Classical Logic in Lean

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Aims and Learning Objectives

- To be able to understand the difference between classical logic and intuitionistic logic;
- To be able to understand and prove the de Morgans Laws in Lean;
- To be able to understand and use the law of the excluded middle in the proof in Lean.
- To be able to construct proofs in Lean using the indirect proof method.

Reading

- Thorsten Altenkirch, *Introduction to Formal Reasoning*, 2023.
  - Chapter 3. Classical Logic

Classical Logic and Intuitionistic Logic

- We stick to propositional logic and discuss a difference between the logic based on truth and the logic based on evidence.
- The truth based logic is called *classical logic*.
- The evidence based logic is called *intuitionistic logic*.

The de Morgan Laws

- The de Morgan laws state that if you negate a disjunction or conjunction, this is equivalent to the negation of their components with the disjunction replaced by conjunction and vice versa.
- More precisely:

$$\neg(P \vee Q) \;\leftrightarrow\; \neg P \wedge \neg Q$$

$$\neg(P \wedge Q) \;\leftrightarrow\; \neg P \vee \neg Q$$

Prove the first de Morgan Law

How to prove $\neg(P \vee Q) \leftrightarrow \neg P \wedge \neg Q$ in Lean?

```
variables P Q : Prop

theorem dm1 : ¬ (P ∨ Q) ↔ ¬ P ∧ ¬ Q :=
begin

end
```

Prove the second de Morgan Law

How to prove $\neg(P \wedge Q) \leftrightarrow \neg P \vee \neg Q$ in Lean?

```
variables P Q : Prop

theorem dm2 : ¬ (P ∧ Q) ↔ ¬ P ∨ ¬ Q :=
begin

end
```

An Example

- *It is not the case that I have a cat and that I have a dog*, can we conclude that *I don't have a cat* or *I don't have a dog*?
- No, because we do not know which one is the case, this is, we do not have evidence for either.

The Law of the Excluded Middle

- The law of the excluded middle expresses the idea that every proposition is either true or false.
- To speak with Shakespeare *To P or not to P*, that is, $P \vee \neg P$ for any proposition P.
- In latin, this law is called *Tertium non datur*, which translates to *the 3rd is not given*.

The Law of the Excluded Middle in Lean

- In Lean, we access the axiom *the law of the excluded middle* by:

```
variable P : Prop

open classical

#check em P
```

- The command *#check* checks the type of a term.
- For any proposition P, *em P* proves $P \vee \neg P$.

Using *em P*

- Using *em P*, we can complete the missing direction of the 2nd de Morgan law:

```
variables P Q : Prop

open classical

theorem dm2_em : ¬ (P ∧ Q) → ¬ P ∨ ¬ Q :=
begin

end
```

- The idea of the proof is that we look at both cases of $P \vee \neg P$.

Indirect Proof 1

- There is another law which is equivalent to the principle of excluded middle, and this is the *principle of indirect proof*, or in latin *reduction ad absurdo* (reduction to the absurd).
- This principle tells that to prove P, it is sufficient to show that $\neg P$ is impossible.

Indirect Proof 2

How to prove $\neg\neg P \to P$?

```
variable P : Prop
open classical

theorem raa : ¬ ¬ P → P :=
begin

end
```

Indirect Proof 3

- How to prove $\neg\neg P \to P$?

```
variable P : Prop
open classical

theorem raa : ¬ ¬ P → P :=
begin

end
```

- The idea is to assume $\neg\neg P$ and then prove P by analysing $P \vee \neg P$.

Exercise 1

How to prove $\neg\neg(P \vee \neg P)$ without using classical logic in Lean?

```
variables P Q : Prop

theorem nn_em : ¬ ¬ (P ∨ ¬ P) :=
begin

end
```

A Story about Time Travel

There was a magician who could time travel and wanted to marry the daughter of a king. There was no gold in the country but people were not sure whether diamonds exist. The king set the magician the task to either find a diamond or to produce a way to turn diamond into gold. The magician went for the 2nd option and gave the king an empty box so he could marry the daughter of the king. However, if the king would get hold a diamond at some point and the magician's lie would become obvious, then the magician would just take the diamond, travel back in time and go for the first option.

Proving *em*

- If we assume we have a constant proving *raa*, we can show *em*.
- How to prove the theorem *em*?

```
variables P Q : Prop

constant raa : ¬ ¬ P → P

theorem em : P ∨ ¬ P :=
begin

end
```

em and *raa*

- While *em* and *raa* are equivalent as global principles, this is not the case for individual propositions.
- If we assume $P \vee \neg P$, we can prove $\neg\neg P \rightarrow P$ for the same proposition P.
- However, if we assume $\neg\neg P \rightarrow P$, we cannot prove $P \vee \neg P$ for that proposition, but we actually need a different instance of *raa*, namely, $\neg\neg(P \vee \neg P) \rightarrow P \vee \neg P$.

Intuitionistic Logic vs. Classical Logic

- Should we always assume *em* (or alternatively *raa*), hence should we always work in classical logic?
- There are a philosophical argument and a pragmatic argument in favour of avoiding it and using intuitionistic logic.

The Philosophical Argument

- While facts about the real world are true or false, even if we do not know them, this is not so obvious about mathematical constructions which take place in our head.
- The set of all numbers does not exist in the real world, it is like a story we share and we do not know whether anything we make up is either true or false.
- However, we can talk about evidence without needing to assume that.

Platonism and Intuitionism

- The idea that the world of ideas is somehow real, and that the real world is just a poor shadow of the world of ideas was introduced by the greek philosopher Plato and hence is called *Platonism*.
- In constrast, that our ideas are just constructions in our head is called *Intuitionism*.

The Pragmatic Argument

- The pragmatic argument is maybe more important.
- Intuitionistic logic is constructive, indeed in a way that is dear to computer scientists: whenever we show that something exists we are actually able to compute it.
- As a consequence, intuitionistic logic introduces many distinctions which are important, especially in computer science.
- For example, we can distinguish decidable properties from properties in general. By a function, we mean something we can compute like in a (functional) programming language.

A Famous Example

- Here is a famous example to show that the principle of excluded middle destroys constructivity.
- We want to show that there are two irrational numbers p and q such that their power p^q is rational .
  - We know $\sqrt{2}$ is irrational. Now what is $\sqrt{2}^{\sqrt{2}}$? Using the excluded middle, it is either rational or irrational.
  - If it is rational, then we are done, $p = q = \sqrt{2}$.
  - Otherwise, we use $p = \sqrt{2}^{\sqrt{2}}$, which we now assume to be irrational and $q = \sqrt{2}$. Now a simple calculation shows $p^q = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = \sqrt{2}^2 = 2$ which is rational.
- However, after this proof, we still do not know two irrational numbers whose power is rational.

Further Notes 1

- In the homework, we will distinguish proofs that use excluded middle and which do not.
- We would ask you to prove propositions intuitionistically where possible, and only use classical reasoning where necessay.

Further Notes 2

- How to see whether a proposition is only provable classically?
- For example, why can we prove all the first de Morgan law and one direction of the second but not $\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$?
- The reason is that the right hand side contains some information, i.e., which of the $\neg P$ or $\neg Q$ is true, while $\neg(P \wedge Q)$ is a *negative* proposition and hence does contain no information.
- In contrast, the both sides of the first de Morgan law $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$ are negative, i.e., contain no information.