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The Brownian Bridge Does Not Offer a Consistent Advantage in Quasi-Monte Carlo Integration

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The Brownian bridge has been suggested as an effective method for reducing the quasi-Monte Carlo error for problems in finance. We give an example of a digital option where the Brownian bridge performs worse than the standard discretization. Hence, the Brownian bridge does not offer a consistent advantage in quasi-Monte Carlo integration. We consider integrals of functions of d variables with Gaussian weights such as the ones encountered in the valuation of financial derivatives and in risk management. Under weak assumptions on the class of functions, we study quasi-Monte Carlo methods that are based on different covariance matrix decompositions. We show that different covariance matrix decompositions lead to the same worst case quasi-Monte Carlo error and are, therefore, equivalent.

Key Words: multi-dimensional integration; quadrature; Monte Carlo methods; low discrepancy sequences; quasi-Monte Carlo methods.

1. INTRODUCTION

Monte Carlo simulation is an important tool for pricing and risk management of complex financial instruments. The reason is that many problems require the numerical evaluation of high dimensional integrals and Monte Carlo can approximate these integrals with expected error $O(n^{-1/2})$, independently of the number of dimensions.

Quasi-Monte Carlo methods, which use points from low discrepancy sequences [6, 13] instead of random numbers, have been found significantly superior to Monte Carlo for many high dimensional problems not only in finance but also in physics [1, 2, 4, 5, 9, 10, 14–20, 24]. The worst

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case quasi-Monte Carlo error is $O(\log^d n/n)$, which even though is asymptotically superior to $O(n^{-1/2})$ does not explain the success of quasi-Monte Carlo in practice, where d is large and n is relatively small. There are a number of papers claiming practical limitations of quasi-Monte Carlo in high dimensions, for example, see [3, 8]. This has led some to suggest that perhaps finance problems are not really high dimensional; that they depend on only a few important variables. Although there is no universally accepted explanation, Sloan and Woźniakowski [22] may offer a possible answer to the question of why quasi-Monte Carlo beats Monte Carlo for high dimensional integrals in finance.

In this paper we study techniques aimed at speeding up quasi-Monte Carlo. They have been applied to integrals with Gaussian weights, which are typical in finance, and have been found successful in certain cases [1, 5, 9, 10]. They are motivated by the fact that some researchers believe that in high dimensions (for example, when the dimension is greater than 50) the first coordinates of low discrepancy points will be more uniformly distributed than the rest [1, 7, 9]. Roughly speaking, the techniques aim to modify the way that paths are simulated, from multidimensional Gaussian samples, so that the resulting integrands depend heavier on the presumably more uniform coordinates of these samples.

The Brownian bridge construction for pricing of financial derivatives is one of these techniques. It was suggested by Caflisch and Moskowitz [4] and by Morokoff and Caflisch [9]. The authors of [1] state that "It attempts to use the best coordinates of each point to determine most of the structure of a path." The principal components construction is based on similar ideas, and was recently proposed by Ackworth *et al.* [1] for pricing of financial derivatives.

These constructions rely on properties of the multivariate Gaussian distribution. They are derived from different decompositions of its covariance matrix, and deal with the way that asset prices are simulated from a sample of d-dimensional points. In particular, they deal with the simulation of Brownian motion paths. They are not concerned with the way the asset prices are combined by the payoff function.

Although the Brownian bridge (or, another similar construction) can enjoy an advantage in a number of interesting cases we shall show that it is not a panacea because there are integrals for which it does not perform well. We provide an example of a digital option, where the underlying asset is lognormally distributed. For this option, the price obtained using the Brownian bridge is considerably worse than that obtained using the standard discretization (i.e., the one reflecting the Cholesky decomposition of the covariance matrix of the Gaussian distribution), not only when the dimension is large but also when the dimension is as small as two. This leads to the following conclusions:

• The Brownian bridge does not offer a consistent advantage in quasi-Monte Carlo integration for lognormally distributed asset prices.

• The argument attributing its success in a number of cases to its ability to reduce the effective dimension of a problem requires clarification [4, 5, 9, 10].

• A covariance matrix decomposition can be interpreted as a change to the integrand or to the sample points. Such a change may yield a harder problem relative to a fixed set of sample points.

Under weak assumptions, we propose a model for a class of integrands, which can be used for functions in finance, and study quasi-Monte Carlo integration with respect to different covariance matrix decompositions. Our goal is not to dismiss any of the potential benefits of the Brownian bridge, or any other construction, but to point out that the advantages of a method depend on the structure of both the integrands and the sample points. In particular, we show that the worst case error of any two quasi-Monte Carlo methods that use points from the same low discrepancy sequence but rely on different covariance matrix decompositions is the same and, therefore, covariance matrix decompositions are equivalent for quasi-Monte Carlo integration.

In the remaining sections we use the terms construction, discretization and decomposition interchangeably. We do so when we refer to a particular Gaussian covariance matrix decomposition, or a method that simulates a given d-dimensional Gaussian distribution using d normal random variables with zero mean and unit variance.

2. SIMULATION OF GAUSSIAN PROCESSES

Let $\{X_t,\ 0 \leqslant t \leqslant T\}$ be a Gaussian Markov process, which is sampled at d times $0 \leqslant t_1 \leqslant \cdots \leqslant t_d \leqslant T$. This results to a random vector $X = (X_{t_1}, \ldots, X_{t_d})$. X is normally distributed and we assume that its mean is zero and its covariance matrix is C. Let $f: \mathbb{R}^d \to \mathbb{R}$ be a given function and let $I(f) = E[f(X)]$ be the integral we want to compute. Then X can be simulated using $z = (z_1, \ldots, z_d)$, where z_j are independent normal random variables with mean zero and variance one. Indeed,

$$I_d(f) = E[f(x)] = (2\pi)^{-d/2} |C|^{-1/2} \int_{\mathbb{R}^d} f(x)\, e^{-\langle C^{-1}x,\, x\rangle/2}\, dx$$

$$= (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(Az)\, e^{-\|z\|^2/2}\, dz, \tag{1}$$

where $|C|$ denotes the determinant of the matrix C, $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^d$, A is any real $d \times d$ matrix that satisfies $AA^T = C$, and f is any integrable function.

For example, when paths of the Brownian motion are required, we have $X_t = W_t$. The standard construction generates the W_{t_j} by

$$W_{t_{j+1}} = W_{t_j} + \sqrt{t_{j+1} - t_j}\, z_{j+1}, \qquad j = 0, \ldots, d-1, \tag{2}$$

where $W_0 = 0$. Equivalently, in matrix notation we have

$$\begin{pmatrix} W_{t_1} \\ W_{t_2} \\ \vdots \\ W_{t_d} \end{pmatrix} = \mathscr{A} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_d \end{pmatrix},$$

where $\mathscr{A}$ is obtained from the Cholesky factorization of the matrix $C = \{\min(t_i, t_j)\}_{i,j=1}^d = \mathscr{A}\mathscr{A}^T$. When $\Delta t = T/d = t_{j+1} - t_j$, $j = 0, \ldots, d-1$, which is often the case in practice, the matrix $\mathscr{A}$ is given by

$$\mathscr{A} = \sqrt{\Delta t} \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 1 \end{pmatrix}. \tag{3}$$

The expectation of any integrable function of the discretized path of the Brownian motion is given by

$$E[f(W_{t_1}, \ldots, W_{t_d})] = E[f(Az)], \qquad z = (z_1, \ldots, z_d)^T.$$

The Brownian bridge construction first generates W_T, then using this value, and $W_0 = 0$, it generates $W_{T/2}$. It generates $W_{T/4}$ using W_0 and $W_{T/2}$, and it generates $W_{3T/4}$ using $W_{T/2}$ and W_T. The construction proceeds recursively filling in the mid points of the subintervals. Thus, the discretely sampled Brownian path is generated by determining its values at $T, T/2, T/4, 3T/4, \ldots, (d-1)\,T/d$ according to

$$
\begin{aligned}
W_T &= \sqrt{T}\, z_1 \\
W_{T/2} &= \frac{1}{2} W_T + \frac{\sqrt{T}}{2} z_2 \\
W_{T/4} &= \frac{1}{2} W_{T/2} + \frac{\sqrt{2T}}{4} z_3 \\
W_{3T/4} &= \frac{1}{2} (W_{T/2} + W_T) + \frac{\sqrt{2T}}{4} z_4 \\
&\vdots \\
W_{(d-1)\,T/d} &= \frac{1}{2} (W_{(d-2)\,T/d} + W_T) + \sqrt{\frac{T}{2d}}\, z_d \,.
\end{aligned}
$$

This results in a matrix B different from that of the Cholesky factorization, where $BB^T = C$. The Brownian bridge can be generalized to include unequal length intervals. For $t_{j+1} = t_j + \Delta t$, $j = 0, \ldots, d-1$, $\Delta t = T/d$, we can simulate a future value W_{t_k}, $k > j$, (given the value W_{t_j}) according to

$$
W_{t_k} = W_{t_j} + \sqrt{(k-j)\,\Delta t}\; z, \tag{4}
$$

where z follows the normal distribution $N(0, 1)$. We can simulate W_{t_i} at any intermediate point $t_j < t_i < t_k$ (given the values W_{t_j} and W_{t_k}) according to the Brownian bridge formula

$$
W_{t_i} = (1-\gamma)\, W_{t_j} + \gamma W_{t_k} + \sqrt{\gamma(1-\gamma)(k-j)\,\Delta t}\; z, \tag{5}
$$

where z follows the normal distribution $N(0, 1)$ and $\gamma = (i-j)/(k-j)$.

Recall that $(2\pi)^{-d/2} \int_{\mathbb{R}^d} g(Ax)\, e^{-\|x\|^2/2}\, dx = (2\pi)^{-d/2} \int_{\mathbb{R}^d} g(Bx)\, e^{-\|x\|^2/2}\, dx$, for all $d \times d$ matrices A, B such that $AA^T = BB^T = C$, and for any function g, g: $\mathbb{R}^d \to \mathbb{R}$, for which the integral is well defined.

Returning to the integral $I_d(f)$ of equation (1), the choice of the covariance matrix decomposition cannot affect the Monte Carlo error because it depends on the first and second moments of f which remain invariant under the different decompositions. In fact, any method with error depending on moments of f is unaffected by the choice of the decomposition of C.

On the other hand, the choice of a matrix A, $AA^T = C$, affects quasi-Monte Carlo. It can be interpreted as a change in the integrand or as a change in the sample points. The deterministic error bound of quasi-Monte Carlo depends on the integrand and on the discrepancy of the sample points and it is important to consider both factors in choosing A.

As we have already mentioned, a number of recent papers show that the Brownian bridge construction has advantages over the standard discretization for a number of problems; some of them require the calculation of integrals in 360 dimensions. However, in the next section we show a simple problem for which the opposite is true, for different values of d, both small and large.

3. A DIGITAL OPTION

We now show an example of an integrand for which the quasi-Monte Carlo convergence using the Brownian bridge construction is worse than that using the standard construction (or discretization), i.e., the one corresponding to the Cholesky decomposition of the covariance matrix C. We consider lognormally distributed asset prices and define the function we want to integrate. It is a rather simple option and its price can be computed analytically. We generate the asset prices according to the Brownian bridge and the standard discretization using low discrepancy sequences. We compare the convergence of the two constructions.

Assume that an asset price S follows the geometric Brownian motion

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t, \tag{6}$$

where the drift μ and the volatility σ are given constants, W is the Wiener process, and S_0 is the present price of the asset. Consider a time interval $[0, T]$ and, for simplicity, assume that it has been discretized at equally spaced the points $t_j = j\,\Delta t$, $j = 0, \ldots, d$, $\Delta t = T/d$. We generate the asset prices by simulating the Brownian motion using equation (2) in the case of the standard construction, and Eqs. (4), (5) in the case of the Brownian bridge.

We define the function

$$P(S_1, \ldots, S_d) = \frac{1}{d} \sum_{j=1}^{d} (S_j - S_{j-1})_+^0\, S_j, \tag{7}$$

where $(x)_+^0$ is equal to 1 if $x > 0$ and is 0 otherwise, $x \in \mathbb{R}$. This is the payoff function of a digital option, (see, [11] for the case $d = 1$). For $d > 1$, it can be viewed as a portfolio of digital options, or as a *ratchet* option since it allows an investor to lock in a gain [12]. Ratchet options are useful to fund managers, for instance, they use them to hedge *equity-linked index* annuities.

The expected value of P is given by

$$I_d(P) = E[P(S_1, \ldots, S_d)] = \frac{S_0}{d} \Phi(\beta(\mu, \sigma, \Delta t) + \sigma \sqrt{\Delta t}\,) \sum_{j=1}^{d} e^{\mu j \Delta t},$$

where $\beta(\mu, \sigma, \Delta t) = (\mu - \sigma^2/2)\sqrt{\Delta t}\,/\sigma$, and Φ denotes the cumulative distribution function of the standard normal distribution with mean zero and variance one. In particular, for $S_0 = 100$, $\mu = 4.5\%$, $\sigma = 30\%$, and $T = 1$, $I_2(P) = 60.40825$, $I_{64}(P) = 52.69044$ and $I_{128}(P) = 52.23314$.

The variance of P, $\mathrm{var}_d(P) = I_d(P^2) - I_d^2(P)$, can be derived by observing that

$$E[(S_j - S_{j-1})_+^0 (S_i - S_{i-1})_+^0 S_j S_i]$$
$$= S_0^2 \Phi(\beta(\mu, \sigma, \Delta t) + \sigma \sqrt{\Delta t}\,) \Phi(\beta(\mu, \sigma, \Delta t) + 2\sigma \sqrt{\Delta t}\,) e^{\mu(i+j)\Delta t} e^{\sigma^2 i \Delta t}, \qquad j > i$$

and

$$E[(S_j - S_{j-1})_+^0 S_j^2] = S_0^2 \Phi(\beta(\mu, \sigma, \Delta t) + 2\sigma \sqrt{\Delta t}\,) e^{2\mu j \Delta t} e^{\sigma^2 j \Delta t}.$$

In particular, for $S_0 = 100$, $\mu = 4.5\%$, $\sigma = 30\%$, and $T = 1$, we have $\mathrm{var}_2^{1/2}(P) = 48.24015$, $\mathrm{var}_{64}^{1/2}(P) = 14.67443$ and $\mathrm{var}_{128}^{1/2}(P) = 12.87541$. These quantities can be used to obtain the Monte Carlo error in each of the cases.

We used the Sobol and the generalized Faure low discrepancy sequences from FinDer[2] to carry out the simulations for $d = 2, 64, 128$. The generalized Faure low discrepancy sequence is due to Tezuka, see [23]. We found that the standard construction has consistently smaller error and converges faster than the Brownian bridge.

We first discuss the case $d = 2$. The performance of the Brownian bridge and the standard construction is significantly different. The standard construction converges very fast. Its relative error becomes less than 10^{-3} using about 1000 points. It becomes less than $5 \cdot 10^{-5}$ when the sample size is about $5 \cdot 10^4$ points. Both low discrepancy sequences perform equally well and their good convergence is maintained with and without skipping an initial part of the sequence.

The Brownian bridge, on the other hand, converges slower for both sequences. Without skipping, the relative error of the Sobol sequence does not become less than 10^{-3} even for sample size greater than $2.5 \cdot 10^5$ points. Skipping a number of terms improves its performance but it does not make it comparable to that of the standard discretization. An interesting case

[2] FinDer is a Columbia University software system.

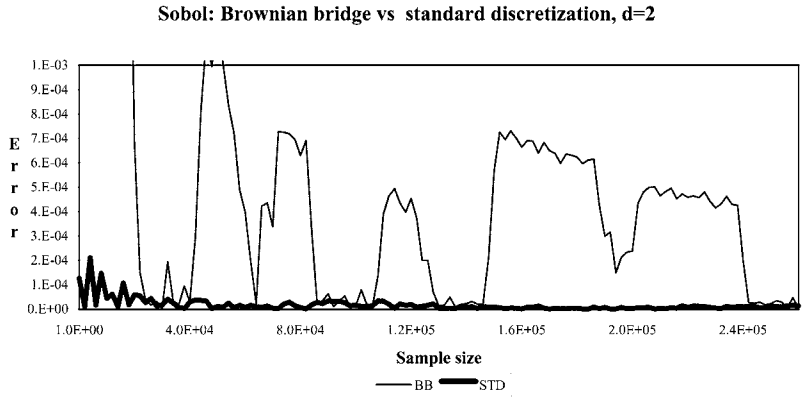

FIG. 1. The Sobol sequence converges slower with the Brownian bridge than with the standard discretization $d=2$.

occurs when $2^{17} = 131072$ terms of the sequence are skipped. Then the relative error oscillates according to the pattern shown in Figure 1. In general, the relative error of the Sobol sequence with the Brownian bridge is at least two to three times larger than the corresponding one using the standard construction.

The generalized Faure sequence with the Brownian bridge also leads to slower convergence than with the standard construction. A relative error of 10^{-3} requires at least 4000 sample points, while an error of 10^{-4} requires about 10^5 sample points.

Figures 1 and 2 compare the Brownian bridge to the standard construction for $d = 2$, using the Sobol and generalized Faure sequences,

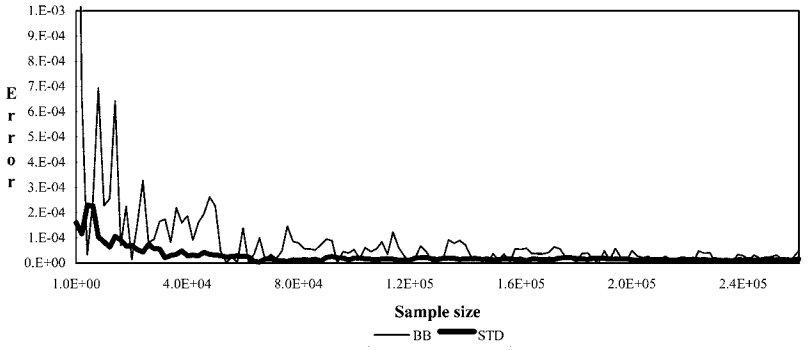

FIG. 2. The generalized Faure sequence converges slower with the Brownian bridge than with the standard discretization $d=2$.

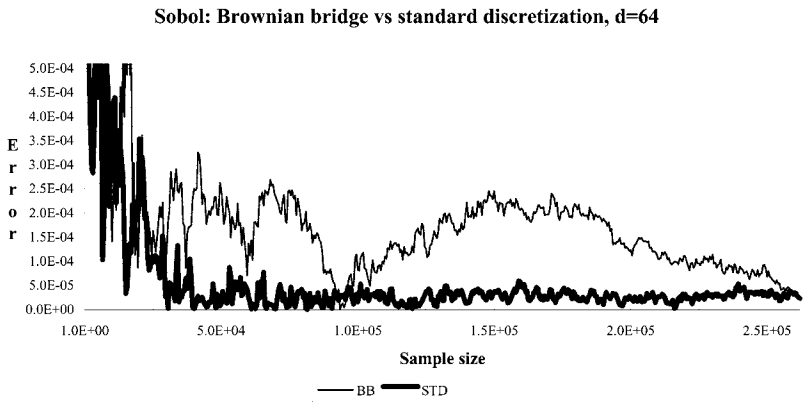

FIG. 3. The Sobol sequence converges slower with the Brownian bridge than with the standard discretization $d=64$.

respectively. The horizontal axis shows the sample size and the vertical axis the corresponding relative error.

For $d=64$ the situation is similar. The Sobol and generalized Faure sequences with the standard construction have error less than 10^{-3} with about 1500 points, and error 10^{-4} with about $5\cdot 10^4$ points. The Brownian bridge requires 6000 points for error 10^{-3}, and while the generalized Faure requires about $1.5\cdot 10^5$ points for error 10^{-4}, Sobol achieves this with about $2.2\cdot 10^5$ points.

Figures 3 and 4 compare the Brownian bridge to the standard construction for $d=64$, using the Sobol and generalized Faure sequence,

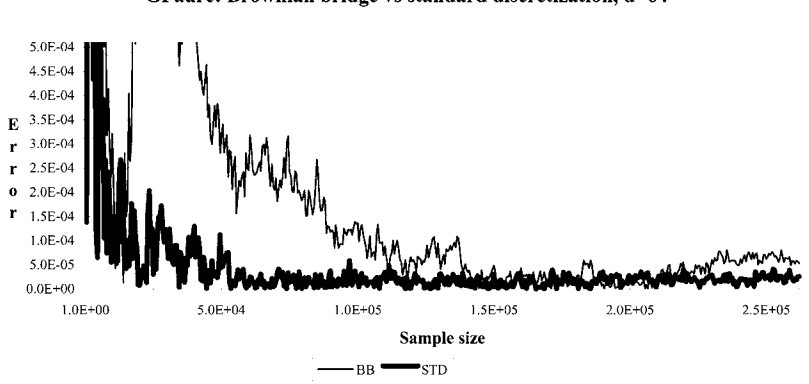

FIG. 4. The generalized Faure sequence converges slower with the Brownian bridge than with the standard discretization $d=64$.

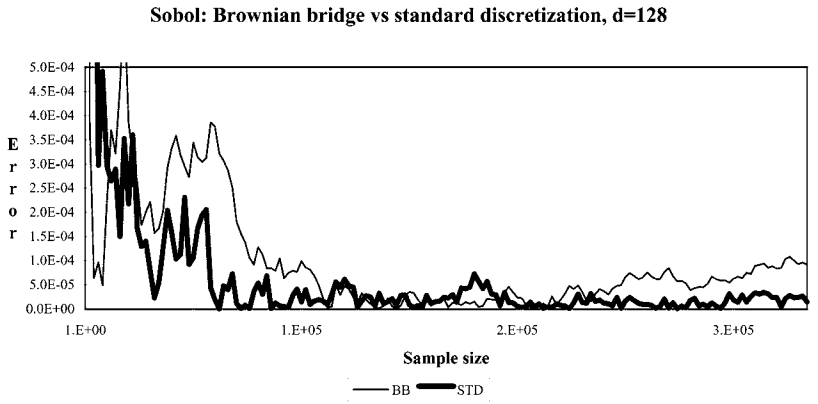

FIG. 5. The Sobol sequence converges slower with the Brownian bridge than with the standard discretization $d=128$.

respectively. The horizontal axis shows the sample size and the vertical axis the corresponding relative error.

For $d=128$ either sequence with either construction achieves accuracy 10^{-3} with about 6000 points. However, when the accuracy demand increases to 10^{-4} we see both Sobol and generalized Faure with the standard discretization requiring about $5\cdot 10^4$ points, while with the Brownian bridge requiring about $3\cdot 10^5$ points. It is also important to point out that both sequences with the Brownian bridge are sensitive to the number of terms that are skipped. The convergence of the standard discretization does not show this sensitivity.

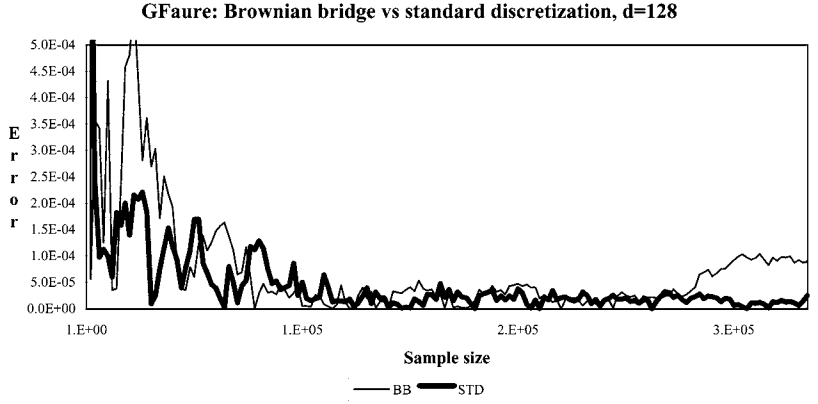

FIG. 6. The generalized Faure sequence converges slower with the Brownian bridge than with the standard discretization $d=128$.

Figures 5 and 6 compare the Brownian bridge to the standard construction for $d = 128$, using the Sobol and generalized Faure sequence, respectively. The horizontal axis shows the sample size and the vertical axis the corresponding relative error.

Our tests show that the convergence of the Brownian bridge is much slower than that of the standard discretization for $d = 2, 64, 128$. Both low discrepancy sequences lead to fast convergence with the standard discretization. There are clear differences in the convergence between the two low discrepancy sequences when the Brownian bridge is used.

Hence, for the approximation of multi dimensional integrals derived from problems in finance with lognormally distributed assets, the Brownian bridge does not offer a consistent advantage in quasi-Monte Carlo integration.

The Brownian bridge gave consistently worse results compared to those of the standard discretization. It made the problem harder and introduced an undesirable sensitivity with respect to the low discrepancy sequence and the number of terms that are skipped.

4. EQUIVALENCE OF COVARIANCE DECOMPOSITIONS

The authors of [9] describing the Brownian bridge state that, "This reduces the effective dimension of the random walk simulation, which increases the accuracy of quasi-Monte Carlo." However, as we will see below, the knowledge of the underlying Gaussian process and the fact that equation (1) is independent of the covariance matrix decomposition are not sufficient to establish that a given decomposition is better than another. Platen [21] points out that in a variety of practical problems, such as integration, a pathwise approximation of the solution of a stochastic differential equation is not required and that a lot of computational effort has been wasted on simulations by missing this point.

We define a class of functions, that can be used to represent problems in finance, and show that the worst case error of quasi-Monte Carlo is independent of the decomposition. Therefore, all decompositions are equivalent in the worst case.

Since the Black Scholes model is important in finance, to motivate our discussion let us consider lognormally distributed asset prices (6). Let us also assume that time has been discretized at equally spaced moments.

The payoff function P of a path dependent financial derivative combines the simulated asset prices S_j, $j = 1, \ldots, d$, along a path of length d (i.e., a set of d different time moments) to obtain the simulated price of the financial derivative, i.e., $P(S_1, \ldots, S_d)$; see, the payoff function of the digital option (7).

For instance, when the standard discretization (2) is used, the asset prices are generated according to

$$S_{j+1} = S_j e^{(\mu-\sigma^2/2)\,\Delta t+\sigma z_{j+1}\sqrt{\Delta t}}, \qquad j=0, \ldots, d-1,$$

where S_0, μ, and σ are given, and z_j, $j=1, \ldots, d$, are independent normal random variables with mean zero and variance one. Using matrix notation we have

$$S_j = S_0 e^{(\mu-\sigma^2/2)\,j\,\Delta t+\sigma\langle \mathscr{A} z, e_j\rangle}, \tag{8}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^d$, $\mathscr{A}$ is the matrix of Eq. (3), e_j is the unit vector in $\mathbb{R}^d$ with 1 in coordinate j, and $z=(z_1, \ldots, z_d)$. To emphasize the dependence of S_j on the vectors z, e_j, and the matrix $\mathscr{A}$ we write $S_j = S_j(\langle \mathscr{A} z, e_j\rangle)$, $j=1\ldots, d$.

The price of the financial derivative P, $P\colon \mathbb{R}^d \to \mathbb{R}$, is then given by

$$\begin{aligned} I_d(P) &= E[P(S_1, \ldots, S_d)] \\ &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} P(S_1(\langle \mathscr{A} z, e_1\rangle), \ldots, S_d(\langle \mathscr{A} z, e_d\rangle))\, e^{-\|z\|^2/2}\, dz \\ &= (2\pi)^{-d/2}\, |C|^{-1/2} \int_{\mathbb{R}^d} P(S_1(\langle x, e_1\rangle), \ldots, S_d(\langle x, e_d\rangle))\, e^{-\langle C^{-1}x,\quad x\rangle/2}\, dx, \end{aligned}$$

where $C = \mathscr{A}\mathscr{A}^T$ and $|C|$ denotes the determinant of the matrix C.

Observe that the above equation is a special case of (1) and has been derived for functions that correspond to payoff functions of financial derivatives, where the underlying asset is lognormally distributed. It is the integral of a function of the form

$$g(x) = \mathscr{G}(\langle x, \xi_1\rangle, \ldots, \langle x, \xi_d\rangle), \qquad x \in \mathbb{R}^d, \tag{9}$$

where $\mathscr{G}\colon \mathbb{R}^d \to \mathbb{R}$ is a given function, and $\xi_j \in \mathbb{R}^d$ are given vectors, $j=1, \ldots, d$.

Let $g\colon \mathbb{R}^d \to \mathbb{R}$ be a fixed function, integrable with respect to the d-dimensional Gaussian distribution with mean zero and covariance $C = C^T > 0$. Recall Eq. (1) for the definition of $I_d(g)$. Consider a fixed $d \times d$ matrix A such that $AA^T = C$. It is easy to show that: $B = AU^T$ for some orthonormal matrix U, i.e., $U^T U = I$, iff $BB^T = C$. Thus any other decomposition of C can be represented as replacing A by AU for some orthonormal $d \times d$ matrix U. We define the class of functions F_g by

$$F_g = \{f\colon \mathbb{R}^d \to \mathbb{R} \mid f(x) = g(AUA^{-1}x),\ x \in \mathbb{R}^d, \text{ where } U^T U = I\}.$$

We will now define the quasi-Monte Carlo methods that we wish to analyze, and the error criterion. Since our goal is to examine the effect of different covariance matrix decompositions we will assume that the sample points are arbitrary but fixed and study the error of quasi-Monte Carlo with respect to the different decompositions $BB^T = C$.

We consider quasi-Monte Carlo methods that approximate the integral $I_d(f)$, $f \in F_g$, by the average

$$Q_{\{x_i\},n,d,B}(f) = \frac{1}{n}\sum_{i=1}^{n} f(Bx_i),$$

where B is a matrix such that $BB^T = C$, and $x_i \in \mathbb{R}$, $i = 1, \ldots, n$, are given sample points.

For $f \in F_g$ we define the error of the method $Q_{\{x_i\},n,d,B}$ by

$$e(Q_{\{x_i\},n,d,B}, f) = |I_d(f) - Q_{\{x_i\},n,d,B}(f)|.$$

We define the worst case error of the method $Q_{\{x_i\},n,d,B}$ in the class F_g by

$$e(Q_{\{x_i\},n,d,B}, F_g) = \sup_{f \in F_g} e(Q_{\{x_i\},n,d,B}, f).$$

THEOREM 4.1. *The quantity $e(Q_{\{x_i\},n,d,B}, F_g)$ is independent of the matrix B.*

Proof. Let $f \in F_g$ then $f(x) = g(AUA^{-1}x)$, for some orthonormal matrix U, $AA^T = C$, $x \in \mathbb{R}^d$. From equation (1) and the change of variable $x = Ut$ we obtain

$$I_d(g) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} g(Ax)\, e^{-\|x\|^2/2}\, dx = (2\pi)^{-d/2}\int_{\mathbb{R}^d} g(AUt)\, e^{-\|t\|^2/2}\, dt$$

$$= (2\pi)^{-d/2}\int_{\mathbb{R}^d} g(AUA^{-1}At)\, e^{-\|t\|^2/2}\, dt$$

$$= (2\pi)^{-d/2}\int_{\mathbb{R}^d} f(At)\, e^{-\|t\|^2/2}\, dt = I_d(f).$$

Thus $I_d(f) = I_d(g)$, $\forall f \in F_g$.

Consider any matrices B_1 and B_2, $B_1B_1^T = B_2B_2^T = C$. There exist orthogonormal matrices V_1 and V_2 such that $B_1 = AV_1^T$ and $B_2 = AV_2^T$. Let $f_1 \in F_g$. Then $f_1(B_1x) = g(AUA^{-1}B_1x) = g(AUV_1^Tx) = g(AUV_1^TB_2^{-1}B_2x)$, $x \in \mathbb{R}^d$. The function f_2 defined by

$$f_2(x) = g(AUV_1^TB_2^{-1}x) = g(AUV_1^TV_2A^{-1}x), \qquad x \in \mathbb{R}^d,$$

belongs to F_g since $UV_1^TV_2$ is orthonormal. Thus $f_1(B_1x) = f_2(B_2x)$, $x \in \mathbb{R}^d$, $I(f_1) = I_d(f_2)$ and $e(Q_{\{x_i\},n,d,B_1}, f_1) = e(Q_{\{x_i\},n,d,B_2}, f_2)$.

Since f_2 has been derived using f_1 the above implies that

$$e(Q_{\{x_i\},n,d,B_1}, F_g) \leqslant e(Q_{\{x_i\},n,d,B_2}, F_g).$$

Similarly we have

$$f_1(B_2x) = g(AUA^{-1}B_2x) = g(AUV_2^Tx) = g(AUV_2^TB_1^{-1}B_1x), \qquad x \in \mathbb{R}^d.$$

The function f_3 defined by $f_3(x) = g(AUV_2^TB_1^{-1}x) = g(AUV_2^TV_1A^{-1}x)$, belongs to F_g since $UV_2^TV_1$ is orthonormal. Thus $f_1(B_2x) = f_3(B_1x)$, $x \in \mathbb{R}^d$, $I_d(f_1) = I_d(f_3)$ and $e(Q_{\{x_i\},n,d,B_2}, f_1) = e(Q_{\{x_i\},n,d,B_1}, f_3)$.

Since f_3 has been derived using f_1 the above implies that

$$e(Q_{\{x_i\},n,d,B_2}, F_g) \leqslant e(Q_{\{x_i\},n,d,B_1}, F_g),$$

which completes the proof. ▮

A larger class of functions for which the same result holds can be defined by

$$F_{g,\mathcal{M}} = \{f: \mathbb{R}^d \to \mathbb{R} \mid f(x) = g(MAUA^{-1}x),\ x \in \mathbb{R}^d, \text{ where } U^TU = I,\ M \in \mathcal{M}\},$$

where $\mathcal{M}$ is a given compact set of matrices.

COROLLARY 4.1. *The quantity $e(Q_{\{x_i\},n,d,A}, F_{g,\mathcal{M}})$ is independent of the matrix A.*

Proof. The proof follows directly from Theorem 1 and we omit the details. ▮

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