AE1MCS: Mathematics for Computer Scientists

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Aim and Learning Objectives

- To gain a good understanding of the definitions of proposition, propositional variable and logical operators;
- To gain a good understanding of other definitions: tautology, contradiction, contingency, logical equivalence, converse, contrapositive and inverse.
- To be able to draw truth tables and use them as a tool to solve logical problems;
- To be able to apply important logical equivalences to solve logical problems.

Proposition

Definition

A *proposition* is a statement that is either true or false.

The area of logic that deals with propositions is called the *propositional logic* or *propositional calculus*.

Is it a proposition?

1. Beijing is the capital of China. ✓ P
2. $1 + 1 = 2.$ ✓
3. $2 + 2 = 3.$ ✓
4. What time is it? ✗
5. Read this sentence carefully. ✗
6. $x + 1 = 2.$ ✗
7. $x + y = z.$ ✗
8. If $x > 0$, then $x > 1$. ✗

Is a claimed proposition true or false?

Unfortunately, it is not always easy to decide if a claimed proposition is true or false.

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3. $313(x^3 + y^3) = z^3$ has no positive integer solutions.

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4. Every map can be colored with 4 colors so that adjacent regions have different colors [Four Color Theorem].

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4. Every map can be colored with 4 colors so that adjacent regions have different colors [Four Color Theorem].
5. Every even integer greater than 2 is the sum of two primes [Goldbach's conjecture, 1742].

Propositional Variable

- a variable that represents a proposition
- denoted using a letter $p, q, r, s, \ldots$
- truth value: T (true); F (false)

Logical Operators

- **Compound Proposition:** formed from existing propositions using logical operators
- Logical Operators
  - Negation not
  - Conjunction and
  - Disjunction or
  - Implication if, then
  - ...

Negation

Definition (Negation)

Let p be a proposition. The *negation* of p, denoted by $\neg p$, is the statement

'It is not the case that p'.

The proposition $\neg p$ is read 'not p'. The truth value of $\neg p$ is the opposite of the truth value of p.

p	$\neg p$
T	F
F	T

truth table

Conjunction

Definition (Conjunction)

Let p and q be propositions. The *conjunction* of p and q, denoted by $p \wedge q$, is the proposition

$$\text{‘}p \text{ and } q\text{’.}$$

The proposition $p \wedge q$ is true when both p and q are true and is false otherwise.

Truth table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

$p \wedge q \wedge r$

2 2 2

Disjunction

Definition (Disjunction)

Let p and q be propositions. The *disjunction* of p and q, denoted by $p \vee q$, is the proposition

$$\text{`}p \text{ or } q\text{'}.$$

The proposition $p \vee q$ is false when both p and q are false and is true otherwise.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive Or

Definition (Exclusive Or)

Let p and q be propositions. The *exclusive or* of p and q, denoted by $p \oplus q$, is the proposition that is true when *exactly one* of p and q is true and is false otherwise.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication

p implies q
p to q

Definition (Implication)

Let p and q be propositions. The *implication* $p \to q$ is the proposition

'if p, then q'.

The proposition $p \to q$ is false when p is true and q is false, and true otherwise. p is called the hypothesis or premise and q is called the conclusion or consequence.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

The proposition $p \to q$ is true, if p is false or q is true.

Examples

- If Goldbach's Conjecture is true, P
  then $x^2 \geq 0$ for every real number x. q true. > True.
- If pigs fly, then your account will not get hacked.

P q

F

T

Bi-Implication

Definition (Bi-Implication)

$(p \to q) \wedge (q \to p)$

Let p and q be propositions. The *bi-implication* $p \leftrightarrow q$ is the proposition

T T

'p if and only if q'.

The bi-implication $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise.

The words 'if and only if' are sometimes abbreviated 'iff'.

$p \quad q \quad p \to q \quad q \to p \quad p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

$p \leftrightarrow q$ is true when both $p \to q$ and $q \to p$ are true, and is false otherwise.

More Definitions

- Tautology, Contradiction and Contingency
- Logical Equivalence
- Converse, Contrapositive and Inverse

Tautology, Contradiction and Contingency

Definition (Tautology, Contradiction and Contingency)

A compound proposition that is *always true*, no matter what the truth values of the propositions that occur in it, is called a **tautology**. A compound proposition that is *always false* is called a **contradiction**. A compound proposition that is neither a tautology nor a contradiction is called a **contingency**.

Exercise

How to construct a tautology, a contradiction and a contingency using just one propositional variable?

p

Exercise Answer

How to construct a tautology, a contradiction and a contingency using just one propositional variable?

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$	$p \rightarrow \neg p$
T	F	T	F	F
F	T	T	F	T

[There are some other ways not shown in the table above...]

Logical Equivalence

Definition (Equivalence)

The compound propositions p and q are logically equivalent, if they always have the same truth value (i.e. $p \leftrightarrow q$ is a tautology). The notation $p \equiv q$ denotes that p and q are logically equivalent.

$p \equiv q$

$p \Leftrightarrow q$ True

Exercise

- Are $\neg(p \vee q)$ and $\neg p \wedge \neg q$ logically equivalent? Why?
- Are $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent? Why?

[Hint: construct truth tables]

p	q	$\neg(p \vee q)$	$\neg p \wedge \neg q$

Exercise Answer

Are $\neg(p \vee q)$ and $\neg p \wedge \neg q$ logically equivalent? Why?

Answer: Yes. As shown in the truth table below, $\neg(p \vee q)$ and $\neg p \wedge \neg q$ always have the same truth value. Thus, they are logical equivalent.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Exercise Answer

Are $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent? Why?

Answer: Yes. As shown in the truth table below, $p \rightarrow q$ and $\neg p \vee q$ always have the same truth value. Thus, they are logical equivalent.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Converse, Contrapositive and Inverse

- The **converse** of $p \rightarrow q$ is the proposition $q \rightarrow p$.
- The **contrapositive** of $p \rightarrow q$ is the proposition $\neg q \rightarrow \neg p$.
- The **inverse** of $p \rightarrow q$ is the proposition $\neg p \rightarrow \neg q$.

Which pairs of the following propositions are equivalent? Why?

- a conditional statement and its converse ✗ $p \to q$ $q \to p$.
- a conditional statement and its contrapositive ✓
- a conditional statement and its inverse ✗

p q $p \to q$ $q \to p$

Some Important Logical Equivalences

	Equivalence	Name
1	$p \wedge T \equiv p$	Identity laws
2	$p \vee F \equiv p$	
3	$p \vee T \equiv T$	Domination laws
4	$p \wedge F \equiv F$	
5	$p \vee p \equiv p$	Idempotent laws
6	$p \wedge p \equiv p$	
7	$\neg(\neg p) \equiv p$	Double negation law
8	$p \vee q \equiv q \vee p$	Commutative laws
9	$p \wedge q \equiv q \wedge p$	

Some Important Logical Equivalences

	Equivalence	Name
10	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	Associative laws
11	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	
12	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	Distributive laws
13	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	
14	$\neg(p \wedge q) \equiv \neg p \vee \neg q$	De Morgan's laws
15	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	
16	$p \vee (p \wedge q) \equiv p$	Absorption laws
17	$p \wedge (p \vee q) \equiv p$	
18	$p \vee \neg p \equiv T$	Negation laws
19	$p \wedge \neg p \equiv F$	

Logical Equivalences involving Implications

20	$p \to q \equiv \neg p \vee q$
21	$p \to q \equiv \neg q \to \neg p$
22	$p \vee q \equiv \neg p \to q$
23	$p \wedge q \equiv \neg(p \to \neg q)$
24	$\neg(p \to q) \equiv p \wedge \neg q$
25	$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
26	$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
27	$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
28	$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

Logical Equivalences involving Bi-Implications

29	$p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p)$
30	$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
31	$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$
32	$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Using De Morgan's Laws

Use De Morgan's laws to express the negations of the following sentences.

- Tony has a cellphone and he has a laptop computer.
- Heather will go to the concert or Steve will go to the concert.

Constructing New Logical Equivalences

- A proposition in a compound proposition can be replaced by a compound proposition that is logically equivalent to it without changing the truth value of the original compound proposition.
- Prove two propositions are logically equivalent by developing a series of logical equivalences.

Exercise

- Show that $\neg(p \to q)$ and $p \wedge \neg q$ are logically equivalent.
- Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.
- Show that $(p \wedge q) \to (p \vee q)$ is a tautology.

Exercise Answer

Show that $\neg(p \to q)$ and $p \wedge \neg q$ are logically equivalent.

Answer:

$$
\begin{aligned}
& \neg(p \to q) && \text{by law 20} \\
\equiv\ & \neg(\neg p \vee q) && \text{by De Morgan's law.} \\
\equiv\ & \neg(\neg p) \wedge \neg q && \text{By double negation} \\
\equiv\ & p \wedge \neg q.
\end{aligned}
$$

Exercise Answer

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

Answer:

$$\begin{aligned}
& \neg(p \vee (\neg p \wedge q)) \\
\equiv\ & \neg p \wedge \neg(\neg p \wedge q) && \text{by De Morgan's} \\
\equiv\ & \neg p \wedge (\neg(\neg p) \vee \neg q) && \text{by De Morgan's} \\
\equiv\ & \neg p \wedge (p \vee \neg q) && \text{by double negation.} \\
\equiv\ & (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{distributive law.} \\
\equiv\ & F \vee (\neg p \wedge \neg q) && \neg p \wedge p \equiv F.\ \text{by negation law} \\
\equiv\ & \neg p \wedge \neg q. && \text{by identity law.}
\end{aligned}$$

Exercise Answer

Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Answer:

$$
\begin{aligned}
& (p \wedge q) \rightarrow (p \vee q) \\
\equiv\ & \neg(p \wedge q) \vee (p \vee q) \\
\equiv\ & (\neg p \vee \neg q) \vee (p \vee q) \\
\equiv\ & (p \vee \neg p) \vee (q \vee \neg q) \\
\equiv\ & T \vee T \equiv T
\end{aligned}
$$

List of Symbols

SYMBOL	MEANING	PAGE
$\neg p$	negation of p	3
$p \wedge q$	conjunction of p and q	4
$p \vee q$	disjunction of p and q	4
$p \oplus q$	exclusive or of p and q	6
$p \rightarrow q$	the implication p implies q	6
$p \leftrightarrow q$	biconditional of p and q	9
$p \equiv q$	equivalence of p and q	23
$\mathbf{T}$	tautology	23
$\mathbf{F}$	contradiction	23
$P(x_1, \ldots, x_n)$	propositional function	36
$\forall x P(x)$	universal quantification of $P(x)$	38
$\exists x P(x)$	existential quantification of $P(x)$	40
$\exists! x P(x)$	uniqueness quantification of $P(x)$	41
$\therefore$	therefore	64
$p\{S\}q$	partial correctness of S	364

Expected Learning Outcomes

- To gain a good understanding of the definitions of proposition, propositional variable and logical operators;
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Reading

Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

- Sections 1.1-1.3.