Question 1

$$\begin{aligned}
2w - x + 2y &= 9 \\
w - x + y + 2z &= 2 \\
4w + 2x - 3y &= 1 \\
3w - 2x &= 4
\end{aligned}$$

The system can be re-written as Ax = b where

$$A = \begin{pmatrix} 2 & -1 & 2 & 0 \\ 1 & -1 & 1 & 2 \\ 4 & 2 & -3 & 0 \\ 3 & -2 & 0 & 0 \end{pmatrix}$$

$$x = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$$

$$b = \begin{pmatrix} 9 \\ 2 \\ 1 \\ 4 \end{pmatrix}$$

Hence, the complete matrix is as belowed:

$$A|b = \left(\begin{array}{cccc|c} 2 & -1 & 2 & 0 & 9 \\ 1 & -1 & 1 & 2 & 2 \\ 4 & 2 & -3 & 0 & 1 \\ 3 & -2 & 0 & 0 & 4 \end{array}\right)$$

Question 2

According to the Question 1:

$$A = \begin{pmatrix} 2 & -1 & 2 & 0 \\ 1 & -1 & 1 & 2 \\ 4 & 2 & -3 & 0 \\ 3 & -2 & 0 & 0 \end{pmatrix}$$

$$A^c = \left(\begin{array}{cccc|c} 2 & -1 & 2 & 0 & 9 \\ 1 & -1 & 1 & 2 & 2 \\ 4 & 2 & -3 & 0 & 1 \\ 3 & -2 & 0 & 0 & 4 \end{array}\right)$$

$$\det(A) = \begin{pmatrix} 2 & -1 & 2 & 0 \\ 1 & -1 & 1 & 2 \\ 4 & 2 & -3 & 0 \\ 3 & -2 & 0 & 0 \end{pmatrix} = -62$$

Since, $\det(A) = -62$, $\rho_A = 4$.
It follows that $\rho_{A^c} = 3$ since a non-singular 4×4 submatrix can be extracted (A) and a 5×5 submatrix cannot be extracted since the size of A^c is 4×5.
Hence, $\rho_A = \rho_{A^c} = m = n = 4$(case 1). This system of linear equations is compatible.

Question 3

Use Cramer's Method to find the solution for x.
The system can be re-written as Ax = b where:

$$A = \begin{pmatrix} 2 & -1 & 2 & 0 \\ 1 & -1 & 1 & 2 \\ 4 & 2 & -3 & 0 \\ 3 & -2 & 0 & 0 \end{pmatrix} b = \begin{pmatrix} 9 \\ 2 \\ 1 \\ 4 \end{pmatrix}$$

Now, let's calculate the determinants:

$$\because \det(A) = \begin{pmatrix} 2 & -1 & 2 & 0 \\ 1 & -1 & 1 & 2 \\ 4 & 2 & -3 & 0 \\ 3 & -2 & 0 & 0 \end{pmatrix}$$

$$\det(A) = 2(-1)^{1+1}\det\begin{pmatrix} -1 & 1 & 2 \\ 2 & -3 & 0 \\ -2 & 0 & 0 \end{pmatrix} + (-1)(-1)^{1+2}\det\begin{pmatrix} 1 & 1 & 2 \\ 4 & -3 & 0 \\ 3 & 0 & 0 \end{pmatrix} + 2(-1)^{1+3}\det\begin{pmatrix} 1 & -1 & 2 \\ 4 & 2 & 0 \\ 3 & -2 & 0 \end{pmatrix} + 0$$

$$\therefore \det(A) = 2\det\begin{pmatrix} -1 & 1 & 2 \\ 2 & -3 & 0 \\ -2 & 0 & 0 \end{pmatrix} - (-1)\det\begin{pmatrix} 1 & 1 & 2 \\ 4 & -3 & 0 \\ 3 & 0 & 0 \end{pmatrix} + 2\det\begin{pmatrix} 1 & -1 & 2 \\ 4 & 2 & 0 \\ 3 & -2 & 0 \end{pmatrix} + 0$$

$$2\det\begin{pmatrix} -1 & 1 & 2 \\ 2 & -3 & 0 \\ -2 & 0 & 0 \end{pmatrix} = 2[-1(-3*0-0)-1(2*0-(-2)*0)+2(2*0-(-2)*(-3))] = -24$$

$$-1\det\begin{pmatrix} 1 & 1 & 2 \\ 4 & -3 & 0 \\ 3 & 0 & 0 \end{pmatrix} = (-1)[1(-3*0-0)-1(4*0-3*0)+2(4*0-3*(-3))] = -18$$

$$2\det\begin{pmatrix} 1 & -1 & 2 \\ 4 & 2 & 0 \\ 3 & -2 & 0 \end{pmatrix} = 2[1(2*0-(-2)*0)-(-1)(4*0-3*0)+2(4*(-2)-3*2)] = -56$$

$\therefore \det(A) = -24 - (-18) - 56 + 0 = -62.$

$$\because \det(A_2) = \begin{pmatrix} 2 & 9 & 2 & 0 \\ 1 & 2 & 1 & 2 \\ 4 & 1 & -3 & 0 \\ 3 & 4 & 0 & 0 \end{pmatrix}$$

$$\det(A_2) = 2(-1)^{1+1}\det\begin{pmatrix} 2 & 1 & 2 \\ 1 & -3 & 0 \\ 4 & 0 & 0 \end{pmatrix} + 9(-1)^{1+2}\det\begin{pmatrix} 1 & 1 & 2 \\ 4 & -3 & 0 \\ 3 & 0 & 0 \end{pmatrix} + 2(-1)^{1+3}\det\begin{pmatrix} 1 & 2 & 2 \\ 4 & 1 & 0 \\ 3 & 4 & 0 \end{pmatrix} + 0$$

$$\therefore \det(A_2) = 2\det\begin{pmatrix} 2 & 1 & 2 \\ 1 & -3 & 0 \\ 4 & 0 & 0 \end{pmatrix} - 9\det\begin{pmatrix} 1 & 1 & 2 \\ 4 & -3 & 0 \\ 3 & 0 & 0 \end{pmatrix} + 2\det\begin{pmatrix} 1 & 2 & 2 \\ 4 & 1 & 0 \\ 3 & 4 & 0 \end{pmatrix} + 0$$

$$2\det\begin{pmatrix}-1 & 1 & 2\\ 2 & -3 & 0\\ -2 & 0 & 0\end{pmatrix} = 2[2(-3*0-0)-1(1*0-4*0)+2(1*0-4*(-3))] = 48$$

$$9\det\begin{pmatrix}1 & 1 & 2\\ 4 & -3 & 0\\ 3 & 0 & 0\end{pmatrix} = 9[1(-3*0-0)-1(4*0-3*0)+2(4*0-3*(-3))] = 162$$

$$2\det\begin{pmatrix}1 & 2 & 2\\ 4 & 1 & 0\\ 3 & 4 & 0\end{pmatrix} = 2[1(1*0-4*0)-2(4*0-3*0)+2(4*4-3*1)] = 52$$

$\therefore \det(A) = 48 - 162 + 52 + 0 = -62$.
Then, use Cramer's Rule to find the solutions for x:

$$x = \frac{\det(A_2)}{\det(A)} = \frac{-62}{-62} = 1$$

Question 4

Given the matrix X:

$$X = \begin{pmatrix} a & b & c & d & 2 \\ 2 & 1 & 1 & 2 & 3 \\ 1 & e & f & g & h \end{pmatrix}$$

For example, my student ID is 20514470, then a = 2, b = 0, c = 5, d = 1, e = 4, f = 4, g = 7 and h = 0.

$$X = \begin{pmatrix} 2 & 0 & 5 & 1 & 2 \\ 2 & 1 & 1 & 2 & 3 \\ 1 & 4 & 4 & 7 & 0 \end{pmatrix}$$

$$X^T = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 4 \\ 5 & 1 & 4 \\ 1 & 2 & 7 \\ 2 & 3 & 0 \end{pmatrix}$$

$$X^T X = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 4 \\ 5 & 1 & 4 \\ 1 & 2 & 7 \\ 2 & 3 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & 5 & 1 & 2 \\ 2 & 1 & 1 & 2 & 3 \\ 1 & 4 & 4 & 7 & 0 \end{pmatrix} = \begin{pmatrix} 9 & 6 & 16 & 13 & 10 \\ 6 & 17 & 17 & 30 & 3 \\ 16 & 17 & 42 & 35 & 13 \\ 13 & 30 & 35 & 54 & 8 \\ 10 & 3 & 13 & 8 & 13 \end{pmatrix}$$

Because I get a 5×5 matrice, the matrice of I is as belowed:

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$10I = \begin{pmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 10 \end{pmatrix}$$

Therefore,

$$A = X^T X + 10I = \begin{pmatrix} 19 & 6 & 16 & 13 & 10 \\ 6 & 27 & 17 & 30 & 3 \\ 16 & 17 & 52 & 35 & 13 \\ 13 & 30 & 35 & 64 & 8 \\ 10 & 3 & 13 & 8 & 23 \end{pmatrix}$$

Question 4

$$A = \begin{pmatrix} 19 & 6 & 16 & 13 & 10 \\ 6 & 27 & 17 & 30 & 3 \\ 16 & 17 & 52 & 35 & 13 \\ 13 & 30 & 35 & 64 & 8 \\ 10 & 3 & 13 & 8 & 23 \end{pmatrix}$$

$$b = \begin{pmatrix} -10 \\ 20 \\ 50 \\ 0 \\ 50 \end{pmatrix}$$

$$A|b = \left(\begin{array}{ccccc|c} 19 & 6 & 16 & 13 & 10 & -10 \\ 6 & 27 & 17 & 30 & 3 & 20 \\ 16 & 17 & 52 & 35 & 13 & 50 \\ 13 & 30 & 35 & 64 & 8 & 0 \\ 10 & 3 & 13 & 8 & 23 & 50 \end{array}\right)$$

$$A^{c(1)} = A|b = \left(\begin{array}{ccccc|c} 19 & 6 & 16 & 13 & 10 & -10 \\ 6 & 27 & 17 & 30 & 3 & 20 \\ 16 & 17 & 52 & 35 & 13 & 50 \\ 13 & 30 & 35 & 64 & 8 & 0 \\ 10 & 3 & 13 & 8 & 23 & 50 \end{array}\right)$$

Let us apply the Gaussian transformations to move to step (2):

$$\begin{aligned} {r_1}^{(2)} &= {r_1}^{(1)} \\ {r_2}^{(2)} &= {r_2}^{(1)} - \frac{6}{19}{r_1}^{(1)} \\ {r_3}^{(2)} &= {r_3}^{(1)} - \frac{16}{19}{r_1}^{(1)} \\ {r_4}^{(2)} &= {r_4}^{(1)} - \frac{13}{19}{r_1}^{(1)} \\ {r_5}^{(2)} &= {r_5}^{(1)} - \frac{10}{19}{r_1}^{(1)} \end{aligned}$$

thus obtaining the following complete matrix:

$$A^{c(2)} = \left(\begin{array}{ccccc|c} 19 & 6 & 16 & 13 & 10 & -10 \\ 0 & \frac{477}{19} & \frac{227}{19} & \frac{492}{19} & \frac{-3}{19} & \frac{440}{19} \\ 0 & \frac{227}{19} & \frac{732}{19} & \frac{457}{19} & \frac{87}{19} & \frac{1110}{19} \\ 0 & \frac{492}{19} & \frac{457}{19} & \frac{1047}{19} & \frac{22}{19} & \frac{130}{19} \\ 0 & \frac{-3}{19} & \frac{87}{19} & \frac{22}{19} & \frac{337}{19} & \frac{1050}{19} \end{array}\right)$$

Let us apply the Gaussian transformations to move to step (3):

$$\begin{aligned}
{r_1}^{(3)} &= {r_1}^{(2)} \\
{r_2}^{(3)} &= {r_2}^{(2)} \\
{r_3}^{(3)} &= {r_3}^{(2)} - \frac{227}{477}{r_2}^{(2)} \\
{r_4}^{(3)} &= {r_4}^{(2)} - \frac{164}{159}{r_2}^{(2)} \\
{r_5}^{(3)} &= {r_5}^{(2)} - \frac{-3}{477}{r_2}^{(2)}
\end{aligned}$$

thus obtaining the following complete matrix:

$$A^{c(3)} = \left(\begin{array}{ccccc|c}
19 & 6 & 16 & 13 & 10 & -10 \\
0 & \frac{477}{19} & \frac{227}{19} & \frac{492}{19} & \frac{-3}{19} & \frac{440}{19} \\
0 & 0 & \frac{15665}{477} & \frac{1865}{159} & \frac{740}{159} & \frac{22610}{477} \\
0 & 0 & \frac{1865}{159} & \frac{1505}{53} & \frac{70}{53} & \frac{-2710}{159} \\
0 & 0 & \frac{740}{159} & \frac{70}{53} & \frac{940}{53} & \frac{8810}{159}
\end{array}\right)$$

Let us apply the Gaussian transformations to move to step (4):

$$\begin{aligned}
{r_1}^{(4)} &= {r_1}^{(3)} \\
{r_2}^{(4)} &= {r_2}^{(3)} \\
{r_3}^{(4)} &= {r_3}^{(3)} \\
{r_4}^{(4)} &= {r_4}^{(3)} - \frac{1119}{3133}{r_3}^{(3)} \\
{r_5}^{(4)} &= {r_5}^{(3)} - \frac{444}{3133}{r_3}^{(3)}
\end{aligned}$$

thus obtaining the following complete matrix:

$$A^{c(4)} = \left(\begin{array}{ccccc|c}
19 & 6 & 16 & 13 & 10 & -10 \\
0 & \frac{477}{19} & \frac{227}{19} & \frac{492}{19} & \frac{-3}{19} & \frac{440}{19} \\
0 & 0 & \frac{15665}{477} & \frac{1865}{159} & \frac{740}{159} & \frac{22610}{477} \\
0 & 0 & 0 & \frac{75840}{3133} & \frac{-1070}{3133} & \frac{-106440}{3133} \\
0 & 0 & 0 & \frac{-1070}{3133} & \frac{53500}{3133} & \frac{152550}{3133}
\end{array}\right)$$

Let us apply the Gaussian transformations to move to step (5):

$$\begin{aligned}
{r_1}^{(5)} &= {r_1}^{(4)} \\
{r_2}^{(5)} &= {r_2}^{(4)} \\
{r_3}^{(5)} &= {r_3}^{(4)} \\
{r_4}^{(5)} &= {r_4}^{(4)} \\
{r_5}^{(5)} &= {r_5}^{(4)} - \frac{-107}{7584}{r_4}^{(4)}
\end{aligned}$$

thus obtaining the following complete matrix:

$$A^{c(5)} = \left(\begin{array}{ccccc|c} 19 & 6 & 16 & 13 & 10 & -10 \\ 0 & \frac{477}{19} & \frac{227}{19} & \frac{492}{19} & \frac{-3}{19} & \frac{440}{19} \\ 0 & 0 & \frac{15665}{477} & \frac{1865}{159} & \frac{740}{159} & \frac{22610}{477} \\ 0 & 0 & 0 & \frac{75840}{3133} & \frac{-1070}{3133} & \frac{-106440}{3133} \\ 0 & 0 & 0 & 0 & \frac{64735}{3792} & \frac{15235}{316} \end{array}\right)$$

The last row $(0\ 0\ 0\ 0\ \frac{64735}{3792} \mid \frac{15235}{316})$ represents the linear equation $\frac{64735}{3792}x_5 = \frac{15235}{316}$, hence $x_5 = \frac{3324}{1177}$.

Then the 4th row$(0\ 0\ 0\ \frac{75840}{3133}\ \frac{-1070}{3133} \mid \frac{-106440}{3133})$ represents $\frac{75840}{3133}x_4 + \frac{-1070}{3133}x_5 = \frac{-106440}{3133}$. Substitute $x_5 = \frac{3324}{1177}$ to get $x_4 = \frac{-15}{11}$.

The 3rd row$(0\ 0\ \frac{15665}{477}\ \frac{1865}{159}\ \frac{740}{159} \mid \frac{22610}{477})$ represents $\frac{15665}{477}x_3 + \frac{1865}{159}x_4 + \frac{740}{159}x_5 = \frac{22610}{477}$. Substituting $x_4 = \frac{-15}{11}$ and $x_5 = \frac{3324}{1177}$ gives $x_3 = \frac{1801}{1177}$.

The 2nd row$(0\ \frac{477}{19}\ \frac{227}{19}\ \frac{492}{19}\ \frac{-3}{19} \mid \frac{440}{19})$ represents $\frac{477}{19}x_2 + \frac{227}{19}x_3 + \frac{492}{19}x_4 + \frac{-3}{19}x_5 = \frac{440}{19}$. Substituting $x_3 = \frac{1801}{1177}$, $x_4 = \frac{-15}{11}$ and $x_5 = \frac{3324}{1177}$ gives $x_2 = \frac{1905}{1177}$.

The 1st row (19 6 16 13 10 | -10) represents $19x_1 + 6x_2 + 16x_3 + 13x_4 + 10x_5 = -10$. Substituting $x_2 = \frac{1905}{1177}$, $x_3 = \frac{1801}{1177}$, $x_4 = \frac{-15}{11}$ and $x_5 = \frac{3324}{1177}$ gives $x_1 = \frac{-64391}{22363}$.

Therefore, $x_1 = \frac{-64391}{22363}$, $x_2 = \frac{1905}{1177}$, $x_3 = \frac{1801}{1177}$, $x_4 = \frac{-15}{11}$ and $x_5 = \frac{3324}{1177}$.