Regression Models for Quantitative and Qualitative Predictors

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Regression Analysis

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- We can no longer interpret β_1 and β_{11} "as usual." Cannot hold x_1 constant and increase x_1^2 by one unit, or vice-versa!
- Can be done easily in R, e.g. `lm(y ~ x + x*x)`.

General notes on fitting polynomials

- Predictors can be first centered:

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where $\bar{x}_j = \frac{1}{n}\sum_{i=1}^{n} x_{ij}$.

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- Polynomials of degree ≥ 4 should rarely be used. High-degree polynomials have wiggly behavior and can provide extremely poor out of sample prediction. Extrapolation is particularly dangerous.

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- "Hierarchical model building," (p. 299) stipulates that a model containing a particular term should also contain all terms of lower order including the cross-product terms.
- Degree of cross-product term is obtained by summing power for each predictor. E.g. the degree of $\beta_{1123} x_{i1}^2 x_{i2} x_{i3}$ is $2 + 1 + 1 = 4$.

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- Added variable plots are a refined plot to help figure out if the "non-linear" pattern is there when other variables are added (Section 10.1)
- With lots of predictors, say $k \geq 5$, it is easier to reduce to important first-order predictors, look for possible pairwise interactions (if necessary), and then see if any of the residual plots look curved; if so, add quadratic term(s).

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  finally leaving only the first-order terms as important:

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How the mean changes depends on the other variable.

Interactions

A model with no interactions is like a sheet of paper held "flat." Adding a pairwise interaction is like twisting the two ends of the paper.

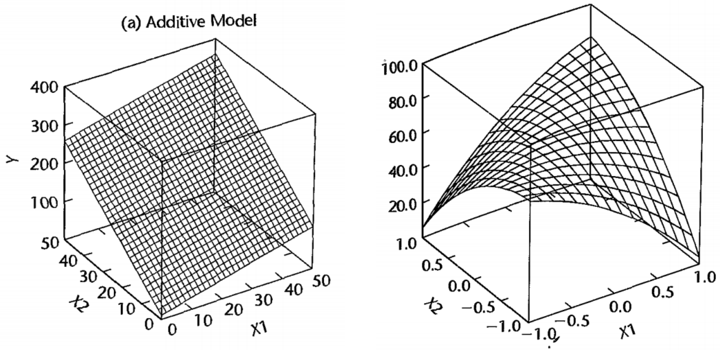

Figure: No interacations (left), With interactions (right)

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- A researcher will often have "an idea" of which variables might interact. This can be helpful.
- Can also start with a first-order model, then add interactions one at a time (forward selection!) using R.

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In R, there is a way to define categorical variables; then calling `lm(y ~ c)` will create and handle all dummy variables automatically.

Partial F-tests can be used to see whether an entire categorical predictor can be dropped from the model (all of the dummy variables at once).

Creating zero-one dummies

Define $z_1, z_2, \ldots, z_{l-1}$ as follows:

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which gives

$$\begin{aligned} \mathrm{E}[Y] &= \beta_0 + \beta_1 && \text{when} \quad c = \text{category}_1 \\ \mathrm{E}[Y] &= \beta_0 + \beta_2 && \text{when} \quad c = \text{category}_2 \\ \mathrm{E}[Y] &= \beta_0 && \text{when} \quad c = \text{category}_3 \end{aligned}$$

β_1 and β_2 are *offsets to baseline* mean.

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Example: Say we have two categorical predictors, $x \in \{1, 2, 3\}$ and $z \in \{1, 2, 3, 4\}$. An additive model is

$$\begin{aligned}\mathrm{E}[Y] = \beta_0 + \beta_1 \mathbb{I}[x = 1] + \beta_2 \mathbb{I}[x = 2] \\ + \beta_3 \mathbb{I}[z = 1] + \beta_4 \mathbb{I}[z = 2] + \beta_5 \mathbb{I}[z = 3].\end{aligned}$$

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Example: Insurance innovation

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```
> ins_data = read.csv("path/to/insurance.csv", header=FALSE)
> colnames(ins_data) = c('months', 'size', 'type')
> head(ins_data)
  months size   type
1     28  164  stock
2     31   85  stock
3     30  124  stock
4     21  175 mutual
5     12  210 mutual
6     15  272  stock
```

8.5 Categorical and quantitative interaction

Consider the following model

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \epsilon$$

where x_1 is size of firm and $x_2 = 1$ if stock firm and $x_2 = 0$ otherwise.

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Then

$$\begin{aligned} \mathrm{E}[Y] &= (\beta_0 + \beta_2) + (\beta_1 + \beta_{12}) x_1 && \text{for stock firm} \\ \mathrm{E}[Y] &= \beta_0 + \beta_1 x_1 && \text{otherwise} \end{aligned}$$

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Have different intercepts and different slopes.

8.5 Categorical and quantitative interaction

If we instead fit an additive model

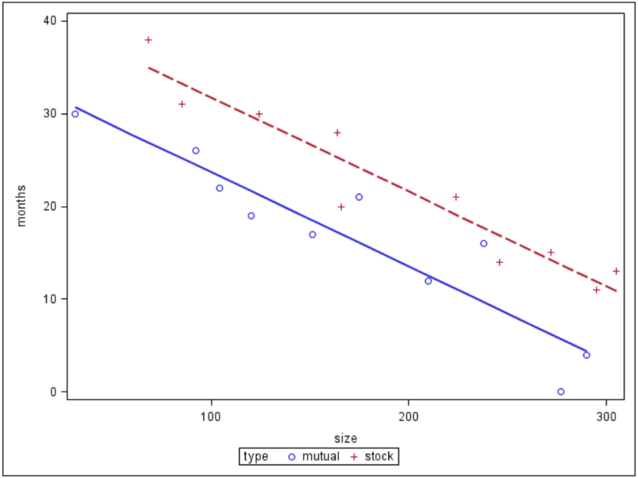
$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \epsilon$$

then

$$\begin{aligned} \mathrm{E}[Y] &= (\beta_0 + \beta_2) + \beta_1 x_1 && \text{for stock firm} \\ \mathrm{E}[Y] &= \beta_0 + \beta_1 x_1 && \text{otherwise} \end{aligned}$$

These are two parallel lines; the slope is the same. β_2 is how much better (or worse) stock firms do *at any firm size*

Insurance innovation



Do these look parallel?

Insurance innovation

```
> m1 = lm(months ~ size + type + size * type, data=ins_data)
> summary(m1)
...
Coefficients:
                 Estimate Std. Error t value Pr(>|t|)
(Intercept)    33.8383695  2.4406498  13.864 2.47e-10 ***
size           -0.1015306  0.0130525  -7.779 7.97e-07 ***
typestock       8.1312501  3.6540517   2.225   0.0408 *
size:typestock -0.0004171  0.0183312  -0.023   0.9821

Residual standard error: 3.32 on 16 degrees of freedom
Multiple R-squared:  0.8951,Adjusted R-squared:  0.8754
F-statistic: 45.49 on 3 and 16 DF,  p-value: 4.675e-08
```

Insurance innovation

Do the following yourself:

- Look at standard diagnostic plots.
- Test whether a quadratic in firm size is necessary.
- Test whether an interaction between firm size and type is necessary.
- Interpret the model.