

## CS/ECE/ME532 Assignment 5

1. Here we continue the problem studied in Activity 11. Let a 4-by-2 matrix  $\mathbf{X}$  have

$$\text{SVD } \mathbf{X} = \mathbf{U}\mathbf{S}\mathbf{V}^T \text{ where } \mathbf{U} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}, \mathbf{S} = \begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix}, \text{ and } \mathbf{V} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\text{Let } \mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

- a) The ratio of the largest to the smallest singular values is termed the condition number of  $\mathbf{X}$ . Find the condition number if  $\gamma = 0.1$ , and  $\gamma = 10^{-8}$ . Solve  $\mathbf{X}\mathbf{w} = \mathbf{y}$  for  $\mathbf{w}$  and find  $\|\mathbf{w}\|_2^2$  for these two values of  $\gamma$ .

- b) A system of linear equations with a large condition number is said to be “ill-conditioned”. One consequence of an ill-conditioned system of equations is solutions with large norms as you found in the previous part of this problem. A second consequence is that the solution is very sensitive to small errors in  $\mathbf{y}$  such as may

result from measurement error or numerical error. Suppose  $\mathbf{y} = \begin{bmatrix} 1 + \epsilon \\ 0 \\ 0 \\ 1 \end{bmatrix}$ . Write

$\mathbf{w} = \mathbf{w}_o + \mathbf{w}_\epsilon$  where  $\mathbf{w}_o$  is the solution for arbitrary  $\gamma$  when  $\epsilon = 0$  and  $\mathbf{w}_\epsilon$  is the perturbation in that solution due to some error  $\epsilon \neq 0$ . How does the norm of the perturbation due to  $\epsilon \neq 0$ ,  $\|\mathbf{w}_\epsilon\|_2^2$ , depend on the condition number? Find  $\|\mathbf{w}_\epsilon\|_2^2$  for  $\epsilon = 0.01$  and  $\gamma = 0.1$  and  $\gamma = 10^{-8}$ .

- c) Now consider a “low-rank” inverse. Instead of writing

$$(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T = \sum_{i=1}^p \frac{1}{\sigma_i} \mathbf{v}_i \mathbf{u}_i^T$$

where  $p$  is the number of columns of  $\mathbf{X}$  (assumed less than the number of rows), we approximate

$$(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \approx \sum_{i=1}^r \frac{1}{\sigma_i} \mathbf{v}_i \mathbf{u}_i^T$$

In this approximation we only invert the largest  $r$  singular values, and ignore all of them smaller than  $\sigma_r$ . Use  $r = 1$  in the low-rank inverse to find  $\mathbf{w} = \mathbf{w}_o + \mathbf{w}_\epsilon$

where  $\mathbf{y} = \begin{bmatrix} 1 + \epsilon \\ 0 \\ 0 \\ 1 \end{bmatrix}$  as in part b). Compare the results to part b).