Simultaneous Inference

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Regression Analysis

Main Idea

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If a 95% CI is created for β_0 and another 95% CI for β_1, we cannot say that we are 95% confident that these two confidence intervals are *simultaneously both* correct.

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If each H_{0i} is conducted independently, then the overall significance level is

$$\alpha^* = \Pr(\text{Reject } \mathrm{H}_0 \mid \mathrm{H}_0 \text{ is true}) = 1 - \Pr(\text{FTR } \mathrm{H}_0 \mid \mathrm{H}_0)$$

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Bonferroni Correction

If we do not know whether the tests are independent,

$$
\begin{aligned}
1-\alpha^* &= \Pr(\text{FTR } \mathrm{H}_0 \mid \mathrm{H}_0) = \Pr(\cap_{i=1}^{g} \text{FTR } \mathrm{H}_{0i} \mid \mathrm{H}_0) \\
&= 1 - \Pr(\cup_{i=1}^{g} \text{Rej } \mathrm{H}_{0i} \mid \mathrm{H}_0) \geq 1 - \sum_{i=1}^{g} \underbrace{\Pr(\text{Rej } \mathrm{H}_{0i} \mid \mathrm{H}_0)}_{\alpha} \\
&= 1 - g\alpha \quad \Rightarrow \quad 1-\alpha^* \geq 1 - g\alpha \quad \Rightarrow \quad \alpha^* \leq g\alpha
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Hence, $\alpha^* \leq g\alpha$, and by choosing individual significance levels as

$$
\boxed{\alpha = \frac{\alpha^*}{g}}
$$

for a *given* α^* we guarantee that the overall significance level will not exceed α^*.

Simultaneous Estimation of Mean Responses

- Bonferroni: Can be used for g simultaneous CIs, each with $100 \cdot (1 - \alpha/g)\%$ confidence level:

$$\hat{Y} \pm t_{1-\alpha/(2g),n-2} \cdot \mathrm{s}[\hat{Y}]$$

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- Working-Hotelling: A confidence band is created for the entire regression line that can be used for any number of confidence intervals for means simultaneously:

  $$\hat{Y} \pm \sqrt{2F_{1-\alpha;2,n-2}} \cdot \mathrm{s}[\hat{Y}]$$

Simultaneous Predictions

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- Scheffé: Widely used method. Like the Bonferroni, the width increases as g increases

  $$\hat{Y} \pm \sqrt{gF_{1-\alpha;g;n-2}} \cdot \mathrm{s}[b_0 + b_1 x + \epsilon]$$