CS/ECE/ME 532
Matrix Methods in Machine Learning

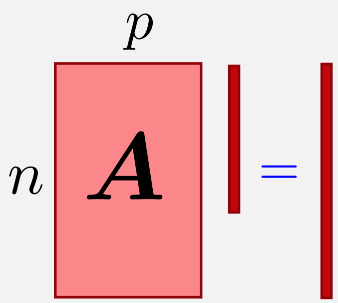
Welcome!

Activity 6

unknown

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$$

(usually) no solution!



- Usually what happens with real data
- The next best thing: find approximate solution

$$\boldsymbol{x}^* = \arg\min_{\boldsymbol{x}} ||\boldsymbol{A}\boldsymbol{x} - \boldsymbol{y}||_2^2$$

this is a called *a least squares solution*

$$\boldsymbol{x}^* = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{y}$$

when does this inverse exist?

Note: $\boldsymbol{X}$ must be a square matrix!

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X^(-1) @ X = I
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$$\boldsymbol{X}\boldsymbol{X}^{-1} = \boldsymbol{I}$$

full rank

if $\text{rank}(\boldsymbol{A}) = p$, then $\boldsymbol{A}^T\boldsymbol{A}$ is invertible

$\Downarrow$

The least squares solution is unique

Positive definiteness (P.D.)

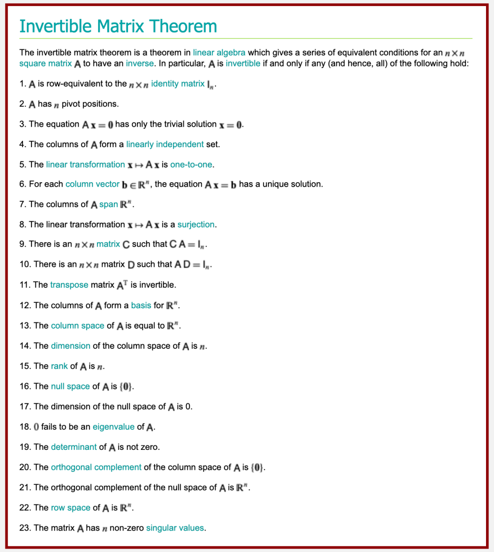
$$\boldsymbol{Q} \succ 0$$

$\Updownarrow$

$$\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} > 0 \text{ for all } \boldsymbol{x} \neq 0$$

$\Downarrow$

$\boldsymbol{Q}$ is invertible



Invertible Matrix Theorem

The invertible matrix theorem is a theorem in linear algebra which gives a series of equivalent conditions for an $n \times n$ square matrix A to have an inverse. In particular, A is invertible if and only if any (and hence, all) of the following hold:

1. A is row-equivalent to the $n \times n$ identity matrix I_n.
2. A has n pivot positions.
3. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.
4. The columns of A form a linearly independent set.
5. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
6. For each column vector $\mathbf{b} \in \mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution.
7. The columns of A span $\mathbb{R}^n$.
8. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is a surjection.
9. There is an $n \times n$ matrix C such that $CA = I_n$.
10. There is an $n \times n$ matrix D such that $AD = I_n$.
11. The transpose matrix A^T is invertible.
12. The columns of A form a basis for $\mathbb{R}^n$.
13. The column space of A is equal to $\mathbb{R}^n$.
14. The dimension of the column space of A is n.
15. The rank of A is n.
16. The null space of A is $\{\mathbf{0}\}$.
17. The dimension of the null space of A is 0.
18. 0 fails to be an eigenvalue of A.
19. The determinant of A is not zero.
20. The orthogonal complement of the column space of A is $\{\mathbf{0}\}$.
21. The orthogonal complement of the null space of A is $\mathbb{R}^n$.
22. The row space of A is $\mathbb{R}^n$.
23. The matrix A has n non-zero singular values.