Languages and Computation (COMP 2049) Lab 08
Closure Properties of Context-Free Languages

In the previous lab, we learned that the class of context-free languages (CFLs) is closed under union, concatenation, and star-closure. Through the following exercises, we will learn that the class of CFLs is not closed under intersection or complementation.

Consider the following languages over the alphabet $\Sigma := \{a, b, c\}$:

- $L_1 := \{a^n b^n c^m \mid m \geq 0, n \geq 0\}$,
- $L_2 := \{a^m b^n c^n \mid m \geq 0, n \geq 0\}$.

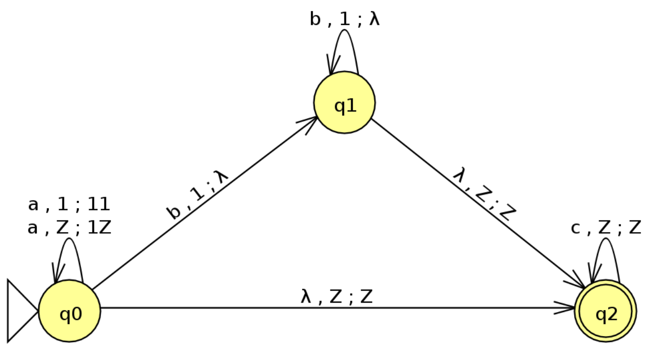
In the lecture slides for Chapter 5 (Context-free languages) you may find two context-free grammars (CFGs) that generate these two languages.

(a) Draw the transition graphs of two pushdown automata (PDAs) M_1 and M_2 such that $L_1 = L(M_1)$ and $L_2 = L(M_2)$.

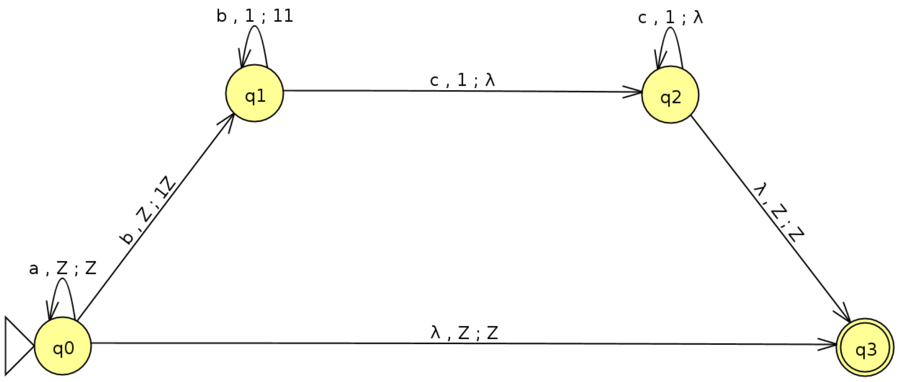
(b) Implement M_1 and M_2 in JFLAP and experiment with them.

Solution

The following is a possible implementation of M_1:



The following is a possible implementation of M_2:



It can be proven that the language $A^nB^nC^n := \{a^nb^nc^n \mid n \geq 0\}$ is not context-free. We did not discuss this in our lectures, but for those who are interested, you may find a proof in [LR23, Example 8.1], which uses the Pumping Lemma for Context-Free Languages [LR23, Theorem 8.1].

(c) Using the fact that $A^nB^nC^n$ is not context-free, prove that the class of CFLs is not closed under intersection.

Solution

We have $A^nB^nC^n = L_1 \cap L_2$. We know that both L_1 and L_2 are context-free.

(d) Prove that the class of CFLs is not closed under complementation. (Hint. Use De Morgan's laws.)

Solution

Note that:

$$A^nB^nC^n = L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}} \tag{1}$$

We know that the class of CFLs is closed under the union operation. Let us now assume that the class of CFLs is closed also under complementation. This means that, by (1), the language $L_1 \cap L_2$ must be context-free. This, in turn, implies that $A^nB^nC^n$ is context-free, which is a contradiction.

References

[LR23] Linz, P. and Rodger, S. H. An Introduction to Formal Languages and Automata. 7th ed. Jones & Bartlett Learning, 2023.