Multiple Regression I

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Regression Analysis

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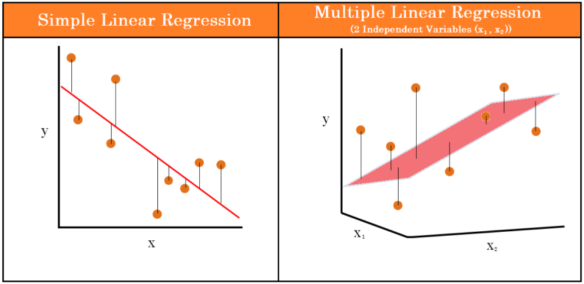

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Generally, for $k = p - 1$ predictors $x_1, \ldots, x_k$ our model is

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- β_0 is mean response when all predictors equal zero (if this makes sense).
- β_j is the change in mean response when x_j is increased by one unit *but the remaining predictors are held constant*.
- We will assume normal errors:

$$\epsilon_1, \ldots, \epsilon_n \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma^2).$$

Example: Dwayne Portrait Studio data

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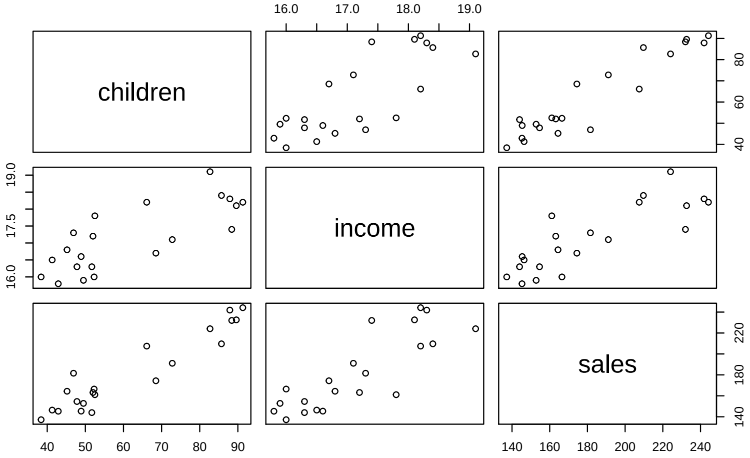
Dwaine Studios, Inc., operates portrait studios in 21 cities of medium size. These studios specialize in portraits of children. The company is considering an expansion into other cities of medium size and wishes to investigate whether sales (Y) in a community can be predicted from the number of persons aged 16 or younger in the community (x_1) and the per capita disposable personal income in the community (x_2).

Assume the linear model is appropriate. One way to check marginal relationships is through a scatterplot matrix.

Scatterplot matrix

https://github.com/zh3nis/MATH440/blob/main/chp06/scatter_matrix.R



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Qualitative predictors

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$$\mathrm{E}[Y] = \begin{cases} \beta_0 + \beta_1 \cdot 0 + \beta_2 x_2 & \text{males} \\ \beta_0 + \beta_1 \cdot 1 + \beta_2 x_2 & \text{females.} \end{cases}$$

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Response functions are two parallel lines, shifted by β_1 units.

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$$\log Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \epsilon.$$

Let $Y^* = \log Y$ and get general linear model.

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All of these models are *linear in the coefficients*, the β_j terms. An example of a model that is *not* in general linear model form is exponential growth:

$$Y = \beta_0 \exp(\beta_1 x) + \epsilon.$$

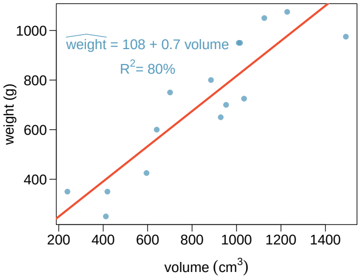
Another example with a binary predictor – weights of books

	weight (g)	volume (cm^3)	cover
1	800	885	hc
2	950	1016	hc
3	1050	1125	hc
4	350	239	hc
5	750	701	hc
6	600	641	hc
7	1075	1228	hc
8	250	412	pb
9	700	953	pb
10	650	929	pb
11	975	1492	pb
12	350	419	pb
13	950	1010	pb
14	425	595	pb
15	725	1034	pb

(From: Maindonald, J.H. and Braun, W.J. (2nd ed., 2007) "Data Analysis and Graphics Using R")

Weights of books (cont.)

The scatterplot shows the relationship between weights and volumes of books as well as the regression output. Which of the below is correct?



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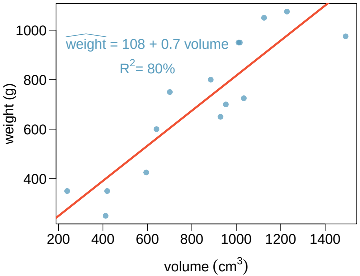

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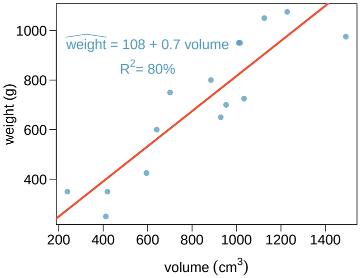

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(b) Books that are 10 cm^3 over average are expected to weigh 7 g over average.

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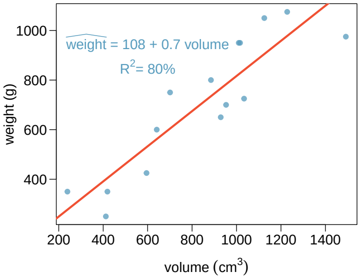

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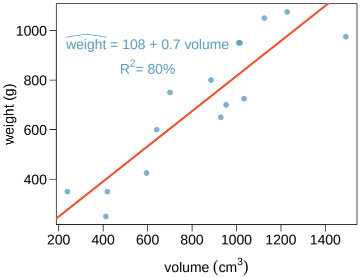

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(d) The model underestimates the weight of the book with the highest volume.

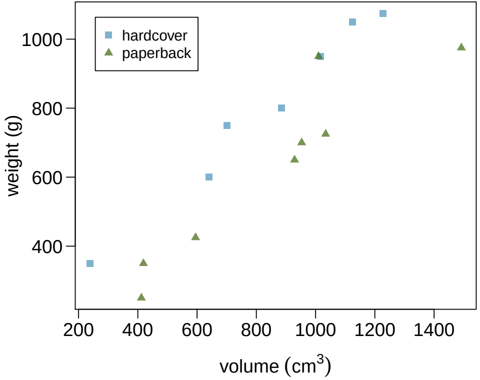
Modeling weights of books using only volume

```
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 107.67931   88.37758   1.218    0.245
volume        0.70864    0.09746   7.271 6.26e-06

Residual standard error: 123.9 on 13 degrees of freedom
Multiple R-squared: 0.8026,Adjusted R-squared: 0.7875
F-statistic: 52.87 on 1 and 13 DF,  p-value: 6.262e-06
```

Weights of hardcover and paperback books

Can you identify a trend in the relationship between volume and weight of hardcover and paperback books?



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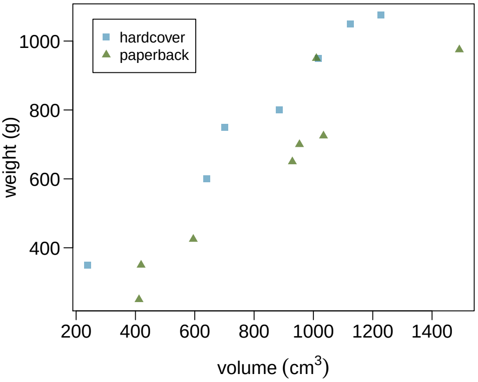
Paperbacks generally weigh less than hardcover books after controlling for the books volume.



Modeling weights of books using volume and cover type

```
Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  197.96284   59.19274   3.344 0.005841 **
volume         0.71795    0.06153  11.669  6.6e-08 ***
cover:pb    -184.04727   40.49420  -4.545 0.000672 ***

Residual standard error: 78.2 on 12 degrees of freedom
Multiple R-squared: 0.9275,Adjusted R-squared: 0.9154
F-statistic: 76.73 on 2 and 12 DF,  p-value: 1.455e-07
```

Determining the reference level

Based on the regression output below, which level of `cover` is `the reference level? Note that pb: paperback.`

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	197.9628	59.1927	3.34	0.0058
volume	0.7180	0.0615	11.67	0.0000
cover:pb	-184.0473	40.4942	-4.55	0.0007

(a) `paperback`

(b) `hardcover`

Linear model

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	197.96	59.19	3.34	0.01
volume	0.72	0.06	11.67	0.00
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$$\widehat{weight} = 197.96 + 0.72 \; volume - 184.05 \; cover : pb$$

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1. For *hardcover* books: plug in 0 for `cover`

$$\widehat{weight} = 197.96 + 0.72\ \ volume - 184.05 \times \color{red}{0}$$

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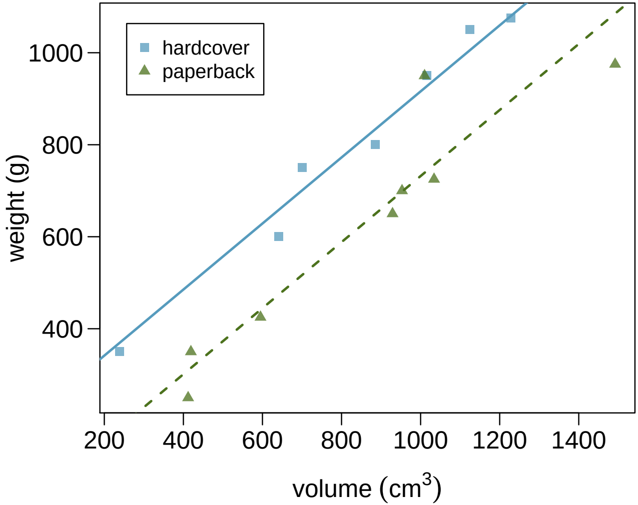
$$\begin{aligned}\widehat{weight} &= 197.96 + 0.72\ volume - 184.05 \times 1 \\ &= 13.91 + 0.72\ volume\end{aligned}$$

Visualising the linear model



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- β_0: Hardcover books with no volume are expected on average to weigh 198 grams.
  - Obviously, the intercept does not make sense in the context.

Prediction

Which of the following is the correct calculation for the predicted weight of a paperback book that is 600 cm^3?

	Estimate	Std. Error	t value	Pr(>|t|)
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(a) 197.96 + 0.72 * 600 - 184.05 * 1

(b) 184.05 + 0.72 * 600 - 197.96 * 1

(c) 197.96 + 0.72 * 600 - 184.05 * 0

(d) 197.96 + 0.72 * 1 - 184.05 * 600

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(a) 197.96 + 0.72 * 600 - 184.05 * 1 = 445.91 grams

(b) 184.05 + 0.72 * 600 - 197.96 * 1

(c) 197.96 + 0.72 * 600 - 184.05 * 0

(d) 197.96 + 0.72 * 1 - 184.05 * 600

6.2 General linear model in matrix terms

Response vector:

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}$$

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$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}$$

Design matrix:

$$\mathbf{X} = \begin{bmatrix} 1 & x_{11} & x_{12} & \cdots & x_{1k} \\ 1 & x_{21} & x_{22} & \cdots & x_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & x_{n2} & \cdots & x_{nk} \end{bmatrix}$$

The first column is a place-holder for the intercept term.

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The first column is a place-holder for the intercept term. What does each column represent? What does each row represent?

General linear model in matrix terms

(Unknown) regression coefficients:

$$\boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}$$

(Unobserved) error vector:

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}$$

General linear model in matrix terms

The general linear model is written in matrix terms as

$$\mathbf{Y} = \underbrace{\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}}_{n\times 1} = \underbrace{\begin{bmatrix} 1 & x_{11} & x_{12} & \cdots & x_{1k} \\ 1 & x_{21} & x_{22} & \cdots & x_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & x_{n2} & \cdots & x_{nk} \end{bmatrix}}_{n\times p} \underbrace{\begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}}_{p\times 1} + \underbrace{\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}}_{n\times 1},$$

where $p = k + 1$, or in short as

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \epsilon.$$

General linear model in matrix terms

Minimal assumptions about the random error vector $\boldsymbol{\epsilon}$ are

$$\mathrm{E}[\boldsymbol{\epsilon}] = \mathbf{0} \quad \text{and} \quad \mathrm{Cov}[\boldsymbol{\epsilon}] = \sigma^2 \mathbf{I}_n,$$

where $\mathbf{I}_n$ is the $n \times n$ identity matrix.

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This allows us to construct t and F tests, obtain confidence intervals, etc.

Fitting the model

Estimating $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_k)^\top$

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Recall the least-squares method:

$$Q(\boldsymbol{\beta}) = \sum_{i=1}^{n} [Y_i - (\beta_0 + \beta_1 x_{i1} + \ldots + \beta_k x_{ik})]^2 \longrightarrow \min_{\boldsymbol{\beta}}$$

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Using matrix calculus we can show that the LSEs are

$$\mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_k \end{bmatrix} = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{Y}$$

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These are also MLEs.

Fitted values, residuals, and hat matrix

The *fitted values* are

$$\hat{\mathbf{Y}} = \begin{bmatrix} \hat{Y}_1 \\ \hat{Y}_2 \\ \vdots \\ \hat{Y}_n \end{bmatrix} = \mathbf{Xb}$$

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$\mathbf{H} = \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top$ is called the **hat matrix** or **projection matrix**. We'll use it shortly when we talk about diagnostics. Notice that $\mathbf{e} = (\mathbf{I} - \mathbf{H})\mathbf{Y}$.

Example: Dwayne Studios

In the **Dwayne Portrait Studio data,** we have

$$\mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -68.857 \\ 1.455 \\ 9.366 \end{bmatrix},$$

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- b_1: For 1000 increase in number of children, mean sales increase by \$1,455 holding per capita income constant.
- b_2: For each \$1000 increase in per capita income, mean sales increase by \$9,366, holding the number of children constant.

Partitioning SST

Recall the decomposition

$$\underbrace{\sum_{i=1}^{n}(Y_i - \bar{Y})^2}_{\text{SST}} = \underbrace{\sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2}_{\text{SSE}} + \underbrace{\sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})^2}_{\text{SSR}}$$

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Notice that

$$\text{SSE} = (\mathbf{Y} - \hat{\mathbf{Y}})^\top(\mathbf{Y} - \hat{\mathbf{Y}}) = \mathbf{Y}^\top(\mathbf{I} - \mathbf{H})^\top(\mathbf{I} - \mathbf{H})\mathbf{Y} = \mathbf{Y}^\top(\mathbf{I} - \mathbf{H})\mathbf{Y}$$

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where $\frac{1}{n}\mathbf{J} = \mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top$

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where $\frac{1}{n}\mathbf{J} = \mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top$, and we used symmetry and idempotence of $\mathbf{I} - \mathbf{H}$ and $\mathbf{H} - \frac{1}{n}\mathbf{J}$ (Ch 5).

Distribution of SSE

Theorem. $\frac{\text{SSE}}{\sigma^2} \sim \chi^2_{n-p}$

Proof.
$\mathbf{Y} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I})$

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&= n - \mathrm{trace}[(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{X}] = n-p, \\
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&= \frac{1}{\sigma^2}\boldsymbol{\beta}^\top\mathbf{X}^\top[\mathbf{I}-\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top]\mathbf{X}\boldsymbol{\beta}
\end{aligned}
$$

Distribution of SSE

Theorem. $\frac{\text{SSE}}{\sigma^2} \sim \chi^2_{n-p}$

Proof.

$\mathbf{Y} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}) \quad \Rightarrow \quad \sigma^{-1}\mathbf{Y} \sim \mathcal{N}_n(\sigma^{-1}\mathbf{X}\boldsymbol{\beta}, \mathbf{I})$. Hence,

$$
\begin{aligned}
\frac{\text{SSE}}{\sigma^2} &= (\sigma^{-1}\mathbf{Y})^\top(\mathbf{I}-\mathbf{H})(\sigma^{-1}\mathbf{Y}) \sim \chi^2_{\mathrm{rank}[\mathbf{I}-\mathbf{H}]}(\lambda), \\
\text{where} \quad \mathrm{rank}[\mathbf{I}-\mathbf{H}] &= \mathrm{trace}[\mathbf{I}-\mathbf{H}] = \mathrm{trace}[\mathbf{I}-\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top] \\
&= \mathrm{trace}\mathbf{I} - \mathrm{trace}[\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top] \\
&= n - \mathrm{trace}[(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{X}] = n-p, \\
\text{and} \quad \lambda &= (\sigma^{-1}\mathbf{X}\boldsymbol{\beta})^\top(\mathbf{I}-\mathbf{H})(\sigma^{-1}\mathbf{X}\boldsymbol{\beta}) \\
&= \frac{1}{\sigma^2}\boldsymbol{\beta}^\top\mathbf{X}^\top[\mathbf{I}-\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top]\mathbf{X}\boldsymbol{\beta} \\
&= \frac{1}{\sigma^2}\boldsymbol{\beta}^\top[\mathbf{X}^\top\mathbf{X}-\mathbf{X}^\top\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{X}]\boldsymbol{\beta}
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\text{and} \quad \lambda &= (\sigma^{-1}\mathbf{X}\boldsymbol{\beta})^\top(\mathbf{I}-\mathbf{H})(\sigma^{-1}\mathbf{X}\boldsymbol{\beta}) \\
&= \frac{1}{\sigma^2}\boldsymbol{\beta}^\top\mathbf{X}^\top[\mathbf{I}-\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top]\mathbf{X}\boldsymbol{\beta} \\
&= \frac{1}{\sigma^2}\boldsymbol{\beta}^\top[\mathbf{X}^\top\mathbf{X} - \mathbf{X}^\top\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{X}]\boldsymbol{\beta} = 0
\end{aligned}
$$

Some properties of the hat matrix

Lemma. $\mathbf{H1} = \mathbf{1}$

Proof.

$\square$

Remark.

Some properties of the hat matrix

Lemma. $\mathbf{H}\mathbf{1} = \mathbf{1}$

Proof.

$$\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{X} = \mathbf{X}$$

$\square$

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$$\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{X} = \mathbf{X}$$

$$\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top \begin{bmatrix}\mathbf{1} & \tilde{\mathbf{X}}\end{bmatrix} = \begin{bmatrix}\mathbf{1} & \tilde{\mathbf{X}}\end{bmatrix}$$

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\begin{bmatrix}\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{1} & \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\tilde{\mathbf{X}}\end{bmatrix} &= \begin{bmatrix}\mathbf{1} & \tilde{\mathbf{X}}\end{bmatrix}
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□

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\end{aligned}$$

Hence, $\mathbf{H1} = \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{1} = \mathbf{1}$. □

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\begin{bmatrix}\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{1} & \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\tilde{\mathbf{X}}\end{bmatrix} &= \begin{bmatrix}\mathbf{1} & \tilde{\mathbf{X}}\end{bmatrix}
\end{aligned}$$

Hence, $\mathbf{H}\mathbf{1} = \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{1} = \mathbf{1}$. $\square$

Remark. $\mathbf{H}\tilde{\mathbf{X}} = \tilde{\mathbf{X}}$

Some properties of the hat matrix

Lemma. $\mathbf{H}\mathbf{1} = \mathbf{1}$

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\begin{bmatrix}\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{1} & \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\tilde{\mathbf{X}}\end{bmatrix} &= \begin{bmatrix}\mathbf{1} & \tilde{\mathbf{X}}\end{bmatrix}
\end{aligned}$$

Hence, $\mathbf{H}\mathbf{1} = \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{1} = \mathbf{1}$. $\square$

Remark. $\mathbf{H}\tilde{\mathbf{X}} = \tilde{\mathbf{X}}$, $\mathbf{1}^\top\mathbf{H} = \mathbf{1}^\top$

Some properties of the hat matrix

Lemma. $\mathbf{H}\mathbf{1} = \mathbf{1}$

Proof.

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\begin{bmatrix}\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{1} & \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\tilde{\mathbf{X}}\end{bmatrix} &= \begin{bmatrix}\mathbf{1} & \tilde{\mathbf{X}}\end{bmatrix}
\end{aligned}$$

Hence, $\mathbf{H}\mathbf{1} = \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{1} = \mathbf{1}$. $\square$

Remark. $\mathbf{H}\tilde{\mathbf{X}} = \tilde{\mathbf{X}}$, $\mathbf{1}^\top\mathbf{H} = \mathbf{1}^\top$, $\tilde{\mathbf{X}}^\top\mathbf{H} = \tilde{\mathbf{X}}^\top$

Distribution of SSR

Theorem. $\frac{\mathrm{SSR}}{\sigma^2} \sim \chi^2_{p-1}(\lambda)$

Distribution of SSR

Theorem. $\frac{\text{SSR}}{\sigma^2} \sim \chi^2_{p-1}(\lambda)$ with $\lambda = \frac{1}{\sigma^2}\tilde{\boldsymbol{\beta}}^\top \bar{\mathbf{X}}^\top \bar{\mathbf{X}} \tilde{\boldsymbol{\beta}}$,

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$$\tilde{\boldsymbol{\beta}} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{bmatrix}, \quad \bar{\mathbf{X}} = \begin{bmatrix} x_{11} - \bar{x}_1 & x_{12} - \bar{x}_2 & \cdots & x_{1k} - \bar{x}_k \\ x_{21} - \bar{x}_1 & x_{22} - \bar{x}_2 & \cdots & x_{2k} - \bar{x}_k \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} - \bar{x}_1 & x_{n2} - \bar{x}_2 & \cdots & x_{nk} - \bar{x}_k \end{bmatrix}$$

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and $\bar{x}_j := \frac{1}{n}\sum_{i=1}^{n} x_{ij}$ is the average value of the j^{th} predictor.

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Distribution of SSR

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Proof.
The proof is analogous to the case of $k = 1$. Let's focus on λ.

Let $\tilde{\mathbf{X}} = \begin{bmatrix} x_{11} & \cdots & x_{1k} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nk} \end{bmatrix}$, then $\mathbf{X}\boldsymbol{\beta} = \begin{bmatrix} \mathbf{1} & \tilde{\mathbf{X}} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \tilde{\boldsymbol{\beta}} \end{bmatrix} = \beta_0 \mathbf{1} + \tilde{\mathbf{X}}\tilde{\boldsymbol{\beta}}$

Distribution of SSR

Proof (cont'd).

Since $\frac{\mathrm{SSR}}{\sigma^2} = \left(\frac{1}{\sigma}\mathbf{Y}\right)^\top \left(\mathbf{H} - \frac{1}{n}\mathbf{J}\right)\left(\frac{1}{\sigma}\mathbf{Y}\right)$ and $\frac{1}{\sigma}\mathbf{Y} \sim \mathcal{N}_n\left(\frac{1}{\sigma}\mathbf{X}\boldsymbol{\beta}, \mathbf{I}\right)$,

Distribution of SSR

Proof (cont'd).

Since $\frac{\text{SSR}}{\sigma^2} = (\frac{1}{\sigma}\mathbf{Y})^\top \left(\mathbf{H} - \frac{1}{n}\mathbf{J}\right) (\frac{1}{\sigma}\mathbf{Y})$ and $\frac{1}{\sigma}\mathbf{Y} \sim \mathcal{N}_n(\frac{1}{\sigma}\mathbf{X}\boldsymbol{\beta}, \mathbf{I})$,

$$\lambda = \frac{1}{\sigma^2}\boldsymbol{\beta}^\top\mathbf{X}^\top[\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top - \mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top]\mathbf{X}\boldsymbol{\beta}$$

Distribution of SSR

Proof (cont'd).

Since $\frac{\text{SSR}}{\sigma^2} = (\frac{1}{\sigma}\mathbf{Y})^\top \left(\mathbf{H} - \frac{1}{n}\mathbf{J}\right) \left(\frac{1}{\sigma}\mathbf{Y}\right)$ and $\frac{1}{\sigma}\mathbf{Y} \sim \mathcal{N}_n(\frac{1}{\sigma}\mathbf{X}\boldsymbol{\beta}, \mathbf{I})$,

$$\begin{aligned}
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&= \frac{1}{\sigma^2}\begin{bmatrix}\beta_0 & \tilde{\boldsymbol{\beta}}^\top\end{bmatrix}\begin{bmatrix}\mathbf{1}^\top \\ \tilde{\mathbf{X}}^\top\end{bmatrix}[\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top - \mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top]\begin{bmatrix}\mathbf{1} & \tilde{\mathbf{X}}\end{bmatrix}\begin{bmatrix}\beta_0 \\ \tilde{\boldsymbol{\beta}}\end{bmatrix}
\end{aligned}$$

Distribution of SSR

Proof (cont'd).

Since $\frac{\text{SSR}}{\sigma^2} = (\frac{1}{\sigma}\mathbf{Y})^\top \left(\mathbf{H} - \frac{1}{n}\mathbf{J}\right)\left(\frac{1}{\sigma}\mathbf{Y}\right)$ and $\frac{1}{\sigma}\mathbf{Y} \sim \mathcal{N}_n(\frac{1}{\sigma}\mathbf{X}\boldsymbol{\beta}, \mathbf{I})$,

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&= \frac{1}{\sigma^2}\begin{bmatrix}\beta_0 & \tilde{\boldsymbol{\beta}}^\top\end{bmatrix}\begin{bmatrix}\mathbf{1}^\top\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top - \mathbf{1}^\top\mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top \\ \tilde{\mathbf{X}}^\top\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top - \tilde{\mathbf{X}}^\top\mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top\end{bmatrix}\begin{bmatrix}\mathbf{1} & \tilde{\mathbf{X}}\end{bmatrix}\begin{bmatrix}\beta_0 \\ \tilde{\boldsymbol{\beta}}\end{bmatrix}
\end{aligned}$$

Distribution of SSR

Proof (cont'd).

Since $\frac{\text{SSR}}{\sigma^2} = (\frac{1}{\sigma}\mathbf{Y})^\top \left(\mathbf{H} - \frac{1}{n}\mathbf{J}\right) \left(\frac{1}{\sigma}\mathbf{Y}\right)$ and $\frac{1}{\sigma}\mathbf{Y} \sim \mathcal{N}_n(\frac{1}{\sigma}\mathbf{X}\boldsymbol{\beta}, \mathbf{I})$,

$$
\begin{aligned}
\lambda &= \frac{1}{\sigma^2}\boldsymbol{\beta}^\top\mathbf{X}^\top[\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top - \mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top]\mathbf{X}\boldsymbol{\beta} \\
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&= \frac{1}{\sigma^2}\begin{bmatrix}\beta_0 & \tilde{\boldsymbol{\beta}}^\top\end{bmatrix}\begin{bmatrix}\mathbf{1}^\top\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top - \mathbf{1}^\top\mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top \\ \tilde{\mathbf{X}}^\top\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top - \tilde{\mathbf{X}}^\top\mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top\end{bmatrix}\begin{bmatrix}\mathbf{1} & \tilde{\mathbf{X}}\end{bmatrix}\begin{bmatrix}\beta_0 \\ \tilde{\boldsymbol{\beta}}\end{bmatrix} \\
&= \frac{1}{\sigma^2}\begin{bmatrix}\beta_0 & \tilde{\boldsymbol{\beta}}^\top\end{bmatrix}\begin{bmatrix}\mathbf{1}^\top - \mathbf{1}^\top \\ \tilde{\mathbf{X}}^\top - \tilde{\mathbf{X}}^\top\mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top\end{bmatrix}\begin{bmatrix}\mathbf{1} & \tilde{\mathbf{X}}\end{bmatrix}\begin{bmatrix}\beta_0 \\ \tilde{\boldsymbol{\beta}}\end{bmatrix}
\end{aligned}
$$

Distribution of SSR

Proof (cont'd).

Since $\frac{\text{SSR}}{\sigma^2} = \left(\frac{1}{\sigma}\mathbf{Y}\right)^\top \left(\mathbf{H} - \frac{1}{n}\mathbf{J}\right)\left(\frac{1}{\sigma}\mathbf{Y}\right)$ and $\frac{1}{\sigma}\mathbf{Y} \sim \mathcal{N}_n\left(\frac{1}{\sigma}\mathbf{X}\boldsymbol{\beta}, \mathbf{I}\right)$,

$$
\begin{aligned}
\lambda &= \frac{1}{\sigma^2}\boldsymbol{\beta}^\top\mathbf{X}^\top[\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top - \mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top]\mathbf{X}\boldsymbol{\beta} \\
&= \frac{1}{\sigma^2}\begin{bmatrix}\beta_0 & \tilde{\boldsymbol{\beta}}^\top\end{bmatrix}\begin{bmatrix}\mathbf{1}^\top \\ \tilde{\mathbf{X}}^\top\end{bmatrix}[\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top - \mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top]\begin{bmatrix}\mathbf{1} & \tilde{\mathbf{X}}\end{bmatrix}\begin{bmatrix}\beta_0 \\ \tilde{\boldsymbol{\beta}}\end{bmatrix} \\
&= \frac{1}{\sigma^2}\begin{bmatrix}\beta_0 & \tilde{\boldsymbol{\beta}}^\top\end{bmatrix}\begin{bmatrix}\mathbf{1}^\top\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top - \mathbf{1}^\top\mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top \\ \tilde{\mathbf{X}}^\top\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top - \tilde{\mathbf{X}}^\top\mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top\end{bmatrix}\begin{bmatrix}\mathbf{1} & \tilde{\mathbf{X}}\end{bmatrix}\begin{bmatrix}\beta_0 \\ \tilde{\boldsymbol{\beta}}\end{bmatrix} \\
&= \frac{1}{\sigma^2}\begin{bmatrix}\beta_0 & \tilde{\boldsymbol{\beta}}^\top\end{bmatrix}\begin{bmatrix}\mathbf{1}^\top - \mathbf{1}^\top \\ \tilde{\mathbf{X}}^\top - \tilde{\mathbf{X}}^\top\mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top\end{bmatrix}\begin{bmatrix}\mathbf{1} & \tilde{\mathbf{X}}\end{bmatrix}\begin{bmatrix}\beta_0 \\ \tilde{\boldsymbol{\beta}}\end{bmatrix} \\
&= \frac{1}{\sigma^2}\begin{bmatrix}\beta_0 & \tilde{\boldsymbol{\beta}}^\top\end{bmatrix}\begin{bmatrix}0 & \mathbf{0}^\top \\ \mathbf{0} & \tilde{\mathbf{X}}^\top\tilde{\mathbf{X}} - \tilde{\mathbf{X}}^\top\mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top\tilde{\mathbf{X}}\end{bmatrix}\begin{bmatrix}\beta_0 \\ \tilde{\boldsymbol{\beta}}\end{bmatrix}
\end{aligned}
$$

Distribution of SSR

Proof (cont'd).

Since $\frac{\text{SSR}}{\sigma^2} = (\frac{1}{\sigma}\mathbf{Y})^\top \left(\mathbf{H} - \frac{1}{n}\mathbf{J}\right)\left(\frac{1}{\sigma}\mathbf{Y}\right)$ and $\frac{1}{\sigma}\mathbf{Y} \sim \mathcal{N}_n(\frac{1}{\sigma}\mathbf{X}\boldsymbol{\beta}, \mathbf{I})$,

$$\begin{aligned}
\lambda &= \frac{1}{\sigma^2}\boldsymbol{\beta}^\top\mathbf{X}^\top[\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top - \mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top]\mathbf{X}\boldsymbol{\beta} \\
&= \frac{1}{\sigma^2}\begin{bmatrix}\beta_0 & \tilde{\boldsymbol{\beta}}^\top\end{bmatrix}\begin{bmatrix}\mathbf{1}^\top \\ \tilde{\mathbf{X}}^\top\end{bmatrix}[\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top - \mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top]\begin{bmatrix}\mathbf{1} & \tilde{\mathbf{X}}\end{bmatrix}\begin{bmatrix}\beta_0 \\ \tilde{\boldsymbol{\beta}}\end{bmatrix} \\
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&= \frac{1}{\sigma^2}\tilde{\boldsymbol{\beta}}^\top[\tilde{\mathbf{X}}^\top\tilde{\mathbf{X}} - \tilde{\mathbf{X}}^\top\mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top\tilde{\mathbf{X}}]\tilde{\boldsymbol{\beta}}
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Distribution of SSR

Proof (cont'd).

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\end{aligned}
$$

Means of SSE and SSR

Theorem. Let $\mathsf{MSE} := \frac{\mathsf{SSE}}{n-p}$, then $\mathrm{E}[\mathsf{MSE}] = \sigma^2$.

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Error	$\mathrm{SSE} = \sum(Y_i - \hat{Y})^2$	$n-p$	$\frac{\mathrm{SSE}}{n-p}$	σ^2
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Here, QF stands for "quadratic form" and is given by

$$\mathrm{QF} = \frac{1}{p-1}\tilde{\boldsymbol{\beta}}^\top \bar{\mathbf{X}}^\top \bar{\mathbf{X}}\tilde{\boldsymbol{\beta}} = \frac{1}{k}\sum_{j=1}^{k}\sum_{s=1}^{k}\beta_j\beta_s\sum_{i=1}^{n}(x_{ij}-\bar{x}_j)(x_{is}-\bar{x}_s) \geq 0.$$

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Note that $\mathrm{QF} = 0 \quad \Leftrightarrow \quad \beta_1 = \beta_2 = \ldots = \beta_k = 0.$

Overall F-test for a regression relationship (p. 226)

In multiple regression, our F-test based on $F^* = \frac{\text{MSR}}{\text{MSE}}$ tests whether the *entire set* of predictors $x_1, \ldots, x_k$ explains a significant amount of variation in Y.

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If $F^* > F_{p-1,n-p,1-\alpha}$, we reject H_0 and conclude that *something* is going on, there is *some* relationship between one or more of the $x_1, \ldots, x_k$ and Y. R provides a p-value for this test.

R^2 in case of multiple regression

The **coefficient of multiple determination** is

$$R^2 = \frac{\mathrm{SSR}}{\mathrm{SST}} = 1 - \frac{\mathrm{SSE}}{\mathrm{SST}}.$$

measures the proportion of sample variation in Y explained by its *linear* relationship with the predictors $x_1, \dots, x_k$. As before, $0 \leq R^2 \leq 1$.

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The **adjusted** R^2

$$R_a^2 = 1 - \frac{\mathrm{SSE}/(n-p)}{\mathrm{SST}/(n-1)}$$

accounts for the number of predictors in the model. It may decrease when we add useless predictors to the model.

Dwayne Studio Regression Output

https://github.com/zh3nis/MATH440/blob/main/chp06/mlr.R

```
> m = lm(sales ~ children + income, data=dwayne)
> summary(m)
...
Multiple R-squared:  0.9167,Adjusted R-squared:  0.9075
F-statistic:  99.1 on 2 and 18 DF,  p-value: 1.921e-10

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Analysis of Variance Table

Response: sales
          Df  Sum Sq Mean Sq  F value    Pr(>F)
children   1 23371.8 23371.8 192.8962 4.64e-11 ***
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Conclusions?

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Conclusions?
We reject $H_0 : \beta_1 = \beta_2 = 0$ at any reasonable significance level α.
About 92% of the total variability in the data is explained by the linear regression model.

Inference about individual regression parameters

The overall F-test concerns the *entire set* of predictors $x_1, \ldots, x_k$.

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We will talk about this shortly, but the main methods are forward selection, backwards elimination, stepwise procedures, C_p, R_a^2, LASSO, etc.

Mean and covariance matrix of a vector

Recall: If $\mathbf{Y}$ is a random vector, then its expected value is also a vector

$$\mathrm{E}[\mathbf{Y}] = \begin{bmatrix} \mathrm{E}[Y_1] \\ \mathrm{E}[Y_2] \\ \vdots \\ \mathrm{E}[Y_n] \end{bmatrix}$$

The random vector $\mathbf{Y}$ also has a covariance matrix

$$\mathrm{Cov}[\mathbf{Y}] = \begin{bmatrix} \mathrm{Cov}[Y_1, Y_1] & \mathrm{Cov}[Y_1, Y_2] & \cdots & \mathrm{Cov}[Y_1, Y_n] \\ \mathrm{Cov}[Y_2, Y_1] & \mathrm{Cov}[Y_2, Y_2] & \cdots & \mathrm{Cov}[Y_2, Y_n] \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}[Y_n, Y_1] & \mathrm{Cov}[Y_n, Y_2] & \cdots & \mathrm{Cov}[Y_n, Y_n]. \end{bmatrix}$$

Multivariate normal

The **multivariate normal** density is given by

$$f(\mathbf{y}) = |2\pi\boldsymbol{\Sigma}|^{-1/2} \exp\{-0.5(\mathbf{y}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu})\},$$

where $\mathbf{y} \in \mathbb{R}^n$. We write

$$\mathbf{Y} \sim \mathcal{N}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}).$$

Then $\mathrm{E}[\mathbf{Y}] = \boldsymbol{\mu}$ and $\mathrm{Cov}[\mathbf{Y}] = \boldsymbol{\Sigma}$.

For the general linear model,

$$\mathbf{Y} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}_n).$$

Error vector

Note that along the diagonal of $\mathrm{Cov}[\mathbf{Y}]$, $\mathrm{Cov}[Y_i, Y_i] = \mathrm{Var}[Y_i]$.

For the general linear model,

$$\mathrm{E}[\boldsymbol{\epsilon}] = \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$\mathrm{Cov}[\boldsymbol{\epsilon}] = \sigma^2 \mathbf{I}_n = \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix}.$$

Inference for $\boldsymbol{\beta}$

$$\mathrm{Cov}[\mathbf{Y}] = \mathrm{Cov}[\underbrace{\mathbf{X}\boldsymbol{\beta}}_{\text{fixed}} + \underbrace{\boldsymbol{\epsilon}}_{\text{random}}] = \mathrm{Cov}[\boldsymbol{\epsilon}]$$

Fact: If $\mathbf{A}$ is a constant matrix, $\mathbf{a}$ is a constant vector, and $\mathbf{Y}$ is any random vector, then

$$\mathrm{E}[\mathbf{AY} + \mathbf{a}] = \mathbf{A}\mathrm{E}[\mathbf{Y}] + \mathbf{a},$$

$$\mathrm{Cov}[\mathbf{AY} + \mathbf{a}] = \mathbf{A}\mathrm{Cov}[\mathbf{Y}]\mathbf{A}^\top.$$

Back to the general linear model

For $\hat{\mathbf{Y}} = \mathbf{HY}$,

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For $\mathbf{e} = (\mathbf{I}_n - \mathbf{H})\mathbf{Y}$,

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(Why?)

Mean and variance of $\mathbf{b}$

Finally, $\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$ is *unbiased*

$$\mathrm{E}[\mathbf{b}] = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathrm{E}[\mathbf{Y}] = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}\boldsymbol{\beta} = \boldsymbol{\beta},$$

and has covariance matrix

$$\begin{aligned}\mathrm{Cov}[\mathbf{b}] &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathrm{Cov}[\mathbf{Y}][(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']' \\ &= \sigma^2(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']' \\ &= \sigma^2(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\ &= \sigma^2(\mathbf{X}'\mathbf{X})^{-1}.\end{aligned}$$

Hence,

$$\mathbf{b} \sim \mathcal{N}_p(\boldsymbol{\beta}, \sigma^2(\mathbf{X}'\mathbf{X})^{-1}).$$

Table of regression effects

From the previous slide, the jth estimated coefficient β_j,

$$\mathrm{Var}[b_j] = \sigma^2 c_{jj},$$

where c_{jj} is the jth diagonal element of $(\mathbf{X}'\mathbf{X})^{-1}$.

As usually, we *estimate* the standard deviation of b_j by $\mathrm{s}[b_j] = \sqrt{\mathrm{MSE} \cdot c_{jj}}$ yielding

$$\frac{b_j - \beta_j}{\mathrm{s}[b_j]} \sim t_{n-p}$$

Note: R gives each $\mathrm{s}[b_j]$ as well as b_j, $t_j^* = b_j/\mathrm{s}[b_j]$, and a p-value for testing each $\mathrm{H}_0 : \beta_j = 0$.

Dwayne output

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -68.8571    60.0170  -1.147   0.2663
children      1.4546     0.2118   6.868    2e-06 ***
income        9.3655     4.0640   2.305   0.0333 *
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 11.01 on 18 degrees of freedom
Multiple R-squared:  0.9167,Adjusted R-squared:  0.9075
F-statistic:  99.1 on 2 and 18 DF,  p-value: 1.921e-10
```

We reject $\mathrm{H}_0 : \beta_1 = 0$ at the $\alpha = 0.01$ level and $\mathrm{H}_0 : \beta_2 = 0$ at the $\alpha = 0.05$ level.

Individual tests of $\mathrm{H}_0 : \beta_j = 0$

Note: A test of $\mathrm{H}_0 : \beta_j = 0$ versus $\mathrm{H}_a : \beta_j = 0$ – available in the table of regression coefficients – is a test of whether predictor x_j is necessary in a model *with the other remaining predictors included*.

For the Dwayne Studio Data:

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- The R summary gives us $F^* = \mathrm{MSR}/\mathrm{MSE} = 99.10$ with associated p-value < 0.0001 (it is actually 2×10^{-10}!). We strongly reject $\mathrm{H}_0 : \beta_1 = \beta_2 = 0$.

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- For example, we are 95% confident that mean sales increases by \$1010 to \$1900 for every 1000 increase in kids 16 and under, holding income constant.
- For $\mathrm{H}_0 : \beta_1 = 0$ we get $p < 0.0001$; for $\mathrm{H}_0 : \beta_2 = 0$ we get $p = 0.03$. Are people under 16 (x_1) and income (x_2) important in the model?

CI for mean response and PI for new response

Let's construct a CI for the mean response corresponding to a set of values

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We want to make inferences about

$$\mathrm{E}[Y_{\text{new}}] = \mathbf{x}_{\text{new}}^{\top} \boldsymbol{\beta} = \beta_0 + \beta_1 x_{\text{new},1} + \ldots + \beta_k x_{\text{new},k}.$$

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- A point estimate is $\hat{Y}_{\text{new}} = \widehat{\mathrm{E}[Y_{\text{new}}]} = \mathbf{x}_{\text{new}}\mathbf{b}$.

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- An *approximate* $100 \cdot (1-\alpha)\%$ prediction interval for a new response $Y_{\text{new}} = \mathbf{x}'_{\text{new}}\boldsymbol{\beta} + \boldsymbol{\epsilon}_{\text{new}}$ is

$$\hat{Y}_{\text{new}} \pm t_{n-p,1-\alpha/2}\sqrt{\mathrm{MSE} \cdot [1 + \mathbf{x}'_{\text{new}}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}_{\text{new}}]},$$

Dwayne Studios

https://github.com/zh3nis/MATH440/blob/main/chp06/dwayne_ci_pi.R

Say we want to estimate mean sales in cities with $x_1 = 65.4$ thousand children and per capita disposable income of $x_2 = 17.6$ thousand dollars.

```
> new_x = data.frame(children=65.4, income=17.6)
> predict.lm(m, new_x, interval="confidence", level=0.95)
       fit      lwr      upr
1 191.1039 185.2911 196.9168
> predict.lm(m, new_x, interval="prediction", level=0.95)
       fit      lwr     upr
1 191.1039 167.2589 214.949
```

Checking model assumptions

The general linear model assumes the following:

1. A linear relationship between $\mathrm{E}[Y]$ and associated predictors $x_1, \ldots, x_k$.
2. The errors have constant variance.
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Linearity b/w mean response and predictors

- Scatterplots of $\{(x_{ij}, Y_i)\}_{i=1}^{n}$ for each predictor $j = 1, \ldots, k$. Look for "nonlinear" patterns. These are *marginal* relationships, and do not get at the simultaneous relationship among variables.

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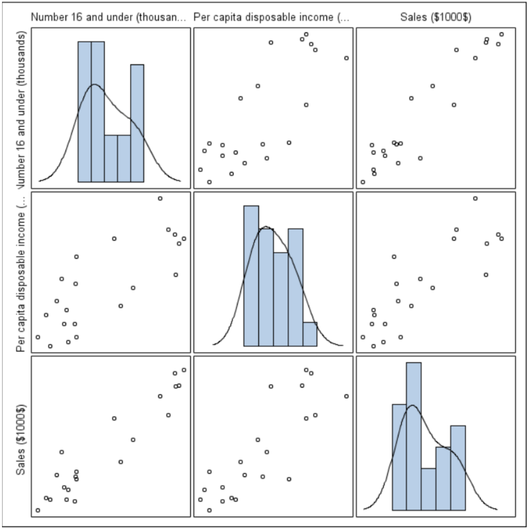
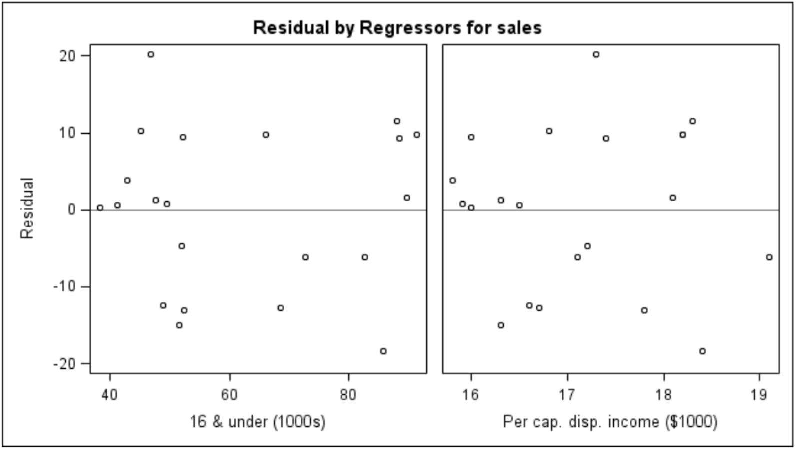
- Look for non-random (especially curved) pattern in the residual plots, indicating violation of linear mean.
- Remedies: (i) choose different functional form of model, (ii) transformation of one or more predictor variables.

Scatterplot matrix



Residuals vs predictors



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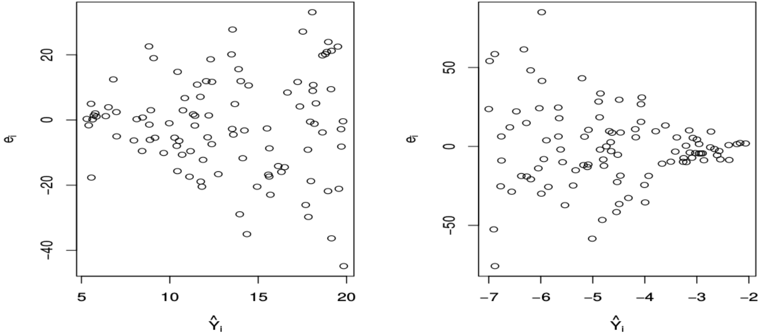

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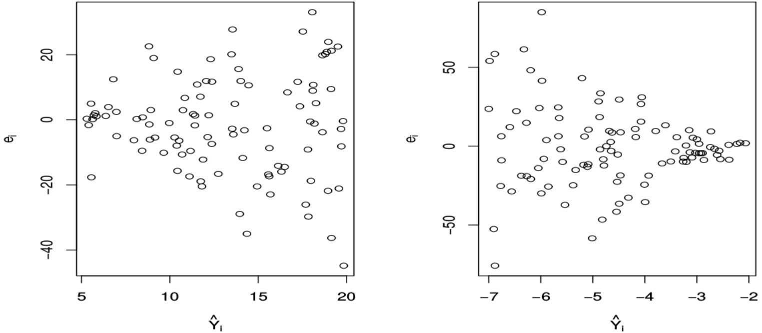

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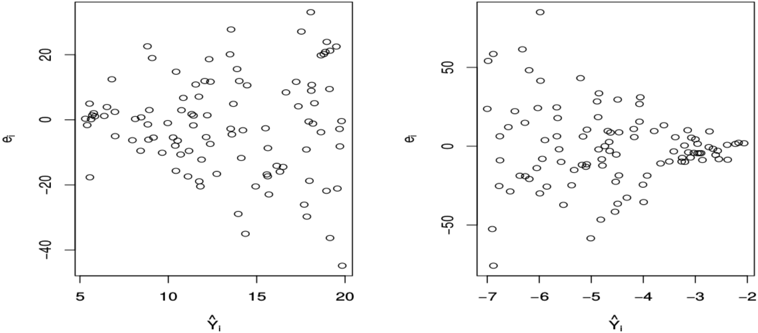

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- *A more advanced method: weighted least squares (Chapter 11).

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- Graphical methods have advantage of checking for *general violations*, not just violation of a specific type.

Breusch Pagan test in R

```
> library(lmtest)
> dwayne = read.table("path/to/CH06FI05.txt", header=FALSE)
> colnames(dwayne) = c("children", "income", "sales")
> m = lm(sales ~ children + income, data=dwayne)
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With p-value = .3774 we do not reject $\mathrm{H}_0 : \sigma_i = \sigma$ at $\alpha = 0.05$, no evidence of non-constant variance.

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Diagnostics include...

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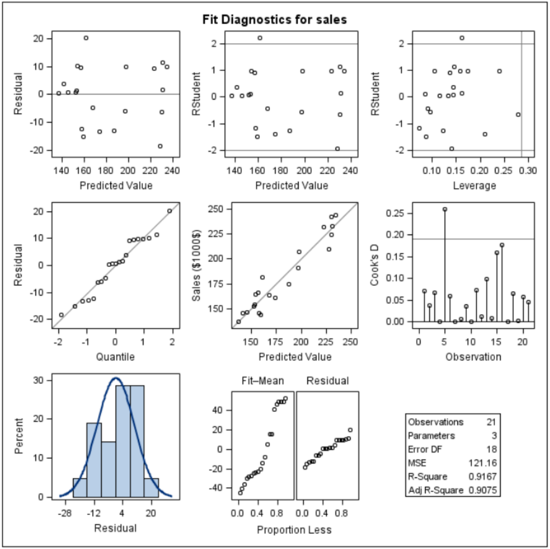
- Remedy: transformation of Y or any of $x_1, \ldots, x_k$, *nonparametric methods (e.g. additive models), *robust regression (least sum of absolute distances), *median regression.

Standard diagnostics



Test for normal residuals in Portrait data

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                    Tests for Normality

Test                  --Statistic---    -----p Value------
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Cramer-von Mises      W-Sq   0.066901    Pr > W-Sq    >0.2500
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```

We do not reject $\mathrm{H}_0 : \epsilon_1, \ldots, \epsilon_n$ are normally distributed

The Anderson-Darling test looks primarily for evidence of non-normal data in the tails of a distribution; the Shapiro-Wilk emphasizes lack of symmetry in the distribution; i.e. less emphasis placed on the tails.

Comments

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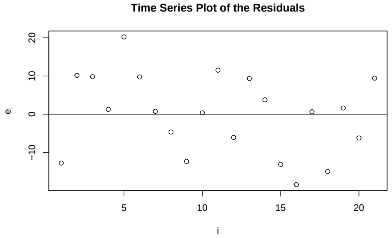
- The formal test will not tell you the *type* of departure from normality (e.g. bimodal, skew, heavy or light tails, etc.).
- Q-Q plots help answer these questions (if the mean is specified correctly).

Independence

As in the case of SLR, plot of e_i vs i:



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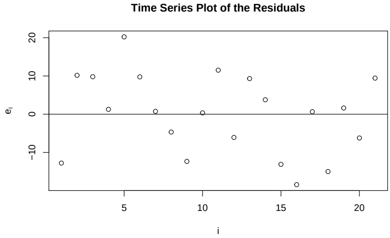

and Runs test of independence:

```
> res = m$residuals
> library(lawstat)
> runs.test(res)

Runs Test - Two sided

data:  res
Standardized Runs Statistic = -0.66258, p-value = 0.5076
```

Transformations of variables

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- A nonlinear relationship *could* show itself in the scatterplot matrix of Y_i versus x_{ij} for $j = 1, \ldots, k$, or the residuals e_i versus x_{ij}.
- The chosen transformation should roughly mimic the relationship seen in the plot.

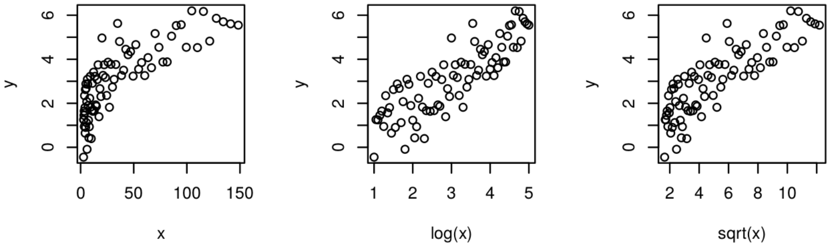
Transformations for $x_{i1}, \ldots, x_{ik}$

Examples of transformations for predictors are:

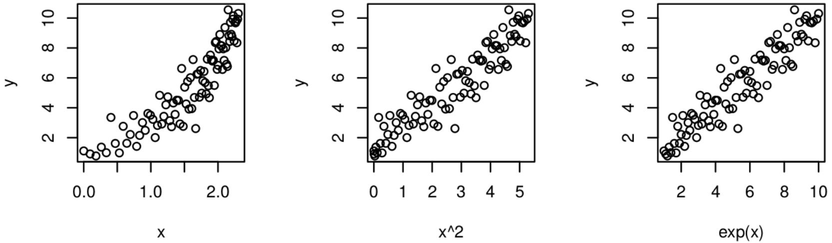
- $x^* = \log(x)$
- $x^* = \sqrt{x}$
- $x^* = 1/x$
- $x^* = \exp(x)$ or $x^* = \exp(-x)$

We will examine *marginal* relationships and transformation "fixes".

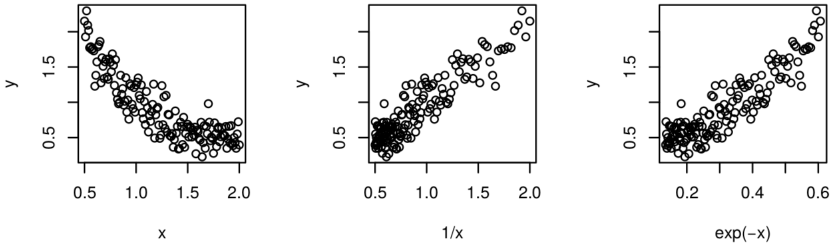
Example 1: transforming a predictor



Example 2: transforming a predictor



Example 3: transforming a predictor



Transforming a response

If there is evidence of nonconstant error variance, a transformation of Y can often fix things. Examples include:

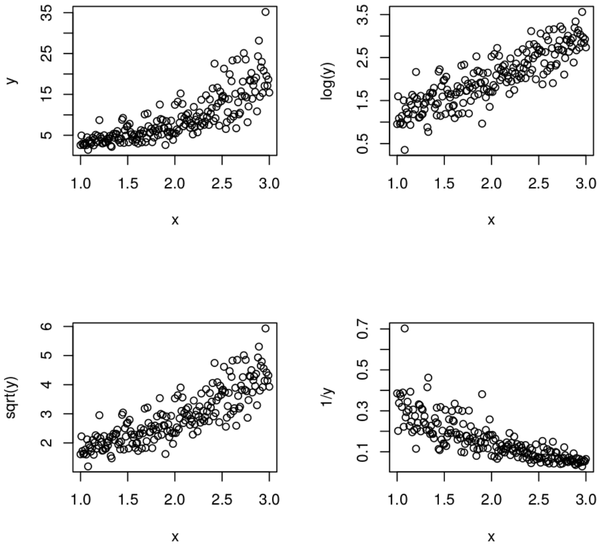
- $Y^* = \log(Y)$
- $Y^* = \sqrt{Y}$
- $Y^* = 1/Y$

See Figure 3.15, page 132.

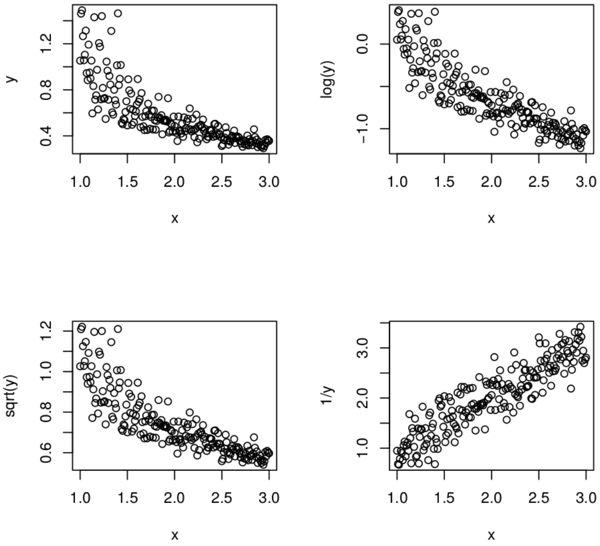
All of these are included in the Box-Cox family of transformations.

For some data, a transformation in Y may be followed by one or more transformations in the $x_{i1}, \ldots, x_{ik}$.

Example 4: transforming the response



Example 5: transforming the response



Box-Cox transformations

Box-Cox transformations are of the type

$$Y^* = \frac{Y^\lambda - 1}{\lambda}$$

where λ is estimated from the data, typically $-3 \leq \lambda \leq 3$. These include

$$\begin{aligned}
\lambda = 2 & \qquad Y^* = (Y^2 - 1)/2 \sim Y^2 \\
\lambda = 1 & \qquad Y^* = Y - 1 \sim Y \\
\lambda = 0 & \qquad Y^* = \log(Y) \\
\lambda = -1 & \qquad Y^* = 1 - 1/Y \sim 1/Y \\
\lambda = -2 & \qquad Y^* = 1/2 - 1/(2Y^2) \sim 1/Y^2
\end{aligned}$$

R will help you to pick λ automatically.