Linear Regression with One Predictor Variable

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Regression Analysis

Introduction

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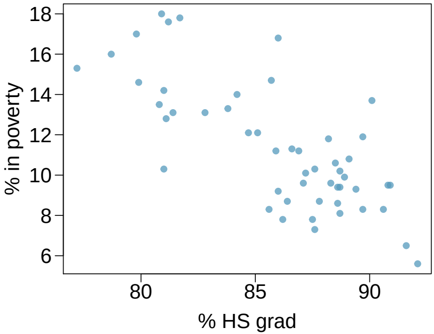
The goal is to tweak θ so that $y = f(x; \theta)$ fits the data in the best possible way.

Poverty vs. HS graduate rate

The *scatterplot* below shows the relationship between HS graduate rate in all 50 US states and DC and the % of residents who live below the poverty line (income below $23,050 for a family of 4 in 2012).



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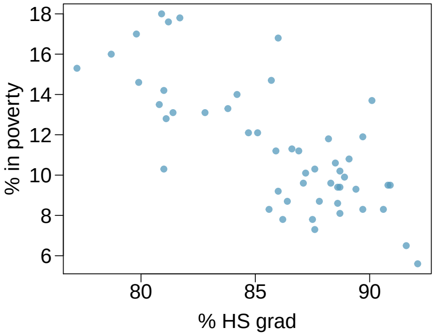

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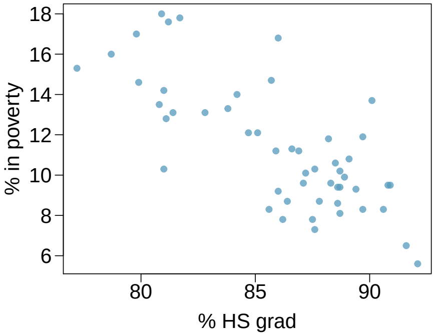

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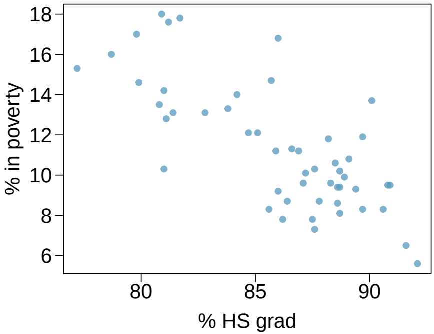

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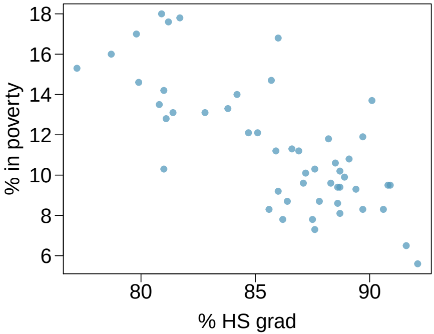

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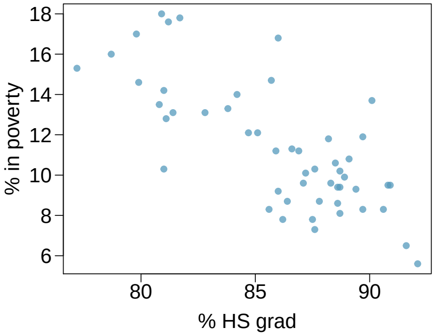

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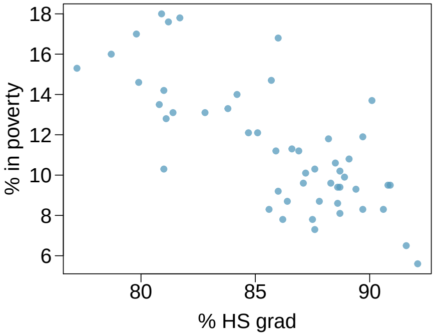

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Relationship
linear, negative, moderately strong

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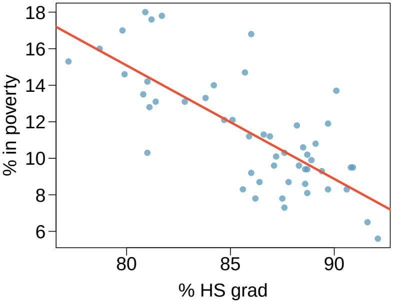
Interpretation of β_0 and β_1

Suppose we somehow estimated β_0 and β_1 in the Poverty vs HS grads example:

$$Y_i = 64.68 - 0.62 \cdot x_i + \epsilon_i$$



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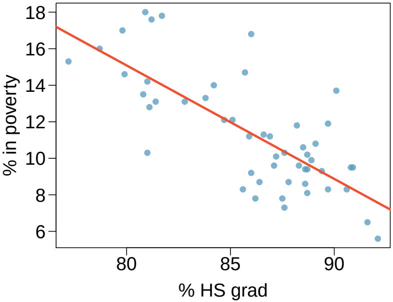

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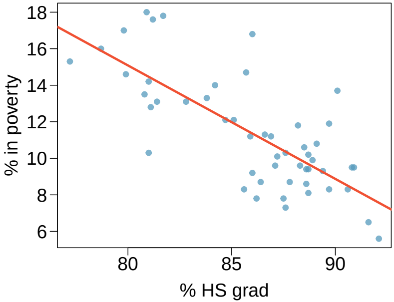

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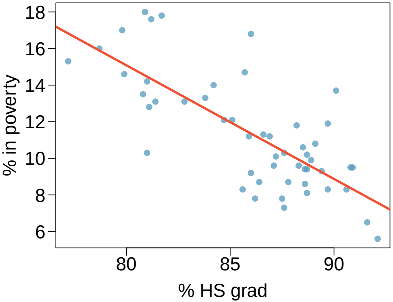

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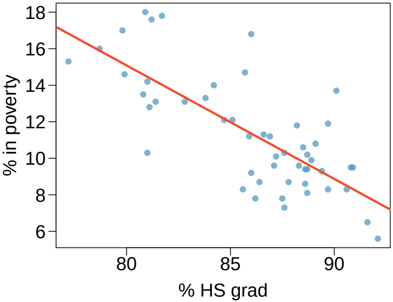

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How is 0.62 interpreted here? Increasing % of high-school grads by 1% is associated with 0.62% decrease in poverty rate *on average*.

Simple linear regression using matrices

Note the simple linear regression model for all examples

$$
\begin{aligned}
Y_1 &= \beta_0 + \beta_1 x_1 + \epsilon_1, \\
&\vdots \\
Y_n &= \beta_0 + \beta_1 x_n + \epsilon_n
\end{aligned}
$$

can be written in matrix terms as

$$
\underbrace{\begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}}_{\mathbf{X}} \underbrace{\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}}_{\boldsymbol{\beta}} + \underbrace{\begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_n \end{bmatrix}}_{\boldsymbol{\epsilon}},
$$

or equivalently

$$
\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}
$$

Remark on $\mathbf{X}^\top \mathbf{X}$

Notice that

$$\begin{aligned}\mathbf{X}^\top \mathbf{X} &= \begin{bmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_n \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \\ &= \begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix}\end{aligned}$$

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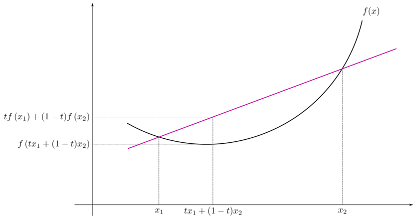

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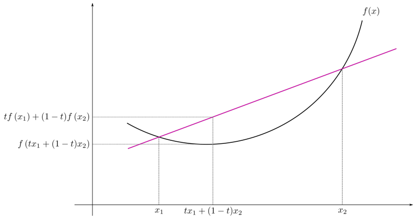

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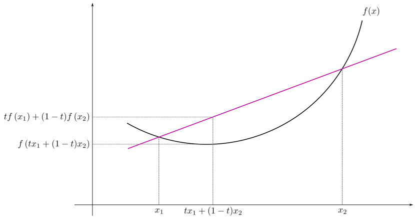

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Theorem. If $f'' \geq 0$, then f is convex.

Hessian

The **Hessian** matrix of $f : \mathbb{R}^d \to \mathbb{R}$ is a matrix of second-order partial derivatives:

$$\mathbf{H}(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_d \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_d^2} \end{bmatrix} \qquad \text{i.e.} \qquad \mathbf{H}_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

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If the partial derivatives are continuous, the order of differentiation can be interchanged, so the Hessian matrix will be symmetric.

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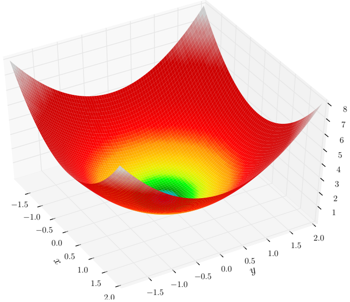

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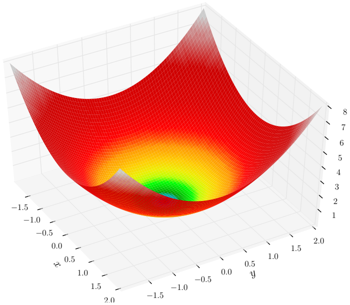

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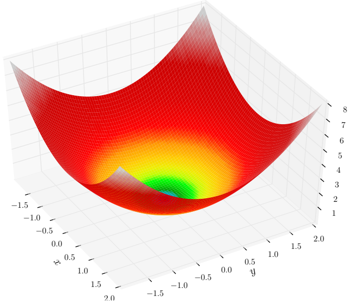

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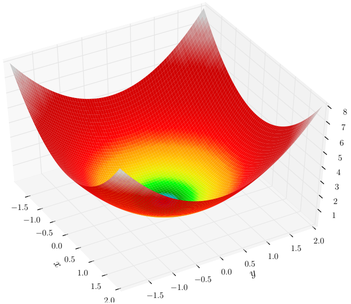

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$\mathbf{H} \succeq \mathbf{0}$ denotes that $\mathbf{H}$ is a **positive semi-definite** matrix. What does this mean? $\mathbf{a}^\top \mathbf{H} \mathbf{a} \geq 0$ for any $\mathbf{a} \in \mathbb{R}^d$.

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What is a stationary point? A point $\mathbf{x}_0$ is called **stationary** if $\nabla f(\mathbf{x}_0) = 0$.

Model

Convexity

Parameter Estimation

Least Squares Estimation (LSE)

Maximum Likelihood Estimation (MLE)

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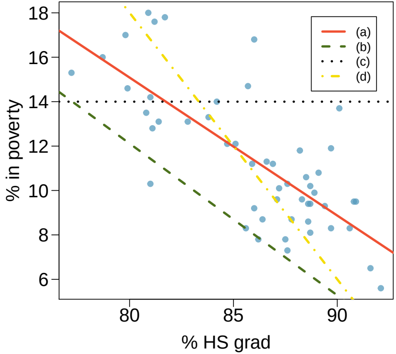

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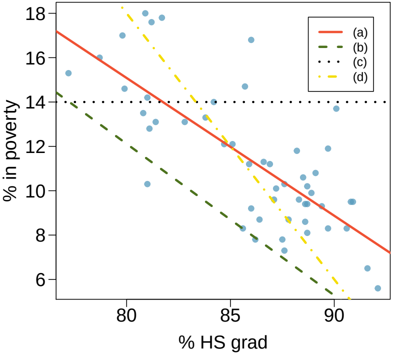

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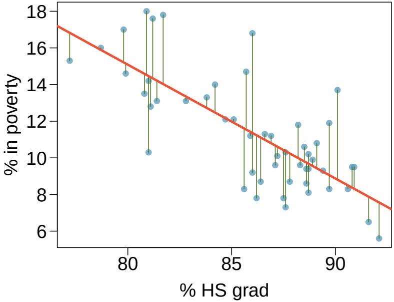

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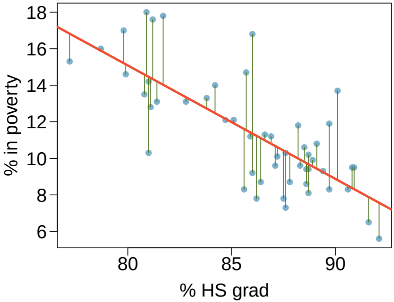

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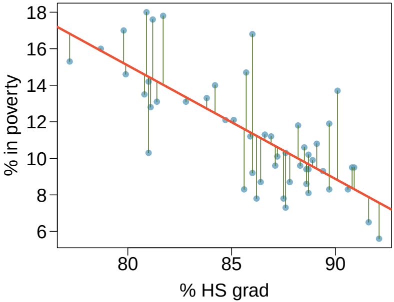

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This is called **least squares estimation (LSE)**.

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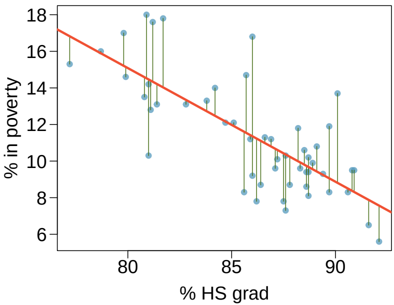

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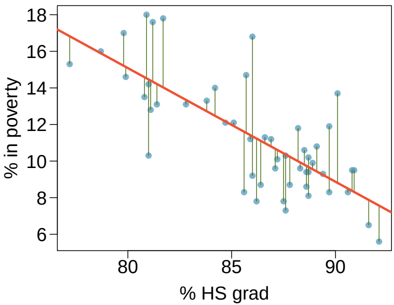

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Least squares estimation of β_0 and β_1

Theorem. The function $Q(\beta_0, \beta_1) = \sum_{i=1}^{n}[Y_i - (\beta_0 + \beta_1 x_i)]^2$ has the global minimum at

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Two equations in two unknowns

Setting these equal to zero, we have

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$$= 2(\mathbf{X}\mathbf{a})^\top(\mathbf{X}\mathbf{a}) = 2\|\mathbf{X}\mathbf{a}\|^2 \geq 0 \quad \Rightarrow \quad \mathbf{H} \succeq \mathbf{0}$$

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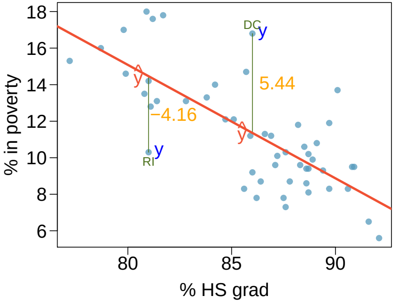

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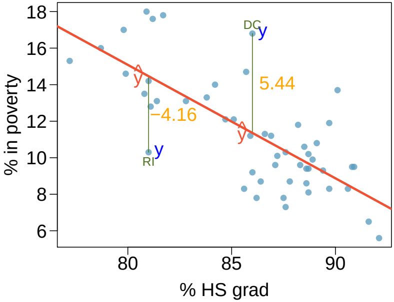

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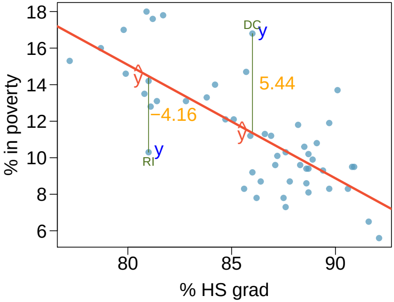

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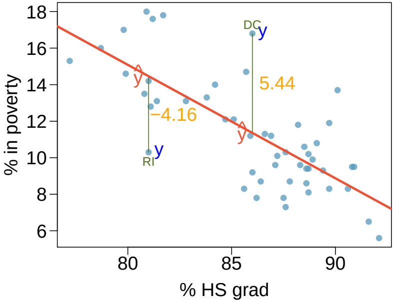

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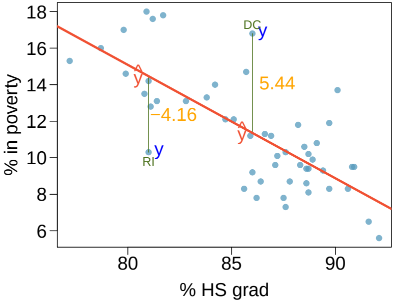

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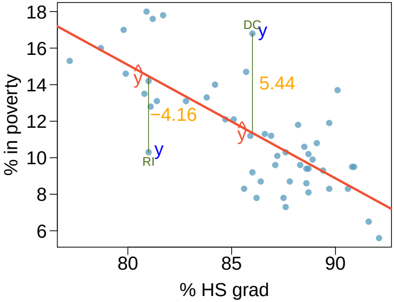

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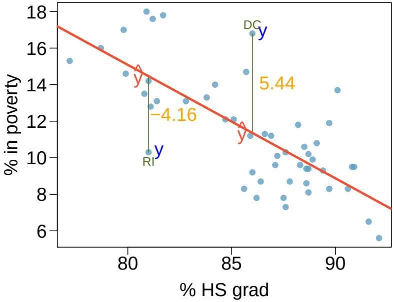

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$\Rightarrow$ MLE is a probabilistic justification for the LSE.

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$$\frac{n}{n-2}\hat{\sigma}^2 = \frac{1}{n-2}\sum_{i=1}^n e_i^2 = \frac{1}{n-2}\sum_{i=1}^n (Y_i - b_0 - b_1 x_i)^2 := \mathrm{MSE}.$$

which is referred to as **mean squared error** (**MSE**).

Parameter estimation in R: Poverty vs HS grad rate

https://raw.githubusercontent.com/zh3nis/MATH440/main/chp01/poverty.R

```
Coefficients:
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(Intercept) 64.78097
Graduates   -0.62122

Residual standard error: 2.082
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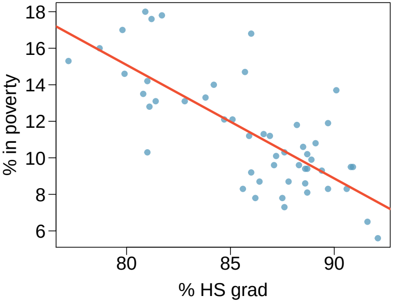

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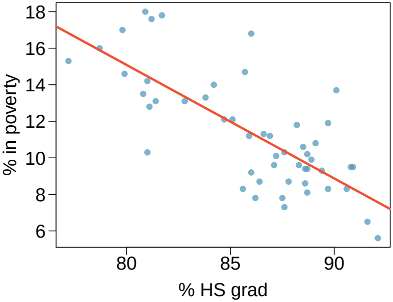

The regression line is $y = \underbrace{64.78}_{b_0} - \underbrace{0.62}_{b_1} x$.

$\sqrt{\mathrm{MSE}} = 2.08$ is an estimate of σ.