In [ ]:

```
import autograd.numpy as anp
from autograd import grad
import matplotlib.pyplot as plt
```

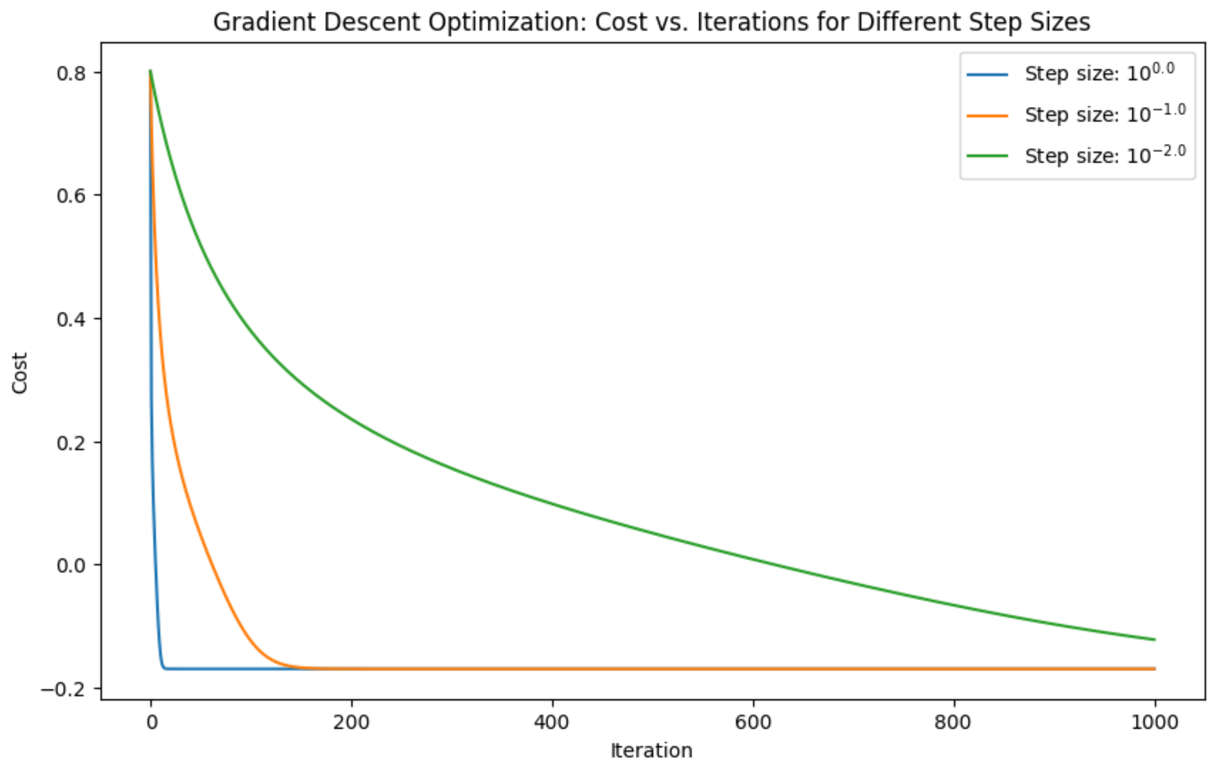
Question 3.5

In [ ]:

```
def gradient_descent(alpha, max_its, w_init):

    # Define the cost and its gradient functions
    def g(w):
        return 1/50 * (w**4 + w**2 + 10*w)

    def grad_g(w):
        return 1/50 * (4*w**3 + 2*w + 10)

    # Initialize variables
    w = w_init
    cost_history = [g(w)]

    for _ in range(max_its):
        w -= alpha * grad_g(w)  # Update step
        cost_history.append(g(w))  # Store the cost

    return cost_history
```

In [ ]:

```
initial_weight = 2.0
iterations = 1000
step_sizes = [10**0, 10**(-1), 10**(-2)]
histories = []

for step_size in step_sizes:
    histories.append(gradient_descent(step_size, iterations, initial_weight)

plt.figure(figsize=(10, 6))
for index, history in enumerate(histories):
    plt.plot(history, label=f'Step size: $10^{{{np.log10(step_sizes[index])}

plt.xlabel('Iteration')
plt.ylabel('Cost')
plt.title('Gradient Descent Optimization: Cost vs. Iterations for Different
plt.legend()
plt.show()
```



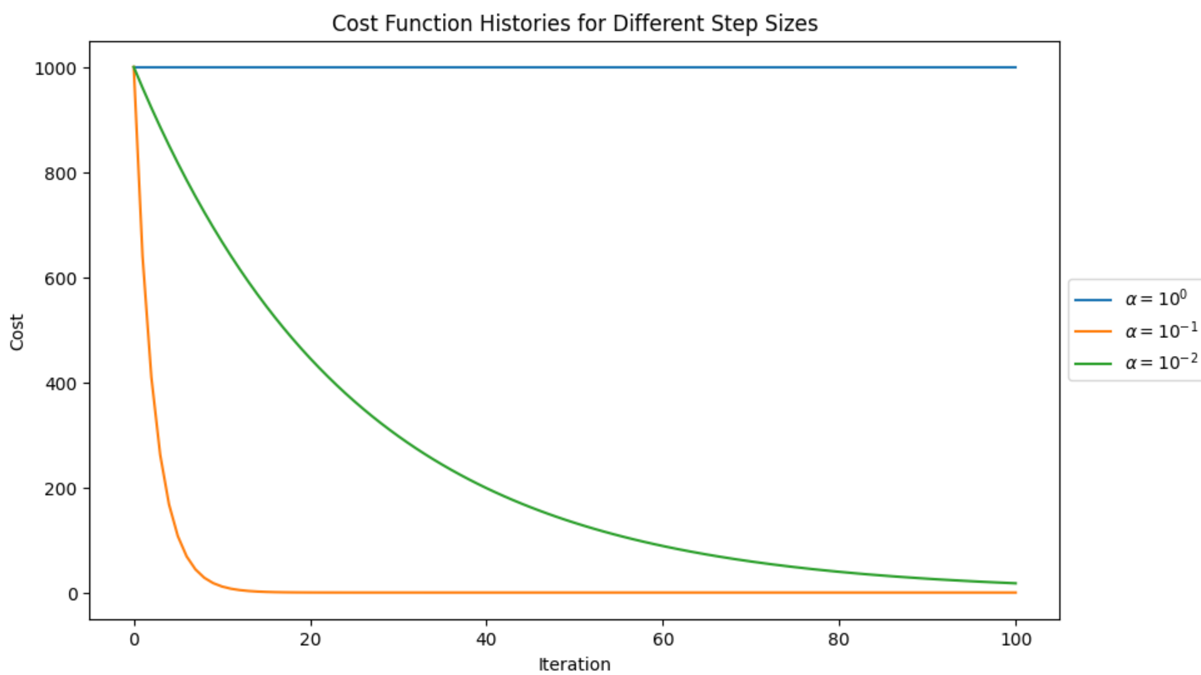
Question 3.8

In [ ]:
```
def gradient_descent(function_to_optimize, step_size, max_iterations, initia

    compute_gradient = grad(function_to_optimize)

    weights_history = [initial_weight]
    costs_history = [function_to_optimize(initial_weight)]

    for _ in range(max_iterations):
        current_gradient = compute_gradient(weights_history[-1])

        updated_weight = weights_history[-1] - step_size * current_gradient

        weights_history.append(updated_weight)
        costs_history.append(function_to_optimize(updated_weight))

    return weights_history, costs_history
```

In [ ]:
```
def squared_norm(w):

    return anp.dot(w.T, w)[0, 0]
```

In [ ]:
```
def g(w):
    return squared_norm(w)

N = 10
initial_weight = 10 * anp.ones((N, 1))
max_iterations = 100
step_sizes = [10**0, 10**(-1), 10**(-2)]
```

```
results = []
for alpha in step_sizes:
    weights_history, costs_history = gradient_descent(g, alpha, max_iteratio
    results.append(costs_history)

def plot_cost_histories(cost_histories, labels):

    plt.figure(figsize=(10, 6))
    for history, label in zip(cost_histories, labels):
        plt.plot(history, label=label)
    plt.legend(loc='center left', bbox_to_anchor=(1, 0.5))
    plt.xlabel('Iteration')
    plt.ylabel('Cost')
    plt.title('Cost Function Histories for Different Step Sizes')
    plt.show()

labels = [r'$\alpha = 10^{0}$', r'$\alpha = 10^{-1}$', r'$\alpha = 10^{-2}$'

plot_cost_histories(results, labels)
```



4.1

(a) For any vector z, the curvature function of a matrix C with non-negative eigen values d_n is

$$\psi(z) = z^T C z = z^T \left(\sum_{n=1}^{N} e_n e_n^T d_n \right) z = \sum_{n=1}^{N} (e_n^T z)^2 d_n$$

$$\psi(z) \geqslant 0$$

This expression is non-negative due to the non-negativity of $(e_n^T z)^2$ and d_n, confirming that C is positive semidefinite.

(b) Conversely, if C is positive semidefinite, its curvature function $\psi(z)$ must satisfy:

$$\psi(z) = z^T C z \geqslant 0$$

This requirement is inherently fulfilled by sum of squares form in its eigen decomposition, which implies that all eigenvalues d_n are non-negative. Any negative eigenvalue would violate the positive semidefiniteness by producing a negative curvature function for z aligned with its corresponding eigenvector

(c) The matrix $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, possessing eigenvalues $d_1 = 0$, $d_2 = 2$, is thus positive semidefinite, indicating the convexity of the associated function g.

(d) Enhancing matrix C with $\lambda I_{n\times n}$, where λ is a scalar, modifies the curvature function to:

$$\varphi(z) = \sum_{n=1}^{N} (e_n^T z)^2 (d_n + \lambda)$$

To ensure semipositivity, λ must be chosen to offset the smallest negative eigenvalue of C.

This augmentation guarantees that $\varphi(z)$ remains non-negative for any z, solidifying the positive semidefinite status of $C + \lambda I_{n\times n}$

4.5

(a) set $g(w) = \log(1+e^{w^Tw})$ to zero

$$\nabla g(w) = \frac{2e^{w^Tw}}{1+e^{w^Tw}} = 0_{N\times 1}$$

As $\frac{2e^{w^Tw}}{1+e^{w^Tw}} \geq 1$ only when $w = 0_{N\times 1}$

(b) The Hessian of g,

$$\nabla^2 g(w) = \frac{4e^{w^Tw}}{(1+e^{w^Tw})^2} ww^T + \frac{2e^{w^Tw}}{1+e^{w^Tw}} I_{N\times N}.$$

Fixing w

$$z^T\nabla^2 g(w) z = z^T\left(\frac{4e^{w^Tw}}{(1+e^{w^Tw})^2} ww^T + \frac{2e^{w^Tw}}{1+e^{w^Tw}} I_{N\times N}\right) z$$

$$= z^T\nabla^2 g(w) z = z^T \frac{4e^{w^Tw}}{(1+e^{w^Tw})^2} ww^T z + z^T \frac{2e^{w^Tw}}{1+e^{w^Tw}} I_{N\times N} z$$

$$= \frac{4e^{w^Tw}}{(1+e^{w^Tw})^2} z^T ww^T z + \frac{2e^{w^Tw}}{1+e^{w^Tw}} z^T z$$

$$= \frac{4e^{w^Tw}}{(1+e^{w^Tw})^2} (z^T w)^2 + \frac{2e^{w^Tw}}{1+e^{w^Tw}} \|z\|_2^2$$

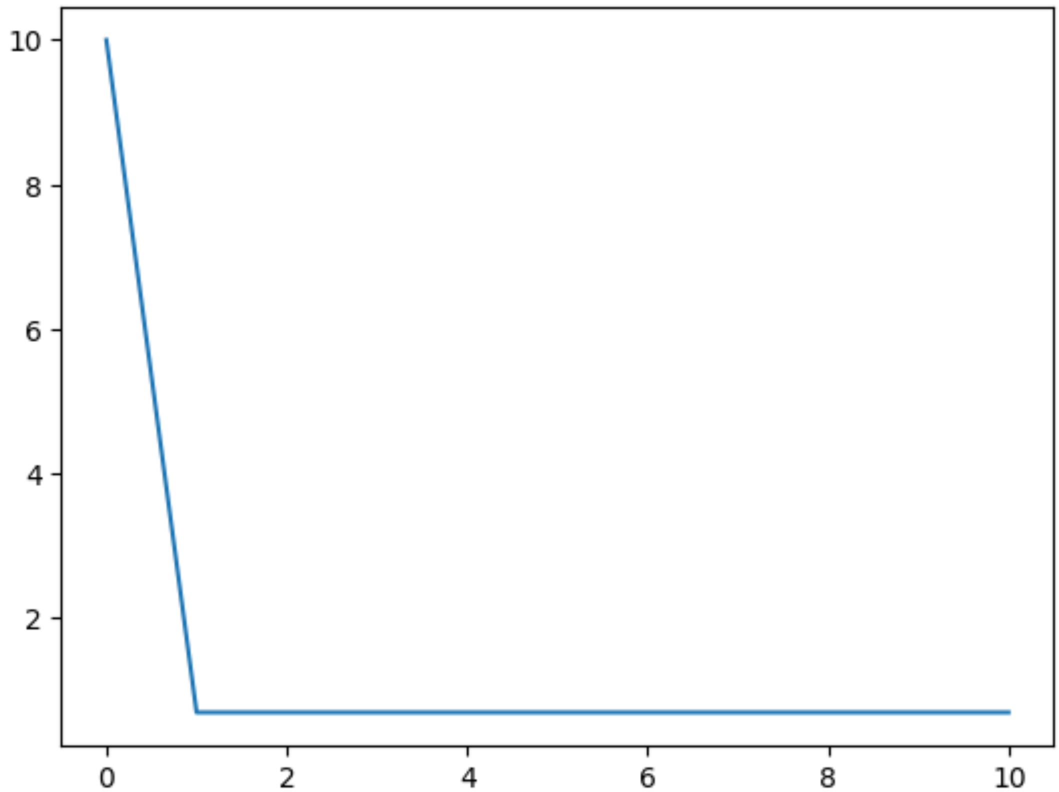
all component is nonnegative whatever z is

so $z^T \nabla^2 g(w) z \geq 0$

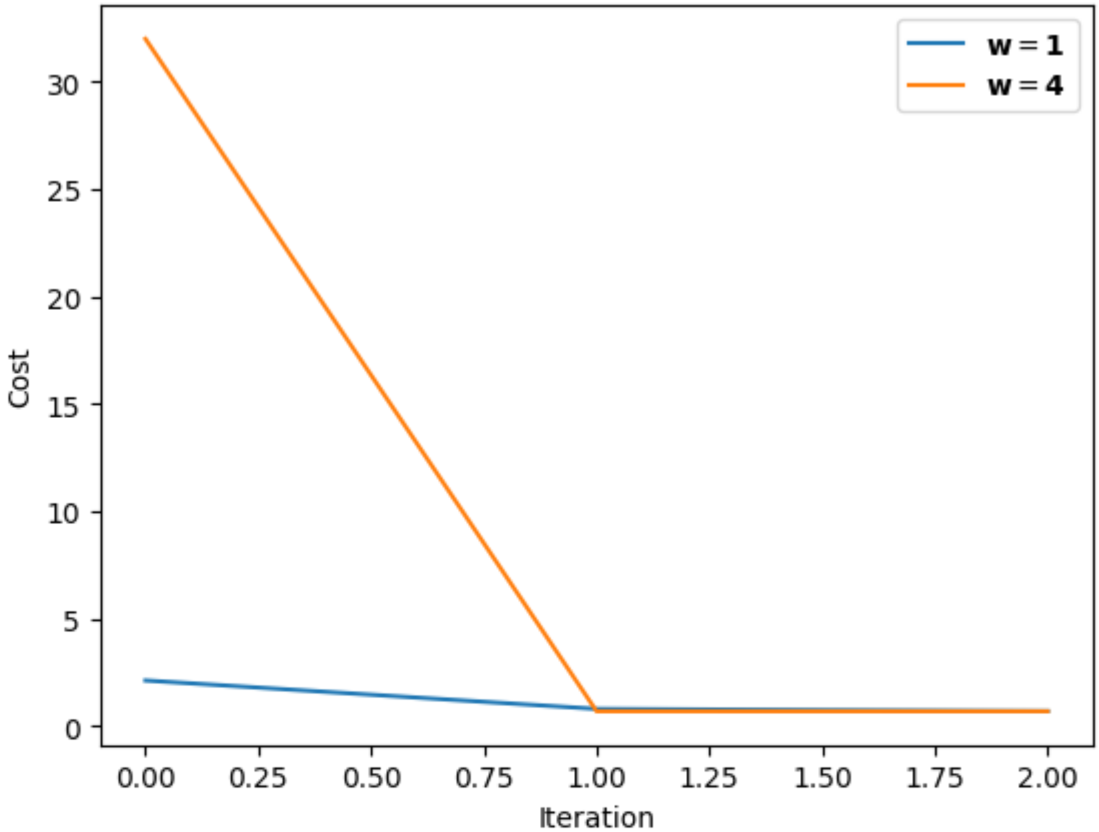
for all w and z, g is indeed convex

Question4.5

In [ ]:
```
from autograd import grad
from autograd import hessian
import matplotlib.pyplot as plt
from autograd import numpy as np

def g(w):
    return np.log(1+np.exp(np.dot(w.T, w)))

def newton_method(g, max_its, w, **kwargs):
    gradient = grad(g)
    hess = hessian(g)
    epsilon = 10**(-7)
    if 'epsilon' in kwargs:
        epsilon = kwargs['epsilon']
    weight_history = [w]
    cost_history = [g(w)]
    for k in range(max_its):
        grad_eval = gradient(w)
        hess_eval = hess(w)
        hess_eval.shape = (int((np.size(hess_eval))**(0.5)), int((np.size(he
        A = hess_eval + epsilon*np.eye(w.size)
        b = grad_eval
        w = np.linalg.solve(A, np.dot(A, w) - b)
        weight_history.append(w)
        cost_history.append(g(w))
    cost_history = [np.squeeze(val) for val in cost_history]
    return weight_history, cost_history

N = 10
w0 = np.ones((N, 1))
weights, cost_history = newton_method(g, 10, w0)
plt.plot(cost_history)
plt.show()
```



In [ ]:
```
import matplotlib.pyplot as plt

def plot_cost_histories(cost_histories, labels, start=0, points=False):
    for cost_history, label in zip(cost_histories, labels):
        if points:
            plt.scatter(range(start, len(cost_history)), cost_history[start:
        else:
            plt.plot(range(start, len(cost_history)), cost_history[start:],

    plt.xlabel('Iteration')
    plt.ylabel('Cost')
    plt.legend()
    plt.show()


w = np.ones((2,)); max_its = 2;
weight_history,cost_history = newtons_method(g,max_its,w)

w = 4*np.ones((2,)); max_its = 2;
weight_history_2,cost_history_2 = newtons_method(g,max_its,w)

plot_cost_histories([cost_history, cost_history_2], labels=[r'$\mathbf{w}=\m
```



w = 1
w = 4
30
25
20
15
10
5
0
Cost
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
Iteration