Hypatia 2022 Solutions

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Question 3: A sequence is created in such a way that

1. a real number is chosen as the first number in the sequence, and
2. each of the following numbers in the sequence is generated by applying a function to the previous number in the sequence

For example, if the first number in a sequence is 1 and the following number are generated by the function $x^2 - 5$, then the first three numbers in the sequence are 1, -4, an1 11 since $1^2 - 5 = -4$ and $(-4)^2 - 5 = 11$.

(a) The first number in a sequence is 3 and the sequence is generated by the function $x^2 - 3x + 1$. What are the first four number in the sequence?

$$\begin{aligned}
a_1 &= 3 \\
a_2 &= 3^2 - 3(3) - 1 = -1 \\
a_3 &= (-1)^2 - 3(-1) + 1 = 3 \\
a_4 &= -1
\end{aligned}$$

(b) The number 7 is the third number in a sequence generated by the function $x^2 - 4x + 7$. What are all possible first numbers in the sequence?

Let $a_1 = x$, then

$$a_2 = x^2 - 4x + 7$$

$$7 = (x^2 - 4x + 7)^2 - 4(x^2 - 4x + 7) + 7$$

$$(x^2 - 4x + 7)(x^2 - 4x - 3) = 0$$

The first quadratic has a negative discriminant, and the second factors to $(x-1)(x-3)$. Thus, x is either 1 or 3.

(c) The first number in a sequence is c and the sequence is generated by the function $x^2 - 7x - 48$. If all numbers in the sequence are equal to c, determine all possible values of c

$$c = c^2 - 7c - 48$$

$$c^2 - 8c - 48 = 0$$

$$(c - 12)(c + 4) = 0$$

$$c = 12, -4$$

(d) A sequence generated by the function $x^2 - 12x + 39$ alternates between two different numbers. That is, the sequence is a,b,a,b,..., with $a \neq b$. Determine all possible values of a.

Completing the square on the equation gives $(x-6)^2 + 3$. Now, we have the two equations

$$(a-6)^2 + 3 = b$$

$$(b-6)^2 + 3 = a$$

Now we could just attempt to solve the messy quartic, but during contest the faster way would be to subtract the two, giving us a difference of squares.

$$\begin{aligned}(a-6)^2 - (b-6)^2 &= b - a \\ (a-6+b-6)(a-6-b+6) &= b - a \\ (a+b-12)(a-b) &= b - a \\ (a+b-12) &= 1 \\ b &= 11 - a\end{aligned}$$

Subbing this back, we get

$$a^2 - 12a + 39 = 11 - a$$

$$a^2 - 11a + 28 = 0$$

$$(a-7)(a-4) = 0$$

$$a = 7, 4$$

Question 4 Every integer $N > 1$ can be written as $N = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$, where k is a positive integer, $p_1 < p_2 < \dots p_k$ are prime numbers, and $a_1, a_2, \dots a_k$ are positive integers. For example, $1400 = 2_3 5^2 7^1$ The number of positive divisors of N is denoted by $f(N)$. It is known that

$$f(N) = (1 + a_1)(1 + a_2) \dots (1 + a_k)$$

(a) Evaluate $f(240)$

$$240 = 2^4 * 3 * 5$$

$$f(240) = (4+1)(1+1)(1+1)$$

$$f(240) = 20$$

(b) Define an integer $N > 1$ to be $refactorable$ if it is divisible by $f(N)$. For example, boht 6 and 8 have 4 positive divisors, so 8 is refactorable and 6 is not refactorable. This is because 8 is divisible by 4, but 8 is not. Determine all refatorable number N with $f(N) = 6$

In order for $6|f(N)$, the prime factorization of $f(N)$ must contain at least one 2 and one 3. Additionally, since $6 = 2 * 3$, there must be one factor raised to the power of 1 and another raised to the power of 2. Thus, the only options are

$$2 * 3^2 = 18$$

$$2^2 * 3 = 12$$

(c) Define the smallest refactorable number N with $f(N) = 256$.

Since $256 = 2^8$, we know that N must contain at least 2^8. However, since $8 + 1 = 9$, N is minimized when it contains $2^1 5$. $256/16 = 16$, so the remaining 16 is optimally added with $3^3 * 5^3$.

$$N = 2^{15} * 3^3 * 5^3$$

$$f(N) = (15+1)(3+1)(3+1)$$
$$f(N) = 256$$

(d) Show that for every integer $m > 1$, there exists a refactorable number N such that $f(N) = m$

We know that $N = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$. Thus, set N to be

$$N = p_1^{p_1^{a_1}-1} + p_2^{p_2^{a_2}-1} + \dots + p_k^{p_k^{a_k}-1}$$

This has cardinality

$$N = (p_1^{a_1} - 1 + 1)(p_2^{a_2} - 1 + 1) \dots (p_k^{a_k} - 1 + 1)$$
$$N = (p_1^{a_1})(p_2^{a_2}) \dots (p_k^{a_k})$$

Finally, all that remains to prove is that

$$p_1^{a_1} - 1 \geq a_1$$

Recall though, that we are given

$$2^n \geq n + 1$$

Rearranging, we get

$$2^n - 1 \geq n$$

Since 2 is the smallest prime and thus the smallest possible value of a_1, this holds true.