Chapter 4: Properties of Regular Languages

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Learning Outcomes

Learning outcomes

At the conclusion of this chapter, the students are expected to be able to:

- State the closure properties applicable to regular languages.
- Prove that regular languages are closed under union, concatenation, star-closure, complementation, and intersection.
- Prove that regular languages are closed under reversal.
- Describe a membership algorithm for regular languages.
- Describe an algorithm to determine if a regular language is empty, finite, or infinite.
- Describe an algorithm to determine if two regular languages are equal.
- Apply the pumping lemma to show that a language is not regular.

Introduction

Review

- So far:
    - We have defined regular languages as those accepted by finite automata.
    - We have studied some ways in which they can be represented:
        - Finite Automata.
        - Regular Expressions.
        - Regular Grammars.
    - We have seen a few examples of their usefulness:
        - Pattern matching and search.
        - Specification of programming language constructs.
- Now, we will investigate some general properties of regular languages.

Properties of Regular Languages

- The followings are some general properties we may want to investigate:
  1. What happens when we perform operations (such as union, complementation, etc.) on regular languages?
     - For example, is it true that for every regular language L the complement $\overline{L}$ is also regular?
  2. Are there any algorithms that we can use to tell whether a given regular language is empty, non-empty, finite, or infinite?
     - For example, if a regular language L is given by a DFA M which accepts it, can we answer any of these questions just by inspecting M?
  3. Is every formal language regular? In other words, for any given $L \subseteq \Sigma^*$, is it possible to design a finite automaton M which accepts L?
     - You know the answer to this question already, but we have not provided a mathematical proof of the fact that $a^n b^n$ is not regular. In this chapter, we will.
     - More importantly, are there any methodical ways of telling whether a given language is regular or not?

Closure Properties

Closure Properties

- **Theorem 4.1:** If L_1 and L_2 are regular languages, then so are $L_1 \cup L_2$, $L_1 \cap L_2$, L_1L_2, $\overline{L_1}$, L_1^*. We say that the family of regular languages is closed under union, intersection, concatenation, complementation and star-closure.

- How to prove it?

Proof of the Closure Properties

- Suppose L_1 and L_2 are regular languages, then there existis regular expressions r_1 and r_2 such that $L_1 = L(r_1)$ and $L_2 = L(r_2)$.

- **It is not difficult to prove the closure properties for union, concatenation and start closure:**

- Then it is obvious that $L_1 \cup L_2 = L(r_1) \cup L(r_2) = L(r_1 + r_2)$, as $r_1 + r_2$ is a regular expression, $L_1 \cup L_2$ is also regular.
- Also, $L_1L_2 = L(r_1)L(r_2) = L(r_1r_2)$, as r_1r_2 is a regular expression, L_1L_2 is therefore a regular language.
- Similarly, $L_1^* = L(r_1)^* = L(r_1^*)$, obviously L_1^* is a regular language.
- **Question:** How to prove complementation and intersection?

Proof of the Closure Properties

- Proof for the complementation $\overline{L}$:
    - **Hint:** remember how we construct a DFA that accpet the strings which does not have substring 001.

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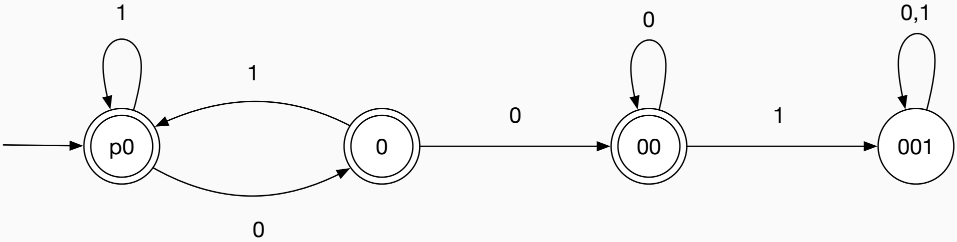


Proof of the Closure Properties

- Proof:
  - Assume there exists a DFA $M = (Q, \Sigma, \delta, q_0, F)$ that accepts L_1.
  - We can construct another DFA $\overline{M}$ that accepts the complement of L_1 as follows:
    - $\overline{M}$ has the same states, alphabet, transition function, and start state as M.
    - The final states in M become non-final states in $\overline{M}$, while non-final states in M becomes final states in $\overline{M}$.
    - That is $\overline{M} = (Q, \Sigma, \delta, q_0, Q - F)$
  - Since $\overline{M}$ accepts precisely the strings that M rejects, and $\overline{M}$ rejects precisely the strings that M accepts, then $\overline{M}$ accepts the complement of L_1.
  - Thus, $\overline{L_1}$ is regular.

Closure under Intersection

- To prove that the intersection of two regular languages L_1 and L_2 is also regular, two basic approaches exist:

- **An elegant logical approach:** Use DeMorgan's law to show that the intersection of L_1 and L_2 can be obtained by applying union and complementation:
$$L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$$
- Since the union and complementation operations have been shown to produce regular languages, the intersection of L_1 and L_2 must also produce a regular language:
L_1 and L_2 are regular $\Rightarrow \overline{L_1}$ and $\overline{L_2}$ are regular $\Rightarrow \overline{L_1} \cup \overline{L_2}$ is regular $\Rightarrow \overline{\overline{L_1} \cup \overline{L_2}}$ is regular.

Closure under Intersection

- **A constructive approach:** Given a DFA $M_1 = (Q, \Sigma, \delta_1, q_0, F_1)$ that accepts L_1 and a DFA $M_2 = (P, \Sigma, \delta_2, p_0, F_2)$ that accepts L_2, construct a new DFA M' with states and transition function resulting from a combination of the states and transition functions from M_1 and M_2:

$$M' = (Q', \Sigma, \delta', (q_0, p_0), F')$$

- The state set Q' consists of pairs (q_i, p_j), where $q_i \in Q$ and $p_j \in P$

$$Q' = Q \times P$$

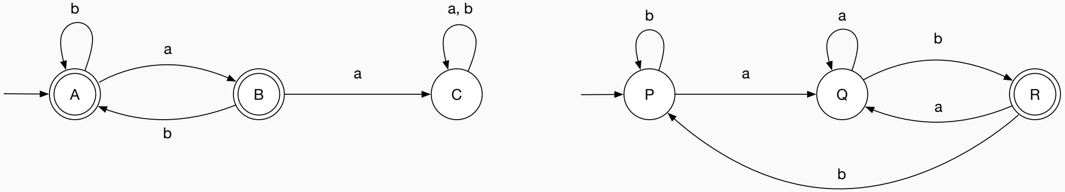
- The transition function δ' is such that M' is in state (q_i, p_j) whenever M_1 is in state q_i and M_2 is in state p_j.

$$\delta'((q_i, p_j), a) = (\delta_1(q_i, a), \delta_2(p_j, a))$$

- F' is defined as the set of all (q_i, p_j) such that $q_i \in F_1$ **and** $p_j \in F_2$.

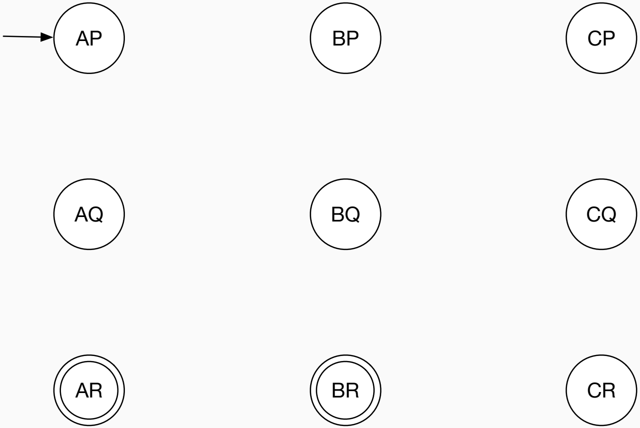
Example: Intersection of two languages

- Given the following two DFAs L_1 AND L_2 and their corresponding transition graphs, construct a DFA for the language $L_1 \cap L_2$.
  - $L_1 = \{w \in \{a, b\}^* \mid aa$ is not a substring of w $\}$
  - $L_2 = \{w \in \{a, b\}^* \mid w$ ends with $ab\}$



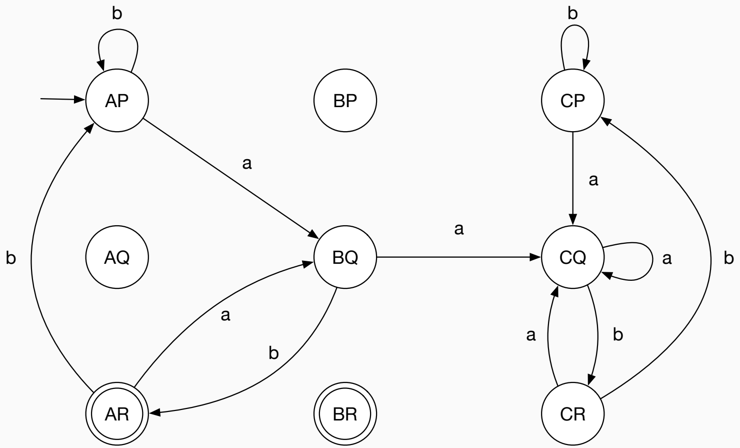
Example: Intersection of two languages

- Generate all combinations and label the initial state and final states.



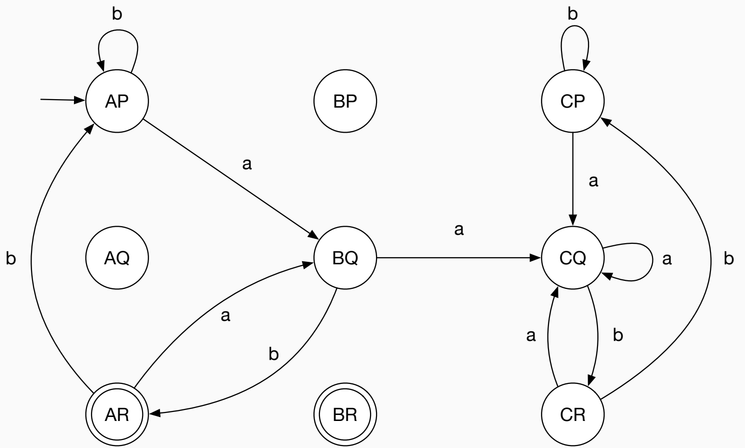
Example: Intersection of two languages

- Add the transitions between states.



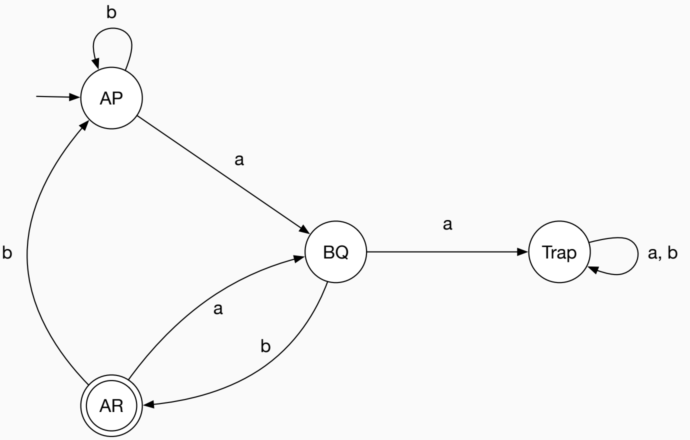
Example: Intersection of two languages

- How to improve it?



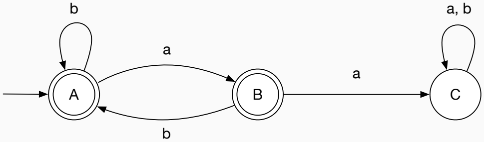
Example: Intersection of two languages

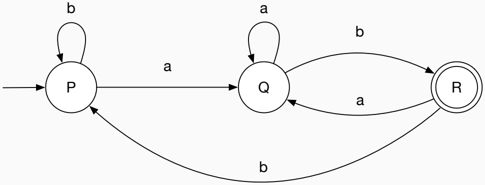
- Remove AQ, BP and BR.
- merge CP, CQ and CR to a single trap state.



Exercise

- Given the same DFAs L_1 and L_2
  - Construct a DFA which accepts $L_1 \cup L_2$.
  - Construct a DFA which accepts $L_1 - L_2$.





Closure under reversal

- **Theorem 4.2** If L is a regular language, so is L^R.
- To prove closure under reversal, we can assume the existence of a nondeterministic finite automaton M with a single final state that accepts L.

- Given the transition graph for M, to construct an NFA M^R that accepts L^R:
- How?

Closure under reversal

- **Theorem 4.2** If L is a regular language, so is L^R.
- To prove closure under reversal, we can assume the existence of a nondeterministic finite automaton M with a single final state that accepts L.

- Given the transition graph for M, to construct an NFA M^R that accepts L^R:
  - The start state in M becomes the final state in M^R.
  - The final state in M becomes the start state in M^R.
  - The direction of all transition edges in M is reversed in M^R.

Exercise

- Given the fact that $L = \{ a^n b^m \mid n, m \geq 0\}$ is a regular language. Prove $\overline{L}$ is also a regular language.

Elementary Questions about Regular Languages

Elementary Questions

- The following are fundamental questions that should be asked about any class of formal languages:
    - Given a regular language L and an arbitrary string w, is there an algorithm to determine whether or not $w \in L$?
    - Given a regular language L, is there an algorithm to determine if L is empty, finite, or infinite?
    - Given two regular languages L_1 and L_2, is there an algorithm to determine whether or not $L_1 = L_2$?

Membership Algorithm

- We say that a regular language is given in a **standard representation** if and only if it is described by one of the following:
  - a finite automaton
  - a regular expression
  - a regular grammar

- **Theorem 4.5** Given a standard representation of any regular language L on L and $w \in \Sigma^*$, there exists an algorithm to determine whether or not $w \in L$.

- How to prove it?

Membership Algorithm

- We say that a regular language is given in a **standard representation** if and only if it is described by one of the following:
  - a finite automaton
  - a regular expression
  - a regular grammar

- **Theorem 4.5** Given a standard representation of any regular language L on L and $w \in \Sigma^*$, there exists an algorithm to determin whether or not $w \in L$.
- **Proof:** To determine if an arbitrary string w is in a regular language L, we assume we are given a standard representation of L, which we then convert to a DFA that accepts L.
- Simulate the operation of the DFA while processing w as the input string.
- If the machine halts in a final state after processing w, then $w \in L$, otherwise, $w \notin L$.

Empty, Finite or Infinite

- **Theorem 4.6** There exists an algorithm for determining whether a regular language, given in the standard representation, is empty, finite or infinite.
- How to prove it?

Empty, Finite or Infinite

- **Theorem 4.6** There exists an algorithm for determining whether a regular language, given in the standard representation, is empty, finite or infinite.

- The problem can be reduced to very simple and intuitive equivalent problems on graphs.

- Given the transition graph of a DFA that accepts L:
    - If there is a simple path from the start state to any final state, L is not empty (since it contains, at least, the corresponding string).
    - If a path from the start state to a final state includes a vertex which is the base of some cycle, L is infinite (otherwise, L is finite).

Equivalence

- The question of the equality of two languages is not just of theoretical interest.
- It is also an important practical issue.

- For example, often several definitions of a construct in a programming language exist, and we need to know whether, in spite of their different appearances, they specify the same language.
- Even for regular languages the argument is not obvious.
- How to do it?

Equivalence

- For finite languages, equality can be determined by performing a comparison of the individual strings.
- However this cannot be done over infinite languages in finite time.
- Nor is it easy to see the answer by looking at the regular expressions, grammars, or DFAs.
- So, is it at all possible to test the equality of two regular languages in an algorithmic way?

Equivalence

- **Theorem 4.7** Given the standard representation of two regular languages L_1 and L_2, there exists an algorithm to determine whether or not $L_1 = L_2$.
- There is an elegant solution uses the already established closure properties.
    - Define the language $L = (L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2)$
    - By closure properties, L is regular.
    - So, we construct a DFA M to accept it, and by Theorem 4.6, we can determine whether L is empty or not.
    - L_1 and L_2 are equal if and only if L is empty.

Identify Non-Regular Languages

Identifying Non-Regular Languages

- **Pigeonhole Principle:** If we put n objects into m boxes (pigeonholes), and if $n > m$, then at least one box must have more than one item in it.
- This simple principle is the basis of most of the methods for proving non-regularity of languages
- **Basic observation:** Although regular languages can be infinite, their associated automata have finite memory.

Prove A^nB^n is not regular: by Pigeonhole Principle

- We prove A^nB^n is not regular by directly applying the pigeonhole principle.
- Proof by contradiction: Assume that $A^nB^n = \{a^nb^n | n \geq 0\}$ is regular.
- Then, there must be a DFA $M = (Q, \{a, b\}, \delta, q_0, F)$ which accepts A^nB^n.
- Now, assume that Q has m elements, i.e., the DFA has m states, and consider the set:

$$X = \{a, a^2, a^3, \ldots, a^m, a^{m+1}\}$$

- Then, consider the set of states:

$$Y = \{\delta^*(q_0, a), \delta^*(q_0, a^2), \ldots, \delta^*(q_0, a^m), \delta^*(q_0, a^{m+1})\}$$

- By the pigeonhole principle, for some $i \neq j$ we must have:

$$\delta^*(q_0, a^i) = \delta^*(q_0, a^j)$$

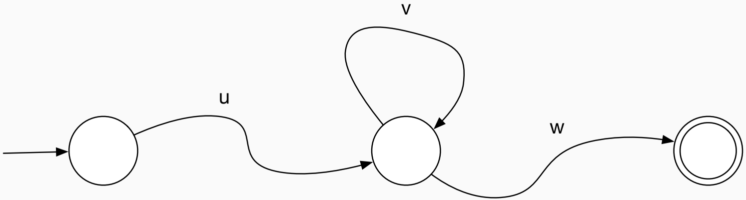
Prove A^nB^n is not regular: by Pigeonhole Principle

- So, we can assume for some $q \in Q$, we have $\delta^*(q_0, a^i) = \delta^*(q_0, a^j) = q$
- As $a^ib^i \in A^nB^n$, we must have $\delta^*(q, b^i) \in F$.
- However, we also had $\delta^*(q_0, a^j) = q$, which, together with $\delta^*(q, b^i) \in F$, implies that $a^jb^i \in A^nB^n$, which is a contradiction because $i \neq j$.
- Therefore, there cannot be any DFA accepting A^nB^n, and the language is not regular.

The Pumping Lemma

- Suppose that $M = (Q, \Sigma, \delta, q_0, F)$ is a DFA with n states that accepts a language L:
    - If it accepts a string x such that $|x| \geq n$, then, by the time n symbols have been read, M must have entered some state more than once.
        - Again, this is a consequence of the pigeonhole principle.
    - In other words, there must be two different prefixes u and uv such that:

$$\delta^*(q_0, u) = \delta^*(q_0, uv)$$



The Pumping Lemma

- This implies that there are many more strings in L, because we can traverse the loop v any number of times (including leaving it out altogether).
- In other words, all of the strings uv^iw for $i \geq 0$ are in L.
- This fact is known as the **Pumping Lemma for Regular Languages**.

The Pumping Lemma

- **Theorem:** Suppose that L is a language over Σ. If L is accepted by the DFA $M = (Q, \Sigma, \delta, q_0, F)$, then there is an integer n so that for every x in satisfying $|x| \geq n$, there are three strings u, v, and w such that $x = uvw$ and:
  - $|uv| \leq n$
  - $|v| > 0$, i.e., $v \neq \lambda$
  - For every $i \geq 0$, the string uv^iw belongs to L.
- The way we found n was to take the number of states in an FA accepting L.
- In typical applications, we do not need to know this, only that there exists such an n.

The Pumping Lemma

- The statement of the Pumping lemma is as follows:
  - If L is regular $\Rightarrow$ the pumping property holds.
- In practice, we use the contrapositive:
  - If the pumping property does not hold $\Rightarrow$ the language L is not regular.
- Thus, the most common application of the pumping lemma is to show that a language is not regular.
- The proof is by contradiction. We suppose that the language can be accepted by an FA, and we let n be the integer in the pumping lemma.
- Then we choose a string x with $|x| \geq n$ to which we can apply the lemma to get a contradiction.

Prove A^nB^n is not regular: by Pumping Lemma

- Let us prove, using the Pumping lemma, that $L = A^nB^n$ cannot be accepted by an FA.
- Suppose, for the sake of contradiction, that A^nB^n is accepted by an FA; let n be as in the pumping lemma;
- Choose $x = a^nb^n$, then $x \in A^nB^n$ and $|x| \geq n$;
- Therefore, by the Pumping lemma, there are strings u, v and w such that $x = uvw$ and the 3 conditions hold;
- Because $|uv| \leq n$ and x starts with n $a's$, all the symbols in u and v are $a's$; Therefore, $v = a^k$, for some $k > 0$;
- The pumping property implies that $uvvw \in A^nB^n$, so $a^{n+k}b^n \in A^nB^n$.
- This is a contradiction, and we conclude that A^nB^n cannot be accepted by an FA. So, A^nB^n is not regular.

Exercise

- Prove that $WW^R = \{ww^R \mid w \in \{a, b\}^*\}$ is not regular.

The Pumping Lemma

- There are other languages that are not accepted by any FA, among them:
  - Balanced, the set of balanced strings of parentheses;
  - Expr, the language of simple algebraic expressions;
  - The set L of legal C programs.
- In all three examples, because of the nature of these languages, a proof using the pumping lemma might look a lot like the proof for $A^n B^n$ in our first example.

- Note that the Pumping lemma only provides a necessary condition, but not a sufficient one.
  - If L is regular $\Rightarrow$ the pumping property holds
- In other words, it does not say anything about the pumping property when L is not regular.
- In fact, there are languages that do satisfy the pumping lemma, but are not regular, e.g.:
  $L = \{a^i b^j c^j \mid i \geq 1 \text{ and } j \geq 0\} \cup \{b^j c^k \mid j \geq 0 \text{ and } k \geq 0\}$.