Arbitrary Real Bases

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Hypothesis

Any real number other than 1, 0, and -1 can be used as a base in a numbering system in order to portray any desired real number in the same capacity as decimal.

Note

I will be portraying all numbers in this paper in base 10 (decimal) for simplicity unless expressly stated otherwise. I will also be using the term “decimal” to represent values following a radix point, such .1415. In standard Arabic numbering systems, every base is base 10, assuming the 10 is written in the given base, which is why I am making this distinction.

Background

A base in a numbering system is the number that each digit is multiplied by to the power of the position of the digit. In a numbering system that uses standard digits, the base will always be represented as 10, or $1*base^1 + 0*base^0$. The “least significant digit”, or the farthest right digit of the integer portion of the number will always be to the power of 0, and going to the left of that by one digit will increase the power of the number by 1, while going to the right will decrease the power by 1. An example of this could be 10.1, where the value is $1*base^1 + 0*base^0 + 1*base^{-1}$.

1. Natural Bases

The digit set for integer bases is trivial. You use the same amount of digits as the base denotes, from 0 to base-1. For instance, base 2, or binary, you can use 2 digits, 0 and 1. With this property you can deduce that a 1 in the second digit should have a higher value than a (base-1) in the first digit. As an example, 9 is less than 10. This can be expanded easily in integer bases by stating that a 1 in the Nth place followed by 0s to the radix point will always be greater than a number that is (N-1) (base-1)s, specifically by exactly 1. As an example in decimal, 1000 is exactly 1 greater than 999. In a binary system, 100 (4 in decimal) is exactly 1 higher than 11 (3 in decimal). As can be imagined, this creates a system where you can trivially display any integer in 1 way, as each integer power greater than or equal to 0 will result in an integer if performed on an integer. Adding a combination of these integers will result in an integer.

This is where the trivial facts stop, as now we must deal with portraying non-integer real numbers. The principle is very similar to portraying integers, but there are limitations which create confusion. There are some trivial fractions to display, such that in a given base, 1/base will be equal to 0.1 and 1/(base-1) will be equal to $0.\overline{1}$. For even bases, ½ in decimal would be equal to 0.(base/2). For bases which are multiples of 3, ⅓ in decimal would be 0.(base/3). Naturally, this can be expanded to arbitrary values. We can say that in a given base, if said base is a multiple of N then 1/N in decimal would be equivalent to 0.(base/N). Also, If 1/X and 1/Y are both terminating, then 1/(Y*X) will also terminate. These rules will help you find all terminating values of 1/N.

All numbers which can be displayed as fractions will have either a terminating decimal or a repeating decimal in any integer base. Therefore, if a 1/N fraction does not follow the prior rules it will be a repeating decimal. In order to find values of X/N, you can add the values of 1/N until you arrive at X/N.

As for representing negative values, there are multiple methods. Computers tend to use a “sign bit” which is a bit that is 0 for positive and 1 for negative, while human systems tend to use a dedicated symbol to denote negative values, typically the minus sign or dash, “-”. Adding this rule we can now represent all rational numbers.

The final piece of the integer bases puzzle is irrational numbers. These can never be truly represented in integer bases, but they can be calculated with arbitrary accuracy by continuously expanding the decimal to just below the true value of the number you wish to represent. To illustrate this point with pi in decimal, you would start with the integer value of the irrational, 3, and go from there. You would increment the next digit to the right until you are over the value of pi, then decrement one and repeat the process with the next digit. The first decimal digit is 1, so you would start at 0, realize that you are under the value, add one to get one, realize you are still under the value, add one, notice that you have surpassed the value of pi, so you decrement to one and move on, repeat the process of 0, 1, 2, 3, 4, 5, decrement to get 4, and so on until you get as close as you need to be. This is obviously a very drawn out and slow process, but it does allow you to get arbitrarily close to irrational numbers, given the desired number has enough known digits to be converted accurately.

2. Positive 1/N Bases

Fractional bases in the form 1/N are the next topic that will be covered. This base is almost trivial, as in a 1/N base, you can just pivot the same number in base N around the 0th power while keeping the radix point after the 0th term. As an example, 314.15926535 in base 10 is equal to 535629514.13 in base 1/10. This leads to an interesting scenario with irrational numbers, as instead of having a non repeating and non terminating decimal, you have the same properties in the integer portion. This is no more of an issue than with integer bases, however, as you can get arbitrarily close to the desired value. Otherwise you can follow the same rules as integer bases as long as you remember to perform the pivot.

3. Natural Bases, Cotd.

Throwing it all away

The prior entry was following typical conventions of bases. If you expand your definition of a base, non-integer bases become much easier to work with. To be more specific, you can technically use more digits than the typical definition of a base would allow, it would just cause ambiguity in integer bases because multiple values could be represented in the same way. For instance, in binary 1011 is equal to 11 in decimal, but with our newly expanded definition of a base, so is 51.

4. Natural Square Root Bases

The prior paragraph may seem confusing and highly arbitrary, which admittedly it may be, but it leads very well into the next topic, bases that are the square root of a natural number. The most well known example of this is the base sqrt(2), but any square root of a natural number would work in the same capacity. Square roots of perfect are easy to work with, as they are just integers, which were previously described. Non-perfect roots are much more interesting, yet still surprisingly intuitive. You can convert any base into its square root counterpart by inserting a zero before each digit. For example, 109 in decimal becomes 10009 in base sqrt(10). Logically, this makes sense, as if you preserve the even powers and set the odd powers to 0, you are calculating the number in the exact same way you would in an integer base. This is obviously only roughly half as efficient as the integer counterpart, but you can represent the root itself with only 2 digits, which is pretty nice. For an irrational example, pi in base sqrt(10) is approximately 3.01040105090206050305. You can also play around with the digits with odd powers to experiment with irrational numbers, assuming N is not a perfect square, but this can be avoided to make life easier. You can also opt to use values between the zeros, giving you free range to use values relating to the square root value, but it obviously gets quite complicated very quickly.

The reason I brought up using numbers larger than the base to represent it is that it ties together nicely with the bases that are the square roots of perfect squares.

Rather than converting a perfect square root into an integer, you can follow the same conventions used for irrational roots. For example, 4 is 2 squared, so you could just do base 2 instead of base sqrt(4), but that doesn't mean you can't use base sqrt(4). As an example, if you wanted to represent the decimal value 14 in base 4, you would write it as 32. Following the irrational base protocol, you can represent the same value in base sqrt(4) as 302. Recalling that sqrt(4) is equal to 2, you can see that 302 is a perfectly valid way of representing 14 in binary, or base 2, as long as you accept the expanded definition of a base.

5. Negative Integer Bases

If everything so far has not made sense, then be warned that it only gets worse from here. Our next challenge to tackle are negative integer bases. If only following the strict definition of a base first mentioned, then there are still two ways to represent any value in a negative integer base. As an example in base -10, the decimal value 10 can be represented as either -10 or 190. This can be explained through the oscillating nature of powers of negative numbers. The odd powers are all negative while the even powers are all positive. You can use this to find all integers. When looking for 10 in base 10, you look for 2 values, a positive and a negative. If the number is on the same level of an odd power of the base, I.E. 10 is 10^1 and 1000 is 10^3, then it is typically easier to look for the negative value. 20 in base 10 is just -20 in base -10. For the other value, you find the lowest positive power that is higher than the number you are looking for, set that to 1, then set the next digit to base-(the number you are looking for). An example of this is, again with 20 in base 10, you can set the 2nd power to 1 and the next value to 10-2, or 8, and get 180 as a valid result.

The oscillating pattern of negative bases can make irrational numbers a pain to deal with, but they are possible nevertheless. As it is the most well known irrational number, I will use pi in base -10 as an example. In base -10, pi can be represented as either 4.959613467610207 or -17.262408754591813. In order to come to these values, you have to decide which one you will look for first, the positive value or the negative value. I will use the positive value as an example. You start with the integer portion of the number you are attempting to show and add one, for reasons that will be clear as we continue. The first digit after the decimal place will be a 1 in base 10, so since -10^-1 is negative, you have to think of it as abs(base)-(the value that you are looking for), or in this case 10-1, 9. The next number is a positive number because -10^-2 is positive, so you just transcribe the number. The next number is once again negative, so you find the 10-N value and add one to the previous number. This continues to the desired degree of accuracy. The process is similar for the negative value. Since you are able to find irrational numbers, you are able to find rational numbers just as easily.

If you find the idea of multiple representations for a single value, then the standard decimal base system is not fit for you either. Take, for instance, the equality of 1 and 0.999…, two representations of the same number.

6. Negative Natural Square Root Bases

Not to be confused with square roots of negative numbers, this section is talking about scenarios such as sqrt(2)*-1. This base is surprisingly easy to deal with, if you understand positive natural square root bases. That is because the implementation is exactly the same. Since in both square roots and negative numbers as bases you try to focus on the digits with even powers, you can interpret a number in precisely the same way in both positive and negative square root bases. You will obviously find curiosities in the odd powered digits, but it is perfectly reasonable to just ignore them if you want to make things less confusing for yourself.

7. All Natural Root Bases

The title on this section may be slightly confusing, but it essentially serves as a generalization to sections 4 and 6. While those two sections focus specifically on square roots, this section focuses on any natural root of a natural number, both positive and negative. All of the rules for square roots apply here as well, the only difference is that the number of 0s preceding each valuable digit is N-1 rather than just 1. While pi in base sqrt(10) is approximately equal to 3.01040105090206050305, pi in base $10^{1/3}$ is 3.001004001005009002006005003005. This is all perfectly applicable to the negative roots as well, -(root(N)), but only when the root you are applying is an even number. When the root is odd, I.E. cubic roots, negative values become slightly more difficult. Instead of using the standard values of a number in base N and adding the appropriate number of zeros, you must take the value of the number in base negative N. Therefore, pi in base $-(10^{1/3})$ is approximately 4.009005009006001003004006007005. This can obviously also be derived from the negative value of pi in base -10 to get a second value.

As you can probably already tell by the given examples, N root bases become quite unwieldy rather quickly. The same rule applies as in square root bases, you can fiddle with the 0 digits, but it is generally confusing and ill advised.

8. Negative 1/N Bases

This section predictably relies heavily on sections 2 and 5. You can effectively represent any number in base -1/N if you know how to represent said number in base

-N and have an understanding of 1/N bases. The easiest way to perform the conversion is by first finding the value you want in base -N. After this, you treat the number much like a positive 1/N base, you pivot about the 0th power term, keeping the radix point after said term. Much like negative integer bases, you can also represent a number in 2 ways, one positive and one negative. As an irrational example, pi in base -1/10 can be either -3181954578042627.1 or 7020167643169594. Again being related to base 1/N, irrational numbers become interesting to represent, as they become a non-terminating integer value rather than a non-terminating decimal. The same reasoning applies that this effectively does not matter, as you can get just as arbitrarily close to representing the value as you can in an integer base.

9. (+/-)1/(Nth Root(X)) Bases

In this terribly titled section, the topic is positive and negative bases in the form of the inverse of the Nth root of X, where N and X are both natural numbers. This is surprisingly easy, given you understand prior topics. All of the respective rules to prior topics remain the same, most importantly sections 7 and 2. You find the values the same way that you would in section 7, and then pivot them about the 0th term, just like in section 2. As an extreme example to show that irrational numbers can be shown in this way, in base $-1/10^{1/3}$, pi is approximately either 1006009005009004 or its negative equivalent.

10. Positive Irrational Bases

Irrational numbers are numbers that can not be expressed as a ratio of two integers in standard bases. In other words, there is no fraction that can completely describe an irrational number. This naturally means that all irrational numbers are non-terminating and non-repeating decimals. A few examples of this in base 10 are π and √2. This property makes irrational number bases particularly unwieldy in calculating just about any value other than the number itself. That being said, it is still possible. As my first example, I will use base π. In an irrational system, base 10 integers and other typically easy to display values become irrational. Keep this in mind as examples are given, they are not the entire answer, they are just approximations to the accuracy of python floating point arithmetic. This is also mostly untreated territory, so I am taking the liberty to do as I please as long as the math checks out in the end. While I could use the same process outlined in section 3 to easily represent integer values, I will not for the sake of demonstration. The integers 1, 2, and 3 in base 10 remain the same in base π, but after that they get much more interesting. 5 in base π is 11.220122021121110301000010110010 and 10 is 100.010221222211211220011112102020. π is by definition 10, and by the same logic, τ is 20. One half (base 10) is equal to

0.112112021020122300010100102110 in base π and finally to represent another irrational number, φ is 1.123000030100021212100211200122. This logic, as you would expect, is not limited to π and extends to any irrational number, I just do not see the necessity in showing all of the same values in another base. There is not an easy way to calculate values in irrational bases other than basically by hand, so there is not any good reason to use these bases other than base e, but they are, in my opinion, fairly interesting. Base e is theoretically useful because it has the best radix economy of any base, but again the calculations make it practically useless.

11. Non 1/N Fraction Bases

This section is harder than would probably be expected, considering how relatively simple other fractional bases are. However, there is not a general rule to follow with these bases. Much like most of the prior categories, I will be taking several liberties to make the work manageable. Numbers will also probably not look particularly nice. To start with, I will use base 10/7. There is no particular reason for this, it was just the first value to come to my mind. Once again I will not be taking the section 3 shortcut. 10/7 comes out to 1.428571 repeating, meaning that for numerals I will only be using 0, and 1, just like binary. This makes representing 1 trivial, as it is the same as in decimal. 1.5 is 10 in base 10/7, again by definition. 2 is the first nontrivial integer, with a value in base 10/7 of approximately 10.0100000010010000100001000010001. 7 is roughly 100001.00000000100010000000000000001, and 10 is around 1000010.000000010001000000000000001000. For a fractional example, 0.5 is near 0.0100000000000100000000010000010. Finally, an irrational number example with π being approximately 1000.000010010000000000000001000000010010000010000001000100. Honestly I was unable to find any rhyme or reason to these values other than the fact that they work mathematically, so there is not a set pattern to follow other than to just do the math out.

0. Incomplete Bases

There are three real bases that are definitely incapable of representing the reals, 1, -1, and 0.

In base 1, you can represent any rational number. Any radix point in base 1 is completely useless, as 1 to any negative power is still 1. 0s are also completely useless in this system. As for how you use the base, in order to portray a given integer you must write that amount of 1s, it is more of a tallying system than a numerical system. If the integer you wish to portray is negative then you can just negate the tally, such as 3 being 111 and -3 being -111. Going further, you can represent any rational number as a ratio of integers, by definition. 1/11 in base 1 is equally as valid as 0.5 in base 10. Using rational numbers you can approximate irrational numbers, but you can not do any better than that.

Base -1 is technically just as capable as base 1, being able to portray any rational number. That being said, it is more complicated. In order to count up, you start with a 1 in the 0th power and place a 1 in each even power until you get the desired amount of 1s. In order to get negative numbers, you can either negate the positive value or have a 1 in each odd power up to the desired value. You can clearly also negate this value to achieve a positive integer. The radix point in base -1 is also meaningless, but you can once again use ratios to express rational numbers. These properties make it worse than base 1 in just about every way, but at least it doesn't question mathematics at a fundamental level, unlike the next and final entry.

The final base in this category is the “impossible” base, base 0. If you can not see the problem with this base, I will point it out. Technically speaking, you can either get all rational values or no values out of this strange base, depending on your view on mathematics as a whole. This discrepancy comes from the definition of exponentiation. There are justifications for two different answers to 0 to the power of 0. Some argue that the expression is undefined, while others argue that the answer is 1. If you fall into the former category, then you will not get any use out of base 0, as the least significant digit involves multiplying by 0 to the power of 0, an indeterminate form under this definition. On the other hand, you can get any rational value (following the logic that was introduced in section 3) because the least significant term will still be 0 to the power of 0, or 1 under this understanding. You can thus set any integer to this digit. This can then be negated to get any negative integer value. You can then use ratios of these integers to get any rational number. Outside of this one curiosity, every other digit is meaningless, as 0 to the power of any nonzero natural number is universally accepted to be 0. On the other side of the radix point, you physically can not get any values, because 0 can not be raised to a negative power, as it results in division by 0. This is, of course, another indeterminate form.