CS/ECE/ME532 Activity 15

1. Consider the 4-by-3 matrix defined as $\boldsymbol{V} = \begin{bmatrix} 1 & \text{X} & \text{X} \\ \text{X} & 2 & 4 \\ -1 & 2 & \text{X} \\ \text{X} & -2 & \text{X} \end{bmatrix}$ where X denotes missing entries. Assume $\boldsymbol{V}$ is a rank-1 matrix.

a) Use what you know about the structure of rank-1 matrices to find the missing entries.

b) What is the minimum number of missing entries for which you cannot complete a 4-by-3 rank 1 matrix? Where are the missing entries in this case?

2. A data file is available that contains a rank-2, 16-by-16 matrix `Xtrue` with integer entries and three versions of this matrix (`Y1`, `Y2`, and `Y3`) with differing numbers of missing entries. The missing entries are indicated by `NaN`.

A script is provided to complete a matrix using iterative singular value thresholding. The script contains a function that requires two inputs: *i)* the matrix with missing entries, and *ii)* the rank.

a) Apply the iterative singular value thresholding function (provided in the script) to the three incomplete matrices assuming the rank is 2. You will first need to complete the line of code in the function. Compare your recovered completed matrices to `Xtrue` (Note: compare the output by subtracting the completed matrix from the original matrix, and then displaying them). Does the number of missing entries affect the accuracy of the completed matrix?

b) Now apply your routine to the three incomplete matrices assuming the rank is 3. Compare your recovered completed matrices to `Xtrue`. Comment on the impact of using the incorrect rank in the completion process.

3. Optional - Preview of Activity 16. The squared-error cost function $f(\boldsymbol{w}) = ||\boldsymbol{y} - \boldsymbol{X}\boldsymbol{w}||_2^2 = (\boldsymbol{y} - \boldsymbol{X}\boldsymbol{w})^T(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{w})$ may be rewritten as a perfect square in the form

$$f(\boldsymbol{w}) = (\boldsymbol{w} - \boldsymbol{w}_{LS})^T \boldsymbol{X}^T \boldsymbol{X} (\boldsymbol{w} - \boldsymbol{w}_{LS}) + c$$

where $\boldsymbol{w}_{LS} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{y}$ and $c = \boldsymbol{y}^T\boldsymbol{y} - \boldsymbol{y}^T\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{y}$. This assumes the n-by-p ($p < n$) matrix $\boldsymbol{X}$ is full rank. $f(\boldsymbol{w})$ is called a "quadratic form" in $\boldsymbol{w}$ since it is a quadratic function of $\boldsymbol{w}$.

a) Prove that the minimum value of $f(\boldsymbol{w}) = c$ when $\boldsymbol{w} = \boldsymbol{w}_{LS}$ and all other $\boldsymbol{w}$ result in higher values of $f(\boldsymbol{w})$.

b) Suppose $\boldsymbol{y} = \begin{bmatrix} 1 \\ 1/2 \\ 1 \\ 0 \end{bmatrix}$ and the 4-by-2 $\boldsymbol{X} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$ has singular value decomposition $\boldsymbol{U} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$, and $\boldsymbol{V} = \boldsymbol{I}$. Sketch a contour plot of $f(\boldsymbol{w})$ in the w_1-w_2 plane.

c) Suppose $\boldsymbol{y} = \begin{bmatrix} 1 \\ 1/5 \\ 1 \\ 0 \end{bmatrix}$, $\boldsymbol{U} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0 \\ 0 & 1/5 \end{bmatrix}$, and $\boldsymbol{V} = \boldsymbol{I}$. Sketch a contour plot of $f(\boldsymbol{w})$ in the w_1-w_2 plane. How do the singular values of $\boldsymbol{X}$ affect the shape of the contours?

To be continued...