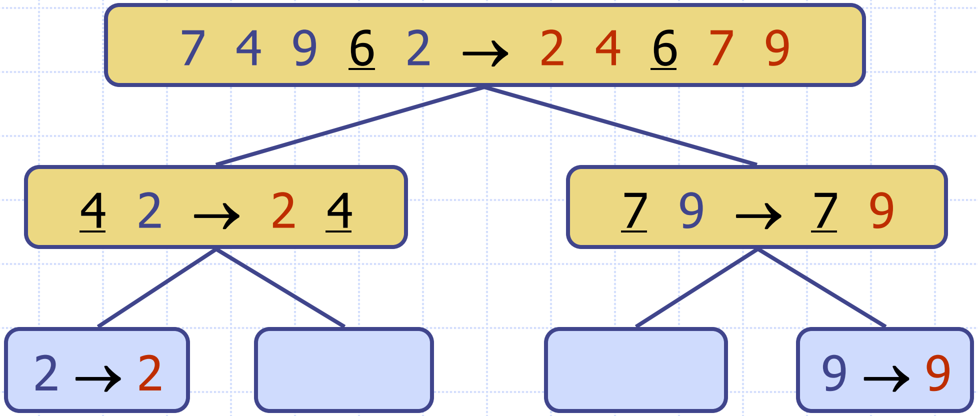
Presentation for use with the textbook Data Structures and Algorithms in Java, 6th edition, by M. T. Goodrich, R. Tamassia, and M. H. Goldwasser, Wiley, 2014

# Quick-Sort



# Aim and Learning Objectives

- To be able to *understand* and *describe* the quick sort algorithm
- To be able to *analyze* the complexity of the quick sort algorithm
- To be able to *implement* the quick sort algorithm and *apply* it to solve problems

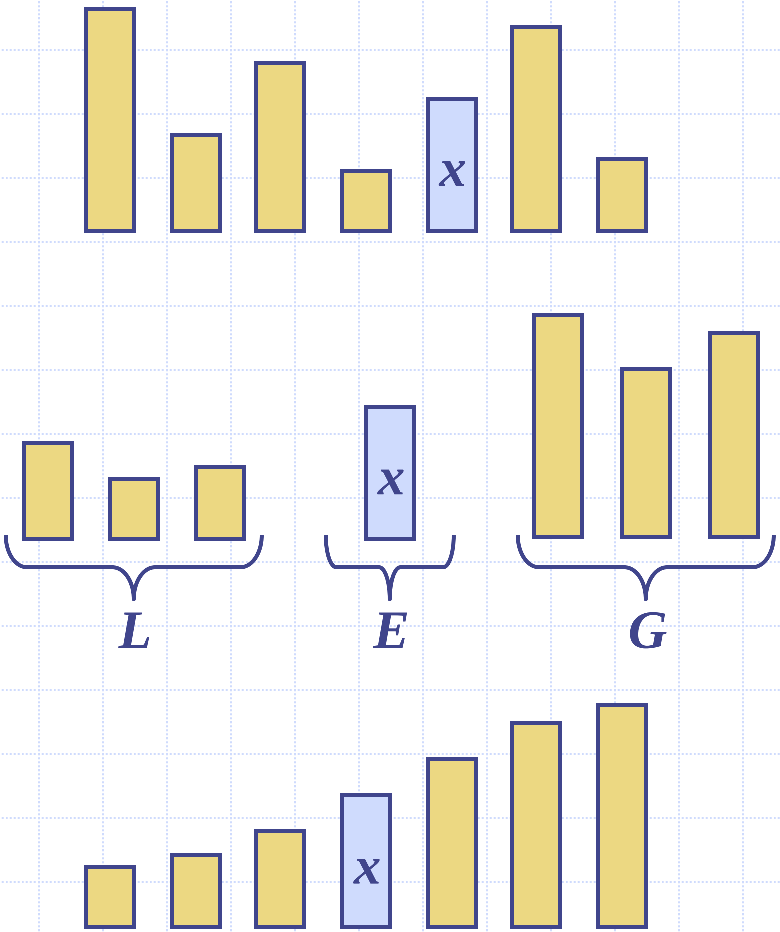
# Reading

**M. T. Goodrich, R. Tamassia and M. H. Goldwasser, *Data Structures and Algorithms in Java*, 6th Edition, 2014.**

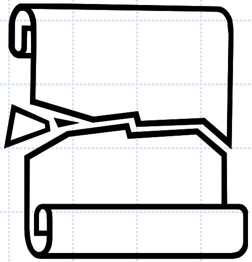
- **Chapter 13. Sorting and Selection**

# Quick-Sort

- Quick-sort is a randomized sorting algorithm based on the divide-and-conquer paradigm:
  - Divide: pick a random element $x$ (called pivot) and partition $S$ into
    - $L$ elements less than $x$
    - $E$ elements equal $x$
    - $G$ elements greater than $x$
  - Recur: sort $L$ and $G$
  - Conquer: join $L$, $E$ and $G$



# Partition



- We partition an input sequence as follows:
  - We remove, in turn, each element $y$ from $S$ and
  - We insert $y$ into $L$, $E$ or $G$, depending on the result of the comparison with the pivot $x$
- Each insertion and removal is at the beginning or at the end of a sequence, and hence takes $O(1)$ time
- Thus, the partition step of quick-sort takes $O(n)$ time

```
Algorithm partition(S, p)
    Input sequence S, position p of pivot
    Output subsequences L, E, G of the
        elements of S less than, equal to,
        or greater than the pivot, resp.
    L, E, G ← empty sequences
    x ← S.remove(p)
    while ¬S.isEmpty()
        y ← S.remove(S.first())
        if y < x
            L.addLast(y)
        else if y = x
            E.addLast(y)
        else { y > x }
            G.addLast(y)
    return L, E, G
```

# Quick Sort

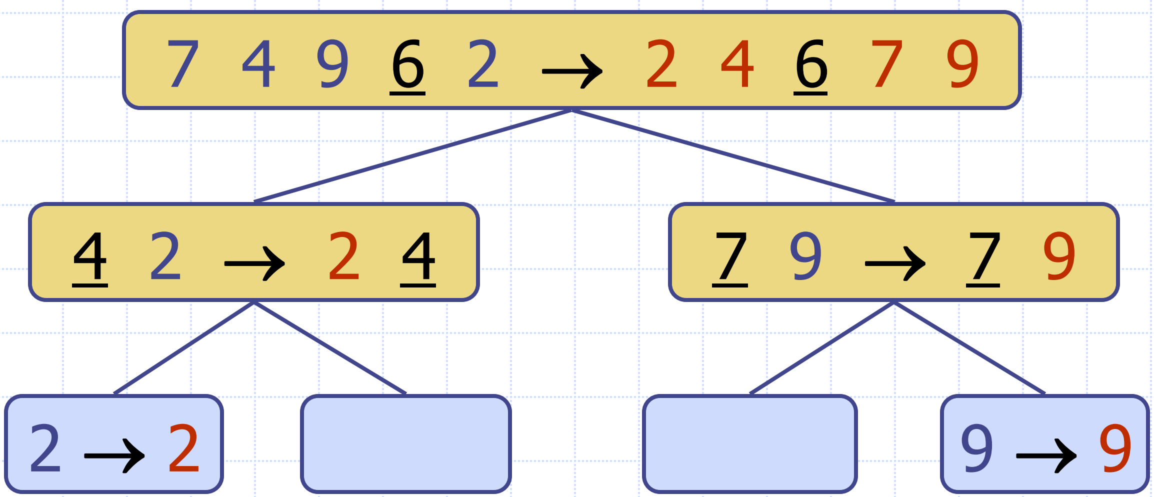
```
Algorithm quickSort(S)
    Input sequence S with n
          elements
    Output sequence S sorted
        according to C
    if S.size() > 1
        p ← getPivot(S)
        (L, E, G) ← partition(S, p)
        quickSort(L)
        quickSort(G)
        S ← merge(L, E, G)
```

How to define the algorithm merge(L, E, G)?

Write down the pseudocode by yourself.

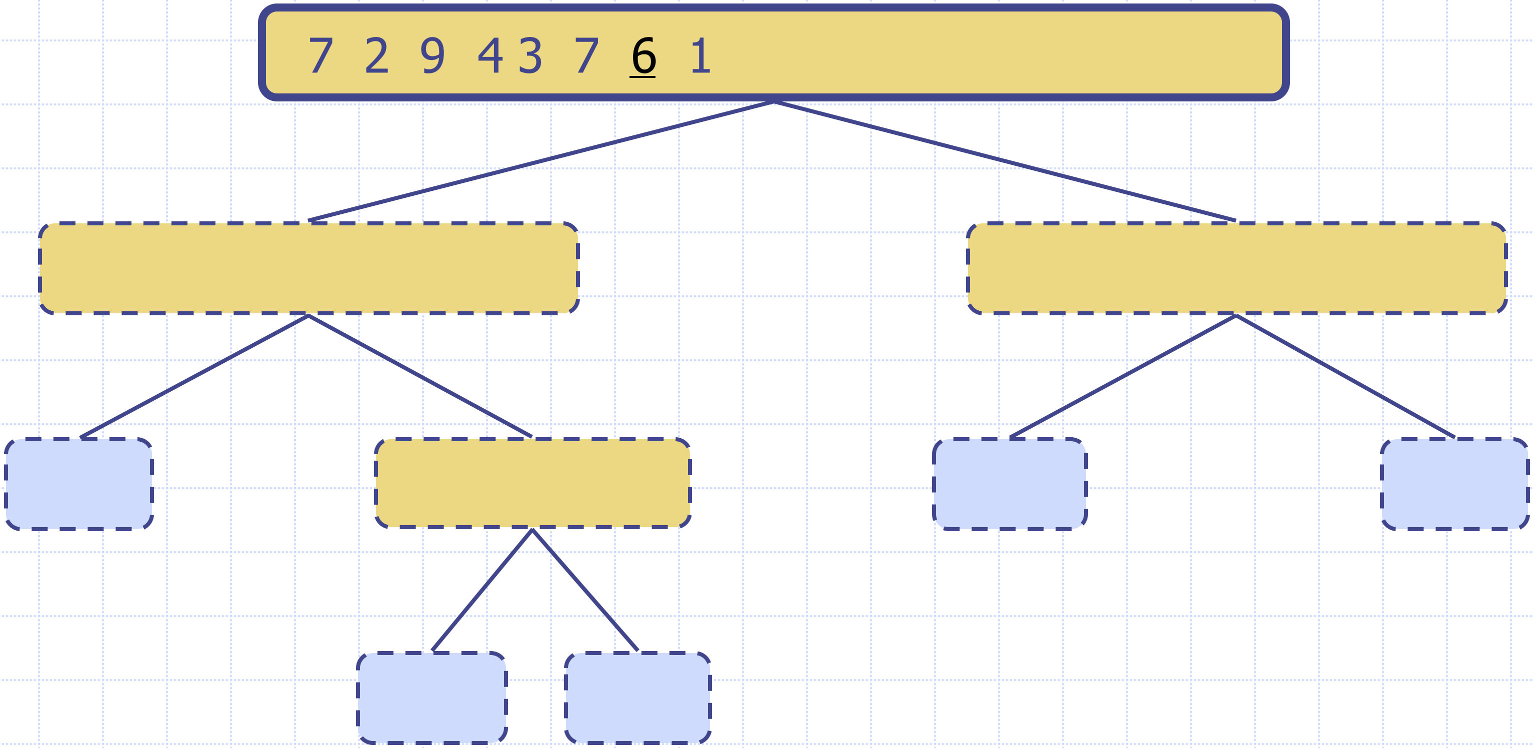
# Quick-Sort Tree

- An execution of quick-sort is depicted by a binary tree
  - Each node represents a recursive call of quick-sort and stores
    - Unsorted sequence before the execution and its pivot
    - Sorted sequence at the end of the execution
  - The root is the initial call
  - The leaves are calls on subsequences of size 0 or 1



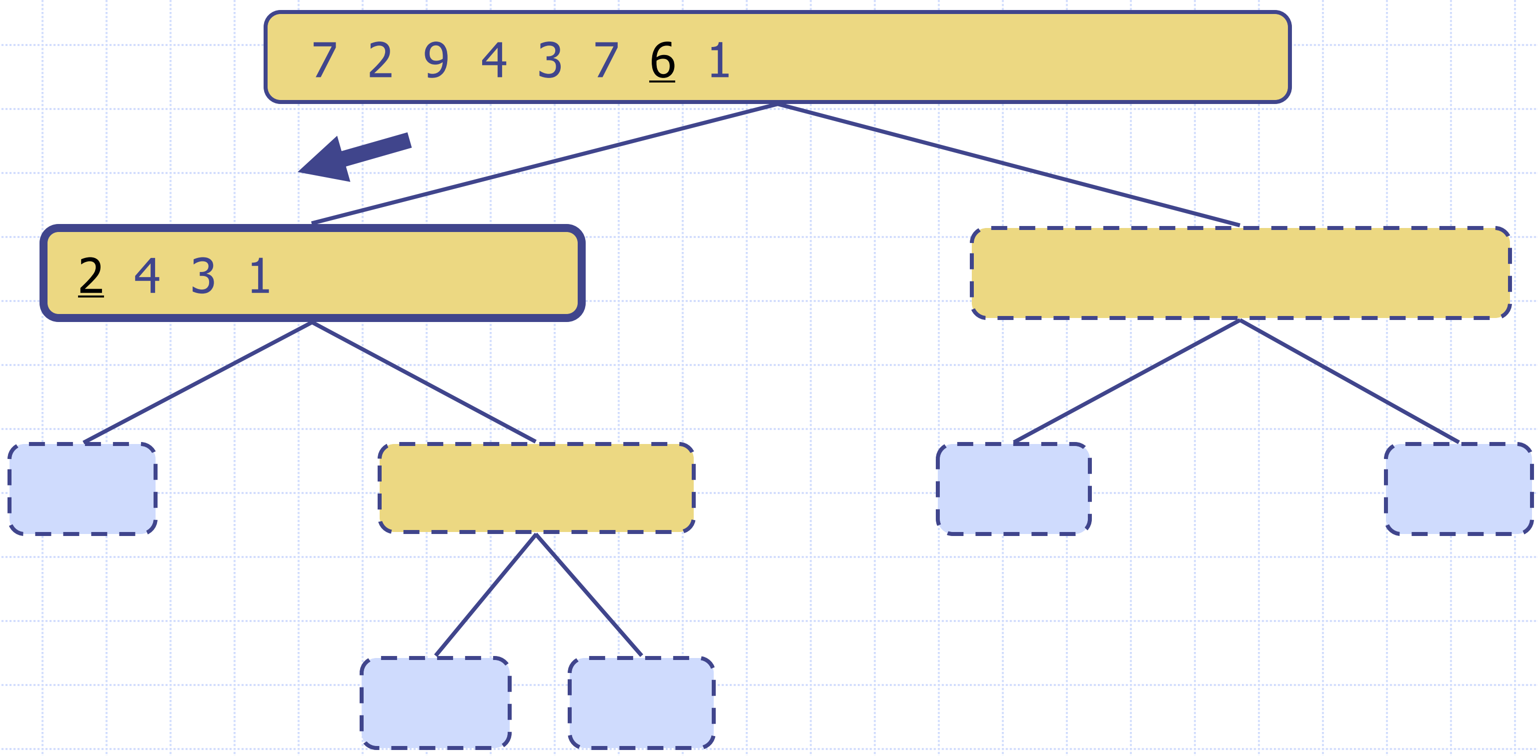
# Execution Example

- Pivot selection



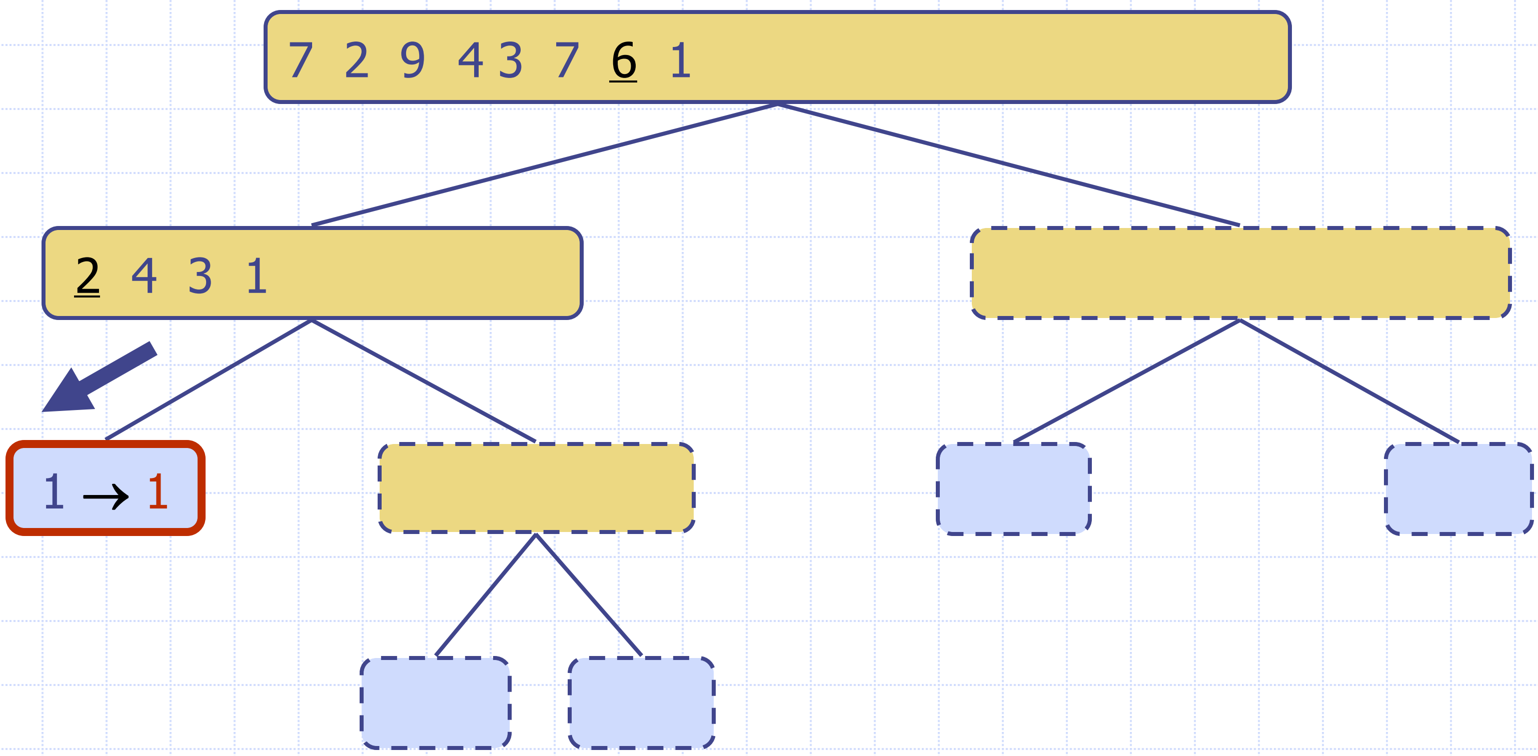
# Execution Example (cont.)

- Partition, recursive call, pivot selection



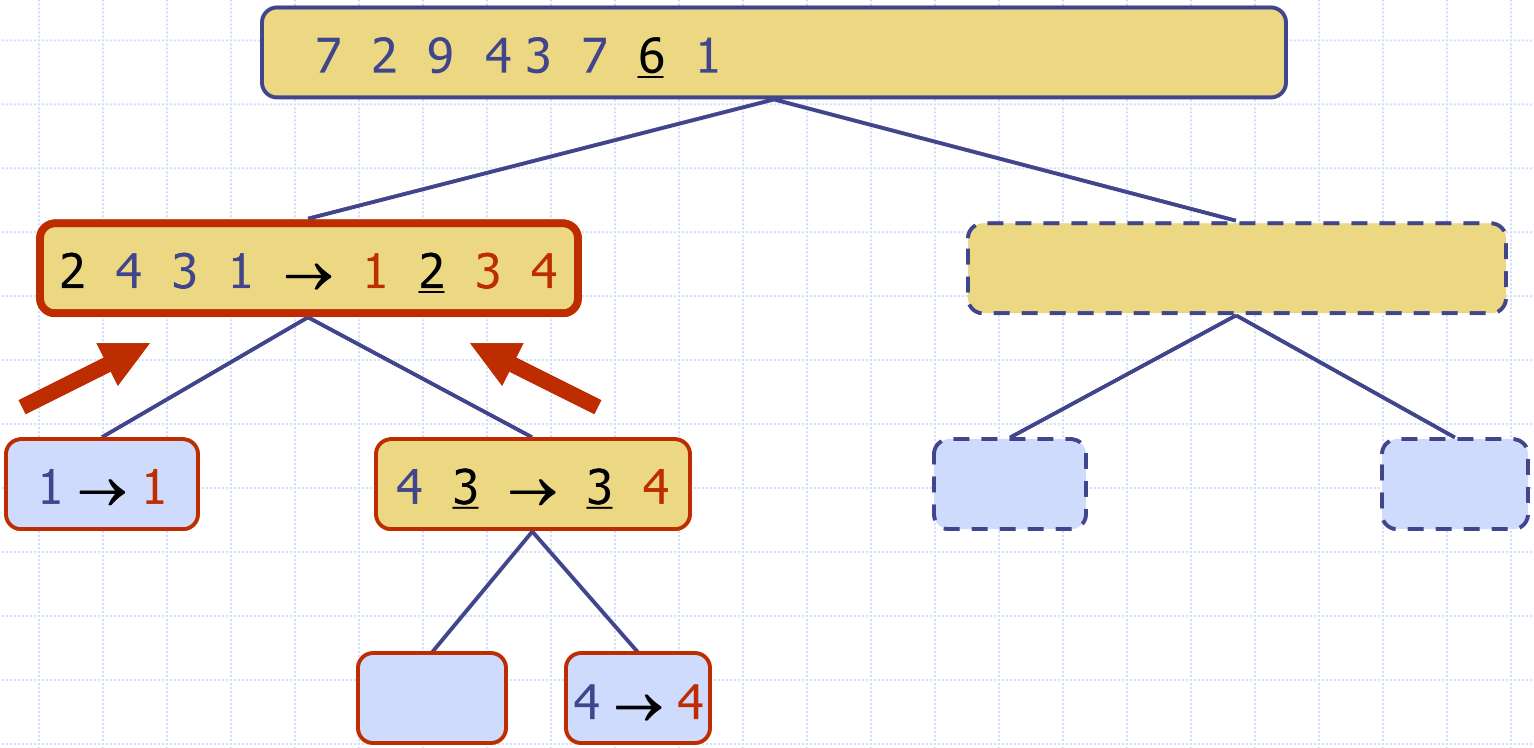
# Execution Example (cont.)

- Partition, recursive call, base case



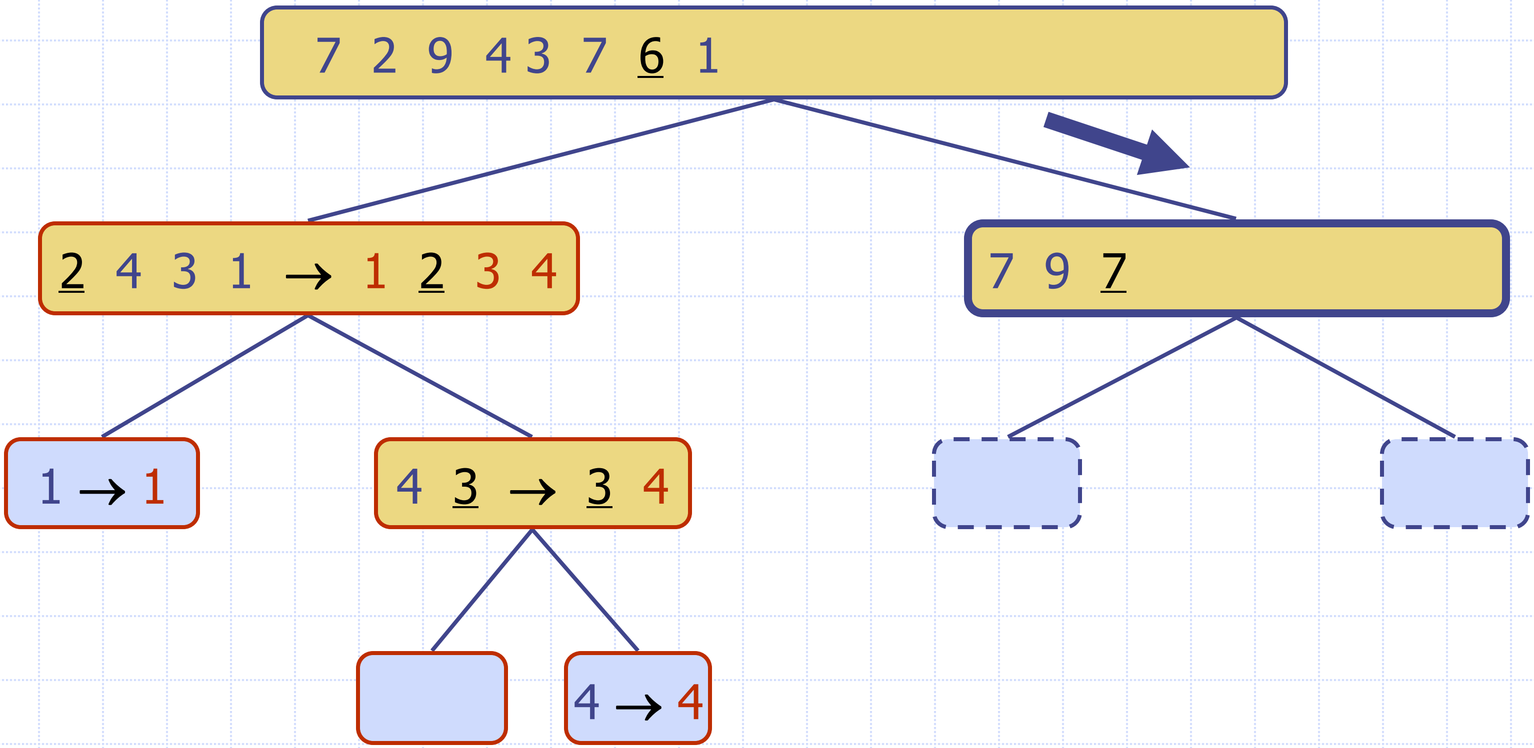
# Execution Example (cont.)

- Recursive call, …, base case, join



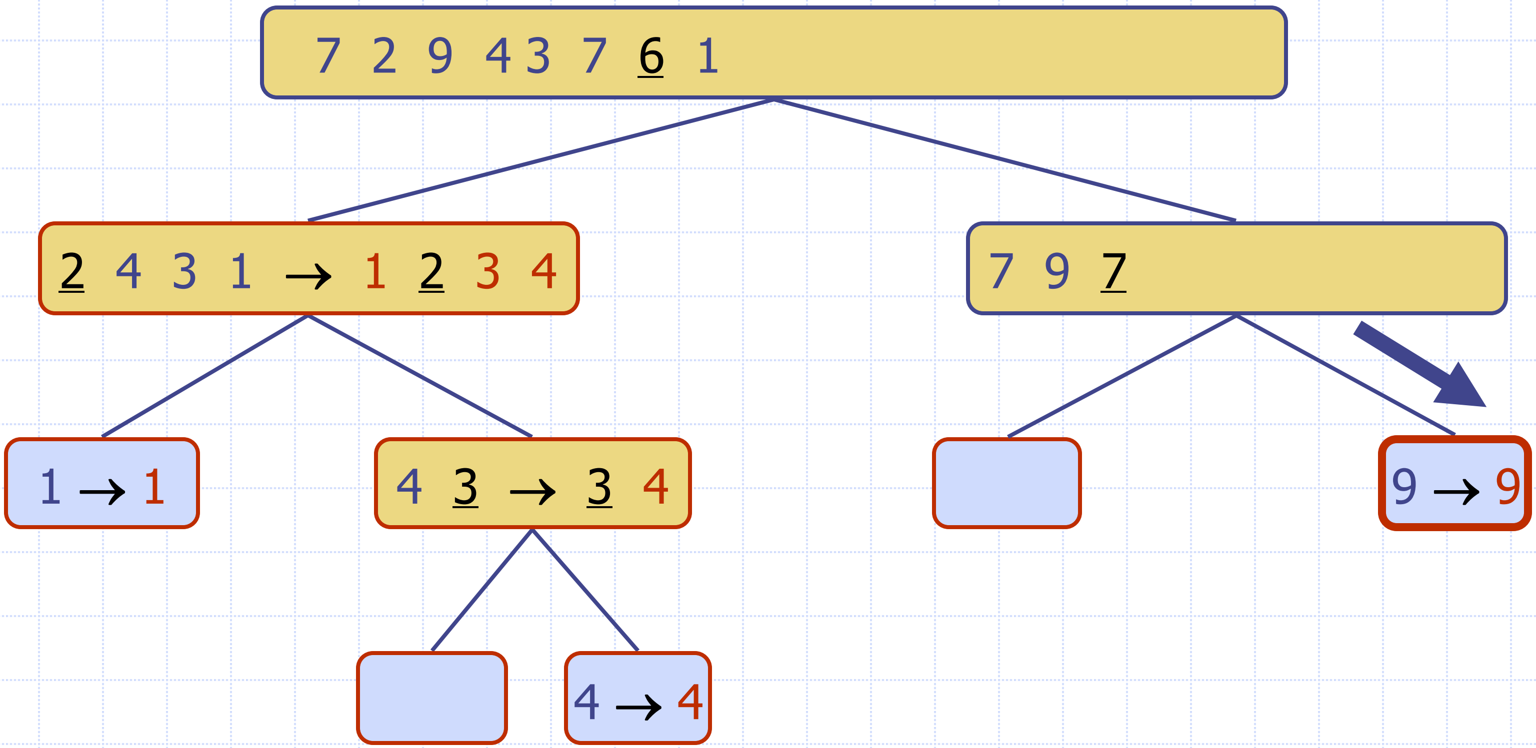
# Execution Example (cont.)

- Recursive call, pivot selection



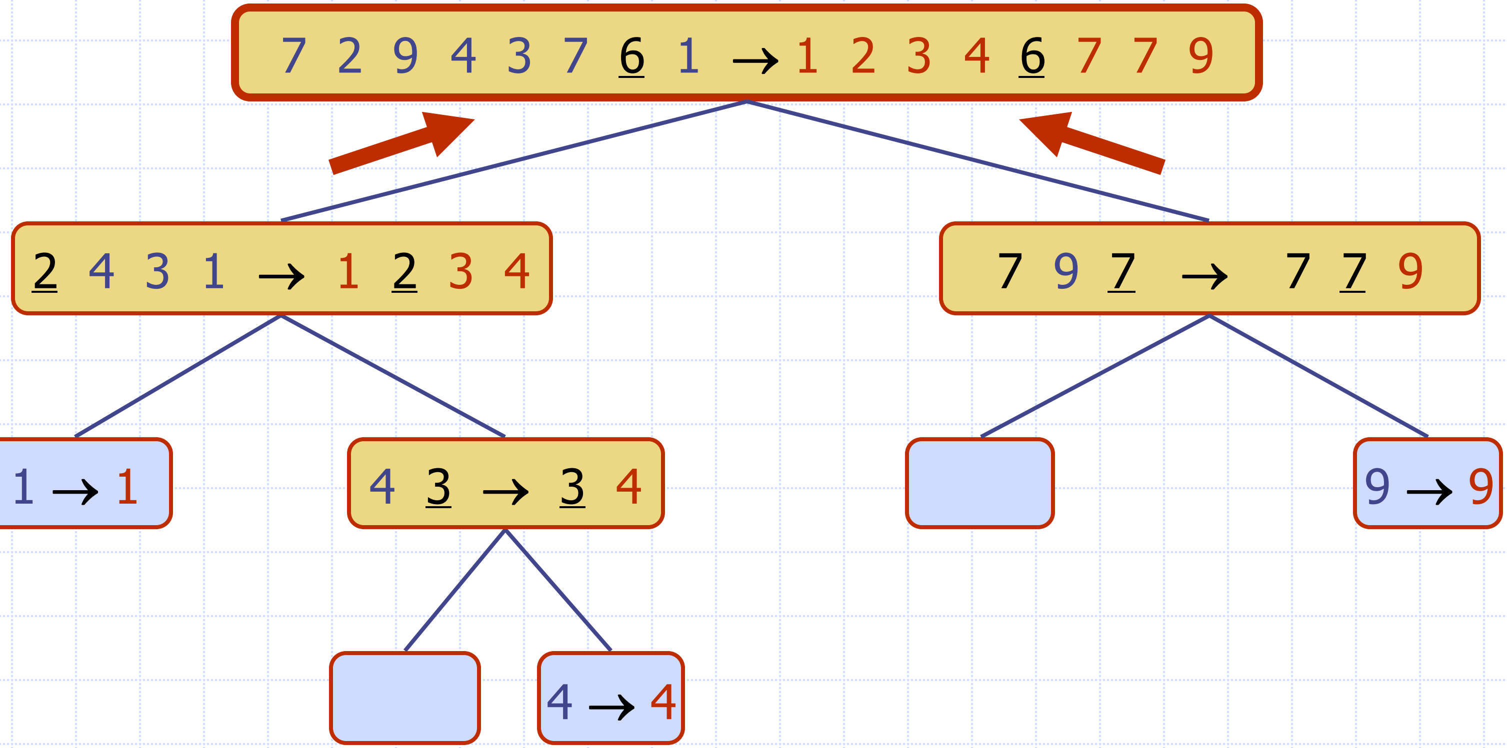
# Execution Example (cont.)

- Partition, …, recursive call, base case



# Execution Example (cont.)

- Join, join



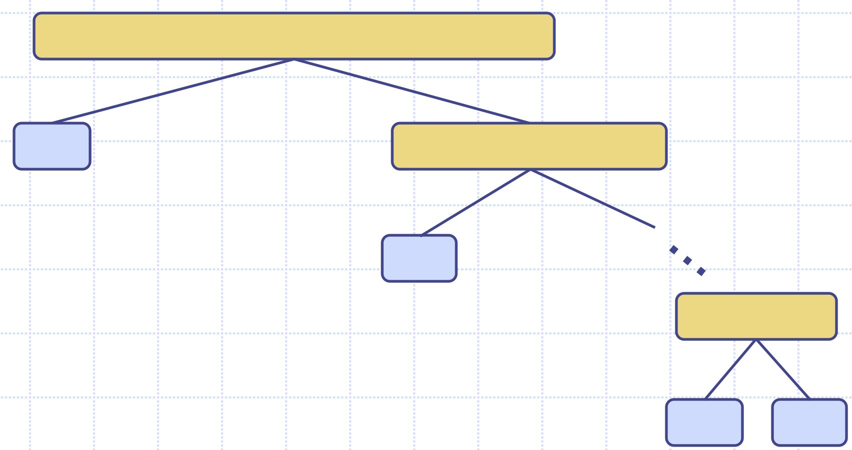
# Worst-case Running Time

- The worst case for quick-sort occurs when the pivot is the unique minimum or maximum element
- One of $L$ and $G$ has size $n - 1$ and the other has size $0$
- The running time is proportional to the sum

$$n + (n - 1) + \ldots + 2 + 1$$

- Thus, the worst-case running time of quick-sort is $O(n^2)$

| depth | time |
|---|---|
| $0$ | $n$ |
| $1$ | $n - 1$ |
| … | … |
| $n - 1$ | $1$ |



# Best-case Running Time

- The best case for quick-sort occurs when the pivot is the median element
- The *L* and *G* parts are equal – the sequence is split in halves, like in merge sort
- Thus, the best-case running time of quick-sort is $O(n \log n)$

**最优情况**：快速排序的最优情况发生在主元（pivot）是序列的**中位数**时。在这种情况下，主元能将序列平衡地分成两个子序列，类似于**归并排序**。
**L 和 G 的大小相等**：在最优情况下，序列 S 被分割成两个大小相等的子序列 L 和 G，其中：L 是小于主元的元素；G 是大于主元的元素。
**最优情况的时间复杂度**：当每次分割都能平衡地分割序列时，快速排序的时间复杂度是 O(n \log n)。

# Average-case Running Time

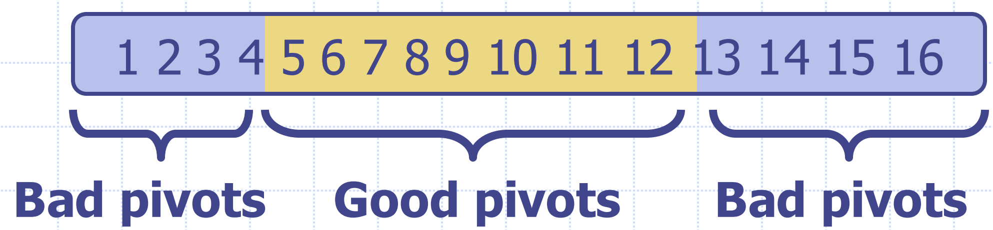
- The average case for quick-sort: half of the times, the pivot is roughly in the middle
- Thus, the average-case running time of quick-sort is $O(n \log n)$ again
- The detailed proof is in the textbook.

# Expected Running Time

- Consider a recursive call of quick-sort on a sequence of size $s$
  - **Good call:** the sizes of $L$ and $G$ are each less than $3s/4$
  - **Bad call:** one of $L$ and $G$ has size greater than $3s/4$

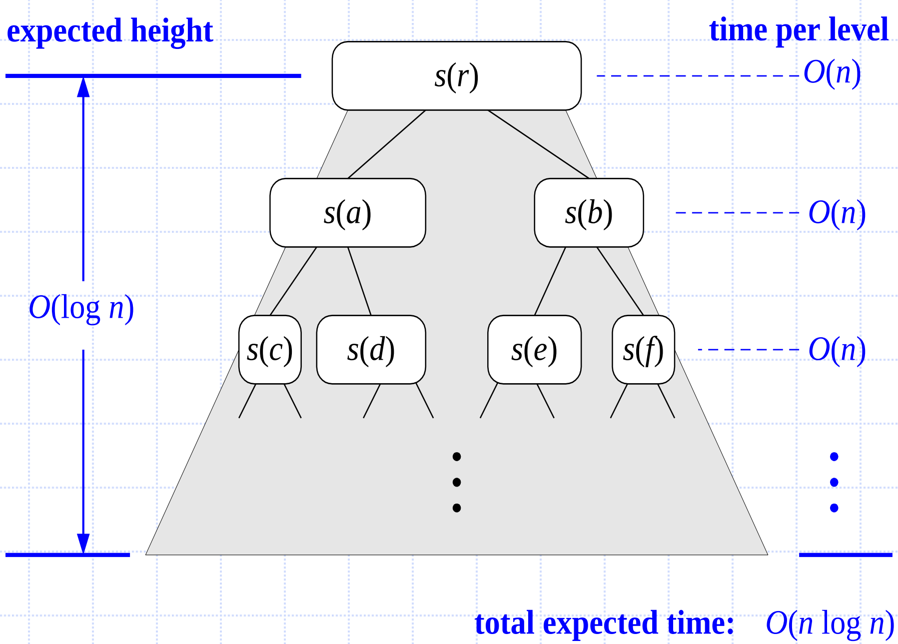

**Good call** **Bad call**

- A call is good with probability $1/2$
  - 1/2 of the possible pivots cause good calls:



# Expected Running Time, Part 2

- Probabilistic Fact: The expected number of coin tosses required in order to get $k$ heads is $2k$
- For a node of depth $i$, we expect
  - $i/2$ ancestors are good calls
  - The size of the input sequence for the current call is at most $(3/4)^{i/2}n$
- Therefore, we have
  - For a node of depth $2\log_{4/3}n$, the expected input size is one
  - The expected height of the quick-sort tree is $O(\log n)$
    (prove $2\log_{4/3}n$ is in $O(\log n)$)
- The amount of work done at the nodes of the same depth is $O(n)$
- Thus, the expected running time of quick-sort is $O(n \log n)$



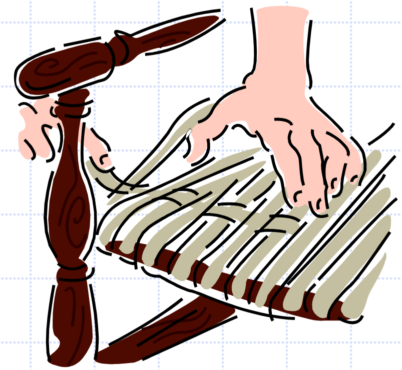
# In-place

- An algorithm is ***in-place*** if it uses only a small amount of memory in addition to that needed for the original input.
- Among all the sorting algorithms we covered so far, which of them are in-place? 典型的就地排序算法包括**快速排序**（Quick Sort）**和堆排序**（Heap Sort）。

**就地算法**：一个算法被称为**就地算法**（In-place），如果它除了用于原始输入所需的内存外，只使用了少量额外的内存。
也就是说，算法仅用极少的额外空间来完成操作，通常是常数级别的空间 O(1)。

小于主元的元素的排名小于 h；
等于主元的元素的排名在 h 和 k 之间；
大于主元的元素的排名大于 k。

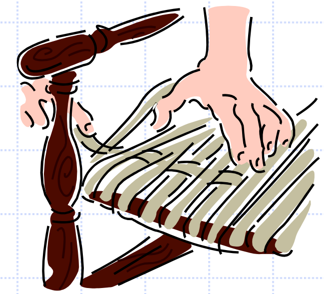
# In-Place Quick-Sort



- Quick-sort can be implemented to run in-place
- In the partition step, we use replace operations to rearrange the elements of the input sequence such that
  - the elements less than the pivot have rank less than $h$
  - the elements equal to the pivot have rank between $h$ and $k$
  - the elements greater than the pivot have rank greater than $k$
- The recursive calls consider
  - elements with rank less than $h$
  - elements with rank greater than $k$

**Algorithm** *inPlaceQuickSort*($S$, $l$, $r$)
  **Input** sequence $S$, ranks $l$ and $r$
  **Output** sequence $S$ with the elements of rank between $l$ and $r$ rearranged in increasing order
  **if** $l \geq r$
    **return**
  $i \leftarrow$ a random integer between $l$ and $r$
  $x \leftarrow S.elemAtRank(i)$
  $(h, k) \leftarrow inPlacePartition(x)$
  $inPlaceQuickSort(S, l, h - 1)$
  $inPlaceQuickSort(S, k + 1, r)$

# In-Place Partitioning



- Perform the partition using two indices to split S into L and E U G (a similar method can split E U G into E and G).

j k

3 2 5 1 0 7 3 5 9 2 7 9 8 9 7 **6** 9 (pivot = 6)

- Repeat until j and k cross:
  - Scan j to the right until finding an element $\geq$ x.
  - Scan k to the left until finding an element < x.
  - Swap elements at indices j and k

重复过程直到 j 和 k 相遇：
**扫描 j 向右移动**，直到找到一个大于主元 x 的元素；
**扫描 k 向左移动**，直到找到一个小于主元 x 的元素；
**交换 j 和 k 位置的元素**，直到 j 和 k 相遇。

j k

3 2 5 1 0 $\boxed{7}$ 3 5 9 $\boxed{2}$ 7 9 8 9 7 **6** 9

# Summary of Sorting Algorithms

| Algorithm | Time | Notes |
|---|---|---|
| selection-sort | $O(n^2)$ | ▪ in-place<br>▪ slow (good for small inputs) |
| insertion-sort | $O(n^2)$ | ▪ in-place<br>▪ slow (good for small inputs) |
| quick-sort | $O(n \log n)$ expected | ▪ in-place, randomized<br>▪ fastest (good for large inputs) |
| heap-sort | $O(n \log n)$ | ▪ in-place<br>▪ fast (good for large inputs) |
| merge-sort | $O(n \log n)$ | ▪ sequential data access<br>▪ fast (good for huge inputs) |

# Reading

**M. T. Goodrich, R. Tamassia and M. H. Goldwasser, *Data Structures and Algorithms in Java*, 6th Edition, 2014.**

- **Chapter 13. Sorting and Selection**