Nonlinear Dynamics

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Machine Learning for Signal Processing - Fall 2021

Notation

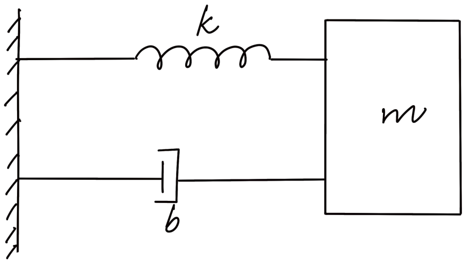
- Dots over variables denote time derivatives.

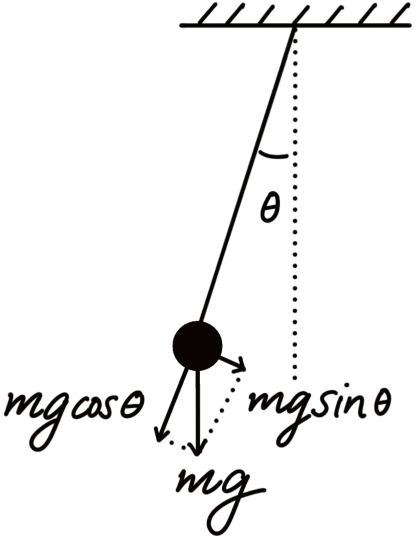
$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x}$$

- Will typically not denote vectors as bold letters. They will be in a normal font. Quantity being a scalar or a vector will be understood from context.

Where do dynamical systems appear?

Mechanical Systems





$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{b}{m}x_2$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Where do dynamical systems appear?

Population Dynamics and Epidemiology

$$\frac{d\text{🐇}}{dt} = \alpha\text{🐇} - \beta\text{🐇}\text{🐺}$$

Exponential growth

Gets eaten by wolves

$$\frac{d\text{🐺}}{dt} = \delta\text{🐇}\text{🐺} - \gamma\text{🐺}$$

Increases with more food

Decreases with competition

Lotka-Volterra aka
Predator-Prey

$$\frac{dS}{dt} = -\frac{\beta S I}{N}$$

$$\frac{dI}{dt} = \frac{\beta S I}{N} - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$

Susceptible-Infectious-Removed/Recovered

Where do dynamical systems appear?

Weather Modeling and Prediction

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x),$$

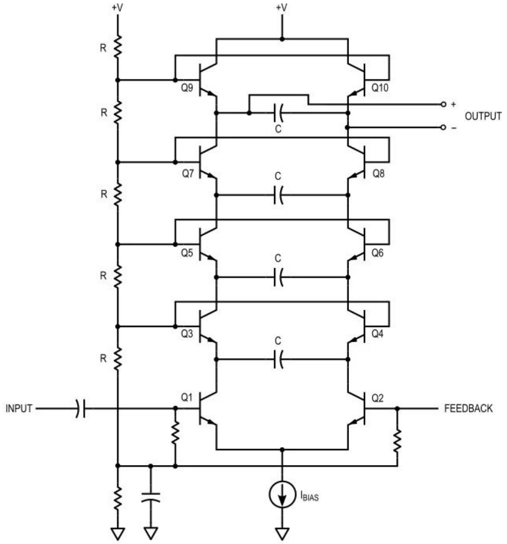
$$\frac{\mathrm{d}y}{\mathrm{d}t} = x(\rho - z) - y,$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = xy - \beta z.$$

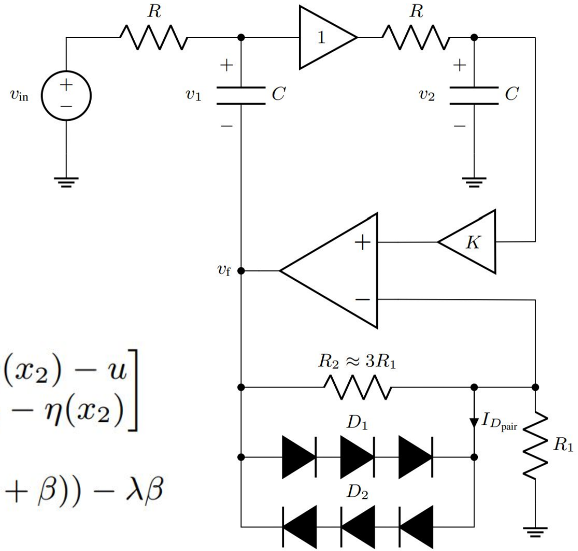
Lorenz system appears in weather modeling. Typical example of chaos. Reason why weather prediction is so hard.

Where do dynamical systems appear?

Electrical Systems



$$\frac{1}{\omega}\dot{\mathbf{x}} = \begin{bmatrix} -\tanh(x_1) + \tanh(u - \alpha^4 x_4) \\ -\tanh(x_2) + \tanh(x_1) \\ -\tanh(x_3) + \tanh(x_2) \\ -\tanh(x_4) + \tanh(x_3) \end{bmatrix}$$



$$\frac{1}{\omega}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_1 - \alpha x_2 + \eta(x_2) - u \\ x_1 + (\alpha - 1)x_2 - \eta(x_2) \end{bmatrix}$$

$$\eta(x_2) = \lambda W(\beta \exp(3\lambda\alpha x_2/4 + \beta)) - \lambda\beta$$

Moog 4-pole Ladder filter

Korg35 Filter (Revision 2 in the MS20 keyboard)

Where do dynamical systems appear?

Machine Learning

$$\mathbf{x}_{n+1} = \mathbf{x}_n - \gamma_n \nabla F(\mathbf{x}_n),\ n \geq 0.$$

Gradient Descent

$$m_w^{(t+1)} \leftarrow \beta_1 m_w^{(t)} + (1 - \beta_1) \nabla_w L^{(t)}$$
$$v_w^{(t+1)} \leftarrow \beta_2 v_w^{(t)} + (1 - \beta_2) (\nabla_w L^{(t)})^2$$

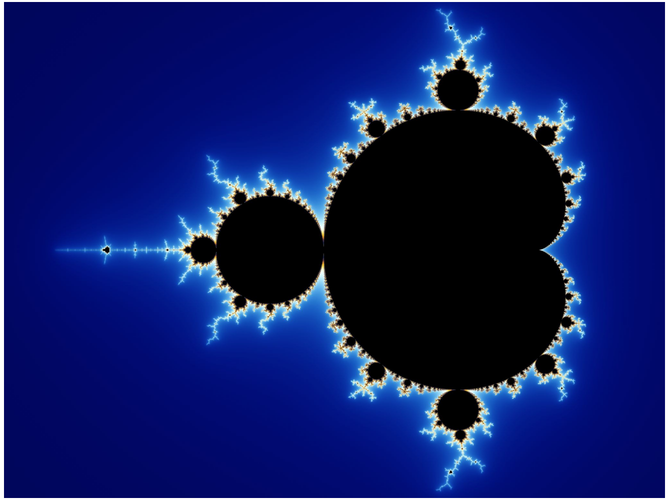
$$\hat{m}_w = \frac{m_w^{(t+1)}}{1 - \beta_1^{t+1}}$$
$$\hat{v}_w = \frac{v_w^{(t+1)}}{1 - \beta_2^{t+1}}$$

$$w^{(t+1)} \leftarrow w^{(t)} - \eta \frac{\hat{m}_w}{\sqrt{\hat{v}_w} + \epsilon}$$

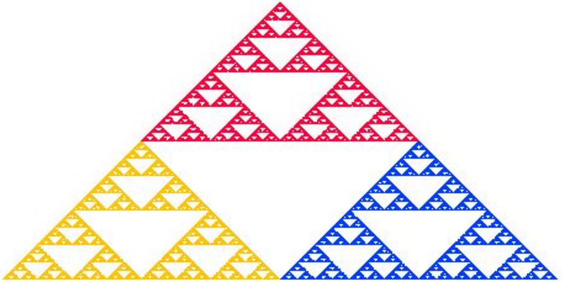
Adaptive Moment Estimation (ADAM)

Where do dynamical systems appear?

Fractals and Chaos



$$z_{n+1} = {z_n}^2 + c$$



Described using Iterated Function Systems.
Ideas useful in image compression and reconstruction.

Dynamical Systems

- A set, T, representing time.
- A set, X, representing all the possible states.
- An evolution function, $\Phi(t, x)$, which is the state of the system at time, t, when the system has its initial state (at $t = 0$) as x.

Dynamical Systems

$$\Phi(0,\, x) = x$$
$$\Phi(t_1 + t_0,\, x) = \Phi(t_1,\, \Phi(t_0,\, x))$$

- State of system at t = 0 is the initial state.
- State at any time can be found by finding state at some intermediate point and using this intermediate state as the new initial state.

Dynamical Systems

$$\Phi(0,\ x) = x$$

$$\Phi(t_1 + t_0,\ x) = \Phi(t_1,\ \Phi(t_0,\ x))$$

- For this to make sense, only constraints on the set of time variables are
  - The + operation must make sense in *T*.
  - There must be an identity element, with respect to +, in *T*.
- (Such a *T* is called a monoid.)

Continuous Time Dynamical Systems - Flows

- In this lecture, we'll mostly look at a special class of dynamical systems that can be described as a system of differential equations.

$$\dot{x}(t) = f(t,\, x(t)),\, x(0) = x_0$$
$$\text{where } x \in \mathbb{R}^n \text{ and } f : T \times \mathbb{R}^n \to \mathbb{R}^n$$

- The state is a vector of dimension *n* and *f* is a vector field that takes the *n*-dimensional state and spits out an *n*-dimensional vector which is the time derivative of the state.

$$T = \mathbb{R},\, \mathcal{X} = \mathbb{R}^n,\, \Phi(t,\, x_0) = x(t) \text{ where } x(0) = x_0$$

Discrete Time Dynamical Systems - Cascades/Iterated Maps

- Discrete time dynamical systems can be described by iterated maps.

$$x_{t+1} = f(t,\, x_t),\ x_0 \text{ is the initial state,}$$
$$\text{where } x \in \mathbb{R}^n \text{ and } f : T \times \mathbb{R}^n \to \mathbb{R}^n$$

- The state is a vector of dimension *n* and *f* is a vector field that takes the *n*-dimensional state and spits out an *n*-dimensional vector which is the next state.

$$T = \mathbb{Z},\ \mathcal{X} = \mathbb{R}^n,\ \Phi(t,\, x_0) = x_t \text{ where } x_0 \text{ is the initial state}$$

Existence and Uniqueness - Discrete Time

- The iterated map is

$$x_{n+1} = f(n, x_n),\ x_0 \textbf{ is the initial state}$$

- A solution is simply the sequence below, so it exists as long as *f* maps time and the state space to the state space.

$$x_0, f(0, x_0), f(1, f(0, x_0)), \ldots$$

- For uniqueness of the solution, *f* should be single-valued.
- Rather trivial conditions. Existence and uniqueness usually never mentioned for deterministic discrete time systems for this reason.

Existence and Uniqueness - Continuous Time

- Situation more complicated. Possible that a solution of the ODE does not exist, or if it exists, it may not be unique.

$$\dot{x}(t) = f(t, x(t)),\ x(0) = x_0$$

- For instance, the ODE below has multiple solutions.

$$\dot{x}(t) = x^{1/3}$$

$$\implies x(t) = 0 \text{ or } x(t) = \pm\left(\frac{2}{3}t\right)^{3/2}$$

Lipschitz Continuity

- Lipschitz on a set, *W*, if there is a nonnegative *L* so that for any *x*, *y* in *W*,

$$\|f(x) - f(y)\| \leq L\|x - y\|$$

- Locally Lipschitz on a set, *X*, if for any point, *x* in *X*, there is a neighborhood around *x* where *f* is Lipschitz on that neighborhood with Lipschitz constant *L* that may depend on the point, *x*.
- Globally Lipschitz if Lipschitz on the entire space where the function is defined.

Existence and Uniqueness - Continuous Time

$$\dot{x}(t) = f(t, x(t)),\ x(t_0) = x_0$$

- Solution **exists** if *f*(*t, x*) is continuous in *t* and *x* (Peano's theorem). Can even weaken this to piecewise continuous (Cartheodary's/Filipov's Theorem).
- Solution is also **unique** if *f*(*t*, *x*) is also locally Lipschitz continuous in *x* (Cauchy/Picard-Lindeloff Theorem).

Existence and Uniqueness - Continuous Time

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- Solution is also **unique** if *f*(*t*, *x*) is also locally Lipschitz continuous in *x* (Cauchy/Picard-Lindeloff Theorem).
- The theorem guarantees the existence and uniqueness of the solution *x*(*t*) only for a small interval of time $[t_0, t_0+\delta)$.
- Duration of time, δ, for which the solution exists and is unique in the entire state space is equal to the inverse of the Lipschitz constant in the neighborhood of x_0.

Example System

$$\dot{x}(t) = Ax(t),\, x(t_0) = x_0$$

$$f(t,\, x) = Ax$$

- *f* is clearly continuous in time.
- Is *f* locally Lipschitz?
  - Yes!
- Theorem says solution exists for a short duration of time. Can we do better?
  - Yes!

- In general, if *f*(*t*, *x*) is globally Lipschitz in *x*, then the solution exists and is unique for all time after the initial time, t_0. In general hard to show global Lipschitz.

Conditions that Imply Lipschitz Continuity

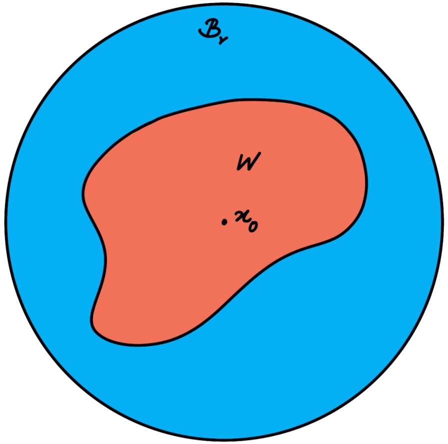
$$\dot{x}(t) = f(t, x(t)),\ x(t_0) = x_0$$

- Continuously differentiable in *x* implies locally Lipschitz.
- Continuously differentiable in *x* and derivative on *x* bounded on a set implies Lipschitz on that set.
- Continuously differentiable in *x* and derivative on *x* bounded everywhere implies globally Lipschitz.

- Bound should be independent of time.

Another Condition for Existence

- If you can show that the solution cannot ever escape some closed and bounded set, then you can claim that the solution exists for all time after the initial time.
- Still need piecewise continuity in time and locally Lipschitz in state.



Autonomous System

- Change of state function has no explicit dependence on time.
- Continuous Time

$$\dot{x}(t) = f(x(t)),\ x(0) = x_0$$
$$\textbf{where } x \in \mathbb{R}^n \textbf{ and } f : \mathbb{R}^n \to \mathbb{R}^n$$

- Discrete Time

$$x_{t+1} = f(x_t),\ x_0 \textbf{ is the initial state,}$$
$$\textbf{where } x \in \mathbb{R}^n \textbf{ and } f : \mathbb{R}^n \to \mathbb{R}^n$$

Fixed/Critical/Stationary/Equilibrium Points

$$\Phi(t,\, x_{\mathrm{FP}}) = x_{\mathrm{FP}}\,\forall\, t$$

- States where the dynamics don't change at all.
- Continuous time ODE, can solve for all the fixed points.

$$x(t) = x_{\mathrm{FP}}\,\forall\, t \implies \dot{x}(t) = 0 \implies f(x(t)) = 0 \implies \boxed{f(x_{\mathrm{FP}}) = 0}$$

- Discrete time cascade, can also solve for all the fixed points.

$$x_n = x_{\mathrm{FP}}\,\forall\, n \implies x_{\mathrm{FP}} = x_{n+1} = f(x_n) = f(x_{\mathrm{FP}}) \implies \boxed{x_{\mathrm{FP}} = f(x_{\mathrm{FP}})}$$

Linear Time Invariant (LTI) Systems

$$\dot{x}(t) = Ax(t),\ x(t_0) = x_0$$

- *f*(*x*) is linear in the state *x*.
- Time invariance implies the vector field of the system has no explicit dependence on time.
- What are the fixed points?
- What is the solution? What does the solution look like?

$$x(t) = e^{A(t-t_0)}x_0$$

LTI Systems

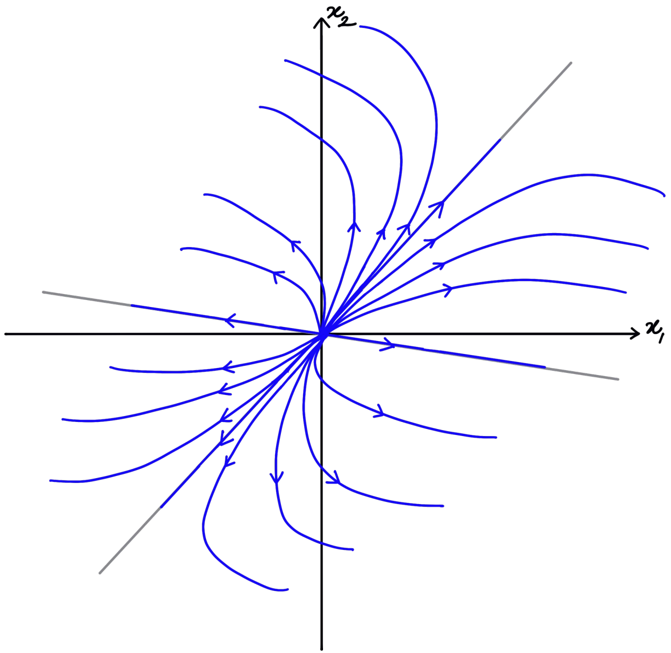
- If A is diagonalizable, then

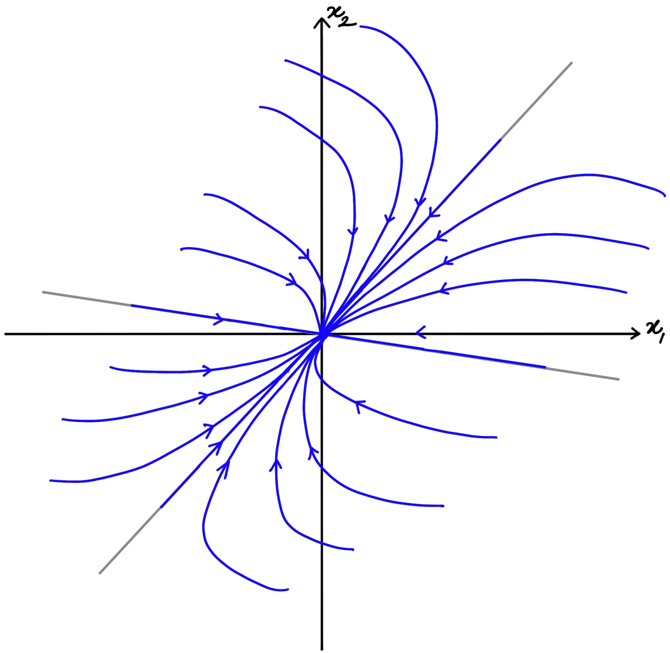
$$A = V\Lambda V^{-1}$$

$$
\begin{aligned}
\dot{x} &= V\Lambda V^{-1}x \\
\implies \frac{\mathrm{d}}{\mathrm{d}t}\left(V^{-1}x\right) &= \Lambda\left(V^{-1}x\right) \\
\implies \dot{z} &= \Lambda z \\
\implies \dot{z}_i &= \lambda_i z_i \text{ for each } i
\end{aligned}
$$

- Eigenvalues describe behavior of solution!

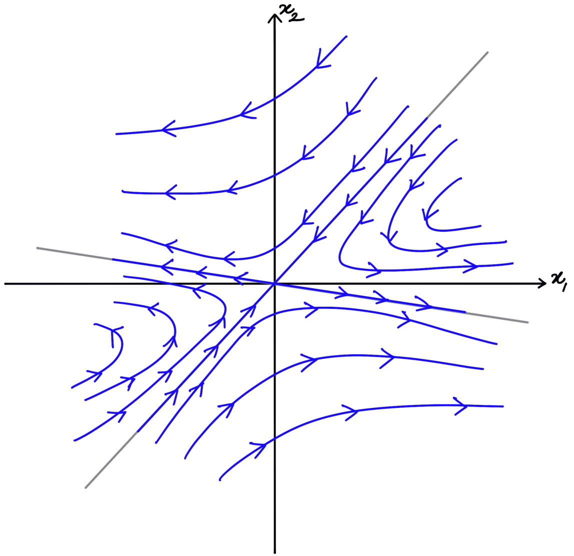
LTI Systems - 2D systems





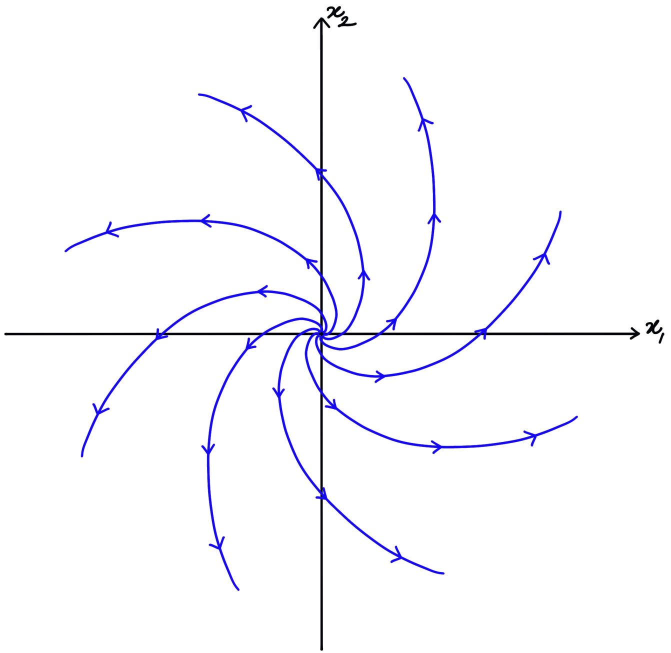
- 2 Eigenvalues.
- If both are real and have the same sign, called a Node. Left, both +, Right, both -.

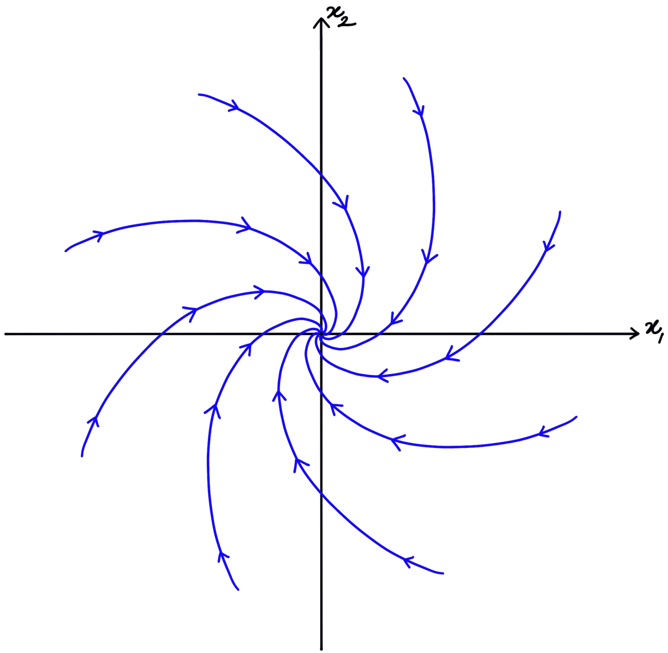
LTI Systems - 2D systems



- 2 Eigenvalues.
- If both are real and have opposite signs, called a Saddle.

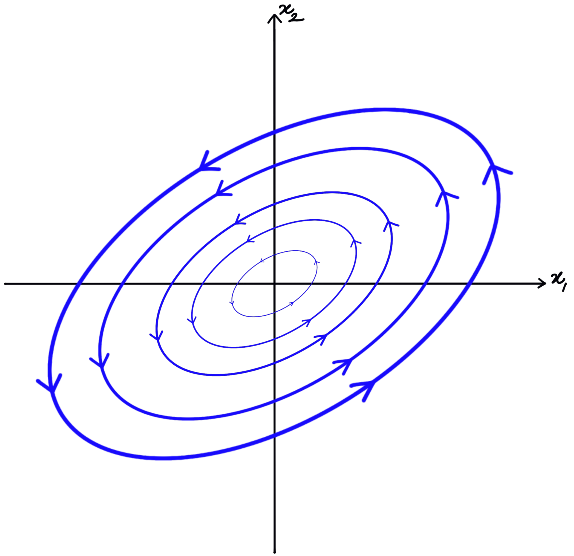
LTI Systems - 2D systems





- 2 Eigenvalues.
- If both are complex with non-zero real part, called a Focus/Spiral. Left, real parts +, Right, real parts -.

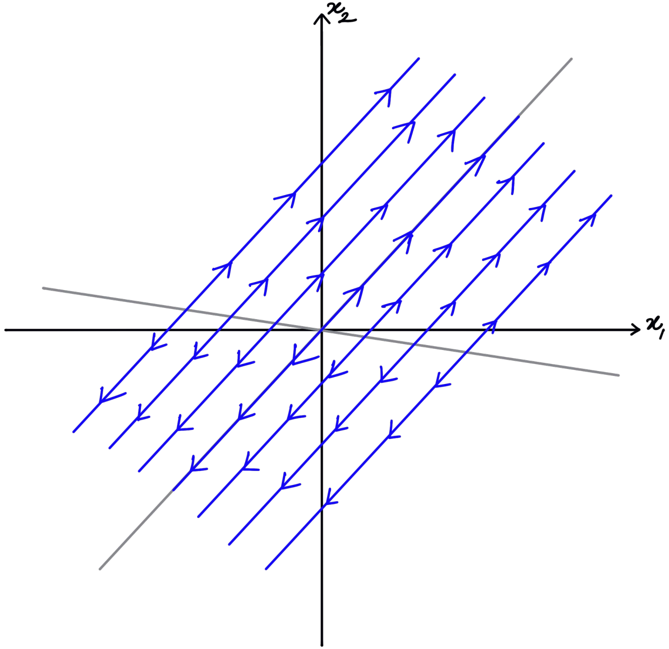
LTI Systems - 2D systems

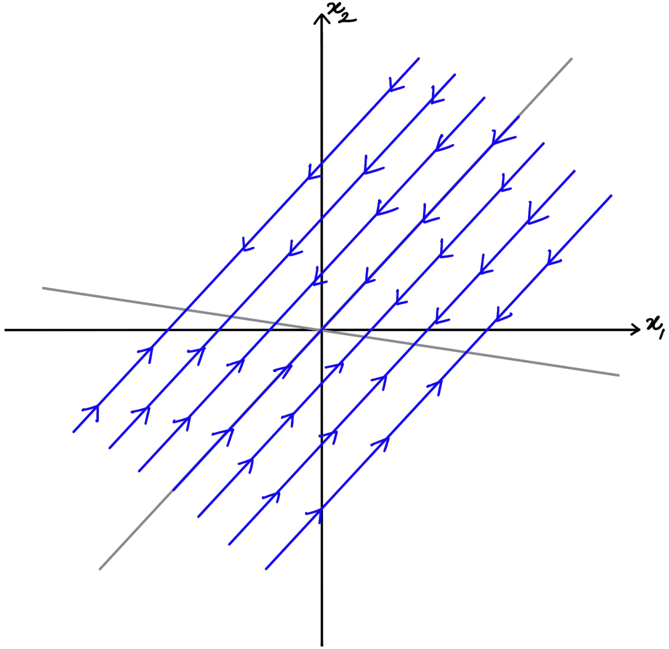


- 2 Eigenvalues.
- If both are purely imaginary, called a center.

LTI Systems - 2D

- What happens when one of the eigenvalues is zero?
- *A* not full rank. A line of equilibrium points.





LTI Systems - 2D

- Another case where the eigenvalues are equal, but the matrix is not diagonalizable.
- Happens when matrix of eigenvectors is not invertible. Degeneracy in eigenspace.
- Dimensionality of Eigenspace < Algebraic Multiplicity of Eigenvalue.

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

$$\implies x(t) = e^{At} x_0 = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} x_0 = \begin{bmatrix} e^{\lambda t}(x_{10} + t x_{20}) \\ e^{\lambda t} x_{20} \end{bmatrix}$$

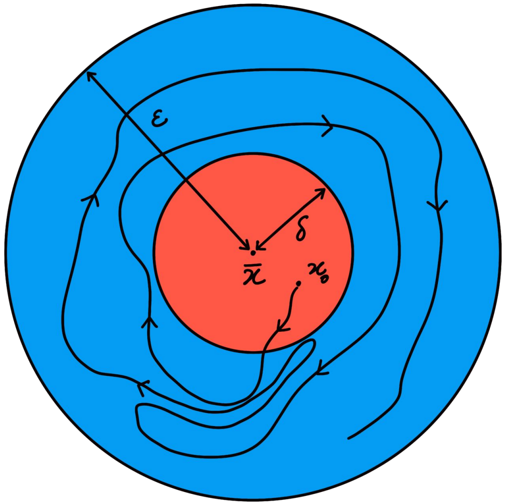
LTI Systems - 2D

- Eigenvalues were able to tell us the nature of the solutions.
- The signs of the real parts also told us whether they would converge to an equilibrium point, or diverge away to infinity.
- If any eigenvalue have a positive real part, the solution tends to infinity as time went on.
- If **all** eigenvalues have a negative real part, the solution converges to the fixed point at the origin.

Stability for a Nonlinear System

- Nonlinear systems can have many fixed points.
- Stability is a property of a fixed point.

$$\bar{x} \text{ such that } f(\bar{x}) = 0 \text{ is stable if}$$
$$\forall \varepsilon > 0 \, \exists \, \delta_\varepsilon > 0 \text{ such that } \|x(0) - \bar{x}\| < \delta_\varepsilon \implies \|x(t) - \bar{x}\| < \varepsilon \, \forall \, t \geq 0$$



Asymptotic Stability

- Fixed point is Asymptotically Stable (AS) if it is stable and

$$\exists\, \delta > 0 \text{ such that } \|x(0) - \bar{x}\| < \delta \implies \lim_{t\to\infty} x(t) = \bar{x}$$

Basin of Attraction and Global AS

- Basin of attraction of a fixed point = set of all points that tend to that asymptotically stable fixed point.
- Fixed point is Globally Asymptotically Stable (GAS) if Basin of Attraction is the entire space.

- Linear Systems: Stable nodes and stable foci are always GAS. Centers are only stable, not AS (and therefore not GAS).

Exponential Stability

- Distance between solution and fixed point decays exponentially when initial state is within some ball around initial state.

- GES if exponential decay regardless of initial state.

- For a linear system, AS equivalent to GAS equivalent to ES equivalent to GES.

Stability of Nonlinear Systems - Linearization

$$\dot{x} = f(x),\ f(\bar{x}) = 0$$

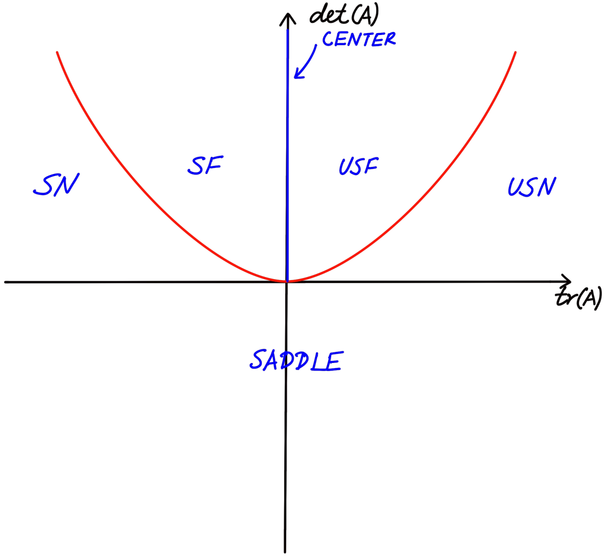
$$x = \bar{x} + \xi$$

$$\begin{aligned}
\implies \frac{\mathrm{d}}{\mathrm{d}t}(\bar{x} + \xi) &= f(\bar{x} + \xi) \\
\implies \dot{\xi} &= f(\bar{x}) + \frac{\partial f}{\partial x}(\bar{x})\xi + \text{h.o.t.} \\
\implies \dot{\xi} &\approx \frac{\partial f}{\partial x}(\bar{x})\xi
\end{aligned}$$

$$\dot{\xi} = A\xi$$

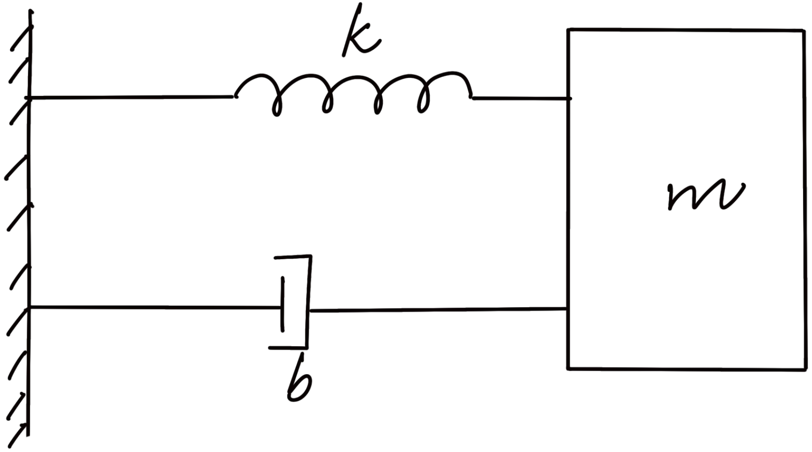
Stability of Nonlinear Systems - Linearization

- Stability is a local property, so linear analysis should work.
- Eigenvalues of the Jacobian of *f* can tell us the nature of the fixed point.
- Nodes, Foci, and Saddles are conclusive.
- Beware Centers! Linearization inconclusive for centers.



Alternative Stability Analysis

- Example Spring Mass Damper system.



Spring-Mass-Damper

$$m\ddot{y} = -b\dot{y} - ky$$

- Convert this to state space form by choosing state variables

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \end{aligned} \implies \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m}x_1 - \frac{b}{m}x_2 \end{aligned}$$

- Energy is kinetic energy + potential energy

$$E = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}ky^2 = \frac{1}{2}mx_2^2 + \frac{1}{2}kx_1^2$$

Properties of Energy Function

$$E = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}ky^2 = \frac{1}{2}mx_2^2 + \frac{1}{2}kx_1^2$$

- Positive for all state not equal to the equilibrium state.
- Zero at equilibrium state.
- How does it change with time?

Properties of Energy Function

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- How does it change with time?

$$\boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m}x_1 - \frac{b}{m}x_2 \end{aligned}}$$

$$\begin{aligned} \dot{E} &= mx_2\dot{x}_2 + kx_1\dot{x}_1 \\ &= mx_2\left(-\frac{b}{m}x_2 - \frac{k}{m}x_1\right) + kx_1(x_2) \\ &= -bx_2^2 \leq 0 \end{aligned}$$

Positive Definite Function

- $V(x)$ is positive definite on a region D that contains zero if
  - $V(x) > 0$ for all nonzero x in D
  - $V(x) = 0$ for $x = 0$

- To use this, we will typically assume the fixed point of interest is at zero. Can always translate the state space to make it so.

Lyapunov Stability Theorem

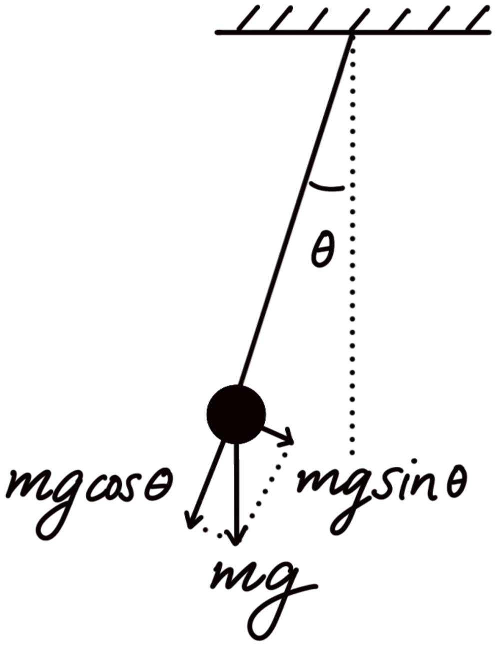
$$\dot{x} = f(x) \text{ and } f(0) = 0$$

- Let $V(x)$ be a continuously differentiable function defined on a region D containing zero such that $V(x)$ is positive definite.
- If $dV(x)/dt \leq 0$ for all x in D, then the FP at zero is stable.
- If $dV(x)/dt < 0$ for all nonzero x in D, then the FP at zero is asymptotically stable.

- These conditions are merely sufficient, not necessary. If a candidate positive definite function does not satisfy this, can't say unstable. Linearization good way to check instability. There are also Lyapunov instability theorems.

Question

Simple Pendulum - With and Without Friction



$$ml\ddot{\theta} = -mg\sin\theta - bl\dot{\theta}$$

$$\begin{aligned} x_1 &= \theta \\ x_2 &= \dot{\theta} \end{aligned} \implies \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 - \frac{b}{m}x_2 \end{aligned}$$

Equilibrium points?

Stability Analysis - Without Friction

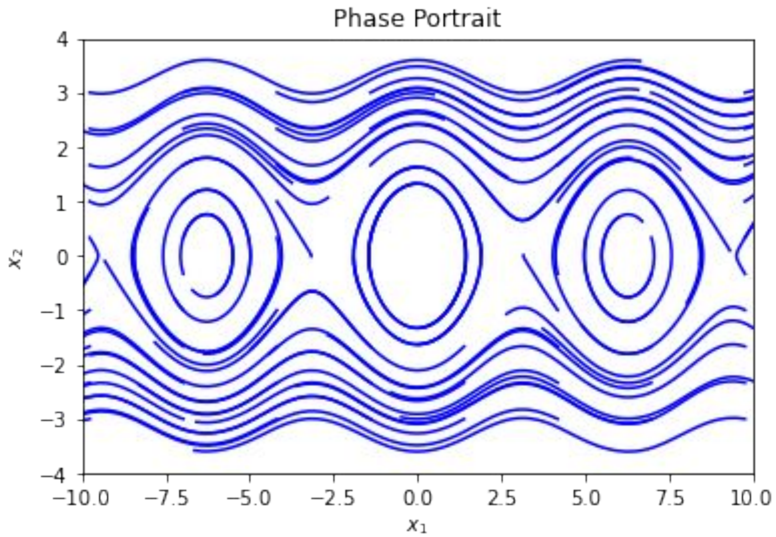
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1$$

- Linear analysis says FP at π is a saddle (and therefore unstable).
- Linear analysis says FP at 0 is a center. Inconclusive! Need Lyapunov analysis.
- Candidate Lyapunov function: energy of the system

Stable!

$$\begin{aligned} V(x) &= \frac{1}{2}x_2^2 + \frac{g}{l}\int_0^{x_1} \sin\xi \, \mathrm{d}\xi \\ &= \frac{1}{2}x_2^2 + \frac{g}{l}(1 - \cos x_1) \end{aligned} \implies \dot{V}(x) = 0$$

Simple Pendulum - Without Friction



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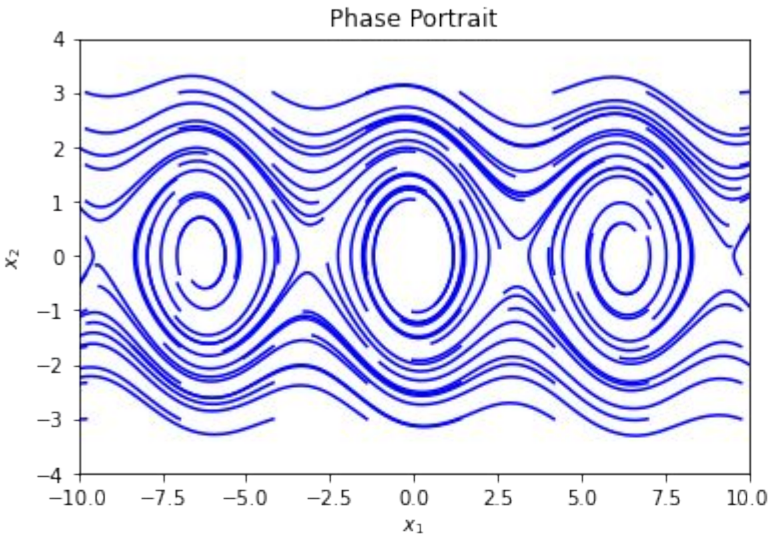
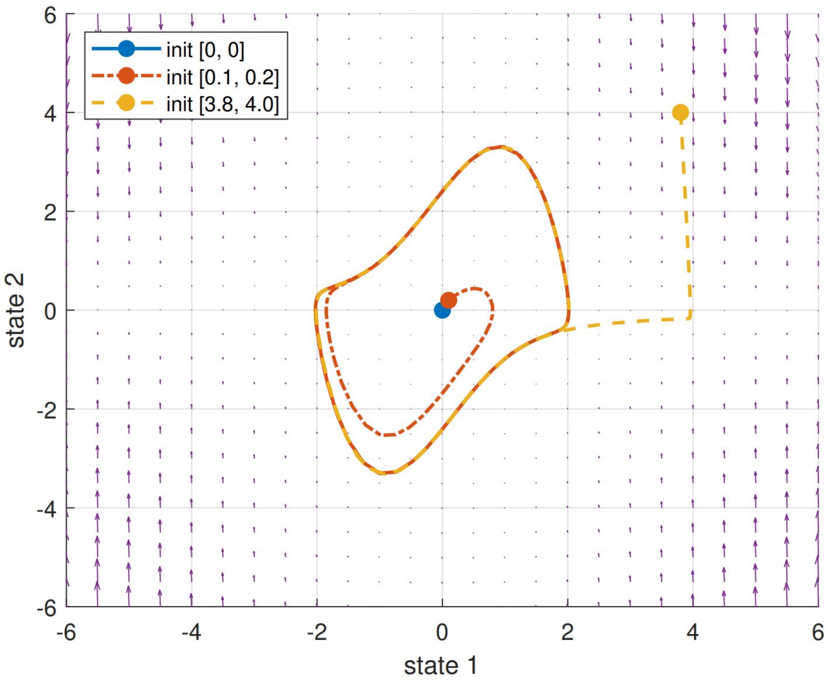
Simple Pendulum - With Friction



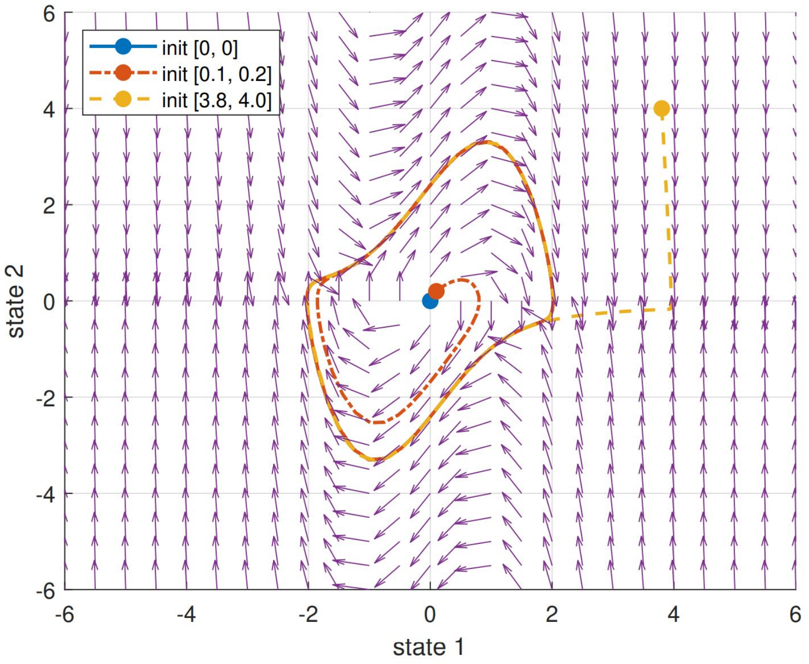
Limit Cycles



- Isolated Periodic Orbits
- Unlike centers, they **are** structurally stable! Perturbations of system do not affect local stability or instability of these

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - \epsilon h'(x_1) x_2$$

Limit Cycles



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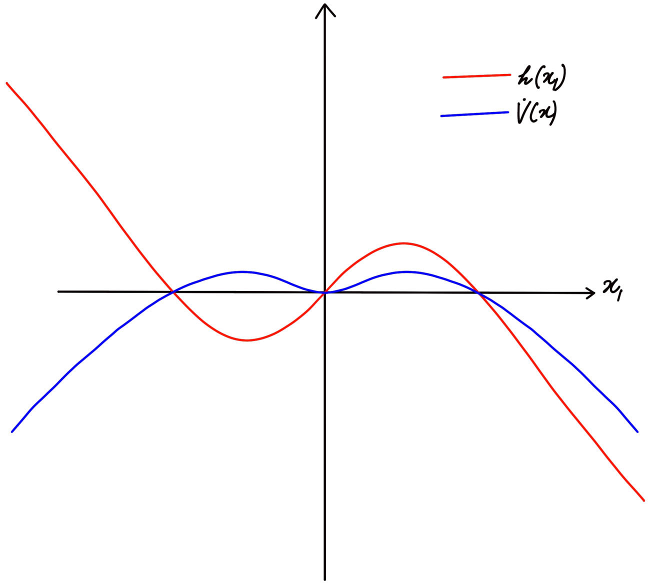

Poincaré-Bendixson Criterion

- Only holds for 2 dimensional systems
- Let *M* be a closed bounded set which the solution never leaves, such that *M* contains no fixed points, or contains exactly one **unstable** fixed point.
- Then *M* contains a periodic orbit.

- If you can also find some curve within *M* so that the vector field *f*(*x*) always points outwards, there must be a limit cycle somewhere in the region between this curve and the boundary of *M*.

Learning Nonlinear Dynamics

Learning Nonlinear Dynamics

- Nonlinear Dynamics of the form

$$\dot{x}(t) = f(x(t)),\ x(0) = x_0$$
$$\textbf{where } x \in \mathbb{R}^n \textbf{ and } f : \mathbb{R}^n \to \mathbb{R}^n$$

- Unknown *f*. Want to learn it from data.
- Data is states (and maybe derivatives of state).
- How?

NEURAL NETWORKS

(Look up Neural ODEs and Universal ODEs)


- We don't do that here.

SINDy

Sparse

Identification of

Nonlinear

Dynamics

Brunton et al. “Discovering governing equations from data by sparse identification of nonlinear dynamical systems”

Data

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}^T(t_1) \\ \mathbf{x}^T(t_2) \\ \vdots \\ \mathbf{x}^T(t_m) \end{bmatrix} = \begin{array}{cc} \overset{\text{state}}{\longrightarrow} & \\ \begin{bmatrix} x_1(t_1) & x_2(t_1) & \cdots & x_n(t_1) \\ x_1(t_2) & x_2(t_2) & \cdots & x_n(t_2) \\ \vdots & \vdots & \ddots & \vdots \\ x_1(t_m) & x_2(t_m) & \cdots & x_n(t_m) \end{bmatrix} & \downarrow\text{time} \end{array}$$

$$\dot{\mathbf{X}} = \begin{bmatrix} \dot{\mathbf{x}}^T(t_1) \\ \dot{\mathbf{x}}^T(t_2) \\ \vdots \\ \dot{\mathbf{x}}^T(t_m) \end{bmatrix} = \begin{bmatrix} \dot{x}_1(t_1) & \dot{x}_2(t_1) & \cdots & \dot{x}_n(t_1) \\ \dot{x}_1(t_2) & \dot{x}_2(t_2) & \cdots & \dot{x}_n(t_2) \\ \vdots & \vdots & \ddots & \vdots \\ \dot{x}_1(t_m) & \dot{x}_2(t_m) & \cdots & \dot{x}_n(t_m) \end{bmatrix}.$$

From Brunton et al. “Discovering governing equations from data by sparse identification of nonlinear dynamical systems”

Dictionary of Functions

- Build a dictionary of functions

$$\Theta(\mathbf{X}) = \begin{bmatrix} | & | & | & | & & | & | & \\ 1 & \mathbf{X} & \mathbf{X}^{P_2} & \mathbf{X}^{P_3} & \cdots & \sin(\mathbf{X}) & \cos(\mathbf{X}) & \cdots \\ | & | & | & | & & | & | & \end{bmatrix}.$$

- We want to find sparse Ξ such that

$$\dot{\mathbf{X}} = \Theta(\mathbf{X})\Xi.$$

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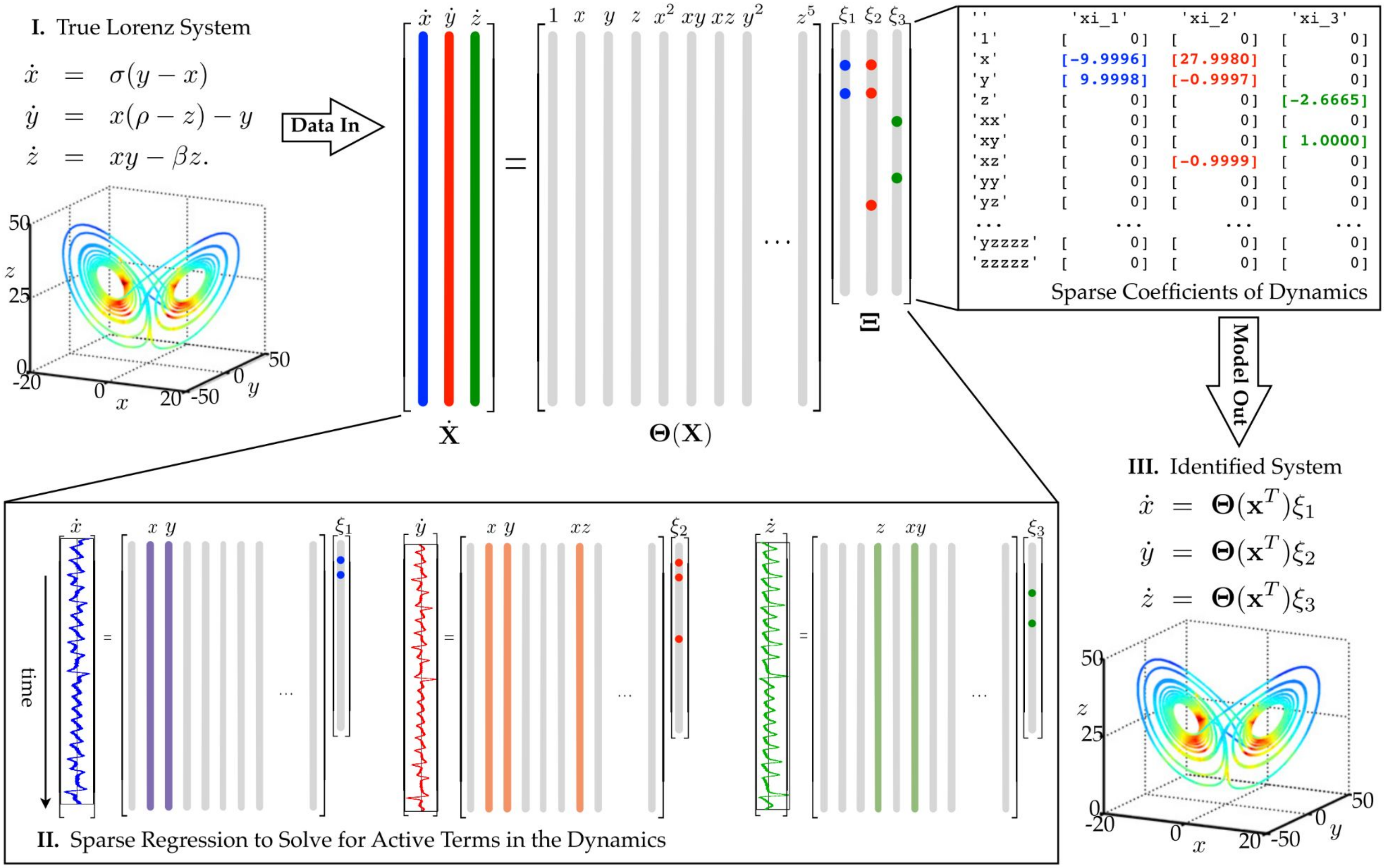
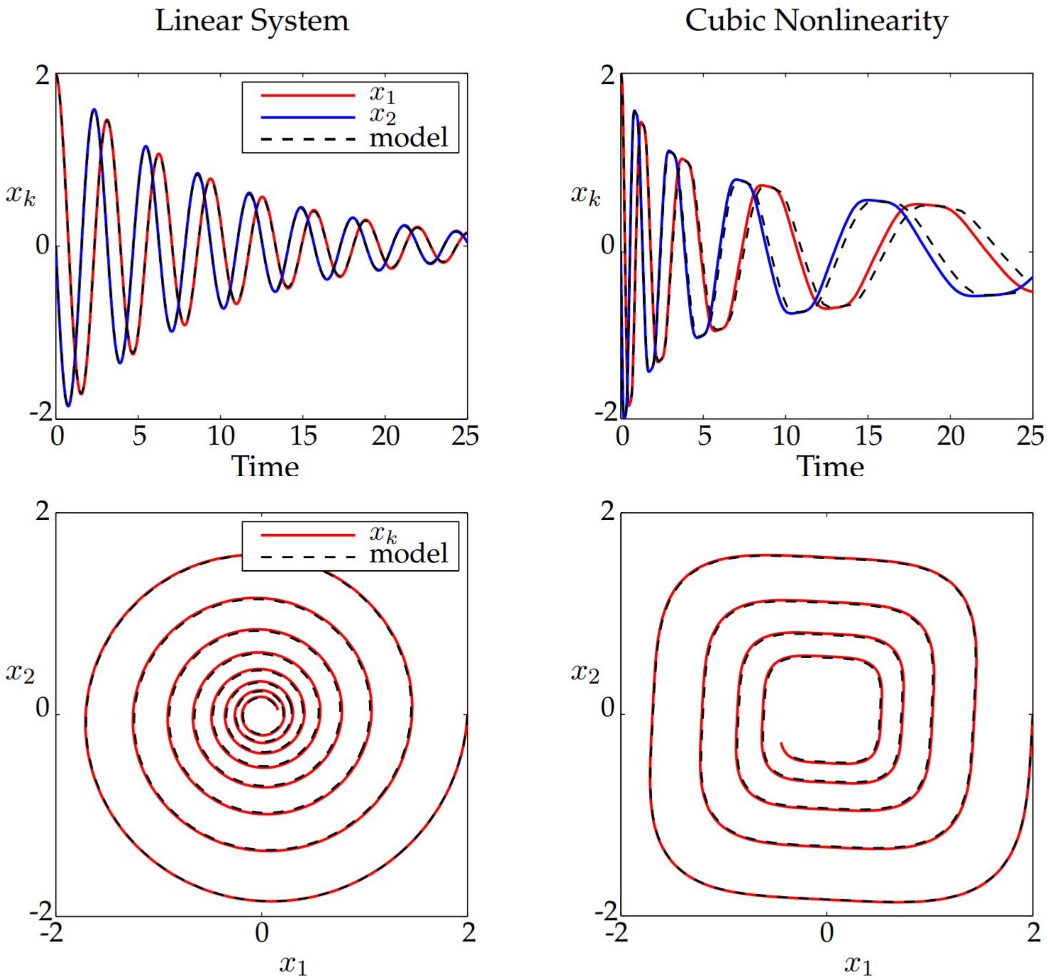
- LASSO works!

From Brunton et al. “Discovering governing equations from data by sparse identification of nonlinear dynamical systems”



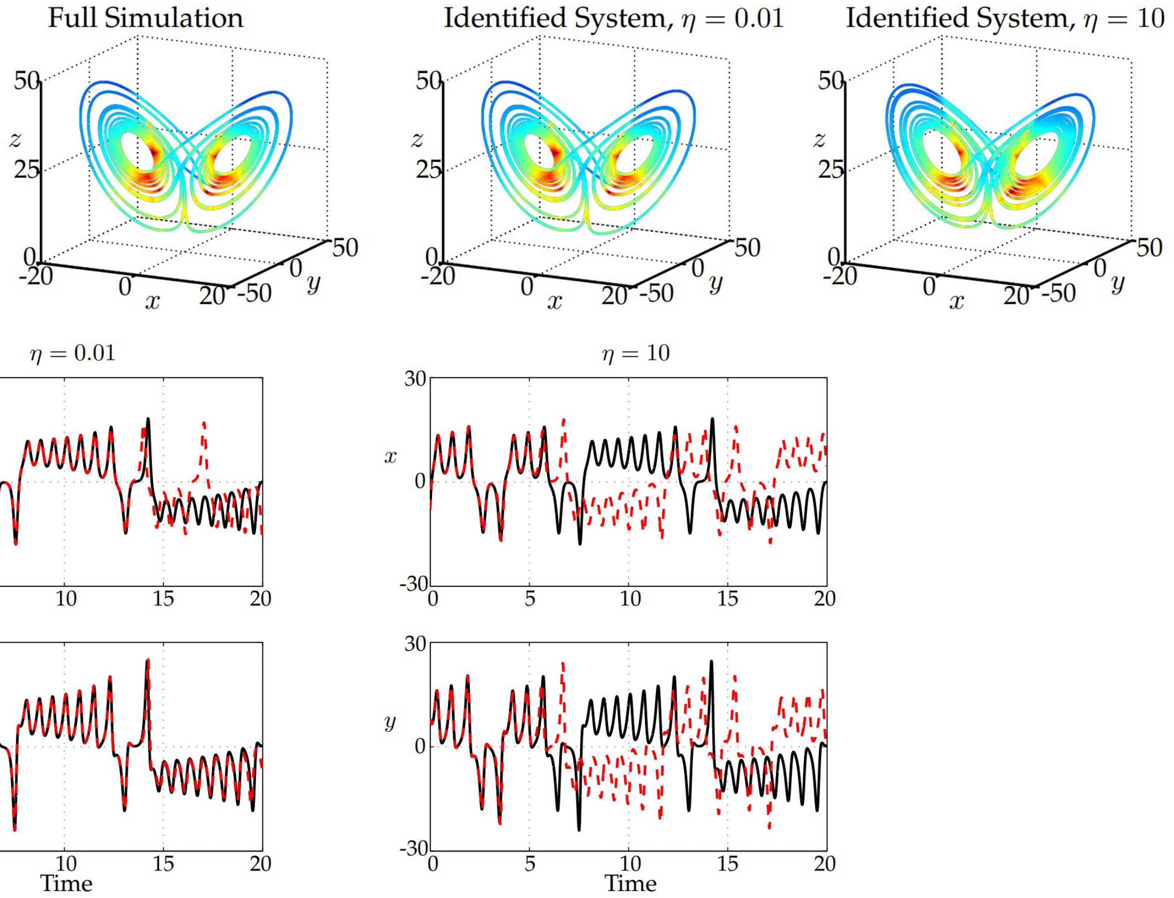
''	'xi_1'	'xi_2'	'xi_3'
'1'	[0]	[0]	[0]
'x'	[-9.9996]	[27.9980]	[0]
'y'	[9.9998]	[-0.9997]	[0]
'z'	[0]	[0]	[-2.6665]
'xx'	[0]	[0]	[0]
'xy'	[0]	[0]	[1.0000]
'xz'	[0]	[-0.9999]	[0]
'yy'	[0]	[0]	[0]
'yz'	[0]	[0]	[0]
...	...	...	...
'yzzzz'	[0]	[0]	[0]
'zzzzz'	[0]	[0]	[0]

From Brunton et al. “Discovering governing equations from data by sparse identification of nonlinear dynamical systems”



From Brunton et al. “Discovering governing equations from data by sparse identification of nonlinear dynamical systems”

$$
\begin{aligned}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= x(\rho - z) - y \\
\dot{z} &= xy - \beta z.
\end{aligned}
$$



From Brunton et al. “Discovering governing equations from data by sparse identification of nonlinear dynamical systems”

High Dimensional Systems

- Such as fluid dynamics, acoustics systems, etc.
- Use SVD

$$\mathbf{X}^T = \mathbf{\Psi\Sigma V}^*.$$

- Keep first *r* columns of basis matrix. *a* are coordinates in this basis.

$$\mathbf{x} \quad \approx \quad \mathbf{\Psi}_r \mathbf{a},$$

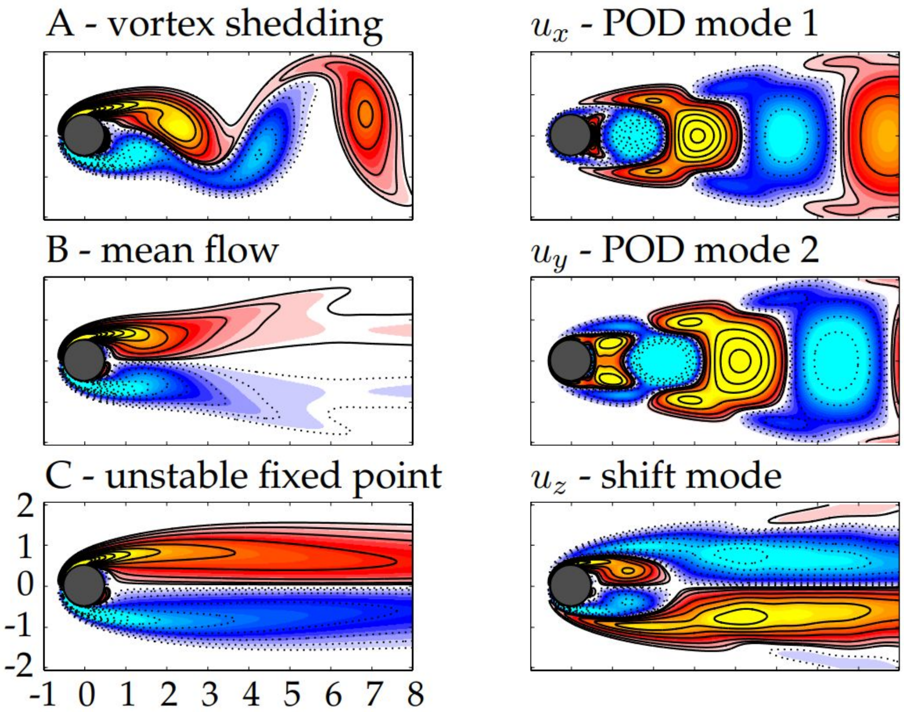
- Dynamics itself projected into lower dimensional space.

$$\dot{\mathbf{a}} \quad = \quad \mathbf{f}_P(\mathbf{a}).$$

- Other methods exist to project into other lower dimensional manifolds.

From Brunton et al. “Discovering governing equations from data by sparse identification of nonlinear dynamical systems”

Cylinder in a Fluid



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