Remedial Measures

Zhenisbek Assylbekov

Department of Mathematics

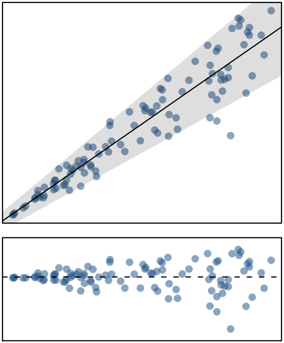
Regression Analysis

Weighted Least Squares

Ridge Regression

Robust Regression

Non-homogeneous variance



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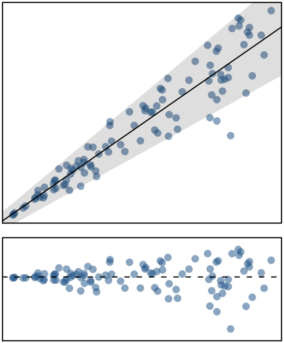

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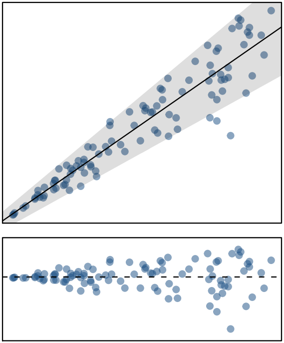

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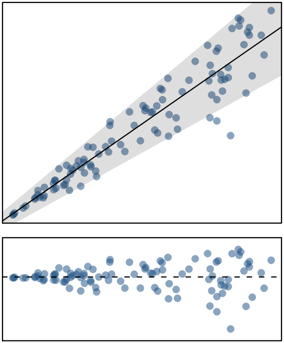

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- Model is as before

$$Y_i = \beta_0 0 + \beta_1 \cdot x_{i1} + \ldots + \beta_k \cdot x_{ik} + \epsilon_i,$$

- Except that

$$\epsilon_i \overset{\text{ind}}{\sim} \mathcal{N}(0, \sigma_i^2)$$

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- Or in matrix notation:

$$\min_{\boldsymbol{\beta}} (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})^\top \boldsymbol{\Omega} (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})$$

where $\boldsymbol{\Omega} = \mathrm{diag}[\omega_1, \ldots, \omega_n] \in \mathbb{R}^{n \times n}$.

Solving WLS

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- Or, e.g., if $|\epsilon_i|$ increases linearly w.r.t. x_{i4} only, then we'll fit $|e_i| = \alpha_0 + \alpha_1 \cdot x_{i4} + \delta_i$

Putting it all together

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3. Let $\widehat{|e_i|}$ be the fitted values for the regression in 2.
4. Define $\omega_i = 1/\widehat{|e_i|}^2$ and feed them into `lm` command using the `weights` parameter.

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We are interested in studying the relationship between diastolic blood pressure and age among healthy adult women 20 to 60 years old.

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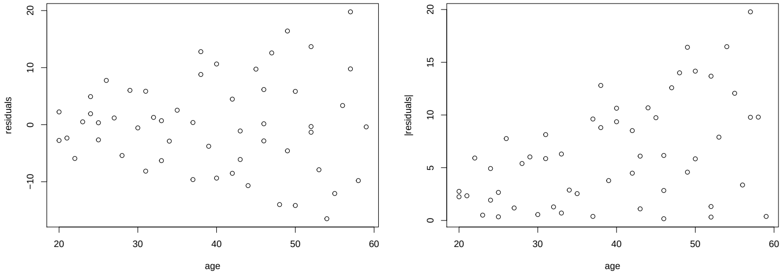
Fitting an OLS $\text{dbp}_i = \beta_0 + \beta_1 \cdot \text{age}_i + \epsilon_i$ gives:



OLS vs WLS

https://github.com/zh3nis/MATH440/blob/main/chp11/dbp.R

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> summary(ols)

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 56.15693    3.99367  14.061  < 2e-16 ***
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Coefficients:
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(Intercept) 55.56577    2.52092  22.042  < 2e-16 ***
age          0.59634    0.07924   7.526 7.19e-10 ***
---
Residual standard error: 1.213 on 52 degrees of freedom
Multiple R-squared:  0.5214,Adjusted R-squared:  0.5122
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- R^2 is no longer interpreted the same way in terms of amount of total variability explained by model.
- In WLS, standard inferences about coefficients may not be valid for small sample sizes when weights are estimated from the data.
- If MSE of the WLS regression is near 1, then our estimation of the σ_i function is okay. Here it's 1.21.

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Exercise: Show that for $\mathbf{X} \in \mathbb{R}^{n \times d}$ with $n \geq d$

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This is bad for OLS, because $\mathbf{X}^\top \mathbf{X}$ will not be invertible.

Simple solution: add penalty term into the cost function:

$$Q(\boldsymbol{\beta}) = \|\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}\|^2 + \lambda \|\boldsymbol{\beta}\|^2,$$

where λ is a *hyperparameter*, to be chosen through a criterion like PRESS or training/validation approach.

Solving Ridge Regression

Theorem. The function $Q(\boldsymbol{\beta}) = \|\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}\|^2 + \lambda\|\boldsymbol{\beta}\|^2$ reaches its minimum at

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Proof.

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$\square$

Solving Ridge Regression

Theorem. The function $Q(\boldsymbol{\beta}) = \|\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}\|^2 + \lambda\|\boldsymbol{\beta}\|^2$ reaches its minimum at

$$\hat{\boldsymbol{\beta}}_{\mathrm{R}} = (\mathbf{X}^\top\mathbf{X} + \lambda\mathbf{I})^{-1}\mathbf{X}^\top\mathbf{Y}$$

Proof.

$$\begin{aligned}
Q(\boldsymbol{\beta}) &= (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})^\top(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}) + \lambda\boldsymbol{\beta}^\top\boldsymbol{\beta} \\
&= \mathbf{Y}^\top\mathbf{Y} - \boldsymbol{\beta}^\top\mathbf{X}^\top\mathbf{Y} - \mathbf{Y}^\top\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\beta}^\top\mathbf{X}^\top\mathbf{X}\boldsymbol{\beta} + \lambda\boldsymbol{\beta}^\top\mathbf{I}\boldsymbol{\beta} \\
\nabla_{\boldsymbol{\beta}} Q &= -\mathbf{X}^\top\mathbf{Y} - \mathbf{X}^\top\mathbf{Y} + 2\mathbf{X}^\top\mathbf{X}\boldsymbol{\beta} + 2\lambda\mathbf{I}\boldsymbol{\beta} \\
&= 2(\mathbf{X}^\top\mathbf{X} + \lambda\mathbf{I})\boldsymbol{\beta} - 2\mathbf{X}^\top\mathbf{Y} = \mathbf{0}
\end{aligned}$$

$$(\mathbf{X}^\top\mathbf{X} + \lambda\mathbf{I})\boldsymbol{\beta} = \mathbf{X}^\top\mathbf{Y} \quad \Rightarrow \quad \boxed{\hat{\boldsymbol{\beta}}_{\mathrm{R}} = (\mathbf{X}^\top\mathbf{X} + \lambda\mathbf{I})^{-1}\mathbf{X}^\top\mathbf{Y}}$$

$\square$

Exercise. Show that $(\mathbf{X}^\top\mathbf{X} + \lambda\mathbf{I})$ is *always* invertible for $\lambda > 0$.

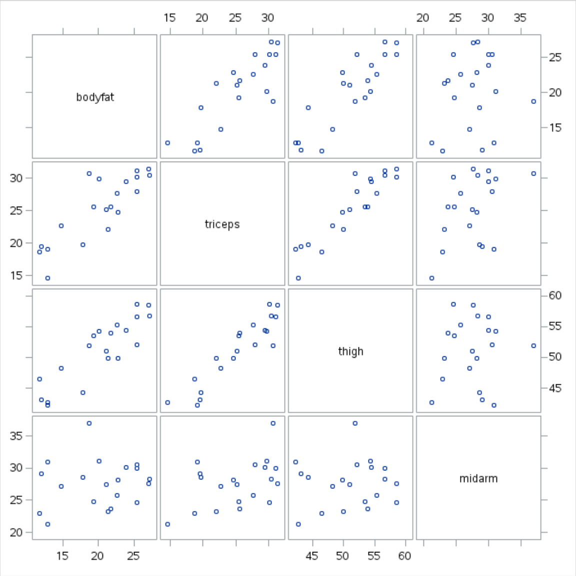
Chapter 7 example: Body fat

$n = 20$ healthy females 25–34 years old.

- x_1 = triceps skinfold thickness (mm)
- x_2 = thigh circumference (cm)
- x_3 = midarm circumference (cm)
- Y = body fat (%)

Obtaining Y_i, the percent of the body that is purly fat, requires immersing a person in water. Want to develop model based on simple body measurements that avoids people getting wet.

Scatterplot



Correlation coefficients

Pearson Correlation Coefficients, N = 20 Prob > |r| under H0: Rho=0			
	triceps	**thigh**	**midarm**
triceps	1.00000	0.92384 <.0001	0.45778 0.0424
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There is high correlation among the predictors. For example $r = 0.92$ for triceps and thigh. These two variables are *essentially carrying the same information*. Maybe only one or the other is really needed.

Effects of multicolinearity

```
lm(formula = bodyfat ~ triceps + thigh + midarm,
   data = bodyfat_data)

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  117.085     99.782   1.173    0.258
triceps        4.334      3.016   1.437    0.170
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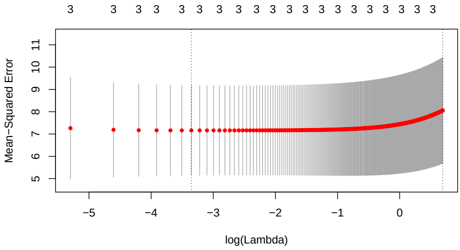
- Holding midarm and triceps constant, increasing the thigh circumference *decreases* bodyfat.
- This may not make sense!

OLS vs Ridge on bodyfat data

https://github.com/zh3nis/MATH440/blob/main/chp11/ridge.R




```
> mean(ols$residuals^2)
[1] 4.920244

> mean((y-ridge_yhat)^2)
[1] 5.340952
```

```
> coef(ols)
(Intercept)     triceps       thigh      midarm
 117.084695    4.334092   -2.856848   -2.186060

> t(coef(ridge))
   (Intercept)   triceps     thigh     midarm
s0   0.6555753 0.8074414 0.1588796 -0.3266744
```

Weighted Least Squares

Ridge Regression

Robust Regression

Back to outliers

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- Outliers are often interesting in their own right and can help in building a better model.
- **Robust regression** weakens the effect of outlying cases on estimation to provide a better fit to the majority of cases.
- Useful in situations when there's no time for "influence diagnostics" or a more careful analysis.

M-estimation

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Iteratively reweighted least squares (IRLS)

Outlying residuals are (iteratively) given less weight in the estimation process.

Algorithm 1: IRLS

1 **Input**: $\{(\mathbf{x}_i, y_i)\}_{i=1}^n$

2 Fit OLS. Let $\mathbf{e} = \mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}_{\text{OLS}}$

3 Initialize weights: $\omega_i \leftarrow \frac{1}{e_i^2}$, $\boldsymbol{\Omega} = \text{diag}[\omega_1, \ldots, \omega_n]$

4 Fit WLS: $\hat{\boldsymbol{\beta}}_{\text{WLS}} \leftarrow (\mathbf{X}^\top \boldsymbol{\Omega} \mathbf{X})^{-1} \mathbf{X}^\top \boldsymbol{\Omega} \mathbf{Y}$

5 Estimate: $\hat{\sigma} \leftarrow \text{median}_i \left\{ \frac{|y_i - \mathbf{x}_i^\top \hat{\boldsymbol{\beta}}_{\text{WLS}}|}{\Phi^{-1}(0.75)} \right\}$

6 Update weights: $\omega_i \leftarrow w\left(\frac{y_i - \mathbf{x}_i^\top \hat{\boldsymbol{\beta}}_{\text{WLS}}}{\hat{\sigma}}\right)$, where

$$w(u) = \begin{cases} 1, & |u| < 1.345 \\ \frac{1.345}{|u|}, & |u| > 1.345 \end{cases}$$

7 Repeat steps 4–6 until $\hat{\sigma}$ and $\hat{\boldsymbol{\beta}}_{\text{WLS}}$ stabilize.

8 **Output**: $\hat{\boldsymbol{\beta}}_{\text{WLS}}$

IRLS example

https://github.com/zh3nis/MATH440/blob/main/chp11/irls.R

```
require(foreign)
require(MASS)

cdata <- read.dta(
    "https://stats.idre.ucla.edu/stat/data/crime.dta")

plot(crime ~ poverty, data=cdata)
ols <- lm(crime ~ poverty, data = cdata)
abline(ols)

irls <- rlm(crime ~ poverty, data=cdata)
abline(irls, col='blue')

summary(ols)
summary(irls)
```

IRLS example