Matrix Approach to SLR

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Regression Analysis

Random Vectors

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It is a function s.t. for any (measurable) $B \subset \mathbb{R}^n$:

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$$\begin{aligned} \Pr(\mathbf{Y} \in B) &= \int_{\mathbf{y} \in B} f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) dy_1 \cdots dy_n \\ &= \int_{\mathbf{y} \in B} f_{\mathbf{Y}}(\mathbf{y}) d\mathbf{y} \end{aligned}$$

Expectation of a Random Vector

The **expectation** of a random vector $\mathbf{Y}$ is

$$\boldsymbol{\mu} = \mathrm{E}[\mathbf{Y}] = \begin{bmatrix} \mathrm{E}[Y_1] \\ \vdots \\ \mathrm{E}[Y_n] \end{bmatrix}$$

Random Vectors

The **covariance matrix** of a random vector $\mathbf{Y}$ is an $n \times n$ matrix defined as

$$\mathrm{Var}[\mathbf{Y}] = \mathrm{E}[(\mathbf{Y} - \mathrm{E}[\mathbf{Y}])(\mathbf{Y} - \mathrm{E}[\mathbf{Y}])^\top]$$

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\vdots & \vdots & \ddots & \vdots \\
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&= \begin{bmatrix} \sigma_1^2 & \sigma_{1,2} & \cdots & \sigma_{1,n} \\ \sigma_{2,1} & \sigma_2^2 & \cdots & \sigma_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n,1} & \sigma_{n,2} & \cdots & \sigma_n^2 \end{bmatrix}
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\end{bmatrix} \\
&= \Sigma \qquad \text{(symmetric)}
\end{aligned}
$$

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Hence for the model

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

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- $\mathrm{E}[\mathbf{Y}] = \mathrm{E}[\mathbf{X}\boldsymbol{\beta}] + \mathrm{E}[\boldsymbol{\epsilon}] = \mathbf{X}\boldsymbol{\beta}$
- $\mathrm{Var}[\mathbf{Y}] = \mathrm{Var}[\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}] = \mathrm{Var}[\boldsymbol{\epsilon}] = \sigma^2 \mathbf{I}_n$

Linear Transform of a Random Vector

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$$= \begin{bmatrix} a_{11} Y_1 + a_{12} Y_2 + \ldots + a_{1n} Y_n \\ a_{21} Y_1 + a_{22} Y_2 + \ldots + a_{2n} Y_n \\ \vdots \\ a_{m1} Y_1 + a_{m2} Y_2 + \ldots + a_{mn} Y_n \end{bmatrix}$$

Mean of a Linear Transform

Theorem. Let $\mathbf{Y}$ be a random n-vector, and $\mathbf{A} \in \mathbb{R}^{m \times n}$. Then $\mathrm{E}[\mathbf{AY}] = \mathbf{A}\mathrm{E}[\mathbf{Y}]$.

Proof.

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&= \mathbf{A}\mathrm{E}[\mathbf{Y}]
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Covariance Matrix of a Linear Transform

Theorem. Let $\mathbf{Y}$ be a random n-vector, and $\mathbf{A} \in \mathbb{R}^{m \times n}$. Then $\mathrm{Var}[\mathbf{AY}] = \mathbf{A}\mathrm{Var}[\mathbf{Y}]\mathbf{A}^\top$.

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&= \mathrm{E}[\mathbf{A}(\mathbf{Y} - \mathrm{E}[\mathbf{Y}])(\mathbf{Y} - \mathrm{E}[\mathbf{Y}])^\top\mathbf{A}^\top]
\end{aligned}$$

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$\forall \mathbf{a} \in \mathbb{R}^n$:

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Proof.
$\forall \mathbf{a} \in \mathbb{R}^n$:

$$\mathbf{a}^\top \mathrm{Var}[\mathbf{Y}]\mathbf{a} = \mathrm{Var}[\mathbf{a}^\top \mathbf{Y}] \geq 0,$$

since $\mathbf{a}^\top \mathbf{Y} = \sum_i a_i Y_i$ is a random variable with values in $\mathbb{R}^1$. $\square$

Pdf of a Linear Transform, $n = 1$

Theorem. Let X be a continuous r.v. with the pdf $f_X(x)$, and $a \neq 0$. Then the pdf of $Y = aX$ is given by

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y}{a}\right)$$

Proof.
For $a > 0$:

$$f_Y(y)$$

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Proof.
For $a > 0$:

$$\begin{aligned} f_Y(y) &= \frac{d}{dy}[F_Y(y)] = \frac{d}{dy}\left[\Pr(Y \leq y)\right] = \frac{d}{dy}\left[\Pr(aX \leq y\right] \\ &= \frac{d}{dy}\left[\Pr\left(X \leq \frac{y}{a}\right)\right] = \frac{d}{dy}\left[F_X\left(\frac{y}{a}\right)\right] = \frac{1}{a} f_X\left(\frac{y}{a}\right) \end{aligned}$$

For $a < 0$ the proof is analogous and is left as *exercise*. $\square$

Pdf of a Linear Transform, General Case

Theorem. Let $\mathbf{X}$ be a random vector with the pdf $f_{\mathbf{X}}(\mathbf{x})$, and $\mathbf{A} \in \mathbb{R}^{n \times n}$ with $\det \mathbf{A} \neq 0$. Then the pdf of $\mathbf{Y} = \mathbf{AX}$ is given by

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{|\det \mathbf{A}|} f_{\mathbf{X}}(\mathbf{A}^{-1}\mathbf{y})$$

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Proof.
For any $B \subset \mathbb{R}^n$,

$\square$

Pdf of a Linear Transform, General Case

Theorem. Let $\mathbf{X}$ be a random vector with the pdf $f_{\mathbf{X}}(\mathbf{x})$, and $\mathbf{A} \in \mathbb{R}^{n \times n}$ with $\det \mathbf{A} \neq 0$. Then the pdf of $\mathbf{Y} = \mathbf{A}\mathbf{X}$ is given by

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{|\det \mathbf{A}|} f_{\mathbf{X}}(\mathbf{A}^{-1}\mathbf{y})$$

Proof.
For any $B \subset \mathbb{R}^n$,

$$\int_{\mathbf{y} \in B} f_{\mathbf{Y}}(\mathbf{y}) d\mathbf{y} = \Pr(\mathbf{Y} \in B)$$

$\square$

Pdf of a Linear Transform, General Case

Theorem. Let $\mathbf{X}$ be a random vector with the pdf $f_{\mathbf{X}}(\mathbf{x})$, and $\mathbf{A} \in \mathbb{R}^{n \times n}$ with $\det \mathbf{A} \neq 0$. Then the pdf of $\mathbf{Y} = \mathbf{AX}$ is given by

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Proof.
For any $B \subset \mathbb{R}^n$,

$$\int_{\mathbf{y} \in B} f_{\mathbf{Y}}(\mathbf{y}) d\mathbf{y} = \Pr(\mathbf{Y} \in B) = \Pr(\mathbf{AX} \in B)$$

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Pdf of a Linear Transform, General Case

Theorem. Let $\mathbf{X}$ be a random vector with the pdf $f_{\mathbf{X}}(\mathbf{x})$, and $\mathbf{A} \in \mathbb{R}^{n \times n}$ with $\det \mathbf{A} \neq 0$. Then the pdf of $\mathbf{Y} = \mathbf{AX}$ is given by

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Proof.
For any $B \subset \mathbb{R}^n$,

$$\int_{\mathbf{y} \in B} f_{\mathbf{Y}}(\mathbf{y}) d\mathbf{y} = \Pr(\mathbf{Y} \in B) = \Pr(\mathbf{AX} \in B) = \int_{\mathbf{Ax} \in B} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$

□

Pdf of a Linear Transform, General Case

Theorem. Let $\mathbf{X}$ be a random vector with the pdf $f_{\mathbf{X}}(\mathbf{x})$, and $\mathbf{A} \in \mathbb{R}^{n\times n}$ with $\det \mathbf{A} \neq 0$. Then the pdf of $\mathbf{Y} = \mathbf{AX}$ is given by

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Proof.
For any $B \subset \mathbb{R}^n$,

$$\begin{aligned}\int_{\mathbf{y}\in B} f_{\mathbf{Y}}(\mathbf{y})d\mathbf{y} = \Pr(\mathbf{Y} \in B) = \Pr(\mathbf{AX} \in B) &= \int_{\mathbf{Ax}\in B} f_{\mathbf{X}}(\mathbf{x})d\mathbf{x} \\ &= \int_{\mathbf{y}\in B} f_{\mathbf{X}}(\mathbf{A}^{-1}\mathbf{y})|\det \mathbf{A}^{-1}|d\mathbf{y}\end{aligned}$$

$\square$

Pdf of a Linear Transform, General Case

Theorem. Let $\mathbf{X}$ be a random vector with the pdf $f_{\mathbf{X}}(\mathbf{x})$, and $\mathbf{A} \in \mathbb{R}^{n \times n}$ with $\det \mathbf{A} \neq 0$. Then the pdf of $\mathbf{Y} = \mathbf{AX}$ is given by

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For any $B \subset \mathbb{R}^n$,

$$\int_{\mathbf{y} \in B} f_{\mathbf{Y}}(\mathbf{y}) d\mathbf{y} = \Pr(\mathbf{Y} \in B) = \Pr(\mathbf{AX} \in B) = \int_{\mathbf{Ax} \in B} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$
$$= \int_{\mathbf{y} \in B} f_{\mathbf{X}}(\mathbf{A}^{-1}\mathbf{y}) |\det \mathbf{A}^{-1}| d\mathbf{y}$$

Noting that $\det \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}$ concludes the proof. $\square$

Pdf of an Affine Transform

Theorem. Let X be a continuous r.v. with the pdf $f_X(x)$, and $a \neq 0$. Then the pdf of $Y = aX + b$ is given by

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

Pdf of an Affine Transform

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Theorem. Let $\mathbf{X}$ be a random vector with the pdf $f_{\mathbf{X}}(\mathbf{x})$, $\mathbf{A} \in \mathbb{R}^{n\times n}$ with $\det \mathbf{A} \neq 0$, and $\mathbf{b} \in \mathbb{R}^b$. Then the pdf of $\mathbf{Y} = \mathbf{AX} + \mathbf{b}$ is given by

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{|\det \mathbf{A}|} f_{\mathbf{X}}(\mathbf{A}^{-1}(\mathbf{y} - \mathbf{b}))$$

Pdf of an Affine Transform

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Proofs are left as exercises.

Multivariate Normal Distribution

Definition. A random vector $\mathbf{Y} = \begin{bmatrix} Y_1 & \dots & Y_n \end{bmatrix}^\top$ is **multivariate normal (Gaussian)** if it can be represented as

$$\mathbf{Y} = \mathbf{AZ} + \boldsymbol{\mu},$$

where

- $\mathbf{Z} = \begin{bmatrix} Z_1 & \dots & Z_k \end{bmatrix}^\top$ with $Z_i \overset{\text{iid}}{\sim} \mathcal{N}(0, 1)$,
- $\mathbf{A} \in \mathbb{R}^{n \times k}$ s.t. $\Sigma = \mathbf{AA}^\top$

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Notation. $\mathbf{Y} \sim \mathcal{N}_n(\boldsymbol{\mu}, \Sigma)$

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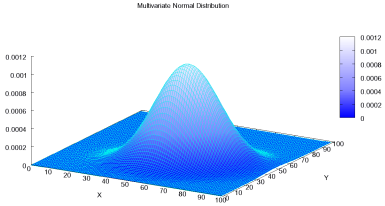

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Multivariate Normal Distribution

Theorem. $\mathbf{Y} \sim \mathcal{N}_n(\boldsymbol{\mu}, \Sigma)$, and $\mathrm{rank}\boldsymbol{\Sigma} = n \quad \Rightarrow \quad$ The pdf of $\mathbf{Y}$ has the form

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{\sqrt{\det \Sigma}(\sqrt{2\pi})^n} e^{-\frac{1}{2}(\mathbf{y}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu})}.$$

Proof.
$\mathbf{Y} \sim \mathcal{N}_n(\boldsymbol{\mu}, \Sigma)$

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Proof.
$\mathbf{Y} \sim \mathcal{N}_n(\boldsymbol{\mu}, \Sigma) \quad \Rightarrow \quad \exists \mathbf{A} \in \mathbb{R}^{n \times k}$, $\mathbf{Z} \in \mathcal{N}_k(\mathbf{0}, \mathbf{I}_k)$:

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$$\mathbf{Y} = \mathbf{AZ} + \boldsymbol{\mu} \quad \text{with } \mathbf{AA}^\top = \Sigma$$

Since $n = \mathrm{rank}\Sigma$

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Proof.

$\mathbf{Y} \sim \mathcal{N}_n(\boldsymbol{\mu}, \Sigma) \quad \Rightarrow \quad \exists \mathbf{A} \in \mathbb{R}^{n\times k}$, $\mathbf{Z} \in \mathcal{N}_k(\mathbf{0}, \mathbf{I}_k)$:

$$\mathbf{Y} = \mathbf{AZ} + \boldsymbol{\mu} \quad \text{with } \mathbf{AA}^\top = \Sigma$$

Since $n = \mathrm{rank}\Sigma = \mathrm{rank}\mathbf{AA}^\top = k$, we have $k = n$.

Also, $\det \Sigma = \det \mathbf{AA}^\top = \det \mathbf{A} \cdot \det \mathbf{A}^\top = (\det \mathbf{A})^2$. Thus,

$$\det \mathbf{A} = \sqrt{\det \Sigma}.$$

Multivariate Normal Distribution

Poof (cont'd).

For the $\mathbf{Z} = \begin{bmatrix} Z_1 & \cdots & Z_n \end{bmatrix}$ with $Z_i \overset{\text{iid}}{\sim} \in \mathcal{N}(0,1)$,

$$f_{\mathbf{Z}}(\mathbf{z})$$

Multivariate Normal Distribution

Poof (cont'd).

For the $\mathbf{Z} = \begin{bmatrix} Z_1 & \cdots & Z_n \end{bmatrix}$ with $Z_i \overset{\text{iid}}{\sim} \in \mathcal{N}(0, 1)$,

$$f_{\mathbf{Z}}(\mathbf{z}) = f_{Z_1,\ldots,Z_n}(z_1, \ldots, z_n)$$

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Multivariate Normal Distribution

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Multivariate Normal Distribution

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Since $\mathbf{Y} = \mathbf{A}\mathbf{Z} + \boldsymbol{\mu}$, and $\Sigma = \mathbf{A}\mathbf{A}^\top$, we have

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Poof (cont'd).

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$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{\det \mathbf{A}} f_{\mathbf{Z}}(\mathbf{A}^{-1}(\mathbf{y}-\boldsymbol{\mu})) = \frac{1}{\sqrt{\det \Sigma}(\sqrt{2\pi})^n} e^{-\frac{1}{2}(\mathbf{y}-\boldsymbol{\mu})^\top (\mathbf{A}^{-1})^\top \mathbf{A}^{-1}(\mathbf{y}-\boldsymbol{\mu})}$$

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Poof (cont'd).

For the $\mathbf{Z} = \begin{bmatrix} Z_1 & \cdots & Z_n \end{bmatrix}$ with $Z_i \overset{\text{iid}}{\sim} \in \mathcal{N}(0,1)$,

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&= \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{1}{2}(z_1^2+\ldots+z_n^2)} = \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{1}{2}\mathbf{z}^\top \mathbf{z}}
\end{aligned}
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Since $\mathbf{Y} = \mathbf{A}\mathbf{Z} + \boldsymbol{\mu}$, and $\Sigma = \mathbf{A}\mathbf{A}^\top$, we have

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f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{\det \mathbf{A}} f_{\mathbf{Z}}(\mathbf{A}^{-1}(\mathbf{y}-\boldsymbol{\mu})) &= \frac{1}{\sqrt{\det \Sigma}(\sqrt{2\pi})^n} e^{-\frac{1}{2}(\mathbf{y}-\boldsymbol{\mu})^\top (\mathbf{A}^{-1})^\top \mathbf{A}^{-1}(\mathbf{y}-\boldsymbol{\mu})} \\
&= \frac{1}{\sqrt{\det \Sigma}(\sqrt{2\pi})^n} e^{-\frac{1}{2}(\mathbf{y}-\boldsymbol{\mu})^\top (\mathbf{A}^\top)^{-1} \mathbf{A}^{-1}(\mathbf{y}-\boldsymbol{\mu})} \\
&= \frac{1}{\sqrt{\det \Sigma|}(\sqrt{2\pi})^n} e^{-\frac{1}{2}(\mathbf{y}-\boldsymbol{\mu})^\top \Sigma^{-1}(\mathbf{y}-\boldsymbol{\mu})}
\end{aligned}
$$

Orthogonal Transform of a Multivariate Gaussian

Theorem. Let $\mathbf{Z} \sim \mathcal{N}_n(\mathbf{0}, \mathbf{I})$, and $\mathbf{Q}$ be an orthogonal $n \times n$ matrix ($\mathbf{Q} \in \mathcal{O}_n$). Then $\mathbf{QZ} \sim \mathcal{N}_n(\mathbf{0}, \mathbf{I})$.

Proof.

$\square$

Orthogonal Transform of a Multivariate Gaussian

Theorem. Let $\mathbf{Z} \sim \mathcal{N}_n(\mathbf{0}, \mathbf{I})$, and $\mathbf{Q}$ be an orthogonal $n \times n$ matrix ($\mathbf{Q} \in \mathcal{O}_n$). Then $\mathbf{QZ} \sim \mathcal{N}_n(\mathbf{0}, \mathbf{I})$.

Proof.
The pdf of $\mathbf{QZ}$ is

$\square$

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Proof.
The pdf of $\mathbf{QZ}$ is

$$f_{\mathbf{QZ}}(\mathbf{w})$$

$\square$

Orthogonal Transform of a Multivariate Gaussian

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Proof.
The pdf of $\mathbf{QZ}$ is

$$f_{\mathbf{QZ}}(\mathbf{w}) = \frac{1}{|\det \mathbf{Q}|} f_{\mathbf{Z}}(\mathbf{Q}^{-1}\mathbf{w})$$

$\square$

Orthogonal Transform of a Multivariate Gaussian

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Proof.
The pdf of $\mathbf{QZ}$ is

$$f_{\mathbf{QZ}}(\mathbf{w}) = \frac{1}{|\det \mathbf{Q}|} f_{\mathbf{Z}}(\mathbf{Q}^{-1}\mathbf{w}) = \frac{1}{1} \cdot \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{1}{2}(\mathbf{Q}^{-1}\mathbf{w})^\top(\mathbf{Q}^{-1}\mathbf{w})}$$

$\square$

Orthogonal Transform of a Multivariate Gaussian

Theorem. Let $\mathbf{Z} \sim \mathcal{N}_n(\mathbf{0}, \mathbf{I})$, and $\mathbf{Q}$ be an orthogonal $n \times n$ matrix ($\mathbf{Q} \in \mathcal{O}_n$). Then $\mathbf{QZ} \sim \mathcal{N}_n(\mathbf{0}, \mathbf{I})$.

Proof.
The pdf of $\mathbf{QZ}$ is

$$
\begin{aligned}
f_{\mathbf{QZ}}(\mathbf{w}) &= \frac{1}{|\det \mathbf{Q}|} f_{\mathbf{Z}}(\mathbf{Q}^{-1}\mathbf{w}) = \frac{1}{1} \cdot \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{1}{2}(\mathbf{Q}^{-1}\mathbf{w})^\top(\mathbf{Q}^{-1}\mathbf{w})} \\
&= \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{1}{2}(\mathbf{Q}^\top\mathbf{w})^\top(\mathbf{Q}^\top\mathbf{w})}
\end{aligned}
$$

$\square$

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The pdf of $\mathbf{QZ}$ is

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Orthogonal Transform of a Multivariate Gaussian

Theorem. Let $\mathbf{Z} \sim \mathcal{N}_n(\mathbf{0}, \mathbf{I})$, and $\mathbf{Q}$ be an orthogonal $n \times n$ matrix ($\mathbf{Q} \in \mathcal{O}_n$). Then $\mathbf{QZ} \sim \mathcal{N}_n(\mathbf{0}, \mathbf{I})$.

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Theorem. Let $\mathbf{Y} \sim \mathcal{N}_n(\boldsymbol{\mu}, \Sigma)$. If $\mathbf{A} \in \mathbb{R}^{m \times n}$ with $\mathrm{rank}\mathbf{A} = \min(m, n)$, then

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Theorem. Let $\mathbf{Z} \sim \mathcal{N}_n(\mathbf{0}, \mathbf{I})$, and $\mathbf{A} \in \mathbb{R}^{n \times n}$ be symmetric idempotent with $\mathrm{rank}\mathbf{A} = r$. Then $\mathbf{Z}^\top \mathbf{A} \mathbf{Z} \sim \chi^2_r$.

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Quadratic Form of a Multivariate Gaussian

Theorem. Let $\mathbf{Y} \sim \mathcal{N}_n(\boldsymbol{\mu}, \mathbf{I})$, and $\mathbf{A} \in \mathbb{R}^{n \times n}$ be symmetric idempotent with $\mathrm{rank}\mathbf{A} = r$. Then $\mathbf{Y}^\top \mathbf{A} \mathbf{Y} \sim \chi^2_r(\lambda)$, with $\lambda = \boldsymbol{\mu}^\top \mathbf{A} \boldsymbol{\mu}$

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Matrix Calculus

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Exercise. Prove the rules above.

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Recall the SLR in matrix form:

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Random Vectors

Multivariate Gaussian Distribution

Quadratic Forms of Multivariate Gaussians

Distributions of SSE, SSR; Independence

Distribution of SSE

Theorem. $\frac{\mathrm{SSE}}{\sigma^2} \sim \chi^2_{n-2}$

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\end{aligned}
$$

Hence, $\text{SSE} = \mathbf{Y}^\top\mathbf{A}^\top\mathbf{A}\mathbf{Y} = \mathbf{Y}^\top\mathbf{A}\mathbf{Y}$.

Distribution of SSE

Proof (cont'd).

Recall, $\mathbf{Y} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I})$

Distribution of SSE

Proof (cont'd).

Recall, $\mathbf{Y} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}) \quad \Rightarrow \quad \sigma^{-1}\mathbf{Y} \sim \mathcal{N}_n(\sigma^{-1}\mathbf{X}\boldsymbol{\beta}, \mathbf{I})$.

Distribution of SSE

Proof (cont'd).

Recall, $\mathbf{Y} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}) \quad \Rightarrow \quad \sigma^{-1}\mathbf{Y} \sim \mathcal{N}_n(\sigma^{-1}\mathbf{X}\boldsymbol{\beta}, \mathbf{I})$. Hence,

$$\frac{\text{SSE}}{\sigma^2} = (\sigma^{-1}\mathbf{Y})^\top \mathbf{A}(\sigma^{-1}\mathbf{Y})$$

$\square$

Distribution of SSE

Proof (cont'd).

Recall, $\mathbf{Y} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}) \quad \Rightarrow \quad \sigma^{-1}\mathbf{Y} \sim \mathcal{N}_n(\sigma^{-1}\mathbf{X}\boldsymbol{\beta}, \mathbf{I})$. Hence,

$$\frac{\mathsf{SSE}}{\sigma^2} = (\sigma^{-1}\mathbf{Y})^\top \mathbf{A}(\sigma^{-1}\mathbf{Y}) \sim \chi^2_{\mathrm{rank}\mathbf{A}}(\lambda),$$

$\square$

Distribution of SSE

Proof (cont'd).

Recall, $\mathbf{Y} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}) \quad \Rightarrow \quad \sigma^{-1}\mathbf{Y} \sim \mathcal{N}_n(\sigma^{-1}\mathbf{X}\boldsymbol{\beta}, \mathbf{I})$. Hence,

$$\frac{\mathsf{SSE}}{\sigma^2} = (\sigma^{-1}\mathbf{Y})^\top \mathbf{A}(\sigma^{-1}\mathbf{Y}) \sim \chi^2_{\mathrm{rank}\mathbf{A}}(\lambda),$$

where $\quad \mathrm{rank}\mathbf{A} = \mathrm{trace}\mathbf{A}$

$\square$

Distribution of SSE

Proof (cont'd).

Recall, $\mathbf{Y} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2 \mathbf{I}) \quad \Rightarrow \quad \sigma^{-1}\mathbf{Y} \sim \mathcal{N}_n(\sigma^{-1}\mathbf{X}\boldsymbol{\beta}, \mathbf{I})$. Hence,

$$\frac{\text{SSE}}{\sigma^2} = (\sigma^{-1}\mathbf{Y})^\top \mathbf{A} (\sigma^{-1}\mathbf{Y}) \sim \chi^2_{\mathrm{rank}\mathbf{A}}(\lambda),$$

where $\quad \mathrm{rank}\mathbf{A} = \mathrm{trace}\mathbf{A} = \mathrm{trace}[\mathbf{I} - \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top]$

$\square$

Distribution of SSE

Proof (cont'd).

Recall, $\mathbf{Y} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}) \quad \Rightarrow \quad \sigma^{-1}\mathbf{Y} \sim \mathcal{N}_n(\sigma^{-1}\mathbf{X}\boldsymbol{\beta}, \mathbf{I})$. Hence,

$$\frac{\mathsf{SSE}}{\sigma^2} = (\sigma^{-1}\mathbf{Y})^\top \mathbf{A}(\sigma^{-1}\mathbf{Y}) \sim \chi^2_{\mathrm{rank}\mathbf{A}}(\lambda),$$

$$\text{where} \quad \mathrm{rank}\mathbf{A} = \mathrm{trace}\mathbf{A} = \mathrm{trace}[\mathbf{I} - \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top]$$
$$= \mathrm{trace}\mathbf{I} - \mathrm{trace}[\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top]$$

$\square$

Distribution of SSE

Proof (cont'd).

Recall, $\mathbf{Y} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}) \quad \Rightarrow \quad \sigma^{-1}\mathbf{Y} \sim \mathcal{N}_n(\sigma^{-1}\mathbf{X}\boldsymbol{\beta}, \mathbf{I})$. Hence,

$$\frac{\text{SSE}}{\sigma^2} = (\sigma^{-1}\mathbf{Y})^\top \mathbf{A}(\sigma^{-1}\mathbf{Y}) \sim \chi^2_{\mathrm{rank}\mathbf{A}}(\lambda),$$

$$\begin{aligned} \text{where} \quad \mathrm{rank}\mathbf{A} &= \mathrm{trace}\mathbf{A} = \mathrm{trace}[\mathbf{I} - \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top] \\ &= \mathrm{trace}\mathbf{I} - \mathrm{trace}[\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top] \\ &= n - \mathrm{trace}[(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{X}] \end{aligned}$$

$\square$

Distribution of SSE

Proof (cont'd).

Recall, $\mathbf{Y} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}) \quad \Rightarrow \quad \sigma^{-1}\mathbf{Y} \sim \mathcal{N}_n(\sigma^{-1}\mathbf{X}\boldsymbol{\beta}, \mathbf{I})$. Hence,

$$\frac{\mathrm{SSE}}{\sigma^2} = (\sigma^{-1}\mathbf{Y})^\top \mathbf{A}(\sigma^{-1}\mathbf{Y}) \sim \chi^2_{\mathrm{rank}\mathbf{A}}(\lambda),$$

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$\square$

Distribution of SSE

Proof (cont'd).

Recall, $\mathbf{Y} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}) \quad \Rightarrow \quad \sigma^{-1}\mathbf{Y} \sim \mathcal{N}_n(\sigma^{-1}\mathbf{X}\boldsymbol{\beta}, \mathbf{I})$. Hence,

$$\frac{\text{SSE}}{\sigma^2} = (\sigma^{-1}\mathbf{Y})^\top \mathbf{A}(\sigma^{-1}\mathbf{Y}) \sim \chi^2_{\mathrm{rank}\mathbf{A}}(\lambda),$$

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\text{where} \quad \mathrm{rank}\mathbf{A} &= \mathrm{trace}\mathbf{A} = \mathrm{trace}[\mathbf{I} - \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top] \\
&= \mathrm{trace}\mathbf{I} - \mathrm{trace}[\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top] \\
&= n - \mathrm{trace}[(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{X}] = n - 2, \\
\text{and} \quad \lambda &= (\sigma^{-1}\mathbf{X}\boldsymbol{\beta})^\top \mathbf{A}(\sigma^{-1}\mathbf{X}\boldsymbol{\beta})
\end{aligned}$$

□

Distribution of SSE

Proof (cont'd).

Recall, $\mathbf{Y} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}) \quad \Rightarrow \quad \sigma^{-1}\mathbf{Y} \sim \mathcal{N}_n(\sigma^{-1}\mathbf{X}\boldsymbol{\beta}, \mathbf{I})$. Hence,

$$\frac{\text{SSE}}{\sigma^2} = (\sigma^{-1}\mathbf{Y})^\top \mathbf{A} (\sigma^{-1}\mathbf{Y}) \sim \chi^2_{\mathrm{rank}\mathbf{A}}(\lambda),$$

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\text{where} \quad \mathrm{rank}\mathbf{A} &= \mathrm{trace}\mathbf{A} = \mathrm{trace}[\mathbf{I} - \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top] \\
&= \mathrm{trace}\mathbf{I} - \mathrm{trace}[\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top] \\
&= n - \mathrm{trace}[(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{X}] = n - 2, \\
\text{and} \quad \lambda &= (\sigma^{-1}\mathbf{X}\boldsymbol{\beta})^\top \mathbf{A} (\sigma^{-1}\mathbf{X}\boldsymbol{\beta}) \\
&= \frac{1}{\sigma^2}\boldsymbol{\beta}^\top\mathbf{X}^\top[\mathbf{I} - \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top]\mathbf{X}\boldsymbol{\beta}
\end{aligned}$$

$\square$

Distribution of SSE

Proof (cont'd).

Recall, $\mathbf{Y} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}) \quad \Rightarrow \quad \sigma^{-1}\mathbf{Y} \sim \mathcal{N}_n(\sigma^{-1}\mathbf{X}\boldsymbol{\beta}, \mathbf{I})$. Hence,

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\text{where} \quad \mathrm{rank}\mathbf{A} &= \mathrm{trace}\mathbf{A} = \mathrm{trace}[\mathbf{I} - \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top] \\
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\text{and} \quad \lambda &= (\sigma^{-1}\mathbf{X}\boldsymbol{\beta})^\top \mathbf{A}(\sigma^{-1}\mathbf{X}\boldsymbol{\beta}) \\
&= \frac{1}{\sigma^2}\boldsymbol{\beta}^\top\mathbf{X}^\top[\mathbf{I} - \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top]\mathbf{X}\boldsymbol{\beta} \\
&= \frac{1}{\sigma^2}\boldsymbol{\beta}^\top[\mathbf{X}^\top\mathbf{X} - \mathbf{X}^\top\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{X}]\boldsymbol{\beta}
\end{aligned}$$

$\square$

Distribution of SSE

Proof (cont'd).

Recall, $\mathbf{Y} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}) \quad \Rightarrow \quad \sigma^{-1}\mathbf{Y} \sim \mathcal{N}_n(\sigma^{-1}\mathbf{X}\boldsymbol{\beta}, \mathbf{I})$. Hence,

$$\frac{\text{SSE}}{\sigma^2} = (\sigma^{-1}\mathbf{Y})^\top \mathbf{A}(\sigma^{-1}\mathbf{Y}) \sim \chi^2_{\mathrm{rank}\mathbf{A}}(\lambda),$$

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\text{where} \quad \mathrm{rank}\mathbf{A} &= \mathrm{trace}\mathbf{A} = \mathrm{trace}[\mathbf{I} - \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top] \\
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&= n - \mathrm{trace}[(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{X}] = n - 2, \\
\text{and} \quad \lambda &= (\sigma^{-1}\mathbf{X}\boldsymbol{\beta})^\top \mathbf{A}(\sigma^{-1}\mathbf{X}\boldsymbol{\beta}) \\
&= \frac{1}{\sigma^2}\boldsymbol{\beta}^\top\mathbf{X}^\top[\mathbf{I} - \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top]\mathbf{X}\boldsymbol{\beta} \\
&= \frac{1}{\sigma^2}\boldsymbol{\beta}^\top[\mathbf{X}^\top\mathbf{X} - \mathbf{X}^\top\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{X}]\boldsymbol{\beta} = 0
\end{aligned}$$

$\square$

Distribution of SSE

Proof (cont'd).

Recall, $\mathbf{Y} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}) \quad \Rightarrow \quad \sigma^{-1}\mathbf{Y} \sim \mathcal{N}_n(\sigma^{-1}\mathbf{X}\boldsymbol{\beta}, \mathbf{I})$. Hence,

$$\frac{\text{SSE}}{\sigma^2} = (\sigma^{-1}\mathbf{Y})^\top \mathbf{A}(\sigma^{-1}\mathbf{Y}) \sim \chi^2_{\mathrm{rank}\mathbf{A}}(\lambda),$$

$$\begin{aligned}
\text{where} \quad \mathrm{rank}\mathbf{A} &= \mathrm{trace}\mathbf{A} = \mathrm{trace}[\mathbf{I} - \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top] \\
&= \mathrm{trace}\mathbf{I} - \mathrm{trace}[\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top] \\
&= n - \mathrm{trace}[(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{X}] = n-2, \\
\text{and} \quad \lambda &= (\sigma^{-1}\mathbf{X}\boldsymbol{\beta})^\top \mathbf{A}(\sigma^{-1}\mathbf{X}\boldsymbol{\beta}) \\
&= \frac{1}{\sigma^2}\boldsymbol{\beta}^\top\mathbf{X}^\top[\mathbf{I} - \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top]\mathbf{X}\boldsymbol{\beta} \\
&= \frac{1}{\sigma^2}\boldsymbol{\beta}^\top[\mathbf{X}^\top\mathbf{X} - \mathbf{X}^\top\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{X}]\boldsymbol{\beta} = 0
\end{aligned}$$

Finally, $\frac{\text{SSE}}{\sigma^2} \sim \chi^2_{n-2}$. $\square$

Distribution of SSR

Theorem. $\frac{\text{SSR}}{\sigma^2} \sim \chi_1^2$ (non-central)

Proof.

Distribution of SSR

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Let $\mathbf{1} := (1, \ldots, 1)$, then $\mathbf{1}^\top \mathbf{1} = n$

Distribution of SSR

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Let $\mathbf{1} := (1, \ldots, 1)$, then $\mathbf{1}^\top \mathbf{1} = n$, and $\bar{Y} = \frac{\mathbf{1}^\top \mathbf{Y}}{\mathbf{1}^\top \mathbf{1}}$.

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Let $\bar{\mathbf{Y}} := (\bar{Y}, \ldots, \bar{Y})$. Then $\bar{\mathbf{Y}} = \mathbf{1}(\mathbf{1}^\top \mathbf{1})^{-1} \mathbf{1}^\top \mathbf{Y}$

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$$\text{SSR} = \sum_{i=1}^{n} (\hat{Y}_i - \bar{Y})^2$$

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Let $\bar{\mathbf{Y}} := (\bar{Y}, \ldots, \bar{Y})$. Then $\bar{\mathbf{Y}} = \mathbf{1}(\mathbf{1}^\top \mathbf{1})^{-1} \mathbf{1}^\top \mathbf{Y}$

$$\text{SSR} = \sum_{i=1}^{n} (\hat{Y}_i - \bar{Y})^2 = (\hat{\mathbf{Y}} - \bar{\mathbf{Y}})^\top (\hat{\mathbf{Y}} - \bar{\mathbf{Y}})$$

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$$
\begin{aligned}
\text{SSR} &= \sum_{i=1}^{n} (\hat{Y}_i - \bar{Y})^2 = (\hat{\mathbf{Y}} - \bar{\mathbf{Y}})^\top (\hat{\mathbf{Y}} - \bar{\mathbf{Y}}) \\
&= [\mathbf{X}(\mathbf{X}^\top \mathbf{X})^{-1}\mathbf{X}^\top \mathbf{Y} - \mathbf{1}(\mathbf{1}^\top \mathbf{1})^{-1}\mathbf{1}^\top \mathbf{Y}]^\top [\mathbf{X}(\mathbf{X}^\top \mathbf{X})^{-1}\mathbf{X}^\top \mathbf{Y} - \mathbf{1}(\mathbf{1}^\top \mathbf{1})^{-1}\mathbf{1}^\top \mathbf{Y}]
\end{aligned}
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Let $\bar{\mathbf{Y}} := (\bar{Y}, \ldots, \bar{Y})$. Then $\bar{\mathbf{Y}} = \mathbf{1}(\mathbf{1}^\top \mathbf{1})^{-1}\mathbf{1}^\top \mathbf{Y}$

$$
\begin{aligned}
\text{SSR} &= \sum_{i=1}^{n} (\hat{Y}_i - \bar{Y})^2 = (\hat{\mathbf{Y}} - \bar{\mathbf{Y}})^\top (\hat{\mathbf{Y}} - \bar{\mathbf{Y}}) \\
&= [\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{Y} - \mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top\mathbf{Y}]^\top [\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{Y} - \mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top\mathbf{Y}] \\
&= \mathbf{Y}^\top [\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top - \mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top]^\top [\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top - \mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top]\mathbf{Y}
\end{aligned}
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Let $\bar{\mathbf{Y}} := (\bar{Y}, \ldots, \bar{Y})$. Then $\bar{\mathbf{Y}} = \mathbf{1}(\mathbf{1}^\top \mathbf{1})^{-1}\mathbf{1}^\top \mathbf{Y}$

$$\begin{aligned}
\text{SSR} &= \sum_{i=1}^{n} (\hat{Y}_i - \bar{Y})^2 = (\hat{\mathbf{Y}} - \bar{\mathbf{Y}})^\top (\hat{\mathbf{Y}} - \bar{\mathbf{Y}}) \\
&= [\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{Y} - \mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top\mathbf{Y}]^\top[\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{Y} - \mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top\mathbf{Y}] \\
&= \mathbf{Y}^\top[\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top - \mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top]^\top[\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top - \mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top]\mathbf{Y}
\end{aligned}$$

The matrix $\mathbf{B} := \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top - \mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top$ is symmetric and idempotent (check).

Distribution of SSR

Theorem. $\frac{\text{SSR}}{\sigma^2} \sim \chi_1^2$ (non-central)

Proof.
Let $\mathbf{1} := (1, \ldots, 1)$, then $\mathbf{1}^\top \mathbf{1} = n$, and $\bar{Y} = \frac{\mathbf{1}^\top \mathbf{Y}}{\mathbf{1}^\top \mathbf{1}}$.

Let $\bar{\mathbf{Y}} := (\bar{Y}, \ldots, \bar{Y})$. Then $\bar{\mathbf{Y}} = \mathbf{1}(\mathbf{1}^\top \mathbf{1})^{-1}\mathbf{1}^\top \mathbf{Y}$

$$\begin{aligned}
\text{SSR} &= \sum_{i=1}^{n} (\hat{Y}_i - \bar{Y})^2 = (\hat{\mathbf{Y}} - \bar{\mathbf{Y}})^\top (\hat{\mathbf{Y}} - \bar{\mathbf{Y}}) \\
&= [\mathbf{X}(\mathbf{X}^\top \mathbf{X})^{-1}\mathbf{X}^\top \mathbf{Y} - \mathbf{1}(\mathbf{1}^\top \mathbf{1})^{-1}\mathbf{1}^\top \mathbf{Y}]^\top [\mathbf{X}(\mathbf{X}^\top \mathbf{X})^{-1}\mathbf{X}^\top \mathbf{Y} - \mathbf{1}(\mathbf{1}^\top \mathbf{1})^{-1}\mathbf{1}^\top \mathbf{Y}] \\
&= \mathbf{Y}^\top [\mathbf{X}(\mathbf{X}^\top \mathbf{X})^{-1}\mathbf{X}^\top - \mathbf{1}(\mathbf{1}^\top \mathbf{1})^{-1}\mathbf{1}^\top]^\top [\mathbf{X}(\mathbf{X}^\top \mathbf{X})^{-1}\mathbf{X}^\top - \mathbf{1}(\mathbf{1}^\top \mathbf{1})^{-1}\mathbf{1}^\top]\mathbf{Y}
\end{aligned}$$

The matrix $\mathbf{B} := \mathbf{X}(\mathbf{X}^\top \mathbf{X})^{-1}\mathbf{X}^\top - \mathbf{1}(\mathbf{1}^\top \mathbf{1})^{-1}\mathbf{1}^\top$ is symmetric and idempotent (check). Hence, $\text{SSR} = \mathbf{Y}^\top \mathbf{B}^\top \mathbf{B}\mathbf{Y} = \mathbf{Y}^\top \mathbf{B}\mathbf{Y}$.

Distribution of SSR

Proof (cont'd).

Recall, $\mathbf{Y} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I})$

Distribution of SSR

Proof (cont'd).

Recall, $\mathbf{Y} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}) \quad \Rightarrow \quad \sigma^{-1}\mathbf{Y} \sim \mathcal{N}_n(\sigma^{-1}\mathbf{X}\boldsymbol{\beta}, \mathbf{I})$.

Distribution of SSR

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Recall, $\mathbf{Y} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}) \quad \Rightarrow \quad \sigma^{-1}\mathbf{Y} \sim \mathcal{N}_n(\sigma^{-1}\mathbf{X}\boldsymbol{\beta}, \mathbf{I})$. Hence,

$$\frac{\mathrm{SSR}}{\sigma^2} = (\sigma^{-1}\mathbf{Y})^\top \mathbf{B}(\sigma^{-1}\mathbf{Y})$$

$\square$

Distribution of SSR

Proof (cont'd).

Recall, $\mathbf{Y} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}) \quad \Rightarrow \quad \sigma^{-1}\mathbf{Y} \sim \mathcal{N}_n(\sigma^{-1}\mathbf{X}\boldsymbol{\beta}, \mathbf{I})$. Hence,

$$\frac{\mathrm{SSR}}{\sigma^2} = (\sigma^{-1}\mathbf{Y})^\top \mathbf{B}(\sigma^{-1}\mathbf{Y}) \sim \chi^2_{\mathrm{rank}\mathbf{B}}(\lambda),$$

$\square$

Distribution of SSR

Proof (cont'd).

Recall, $\mathbf{Y} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}) \quad \Rightarrow \quad \sigma^{-1}\mathbf{Y} \sim \mathcal{N}_n(\sigma^{-1}\mathbf{X}\boldsymbol{\beta}, \mathbf{I})$. Hence,

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where $\quad \mathrm{rank}\mathbf{B} = \mathrm{trace}\mathbf{B}$

$\square$

Distribution of SSR

Proof (cont'd).

Recall, $\mathbf{Y} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}) \quad \Rightarrow \quad \sigma^{-1}\mathbf{Y} \sim \mathcal{N}_n(\sigma^{-1}\mathbf{X}\boldsymbol{\beta}, \mathbf{I})$. Hence,

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$\square$

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Proof (cont'd).

Recall, $\mathbf{Y} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2 \mathbf{I}) \quad \Rightarrow \quad \sigma^{-1}\mathbf{Y} \sim \mathcal{N}_n(\sigma^{-1}\mathbf{X}\boldsymbol{\beta}, \mathbf{I})$. Hence,

$$\frac{\mathrm{SSR}}{\sigma^2} = (\sigma^{-1}\mathbf{Y})^\top \mathbf{B} (\sigma^{-1}\mathbf{Y}) \sim \chi^2_{\mathrm{rank}\mathbf{B}}(\lambda),$$

where

$$\begin{aligned} \mathrm{rank}\mathbf{B} &= \mathrm{trace}\mathbf{B} = \mathrm{trace}[\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top - \mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top] \\ &= \mathrm{trace}[\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top] - \mathrm{trace}[\mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top] \end{aligned}$$

$\square$

Distribution of SSR

Proof (cont'd).

Recall, $\mathbf{Y} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}) \quad \Rightarrow \quad \sigma^{-1}\mathbf{Y} \sim \mathcal{N}_n(\sigma^{-1}\mathbf{X}\boldsymbol{\beta}, \mathbf{I})$. Hence,

$$\frac{\text{SSR}}{\sigma^2} = (\sigma^{-1}\mathbf{Y})^\top \mathbf{B}(\sigma^{-1}\mathbf{Y}) \sim \chi^2_{\mathrm{rank}\mathbf{B}}(\lambda),$$

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\mathrm{rank}\mathbf{B} &= \mathrm{trace}\mathbf{B} = \mathrm{trace}[\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top - \mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top] \\
&= \mathrm{trace}[\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top] - \mathrm{trace}[\mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top] \\
&= \mathrm{trace}[(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{X}] - \mathrm{trace}[(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top\mathbf{1}]
\end{aligned}$$

$\square$

Distribution of SSR

Proof (cont'd).

Recall, $\mathbf{Y} \sim \mathcal{N}_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}) \quad \Rightarrow \quad \sigma^{-1}\mathbf{Y} \sim \mathcal{N}_n(\sigma^{-1}\mathbf{X}\boldsymbol{\beta}, \mathbf{I})$. Hence,

$$\frac{\mathrm{SSR}}{\sigma^2} = (\sigma^{-1}\mathbf{Y})^\top \mathbf{B}(\sigma^{-1}\mathbf{Y}) \sim \chi^2_{\mathrm{rank}\mathbf{B}}(\lambda),$$

where

$$\begin{aligned}
\mathrm{rank}\mathbf{B} &= \mathrm{trace}\mathbf{B} = \mathrm{trace}[\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top - \mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top] \\
&= \mathrm{trace}[\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top] - \mathrm{trace}[\mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top] \\
&= \mathrm{trace}[(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{X}] - \mathrm{trace}[(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top\mathbf{1}] \\
&= \mathrm{trace}\mathbf{I}_2 - \mathrm{trace}1 = 2 - 1 = 1
\end{aligned}$$

$\square$

Distribution of SSR

Proof (cont'd).

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Finally, $\frac{\mathrm{SSR}}{\sigma^2} \sim \chi^2_1$ (non-central). $\square$

Independence between SSR and SSE

Theorem. Let $\mathbf{Y} \sim \mathcal{N}_n(\boldsymbol{\mu}, \Sigma)$, and $\Sigma \succ \mathbf{0}$. Then two quadratic forms $\mathbf{Y}^\top \mathbf{A} \mathbf{Y}$ and $\mathbf{Y}^\top \mathbf{B} \mathbf{Y}$ are independent $\quad \Leftrightarrow \quad \mathbf{A} \Sigma \mathbf{B} = \mathbf{0}$.

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$\mathbf{A}\sigma^2\mathbf{I}\mathbf{B} = \sigma^2 \mathbf{A}\mathbf{B}$

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$$\mathbf{A} \sigma^2 \mathbf{I} \mathbf{B} = \sigma^2 \mathbf{A} \mathbf{B}$$

$$= \sigma^2 [\mathbf{I} - \mathbf{X}(\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top][\mathbf{X}(\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top - \mathbf{1}(\mathbf{1}^\top \mathbf{1})^{-1} \mathbf{1}^\top]$$

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$$\mathbf{A}\sigma^2 \mathbf{I}\mathbf{B} = \sigma^2 \mathbf{A}\mathbf{B}$$
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$$= \sigma^2 [\mathbf{X}(\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top - \mathbf{X}(\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top - \mathbf{1}(\mathbf{1}^\top \mathbf{1})^{-1} \mathbf{1}^\top \mathbf{X}(\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top]$$

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$$
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&\mathbf{A}\sigma^2\mathbf{I}\mathbf{B} = \sigma^2\mathbf{A}\mathbf{B} \\
&= \sigma^2[\mathbf{I} - \mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top][\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top - \mathbf{1}(\mathbf{1}^\top\mathbf{1})^{-1}\mathbf{1}^\top] \\
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&= \mathbf{0}
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&= \mathbf{0}
\end{aligned}
$$

Hence, SSE and SSR are independent. $\square$