Leon sur la thorie gnrale des surfaces *Lessons on the general theory of surfaces*

?[1]

2

inter alia

The numbers between the square brackets refer to entries in the Bibliography Index at the end of the book. Let us enunciate three

T
A
$_{1,2}$, $\vec{\mathbf{I}}_3$
1
A
T
M
x, y, z
$\overrightarrow{\mathbf{AM}}$

$$\frac{dM}{dt} = \frac{dA}{dt} + x\frac{d_1}{dt} + y\frac{d_2}{dt} + z\frac{d_3}{dt}.$$

$\omega_i = \xi_i(t)dt, \omega_{ji} = p_{ji}(t)dt, (i = 1, 2, 3; j < i).$

$\mathbf{T}_3$ $\mathbf{T}_3$ $\mathbf{T}_1$ $t=0$ $\mathbf{T}_2$ $t=1$ $\mathbf{T}_3^\star$ $\mathbf{T}_3$ $\omega^\star$ $\mathbf{T}_3^\star$ ω $\mathbf{T}_3$ $\mathbf{T}_1$ $\mathbf{T}_1^\star$ $\mathbf{T}_3$ $\mathbf{T}_3^\star$ $\mathbf{T}_2$ $\mathbf{T}_2^\star$ $\mathbf{T}_2$ $\mathbf{T}_2^\star$ $\mathbf{T}$ $\mathbf{T}^\star$ $\mathbf{T}$ $\mathbf{T}^\star$

Consider a bijective correspondence established between two ρ parameter families of trihedrals $\mathbf{T}$ and $\mathbf{T}^\star$. For a single displacement to suffice to superimpose all the corresponding trihedrals, it is necessary and sufficient that the correspondence in question makes the components ω of infinitesi-

$x(t)$
$y(t)$
$z(t)$
${}^2+$
y'^2+
$z'^2=$
$0,$
$x'x''+$
$y'y''+$
$z'z''=$
$0.$

11

$\mathbf{O}xyz$
$\mathbf{O}$
$''$
$'''$
$\mathbf{O}$

$\mathbf{O}xyz$
$\mathbf{O}xyz$

$\mathbf{A}_{123}$

$$( )_1{}_1^2\times{}_{21}\times{}_{32}\times{}_{12}{}_2^2\times{}_{33}\times{}_{13}\times{}_{23}{}^2 is\,(\,0\,)\,01010100. \tag{16}$$

$${}_1\wedge{}_2 = i_1, {}_2\wedge{}_3 = i_3, {}_3\wedge{}_1 = i_2 \tag{17}$$

$$({}_1\wedge{}_2)\times{}_3 = i. \tag{18}$$

theorem

Every trihedral $\mathbf{A}_{123}$ *satisfying eq:2.3 and eq:2.5 is a right cyclic trihedral.*

${}_1\mathbf{O}_3$
$x\mathbf{O}y$
1
3
$''$
$\mathbf{O}z$
${}_1 =$

13

ω_1

C
C
$\mathbf{T}$
C
$\mathbf{T}$
C
$\mathbf{T}$
$\mathbf{T}$
$\mathbf{T}$
$\mathbf{T}_0$
$\mathbf{T}_0$

$$\omega_1 = \omega_1, \omega_{11} = \omega_{11}, \omega_{21} = \omega_{21},$$

$\mathbf{T}_0$
$\mathbf{T}_0$
C
C
fundamental condition of equality
??

C
C

$\omega_{12} = 0$
$d_1 = \omega_{111}$
14

k
$d\sigma$
ω
??

determining the displacements which superimpose two given oriented minimal curves C *and* C
C
C
??
C
C

$$k(\sigma) = k(\sigma), d\sigma = d\sigma. \tag{28}$$

C
C

$$k(\sigma) = k(\sigma), d\sigma = -d\sigma. \tag{29}$$

C
C

special case
C
C
C

15
P
$(\geq$
$0)$
P
P
P

G

G
$\mathbf{A}_{123}$
1
G
G

Comparison of two zeroth order trihedrals $\mathbf{A}_{123}$ *and* $\mathbf{B}_{123}$.

$\{\mathbf{A} = \mathbf{B} + \rho_1 (thissignifies$

$\overrightarrow{\mathbf{BA}}\rho_1$
$_1 = _1$
$_2 =$
$_2 \cos\theta + _3 \sin\theta,$
$_3 =$
$-_2 \sin\theta + _3 \cos\theta,$
(38) ρ
θ

$\frac{1}{2} {}_1$
$-_2$

six
ω_i
ω_{ji}
$\mathbf{A}_{123}$
two

first order invariant k

$\omega_2 = \bar{\omega}_2 + (\rho + \lambda k)\bar{\omega}_{12},$

ω_2 *first order trihe-*

P

20

$\mathbf{A}_{123}$

$\mathbf{A}$

1

k

k

21
ω_{12}
ω_1+
$k\omega_{21}$

$$\omega_{12} = \omega_1 + k\omega_{21}. \tag{48}$$

$\frac{1}{3}{}_1$
$-{}_3$

*ori-
en-
ta-
tion*

k
*The
sec-
ond
or-
der
con-
tact
prob-
lem*

α $\mathbf{P}$ $\mathbf{P}_{123}$ $k =$ 0 $\mathbf{A}$ $\mathbf{P}$ k k **??**

$$\omega_{12} = dk, \omega_1 + k\omega_{21} = 0. \quad (52)$$

second order trihedral

Second order contact problem. *Consider two generatrices G_0 and G_0 of two surfaces and a displacement realising a contact of order ≥ 1 along these two generatrices: it superimposes one of the first order trihedrals $\mathbf{T}_0$ attached at G_0 to a first order trihedral $\mathbf{T}_0$ attached at G_0. For the order of contact to be greater than 1, it is necessary and*

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$\beta_0\alpha_0, \beta_0\gamma_0, \beta_0\delta_0, \beta_0\mathbf{P}),$

$X =$

$(\alpha_0\beta_0, \alpha_0\gamma_0, \alpha_0\delta_0, \alpha_0\mathbf{P}),$

$Z =$

1.

$\mathbf{M}$

$$X = (\beta\alpha, \beta\gamma, \beta\delta, \beta\mathbf{M}); Y = (\alpha\beta, \alpha\gamma, \alpha\delta, \alpha\mathbf{M}); Z = 1.$$

n

(59) $$\{x' = ax + by + cz, y' = a'x + b'y + c'z, z' = a''x + b''y + c''z,\ abca'b'c'a''b''c'' = 0.$$

a
b
c
$c''.$
(x, y, z)
an analytic point geometrical point
(x, y, z)
T
$\mathbf{M}'(x', y', z')$
$T(T'\mathbf{M}) =$
$(TT')\mathbf{M}$
p

$$\mathbf{M}_1(x_1, y_1, z_1), \mathbf{M}_2(x_2, y_2, z_2), \ldots,$$

$$_1x_1 + a_2x_2 + \ldots + a_px_p = 0,$$
$$a_1y_1 + a_2y_2 + \ldots + a_py_p = 0,$$
$$a_1z_1 + a_2z_2 + \ldots + a_pz_p = 0,$$

$$a_1\mathbf{M}_1 + a_2\mathbf{M}_2 + \ldots + a_p\mathbf{M}_p = 0.$$

$\mathbf{M}_1'$
$\ldots$
$\mathbf{M}_p'$

$$a_1\mathbf{M}_1' + a_2\mathbf{M}_2' + \ldots + a_p\mathbf{M}_p' = 0.$$

$$[\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3]$$

$$x_1y_1z_1x_2y_2z_2x_3y_3z_3$$

$$[\mathbf{M}_1', \mathbf{M}_2', \mathbf{M}_3'] = [\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3].$$

$$\mathbf{A}^0(0, 0, 1), \mathbf{A}_1^0(1, 0, 0), \mathbf{A}_2^0(0, 1, 0).$$

$\mathbf{R}_0$
$\mathbf{R}_0$
$\mathbf{R}$

$$\mathbf{A}(c, c', c''), \mathbf{A}_1(a, a', a''), \mathbf{A}_2(b, b', b'').$$

$\mathbf{A}$
$\mathbf{A}_1$
$\mathbf{A}_2$

$$[\mathbf{A}, \mathbf{A}_1, \mathbf{A}_2] = 1$$

$\mathbf{R}$
X
Y
Z
$\mathbf{R}$
$\mathbf{M}$
$\mathbf{R}$
$\mathbf{R}_0$
$\mathbf{M}$
$\mathbf{P}$
X
Y
Z

$$\mathbf{P} = X\mathbf{A}_1^0 + Y\mathbf{A}_2^0 + Z\mathbf{A}_3^0.$$

$\mathbf{M}$

$$\mathbf{M} = X\mathbf{A}_1 + Y\mathbf{A}_2 + Z\mathbf{A}_3.$$

T_ϵ
ϵ
T_ϵ

(60) $$x_i' = x_i + \epsilon\eta_i(x_1, \ldots, x_n) + \cdots$$

ϵ
x_i

relative components absolute components with a sign change parameters

Example

$x' = ax + b$
$x' = \alpha x + \beta$

$$\alpha = \frac{1}{\alpha}, \beta = -\frac{b}{a}.$$

??

$$\omega_1(a, b, da, db) = \frac{da}{a}, \omega_2(a, b, da, db) = \frac{db}{a}.$$

$_1(\alpha, \beta, d\alpha, d\beta) = -\frac{da}{a} = \frac{d\alpha}{\alpha},$
$_2(\alpha, \beta, d\alpha, d\beta) = -\frac{db}{a} = \alpha\, d\left(\frac{\beta}{\alpha}\right) = d\beta - \frac{\beta}{\alpha} d\alpha.$

25
Consider a moving

27

Therefore every group is holoedrically isomorphic to its parameter group.

Θ_a ξ ξ' *the parameter group is simply transitive.*

ξ Θ_a $\xi' = \Theta_a \xi$ **??**

$$\omega_p(\xi', d\xi') = \omega_p(\xi, d\xi).$$

the parameter group leaves the r *Pfaffian forms* $\omega_p(\xi, d\xi)$ *invariant*

$\Omega(\xi, d\xi)$ $\omega_p(\xi, d\xi)$ **??** $\Omega(\xi, d\xi)$

$$\Omega(\xi, d\xi) = a_1(\xi)\omega_1(\xi, d\xi) + \ldots + a_r(\xi)\omega_r(\xi, d\xi).$$

Θ_a ξ ξ'

$$a_1(\xi)\omega_1(\xi, d\xi) + \ldots + a_r(\xi)\omega_r(\xi, d\xi) = a_1(\xi')\omega_1(\xi', d\xi') + \ldots + a_r(\xi')\omega_r(\xi', d\xi').$$

$\omega_p(\xi, d\xi) = \omega_p(\xi', d\xi')$

$$[a_1(\xi) - a_1(\xi')]\omega_1(\xi, d\xi) + \ldots + [a_r(\xi) - a_r(\xi')]\omega_r(\xi, d\xi) = 0.$$

$\overset{31}{S}$.

$1, S_1, \ldots, S_n, S,$

$S_1, S_1^{-1}S_2, S_2^{-1}S_3, \ldots, S_{n-1}^{-1}S_n, S_n^{-1}S$

?

34

35

$\mathbf{R}$
ω_1
$\mathbf{R}_1$
g_1
G
ω_1
S
ω_1
ω
ω
$S g_1 \mathbf{R}_1$
ω
ω

Example. *practically*

Generalities of the operation called orientation.
G
ω
g_1
g_1'
g_1'
g_1
g_1
ω_1'
$g_1' \mathbf{R}_1$
ω'
G

δ
$\mathbf{A}_1$
r
$n-$
r
x^1_{r+1}
$\cdots$
x^1_n
a
r

a class ω for which G operates transitively is equivalent to a body c_1 defined by the subgroup g_1 leaving an object ω_1 arbitrarily chosen from the class ω invariant.

43

$\mathbf{R} = S\mathbf{R}_0$

$S\mathbf{A}$

S

$\mathbf{R}$

Every simply transitive group is similar to its parameter group.

u_1
u_2
u_{r-n}

a_1
a_r
$(x_1, \ldots, x_n)$
$(x'_1, \ldots, x'_n)$

$$x'_i = \phi_i(x_1, \ldots, x_n; a_1, \ldots, a_r), \tag{97}$$

$(x_1, \ldots, x_n; u_1, \ldots, u_{r-n})$
$(x'_1, \ldots, x'_n)$
x'_1
x'_n
u'_1
u'_{r-n}

$$\{x'_i = \phi_i(x_1, \ldots, x_n; a_1, \ldots, a_r), (i = 1, \ldots, n), u'_j = \psi_j(u_1, \ldots, u_{r-n}; x_1, \ldots, x_n; a_1, \ldots, a_r), (j = 1, \ldots, r-n). \tag{98}$$

(106) $\{\xi_1' = \psi_1(\xi_1, \xi_2, \ldots; a_1, \ldots, a_r), \ldots$
$G(\lambda)$

μ
λ
μ
$G(\mu)$
G
$G(\lambda)$
μ
ξ_1
ξ_2.
λ
ζ_1
ζ_2.
δ
μ

$G(\mu)$

n
n
r

$$x_i' = \phi_i(x_1, \ldots, x_n; a_1, \ldots, a_r), (i = 1, 2, \ldots, n). \tag{114}$$

A simply

Theorem.

$\omega_i(X, dX) = \omega_i(x, dx).$

$(x'_1, x'_2, \dots, x'_n)$
$(x'_1 +$
$dx'_1, \dots, x'_n +$
$dx'_n)$
$(X_i +$
$dX_i)$
(X_i)
(x'_i)

$$dX_i = \alpha_{i1}(X_1, \dots, X_n; x'_1, \dots, x'_n)dx'_1 + \dots + \alpha_{in}(X_1, \dots, X_n; x'_1, \dots, x'_n)dx'_n. \quad (118)$$

$\alpha_{i1}(X_1, \dots, X_n; x_1, \dots, x_n)dx_1 +$
$\dots +$
$\alpha_{in}(X_1, \dots, X_n; x_1, \dots, x_n)dx_n$
$\equiv$
$\alpha_{i1}(X_1, \dots, X_n; x'_1, \dots, x'_n)dx'_1 +$
$\dots +$
$\alpha_{in}(X_1, \dots, X_n; x'_1, \dots, x'_n)dx'_n.$
(X_i)
r

$\alpha_{i1}(X_1, \dots, X_n; x_1, \dots, x_n)dx_1 + \dots + \alpha_{in}(X_1, \dots, X_n; x_1, \dots, x_n)dx_n$

(x_i)
(x'_i)

Remark.

??
$\mathbf{R}_0$
A
(X_i)
$\omega_i(x, dx)$
$(x_i +$
$dx_i)$
(x_i)
a_i
S_a
(x_i)
$\mathbf{R}_0 \equiv$
A
$(X_i +$
$dX_i)$
$(x_i +$
$dx_i)$
S_a

$dX_i = \omega_i(x, dx),$

${}_i =$
$\phi_i(x_1, \dots, x_n; a_1, \dots, a_n),$
$dX_i =$
$\frac{\phi_i(x_1, \dots, x_n; a_1, \dots, a_n)}{x_1} dx_1 +$
$\dots +$
$\frac{\phi_i(x_1, \dots, x_n; a_1, \dots, a_n)}{x_n} dx_n.$
$\omega_i(x, dx)$
a_i

Example:

a
$dX =$
$X dxx$
$X =$
1
dxx
$dx' x' =$
dxx
$x' =$
dx
??
x_1
$x_2.$
x_n
$u_1.$
u_{r-n}
r

$\omega_p(x, u, dx, du).$

$$\left\{ \sum_{p=1}^{r} \omega_p(a, da) X_p f = \sum_{j=1}^{n} \frac{f}{x_j} \delta x_j; \ \sum_{p=1}^{r} \frac{\phi_i(x_1, \dots, x_n; a_1, \dots, a_r)}{a_p} da_p = \sum_{j=1}^{n} \frac{\phi_i(x_1, \dots, x_n; a_1, \dots, a_r)}{x_j} \delta x_j (i = 1, \dots, n). \right. \quad (119)$$

Group
of
pla-
nar
dis-
place-

50 of the system of partial differential equations is equal to the number of classes

51 of the variables u_i, v_l, $\dots$, w_m introduced in paragraph 1 2 3

??

G p p p p $x_1 \dots x_n$ G r

intransitive group n p characterised leaving invariant

p functions $y_1 \dots y_p$ $x_1 \dots x_n$

$G(V)$

G
$G(V)$
$(\xi_1, \ldots, \xi_r)$
$(x_1, \ldots, x_n; u_1, \ldots, u_{r-n})$
V
$x_1 =$
$x_n =$
ω_1
ω_n
V
du_j
n

A first case which can arise
u_j
n

$V^0(x_i^0)$
$V^1(x_i^1)$
G
(x_i^0, u_k^0)
V^0
(x_i^1, u_k^1)
V^1
G
V^0
V^1
$r-$
n
$G(V)$
n
$\omega_i(x, dx)$
V^0
V^1
??
$G(V)$
G

If the preceding case does not arise
ω_1
ω_n
u_1
u_{r-n}

$$\omega_i = \sum_{k=1}^{n} \alpha_{ik}(x, u) dx_k, (i = 1, \ldots, n).$$

u_j
α_{ik}
u_{p+1}
u_{r-n}

con-
tact
el-
e-
ments
λ
V_λ
D
the-
o-
ret-
i-
cally
pos-
si-
ble
??
??
prac-
ti-
cally

$\mathbf{A}$
μ_0
$0 \ldots$
$\mu_P -$
μ_{P-1}
$P \ldots$

The contact element of order P of a point $\mathbf{A}$ of V_λ will be distinguished by the family of frames of order P of this point and by the set of invariants of order $< P$ of the point.

$P+$
1
P

65
P
66

V_λ
Q
$Q+$
1
Q
$Q+$
1
$<$
Q
v_0
V_λ
λ
V_λ
$Q+$
1
Q
Q
$Q+$
2

given any two points of V_λ *there always exists an operation of* G *transforming* V_λ *into itself and transforming the two points into each other?*

$_1\wedge_2 = 1.$

(2)

$\mathbf{R}_{da}$
$\mathbf{R}_0$
$\omega_1(0, da)$
$\omega_2(0, da)$
$\mathbf{A}$
$1+$
$\omega_{11}(0, da)$
$\omega_{12}(0, da)$
$_1$
$\omega_{21}(0, da)$
$1+$
$\omega_{22}(0, da)$
$_2$

$$\omega_{11}(0, da)+\omega_{22}(0, da) = 0.$$

$$\omega_1(0, da), \omega_2(0, da), \omega_{11}(0, da), \omega_{12}(0, da), \omega_{21}(0, da)$$

$\mathbf{R}_0$
$\mathbf{R}_{da}$

??
relative components
$\mathbf{A}_{12}$
ω

$$\begin{cases}\overrightarrow{d\mathbf{A}} = \omega_1\ (a, da)_1 + \omega_2\ (a, da)_2, d_1 = \omega_{11}(a, da)_1 + \omega_{12}(a, da)_2, d_2 = \omega_{21}(a, da)_1 - \omega_{11}(a, da)_2.\end{cases} \quad (3)$$

0
1
1
0
1
frames of order
0
$\mathbf{A}$
$\mathbf{A}_{12}$
$\mathbf{A}$
frames of order
1
$\mathbf{A}$
$\mathbf{A}_{12}$
1
0
1
1

G
$\ell =$
5
0
1
$v_0 =$
3
$v_1 =$
2
0
$r-$
$v_0 =$
2
1
v_0-

73
ω_{00}
ω_{01}
ω_{02}
ω_{10}
ω_{11}
ω_{12}
ω_{20}
ω_{21}

$$\{d\mathbf{A} = \omega_{00}\mathbf{A} + \omega_{01}\mathbf{A}_1 + \omega_{02}\mathbf{A}_2, d\mathbf{A}_1 = \omega_{10}\mathbf{A} + \omega_{11}\mathbf{A}_1 + \omega_{12}\mathbf{A}_2, d\mathbf{A}_2 = \omega_{20}\mathbf{A} + \omega_{21}\mathbf{A}_1 + \omega_{22}\mathbf{A}_2,$$

??$\omega_{00}+$
$\omega_{11}+$
$\omega_{22} =$
0.

0
1
0
1
0
1
0

G
$r =$
8
0
1
$v_0 =$
6
$v_1 =$
5
0
$r-$
$v_0 =$
2
1
v_0-
$v_1 =$
1

0
$d\mathbf{A}$
$\mathbf{A}$
$\omega_{01} =$
$\omega_{02} =$
0
ω_{01}
ω_{02}
0
1
$d\mathbf{A}$
$\mathbf{A}$
$\mathbf{A}_1$
$\omega_{02} =$
0
0
1

$\omega_{01}, \omega_{02}^{0}(= 0)$

2
1
$\mathbf{A}\mathbf{A}_1\mathbf{A}_2$
$\mathbf{B}\mathbf{B}_1\mathbf{B}_2$

$$\{\mathbf{A} = \lambda\mathbf{B}, \mathbf{A}_1 = \lambda_1\mathbf{B}_1 + \mu\mathbf{B}, \mathbf{A}_2 = \lambda_2\mathbf{B}_2 + \rho\mathbf{B}_1 + \sigma\mathbf{B},\ where \lambda\lambda_1\lambda_2 = 1. \tag{3}$$

1
λ
λ_1
1

1
$\mathbf{B}\mathbf{B}_1\mathbf{B}_2$
λ
λ_1
λ_2
$d\mathbf{A}_1$
$\mathbf{A}$
$\mathbf{A}_1$
$\omega_{12} =$
0
ω_{12}
1
b
1
$\omega_{12} =$
$b\omega_0$

74
ω_{01}
ω_{12}
$\bar{\omega}_{01}$
$\bar{\omega}_{12}$
λ
λ_1
λ