

## CS/ECE/ME532 Assignment 3

1. *Polynomial fitting.* Suppose we observe pairs of points  $(a_i, b_i)$ ,  $i = 1, \dots, m$  representing measurements from a scientific experiment. The variables  $a_i$  are the experimental conditions and the  $b_i$  correspond to the measured response in each condition. We fit a degree  $d < m$  polynomial to these data. In other words, we want to find the coefficients of a degree  $p$  polynomial  $w(a)$  so that  $w(a_i) \approx b_i$  for  $i = 1, 2, \dots, m$ .
  - a) Suppose  $w(a)$  is a degree  $p$  polynomial. Write the general expression for  $w(a_i) = b_i$ .
  - b) Express the  $i = 1, \dots, m$  equations as a system in matrix form  $\mathbf{A}\mathbf{x} = \mathbf{d}$  while defining  $\mathbf{A}$  and  $\mathbf{d}$ . What is the form/structure of  $\mathbf{A}$  in terms of the given  $a_i$ ?
  - c) Write a script to find the least-squares model fit to the  $m = 30$  data points in `polydata.mat`. Plot the points and the polynomial fits for  $p = 1, 2, 3$ .
  
2. Least Squares Approximation of Matrices.
  - a) Derive the solution to least-squares problem  $\min_{\mathbf{w}} \|\mathbf{x} - \mathbf{T}\mathbf{w}\|_2^2$  when  $\mathbf{T}$  is an  $n$ -by- $r$  matrix of orthonormal columns. Your solution should not involve a matrix inverse.
  - b) Let  $\mathbf{X} = [\mathbf{x}_1 \ \mathbf{x}_2 \ \cdots \ \mathbf{x}_p]$  be an  $n$ -by- $p$  matrix. Use the least-squares problems  $\min_{\mathbf{w}_i} \|\mathbf{x}_i - \mathbf{T}\mathbf{w}_i\|_2^2$  to find  $\mathbf{W} = [\mathbf{w}_1 \ \mathbf{w}_2 \ \cdots \ \mathbf{w}_p]$  in the approximation  $\mathbf{X} \approx \mathbf{T}\mathbf{W}$ . Your solution should express  $\mathbf{W}$  as a function of  $\mathbf{T}$  and  $\mathbf{X}$ .
  
3. We return to the movies rating problem of Activity 5. The ratings on a scale of 1-10 are:

Movie	Jake	Jennifer	Jada	Theo	Ioan	Bo	Juanita
Star Trek	4	7	2	8	7	4	2
Pride and Prejudice	9	3	5	6	10	5	5
The Martian	4	8	3	7	6	4	1
Sense and Sensibility	9	2	6	5	9	5	4
Star Wars: Empire Strikes	4	9	2	8	7	4	1

A matrix  $\mathbf{X}$  containing this data is available in the file `movie.mat` and the csv file `movie.csv`. Our goal is to approximate  $\mathbf{X}$  using  $r$  “tastes”, the columns of  $\mathbf{T}$ , that

is,  $\mathbf{X} \approx \mathbf{T}\mathbf{W}$  where  $\mathbf{T}$  is 5-by- $r$ . You will use a Gram-Schmidt orthogonalization code to find a set of tastes that approximate the ratings. A script that implements Gram-Schmidt orthogonalization is available.

Define a 5-by- $r$  taste matrix  $\mathbf{T} = [\mathbf{t}_1 \ \mathbf{t}_2 \ \cdots \ \mathbf{t}_r]$  with orthonormal columns and the  $r$ -by-7 weight matrix

$$\mathbf{W} = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{17} \\ w_{21} & w_{22} & \cdots & w_{27} \\ & & \vdots & \\ w_{r1} & w_{r2} & \cdots & w_{r7} \end{bmatrix}$$

- a) In Activity 5 you found the baseline (average) rating for each friend by requiring the first basis vector in the taste matrix to be

$$\mathbf{t}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

You have noticed by now that the first vector in the Gram-Schmidt procedure is a scaled version of the first vector of the matrix, so you decide to define an augmented matrix  $\tilde{\mathbf{X}} = [\mathbf{1} \ \mathbf{X}]$  where  $\mathbf{1}$  is a column vector containing five unity entries. Apply a Gram-Schmidt orthogonalization code to  $\tilde{\mathbf{X}}$  to find a set of orthonormal basis vectors. Is the first basis vector you obtain equal to  $\mathbf{t}_1$ ?

- b) Use your solution to the preceding problem in this homework assignment to find the rank-1 approximation of  $\mathbf{X}$  using only  $\mathbf{t}_1$ . That is, find  $\mathbf{W}$  so that  $\mathbf{X} \approx \mathbf{t}_1\mathbf{W}$ . Use  $\mathbf{W}$  to compute  $\mathbf{t}_1\mathbf{W}$ . This gives you each friend's baseline ratings. Also compute the residual error  $\mathbf{X} - \mathbf{t}_1\mathbf{W}$ .
- c) Now find a rank-2 approximation using  $\mathbf{T} = [\mathbf{t}_1 \ \mathbf{t}_2]$ . That is, find  $\mathbf{W}$  so that  $\mathbf{X} \approx \mathbf{T}\mathbf{W}$ . Use  $\mathbf{W}$  to compute  $\mathbf{T}\mathbf{W}$ . This gives you a rank-2 approximation to the ratings. Also compute the residual error  $\mathbf{X} - \mathbf{T}\mathbf{W}$ . How does  $\mathbf{t}_2$  relate to the distinction between sci-fi and and romance movie preferences?
- d) Now find a rank-3 approximation using  $\mathbf{T} = [\mathbf{t}_1 \ \mathbf{t}_2 \ \mathbf{t}_3]$ . That is, find  $\mathbf{W}$  so that  $\mathbf{X} \approx \mathbf{T}\mathbf{W}$ . Use  $\mathbf{W}$  to compute  $\mathbf{T}\mathbf{W}$ . This gives you a rank-3 approximation to the ratings. Also compute the residual error  $\mathbf{X} - \mathbf{T}\mathbf{W}$ . Qualitatively discuss the effect of increasing the rank of the approximation on the residual error.
- e) Suppose you interchange the order of Jake and Jennifer so that Jennifer's ratings are in the first column of  $\mathbf{X}$  and Jake's ratings are in the second column. Does the

rank-2 approximation change? Why or why not? Does the rank-3 approximation change? Why or why not?

4. Let  $\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ .

a) Is  $\mathbf{Q} \succ 0$ ?

b) Sketch the surface  $y = \mathbf{x}^T \mathbf{Q} \mathbf{x}$  where  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ . If you find 3-D sketching too difficult, you may draw a contour map with labeled contours.

5. Suppose  $\mathbf{P} \succ 0$  and  $\mathbf{Q} \succ 0$  are (symmetric) positive definite  $n \times n$  matrices. Prove that  $\mathbf{QPQ} \succ 0$ .