(a) Proof: we assume $\sqrt{3}$ is a rational number written as $\sqrt{3}=\frac{n}{m}$ where m and n are non-zero integers without a common factor.
We could then have $3m^2=n^2$

Languages and Computation (COMP 2049) Lab 01

Proof Techniques & Basic Concepts

This equation implies n^2 is an odd number, and thus n is also an odd number. We could rewrite n as 3k: $m^2=3k^2$
Therefore, m is odd as well. But from the assumption we made, n and m do not have common factor. Contradiction. Hence, $\sqrt{3}$ is irrational.

(1) Use proof by contradiction to answer the following:

(a) Prove that $\sqrt{3}$ is irrational. 无理数.

(b) Using the fact that $\sqrt{2}$ is irrational, prove that the following numbers are also irrational:

(i) $\sqrt{8}$ $=\sqrt{2\times4}=2\sqrt{2}$ Since $\sqrt{2}$ is irrational, multiplying it by 2 still results in an irrational number. Therefore, $\sqrt{8}$ is irrational.

(ii) $2-\sqrt{2}$

Suppose $2-\sqrt{2}$ is rational. Then we can write it as $2-\sqrt{2}=\frac{m}{n}$, where m and n are integers with no common factors. $\sqrt{2}=2-\frac{m}{n}=\frac{2n-m}{n}$. This expression shows that $\sqrt{2}$ is rational, which contradicts the fact that $\sqrt{2}$ is irrational. Therefore $2-\sqrt{2}$ must be irrational.

(2) True or False:

$\sqrt{2}-\sqrt{2}=0 \leftarrow$ F (a) The sum of every two irrational numbers is irrational.

$\sqrt{2}\times\frac{1}{\sqrt{2}}=1 \leftarrow$ F (b) The product of every two irrational numbers is irrational.

? (c) [Optional] For any two irrational numbers a and b, the number a^b is irrational.

(3) Using mathematical induction, prove that, for any real number $r \neq 1$ and natural number n:

$$\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}.$$

Base case: when $n=0$, $\sum_{i=0} r^i = r^0 = 1$. The formula gives $\frac{1-r^{0+1}}{1-r}=\frac{1-r}{1-r}=1$ So, the base case holds.

Assume the formula holds for $n=k$; $\sum_{i=0}^{k} r^i = \frac{1-r^{k+1}}{1-r}$

Inductive Step: show that $n=k+1$ holds $\sum_{i=0}^{k+1} r^i = \frac{1-r^{k+2}}{1-r} = \sum_{i=0}^{k} r^i + r^{k+1}$

$\sum_{i=0}^{k+1} r^i = \frac{1-r^{k+1}}{1-r} + r^{k+1}$ by inductive hypothesis

$\frac{1-r^{k+1}}{1-r} + r^{k+1} = \frac{\boxed{1-r^{k+1}+r^{k+1}(1-r)}}{1-r}$

$\boxed{1-r^{k+1}+r^{k+1}-r^{k+2}} = 1-r^{k+2}$

$\frac{1-r^{k+2}}{1-r}$

Thus, the formula holds for $n=k+1$, completing the inductive step.

(4) Consider the grammar $G=(V,T,S,P)$, in which $V=\{S\}$, $T=\{a,b\}$, and the set of productions is given by:

$$S \to aSa \mid bSb \mid \lambda$$

(a) Write a derivation of the string *abba*.

(b) Experiment with, and derive a few strings generated by, the grammar G. What language is generated by the grammar? Write down a concise description of $L(G)$. 对称 ✓.

Derivation steps:

1. Start with the start symbol S.
2. Apply the production $S \to aSa$ to begin the derivation with "a" and "a" on either side:

$S \Rightarrow aSa$

$G=(V,T,S,P)$, $V=\{S\}$, $T=\{a,b\}$. $S \to aSa \mid bSb \mid \lambda$ where λ respresents empty string

(a) Start with start symbol S.

3. Now, the symbol S in the middle can be replaced by the production $S \to bSb$ to match the "b" and "b" in the middle of the string:

$aSa \Rightarrow a(bSb)a$

For $S \Rightarrow \lambda$: empty string.

For $S \Rightarrow aSa$: aa, aaaa, etc.

4. We now have bSb in the middle, and to match the "b" and "b" on both sides, we can replace the inner S with λ (the empty string):

For $S \Rightarrow bSb$: bb, bbbb, etc.

For $S \Rightarrow bSb \Rightarrow aSa \Rightarrow \lambda$: ab, ba, abba, baab, aabb, etc.

$a(bSb)a \Rightarrow a(b\lambda b)a \Rightarrow abba$

Thus, the derivation of "abba" is:

$S \Rightarrow aSa \Rightarrow a(bSb)a \Rightarrow abba$

Apply the rule $S \Rightarrow bSb$: $S \Rightarrow bSb$.

Apply the rule $S \Rightarrow aSa$: $bSb \Rightarrow b(aSa)b$.

Apply the rule $S \Rightarrow \lambda$: $b(aSa)b \to ba\lambda ab \Rightarrow abba$

Thus, abba: $S \Rightarrow bSb \Rightarrow b(aSa)b \Rightarrow b\boxed{a}ab \Rightarrow abba$

(b) Experiment with and derive a few strings generated by the grammar G, and describe the language $L(G)$:

Strings generated by the grammar:

- For $S \to \lambda$, the string "empty" or λ (the empty string) is generated.
- For $S \to aSa$, we can generate strings such as "aa", "aaaa", etc.
- For $S \to bSb$, we can generate strings such as "bb", "bbbb", etc.
- For $S \to bSb \to aSa \to \lambda$, we can generate strings like "ab", "ba", "abba", etc.

Language description $L(G)$:

The grammar generates strings in which the number of 'a's and 'b's are balanced and occur symmetrically around the center. Specifically, the grammar generates: 不对 baab

- Any string where there are equal numbers of 'a's and 'b's,
- The 'a's and 'b's appear symmetrically around the center.

Thus, $L(G)$ can be described as the set of strings that are **palindromes** over the alphabet $\{a,b\}$, where the number of 'a's and 'b's are equal.

For example, some strings generated by the grammar include: "", "aa", "bb", "abba", "baab", "aabb", etc.