

Activity 16

$$1. f(w) = (w - w_{LS})^T X^T X (w - w_{LS}) + C$$

a. • when $w = w_{LS}$

$$f(w) = 0 \quad X^T X \quad 0 + C$$

$$f(w) = C$$

• when $w \neq w_{LS}$

$$f(w) = Q X^T X Q > 0$$

$$f(w) = \text{positive definite} > 0$$

$$b. w_{LS} = V \Sigma^{-1} U^T y \quad X = U \Sigma V^T \quad V = I \quad \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{1/2} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{2 \times 4} \underbrace{\begin{bmatrix} 1 \\ 1/2 \\ 1 \\ 0 \end{bmatrix}}_{4 \times 1}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{1/2} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} 2 \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$w_{LS} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X = U \Sigma V^T$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}}_{2 \times 2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underbrace{\quad}_{4 \times 2}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$X^T X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix}$$

$$f(w) = (w - w_{LS})^T X^T X (w - w_{LS}) + C$$

$$= \begin{bmatrix} w_1 - w_{01} \\ w_2 - w_{02} \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} w_1 - w_{01} \\ w_2 - w_{02} \end{bmatrix}$$

$$= \begin{bmatrix} w_1 - w_{01} & w_2 - w_{02} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix}_{2 \times 2} \begin{bmatrix} w_1 - w_{01} \\ w_2 - w_{02} \end{bmatrix}_{2 \times 1}$$

$$= \begin{bmatrix} w_1 - w_{01} & w_2 - w_{02} \end{bmatrix} \begin{bmatrix} w_1 - w_{01} \\ 1/4 w_2 - w_{02} \end{bmatrix}$$

$$= (w_1 - w_{01})^2 + \frac{1}{4} (w_2 - w_{02})^2 + C$$

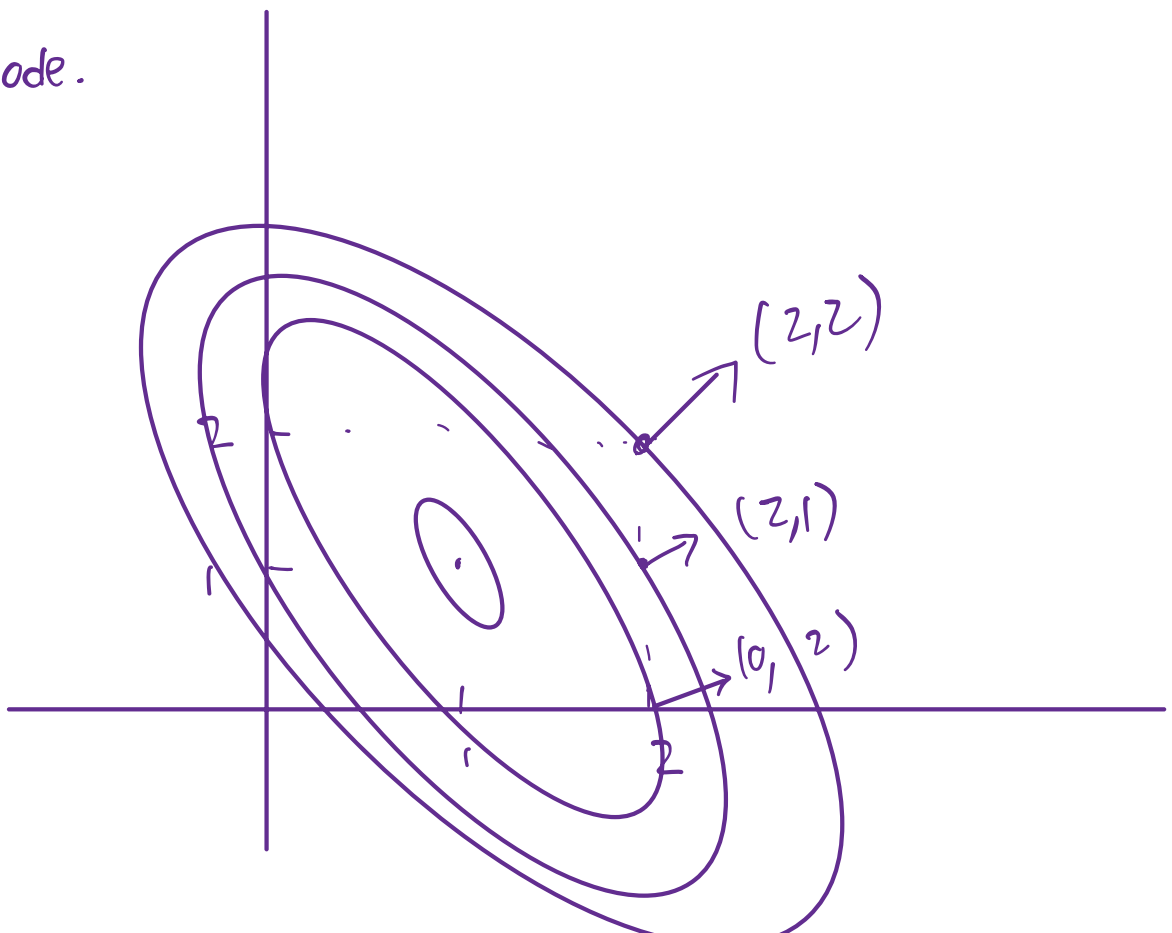
$$f(w) = (w_1 - 1)^2 + \frac{1}{4} (w_2 - 1)^2 + C$$

b. Plot \rightarrow code.

c. code

d. code

e.



2. a. max value for the step τ

$$0 < \tau < \frac{2}{\sigma_i^2}$$

$$0 < \tau < \frac{2}{1}$$

$$0 < \tau < 2$$

period_16_activity_starter

March 21, 2024

```
[1]: import numpy as np
import matplotlib.pyplot as plt
```

```
[2]: ## DO NOT Change
def graddescent(X,y,tau,w_init,it):
    """
    compute 10 iterations of gradient descent starting at w1
    w_{k+1}= w_k - tau*X'*(X*w_k - y)
    """
    W = np.zeros((w_init.shape[0],it))
    W[:,[0]] = w_init
    for k in range(it-1):
        W[:,[k+1]] = W[:,[k]] - tau * X.T @ (X @ W[:,[k]] - y)
    return W
```

1 Question 1b)

```
[3]: U = np.array([[1, 0], [0, 1], [0, 0], [0, 0]])
S = np.array([[1, 0], [0, 0.5]])
Sinv = np.linalg.inv(S)
V = np.eye(2)
X = U @ S @ V.T
y = np.array([[1], [0.5], [1], [0]])

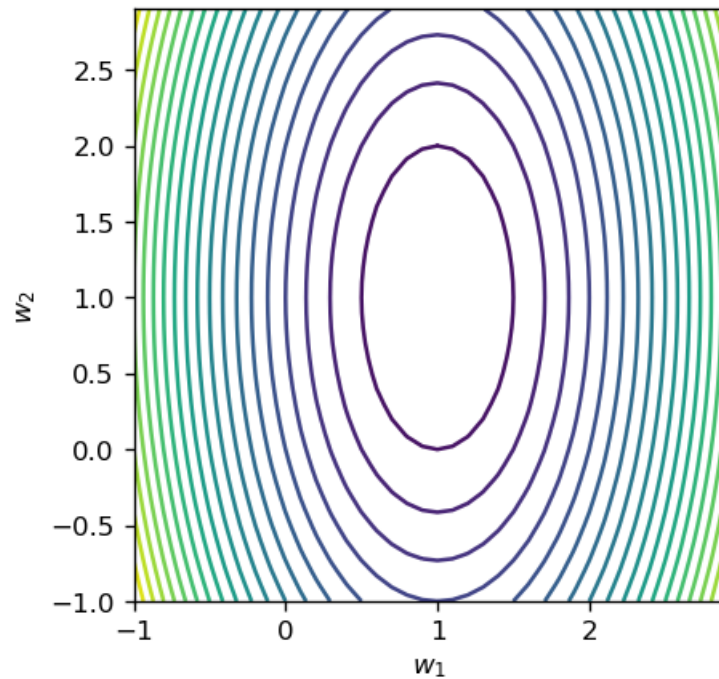
### Find Least Squares Solution
w_ls = V @ Sinv @ U.T @ y
c = y.T @ y - y.T @ X @ w_ls

### Find values of f(w), the contour plot surface for
w1 = np.arange(-1,3,.1)
w2 = np.arange(-1,3,.1)
fw = np.zeros((len(w1), len(w2)))
for i in range(len(w2)):
    for j in range(len(w1)):
        w = np.array([ w1[j], [w2[i]] ])
        fw[i,j] = (w-w_ls).T @ X.T @ X @ (w-w_ls) + c
```

```

### Plot the countours
plt.figure(num=None, figsize=(4, 4), dpi=120)
plt.contour(w1,w2,fw,20)
plt.xlim([-1,3])
plt.ylim([-1,3])
plt.xlabel('$w_1$')
plt.ylabel('$w_2$')
plt.axis('square');

```



2 Question 1c)

The sigma will affect the shape of ellipse (major and minor)

```

[5]: ## Copy and paste code from 1b
U = np.array([[1, 0], [0, 1], [0, 0], [0, 0]])
S = np.array([[1, 0], [0, 0.2]])
Sinv = np.linalg.inv(S)
V = np.eye(2)
X = U @ S @ V.T
y = np.array([[1], [0.2], [1], [0]])

### Find Least Squares Solution
w_ls = V @ Sinv @ U.T @ y

```

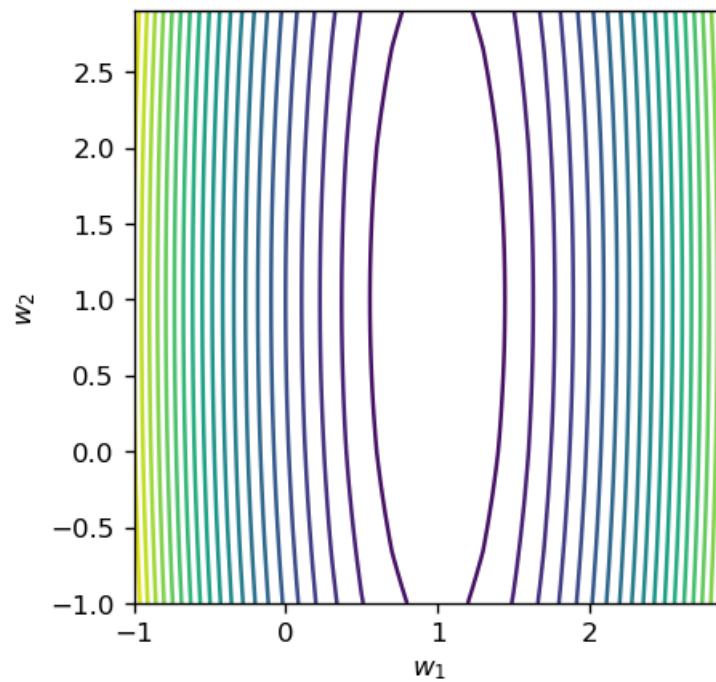
```

c = y.T @ y - y.T @ X @ w_ls

### Find values of  $f(w)$ , the contour plot surface for
w1 = np.arange(-1,3,.1)
w2 = np.arange(-1,3,.1)
fw = np.zeros((len(w1), len(w2)))
for i in range(len(w2)):
    for j in range(len(w1)):
        w = np.array([ w1[j], w2[i] ])
        fw[i,j] = (w-w_ls).T @ X.T @ X @ (w-w_ls) + c

### Plot the contours
plt.figure(num=None, figsize=(4, 4), dpi=120)
plt.contour(w1,w2,fw,20)
plt.xlim([-1,3])
plt.ylim([-1,3])
plt.xlabel('$w_1$')
plt.ylabel('$w_2$')
plt.axis('square');

```



3 Question 1d)

The V matrix will rotate the ellipse

```

[15]: ## Copy and paste code from 1b
      ## Copy and paste code from 1b
      import math
      U = np.array([[1, 0], [0, 1], [0, 0], [0, 0]])
      S = np.array([[1, 0], [0, 0.5]])
      Sinv = np.linalg.inv(S)
      # V = np.eye(2)
      V = np.array([[1/np.sqrt(2), 1/np.sqrt(2)], [1/np.sqrt(2), -1/np.sqrt(2)]])
      print(V.shape)
      X = U @ S @ V.T
      y = np.array([[np.sqrt(2)], [0], [1], [0]])

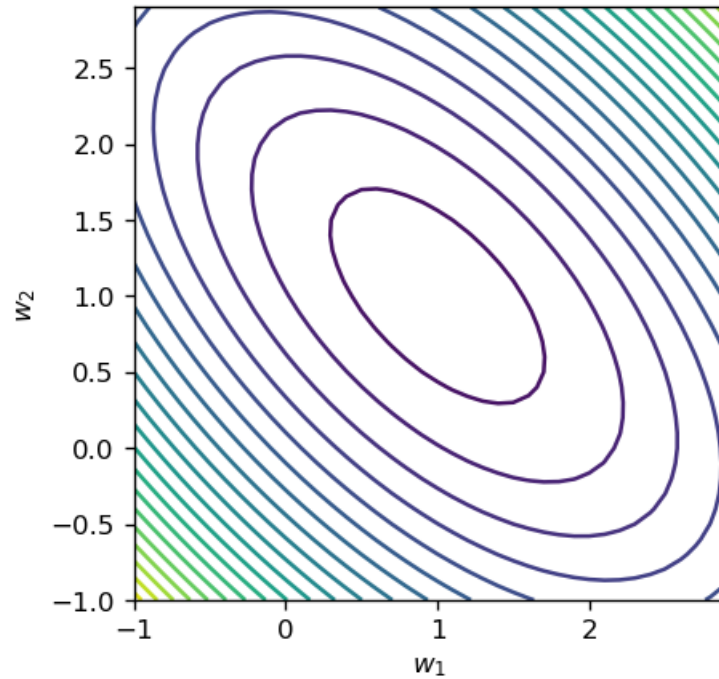
      ### Find Least Squares Solution
      w_ls = V @ Sinv @ U.T @ y
      c = y.T @ y - y.T @ X @ w_ls

      ### Find values of f(w), the contour plot surface for
      w1 = np.arange(-1,3,.1)
      w2 = np.arange(-1,3,.1)
      fw = np.zeros((len(w1), len(w2)))
      for i in range(len(w2)):
          for j in range(len(w1)):
              w = np.array([ w1[j], w2[i] ])
              fw[i,j] = (w-w_ls).T @ X.T @ X @ (w-w_ls) + c

      ### Plot the countours
      plt.figure(num=None, figsize=(4, 4), dpi=120)
      plt.contour(w1,w2,fw,20)
      plt.xlim([-1,3])
      plt.ylim([-1,3])
      plt.xlabel('$w_1$')
      plt.ylabel('$w_2$')
      plt.axis('square');

```

(2, 2)



4 Question 2b)

If start from $(0, 0)$, it is faster to convergence.

```
[16]: U = np.array([[1, 0], [0, 1], [0, 0], [0, 0]])
      S = np.array([[1, 0], [0, 0.5]])
      Sinv = np.linalg.inv(S)
      V = 1/np.sqrt(2)*np.array([[1, 1], [1, -1]])
      X = U @ S @ V.T
      y = np.array([[np.sqrt(2)], [0], [1], [0]])

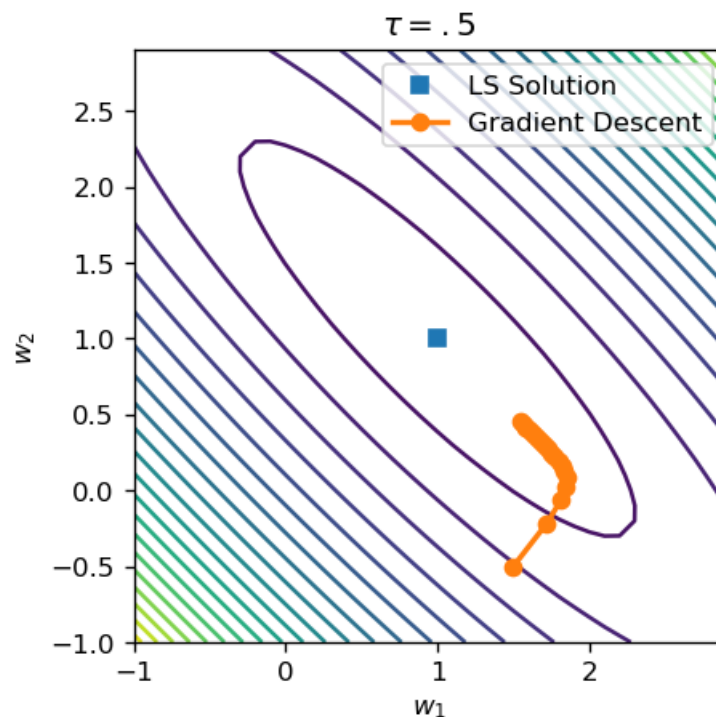
      ### Find Least Squares Solution
      w_ls = V @ Sinv @ U.T @ y
      c = y.T @ y - y.T @ X @ w_ls

      ### Find values of f(w), the contour plot surface for
      w1 = np.arange(-1,3,.1)
      w2 = np.arange(-1,3,.1)
      fw = np.zeros((len(w1), len(w2)))
      for i in range(len(w1)):
          for j in range(len(w2)):
              w = np.array([ w1[i], w2[j]] )
              fw[i,j] = (w-w_ls).T @ X.T @ X @ (w-w_ls) + c
```



```
[32]: w_init = np.array([[1.5], [-0.5]])# complete this line with a 2x1 numpy array
      ↪for the values specified in the activity
      it = 20
      tau = .5
      W = graddescent(X,y,tau,w_init,it);

      ### Create plot
      plt.figure(num=None, figsize=(4, 4), dpi=120)
      plt.contour(w1,w2,fw,20)
      plt.plot(w_ls[0],w_ls[1],"s", label="LS Solution")
      plt.plot(W[0,:],W[1:,:],'o-',linewidth=2, label="Gradient Descent")
      plt.legend()
      plt.xlim([-1,3])
      plt.xlabel('$w_1$')
      plt.ylim([-1,3])
      plt.ylabel('$w_2$')
      plt.title(r'$\tau = .5$');
      plt.axis('square');
```



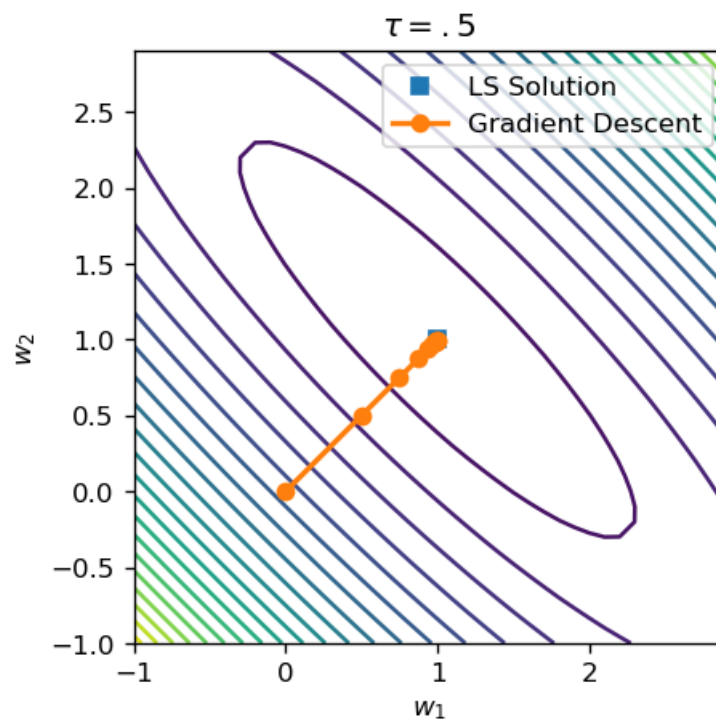
```
[33]: # copy and paste code from above
      w_init = np.array([[0], [0]])# complete this line with a 2x1 numpy array for
      ↪the values specified in the activity
```

```

it = 10
tau = .5
W = graddescent(X,y,tau,w_init,it);

### Create plot
plt.figure(num=None, figsize=(4, 4), dpi=120)
plt.contour(w1,w2,fw,20)
plt.plot(w_ls[0],w_ls[1],"s", label="LS Solution")
plt.plot(W[0,:],W[1,:],'o-',linewidth=2, label="Gradient Descent")
plt.legend()
plt.xlim([-1,3])
plt.xlabel('$w_1$')
plt.ylim([-1,3])
plt.ylabel('$w_2$')
plt.title(r'$\tau = .5$');
plt.axis('square');

```



```

[22]: # copy and paste code from above
w_init = np.array([[0], [2]])# complete this line with a 2x1 numpy array for
    ↳ the values specified in the activity
it = 20
tau = .5

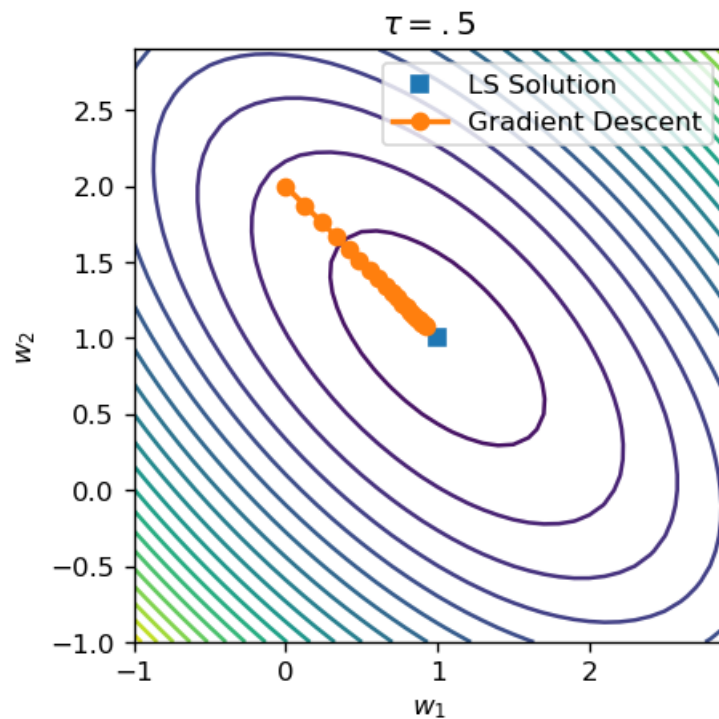
```

```

W = graddescent(X,y,tau,w_init,it);

### Create plot
plt.figure(num=None, figsize=(4, 4), dpi=120)
plt.contour(w1,w2,fw,20)
plt.plot(w_ls[0],w_ls[1],"s", label="LS Solution")
plt.plot(W[0,:],W[1,:],'o-',linewidth=2, label="Gradient Descent")
plt.legend()
plt.xlim([-1,3])
plt.xlabel('$w_1$')
plt.ylim([-1,3])
plt.ylabel('$w_2$')
plt.title(r'$\tau = .5$');
plt.axis('square');

```



5 Question 2c)

It doesn't converge because tau is too big.

```

[25]: w_init = np.array([[1.5], [-0.5]])# complete this line with a 2x1 numpy array
      ↪ for the values specified in the activity
      it = 20

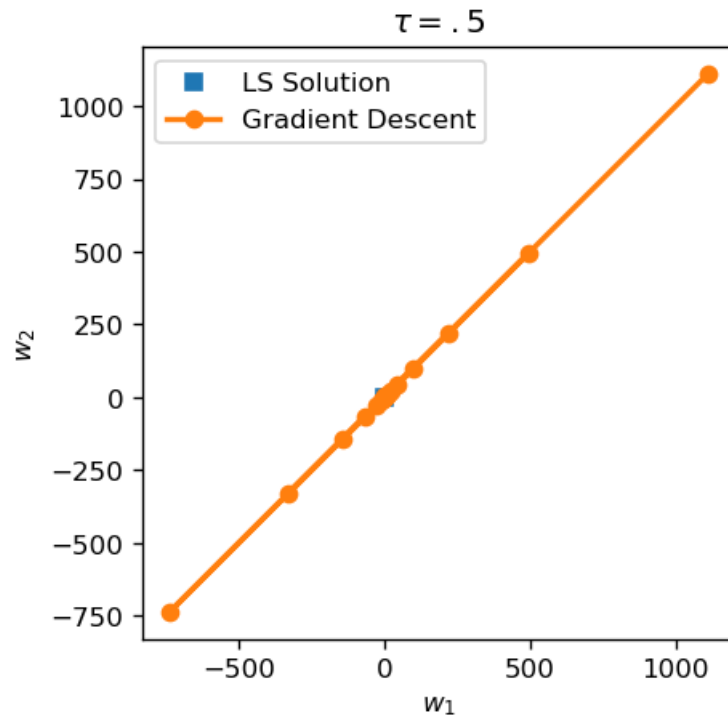
```

```

tau = 2.5
W = graddescent(X,y,tau,w_init,it);

### Create plot
plt.figure(num=None, figsize=(4, 4), dpi=120)
plt.contour(w1,w2,fw,20)
plt.plot(w_ls[0],w_ls[1],"s", label="LS Solution")
plt.plot(W[0,:],W[1,:], 'o-',linewidth=2, label="Gradient Descent")
plt.legend()
plt.xlim([-1,3])
plt.xlabel('$w_1$')
plt.ylim([-1,3])
plt.ylabel('$w_2$')
plt.title(r'$\tau = .5$');
plt.axis('square');

```



6 Question 2d)

As the ellipse major axis is bigger, it takes longer to converge. The sigma will affect the shape of ellipse.

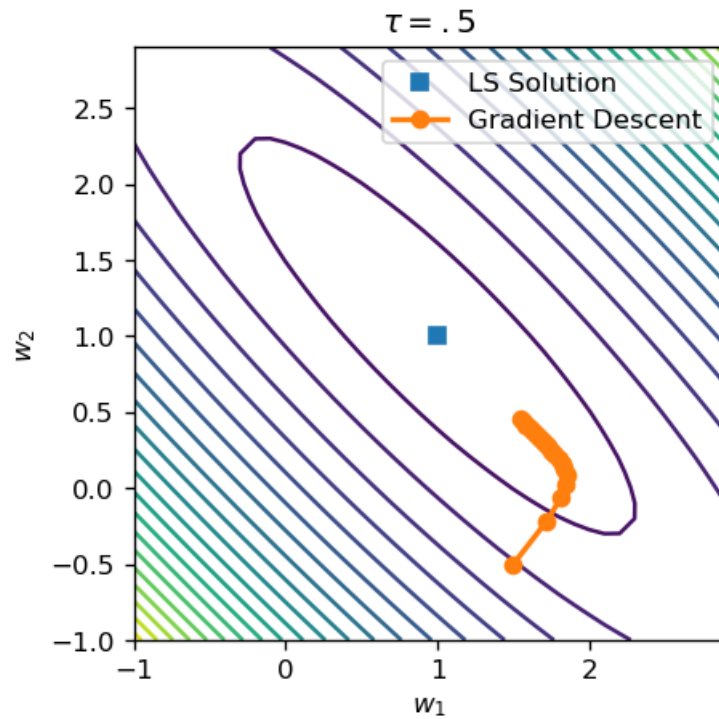
```
[34]: ## Copy and paste code from above
U = np.array([[1, 0], [0, 1], [0, 0], [0, 0]])
S = np.array([[1, 0], [0, 0.25]])
Sinv = np.linalg.inv(S)
V = 1/np.sqrt(2)*np.array([[1, 1], [1, -1]])
X = U @ S @ V.T
y = np.array([[np.sqrt(2)], [0], [1], [0]])

### Find Least Squares Solution
w_ls = V @ Sinv @ U.T @ y
c = y.T @ y - y.T @ X @ w_ls

### Find values of f(w), the contour plot surface for
w1 = np.arange(-1,3,.1)
w2 = np.arange(-1,3,.1)
fw = np.zeros((len(w1), len(w2)))
for i in range(len(w1)):
    for j in range(len(w2)):
        w = np.array([ w1[i], w2[j]] )
        fw[i,j] = (w-w_ls).T @ X.T @ X @ (w-w_ls) + c

[30]: w_init = np.array([[1.5], [-0.5]])# complete this line with a 2x1 numpy array
      ↪for the values specified in the activity
it = 20
tau = .5
W = graddescent(X,y,tau,w_init,it);

### Create plot
plt.figure(num=None, figsize=(4, 4), dpi=120)
plt.contour(w1,w2,fw,20)
plt.plot(w_ls[0],w_ls[1],"s", label="LS Solution")
plt.plot(W[0,:],W[1:], 'o-',linewidth=2, label="Gradient Descent")
plt.legend()
plt.xlim([-1,3])
plt.xlabel('$w_1$')
plt.ylim([-1,3])
plt.ylabel('$w_2$')
plt.title(r'$\tau = .5$');
plt.axis('square');
```



7 Question 2e)

Sigma affects the shape of ellipse. If the ratio between the major and the minor axis of ellipse is huge. It takes longer to converge.

[]: