Inferences in Regression Analysis

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Regression Analysis

Normal Errors Regression

Throughout this chapter we assume

$$Y_i = \beta_0 + \beta_1 x_i + \epsilon_i,$$

$$\epsilon_1, \ldots, \epsilon_n \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma^2).$$

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$$\epsilon_1, \ldots, \epsilon_n \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma^2).$$

This is equivalent to

$$Y_i \overset{\text{ind}}{\sim} \mathcal{N}(\beta_0 + \beta_1 x_i, \sigma^2)$$

Inferences on β_1 and β_0

Inferences on $\mathrm{E}[Y]$ and $\hat{Y}$

Analysis of Variance Approach

Coefficient of Determination

Unbiasedness of b_1

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Expected value of the numerator is

$$\mathrm{E}\left[\sum(x_i - \bar{x})Y_i\right] = \sum(x_i - \bar{x})\mathrm{E}[Y_i] = \sum(x_i - \bar{x})(\beta_0 + \beta_1 x_i)$$
$$= \beta_0 \sum x_i - n\bar{x}\beta_0 + \beta_1 \sum x_i^2 - n\bar{x}^2\beta_1 = \beta_1 \left(\sum x_i^2 - n\bar{x}^2\right)$$

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The denominator is

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Hence, $\mathrm{E}[b_1] = \beta_1$. $\square$

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Remark.

- $\sum_i (x_i - \bar{x})^2 \uparrow \quad \Rightarrow \quad \mathrm{Var}[b_1] \downarrow$

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- $n \uparrow \quad \Rightarrow \quad \mathrm{Var}[b_1] \downarrow$

Distribution of b_1

Rewrite b_1 as

$$b_1 = \frac{\sum_i (x_i - \bar{x}) Y_i}{\sum_i (x_i - \bar{x})^2} = \sum_{i=1}^{n} \left[\frac{(x_i - \bar{x})}{\sum_{j=1}^{n} (x_j - \bar{x})^2} \right] Y_i$$

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$$\frac{b_1 - \beta_1}{\sqrt{\mathrm{Var}[b_1]}} \sim \mathcal{N}(0,1).$$

We never know σ^2, we estimate it by MSE $= \frac{1}{n-2} \sum_i (Y_i - \hat{Y}_i)^2$.

Distribution of $\frac{b_1 - \beta_1}{\mathrm{s}[b_1]}$

Denote $\mathrm{s}[b_1] = \sqrt{\frac{\mathsf{MSE}}{\sum_{i=1}^{n}(x_i - \bar{x})^2}}$.

Distribution of $\frac{b_1-\beta_1}{\mathrm{s}[b_1]}$

Denote $\mathrm{s}[b_1] = \sqrt{\frac{\text{MSE}}{\sum_{i=1}^{n}(x_i-\bar{x})^2}}$.

Theorem. $\frac{b_1-\beta_1}{\mathrm{s}[b_1]} \sim t_{n-2}$.

Proof.

$$\frac{b_1-\beta_1}{\mathrm{s}[b_1]} = \overbrace{\frac{b_1-\beta_1}{\sqrt{\mathrm{Var}[b_1]}}}^{Z} \cdot \frac{\sqrt{\mathrm{Var}[b_1]}}{\mathrm{s}[b_1]} = Z \cdot \sqrt{\frac{\sigma^2}{\frac{1}{n-2}\sum_i e_i^2}}$$

$$= \frac{Z}{\sqrt{\frac{\sum_i e_i^2/\sigma^2}{n-2}}} = \frac{Z}{\sqrt{\frac{\chi^2_{n-2}}{n-2}}} \sim t_{n-2},$$

where we used the fact that $\frac{\sum_i e_i^2}{\sigma^2} \sim \chi^2_{n-2}$ (Ch 5). $\square$

Confidence interval for β_1 and testing $\mathrm{H}_0 : \beta_1 = \beta_{10}$

A $(1-\alpha)\cdot 100\%$ CI for β_1 has endpoints

$$b_1 \pm t_{n-2,1-\alpha/2} \cdot \mathrm{s}[b_1].$$

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In simple linear regression

$$Y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

of particular interest is $\mathrm{H}_0 : \beta_1 = 0$, that Y_i and *does not depend on* x_i.

Expectation, Variance, and Distribution of b_0

Exercise. Show that

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CI and Hypothesis Test for β_0 are as usual.

Table of regression coefficients

Regression output typically produces a table like:

Parameter	Estimate	Standard error	t^*	p-value
Intercept β_0	b_0	$\mathrm{s}[b_0]$	$t_0^* = \frac{b_0}{\mathrm{s}[b_0]}$	$\Pr(\lvert T \rvert > \lvert t_0^* \rvert)$
Slope β_1	b_1	$\mathrm{s}[b_1]$	$t_1^* = \frac{b_1}{\mathrm{s}[b_1]}$	$\Pr(\lvert T \rvert > \lvert t_1^* \rvert)$

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The two p-values in the table test $\mathrm{H}_0 : \beta_0 = 0$ and $\mathrm{H}_1 : \beta_1 = 0$ respectively. The test for zero intercept is usually not of interest.

Regression Output in R: Poverty vs HS grad rate

https://raw.githubusercontent.com/zh3nis/MATH440/main/chp01/poverty.R

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 64.78097    6.80260   9.523 9.94e-13 ***
Graduates   -0.62122    0.07902  -7.862 3.11e-10 ***
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Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 2.082 on 49 degrees of freedom
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We reject $\mathrm{H}_0 : \beta_1 = 0$ at any reasonable significance level. There is a significant linear association between HS graduation and poverty rates.

Inferences on β_1 and β_0

Inferences on $\mathrm{E}[Y]$ and $\hat{Y}$

Analysis of Variance Approach

Coefficient of Determination

Inference on $\mathrm{E}[Y] = \beta_0 + \beta_1 x$

Let x be any value of the *predictor*; we want to estimate the mean of all responses in the *population* that correspond to x. This is given by

$$\mathrm{E}[Y] = \beta_0 + \beta_1 x.$$

Our estimator of $\mathrm{E}[Y]$ is

$$\begin{aligned}
\hat{Y} &= b_0 + b_1 x \\
&= \sum_{i=1}^{n} \left[\frac{1}{n} - \frac{\bar{x}(x_i - \bar{x})}{\sum_{j=1}^{n}(x_j - \bar{x})^2} + \frac{(x_i - \bar{x})x}{\sum_{j=1}^{n}(x_j - \bar{x})^2} \right] Y_i \\
&= \sum_{i=1}^{n} \left[\frac{1}{n} + \frac{(x - \bar{x})(x_i - \bar{x})}{\sum_{j=1}^{n}(x_j - \bar{x})^2} \right] Y_i
\end{aligned}$$

Distribution of $\hat{Y}$

Again we have a linear combination of independent normals as our estimator. This leads, after some math, to

$$b_0 + b_1 x \sim \mathcal{N}\left(\beta_0 + \beta_1 x, \sigma^2 \left[\frac{1}{n} + \frac{(x - \bar{x})^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right]\right). \quad (1)$$

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As before, this leads to a $(1-\alpha)\cdot 100\%$ CI for $\beta_0 + \beta_1 x$

$$b_0 + b_1 x \pm t_{n-2,1-\alpha/2} \cdot \mathrm{s}[b_0 + b_1 x],$$

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where $\mathrm{s}[b_0 + b_1 x] = \sqrt{\mathrm{MSE} \cdot \left[\frac{1}{n} + \frac{(x-\bar{x})^2}{\sum_{i=1}^n (x_i-\bar{x})^2}\right]}$.

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For what value of x is the CI narrowest?

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For what value of x is the CI narrowest? For $x = \bar{x}$.

Exercise. Prove (1).

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$$b_0 + b_1 x \sim \mathcal{N}\left(\beta_0 + \beta_1 x, \sigma^2 \left[\frac{1}{n} + \frac{(x - \bar{x})^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right]\right). \qquad (1)$$

As before, this leads to a $(1-\alpha)\cdot 100\%$ CI for $\beta_0 + \beta_1 x$

$$b_0 + b_1 x \pm t_{n-2,1-\alpha/2} \cdot \mathrm{s}[b_0 + b_1 x],$$

where $\mathrm{s}[b_0 + b_1 x] = \sqrt{\mathrm{MSE} \cdot \left[\frac{1}{n} + \frac{(x-\bar{x})^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right]}$.

For what value of x is the CI narrowest? For $x = \bar{x}$.

Exercise. Prove (1). Solution is on pp. 53–54 of the textbook.

Prediction intervals

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- If we knew β_0, β_1, and σ^2 this would be easy, because

$$Y = \beta_0 + \beta_1 x + \epsilon \sim \mathcal{N}(\beta_0 + \beta_1 x, \sigma^2),$$

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- Unfortunately, we don't know β_0, β_1, and σ, but we can estimate all three of these.

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Variability of $b_0 + b_1 x + \epsilon$

An interval that contains Y with $(1-\alpha)$ probability needs to account for

- the variability of the estimators b_0 and b_1
- the natural variability of response Y built into the model: $\epsilon \sim \mathcal{N}(0, \sigma^2)$. We have

$$\begin{aligned}
\mathrm{Var}[b_0 + b_1 x + \epsilon] &= \mathrm{Var}[b_0 + b_1 x] + \mathrm{Var}[\epsilon] \\
&= \sigma^2 \left[\frac{1}{n} + \frac{(x - \bar{x})^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \right] + \sigma^2 \\
&= \sigma^2 \left[\frac{1}{n} + \frac{(x - \bar{x})^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2} + 1 \right]
\end{aligned}$$

Prediction interval

Estimating σ^2 by MSE we obtain that a $(1-\alpha)\cdot 100\%$ **prediction interval** (PI) for Y is

$$b_0 + b_1 x \pm t_{n-2,1-\alpha/2}\sqrt{\text{MSE}\left[\frac{1}{n} + \frac{(x-\bar{x})^2}{\sum_{i=1}^n (x_i - \bar{x})^2} + 1\right]}.$$

Remark. As $n \to \infty$, we have $b_0 \xrightarrow{\text{P}} \beta_0$, $b_1 \xrightarrow{\text{P}} \beta_1$, $t_{n-2,1-\alpha/2} \to z_{1-\alpha/2}$, and $\text{MSE} \xrightarrow{\text{P}} \sigma^2$.

I.e., as the sample size grows, the PI converges to

$$\beta_0 + \beta_1 x \pm z_{1-\alpha/2}\cdot\sigma.$$

Example: Poverty vs HS Graduation data

- Find a 95% CI for the mean poverty rate $\mathrm{E}[Y]$ in a state with HS graduation rate $x = 80$.
- Find a 95% PI for the poverty rate Y in a state with HS graduation rate $x = 80$.
- `R` code follows. . .

R code

https://raw.githubusercontent.com/zh3nis/MATH440/main/chp02/pov_predict.R

```
> poverty = read.table("path/to/poverty.txt", h = T, sep = "\t")

> my_model = lm(Poverty ~ Graduates, data=poverty)

> new_x = data.frame(Graduates=80)

> predict.lm(my_model, new_x, interval="confidence", level=0.95)
       fit     lwr      upr
1 15.08363 13.9636 16.20365

> predict.lm(my_model, new_x, interval="prediction", level=0.95)
       fit     lwr      upr
1 15.08363 10.7527 19.41455
```

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A 95% CI for $\mathrm{E}[Y]$ given $x = 80$ is $[13.96, 16.20]$.

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```

A 95% CI for $\mathrm{E}[Y]$ given $x = 80$ is $[13.96, 16.20]$.
A 95% PI for Y given $x = 80$ is $[10.75, 19.41]$.

Confidence Band for Regression Line

Working-Hotelling
confidence band:

$$\hat{Y} \pm W \cdot \mathrm{s}[\hat{Y}],$$

where $W^2 = 2 \cdot F_{1-\alpha;2,n-2}$.

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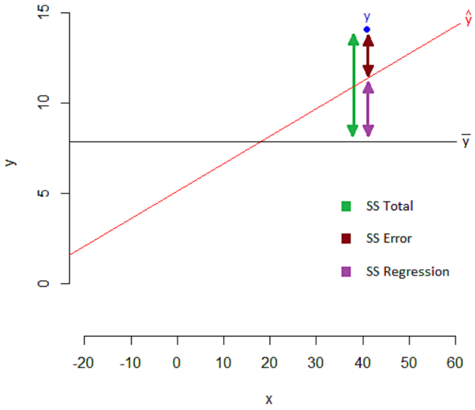
https://raw.githubusercontent.com/zh3nis/MATH440/main/chp02/pov_cb.R

Decomposing $Y_i - \bar{Y}$

Notice that

$$\overbrace{Y_i - \bar{Y}}^{\text{Total}} = \overbrace{Y_i - \hat{Y}_i}^{\text{Error}} + \overbrace{\hat{Y}_i - \bar{Y}}^{\text{Regression}}$$



Partitioning SST

$$\overbrace{Y_i - \bar{Y}}^{\text{Total}} = \overbrace{Y_i - \hat{Y}_i}^{\text{Error}} + \overbrace{\hat{Y}_i - \bar{Y}}^{\text{Regression}} \tag{2}$$

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$$\underbrace{\sum (Y_i - \bar{Y})^2}_{\text{SST}} = \underbrace{\sum (Y_i - \hat{Y}_i)^2}_{\text{SSE}} + \underbrace{\sum (\hat{Y}_i - \bar{Y})^2}_{\text{SSR}}$$

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because (see Chapter 1)

$$\sum (Y_i - \hat{Y}_i)(\hat{Y}_i - \bar{Y}) = \sum e_i(\hat{Y}_i - \bar{Y}) = \sum e_i \hat{Y}_i - \bar{Y} \sum e_i = 0$$

Degrees of Freedom of SS

In Chapter 5 we will show that

- $\frac{\text{SSE}}{\sigma^2} \sim \chi^2_{n-2}$

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- SSE and SSR are indep.

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- SSE and SSR are indep. $\quad \Rightarrow \quad \frac{\text{SST}}{\sigma^2} = \frac{\text{SSE}}{\sigma^2} + \frac{\text{SSR}}{\sigma^2} \sim \chi^2_{n-1}$

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- $\frac{\text{SSR}}{\sigma^2} \sim \chi^2_1$ (noncentral) $\quad \Rightarrow \quad$ SSR has 1 df
- SSE and SSR are indep. $\quad \Rightarrow \quad \frac{\text{SST}}{\sigma^2} = \frac{\text{SSE}}{\sigma^2} + \frac{\text{SSR}}{\sigma^2} \sim \chi^2_{n-1} \quad \Rightarrow$ SST has $n-1$ df

Expectations of SS

Proposition. $\mathrm{E}[\mathsf{MSE}] = \sigma^2$.

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Proof.

$$\mathrm{E}[\mathsf{MSE}] = \mathrm{E}\left[\frac{\mathsf{SSE}}{n-2}\right] = \frac{\sigma^2}{n-2}\mathrm{E}\left[\frac{\mathsf{SSE}}{\sigma^2}\right] = \frac{\sigma^2}{n-2}(n-2) = \sigma^2. \qquad \square$$

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$$\mathrm{E}[\mathsf{MSR}] = \mathrm{E}\left[\frac{\mathsf{SSR}}{1}\right] = \mathrm{E}\left[b_1^2 \sum_i (x_i - \bar{x})^2\right]$$

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\mathsf{SSR} &= \sum_i (\hat{Y}_i - \bar{Y})^2 = \sum_i (b_0 + b_1 x_i - \bar{Y})^2 \\
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\mathrm{E}[\mathsf{MSR}] &= \mathrm{E}\left[\frac{\mathsf{SSR}}{1}\right] = \mathrm{E}\left[b_1^2 \sum_i (x_i - \bar{x})^2\right] \\
&= \sum_i (x_i - \bar{x})^2 (\mathrm{Var}[b_1] + (\mathrm{E}[b_1])^2)
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Analysis of Variance (ANOVA) table

Source	SS	df	MS	$\mathrm{E}[\mathrm{MS}]$
Regression	$\mathrm{SSR} = \sum(\hat{Y}_i - \bar{Y})^2$	1	$\frac{\mathrm{SSR}}{1}$	$\sigma^2 + \beta_1^2 \sum(x_i - \bar{x})^2$
Error	$\mathrm{SSE} = \sum(Y_i - \hat{Y})^2$	$n-2$	$\frac{\mathrm{SSE}}{n-2}$	σ^2
Total	$\mathrm{SST} = \sum(Y_i - \bar{Y})^2$	$n-1$		

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Remark. $\frac{\mathrm{E}[MSR]}{\mathrm{E}[MSE]} = \begin{cases} 1 & \text{if } \beta_1 = 0 \\ > 1 & \text{if } \beta_1 \neq 0. \end{cases}$

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Loosely, we expect MSR to be larger than MSE when $\beta_1 \neq 0$.

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Proposition. Under $\mathrm{H}_0 : \beta_1 = 0$, we have $\frac{\mathrm{MSR}}{\mathrm{MSE}} \sim F_{1,n-2}$.

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because SSR and SSE are statistically independent (Ch 5). $\square$

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Remark. The F-test and t-test for $H_0 : \beta_1 = 0$ vs $H_1 : \beta_1 \neq 0$ are equivalent, since

$$\frac{\text{MSR}}{\text{MSE}} = \frac{b_1^2 \sum (x_i - \bar{x})^2}{\text{MSE}} = \frac{b_1^2}{\text{MSE}/\sum (x_i - \bar{x})^2} = \frac{b_1^2}{\mathrm{s}^2[b_1]} = \left(\frac{b_1 - 0}{\mathrm{s}[b_1]}\right)^2.$$

F-test for β_1

Proposition. Under $H_0 : \beta_1 = 0$, we have $\frac{MSR}{MSE} \sim F_{1,n-2}$.

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$$\frac{MSR}{MSE} = \frac{b_1^2 \sum(x_i - \bar{x})^2}{MSE} = \frac{b_1^2}{MSE/\sum(x_i - \bar{x})^2} = \frac{b_1^2}{s^2[b_1]} = \left(\frac{b_1 - 0}{s[b_1]}\right)^2.$$

and both are generalized likelihood ratio tests (GLRT).

ANOVA F-test in R

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> poverty = read.table("path/to/poverty.txt", h = T, sep = "\t")
> my_model = lm(Poverty ~ Graduates, data=poverty)
> anova(my_model)
Analysis of Variance Table

Response: Poverty
          Df Sum Sq Mean Sq F value    Pr(>F)
Graduates  1 267.88 267.881  61.809 3.109e-10 ***
Residuals 49 212.37   4.334

> summary(my_model)
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 64.78097    6.80260   9.523 9.94e-13 ***
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```

p-values of F-test and t-test for $H_0 : \beta_0 = 0$ are same.

Inferences on β_1 and β_0

Inferences on $\mathrm{E}[Y]$ and $\hat{Y}$

Analysis of Variance Approach

Coefficient of Determination

Coefficient of Determination

Definition. The **coefficient of determination** is

$$R^2 = \frac{\mathrm{SSR}}{\mathrm{SST}} = 1 - \frac{\mathrm{SSE}}{\mathrm{SST}},$$

the proportion of total sample variation in Y that is explained by its linear relationship with x.

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Note:

- $0 \leq R^2 \leq 1$.
- $R^2 = 1 \Rightarrow$ data perfectly linear.
- $R^2 = 0 \Rightarrow$ regression line horizontal ($b_1 = 0$).

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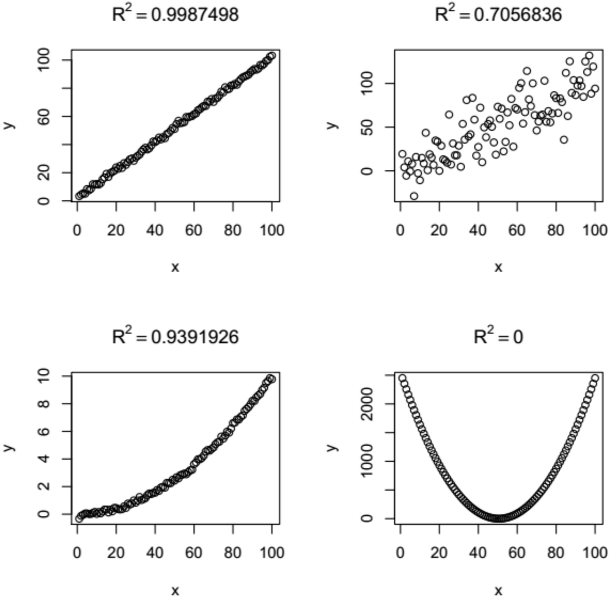
Note:

- $0 \leq R^2 \leq 1$.
- $R^2 = 1 \Rightarrow$ data perfectly linear.
- $R^2 = 0 \Rightarrow$ regression line horizontal ($b_1 = 0$).

The closer R^2 is to one, the greater the linear relationship between x and Y.

R^2 for different data sets



Sample correlation r

Let

$$r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(Y_i - \bar{Y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2 \sum_{i=1}^{n}(Y_i - \bar{Y})^2}}$$

be the sample correlation between x and Y.

Sample correlation r

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be the sample correlation between x and Y.

Exercise. Show that

- $R^2 = r^2$,
- $\mathrm{sgn}(r) = \mathrm{sgn}(b_1)$.

Sample correlation r

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Remark.

- $r \approx 0 \Rightarrow$ little linear association b/w x and Y
- $r \approx 1 \Rightarrow$ strong positive, linear association b/w x and Y
- $r \approx -1 \Rightarrow$ strong negative, linear association b/w x and Y.

Cautions about R^2 and r

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Poverty vs HS Graduation data:

```
> summary(my_model)
...
Residual standard error: 2.082 on 49 degrees of freedom
Multiple R-squared:  0.5578, Adjusted R-squared:  0.5488
F-statistic: 61.81 on 1 and 49 DF,  p-value: 3.109e-10
```

Cautions about regression

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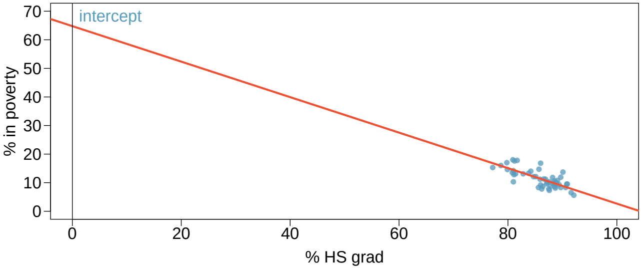
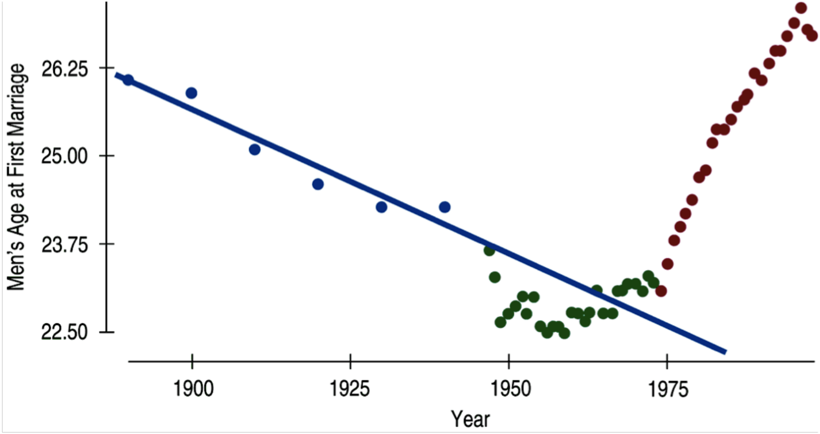


Examples of extrapolation



Examples of extrapolation

Momentous sprint at the 2156 Olympics?

Women sprinters are closing the gap on men and may one day overtake them.

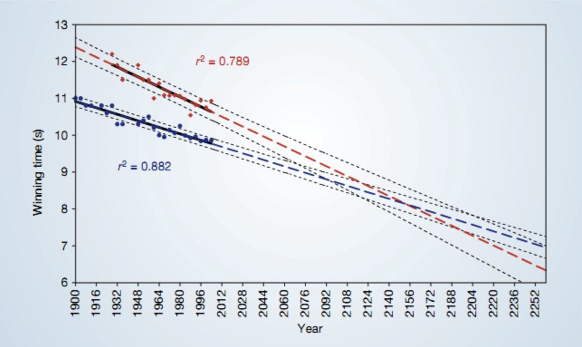


Figure 1 The winning Olympic 100-metre sprint times for men (blue points) and women (red points), with superimposed best-fit linear regression lines (solid black lines) and coefficients of determination. The regression lines are extrapolated (broken blue and red lines for men and women, respectively) and 95% confidence intervals (dotted black lines) based on the available points are superimposed. The projections intersect just before the 2156 Olympics, when the winning women's 100-metre sprint time of 8.079 s will be faster than the men's at 8.098 s.