AE1MCS: Mathematics for Computer Scientists

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Aim and Learning Objectives

- To gain a good understanding of the definitions of proposition, propositional variable and logical operators;
- To gain a good understanding of other definitions: tautology, contradiction, contingency, logical equivalence, converse, contrapositive and inverse.
- To be able to draw truth tables and use them as a tool to solve logical problems;
- To be able to apply important logical equivalences to solve logical problems.

Proposition

Definition

A proposition is a statement that is either true or false.

The area of logic that deals with propositions is called the *propositional logic* or *propositional calculus*.

Is it a proposition?

- Beijing is the capital of China. ✓
- 2 1+1=2. 3 2+2=3.
- What time is it? X
- 5 Read this sentence carefully.
- 6 x + 1 = 2.
- 7 X + y = z.
- 8 If x > 0, then x > 1.

Unfortunately, it is not always easy to decide if a claimed proposition is true or false.

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- 3 313($x^3 + y^3$) = z^3 has no positive integer solutions.
- 4 Every map can be colored with 4 colors so that adjacent regions have different colors [Four Color Theorem].
- 5 Every even integer greater than 2 is the sum of two primes [Goldbach's conjecture, 1742].

Propositional Variable

- a variable that represents a proposition
- \blacksquare denoted using a letter p, q, r, s, \dots
- truth value: T (true); F (false)

Logical Operators

- **Compound Proposition:** formed from existing propositions using logical operators
- Logical Operators

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Negation
Conjunction
Disjunction
Implication
Implication
```

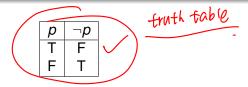
Negation

Definition (Negation)

Let p be a proposition. The *negation* of p, denoted by p is the statement

'It is not the case that p'.

The proposition $\neg p$ is read 'not p'. The truth value of $\neg p$ is the opposite of the truth value of p.



Conjunction

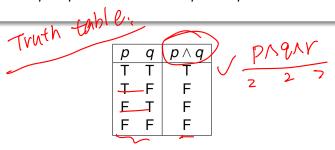
Definition (Conjunction)

Let p and q be propositions. The *conjunction* of p and q, denoted by $p \wedge q$, is the proposition

'p and q'.

The proposition $p \land q$ is true when both p and q are true and is false

otherwise.



Disjunction

Definition (Disjunction)

Let p and q be propositions. The *disjunction* of p and q, denoted by $p \neq q$ is the proposition

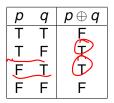
The proposition $p \lor q$ is false when both p and q are false and is true otherwise.

p	q	$p \lor q$	
Т	Т	Т	\vee
T	F	Т	V
F	Τ	Т	\lor
F	F	F	\bigvee

Exclusive Or

Definition (Exclusive Or)

Let p and q be propositions. The *exclusive* or of p and q, denoted by $p \oplus q$, is the proposition that is true when *exactly* one of p and q is true and is false otherwise.



Implication

Definition (Implication)

Pimplies 9 P to 9

Let p and q be propositions. The *implication* p o q is the proposition

'if p, then q'.

The proposition $p \to q$ is false when p is true and q is false, and true otherwise. p is called the hypothesis or premise and q is called the conclusion or consequence.

р	q	p o q
Т	Т	T
Т	F	Ę
F	Ţ	
E	Ē	\

The proposition $p \rightarrow q$ is true, if p is false or q is true,

Examples

If Goldbach's Conjecture is true, then $x^2 \ge 0$ for every real number x. True.

If pigs fly, then your account will not get hacked.

Bi-Implication

Definition (Bi-Implication)

(P>91 (A) (9>P)

Let p and q be propositions. The bi-implication $p \leftrightarrow q$ is the proposition

'p if and only if q'.

The bi-implication $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise.

The words 'if and only if' are sometimes abbreviated 'iff'.

P 9 P39 93P P69

	р	q	$p \leftrightarrow q$	
4	T) T F F	T F T F	T F F T	

 $p \leftrightarrow q$ is true when both $p \rightarrow q$ and $q \rightarrow p$ are true, and is false otherwise.

More Definitions

- Tautology, Contradiction and Contingency
- Logical Equivalence
- Converse, Contrapositive and Inverse

Tautology, Contradiction and Contingency

Definition (Tautology, Contradiction and Contingency)

A compound proposition that is *always true*, no matter what the truth values of the propositions that occur in it, is called a **tautology**. A compound proposition that is *always false* is called a **contradiction**. A compound proposition that is neither a tautology nor a contradiction is called a **contingency**.

Exercise

How to construct a tautology, a contradiction and a contingency using just one propositional variable?



How to construct a tautology, a contradiction and a contingency using just one propositional variable?

		\ /		/
p	$\neg p$	$p \vee \neg p$	$p \land \neg p$	p ightarrow eg p
T	F	T	F	<u>F</u>
F	Т	Т	F	T

[There are some other ways not shown in the table above...]

Logical Equivalence

Definition (Equivalence)

The compound propositions p and q are logically equivalent, if they always have the same truth value (i.e. $p \leftrightarrow q$ is a tautology). The notation p = q denotes that p and q are logically equivalent.

$$P \equiv Q$$

Exercise

- Are $\neg(p \lor q)$ and $\neg p \land \neg q$ logically equivalent? Why?
- Are $p \rightarrow q$ and $\neg p \lor q$ are logically equivalent? Why?

[Hint: construct truth tables]



Are $\neg(p \lor q)$ and $\neg p \land \neg q$ logically equivalent? Why?

Answer: Yes. As shown in the truth table below, $\neg(p \lor q)$ and $\neg p \land \neg q$ always have the same truth value. Thus, they are logical equivalent.

				\sim			
	р	q	$p \lor q$	$\neg(p \lor q)$	$\neg p$	$\neg q$	$\neg p \land \neg q$
	Т	Т		FV	F	F	F
1	Т	F	Т	F 🗸	F	Т	F 🗸
	F	Τ	Т	F 🗸	Т	F	F 🗸
	F,	F,	F	T_/	Т	Т	T //
•	\checkmark	V		V			٩/

Are $p \rightarrow q$ and $\neg p \lor q$ are logically equivalent? Why?

Answer: Yes. As shown in the truth table below, $p \to q$ and $\neg p \lor q$ always have the same truth value. Thus, they are logical equivalent.

			'\/	
p	q	$\neg p$	$\neg p \lor q$	$p \rightarrow q$
Т	Т	F	T	Т
T	F	F	F	F
F	Τ	Т	T	T
F	F	Τ	T	Т

Converse, Contrapositive and Inverse

- The **converse** of $p \rightarrow q$ is the proposition $q \rightarrow p$.
- The **contrapositive** of $p \rightarrow q$ is the proposition $\neg q \rightarrow \neg p$.
- The **inverse** of $p \rightarrow q$ is the proposition $\neg p \rightarrow \neg q$.

Which pairs of the following propositions are equivalent? Why?

- a conditional statement and its converse X
- a conditional statement and its contrapositive
- a conditional statement and its inverse X
- 9, p>9, 9->12

Some Important Logical Equivalences

	Equivalence	Name
1	$p \wedge T \equiv p$	Identity laws
2	$p \lor F \equiv p$	
3	$p \lor T \equiv T$	Domination laws
4	$p \wedge F \equiv F$	
5	$p \lor p \equiv p$	Idempotent laws
6	$p \wedge p \equiv p$	
7	$\neg(\neg p) \equiv p$	Double negation law
8	$p \lor q \equiv q \lor p$	Commutative laws,
9	$p \wedge q \equiv q \wedge p$	

Some Important Logical Equivalences

- 1			
		Equivalence	Name
	10	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	Associative laws
	11	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	
	12	$p \lor (g \land r) \equiv (p \lor q) \land (p \lor r)$	Distributive laws
	13	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	
,	14	$(\neg(p \land q) \equiv \neg p \lor \neg q$	De Morgan's laws
6	15	$ eg(p \lor q) \equiv eg p \land eg q$	
	16	$p \vee (p) \wedge q \equiv p$	Absorption laws
	17	$p \land (p \lor q) \equiv p$	
	18	$ otan \lor eg otan \equiv T $	Negation laws
	19	$p \land \neg p \equiv F$	



Logical Equivalences involving Implications

$$\begin{array}{c|c}
 & p \rightarrow q \equiv \neg p \lor q \\
21 & p \rightarrow q \equiv \neg q \rightarrow \neg p \\
22 & p \lor q \equiv \neg p \rightarrow q \\
23 & p \lor q \equiv \neg p \rightarrow q \\
24 & p \lor q \equiv \neg p \rightarrow q \\
25 & (p \rightarrow q) \land (p \rightarrow r) \equiv p \land \neg q \\
26 & (p \rightarrow r) \land (q \rightarrow r) \equiv (p \lor q) \rightarrow r \\
27 & (p \rightarrow q) \lor (p \rightarrow r) \equiv p \rightarrow (q \lor r) \\
28 & (p \rightarrow r) \lor (q \rightarrow r) \equiv (p \land q) \rightarrow r
\end{array}$$

Logical Equivalences involving Bi-Implications

$$\begin{array}{c|c} 29 & p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p) \\ 30 & p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q \\ 31 & p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q) \\ 32 & \neg (p \leftrightarrow q) \equiv p \leftrightarrow \neg q \\ \end{array}$$

Using De Morgan's Laws

Use De Morgan's laws to express the negations of the following sentences.

- Tony has a cellphone and he has a laptop computer.
- Heather will go to the concert or Steve will go to the concert.

Constructing New Logical Equivalences

- A proposition in a compound proposition can be replaced by a compound proposition that is logically equivalent to it without changing the truth value of the original compound proposition.
- Prove two propositions are logically equivalent by developing a series of logical equivalences.

Exercise

- Show that $\neg(p \rightarrow q)$ and $p \land \neg q$ are logically equivalent.
- Show that $\neg(p \lor (\neg p \land q))$ and $\neg p \land \neg q$ are logically equivalent.
- Show that $(p \land q) \rightarrow (p \lor q)$ is a tautology.

Show that $\neg(p \to q)$ and $p \land \neg q$ are logically equivalent.

Answer:

$$\begin{array}{l} \neg(p \rightarrow q) \\ \equiv \neg(\neg p \lor q) \text{ by law 20} \\ \equiv \neg(\neg p) \land \neg q \text{ by De Morgan's (aw)} \\ \equiv p \land \neg q. \text{ by double negation} \end{array}$$

Show that $\neg(p \lor (\neg p \land q))$ and $\neg p \land \neg q$ are logically equivalent.

Answer:

$$\begin{array}{l} \neg(p \lor (\neg p \land q)) \\ \equiv \neg p \land \neg(\neg p \land q) \\ \equiv \neg p \land (\neg p \land q) \\ \equiv \neg p \land (\neg p \land q) \\ \equiv \neg p \land (p \lor \neg q) \\ \equiv \neg p \land (p \lor \neg q) \\ \equiv (\neg p \land p) \lor (\neg p \land \neg q) \\ \equiv F \lor (\neg p \land \neg q) \\ \equiv \neg p \land \neg q. \\ \text{by identity law}. \end{array}$$

Show that $(p \land q) \rightarrow (p \lor q)$ is a tautology.

Answer:

$$(p \land q) \rightarrow (p \lor q)$$

$$\equiv \neg(p \land q) \lor (p \lor q)$$

$$\equiv (\neg p \lor \neg q) \lor (p \lor q)$$

$$\equiv (p \lor \neg p) \lor (q \lor \neg q)$$

$$= T \lor T = T$$

List of Symbols

SYMBOL	MEANING	PAGE
$\neg p$	negation of p	3
$p \wedge q$	conjunction of p and q	4
$p \vee q$	disjunction of p and q	4
$p \oplus q$	exclusive or of p and q	6
$p \rightarrow q$	the implication p implies q	6
$p \leftrightarrow q$	biconditional of p and q	9
$p \equiv q$	equivalence of p and q	23
ceiling falkening	tautology	23
Fw [m] ho mest	contradiction	23
$P(x_1,\ldots,x_n)$	propositional function	36
$\forall x P(x)$	universal quantification of $P(x)$	38
$\exists x P(x)$	existential quantification of $P(x)$	40
	uniqueness quantification of $P(x)$	41
$\exists !x P(x)$	therefore	64
$P\{S\}q$	partial correctness of S	364

Expected Learning Outcomes

- To gain a good understanding of the definitions of proposition, propositional variable and logical operators;
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Reading

Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

■ Sections 1.1-1.3.