CS/ECE/ME 532
Matrix Methods in Machine Learning

Welcome!

Activity 11: K-means and SVD. Low-rank inversion via SVD

Low-rank decompositions:

taste vectors or patterns

$$\boldsymbol{A} \approx \boldsymbol{T}\boldsymbol{W}^T = \begin{bmatrix} t_{1,1} & t_{1,2} \\ t_{2,1} & t_{2,2} \\ t_{3,1} & t_{3,2} \\ t_{4,1} & t_{4,2} \end{bmatrix} \begin{bmatrix} w_{1,1} & \dots & w_{1,6} \\ w_{2,1} & \dots & w_{2,6} \end{bmatrix}$$

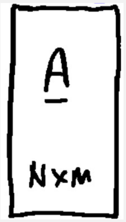
K-Means: cluster centers

SVD for least squares:

economy SVD

$$\min_{\boldsymbol{w}} ||\boldsymbol{A}\boldsymbol{w} - \boldsymbol{y}||^2$$

$$\boldsymbol{w}^* = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{y} \rightarrow \boldsymbol{w}^* = \boldsymbol{V}\Sigma^{-1}\boldsymbol{U}^T\boldsymbol{y}$$



Problems arise when the singular values are small!

- Fix by dropping corresponding singular vectors.

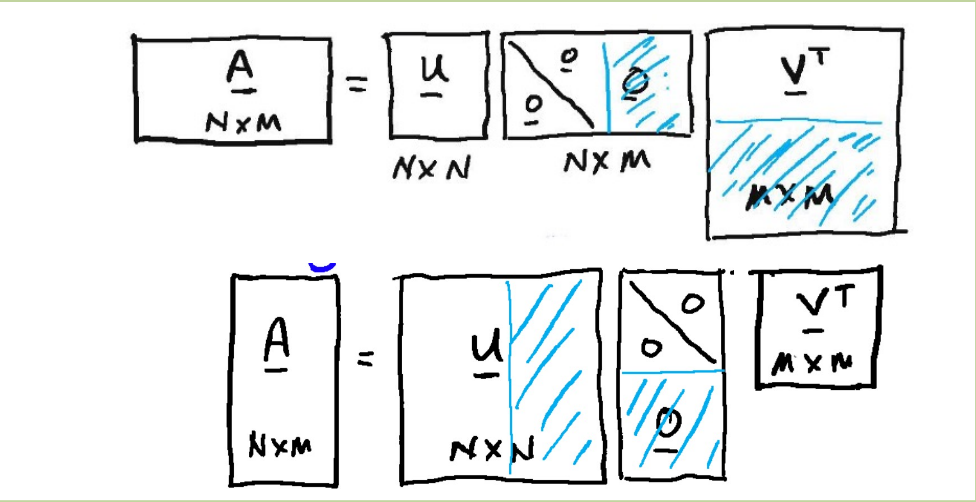
Hint for 1g:

$$\boldsymbol{A} \approx \boldsymbol{T}\boldsymbol{W}^T = \underbrace{\begin{bmatrix} t_{1,1} & t_{1,2} \\ t_{2,1} & t_{2,2} \end{bmatrix}}_{\tilde{\boldsymbol{T}}}\!\!\begin{matrix} \\ t_{3,1} \quad t_{3,2} \\ t_{4,1} \quad t_{4,2} \end{matrix} \begin{bmatrix} w_{1,1} & \dots & w_{1,6} \\ w_{2,1} & \dots & w_{2,6} \end{bmatrix} \qquad \boldsymbol{a}_{\text{JON}} = \begin{bmatrix} \boxed{\begin{matrix} 6 \\ 4 \end{matrix}} \\ ? \\ ? \end{bmatrix} \; (\boldsymbol{y})$$

$$\min_{\boldsymbol{w}_{\text{JON}}} ||\tilde{\boldsymbol{T}}\boldsymbol{w}_{\text{JON}} - \boldsymbol{y}||_2^2$$

$$\hat{\boldsymbol{a}}_{\text{JON}} = \boldsymbol{T}\boldsymbol{w}_{\text{JON}}$$

The Singular Value Decomposition: $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$



- Singular values $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_N \geq 0$
- "Importance" of patterns in U, V ranked by σ_i

SVD gives the 'best' rank-r approximation:

$$\min_{\text{rank } \tilde{\boldsymbol{A}} \leq r} ||\boldsymbol{A} - \tilde{\boldsymbol{A}}||_F \rightarrow \tilde{\boldsymbol{A}} = \sum_{i=1}^{r} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T$$

Frobenius norm

Eckart-Young, 1936