satisfying the structural equations of S. Lie (17). These relations impose conditions on the constants c_{pqs} that they satisfy the third fundamental theorem [c.f. §**207**, relations (8) and (9)]. Paragraph §**203** (p. 221) tells us how to construct r independent Pfaffian forms ω_s as functions of r parameters $a_1, \ldots, a_r$ satisfying the structural equations of E. Cartan (18). By virtue of paragraph §**209** the system

$$dx_i + \sum_{s=1}^{r} \omega_s(a, da)\mathbf{X}_s x_i = 0 \tag{11}$$

is completely integrable [(†)].

Consider the solution of (11) defined by the initial conditions $x_i = x_i^0$ for $a_s = 0$. The point with coordinates x_i will be said to transformed into a point with coordinates x_i^0 by the transformation S_a^{-1} (this definition is suggested by paragraph §**208**). The set of transformations S_a thus defined constitute an r parameter family of transformations, all possessing inverses. This family contains the identity transformation S_0.

If one solution determined by the equations (11) has values a_s for the parameters x_i and b_s for the parameters y_i, then the point x_i is the transformed image of y_i by the transformation $S_a^{-1}S_b$. Let us choose $b_s = a_s + da_s$ and set $y_i = x_i + dx_i$. $S_a^{-1}S_{a+da}$ transforms the point $x_i + dx_i$ into the point x_i, the differentials dx_i being defined by the relations (11). In other words the transformation $S_a^{-1}S_{a+da}$ transforms the point x_i into the point $x_i + \delta x_i$, δx_i defined by the relations

$$\delta x_i = \sum_{s=1}^{r} \omega_s(a, da)\mathbf{X}_s x_i.$$

The moving frame $\mathbf{R}_a = S_a\mathbf{R}_0$ hence possesses relative components. According to the first fundamental theorem (§**78**, p. 90), the transformations S_a constitute a kernel of group.

[(†)] A variant to this reasoning consists in constructing the forms ϖ_s satisfying the structural equations (19) and substitute the system (10) for the system (11).

The group has its kernel of group depending on r parameters. Its infinitesimal transformations are $\mathbf{X}_1, \ldots, \mathbf{X}_r$. It is constructed by the procedures described in paragraphs §**85** and §**86** (p. 98).

Our reasonings are only rigorous if the integration of equations (11) defines transformations S_a that are regular in the space of points x_i. This is the case if the procedures of paragraphs §**85** and §**86** construct regular transformations.

From which we have the following conclusion:

Complement to the second fundamental theorem of group theory (S. Lie). *Consider r linearly independent infinitesimal transformations $\mathbf{X}_1, \ldots, \mathbf{X}_r$, defined in an n dimensional space, $x_1, \ldots, x_n$. For them to generate a r parameter group, it is necessary and sufficient that the following conditions are satisfied:*

1. *The infinitesimal transformations $\mathbf{X}_s f$ satisfy Lie's structure equations*

$$(\mathbf{X}_p \mathbf{X}_q) = \sum_{s=1}^{r} c_{pqs} \mathbf{X}_s, \qquad [c_{pqs} = \text{ constants}^{(\dagger)}];$$

2. *The construction procedures that we have described in paragraphs* §**85** *and* §**86** *furnish bijective transformations in the space $x_1, \ldots, x_n$.*

Remark. Suppose that the transformations $\mathbf{X}_s f$ depend linearly on the variables $x_1, \ldots, x_n$. The transformations generated by them are obtained by integrating a linear system. They are hence bijective linear transformations. The second condition of the statement of the theorem is hence satisfied.

III. Determination of groups and subgroups

212 Subgroups of a given group. Consider a group G, its infinitesimal transformations $\mathbf{X}_1, \ldots, \mathbf{X}_r$ and a connected subgroup g of G. The infinitesimal transformations of g (‡) are linear combinations of $\mathbf{X}_1, \ldots, \mathbf{X}_r$ satisfying the structural equations of S. Lie.

Conversely, if a system of $r-\rho$ linear combinations of $\mathbf{X}_1, \ldots, \mathbf{X}_r$ satisfy the structural equations of S. Lie, the group it generates constitutes a subgroup of G, viewed from the way that it is constructed (§**85** and §**86**).

For example, the transformations $\mathbf{X}_{\rho+1}, \ldots, \mathbf{X}_r$ generate a subgroup of G if and only if

$$(20) \qquad c_{pqs} = 0 \qquad \text{for} \qquad p = \rho+1, \ldots, r, \quad q = \rho+1, \ldots, r, \quad s = 1, \ldots, \rho.$$

Suppose these relations (20) are satisfied. According to paragraph §**169** [c.f. equations (42), p. 184] the system

$$\omega_1 = 0, \qquad \ldots, \qquad \omega_\rho = 0$$

(†)The constants c_{pqs} are the structural constants of the group.

(‡)The subgroup g is assumed to be analytic. E. Cartan has proved [15] that every connected subgroup contained in an analytic group is generated by infinitesimal transformations.

is completely integrable. The $r-\rho$ dimensional integral varieties passing through the origin of the parameter space is the image of a subgroup of G. The transformations of this subgroup is obtained by integrating the system

$$dx_i + \sum_{s=\rho+1}^{r} \omega_s(a, da)\mathbf{X}_s x_i = 0.$$

Its infinitesimal transformations are hence $\mathbf{X}_{\rho+1}, \ldots, \mathbf{X}_r$. This subgroup is g.

Hence *the subgroup generated by the infinitesimal transformations*

$$\mathbf{X}_{\rho+1}, \qquad \ldots, \qquad \mathbf{X}_r$$

(which we suppose to satisfy the structural equations of Lie) is represented in the parameter space by an integral variety of the system

$$\omega_1 = 0, \qquad \ldots, \qquad \omega_\rho = 0.$$

213 The case of isomorphism and similitude. Consider two r parameter groups G_1 and G_2, acting respectively on the points x and y, and having the same Lie's structural equations

$$(\mathbf{X}_p\mathbf{X}_q) = \sum_{s=1}^{r} c_{pqs}\mathbf{X}_s \qquad (\text{for } G_1),$$
$$(\mathbf{Y}_p\mathbf{Y}_q) = \sum_{s=1}^{r} c_{pqs}\mathbf{Y}_s \qquad (\text{for } G_2).$$

These two groups are *isomorphic* by virtue of the isomorphism conditions stated in pages following p. 179 and we can identify their parameter spaces.

Suppose G_1 and G_2 are transitive and we can find two points x and y such that the subgroup g_x of G_1 and the subgroup g_y of G_2 leaving these points invariant have the same combinations of the infinitesimal transformations of $\mathbf{X}_s$ and $\mathbf{Y}_s$, which, for example, we can take to be $\mathbf{X}_{n+1}, \ldots, \mathbf{X}_r$, and $\mathbf{Y}_{n+1}, \ldots, \mathbf{Y}_r$. The subgroup g_x and g_y then have the same images in the parameter space, which is the $r-n$ dimensional variety passing through the origin on which

$$\omega_1 = 0, \qquad \ldots, \qquad \omega_n = 0.$$

G_1 and G_2 are similar to the group of parameters, considered as acting on the integral varieties of this system. G_1 and G_2 are hence *similar.*

We have implicitly assumed that the parameter spaces of G_1 and G_2 are simply connected and g_x and g_y are connected. If we do not make these hypotheses, G_1 and G_2 are not necessarily similar. But it is again possible to identify these two groups by establishing a correspondence between x and y. This correspondence cannot be bijective. In other words *a change of variables $y(x)$ can identify the equations of G_1 to those of G_2.*

214 The search of all groups acting on one variable. Consider a connected and finite dimensional group of analytic transformations operating on one variable x. Let

$$\mathbf{X}_s f = \xi_s(x)\frac{\partial f}{\partial x} \qquad (s = 1, \dots, r)$$

be its infinitesimal transformations. The coefficients $\xi_s(x)$ are supposed to be analytic functions of x. The points where all these functions are zero are exceptional points left fixed by all transformations of the group. Suppose that the point $x = 0$ is not such an exceptional point. A linear transformation with constant coefficients applied to $\mathbf{X}_1, \dots, \mathbf{X}_r$ will lead to the case where we have the truncated expansions

$$\xi_1(x) = 1 + \cdots, \qquad \xi_2(x) = x^\alpha + \cdots, \qquad \xi_3(x) = x^\beta + \cdots, \qquad \cdots$$

0, α, β being a set of increasing integers.

We have

$$(\mathbf{X}_p\mathbf{X}_q)f = \left(\xi_p\frac{\partial \xi_q}{\partial x} - \xi_q\frac{\partial \xi_p}{\partial x}\right)\frac{\partial f}{\partial x}.$$

According to the structural equations of S. Lie, the expressions

$$\left(\xi_p\frac{\partial \xi_q}{\partial x} - \xi_q\frac{\partial \xi_p}{\partial x}\right)$$

are linear combinations of $\xi_1, \dots, \xi_r$. Their truncated expansions hence must being with the terms of exponents 0, or α, or β, ... But

$$\xi_1\frac{\partial \xi_2}{\partial x} - \xi_2\frac{\partial \xi_1}{\partial x} = \alpha x^{\alpha-1} + \cdots, \qquad \xi_1\frac{\partial \xi_3}{\partial x} - \xi_3\frac{\partial \xi_1}{\partial x} = \beta x^{\beta-1} + \cdots, \qquad \cdots$$

The series $\alpha - 1$, $\beta - 1$, ... hence must be part of the series 0, α, β. Consequently $\alpha = 1$, $\beta = 2$, ... We hence have the truncated expansions

$$\xi_s(x) = x^{s-1} + \cdots$$

But

$$\xi_{r-1}\frac{\partial \xi_r}{\partial x} - \xi_r\frac{\partial \xi_{r-1}}{\partial x} = x^{2r-4} + \cdots,$$

the exponent $2r - 4$ must belong to the series 0, 1, ..., $r - 1$. Hence

$$2r - 4 \le r - 1, \qquad \text{or} \qquad r \le 3.$$

Suppose that $r = 1$. We have a group acting on one variable with one parameter. A change of variables allows us to make its infinitesimal transformation $\mathbf{X}_1 f = \xi_1(x)\dfrac{\partial f}{\partial x}$ to $\mathbf{X}_1 f = \frac{\partial f}{\partial x}$. The group generated by this infinitesimal transformation is the translation group of the straight line $x' = x + a$.

Suppose $r = 2$. Since

$$\xi_1\frac{\partial \xi_2}{\partial x} - \xi_2\frac{\partial \xi_1}{\partial x} = 1 + \cdots,$$

we have

$$(\mathbf{X}_1\mathbf{X}_2) = \mathbf{X}_1 + \mu\mathbf{X}_2 \qquad (\mu = \text{ constant}).$$

Let us substitute $\mathbf{X}_1$ with the infinitesimal transformation $\mathbf{X}_1 + \mu\mathbf{X}_2$. The structural equation of the group becomes

$$(\mathbf{X}_1\mathbf{X}_2) = \mathbf{X}_1.$$

All the groups we search for in this case hence have the same structural equation: in all these groups the infinitesimal transformation $\mathbf{X}_2 f = (x + \cdots)\dfrac{\partial f}{\partial x}$ generates the subgroup leaving the point $x = 0$ fixed. According to the preceding paragraph all these groups can be deduced from each other by change of variables on x. To know them all it suffices to determine one of them. But the properties imposed on $\mathbf{X}_1$ and $\mathbf{X}_2$ are satisfied by the infinitesimal transformations

$$\mathbf{X}_1 f = \frac{\partial f}{\partial x}, \qquad \mathbf{X}_2 f = x\frac{\partial f}{\partial x}.$$

According to paragraph §**85** (p. 98), the transformations of the group generated by these two infinitesimal transformations transform the point $x(0)$ into the point $x(1)$, $x(t)$ satisfying a differential equation of the form

$$\frac{dx}{dt} = \lambda_1(t) + x\lambda_2(t).$$

Since this equation is linear, $x(1)$ depends linearly on $x(0)$. We obtain the group of linear transformations. Every group acting on one variable of two parameters hence reduce to, by a change of variables on x, the group of linear transformations

$$x' = ax + b \qquad (a > 0).$$

Now suppose $r = 3$. We have

$$\begin{aligned} \xi_1\frac{\partial \xi_2}{\partial x} - \xi_2\frac{\partial \xi_1}{\partial x} &= 1 + \cdots, \\ \xi_1\frac{\partial \xi_3}{\partial x} - \xi_3\frac{\partial \xi_1}{\partial x} &= 2x + \cdots, \\ \xi_2\frac{\partial \xi_3}{\partial x} - \xi_3\frac{\partial \xi_2}{\partial x} &= x^2 + \cdots. \end{aligned}$$

The structural equations of the group are hence of the type

$$\begin{aligned} (\mathbf{X}_1\mathbf{X}_2) &= \mathbf{X}_1 + \mu\mathbf{X}_2 + \nu\mathbf{X}_3, \\ (\mathbf{X}_1\mathbf{X}_3) &= 2\mathbf{X}_2 + \rho\mathbf{X}_3, \\ (\mathbf{X}_2\mathbf{X}_3) &= \mathbf{X}_3. \end{aligned}$$

Let us substitute $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ infinitesimal transformations of the type $\mathbf{X}_1 + \mu'\mathbf{X}_2 + \nu'\mathbf{X}_3$, $\mathbf{X}_2 + \rho'\mathbf{X}_3$, $\mathbf{X}_3$. We can choose the constants μ', ν', ρ' in a way as to reduce these structural equations to the form

$$(\mathbf{X}_1\mathbf{X}_2) = \mathbf{X}_1, \qquad (\mathbf{X}_1\mathbf{X}_3) = 2\mathbf{X}_2, \qquad (\mathbf{X}_2\mathbf{X}_3) = \mathbf{X}_3,$$

and we always have the truncated expansion

$$\xi_s(x) = x^{s-1} + \cdots .$$

The the group we search for in this case hence have the same structural equations. In all these groups the subgroup generated by $\mathbf{X}_2$ and $\mathbf{X}_3$ is the one that leaves the point $x = 0$ fixed. Every group we search for hence reduce to one of the two by a change of variables on x (c.f. §**213**). But we can choose

$$\mathbf{X}_1 f = \frac{\partial f}{\partial x}, \qquad \mathbf{X}_2 f = x\frac{\partial f}{\partial x}, \qquad \mathbf{X}_3 f = x^2\frac{\partial f}{\partial x}.$$

According to paragraph §**85** the transformations of the group generated by these three infinitesimal transformations transform $x(0)$ into $x(1)$, $x(t)$ satisfying a differential equation of the form

$$\frac{dx}{dt} = \lambda_1(t) + x\lambda_2(t) + x^2\lambda_3(t).$$

This equation is a Riccati equation. $x(1)$ is hence a homographic function of $x(0)$. Every analytic group acting on one variable in three parameters hence reduce to, by a change of variables on x, the group of homographic transformations

$$x' = \frac{ax+b}{cx+d} \qquad (ad - bc = 1).$$

Conclusion. *Every connected and finite dimensional analytic group acting on one variable reduces to, by a change of variables on x, one of the three following groups:*

The group of translations on the straight line

$$x' = x + a;$$

The group of linear transformations

$$x' = ax + b \qquad (a > 0);$$

The group of homographic transformations

$$x' = \frac{ax+b}{cx+d} \qquad (ad - bc = 1).$$

Application. Suppose in the differential geometry of the study of curves, we are to construct in an intrinsic manner a parameter along the curve. This parameter must be either completely defined, or defined up to transformations of a group. This group will necessarily be one of the three groups we just stated. The parameter will be defined up to an additive constant, up to a linear transformation, or un to a homographic transformation (*see* §**184**, p. 202).

Example: *Planar projective geometry.* The invariant k constitutes a completely determined parameter at each point. The projective arc σ constitutes a parameter defined up to an additive constant. The anharmonic ratio of a variable point and three fixed point constitutes a parameter defined up to a homographic transformation.

IV. Adjoint groups and the third fundamental theorem

215 Non-linear adjoint group. Consider a variable transformation S_ξ of a group and a fixed transformation S_a of this group. Let $S_{\xi'} = S_a S_\xi S_a^{-1}$ be S_ξ transformed by S_a. We call T_a the transformation acting on the parameter space which transforms the point ξ into the point ξ'. We say that T_a is homologous to S_a.

T_a^{-1} is homologous to S_a^{-1}, since $S_\xi = S_a^{-1} S_{\xi'} S_a$. $T_b T_a$ is homologous to $S_b S_a$, since the relations

$$S_{\xi'} = S_a S_\xi S_a^{-1}, \qquad S_{\xi''} = S_b S_{\xi'} S_b^{-1}$$

entail

$$S_{\xi''} = (S_b S_a) S_\xi (S_b S_a)^{-1}.$$

The transformation T_a hence constitute a group isomorphic to the group S_a. We call it the *adjoint group* of the group S_a.

This isomorphism is meriedric if there exists a transformation $S_a \neq 1$ such that we have, regardless of what ξ is,

$$S_\xi = S_a S_\xi S_a^{-1}, \qquad \text{i.e.,} \qquad S_\xi S_a = S_a S_\xi.$$

Every transformation S_a that *commutes with all others* is called a *distinguished transformation.* The set of these transformations constitute the *centre* of the group. *The adjoint group is isomorphic and hence similar to the parameter group when the given group does not contain any distinguished transformation.*

216 Linear adjoint group. Consider a connected and finite dimensional group S_a whose infinitesimal transformations are $\mathbf{X}_1, \ldots, \mathbf{X}_r$. Denote by $e_1'\mathbf{X}_1 + \cdots + e_r'\mathbf{X}_r$ the infinitesimal transformation $e_1\mathbf{X}_1 + \cdots + e_r\mathbf{X}_r$ transformed by S_a (c.f. §**69**, p. 80). We give the homogeneous linear transformation transforming the point $e_1, \ldots, e_r$ into the point $e_1', \ldots, e_r'$ the name τ_a. We observe, as in the preceding paragraph, that the transformations τ_a constitute a group. This group is called the *linear adjoint group.* It is isomorphic to the group S_a, meriedrically or holoedrically, according to whether there exists a transformation S_a commuting with all the infinitesimal transformations $\mathbf{X}_s$.

Relations between the two adjoint groups. We can consider an infinitesimal transformation and its transformed image under S_a as being the same geometrical transformation resolved respect to $\mathbf{R}_a$, then with respect to $\mathbf{R}_0$. The transformations generated by these infinitesimal transformations constitute the same geometrical transformations resolved respect to $\mathbf{R}_a$, then with respect to $\mathbf{R}_0$. Hence, if we transform the canonical parameters $e_1, \ldots, e_r$ of a point ξ in the parameter space by the linear transformation τ_a, we obtain the canonical parameters to which correspond the point $\xi' = T_a\xi$.

217 Infinitesimal transformations of the linear adjoint group. According to the paragraph §**205** (p. 225), the transformed image of the infinitesimal transformation

$$e_1\mathbf{X}_1 + \cdots + e_r\mathbf{X}_r$$

by the infinitesimally small transformation $\varepsilon\mathbf{X}_\alpha$ is the infinitesimal transformation

$$\begin{aligned} &e_1\mathbf{X}_1 + \cdots + e_r\mathbf{X}_r + \varepsilon(e_1\mathbf{X}_1 + \cdots + e_r\mathbf{X}_r, \mathbf{X}_\alpha) \\ &= e_1\mathbf{X}_1 + \cdots + e_r\mathbf{X}_r + \varepsilon\sum_{s=1}^{r}\{e_1c_{1\alpha s}\mathbf{X}_s + \cdots + e_rc_{r\alpha s}\mathbf{X}_s\}. \end{aligned}$$

To the infinitesimally small transformation $\varepsilon\mathbf{X}_\alpha$ of the given group hence corresponds in the linear adjoint group the infinitesimally small transformation which inflicts on e_s the increment

$$\delta e_s = \varepsilon\sum_{q=1}^{r} e_q c_{q\alpha s}.$$

In other words, the infinitesimal transformation $\mathbf{X}_s$ of the given group has as its homologue in the linear adjoint group the infinitesimal transformation

$$\mathbf{E}_\alpha f = \sum_{q,s} e_q c_{q\alpha s}\frac{\partial f}{\partial e_s}. \tag{21}$$

But, when a group is isomorphic to the second, the bracket of two infinitesimal transformations of the second group corresponds to the bracket of two homologous infinitesimal transformations of the first group. This follows from the definition of brackets together with the notion of transformed images (§**205**, p. 225).

We have in the present case

$$(\mathbf{X}_\alpha\mathbf{X}_\beta) = \sum_s c_{\alpha\beta s}\mathbf{X}_s.$$

We hence must have, similarly,

$$(\mathbf{E}_\alpha\mathbf{E}_\beta) = \sum_s c_{\alpha\beta s}\mathbf{E}_s. \tag{22}$$

Let us find under what conditions that the infinitesimal transformations of the type (21) satisfy the structural equations (22). We have

$$\begin{aligned} (\mathbf{E}_\alpha\mathbf{E}_\beta) &= \sum_{\gamma,p} e_\gamma c_{\gamma\alpha p}\frac{\partial}{\partial e_p}\left(\sum_{q,s} e_q c_{q\beta s}\frac{\partial f}{\partial e_s}\right) - \sum_{\gamma,p} e_\gamma c_{\gamma\beta p}\frac{\partial}{\partial e_p}\left(\sum_{q,s} e_q c_{q\alpha s}\frac{\partial f}{\partial e_s}\right) \\ &= \sum_{\alpha,p,s}(c_{\gamma\alpha p}c_{p\beta s} + c_{\beta\gamma p}c_{p\alpha s})e_\gamma\frac{\partial f}{\partial e_s}. \end{aligned}$$

Hence the equations (22) are only satisfied if

$$\sum_{p=1}^{r}(c_{\alpha\beta p}c_{p\gamma s} + c_{\gamma\alpha p}c_{p\beta s} + c_{\beta\gamma p}c_{p\alpha s}) = 0. \tag{23}$$

Then, the structural equations of a group satisfy these equations (23). We have just proved again the direct part of the third fundamental theorem.

218 The converse of the third fundamental theorem of S. Lie. The facts expounded in the in the preceding paragraph led S. Lie to the following proof of the converse of his third fundamental theorem (c.f. §**204**, p. 223).

Let us specify constants c_{pqs} satisfying the relations (23) and the relations $c_{pqs}+c_{qps} = 0$. Consider an Euclidean space with coordinates $e_1, \ldots, e_r$ and the infinitesimal transformations (21) of this Euclidean space. The calculations of the preceding paragraph show that these infinitesimal transformations satisfy the structural equations (22). Let us apply to them the complement of the second fundamental theorem (p. 231). The remark following the complement applies also, and the transformations $\mathbf{E}_\alpha f$ are linear in $e_1, \ldots, e_r$. Then, there exists a connected and finite dimensional group Γ whose infinitesimal transformations are $\mathbf{E}_1 f, \ldots, \mathbf{E}_r f$. This group is manifestly a subgroup of the homogeneous linear transformations on r variables.

However, to affirm that the structural equations of Γ are the equations (22), it is necessary to exclude a case which has escaped S. Lie: the infinitesimal transformations $\mathbf{E}_\alpha f$ must be linearly independent, otherwise the number of parameters of Γ is less than r.

The infinitesimal transformations $\mathbf{E}_\alpha f$ are dependent if there exists constants e_α such that

$$\sum_{\alpha=1}^{r} e_\alpha c_{q\alpha s} = 0.$$

We also have

$$\left(X_q, \sum_\alpha e_\alpha \mathbf{X}_\alpha\right) = \sum_{\alpha,s} e_\alpha c_{q\alpha s} \mathbf{X}_s = 0.$$

The infinitesimal transformation $\sum_\alpha e_\alpha \mathbf{X}_\alpha$ is identical to all of its transformed images. We say that this is a *distinguished infinitesimal transformation.*

We hence obtain a still incomplete converse to the third fundamental theorem:

Consider constants c_{pqs} satisfying the relations $c_{pqs} + c_{qps} = 0$ and the equations (23)*. Suppose that these constants are such that a corresponding group cannot have distinguished infinitesimal transformations. Then there exists a group of transformations (a subgroup of the homogeneous linear transformations on r variables) whose structural constants are c_{pqs} (S. Lie).*

E. Cartan [14, 19] was able to prove the converse of the third fundamental theorem completely by using simultaneously its structural equations and this theorem of S. Lie that we have just stated. Thus we could effectively construct a group (in general a non-linear group) admitting the given structural constants. Recently I. Ado has proved the existence of a linear group corresponding to the question [21], and E. Cartan has given this result a simpler proof (in an article that has not yet appeared in the *Journal de Mathématiques pures et appliquées*),

THE END.

Appendix A

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