# Recurrence

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# Aims and Learning Objectives

- To be able to understand and explain the definition of a recurrence.
- To be able to understand and use the substitution method, recursion-tree method and master method to solve recurrences.

# Reading

- Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest and Clifford Stein, *Introduction to Algorithms*, Fourth Edition, 2022.
  - Chapter 4.
- Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest and Clifford Stein, *Introduction to Algorithms*, Third Edition, 2009.
  - Chapter 4.

# Divide-and-conquer 分治法

In divide-and-conquer, we solve a problem recursively, applying three steps:

- **Divide** the problem into one or more subproblems that are smaller instances of the same problem.
- **Conquer** the subproblems by solving them recursively. If the subproblem sizes are small enough, however, just solve the subproblems in a straightforward manner.
- **Combine** the subproblem solutions to form a solution to the original problem.

e.g., merge sort and quick sort

# Recursive Case and Base Case

**递归案例（Recursive Case）**：当子问题足够大，可以递归求解时，称为**递归案例**。也就是说，当前问题还需要通过递归的方式继续分解和求解。
**基本案例（Base Case）**：当子问题足够小，可以直接解决，不需要进一步递归时，称为**基本案例**。基本案例是递归的终止条件，用来防止递归无限进行下去。

- When the subproblems are large enough to solve recursively, we call that the **recursive case**.
- When the subproblems become small enough to solve directly without further recursing, we call that the **base case**.

在递归算法中，**递归案例**是继续调用递归的步骤，而**基本案例**是递归停止的条件。当问题的规模逐渐缩小到基本案例时，递归过程就结束，最终合并结果返回。

# Recurrence

Recurrences give us a natural way to characterize the running times of recursive or divide-and-conquer algorithms mathematically.

## Definition

A recurrence is an equation or inequality that describes a function in terms of its value on smaller inputs.

# Recurrence of Merge Sort

Let $n$ denote the size of input, $T(n)$ denote the worst-case running time of merge sort. Then the recurrence is:

- if $n = 1$, then $T(n) = \Theta(1)$.
- if $n > 1$, then $T(n) = 2T(n/2) + \Theta(n)$.

The solution to the recurrence is $T(n) = \Theta(n \log n)$

# Different Forms of Recurrences

Recurrences can take many forms, e.g.

- $T(n) = T(2n/3) + T(n/3) + \Theta(n)$
- $T(n) = T(n-1) + \Theta(1)$
- $T(n) \leq 2T(n/2) + \Theta(n)$ (states only an upper bound on $T(n)$)
- $T(n) \geq 2T(n/2) + \Theta(n)$ (gives only a lower bound on $T(n)$)

# Methods for Solving Recurrences

Three methods for solving recurrences to obain asymptotic $O$ or $\Theta$ bounds on the solution:

- *substitution method*: guess a bound and then use mathematical induction to prove the guess correct.
- *recursion-tree method*: converts the recurrence into a tree whose nodes represent the costs incurred at various levels of the recursion. We use techniques for bounding summations to solve the recurrence.
- *master method*: provides bounds for recurrences of the form

$$T(n) = aT(n/b) + f(n),$$

where $a \geq 1$, $b > 1$, and $f(n)$ is a given function.

这种方法适用于分治法类型的递归关系，例如归并排序和快速排序等算法。

# A Recurrence of the form $T(n) = aT(n/b) + f(n)$

- A recurrence of the form $T(n) = aT(n/b) + f(n)$ characterizes a divide-and-conquer algorithm that creates $a$ subproblems, each of which is $1/b$ times the size of the original problem, using $f(n)$ time for the divide and combine steps.
- Such recurrences arise frequently.

# Technicalities in recurrences 1

- In practice, we neglect certain technical details when we state and solve recurrences.
- e.g., call merge sort on $n$ elements when $n$ is odd
- Technically, the recurrence describing the worst-case running time of merge sort is really
  - if $n = 1$, then $T(n) = \Theta(1)$
  - if $n > 1$, then $T(n) = T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + \Theta(n)$

**忽略某些技术细节**：在实际中，我们通常在陈述和解决递归关系时忽略一些技术细节。例如，当 n 为奇数时，我们会直接对 n 个元素调用归并排序。
**递归关系的准确形式**：**技术上**，描述归并排序最坏情况运行时间的递归关系实际上是：
当 n = 1 时，T(n) = \Theta(1)。
当 n > 1 时，T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + \Theta(n)，即对 n 个元素的归并排序，我们会递归地对左右两个子序列进行排序，每次递归的规模为 n/2，并在合并阶段花费 \Theta(n) 的时间。

# Technicalities in recurrences 2

- Boundary conditions are typically ignored.
- For convenience, we shall generally omit statements of the boundary conditions of recurrences and assume that $T(n)$ is constant for small $n$.
- e.g., we normally state the recurrence for merge sort as $T(n) = 2T(n/2) + \Theta(n)$, without explicitly giving values for small $n$.

# Technicalities in recurrences 3

- When we state and solve recurrences, we often omit floors, ceilings, and boundary conditions.
- We forge ahead without these details and later determine whether or not they matter.
- They usually do not, but you should know when they do.
- Experience helps, and so do some theorems (e.g., the master theorem) stating that these details do not affect the asymptotic bounds of many recurrences characterizing divide-and-conquer algorithms.

在分析递归关系时，虽然忽略了某些细节（如地板和天花板函数），但这通常不会影响最终的时间复杂度分析。然而，了解何时这些细节可能产生影响是很重要的。

# The Substitution Method for Solving Recurrences

The substitution method for solving recurrences comprises two steps:

- Guess the form of the solution.
- Use mathematical induction to find the constants and show that the solution works.

# The Substitution Method: Step 1

We want to determine an upper bound on the recurrence

$$T(n) = 2T(\lfloor n/2 \rfloor) + n$$

- It is similar to $T(n) = 2T(n/2) + \Theta(n)$.
- We guess that the solution is $T(n) = O(n \log n)$.
- The substitution method requires us to prove that $T(n) \leq cn \log n$ for an appropriate choice of the constant $c > 0$.

# The Substitution Method: Step 2

The substitution method requires us to prove that $T(n) \leq cn \log n$ for an appropriate choice of the constant $c > 0$. A skech proof:

- Assume this bound holds for all positive $m < n$, in particular for $m = \lfloor n/2 \rfloor$.
- Hence $T(\lfloor n/2 \rfloor) \leq c\lfloor n/2 \rfloor \log(\lfloor n/2 \rfloor)$.
- Then $T(n) = 2T(\lfloor n/2 \rfloor) + n \leq 2(c\lfloor n/2 \rfloor \log(\lfloor n/2 \rfloor)) + n$ (substitution)
- After some calculations, it can be proved that $T(n) \leq cn \log n$.

# The Substitution Method: more details of Step 2

$$
\begin{aligned}
T(n) = \quad & 2T(\lfloor n/2 \rfloor) + n \\
& \leq 2(c\lfloor n/2 \rfloor \log(\lfloor n/2 \rfloor)) + n \\
& \leq cn \log(n/2) + n \\
& = cn \log n - cn \log 2 + n \\
& = cn \log n - cn + n \\
& \leq cn \log n \text{ for every } c \geq 1
\end{aligned}
$$

# Base Case

Mathematical induction now requires us to show that our solution holds for the boundary conditions. Typically, we do so by showing that the boundary conditions are suitable as **base cases** for the inductive proof.

- Assume that $T(1) = 1$ is the sole boundary condition of the recurrence. However $T(1) \nleq c1 \log 1 = 0$.
- Note that asymptotic notation requires us only to prove $T(n) \leq cn \log n$ for $n \geq n_0$, where $n_0$ is a constant that we get to choose.
- We can replace $T(1)$ by $T(2)$ and $T(3)$ as the base cases in the inductive proof, letting $n_0 = 2$.

# Base Case: more details

- With $T(1) = 1$, we derive from the recurrence that $T(2) = 4$ and $T(3) = 5$.
- Now we can complete the inductive proof that $T(n) \leq cn \log n$ for some constant $c \geq 1$ by choosing $c$ large enough so that $T(2) \leq c * 2 \log 2$ and $T(3) \leq c * 3 \log 3$.
- Any choice of $c \geq 2$ suffices for the base cases of $n = 2$ and $n = 3$ to hold.

# Making a good guess

- There is no general way to guess the correct solutions to recurrences.
- Guessing a solution takes experience and, occasionally, creativity.
- If a recurrence is similar to one you have seen before, then guessing a similar solution is reasonable. e.g., $T(n) = 2T(\lfloor n/2 \rfloor + 17) + n$
- Another way to make a good guess is to prove loose upper and lower bounds on the recurrence and then reduce the range of uncertainty.
- You can use some heuristics, recursion trees, etc.

# Exercise

- Solve the recurrence $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$ by using the substitution method.
- Solve the recurrence $T(n) = 2T(\lfloor \sqrt{n} \rfloor) + \log n$ by using the substitution method.

For answers, see “Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest and Clifford Stein, *Introduction to Algorithms*, Third Edition, 2009. Chapter 4. ”

# The Recursion-tree Method for Solving Recurrences

- In a recursion tree, each node represents the cost of a single subproblem somewhere in the set of recursive function invocations.
- We sum the costs within each level of the tree to obtain the per-level costs, and then we sum all the per-level costs to determine the total cost of all levels of the recursion.
- A recursion tree is best used to generate a good guess, which you can then verify by the substitution method.

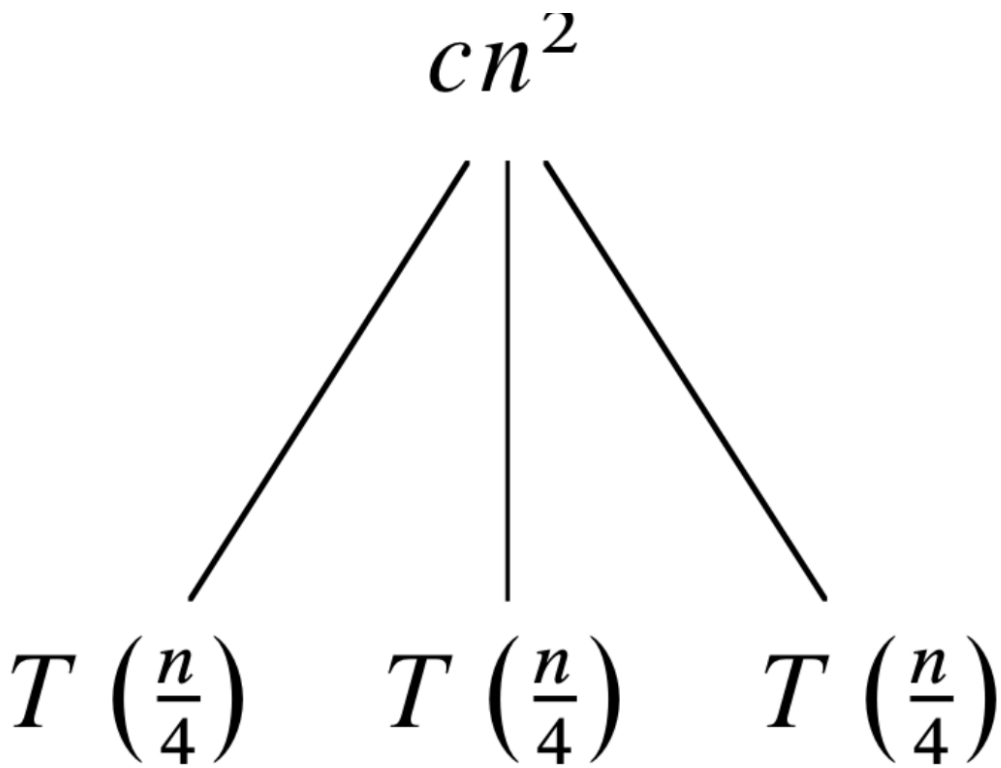
# An Example

Let us see how a recursion tree would provide a good guess for the recurrence $T(n) = 3T(\lfloor n/4 \rfloor) + \Theta(n^2)$

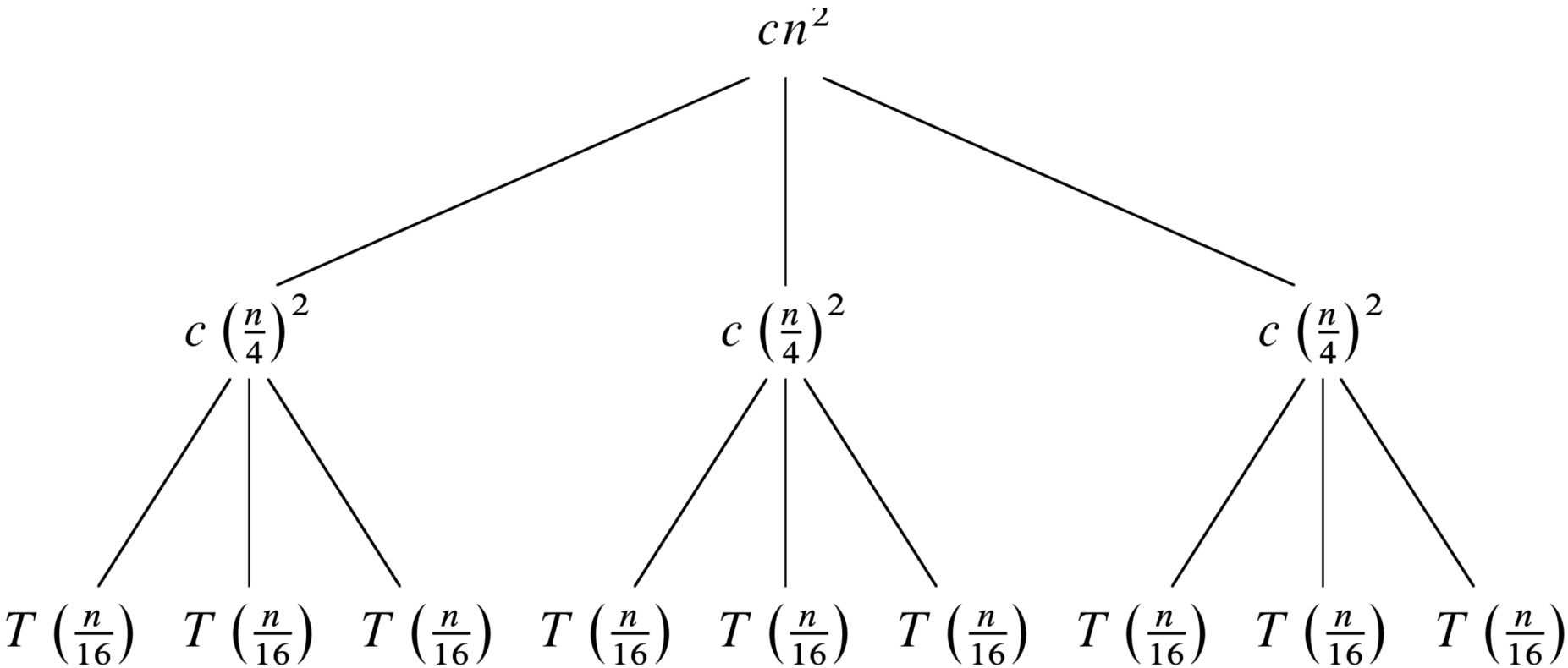
- We start by focusing on finding an upper bound for the solution.
- Floors and ceilings usually do not matter when solving recurrences.
- So we create a recursion tree for the recurrence $T(n) = 3T(n/4) + cn^2$, having written out the implied constant coefficient $c > 0$.
- For convenience, we assume that $n$ is an exact power of 4 (an example of tolerable sloppiness) so that all subproblem sizes are integers.

# Constructing a Recursion Tree 1

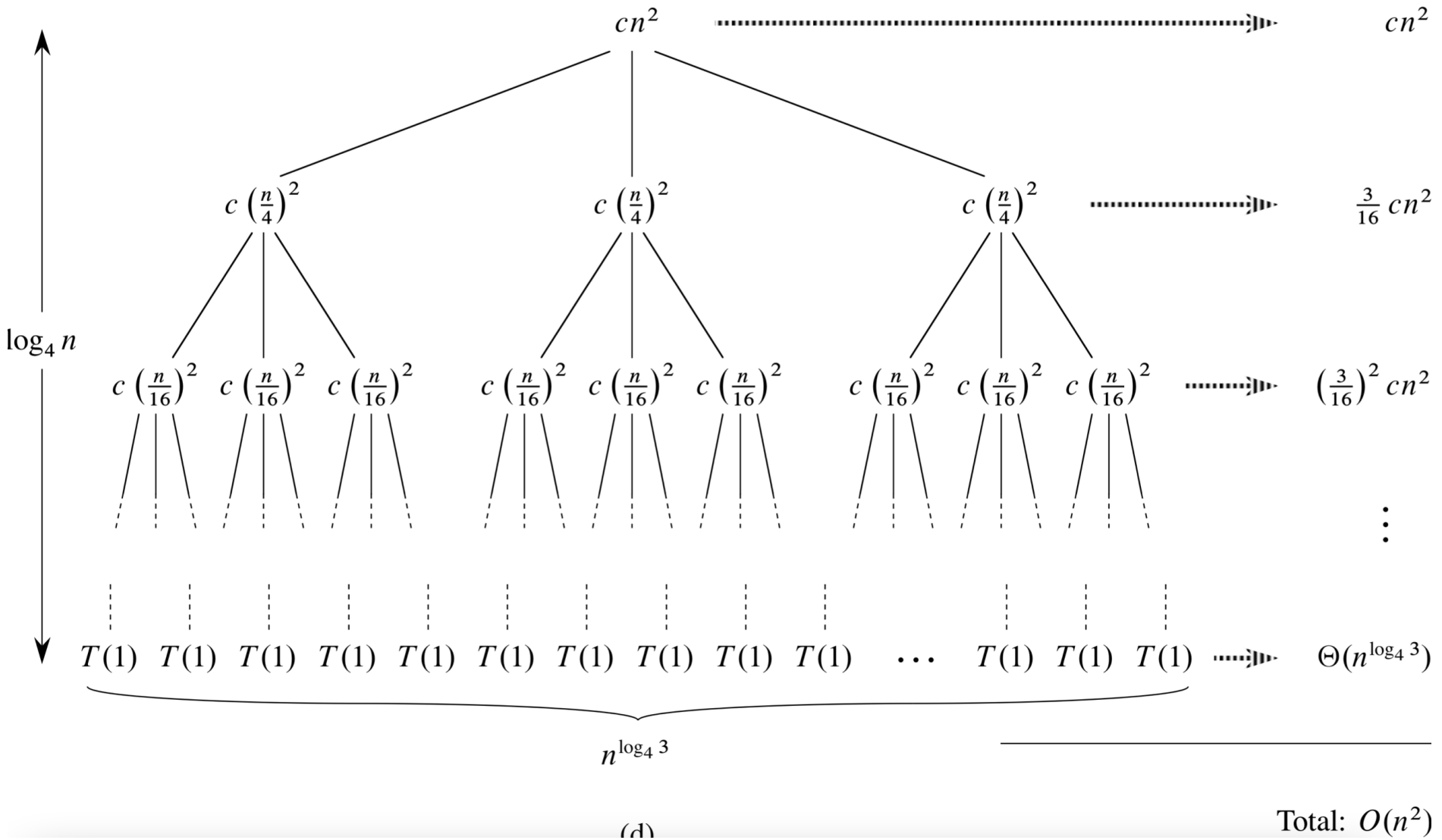
$T(n)$



# Constructing a Recursion Tree 2



# A Fully Expanded Recursion Tree 3



# Cost Analysis 1

- Because subproblem sizes decrease by a factor of 4 each time we go down one level, we eventually must reach a boundary condition.
- The subproblem size for a node at depth $i$ is $n/4^i$.
- The subproblem size is 1 when $n/4^i = 1$, i.e., $i = \log_4 n$.
- Thus, the tree has $\log_4 n + 1$ levels (at depths 0, 1, 2, ..., $\log_4 n$).

# Cost Analysis 2

Now we determine the cost at each level of the tree.

- Each level has three times more nodes than the level above.
- So the number of nodes at depth $i$ is $3^i$.
- Each node at depth $i$, for $i = 0, 1, 2, \ldots, \log_4 n - 1$, has a cost of $c(n/4^i)^2$.
- The total cost over all nodes at depth $i$, for $i = 0, 1, 2, \ldots, \log_4 n - 1$, is $3^i c(n/4^i)^2 = (3/16)^i cn^2$.
- The bottom level, at depth $\log_4 n$, has $3^{\log_4 n} = n^{\log_4 3}$ nodes (to prove they are equal, take $\log_4$ on both sides), each contributing cost $T(1)$, for a total cost of $n^{\log_4 3} T(1)$, which is $\Theta(n^{\log_4 3})$.

# Cost Analysis 3

Now we add up the costs over all levels to determine the cost for the entire tree:

$$
\begin{aligned}
T(n) &= cn^2 + \tfrac{3}{16}cn^2 + (\tfrac{3}{16})^2 cn^2 + \cdots + (\tfrac{3}{16})^{\log_4 n-1} cn^2 + \Theta(n^{\log_4 3}) \\
&= \sum_{i=0}^{\log_4 n-1} (\tfrac{3}{16})^i cn^2 + \Theta(n^{\log_4 3}) \\
&< \sum_{i=0}^{\infty} (\tfrac{3}{16})^i cn^2 + \Theta(n^{\log_4 3}) \\
&= \tfrac{1}{1-(3/16)} cn^2 + \Theta(n^{\log_4 3}) \\
&= \tfrac{16}{13} cn^2 + \Theta(n^{\log_4 3}) \\
&= O(n^2)
\end{aligned}
$$

Thus, we have derived a guess of $T(n) = O(n^2)$ for our original recurrence $T(n) = 3T(\lfloor n/4 \rfloor) + \Theta(n^2)$.

# Cost Analysis 4

- In fact, if $O(n^2)$ is indeed an upper bound for the recurrence (it can be verified using the substitution method), then it must be a tight bound.
- The first recursive call contributes a cost of $\Theta(n^2)$, and so $\Omega(n^2)$ must be a lower bound for the recurrence.

# Exercise

- Use the substitution method to verify that $T(n) = O(n^2)$ is an upper bound for the recurrence $T(n) = 3T(\lfloor n/4 \rfloor) + \Theta(n^2)$.
- Use the recursion tree method to guess a solution of $T(n) = T(n/3) + T(2n/3) + O(n)$, then use the substitution method to verify your guess.

1. $T(n) = O(n^2)$ 是递归关系 $T(n) = 3T(\lfloor n/4 \rfloor) + \Theta(n^2)$ 的上界。
2. 递归关系 $T(n) = T(n/3) + T(2n/3) + O(n)$ 的解是 $T(n) = O(n \log n)$。

# The Master Method for Solving Recurrences

The master method provides a 'cookbook' method for solving recurrences of the form

$$T(n) = aT(n/b) + f(n)$$

where $a \geq 1$ and $b > 1$ are constants and $f(n)$ is an asymptotically positive function.

- The recurrence describes the running time of an algorithm that divides a problem of size $n$ into $a$ subproblems, each of size $n/b$, where $a$ and $b$ are positive constants.
- The $a$ subproblems are solved recursively, each in time $T(n/b)$.
- The function $f(n)$ encompasses the cost of dividing the problem and combining the results of the subproblems.

# Some Notes for the Master Method

- As a matter of technical correctness, the recurrence is not actually well defined, because $n/b$ might not be an integer.
- However, replacing each of the $a$ terms $T(n/b)$ with either $T(\lfloor n/b \rfloor)$ or $T(\lceil n/b \rceil)$ will not affect the asymptotic behavior of the recurrence.

# The Master Theorem

## Theorem (Master Theorem)

*Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence*

$$T(n) = aT(n/b) + f(n)$$

*where we interpret $n/b$ to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ has the following asymptotic bounds:*

1. *If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.*
2. *If $f(n) = \Theta(n^{\log_b a} \log^k n)$ for some constant $k \geq 0$, then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$.*
3. *If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large $n$, then $T(n) = \Theta(f(n))$.*

# Intuition of Master Theorem

在主定理的三个情况中，我们通过比较函数 f(n) 和临界函数 n^{\log_b a}（即“分水岭函数”）来判断递归的复杂度：
**情况 1**：如果 f(n) 的增长速度比 n^{\log_b a} 快得多（即，f(n) 增长更快），那么递归的复杂度将主要由 f(n) 决定，应用**情况 1**。
**情况 2**：如果 f(n) 和 n^{\log_b a} 增长的速率几乎相同（即它们的增长速度相当），那么使用**情况 2**，在这种情况下递归的复杂度会加上一个对数项。
**情况 3**：如果 f(n) 的增长速度比 n^{\log_b a} 更快（即，f(n) 增长的更慢），那么**情况 3**适用，递归的复杂度将由 f(n) 支配。

In each of the three cases, we compare the function $f(n)$ with the function $n^{\log_b a}$ (**the watershed function**).

1. If the watershed function grows asymptotically faster, then Case 1 applies.
2. Case 2 applies if the two functions grow at nearly the same asymptotic rate.
3. If the function $f(n)$ grows asymptotically faster, then Case 3 applies.

# More Explanations 1

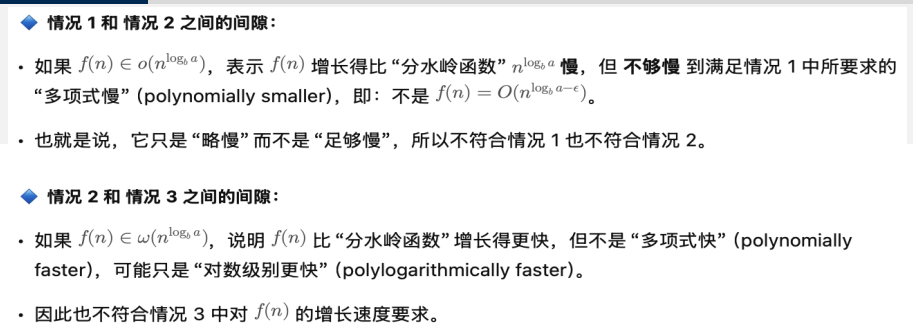
- In Case 1, $f(n)$ should be polynomially smaller (i.e., asymptotically smaller than $n^{\log_b a}$ by a factor of $n^{\epsilon}$ for some constant $\epsilon > 0$).
- In Case 2, $f(n)$ grows faster than $n^{\log_b a}$ by a factor of $\Theta(\log^k n)$, where $k \geq 0$.
- In Case 2, each level of the recursion tree costs approximately the same - $\Theta(n^{\log_b a} \log^k n)$ -and there are $\Theta(\log n)$ levels.
- In practice, the most common situation for Case 2 occurs when $k = 0$.

**情况 1**：f(n) 增长得比 n^{\log_b a} 慢，最终复杂度为 \Theta(n^{\log_b a})。
**情况 2**：f(n) 和 n^{\log_b a} 增长速度相当，复杂度为 \Theta(n^{\log_b a} \log^k n)。
**情况 3**：f(n) 增长得比 n^{\log_b a} 快，复杂度由 f(n) 支配。

# More Explanations 2

- In Case 3, $f(n)$ should be polynomially larger than $n^{\log_b a}$ and also satisfy the 'regularity' condition that $af(n/b) \leq cf(n)$. This condition is satisfied by most of the polynomially bounded functions that we shall encounter.
- For Case 3, if we look at the recursion tree, the cost per level drops at least geometrically from the root to the leaves, and the root cost deominates the cost of all other nodes.

# Gaps between Cases




- There is a gap between cases 1 and 2 when $f(n)$ is in $o(n^{\log_b a})$, yet the watershed function does not grow polynomially faster than $f(n)$.
- There is a gap between cases 2 and 3 when $f(n)$ is in $\omega(n^{\log_b a})$ (i.e., $n^{\log_b a}$ is in $o(f(n))$) and $f(n)$ grows more than polylogarithmically faster than the watershed function, but it does not grow polynomially faster.

If the function $f(n)$ falls into one of these gaps, or if the regularity condition in case 3 fails to hold, then you need to use something other than the master method to solve the recurrence.

# Example 1

| 情况 | 条件 | $f(n)$ 与 $n^{\log_b a}$ 的关系 | 结果 |
|---|---|---|---|
| Case 1 | $f(n) = O(n^{\log_b a - \varepsilon})$ | 明显更小 | $T(n) = \Theta(n^{\log_b a})$ |
| Case 2 | $f(n) = \Theta(n^{\log_b a} \log^k n)$ | 差不多 | $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$ |
| Case 3 | $f(n) = \Omega(n^{\log_b a + \varepsilon})$ 且满足 regularity condition | 明显更大 | $T(n) = \Theta(f(n))$ |

Solve the recurrence $T(n) = 9T(n/3) + n$.

# Example 1: Solution

Solve the recurrence $T(n) = 9T(n/3) + n$.

- $a = 9$, $b = 3$, $f(n) = n$
- $n^{\log_b a} = n^{\log_3 9} = n^2$
- Since $f(n) = O(n^{\log_3 9 - \epsilon})$, where $\epsilon = 1$, we can apply case 1 of the master theorem.
- By case 1 of the master theorem, $T(n) = \Theta(n^2)$.

# Example 2

Solve the recurrence $T(n) = T(2n/3) + 1$.

# Example 2: Solution

Solve the recurrence $T(n) = T(2n/3) + 1$.

- $a = 1$, $b = 3/2$, $f(n) = 1$
- $n^{\log_b a} = n^{\log_{3/2} 1} = n^0 = 1$
- Since $f(n) = \Theta(1)$, we can apply case 2 of the master theorem.
- By case 2 of the master theorem, $T(n) = \Theta(\log n)$

# Example 3

Solve the recurrence $T(n) = 3T(n/4) + n \log n$.

# Example 3: Solution

Solve the recurrence $T(n) = 3T(n/4) + n \log n$.

- $a = 3$, $b = 4$, $f(n) = n \log n$
- $n^{\log_b a} = n^{\log_4 3} = O(n^{0.793})$
- Since $f(n) = \Omega(n^{\log_4 3 + \epsilon})$, where $\epsilon \approx 0.2$, case 3 applies if we can show that the regularity condition holds for $f(n)$.
- For sufficiently large $n$, we have that $af(n/b) = 3(n/4)\log(n/4) \leq (3/4)n \log n = cf(n)$ for $c = 3/4$.
- Hence by case 3, $T(n) = \Theta(n \log n)$.

# Example 4

Solve the recurrence $T(n) = 2T(n/2) + n \log n$.

# Example 4: Solution

Solve the recurrence $T(n) = 2T(n/2) + n \log n$.

- $a = 2$, $b = 2$, $f(n) = n \log n$
- $n^{\log_b a} = n^{\log_2 2} = n^1 = n$
- Case 2 applies since $f(n) = n \log n = \Theta(n^{\log_b a} \log^1 n)$.
- Hence, $T(n) = \Theta(n \log^2 n)$.

# Example 5

Solve the recurrence $T(n) = 2T(n/2) + n/\log n$.

# Example 5: fall into a gap...

Solve the recurrence $T(n) = 2T(n/2) + n/\log n$.

- $a = 2$, $b = 2$
- $n^{\log_b a} = n^{\log_2 2} = n^1 = n$
- $f(n) = n/\log n$ which is $o(n)$, which means that it grows asymptotically more slowly than the watershed function $n$.
- But $n/\log n$ grows only logarithmically slower than $n$, not polynomially slower.
- Neither Case 1 nor Case 2 applies.
- The recurrence falls into the gap between Case 1 and Case 2.

Note that $\log n$ is in $o(n^\epsilon)$ for any constant $\epsilon > 0$.

# Exercises

我们定义：

$n^{\log_b a}$

作为分水岭（**watershed function**）——用来和 $f(n)$ 进行比较：

✅ **Case 2：f(n) 和 n^{log_b a} 增长一样快（差一个对数因子）**

**条件：**

$f(n) = \Theta(n^{\log_b a} \log^k n) \quad \text{for some constant } k \geq 0$

**结论：**

$T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$

**关键词记忆：**

- “**f(n)** 和分水岭一样快”
- “递归和非递归共同贡献复杂度”

Use the master theorem to solve the following recurrences:

1. $T(n) = 2T(n/2) + \Theta(n)$
2. $T(n) = 8T(n/2) + \Theta(n^2)$
3. $T(n) = 7T(n/2) + \Theta(n^2)$

✅ **Case 1：f(n) 明显更小（小一个多项式量级**

**条件：**

$f(n) = O(n^{\log_b a - \varepsilon}) \quad \text{for some } \varepsilon > 0$

**结论：**

$T(n) = \Theta(n^{\log_b a})$

**关键词记忆：**

- “**f(n)** 小很多”
- “递归支配全部复杂度”

✅ **Case 3：f(n) 比 n^{log_b a} 明显更大，而且递归不能赶上**

**条件：**

1. $f(n) = \Omega(n^{\log_b a + \varepsilon}) \quad \text{for some } \varepsilon > 0$
2. 必须满足“**regularity condition**”：

$af(n/b) \leq cf(n) \quad \text{for some } c < 1 \text{ and large enough } n$

**结论：**

$T(n) = \Theta(f(n))$

**关键词记忆：**

- “**f(n)** 明显更大”
- “非递归工作主导复杂度”

# Exercises: Solution

Use the master theorem to solve the following recurrences:

1. Case 2 applies, $T(n) = \Theta(n \log n)$
2. Case 1 applies, $T(n) = \Theta(n^3)$
3. Case 1 applies, $T(n) = \Theta(n^{\log 7})$

Note that for a complete answer, more details should be provided.