AE1MCS: Mathematics for Computer Scientists

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Topics Covered by Discrete Mathamatics

1. Propositional Logic
2. Predicate Logic
3. Inference Rules
4. Proof Techniques
5. Sets
6. Functions
7. Relations
8. Counting
9. Probability

Propositional Logic

- To gain a good understanding of the definitions of proposition, propositional variable and logical operators;
- To gain a good understanding of other definitions: tautology, contradiction, contingency, logical equivalence, converse, contrapositive and inverse.
- To be able to draw **truth tables** and use them as a tool to solve logical problems;
- To be able to apply important logical equivalences to solve logical problems.

Propositional Logic

- Propositions
- Logical Operators
- Tautology, Contradiction and Contingency
- Logical Equivalence
- Converse, Contrapositive and Inverse
- 32 Rules

Some Important Logical Equivalences

	Equivalence	Name
1	$p \wedge T \equiv p$	Identity laws
2	$p \vee F \equiv p$	
3	$p \vee T \equiv T$	Domination laws
4	$p \wedge F \equiv F$	
5	$p \vee p \equiv p$	Idempotent laws
6	$p \wedge p \equiv p$	
7	$\neg(\neg p) \equiv p$	Double negation law
8	$p \vee q \equiv q \vee p$	Commutative laws
9	$p \wedge q \equiv q \wedge p$	

Some Important Logical Equivalences

	Equivalence	Name
10	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	Associative laws
11	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	
12	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	Distributive laws
13	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	
14	$\neg(p \wedge q) \equiv \neg p \vee \neg q$	De Morgan's laws
15	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	
16	$p \vee (p \wedge q) \equiv p$	Absorption laws
17	$p \wedge (p \vee q) \equiv p$	
18	$p \vee \neg p \equiv T$	Negation laws
19	$p \wedge \neg p \equiv F$	

Logical Equivalences involving Implications

20	$p \rightarrow q \equiv \neg p \vee q$
21	$p \rightarrow q \equiv \neg q \rightarrow \neg p$
22	$p \vee q \equiv \neg p \rightarrow q$
23	$p \wedge q \equiv \neg(p \rightarrow \neg q)$
24	$\neg(p \rightarrow q) \equiv p \wedge \neg q$
25	$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
26	$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
27	$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
28	$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Logical Equivalences involving Bi-Implications

29	$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$
30	$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
31	$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$
32	$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Predicate Logic

- To gain a good understanding of the definitions of universal quantification, existential quantification and logical equivalence involving quantifiers;
- To be able to translate between English expressions and quantified expressions.
- To be able to apply De Morgan's laws to negate quantified expressions.
- To be able to apply important logical equivalences to solve logical problems.
- To be able to use predicate logic as a tool to solve problems.

Predicate Logic

- Predicates
- Universal and Existential Quantification
- Logical Equivalences involving Quantifiers
- De Morgan's Laws for Quantifiers

Let $P(x)$,$Q(x)$, and $R(x)$ be the statements “x is a professor”, “x is ignorant,” and ”x is vain,” respectively. Express each of these statements using quantifiers; logical connectives; and $P(x)$,$Q(x)$, and $R(x)$, where the domain consists of all people.

- No professors are ignorant.
- All ignorant people are vain.
- No professors are vain.

A common but not universal pattern:

1. The universal quantifier is very often followed by an implication, because a universal statement is most often of the form ‘given any x, if it has property A then it also has property B’.
2. The existential quantifier is very often followed by a conjunction, because an existing statement is most often of the from ‘there exists an x with property A, which also has property B’.

Inference Rules

- To be able to understand inference rules.
- To be able to use inference rules to solve logical problems.

Inference Rules

- Modus Ponens and Modus Tollens
- Hypothetical syllogism
- Disjunctive syllogism
- Addition
- Simplification
- Conjunction
- Resolution

Inference Rules for Quantified Statements

- Universal instantiation
- Universal generalization
- Existential instantiation
- Existential generalization

Proof Techniques

- To be able to understand different methods of proving theorems.
- To be able to apply different methods to construct proofs.

Proof Techniques

- Direct Proof
- Proof by Contraposition
- Proof by Contradiction
- Proof by Cases
- Proof by Induction

Proof Format (1)

1. A sequence of logical equivalences with reasons for each step
   - Propositional logic (For example, page 35 in slides of "Propositional logic", for the laws without a name, please define it yourself.)
   - Predicate logic (For example, page 33 in slides of "predicate logic")
   - Set (for example, p30 in slides of ?Set?)

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law} \\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law} \\
&\equiv \mathbf{F} \vee (\neg p \wedge \neg q) && \text{because } \neg p \wedge p \equiv \mathbf{F} \\
&\equiv (\neg p \wedge \neg q) \vee \mathbf{F} && \text{by the commutative law for disjunction} \\
&\equiv \neg p \wedge \neg q && \text{by the identity law for } \mathbf{F}
\end{aligned}
$$

Consequently $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

Proof Format(1)

Show that $\neg(p \to q)$ and $p \wedge \neg q$ are logically equivalent.

Answer:

$$
\begin{aligned}
& \neg(p \to q) \\
\equiv\ & \neg(\neg p \vee q) \\
\equiv\ & \neg(\neg p) \wedge \neg q \\
\equiv\ & p \wedge \neg q.
\end{aligned}
$$

Show that $\neg\forall x\,(P(x) \to Q(x))$ and $\exists x\,(P(x) \wedge \neg Q(x))$ are logically equivalent.

Answer:

$$
\begin{aligned}
& \neg\forall x\,(P(x) \to Q(x)) \\
\equiv\ & \exists x\,\neg(P(x) \to Q(x)) \\
\equiv\ & \exists x\,\neg(\neg P(x) \vee Q(x)) \\
\equiv\ & \exists x\,(P(x) \wedge \neg Q(x))
\end{aligned}
$$

Proof Format(1)

Let A, B, and C be sets. Show that

$$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}.$$

Solution: We have

$$\begin{aligned}
\overline{A \cup (B \cap C)} &= \overline{A} \cap (\overline{B \cap C}) && \text{by the first De Morgan law} \\
&= \overline{A} \cap (\overline{B} \cup \overline{C}) && \text{by the second De Morgan law} \\
&= (\overline{B} \cup \overline{C}) \cap \overline{A} && \text{by the commutative law for intersections} \\
&= (\overline{C} \cup \overline{B}) \cap \overline{A} && \text{by the commutative law for unions.}
\end{aligned}$$

Proof Format (2)

2. Steps and reasons: Rules of Inference(for example, page 15, 23, 24 in slides of ?Rule of inferences?)

Show that the premises "If you send me an e-mail message, then I will finish writing the program," "If you do not send me an e-mail message, then I will go to sleep early," and "If I go to sleep early, then I will wake up feeling refreshed" lead to the conclusion "If I do not finish writing the program, then I will wake up feeling refreshed."

Solution: Let p be the proposition "You send me an e-mail message," q the proposition "I will finish writing the program," r the proposition "I will go to sleep early," and s the proposition "I will wake up feeling refreshed." Then the premises are $p \to q$, $\neg p \to r$, and $r \to s$. The desired conclusion is $\neg q \to s$. We need to give a valid argument with premises $p \to q$, $\neg p \to r$, and $r \to s$ and conclusion $\neg q \to s$.

This argument form shows that the premises lead to the desired conclusion.

Step	Reason
1. $p \to q$	Premise
2. $\neg q \to \neg p$	Contrapositive of (1)
3. $\neg p \to r$	Premise
4. $\neg q \to r$	Hypothetical syllogism using (2) and (3)
5. $r \to s$	Premise
6. $\neg q \to s$	Hypothetical syllogism using (4) and (5)

Proof Format (3)

3. Natural Language
   - Propositional logic
   - Predicate logic
   - Set (tutorial examples, please do not use Venn diagram to prove.)
   - Function (tutorial examples)
   - Counting (tutorial examples)
   - Probability (tutorial examples)
   - Relations

Proof Format (3)

Claim proof methods in the beginning, say direct proof, contraposition, contradiction, equivalence, counterexamples, cases, induction, etc.

Prove that $\sqrt{2}$ is irrational.

Proof.

Suppose $\sqrt{2}$ is rational. Then there exist integers p and q with $q \neq 0$ such that $\sqrt{2} = p/q$ and p and q do not have any common factor. Thus, $2 = p^2/q^2$. $p^2 = 2q^2$. Thus, p^2 is even. Since if n is odd, then n^2 is odd (proved in previous slides), p is even. Hence there exists an integer k such that $p = 2k$. Then $p^2 = (2k)^2 = 2q^2$. $q^2 = 2k^2$. Thus q^2 is even, hence q is even. Thus, p and q are both even, which contradicts the fact that p and q do not have any common factor. $\square$

Set

- Set Notations
- Empty Set and Singleton Set
- Venn Diagram
- Subset, Proper Subset, and Equal Sets
- Power Sets, Cartesian products
- Set Operations
- Set Identities

Set Identities

	Identity	Name
1	$A \cap U = A$	Identity laws
2	$A \cup \emptyset = A$	
3	$A \cup U = U$	Domination laws
4	$A \cap \emptyset = \emptyset$	
5	$A \cup A = A$	Idempotent laws
6	$A \cap A = A$	
7	$\overline{(\overline{A})} = A$	Complementation law
8	$A \cup B = B \cup A$	Commutative laws
9	$A \cap B = B \cap A$	

Set Identities

	Identity	Name
10	$A \cup (B \cup C) = (A \cup B) \cup C$	Associative laws
11	$A \cap (B \cap C) = (A \cap B) \cap C$	
12	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
13	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	
14	$\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
15	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	
16	$A \cup (A \cap B) = A$	Absorption laws
17	$A \cap (A \cup B) = A$	
18	$A \cup \overline{A} = U$	Complement laws
19	$A \cap \overline{A} = \emptyset$	

Function

- What is a function?
- One-to-One Function, Onto Functions, One-to-one Correspondence
- Inverse Function, Invertible Function
- Compositions of Functions

Counting

- Basic Counting Principles
  - Product Rule
  - Sum Rule
  - Subtraction Rule (Principle of Inclusion-Exclusion)
  - Division Rule
- The Pigeonhole Principle
- Permutations and Combinations

Relation

- Relations on a Set
- Reflexive, Symmetric, Antisymmetric, Transitive Relations
- Combining Relations
- Composite of Relations
- Equivalence Relations

Probability

- Probability of an event
- Conditional probability
- Bayes' theorem
- Expected value and variance

Some suggestions

- Be well-prepared!
- **Understand** the concepts and definitions.
- Practice
  - Try to solve some problems given in the textbook.
  - Express your answers **clearly** in English.
  - Use **simple** words.
  - Write less but **to the point**!
- Sleep well! :-)

Good Luck!