Gravitational Energy in General Relativity: An Historical-Critical Approach

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Abstract

We present an analysis of the problem of gravitational energy in general relativity (GR) by considering its foundations in an historical-critical fashion. By analyzing the concept of 'gravitational field', and arguing by analogy with arguments deriving the energy density of the classical electric and magnetic fields, we show that any gravitational field energy density must be negative. This raises interpretational questions about how this negative energy density modifies the spacetime, and the additional question of whether or not Einstein's equations as they are resolve such questions. Further, by analyzing the gravitational equivalence principle and the resulting Newton-Cartan chronogeometrized theory of gravitation in comparison to the classical concept of a gravitational field, we argue that, contrary to conventional interpretations of GR, the ontology of the gravitational field ought to be null in any theory that treats gravitation as a manifestation of the interrelations of spacetime points. On this basis we argue that energy cannot be globally conserved, even if it is by construction conserved locally, in any presently-existing geometrized theory of gravitation. We conclude the paper with a discussion of two possible routes of further interpretation of GR given the analysis presented here.

Contents

1 Introduction

In special relativity (SR), matter and energy are postulated to be equivalent. In general relativity (GR), it is stated that all forms of matter/energy content of the spacetime contribute to modifying the spacetime structure itself, and gravitation is described as a manifestation of the structure of spacetime so modified. Prior to GR, our theory of gravity was Newton's Universal Law of Gravitation, which included the concept of gravitational potential energy. The more advanced formulations of Newtonian gravity included the concept of the gravitational field as a local entity giving rise to the gravitational force on test particles. It is well established that, in electromagnetic theories, electromagnetic potential energy is stored in electromagnetic fields, entities analogous to the gravitational field. Although gravitation is described in GR as a manifestation of the structure of spacetime, conventional interpretations of the theory take the gravitational field to be an entity from which the spacetime structure is derived. In the words of Einstein himself: "spacetime does not claim existence of its own, but only as a structural quality of the [gravitational] field" (Stachel , 2014). The conceptual difficulty then arises: how does the energy of the general-relativistic gravitational field, from which spatiotemporal structure is derived, itself modify the spacetime structure, i.e. itself? Where is the energy of the gravitational field located, and how should we account for this "self-action," sometimes said to be responsible for the non-linear nature of Einstein's field equations, in determining fully rigorous solutions of general-relativistic spacetimes? (Weiss-Baez, 2017).

An analysis of this problem at the fully general-relativistic level is difficult because the mathematics underlying GR are difficult and GR's background-independent nature makes any analysis inherently self-referential (i.e., when and where things are and the very meaning of when and where must be solved for immanently and self-consistently). With such considerable challenges, it is perhaps no surprise that there have been numerous contributions to the discussion of gravitational energy, but as yet no consensus. (Xulu , 2003).

It has been argued that emphasizing the logical independence of the gravitational equivalence principle and the postulates of SR can improve the pedagogy of GR in (Stachel , 2006) by introducing the idea of a curved spacetime independently of the fundamental velocity of SR and GR. (Cartan , 1923) demonstrated that we may chronogeometrize gravity while preserving absolute time and Euclidian geometry as postulates of the nature of space and time with the definiton of an affine spacetime. (Stachel , 2006) extended this include non-zero $\Gamma^{(a)}_{(0)(b)}$ and nonzero $\Gamma^{(a)}_{(b)(0)}$ in the tetrad components of the affine connection and used this chronogeometric structure to develop equations that give solutions that can serve as the near-field, low-velocity limit of solutions to GR that include the frame dragging of gravitomagnetism.

Whereas (Stachel , 2006) introduced a gravitomagnetic extension of (Cartan , 1923) with the constants of the equations set up to serve as a first-order solution to Einstein's equations that could then be iteratively improved upon, we here represent his argument only insofar as it is relevant for a spacetime equipped with affine structure, and use this to inform our investigation of gravitational energy in GR. Specifically, we analyze the problem by considering classical concepts of gravitational energy in comparison to the gravitational equivalence principle and the chronogeometrized, non-metric (i.e. without the postulates of SR) theories of gravity founded thereopon. This limits the scope of the analysis to the conceptual consequences of gravitational equivalence principle alone, in the hopes of shedding some light on the problem without tackling it in its entirety.

Before considering the mathematics, we wish to clarify a choice of terms out front. Sometimes, we use the term “geometry”, and by this we mean some relational structure posited over a set of points embedded in an n-dimensional manifold. Sometimes, we use the term “chronogeometry”, or the like, and by this we mean such a structure with 4-dimensions meant to represent a spacetime. We distinguish the two usages because, although the mathematics we will use treats time and the spatial dimensions as no different in essence, we follow Kant in the position that our intuition of time and intuition of space are different and irreducible to one another (Kant , 1977); thus, we find it important to flag that **both** our temporal and our spatial intuitions inform our use of a 4-dimensional manifold to represent reality.

2 The Inertial, Motional Tendencies of Matter

Premised on the experimentally demonstrated equivalence of inertial and gravitational mass, the gravitational equivalence principle would have been conceptually meaningful even before the advent of the special theory of relativity: The fact that the effects of gravitation and the effects of being in an accelerated reference frame are locally indistinguishable forces us to assign these two seemingly disparate processes the same physical significance, independent of any of the considerations informing SR.

In traditional Newtonian theory, for a given gravitational field $\vec{g}$ and test particle,

$$m_I\vec{a} = m_G\vec{g}$$

, where $\vec{a}$ is the 3-acceleration of a test particle w.r.t. the center of mass of the matter giving rise to the gravitational field $\vec{g}$, m_I is a determiner of the 3-acceleration of test particle in response to a given external force, i.e. the test particle's inertial mass, and m_G is the gravitational mass or, in analogy with electromagnetism, gravitational “charge” of the test particle that determines the magnitude of the gravitational force acting on the test particle. Experimentally it has been shown to a high degree of precision that $\vec{a} = \vec{g}$ and thus $m_I = m_G$ for every test particle.

This forces us to reconceptualize the motional tendencies of matter as influenced by the force of gravity, in Newtonian theory, as merely the inertial motional tendencies of matter with respect to so-called inertial frames of reference in Newtonian theory. To be in a frame in which, locally, there is a so-called gravitational force must be reinterpreted as being in a frame which has a non-zero four-acceleration.

First, to understand how we can even begin to formulate this statement mathematically, we must make a minor, albeit abstract, change to how we usually think about time, space, and motion. In Newtonian mechanics, it is taken to be self-evident that there is a global time that is the same for all observers, and things move through space, occupying various positions as a function of this global time. However, we can just as well say that things occupy various positions in space *and* they ‘occupy’ various ‘positions’ in time, i.e. different times.

Although our concept of 'motion' is logically analyzable into a succession of places as a function of time, we can modify this concept and say that, in a different way, objects 'move' through the spacetime, occupying, throughout their histories, different places and different times. The collection of different places and different times an object is at throughout its history is called that object's *world line*. We can then make the statement that the object moves through space and concurrently 'moves' through time by introducing a parameter that corresponds to different points in the worldline as it increases, i.e. some function $X^\mu(\lambda)$ that gives the spatiotemporal situation of the object as a function of the parameter λ, where X^0 will correspond to the time of the object multiplied by some spacetime conversion factor to make all four coordinates of the object commensurate, and $X^i, i = 1, 2, 3$ are the three spatial coordinate positions of the object. The statement that the object 'moves' through space and through time can then be made mathematically that the object has a *four-velocity* $W^\mu := \frac{dX^\mu}{d\lambda}$.

Such considerations do not require us to impose any 4-metric structure on the spacetime − this is only demanded by the need to construct an invariant spacetime interval under Lorentz transformations in order to preserve the fundamental velocity of SR in various bases / coordinate systems. Thus, we are free to reformulate Newtonian theory chronogeometrically with the above explicated insights.

The way we will do this mathematically is by the imposition of non-flat affine structure on a spacetime manifold. A spacetime manifold is a collection of spacetime points, i.e. times and places. In order to describe the interrelations of the points in a manifold with affine structure, we must first define an affine space. A global affine space is an n-dimensional space that maintains all postulates of Euclidian geometry without any metric and hence no concept of length or angles between vectors. Formally, if we consider an n-dimensional vector space $\vec{U}$, and a n-dimensional set of points $\mathbf{A}$, $\mathbf{A}$ is an affine space if for every element $\vec{v}$ and $\vec{w}$ of $\vec{U}$ and for every element p of $\mathbf{A}$, then:

$$p + \vec{v} \in \mathbf{A}$$

$$(p + \vec{v}) + \vec{w} = p + (\vec{v} + \vec{w})$$

$$p + \vec{v} = p \leftrightarrow \vec{v} = \vec{0}$$

and for all p,q $\in \mathbf{A}, \exists\ \vec{v}$ such that $\vec{v} = p - q$

An affine transformation $\mathbf{A} \rightarrow \mathbf{A'}$ preserves these properties, i.e. the coordinates of the new points in **A'**, $X^{\gamma'}$, are related to the coordinates of the points in the original $\mathbf{A}$, X^γ, via

$X^{\gamma'} = \frac{\partial X^{\gamma'}}{\partial X^\gamma} X^\gamma + C^{\gamma'}$

where $\frac{\partial X^{\gamma'}}{\partial X^\gamma}$ defines the transformation matrix, the $X^{\gamma'}$ are linear in the X^γ, and $C^{\gamma'}$ is a translation of the points.

As an aside, throughout this paper, we use the convention that Latin letters indicate enumeration over the three spatial components of 4-vector or 4-tensor, while Greek letters indicate enumeration over all 4 indices. We will also use the Einstein summation convention that repeated indices indicate summation over the elements enumerated.

In a global affine space, parallelism is a well-defined concept for any two vectors in the space – two vectors are parallel if one is a linear rescaling of the other. In a manifold, however, we do not make this assumption. Specifically, **if** $\vec{u}$ and $\vec{v}$ are at different points of the manifold, we do not assume that a linear rescaling $\vec{u}$ of a vector $\vec{v}$ is parallel to $\vec{v}$. We can allow the manifold to have this non-flat structure and still require that the tangent spaces of every point in the manifold are affine and thus have a meaningful concept of distant parallelism. A tangent space is the space spanned by the collectivity of the tangent vectors to any possible curve through that point. We can then take this locally defined tangent space to be centered affine, i.e. an affine space that has a fixed origin, in order to ensure that the space remains a description of the tangent space at the particular point of the manifold at which it is defined. This eliminates translations from the allowable coordinate transformations of the tangent spaces, i.e. we only consider $X^{\gamma'} = \frac{\partial X^{\gamma'}}{\partial X^{\gamma}} X^{\gamma}$. In order to describe the relations of spacetime points in a manifold with affine structure, we need to be able to connect neighboring tangent spaces. This we will do shortly.

We first turn to further characterize what we should require of the manifolds we will consider. In order for a spacetime manifold to be Newtonian, we demand as conditions on the manifold that the rate of any observer's passage through the 0th dimension, i.e. time, is the same regardless of their frame of reference, and that the 3-geometry of hypersurfaces that are transvected by the 0th dimension is affine flat, which is the same as to say Euclidean (although non-metrically).

The basic idea of what we mean by an 'observer' is the usual sense of this word: one who observes nature. Mathematically, then, it is most natural to define an observer by their world-line, i.e. the collection of times and places the observer is at relative to the other processes of nature she observes. The term 'frame of reference' also needs a mathematical definition. Roughly, a frame of reference is what we say an object is moving with respect to, i.e. it is the scheme of naming when and where certain processes occur relative to a certain reference point and reference time. Mathematically, we can define a frame of reference globally, by either the spacetime coordinatization of all points in the manifold or with a field of tetrad vectors, i.e. a set of four linearly independent vectors at every point of the manifold. We can locally define a frame of reference at a point (i.e. single time and place) by a coordinatization of the tangent space of the manifold or specification of a tetrad at that point. We can now describe the *local* frame of reference of an observer by a coordinatization of the tangent spaces at all points or specification of a tetrad for all points on the observer's world-line.

Following our definition of reference frames via tangent spaces or tetrad fields, we can also define a global frame of reference by a coordinatization of the tangent spaces of every point of the manifold. See Figures 1 and 2. A subset of such frames are the traditional Newtonian inertial frames of reference. That is, frames, defined either locally or globally, in which there is no absolute rotation and zero three-acceleration w.r.t. the center of mass of the source matter. We will characterize these frames of reference in terms of our definitions of an affine space shortly.

Now we are ready to characterize our manifold mathematically. In order to represent a spacetime, let us construct a manifold M that is topologically homeomorphic to $\mathbb{R}^4$, and a foliation of M given by a differentiable function $T(X^\mu)$, where $T(X^\mu)$ represents absolute time as a function of the coordinates

X^μ of the points in the manifold. If we introduce the covector $T_\mu = \partial_\mu T$, we can distinguish a 4-vector B^μ as space-like if $B^\mu T_\mu = 0$, future time-like if $B^\mu T_\mu > 0$, and past time-like if $B^\mu T_\mu < 0$. Later, we will make T_μ one of a set of 4 basis covectors, each with units of inverse length. In order for T_μ to have units of inverse length, T must be unitless. Since the X^0 represent the 'temporal coordinates' in a framework that includes absolute time, though, their values will be numerically equal to the values of T up to a linear rescaling and choice of zero-point, i.e. $\partial_0 T = const.$, even though the units of the X^0 are length. It is therefore simplest just to enforce numerical equality between the X^0 and T. From these definitions, the level surfaces $T = const.$ define the level surfaces of global simultaneity.

If we are to allow the tangent spaces of the manifold to differ, then the ordinary concept of a directional derivative $V^\nu \partial_\nu = V^\nu \frac{\partial}{\partial X^\nu}$ is no longer meaningful, as for some vector B^μ at a point in the manifold, if we consider an arbitrary coordinate transformation (i.e. not necessarily affine, since we are considering a manifold and not a global affine space) $X^{\gamma'} = \frac{\partial X^{\gamma'}}{\partial X^\gamma} X^\gamma$ then $\partial_\nu B^\mu$ is related to $\partial_{\nu'} B^{\mu'}$ via

$$\partial_{\nu'} B^{\mu'} = \frac{\partial X^\nu}{\partial X^{\nu'}} \partial_\nu \left(\frac{\partial X^{\mu'}}{\partial X^\mu} B^\mu\right) = \partial_\nu B^\mu \frac{\partial X^{\mu'}}{\partial X^\mu} \frac{\partial X^\nu}{\partial X^{\nu'}} + \left(\frac{\partial^2 X^{\mu'}}{\partial X^{\nu'} \partial X^\mu}\right) B^\mu$$

, and $\frac{\partial^2 X^{\mu'}}{\partial X^{\nu'} \partial X^\mu}$ is in general non-zero if the tangent spaces differ, and thus $\partial_\nu B^\mu$ is not a tensor. In simpler terms, it is not enough to evaluate the change in the components of a vector in a given direction, as the direction represented by an increase of a coordinate can change from point to point in the manifold. In other words, the direction of **basis vectors** naturally adapted to the coordinate system and/or geometric structure changes from point to point in the manifold.

We need to find an operation analogous to ∂_ν that accounts for differences in neighboring tangent spaces. Mathematically, we must search for an expression that is tensorial and a natural extension of ∂_ν. For readers unfamiliar with what it means for a quantity to be tensorial, it means that the $\frac{\partial^2 X^{\mu'}}{\partial X^{\nu'} \partial X^\mu}$ is not present in the expression for how the geometric object considered is described under a new system of coordinates. This is desirable because this term is a relic of differences in the tangent spaces from point to point in the manifold, and not owing to differences in quantities defined over the manifold.

It is easy to show that we can introduce an operation D_ν on a contravector B^μ , called the covariant derivative, which is tensorial and related to $\partial_\nu B^\mu$ by a geometric object $\Gamma^\mu_{\kappa\nu}$ via $D_\nu B^\mu = \partial_\nu B^\mu + \Gamma^\mu_{\kappa\nu} B^\kappa$, where $\Gamma^\mu_{\kappa\nu}$ is required to have the transformation property of

$$\Gamma^{\mu'}_{\kappa'\nu'} = \Gamma^\mu_{\kappa\nu} \frac{\partial X^\nu}{\partial X^{\nu'}} \frac{\partial X^\kappa}{\partial X^{\kappa'}} \frac{\partial X^{\mu'}}{\partial X^\mu} - \left(\frac{\partial^2 X^{\kappa'}}{\partial X^\nu \partial X^\kappa}\right)\left(\frac{\partial X^{\kappa'}}{\partial X^\kappa}\right)\left(\frac{\partial X^\nu}{\partial X^{\nu'}}\right)$$

,

so that $D_\nu B^\mu$ transforms to

$$D_{\nu'} B^{\mu'} = \partial_\nu B^\mu \frac{\partial X^{\mu'}}{\partial X^\mu} \frac{\partial X^\nu}{\partial X^{\nu'}} + \Gamma^\mu_{\kappa\nu} \frac{\partial X^\nu}{\partial X^{\nu'}} \frac{\partial X^\kappa}{\partial X^{\kappa'}} \frac{\partial X^{\mu'}}{\partial X^\mu} B^\gamma \frac{\partial X^{\kappa'}}{\partial X^\gamma}$$

,

which is the transformation property of a tensor. $\Gamma^\mu_{\kappa\nu}$ can be thought of as a geometric object connecting neighboring tangent spaces of the manifold, so that expressions for changes in directional quantities defined over the manifold account for differences among the tangent spaces. Thus, it is called an affine connection.

It may be helpful for understanding this object to think of it in terms of autoparallel transport – a vector in a manifold that 'moves itself' to the neighboring tangent (affine) space of the manifold. In the case of the surface of a sphere, perhaps the easiest to visualize non-flat manifold, a given vector autoparallel transporting itself defines the curve of a great circle on the sphere. If we consider a coordinatization in which a vector on the surface of the sphere takes itself around the so-called equator of the sphere ($\varphi = 0$ to $\varphi = 2\pi$), then the components of this vector will be constant – 0 in the θ (longitudinal) direction and some non-zero constant v^{φ} in the φ (latitudinal) direction. However, if we consider the same great circle after a change of coordinate system that rotates the so-called north pole of the sphere, then neither the θ nor the φ components of the autoparallel transported vector defining this curve will be constant, but the curve will still be just as much an autoparallel in the space. Thus, constancy of the coordinate components of a tangent vector is not a sufficent condition for defining an autoparallel in non-flat manifold. The introduction of $\Gamma^{\mu}_{\kappa\nu}$ makes the above two descriptions of the same autoparallel curve consistent, as it offers a description of parallel vectors from point to point in the manifold, i.e. it connects tangent spaces of the space, in a way that is independent of the coordinatization of that space. See Figure 3 for a visual demonstration of this argument and (Wolfram , 2017) for the components of the connection.

In a spacetime manifold, the equations of motion that are invariant for all spacetime coordinatizations, i.e. all frames of reference, of a four-force-free test-particle with a four-velocity $W^{\kappa} := \frac{dX^{\kappa}}{d\lambda}$ (here $X^{\kappa}(\lambda)$ are the coordinates of the spacetime curve the particle follows, rather than the coordinates of all points in the manifold), are given by the autoparallel condition:

$$W^{\nu} D_{\nu} W^{\kappa} = 0 \rightarrow$$

$$\frac{d^2 X^{\kappa}}{d\lambda^2} + \Gamma^{\kappa}_{\mu\nu} \frac{dX^{\mu}}{d\lambda} \frac{dX^{\nu}}{d\lambda} = 0 \tag{1}$$

,where λ is the preferred parameter of the curve. Conceptually, this states that W^{ν} parallel transports itself through the spacetime, and moves along this curve as we increase λ.

This expression has been traditionally called the geodesic equation. However, its introduction only requries the idea of autoparallel transport in a manifold with affine tangent spaces; it does not require any metric structure. Thus, it is most properly called the autoparallel equation. More generally, $W^{\nu} D_{\nu} W^{\kappa} = A^{\kappa}$ is the four-acceleration of a curve.

2.1 Newton's Three Laws

Newton's First Law as he originally concieved it can now be thought of as the autoparallel condition in a globally flat affine spacetime, in which all $\Gamma^{\mu}_{\kappa\nu} = 0$ and $\frac{d^2 X^{\kappa}}{d\lambda^2} = 0$ for some spacetime coordinate system X^{μ} (here referring to the coordinates of all points in the frame of reference and not the coordinates of a curve), and additionally for every choice of coordinates related to this coordinate system in such a way that $(\frac{\partial^2 X^{\mu'}}{\partial X^{\kappa} \partial X^{\nu}})(\frac{\partial X^{\kappa'}}{\partial X^{\kappa}})(\frac{\partial X^{\nu}}{\partial X^{\nu'}}) = 0$. From this it follows that $\frac{\partial^2 X^{\kappa'}}{\partial (X^0)^2} = 0$, among other conditions on the coordinate transformation. Physically, we can understand this as saying that the first coordinate system corresponds to some frame of reference in which Newton's First Law holds, i.e. some inertial frame of reference, and every coordinate system which is not accelerated w.r.t. this coordinate system also corresponds to an inertial frame of reference in which all $\frac{d^2 X^\kappa}{d\lambda^2} = 0$. See Figure 4. In this sense, Newton's First Law, at bottom, defines the frames of reference in which Newton's Second Law is valid. In the language of geometry, these are the set of spacetime coordinate systems naturally adapted to the affine-flat structure of the spacetime.

Newton's Second Law is then that the deviation of a particle's worldline from autoparallel transport in an affine-flat spacetime is proportional to the net force acting upon that particle and inversely proportional to the particle's mass: $\vec{a} = \frac{\vec{F}}{m}$. See Figure 5. Since the deviation from affine-flat autoparallel transport is the same for all particles being acted upon by the gravitational field, it is simpler to reconceptualize the deviation from affine-flat autoparallel transport as simply the new autoparallel transport of the spacetime, something we will discuss in greater depth in the next section.

Newton's Third Law, combined with his first two, is then that the law of affine-flat autoparallel transport, i.e. momentum conservation, holds for the center of mass of a closed system. When the autoparallels are modified by gravitating matter, the center of mass of a closed system still follows autoparallel spacetime curves, although in curved spacetime. Noteworthy, though, is that a component of Newton's Third Law is reconceptualized: the recipricol nature of gravitational interactions is simply that everything modifies the spacetime structure in exactly the same fashion (i.e. determines the local value of the affine Ricci tensor, as shown in the section 3), rather than two gravitating particles mutually pulling each other from autoparallel transport in an affine flat spacetime.

2.2 Non-Flat Manifolds

In the case where the manifold is non-flat, we are always free to pick a frame of reference along a test particle's world-line which is co-accelerating with the test particle, in which case, locally $\frac{d^2 X^\kappa}{d\lambda^2} = 0$ and thereby all $\Gamma^\kappa_{\nu\mu} = 0$. This defines the only frame of reference which resembles the concept of a Newtonian inertial frame. More generally in this framework, the autoparallel condition, as given by equation (1), is always true, but it is not always true that $\Gamma^\kappa_{\nu\mu} = 0$ *and* $\frac{d^2 X^\kappa}{d\lambda^2} = 0$. That is, one can, by a suitable transformation of coordinates, change what was attributed to the gravitational force in one frame as merely inertia in the transformed frame. While in Newtonian theory, it is posited that there is an underlying inertial structure, and a gravitational field deflecting particles from what would be otherwise inertial tendencies, the gravitational equivalence principle informs us that these are one and the same structure on the spacetime. Instead of being characterized independently by inertial structure and a gravitational field, there is inertiogravitational structure, or an inertiogravitational field if you rather, on the spacetime, definable rigorously in terms of the affine connection.

In order to offer a coordinate-free description of the connection and autoparallel equation, we now introduce a triad of spatial basis vector fields $e^\mu_{(i)}$, with a dual co-basis $e^{(i)}_{\mu}$ defined by $e^{\nu}_{(j)} e^{(i)}_{\nu} = \delta^{(i)}_{(j)}$ and $e^{\nu}_{(i)} e^{(i)}_{\mu} = \delta^{\nu}_{\mu}$ and that span the tangent space and cotagent space at each point of each leaf of the foliation. We then define a time-like vector field that transvects the spatial hypersurfaces and thereby defines a global frame of reference, $e^{\mu}_{(0)}$, with corresponding co-basis vector field $e^{(0)}_{\mu} := \partial_{\mu} T$ such that $e^{\nu}_{(0)} e^{(0)}_{\nu} = 1$. This completes duality conditions between the basis and co-basis, i.e. $e^{\nu}_{(\beta)} e^{(\alpha)}_{\nu} = \delta^{(\alpha)}_{(\beta)}$ and $e^{\nu}_{(\alpha)} e^{(\alpha)}_{\mu} = \delta^{\nu}_{\mu}$

We can then project W^{κ} onto this tetrad, $W^{\kappa} = W^{(\alpha)} e^{\kappa}_{(\alpha)}$, with $W^{(0)} = 1$ and $W^{(i)} = \frac{w^{(i)}}{c}$, i.e. the 'passage of time' is the same for the test-particle as it is for the $e^{\nu}_{(0)}$ which defines the frame of reference. The $w^{(i)}$ are the ordinary Newtonian 3-velocity with units of $\frac{[length]}{[time]}$ and c is simply a conversion factor that also has units of $\frac{[length]}{[time]}$, that we do not need to interpret physically as the fundamental velocity of SR. Making the scalars $W^{(\alpha)}$ unitless ensures that all components of W^{κ} have spatial units, as the tetrad has spatial units.

The autoparallel condition then becomes

$$(W^{(\alpha)} e^{\nu}_{(\alpha)}) D_{\nu} (W^{(\beta)} e^{\kappa}_{(\beta)}) = 0 \rightarrow$$

$$W^{(\alpha)} e^{\nu}_{(\alpha)} (D_{\nu} W^{(\beta)}) e^{\kappa}_{(\beta)} + e^{\nu}_{(\alpha)} D_{\nu} e^{\kappa}_{(\beta)} W^{(\alpha)} W^{(\beta)} = 0$$

If we contract this with $e^{(\gamma)}_{\kappa}$ we have

$$W^{(\alpha)} e^{\nu}_{(\alpha)} (D_{\nu} W^{(\beta)}) e^{\kappa}_{(\beta)} e^{(\gamma)}_{\kappa} + e^{(\gamma)}_{\kappa} e^{\nu}_{(\alpha)} D_{\nu} e^{\kappa}_{(\beta)} W^{(\alpha)} W^{(\beta)} = 0$$

or, defining $\frac{D}{d\lambda} := W^{\nu} D_{\nu}$ and the tetrad components of the affine connection, or t.c.c. for short, as $\Gamma^{(\gamma)}_{(\alpha)(\beta)} := e^{(\gamma)}_{\kappa} e^{\nu}_{(\alpha)} D_{\nu} e^{\kappa}_{(\beta)}$

$$\frac{D\left(W^{(\gamma)}\right)}{d\lambda} + \Gamma^{(\gamma)}_{(\alpha)(\beta)} W^{(\alpha)} W^{(\beta)} = 0 \tag{2}$$

, which is a coordinate-free stipulation of the autoparallel condition. It is easy to show that, if we naturally adapt the basis to the coordinate system, $e^{\mu}_{(\alpha)} = \delta^{\mu}_{(\alpha)}$ and $e^{(\alpha)}_{\mu} = \delta^{(\alpha)}_{\mu}$, then $\Gamma^{(\gamma)}_{(\alpha)(\beta)} = \Gamma^{\mu}_{\kappa\nu}$.

The four scalars $\frac{D\left(W^{(\gamma)}\right)}{d\lambda}$ describe, in the frame of reference defined by the tetrad, the 3-acceleration of the four-force-free test particle as well as its change w.r.t. the affine parameter of the temporal coordinate of the test-particle's four-velocity, the latter of which should be zero by our compatibility conditions.

We now turn to the compatibility conditions between the connection and the chronometry (i.e. absolute time) and the spatial basis vectors, which require Euclidean 3-geometry of the spatial hypersurfaces, to see which t.c.c. are eliminated *a priori*. To require that absolute time holds for all frames of reference is equivalent to demanding that $D_{\nu} T_{\mu} = 0 \leftrightarrow D_{\nu} e^{(0)}_{\mu} = 0$, i.e. the magnitude of the covector defining the separation of the spatial hypersurfaces does not change at any point of the manifold in the direction of any of the vectors.

Now we show that

$$\Gamma^{(0)}_{(\alpha)(\beta)} = e^{(0)}_{\kappa} e^{\nu}_{(\alpha)} D_{\nu} e^{\kappa}_{(\beta)} = e^{\nu}_{(\alpha)} D_{\nu} (e^{(0)}_{\kappa} e^{\kappa}_{(\beta)}) - e^{\kappa}_{(\beta)} e^{\nu}_{(\alpha)} D_{\nu} e^{(0)}_{\kappa} = e^{\nu}_{(\alpha)} D_{\nu} (\delta^{(0)}_{(\beta)}) - e^{\kappa}_{(\beta)} e^{\nu}_{(\alpha)} D_{\nu} e^{(0)}_{\kappa} = 0$$

by the duality conditions of the basis and the compatibility condition requiring an absolute time.

We now impose the Euclidicity of the 3-spaces of the spatial hypersurfaces by demanding that parallel transport constrained to such a hypersurface be independent of the space-like path. Formally we can thus require

$$e^{\nu}_{(a)} D_{\nu} e^{\kappa}_{(b)} = 0$$

from which it follows that $\Gamma^{(\gamma)}_{(a)(b)} = 0$.

Thus requiring absolute Newtonian time and flat Euclidean 3-space only allows for non-zero $\Gamma^{(a)}_{(0)(0)}$, $\Gamma^{(a)}_{(0)(b)}$, and $\Gamma^{(a)}_{(b)(0)}$ and demands that all others be zero.

Because we require that the spatial triad remain dual, we require that

$$e^{(a)}_{\kappa} D_{\nu} e^{\kappa}_{(b)} = -e^{(b)}_{\kappa} D_{\nu} e^{\kappa}_{(a)}$$

and thus

$$e^{(a)}_{\kappa} e^{\nu}_{(0)} D_{\nu} e^{\kappa}_{(b)} = -e^{(b)}_{\kappa} e^{\nu}_{(0)} D_{\nu} e^{\kappa}_{(a)} \rightarrow$$

$$\Gamma^{(a)}_{(0)(b)} = -\Gamma^{(b)}_{(0)(a)}$$

With our compatibility conditions imposed on the t.c.c., if we now look at the (0) component of equation (2), we have

$$\frac{D\left(W^{(0)}\right)}{d\lambda} + \Gamma^{(0)}_{(\alpha)(\beta)} W^{(\alpha)} W^{(\beta)} = 0$$

thus implying

$$\frac{D\left(W^{(0)}\right)}{d\lambda} = 0$$

Now, since we are building in absolute time to this framework, we should require that $e^{\nu}_{(0)} = \delta^{\nu}_{(0)}$, from which it follows that $\frac{dW^0}{d\lambda} = 0$ and so $\frac{dX^0}{d\lambda} = const.$ Since we have numerically identified the X^0 and T, it follows that λ agrees with the absolute time T up to a linear rescaling and choice of origin. For simplicity, therefore, we choose to identify the two, so that $\lambda = X^0 = T$, thereby now defining $W^{\nu} := \frac{dX^{\nu}}{dT}$ and making $W^0 = 1$, where T now has units of length since it carries over by virtue of its numerical identification with X^0. In order for T to have units of length, we relate it to the ordinary absolute time t via $T = ct$, where c is the same space-time conversion factor with units of $\frac{[length]}{[time]}$. Now, consider that a unit increase in ordinary time corresponds to an increase of c units of length in the $e^{\nu}_{(0)}$ direction of the manifold, but we want all $e^{\nu}_{(\alpha)}$ to have a unit length. Thus, if we define $e^{\nu}_{(t)}$ as the contravector that represents a unit increase in ordinary time with a corresponding dual covector such that $e^{(t)}_{\nu} e^{\nu}_{(t)} = 1$, we should relate these to the (0) basis vectors by $e^{\nu}_{(0)} = \frac{1}{c} e^{\nu}_{(t)}$ and $e^{(0)}_{\nu} = c e^{(t)}_{\nu}$.

The equations of four-force-free motion now become

$$\frac{D\left(W^{(m)}\right)}{dT} + \Gamma^{(m)}_{(0)(0)} + \Gamma^{(m)}_{(0)(n)} W^{(n)} + \Gamma^{(m)}_{(n)(0)} W^{(n)} = 0$$

,

or, w.r.t. ordinary absolute time t

$$\frac{1}{c^2}\frac{D\left(W^{(m)}\right)}{dt}+\frac{1}{c^2}\Gamma^{(m)}_{\ (t)(t)}+\frac{1}{c}\Gamma^{(m)}_{\ (t)(n)}\frac{w^{(n)}}{c}+\frac{1}{c}\Gamma^{(m)}_{\ (n)(t)}\frac{w^{(n)}}{c}=0$$

after cancelling out the common $\frac{1}{c^2}$ term we have

$$\frac{D\left(W^{(m)}\right)}{dt}+\Gamma^{(m)}_{\ (t)(t)}+\Gamma^{(m)}_{\ (t)(n)}w^{(n)}+\Gamma^{(m)}_{\ (n)(t)}w^{(n)}=0 \tag{3}$$

As I have demonstrated at length in other work (see the url at the end of the references), the second term corresponds to the Newtonian electric-type gravitational force, while the third and fourth, 3-velocity dependent terms correspond to a magnetic-type gravitational interaction. The $\Gamma^{(m)}_{\ (0)(n)}$ and the $\Gamma^{(m)}_{\ (n)(0)}$ are related by the relation between the t.c.c., the torsion tensor, and the anholonomic object given by (Papapetrou-Stachel 1978). In the case of a torsionless connection, this reduces to

$$\Gamma^{(\gamma)}_{\ [(\alpha)(\beta)]}=-\Omega^{(\gamma)}_{\ (\alpha)(\beta)} \tag{4}$$

where

$$\Omega^{(\gamma)}_{\ (\alpha)(\beta)}=\frac{1}{2}e^{\nu}_{(\alpha)}e^{\mu}_{(\beta)}(e^{(\gamma)}_{\mu,\nu}-e^{(\gamma)}_{\nu,\mu})$$

is the anholonomic object of the basis.

If we consider a holonomic basis, with all $\Omega^{(\gamma)}_{\ (\alpha)(\beta)}=0$, then equation (4) tells us

$$\Gamma^{(m)}_{\ (0)(n)}=\Gamma^{(m)}_{\ (n)(0)}$$

Since our compatibility conditons have imposed that

$$\Gamma^{(m)}_{\ (0)(n)}=-\Gamma^{(n)}_{\ (0)(m)}$$

,

it follows in an holonomic spacetime that

$$\Gamma^{(m)}_{\ (n)(0)}=-\Gamma^{(n)}_{\ (m)(0)}$$

.

Visually, it is easy to see that such a change is a rotation of the spatial triads $e^{\kappa}_{(m)}$ and $e^{\kappa}_{(n)}$, along with the corresponding dual covectors $e^{(m)}_{\kappa}$ and $e^{(n)}_{\kappa}$. See Figure 6. This change in spatial triad occurs in the direction of the manifold defined by $e^{\nu}_{(0)}$, which goes from spatial hypersurface to spatial hypersurface. Thus the $\Gamma^{(m)}_{\ (n)(0)}$ and $\Gamma^{(m)}_{\ (0)(n)}$ represent a rotation *rate* in an holonomic spacetime. From the definition of the $\Gamma^{(m)}_{\ (0)(0)}=e^{(m)}_{\kappa}e^{\nu}_{(0)}D_{\nu}e^{\kappa}_{(0)}$ we see that they are the four-acceleration of a frame of reference defined by the $e^{\nu}_{(0)}$, $A^{\kappa}=e^{\nu}_{(0)}D_{\nu}e^{\kappa}_{(0)}$, projected upon the spatial triad of covectors.

3 Chronogeometrized Gravitational Dynamics - General Case

In order to reformulate Newton's gravitational theory, the question now becomes: how do we treat matter's influence on the spacetime structure we have now defined? We have already seen that the t.c.c. are not physically meaningful, as in a frame with zero 3-acceleration w.r.t. the source matter, we expect that they correspond to the 3-acceleration due to the so-called gravitational force in traditional Newtonian gravitational theory, while in a frame that is locally co-accelerating with a test-particle, they vanish.

It turns out that there is a tensor that describes the the affine curvature, called the affine Ricci tensor $R_{(\mu)(\nu)}$, which is physically meaningful, since it is a tensorial quantity, and that naturally extends Poisson's equation for gravity, $\nabla^2\varphi = 4\pi G\rho$ (Papapetrou-Stachel 1978). We allow the values of this tensor to vary differentiably over the manifold and require that it be determined by the momentum-energy distribution in the spacetime. Traditional Newtonian gravitational theory, with non-vanishing $\Gamma^{(m)}_{(0)(0)}$ only, would lead us to

$$R_{(0)(0)} = (const.)G\rho$$

where $R_{(0)(0)}$ is the (0)(0) tetrad component of the affine Ricci tensor, given by (Papapetrou-Stachel, 1978) as

$$R_{(\lambda)(\mu)} = \Gamma^{(\kappa)}_{(\lambda)(\mu),(\kappa)} - \Gamma^{(\kappa)}_{(\kappa)(\mu),(\lambda)} + \Gamma^{(\kappa)}_{(\kappa)(\rho)}\Gamma^{(\rho)}_{(\lambda)(\mu)} - \Gamma^{(\kappa)}_{(\lambda)(\rho)}\Gamma^{(\rho)}_{(\kappa)(\mu)} + 2\Omega^{(\rho)}_{(\kappa)(\lambda)}\Gamma^{(\kappa)}_{(\rho)(\mu)}$$

where $_{,(\kappa)} = \partial_{(\kappa)} := e^{\rho}_{(\kappa)}\partial_\rho$.

Following from the fact that we have eliminated all but the $\Gamma^{(m)}_{(0)(0)}$, $\Gamma^{(m)}_{(0)(n)}$, and $\Gamma^{(m)}_{(0)(n)}$, we have

$$R_{(0)(0)} = \Gamma^{(m)}_{(0)(0),(m)}$$

$$R_{(0)(n)} = \Gamma^{(m)}_{(0)(n),(m)}$$

$$R_{(n)(0)} = \Gamma^{(m)}_{(n)(0),(m)} + 2\Omega^{(\rho)}_{(m)(n)}\Gamma^{(m)}_{(\rho)(0)}$$

$$R_{(m)(n)} = 0$$

If we require the spacetime to be holonomic, we have $\Gamma^{(m)}_{(0)(n)} = \Gamma^{(m)}_{(n)(0)}$ and also

$$R_{(n)(0)} = \Gamma^{(m)}_{(n)(0),(m)} = R_{(0)(n)}$$

Thus our theory also permits non-zero $R_{(0)(n)}$ and $R_{(n)(0)}$, which are equal for an holonomic spacetime.

It is natural to assume that, in this case, these are given, analogous to magnetic-type Einstein field equations $G_{0n} = \frac{8\pi G}{c^4}T_{0n}$, by

$R_{(0)(n)} = (const.)\rho V^{(n)} = (const.)\rho\frac{v^{(n)}}{c}$

where we have the same constant as in our expression for $R_{(0)(0)}$ and $V^{(n)} = \frac{v^{(n)}}{c}$ is the 3-velocity of the source-matter.

Following (Stachel , 2006), if we assume that the $\Gamma^{(m)}_{(0)(0)}$ and the $\Gamma^{(m)}_{(0)(n)}$ are derivable from a gravitational scalar potential φ and gravitational vector potential $\vec{A}$, respectively, namely

$$\Gamma^{(m)}_{(0)(0)} = \delta^{(m)(j)}\partial_{(j)}\varphi \tag{5}$$

and

$$\Gamma^{(m)}_{(0)(n)} = \delta^{(m)(j)}[\partial_{(j)}A_{(n)} - \partial_{(n)}A_{(j)}], \tag{6}$$

and take $const. = \frac{4\pi}{c^2}$ in the field equations, they reduce to

$$R_{(0)(0)} = \delta^{mj}\partial_{mj}\phi = \nabla^2\varphi = \frac{4\pi G\rho}{c^2} \tag{7}$$

and

$$R_{(0)(n)} = \delta^{mj}\partial_{mj}A_{(n)} = \nabla^2 A_{(n)} = \frac{4\pi G\rho v^{(n)}}{c^3} \tag{8}$$

with the condition $\partial_j(\delta^{mj}A_m) = \nabla\cdot\vec{A} = 0$.

4 Classical Gravitational Potential Energy

Now we review classical gravitational potential energy in comparison to the above, chronogeometrized conception of gravitation. To do so, we must review the interrelated concepts of conservative forces, kinetic energy, work, and potential energy.

Kinetic energy, or energy of motion, can be defined in terms of the amount of force applied on an object over a distance. Specifically,

$$W = \Delta K = \int \vec{F}\cdot d\vec{x} \tag{9}$$

,

over the 3-dimensional path an object takes, where W is the work done on an object, ΔK is the resulting change in kinetic energy of the object considered, $\vec{F}$ is the force applied to the object at a given position and time, and $d\vec{x}$ is the infinitesimal three dimensional displacement of the object along its path. Using the Newtonian expression $\vec{F} = m\vec{a}$, it can then be shown that, if we take an object to have zero kinetic energy when it is at rest in a given frame of reference, then the kinetic energy of that object is numerically given by

$$K = \frac{1}{2}m\vec{v}\cdot\vec{v} = \frac{1}{2}mv^2$$

A conservative force is then defined as a force that has the property that the resulting change in kinetic energy of an object being acted upon by that force is *path independent*, i.e. the kinetic energy an object gains from the force considered getting from point a to point b does not depend on how that object gets from point a to point b. The Newtonian gravitational force and the electrical Coulomb force both satisfy this criterion. If the change in kinetic energy is path independent, it then becomes conceptually helpful and mathematically convenient to introduce another concept, potential energy, U, which is a positionally dependent quantity that keeps track of by how much the kinetic energy of an object changes from moving from position to position. Specifically, the total change in kinetic plus potential energy (often called 'mechanical energy') is stipulated to be zero: $\Delta K + \Delta U = 0$.

The gravitational potential energy in its simplest and most general form is given by

$$U = -\frac{GMm}{r}$$

where G is the universal gravitational constant, M and m are gravitating masses, and r is the straight-line distance between the center of mass of the two masses. It is defined, by convention and for convenience, such that $U \to 0$ when $r \to \infty$, so that the amount of energy that two gravitating masses have that is transformed into kinetic energy when brought from infinitely far away to a distance r can easily be read off as $\Delta K = \frac{GMm}{r}$.

Immediately, there is a tension between the frameworks of (Cartan , 1923) and (Stachel , 2006) and this classical framework. What these works show us is that we can and should regard masses in gravitational free-fall, even when Euclidian space and absolute time are preserved by being specified *a priori*, as following *autoparallel* curves through a spacetime, and that this reformulation is neccessarily a mathematical limit of the full general theory of relativity in appropriate circumstances. If we are to commit to the basic insight of the gravitational equivalence principle that objects in gravitational free fall are, in fact, following the straightest paths possible, we must re-examine the concept of gravitational potential energy at its Newtonian foundation.

In conventional Newtonian gravitational theory, as elaborated upon above, the gravitational potential energy of a system is a quantity dependent on the spatial relations of inertial/gravitational masses, as changing spatial relations of these masses is neccessarily accompanied by the gravitational force being applied on the gravitating masses over infinitesimal spatial intervals, and thus a resulting change in kinetic energy of the system. However, in the non-flat spacetime reformulation of Newtonian gravitation, there is no force – the inertial tendencies, i.e. straight paths through the spacetime, are simply modified by the matter-momentum content of the spacetime. When following a spacetime autoparallel, in general the position of the object will change, and thus be ascribed changing gravitational energy in conventional Newtonian theory. If we are commit to this basic insight allowing us to chronogeometrize gravity, should we really ascribe a changing property to an object which is going straight through the spacetime, i.e. along autoparallels – or even more generally, to a system in which all masses are following autoparallels? Despite the immediate temptation to answer an emphatic "no" to this question, conceptual difficulties remain to be addressed.

If we consider an individual object following an autoparallel in its rest frame, its kinetic energy is by definition not changing. If we consider that this individual object is an observer in one of Einstein's elevators, then she will have no knowledge of her changing position and velocity with respect to the other objects in the spacetime, and thus will be unable to measure either her own kinetic energy or her own gravitational potential energy. Perhaps, then, we can surmount this difficulty by disregarding the physical significance of both kinetic energy and gravitational potential energy when we integrate the equivalence principle into a theory.

Some simple additional considerations introduce difficulties with this option. Consider the example of two objects beginning effectively infinitely far away, with initially no kinetic energy, that begin gravitating together and eventually collide. Just before colliding, in the rest frame of the center of mass, the two objects will have a total kinetic energy of $K = \frac{GMm}{r_{cm}}$, where r_{cm} is the distance between the two components' centers of masses just before the collision. If they then collide and come to rest with respect to each other, all of the kinetic energy will have been converted to thermal energy internal to the system. Just prior to the collision, we could attempt to say that the kinetic energy had no physical significance, as neither observer had any knowledge of it. However, after the collision, we cannot make such a statement about the thermal energy. All observers agree that this increase in energy internal to the two objects has taken place. It is also the case that, during the collision, non-gravitational four-forces must have been applied to bring the objects to rest with respect to each other, such that

$$\int F_i dx^i = -\frac{GMm}{r_{cm}}$$

Perhaps we could adopt the negative of $\int F_i dx^i$ as the definition of the amount of energy introduced into a system, and still regard gravitational potential energy and kinetic energy as mere mathematical conveniences in giving a numerical value for this expression, rather than entities with any ontological standing of their own.

Under this interpretation, what would retain its physical significance is *gravitational assembly energy*, or *gravitational binding energy.* These are numerically equal – the first is the amount of thermal energy transferred into a system of masses by virtue of their having gravitated together into a static arrangement, and the second is the amount of energy required to dissemble such a system of masses. Implicit in both of these processes is that four-forces must have been applied to bring the matter / energy distribution into a *static* arrangement, as in the example above of an object falling on and coming to rest upon a gravitating sphere, for gravitational assembly energy, and that four-forces must be applied to dissemble such a matter / energy distribution in the case of gravitational binding energy. The gravitational assembly energy for a uniform sphere of mass M and radius R can be calculated by assuming matter of a constant density ρ is brought together in concentric spherical shells of increasing radius of width dr of mass dm, and that each time such a differential shell of mass is added to the existing matter concentration m, and brought to rest thereupon, an energy equal to $dE = \frac{Gm(r)dm}{r}$ is added to the distribution. The resulting expression is $E_g = \frac{3GM^2}{5R}$. This energy is unmistakenly added to the system internally, and so must modify the spacetime structure in the field equations.

4.1 Further Conceptual Difficulties - Global Energy Conservation

The immediate objection to this conclusion arises in considering more carefully the system just before the collision considered takes place. In this state, in any frame of reference, there is relative motion of the objects gravitating together, and thus there is kinetic energy internal to this system that was not present prior to the process of gravitation. We attempted to surmount this difficutly by postulating that kinetic energy had no physical significance when purely gravitational dynamics of macroscopic entities were considered, and that energy only had physical significance when the energy considered was thermal energy, as this is energy that all frames of reference agree upon. However, according to the kinetic theory of temperature, thermal energy is simply the kinetic energy of the constituent, microscopic particles of the system. What constitutes a constituent particle of a system, as well as what is "macroscopic", is a matter of an arbitrary choice of scale. Thus, we may just as well say that the thermal energy of the system had increased before the collision occured, as the average of the kinetic energy of the constituents of the system has increased here as well as just after the collision. Thus it is not possible to define energy introduced to a system via gravitational dynamics by non-gravitational work alone, as we previously attempted, without arbitrarily singling out one scale of nature (namely, the one on which we measure the kinematics of the system) as priviledged.

Resolving this problem is why there has been such an effort to define the energy of the gravitational field in GR, in analogy with how the electromagnetic field described by Maxwell's equations can have its energy density defined.

5 Classical Gravitational Field Energy Density

In order to guide the interpretation of any possible expression for the energy of the gravitational field in GR, we analyze what the gravitational field energy density should be in classical gravitation. To do so, we turn first to derivations for the expressions of the field energy densities of the electric and magnetic fields given in introductory electromagnetism courses, and then proceed by analogy.

Classically, charges are the source of electric fields, and currents are the source of magnetic fields. The electric field energy density can be derived by an argument considering the work required to charge a parallel plate capacitor, which is a static arrangement of charges giving rise to a roughly uniform electric field in a localized region of space. The argument is as follows: if positive work is done to separate the positive and negative charges of the capacitor, then positive energy must thereafter reside in the resulting electric field, having been transferred from whatever entity is doing the positive work. Typically this entity is a battery in a circuit, and the internal energy of the battery decreases after having charged the capacitor. This argument derives the expression for the energy density of the electric field of $u_E = \frac{1}{2}\epsilon E^2$, where E is the magnitude of the electric field and ϵ is the permittivity of free space multiplied by the relative static permittivity. Similarly, the magnetic field density of $u_B = \frac{1}{2\mu}B^2$ can be derived by postulating that the total amount of energy that must have been released from a battery charging an inductor to a constant current (analogous to a static distribution of charges: a "static current") is stored in the resulting (approximately) uniform magnetic field within the inductor.

In Newtonian gravitation, positive work can be done to separate "gravitational charges." However, the fact that there is only one kind of "gravitational charge" makes it difficult to imagine any matter distribution that would give rise to a uniform gravitational field localized to one region of spacetime, analagous to the electrical case. What, then, should we look for as the energy characteristic of a distribution of gravitating masses, to make an argument analogous to those made in electromagnetism? As we have seen, the gravitational assembly energy of a distribution of masses keeps track of the increase in the thermal energy internal to that distribution , or the decrease in thermal energy that would accompany dissembling such a distribution. This is the only energy change and process of work being done characteristic of the mass distribution. The only way to find the energy density of the gravitational field is then to postulate that the increase in the thermal energy resulting from gravitational assembly is exactly canceled by a manifestation of negative energy of exactly the same magnitude throughout the gravitational field resulting from the assembled distribution of masses, just as the decrease of the internal energy of a battery charging a capacitor is postulated to become positive energy in the resulting electric field.

Let us consider the gravitational assembly energy of a sphere of mass M and uniform density. The assembly energy is well known to be $E_g = \frac{3GM^2}{5R}$, and, as detailed in Section 4, becomes thermal energy internal to that mass distribution. The gravitational field of such a sphere is well known to be given by

$$\vec{g} = \begin{cases} -\frac{GMr}{R^3}\hat{r} & \text{if } r \leq R \\ -\frac{GM}{r^2}\hat{r} & \text{if } r \geq R \end{cases} \tag{10}$$

If we postulate that the energy density of the gravitational field is of the form $u_g = -k_g g^2$, where k_g is positive definite, the negative sign ensuring negative energy is in the gravitational field, then the total energy of the gravitational field is given by

$$U_g = -k_g \iiint\limits_{AllSpace} g^2 d^3\vec{r} \tag{11}$$

Substituting (10) into (11) gives

$$U_g = -\frac{24\pi}{5}k_g\frac{G^2M^2}{R} \tag{12}$$

If we require that $E_g + U_g = 0$, i.e. require that the total energy change in the process of the sphere being gravitationally assembled is zero, and thus that the negative work done on the constituent masses ends up throughout the gravitational field, this gives $k_g = \frac{1}{8\pi G}$.

This was only worked out for one geometry. As a check, we try another: a shell of radius R, mass M, and uniform density. Of course, no such shell exists in nature, but nonetheless performing the same analysis on such an ideal matter distribution could tell us whether or not the expression we found above of $u_g = -\frac{1}{8\pi G}g^2$ is unique to the matter distribution considered.

As detailed in Section 4, in adding a mass element dm to a spherically symmetric matter distribution of mass m and radial extent R, thermal energy of quantity $dE = \frac{Gmdm}{R}$ is added to the system, this amount of negative work being done on the constituents. Thus we may find the assembly energy of a shell of mass M and radius R simply by

$$E_g = \int_0^M \frac{Gmdm}{R} = \frac{GM^2}{2R} \tag{13}$$

The field of a shell of mass M and radius R is well known to be given by

$$\vec{g} = \begin{cases} \vec{0} & \text{if } r \leq R \\ -\frac{GM}{r^2}\hat{r} & \text{if } r \geq R \end{cases} \tag{14}$$

Substituting (13) into (11) gives $U_g = -4\pi k_g \frac{G^2M^2}{R}$. Requiring once more that $E_g + U_g = 0$ gives $k_g = \frac{1}{8\pi G}$, as before, telling us that we should conclude in general that the energy density of the classical gravitational field should be given by $u_g = -\frac{1}{8\pi G}g^2$. This can also be derived by analogy from the Coloumb law for electrostatics and the energy density of the electric field in vacuum, $u_E = \frac{1}{2}\epsilon_0 E^2$, by making the substitutions $E \to g$ and $\frac{1}{4\pi\epsilon_0} \to -G$, i.e. replacing the prefactor in front of the Coloumb force law with that of Newton's Law of Universal Gravitation and changing the sign to account for the difference that like gravitational charges (which is to say all masses) attract, while like electrical charges repel. Deriving it in the manner we have above demonstrates that the energy density of the field is postulated to be what it is in order to compensate for energy density lost or gained elsewhere in both the gravitational and electromagnetic cases – thermal energy gained via gravitational assembly and internal energy lost from a battery in charging a capacitor. This analysis demonstrates that any classical gravitational field energy density should be negative. Specifically, we show it to be given by $u_g = -\frac{1}{8\pi G}g^2$. Thus, in GR, any "gravitational field" must have negative energy, unless compelling reasons are given that "field energy" or "potential energy" means something much different than it does in the classical case.

6 The Ontology of the Gravitational Field in Spacetime Theories of Gravitation

In formulations of Newtonian mechanics preceding GR, the concept of a field is employed frequently. In essence, a field is a physical entity defined at various points of space and time that interacts with bodies to give rise to observed phenomena. The field concept is employed frequently in modern physics. The foundations of SR come from descriptions of space and time in which two relatively moving observers agree upon their observations of how Maxwell's electromagnetic fields act, both as propogating wave phenomena and as entities acting on test bodies. The field concept is even more pervasive in quantum field theory, in which everything we observe, including particles themselves, is an excited, quantized state of a field defined over all space and time. Since science develops by modifying concepts of predecessor theories, it is advantageous for the construction of future theories to analyze the foundations of concepts, including that of a field. In this spirit, we ask: what is a field in classical theories of physics? We first contend that all physical theories are abstracted from our collective experience and that these physical theories are dependent upon conceptual frameworks and conceptual entities postulated within these frameworks. Given that Newton's Laws served as the foundation of modern physics, we should ask what a field is in relation to Newton's Laws.

Referencing Section 2.1 and Figures 4 and 5, we see that Newton's 1st Law amounts to the law of autoparallel transport holding for a force-free test particle in an affine-flat spacetime. Newton's 2nd Law, then, is that a force acting on a test-particle causes its worldline deviate from global, affine-flat autoparallel transport in proportion to the magnitude of the force and in inverse proportion to its mass, $\vec{a} = \frac{\vec{F}}{m}$. In this framework, a field is an entity, defined locally, that gives rise to a particular force on a test particle, with the particular force depending on the nature of the field and the properties of the test particle. For example, an electric field with a particular magnitude and direction will give rise to a net force on a test particle with charge, with this net force depending on the magnitude and sign of the charge and magnitude and direction of the electric field. In short, a field is something that deflects particles from autoparallel transport.

What of the gravitational field? The nature of the gravitational field is such that all test particles will have the same 3-acceleration. This is the insight that allows us to redescribe Newton's 1st and 2nd Laws combined with his Universal Law of Gravitation as simply the introduction of affine curvature on the spacetime by the matter-content of the spacetime. This redescribes all the same observed phenomena as Newton's 1st and 2nd Laws and the Universal Law of Gravitation. See Figure 7. Significant, though, is that under this redescription, there is no longer any need for the concept of a gravitational field as an entity moving test-particles away from their otherwise inertial tendencies. Rather, the basic insight of the gravitational equivalence principle is that gravitational free-fall replaces the concept of inertial motion.

The local character of the field formulation is sometimes used as a motivation for adopting a view of gravitation informed by the equivalence principle. However, the concept of a gravitational field was only ever a reconceptualization of Newton's Laws and his Universal Law of Gravitation, which is expressedly non-local in character. The reformulation of Newton's Laws as presented by (Cartan , 1923), (Stachel , 2006), and Sections 2-3 here is thus not dependent on the concept of a gravitational field, but rather only on the empirical evidence informing Newton's Laws of Motion and Gravitation, namely: Galileo's experiments showing the mass-independence of gravitational acceleration at the Earth's surface and Tycho Brahe's observations of the positions of the planets and Kepler's mathematical model of elliptical orbits founded thereupon. (Stein, 2010)

Rather than the gravitational field and the non-vanishing of the affine connection $\Gamma^{(r)}_{(0)(0)}$ of a frame of reference being one and the same, we may just as well say that the gravitational field is only an apparent phenomenon of the non-flat affine structure of the spacetime. We take the position that, in light of the gravitational equivalence principle, the ontology of the gravitational field is null in any theory that treats gravitation as a manifestation of the structure of space and time. We justify this by the principle that redundant entities should be eliminated from physical theories: Newton's 1st and 2nd Laws with the Universal Law of Gravitation being synthesized into the introduction of affine curvature on the spacetime by matter content is a simpler framework of physical phenomena, even if more mathematical definitions must first be made to do it. In this framework, the gravitational field is a redundant entity, no longer needed to account for the observations for which it was first introduced.

This argument can also be made by noting that the centrifugal force, or indeed any fictitious force in conventional Newtonian dynamics, has no entity postulated to bring it about. Rather, it is an apparent force that is only present when in a reference frame with a non-zero acceleration, i.e. only felt by an observer with a net force. Likewise with gravitation, in order to feel the gravitational force, one must be acted upon by a non-gravitational force, i.e. be in a reference frame with a non-zero four-acceleration. The reconceptualization of Newtonian gravity in terms of the structure of space and time puts the gravitational force and fictitious forces on equal footing in this way, and so just as no entity is postulated to bring about ficticious forces, it ought to be that no entity is postulated to bring about the gravitational force in any theory of nature that treats gravitation as a manifestation of the structure of spacetime. This interpretation applies to the strictly affine spacetime theories of (Cartan , 1923) and (Stachel , 2006), as well as the metric spacetime theory of GR.

We recognize that this is a novel interpretation of GR – Einstein himself once stated that "spacetime does not claim existence of its own, but only as a structural quality of the field." (Stachel , 2014) That is why Einstein's equations are often referred to as the field equations of GR. However, this position has only ever been stated without justification, with its truth seeming to rest on intuition alone. Without an argument, this position cannot be taken as in any way authoritative. By analyzing the foundations of the concept of the gravitational field in light of the gravitational equivalence principle, the gravitational field becomes a redundant entity, and thus should not be said to exist in any theory of gravitation that treats it as a manifestation of the structure of spacetime. Rather than the spacetime being derived as a structural quality of the gravitational field, with no independent existence of its own, a simpler interpretation is that the gravitational field is only an apparent phenomenon of the structure of the spacetime, with no independent existence of its own.

6.1 Conceptual Consequences

If we are to say that the gravitational field does not exist as an entity of its own, but rather as an apparent phenomenon of the structure of spacetime, we cannot meaningfully ascribe it any quantity of energy. If we ascribe the gravitational field a null ontology, we are forced to conclude that in the theories of Cartan (1923), Stachel (2006), and GR, energy is not globally conserved if energy includes thermal energy of a confined spacetime region. This is a consequence of two facts: (1) the convergent spacetime autoparallels in such theories result in kinetic energy being created and further concentrated in the spacetime; (2) this increase in kinetic/thermal energy cannot be cancelled by negative energy in the gravitational field, as the gravitational field has no fundamental existence and thus cannot be ascribed physical properties. That energy is not conserved globally in GR seems immediately falsifiable by considering that Einstein's equations are constructed with the explicit requirement that they conserve stress-energy-momentum content locally at every point of the spacetime through the conservation law of GR,

$$T^{\mu\nu}_{;\nu} = 0$$

From the definition of the (μ, ν) component of stress-energy tensor $T^{\mu\nu}$ as the flux of the (μ) component of four-momentum through a surface of constant (ν) coordinate, we see that the conservation law is really that the four-momentum of the spacetime is divergenceless. $T^{\mu\nu}_{;\nu} = 0$ essentially says that the "stuff" of the universe is not created from nowhere.

Einstein's equations that satisfy this requirement are

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = -\frac{8\pi G}{c^4}T^{\mu\nu}$$

and themselves give rise to spacetimes that include convergent autoparallels, and convergent autoparallels necessarily result in an increase of kinetic energy further confined to a spacetime region. Thus, with a null ontology of the gravitational field, it is still true that there is an increase in the total energy content of any general-relativistic spacetime specified, with this increase further concentrating itself in the spacetime, even if locally no four-momentum content is spontaneously generated at any spacetime point. Further, because greater amounts of matter are greater amounts of energy following autoparallels of the spacetime, global energy conservation is violated more as larger structures of a spacetime are specified and evolved, i.e. on larger heirarchies of the structure of the natural world. This seems an objectionable statement because global energy conservation is a feature of every other theory of nature. However, every other theory of nature is either rooted in Newtonian dynamics or the conception of energy in SR, neither of which take into consideration the gravitational equivalence principle, but rather which rely on globally flat spacetimes. The gravitational equivalence principle neccessitates we abandon this basic framework, so it is not surprising that it introduces conceptual difficulties to the framework of energy conservation founded thereupon.

Conceptual difficulties such as these may prompt some to reject the chronogeometric interpretation altogether, in the spirit of Richard Feynman, in favor of a field approach analogous to Maxwell's equations, as detailed in (Baryshev , 2008). However, there have been well-accepted experimental tests of chronogeometric entities of the theory: the detection of gravitational redshift shows the presence of non-zero Γ^0_{r0} in the spacetime and the measuring of the geodetic effect by NASA / Stanford's Gravity Probe B shows the presence of non-zero Γ^r_{rr}, neither of which are forces derived from fields, and, if accepted as valid experiments, necessitate the chronogeometric interpretation of gravity on which our argument rests.

7 Concluding Remarks

In this paper, we have advanced the novel interpretation of GR that the ontology of the gravitational field is null, and, rather than being the entity from which spacetime is derived as a structural quality, it is more simply interpreted to be an apparent phenomenon emergent from the structure of the spacetime. We can only see two routes for further inquiry in interpreting GR, the choice of which depends on whether or not the argument for this interpretation is accepted or rejected.

If the argument here is rejected, and one chooses instead to interpret GR as a field theory, then the considerations of Sections 4 and 5 demonstrate that any possible gravitational field energy density can only be negative in order to cancel the increase in kinetic/thermal energy internal to systems gravitating together. If the gravitational field is thought to be a fundamental entity of a theory, with some objective reality of its own and with negative energy density, the question then arises: how does negative energy density modify the spacetime? Should we modify Einstein's equations in order that the negative-definite gravitational field energy density explicitly modifies the spacetime structure as does the nongravitational four-momentum-density of the stress-energy tensor?

If the argument here is accepted, as shown in Section 6, the only appropriate interpretation of GR is that it is not a field theory, and that energy is not conserved globally by Einstein's equations, despite the divergence of the four-momentum content vanishing locally at every spacetime point. We advocate this approach, as experimental tests of GR validate its chronogeometric interpretation, and in any chronogeometric theory of gravity, the gravitational field is a redundant entity. This prompts the further question of whether or not this violation of global energy conservation ought to inform how it is we actually find solutions to the background-independent equations of Einstein.

Given that global energy conservation is violated in an increasing proportion with the scale of the spacetime specified and evolved, it seems reasonable to assume that any rigorous fix of this conceptual diffculty would make predictions of observations of nature that differ from predictions under the current interpretations of solving Einstein's equations in mathematical proportion to the scale of nature observed.

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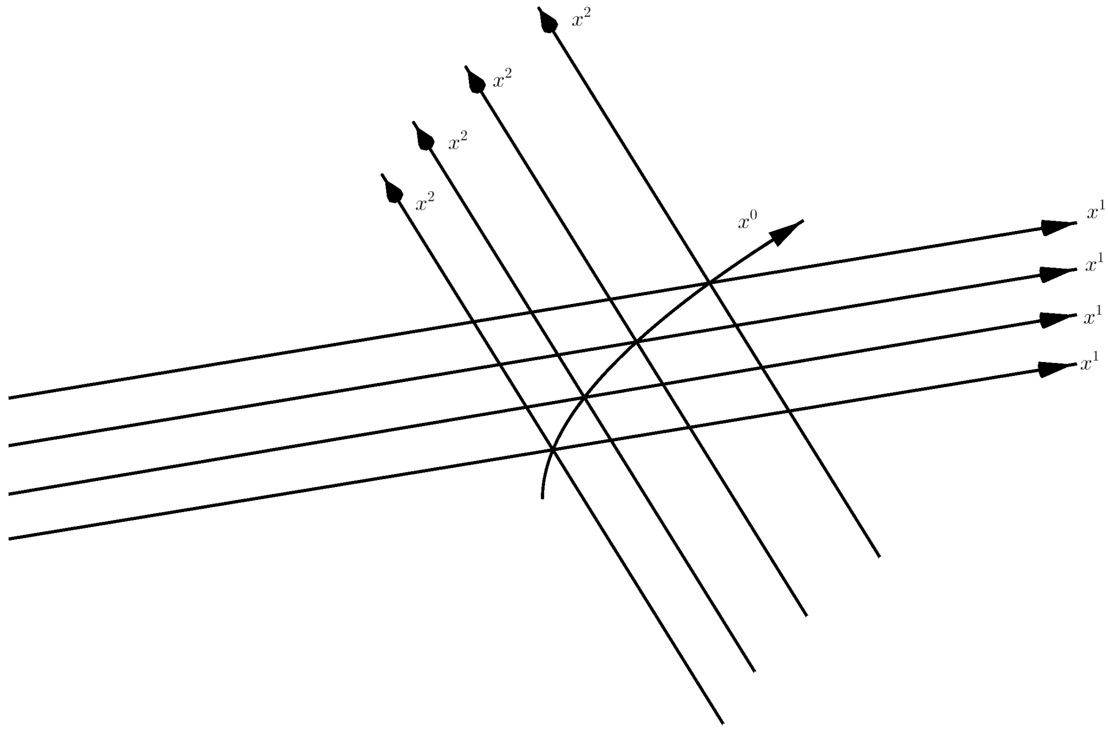

Figure 1: A coordinitization of a 4-space without the $x^{(3)}$ axis shown. Such a coordinatization can extend over the entirety of the manifold, or characterize the tangent space of a given point or set of points of the manifold

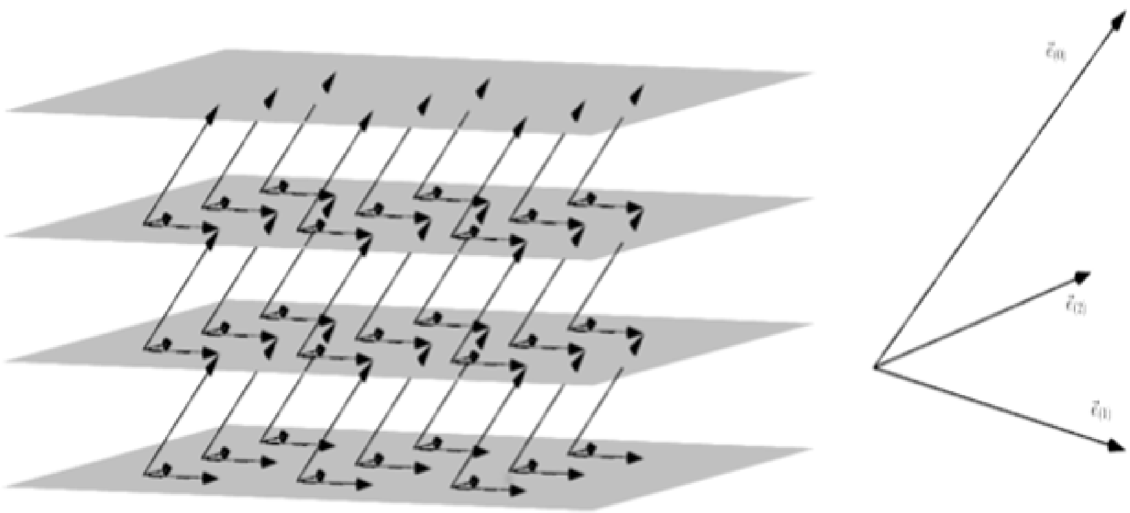

Figure 2: A specification of a tetrad field, without the $e^{\nu}_{(3)}$. The surfaces represent spatial hypersurfaces compressed to two dimensions. Such a tetrad field can be specified over the entirety of the manifold, or over the tangent space of a given point or set of points of the manifold. While this particular tetrad field defines a uniformly translating frame of reference, the $e^{\nu}_{(0)}$ field can be non-constant, which would allow for accelerating or rotating reference frames, and the $e^{\nu}_{(i)}$ field can be non-constant, e.g. for a spherical polar spatial basis.

Figure 3: The same autoparallel curve has constant components under one coordinatization of the sphere (left) but non-constant components under a rotation of the original coordinatization of the sphere (right)

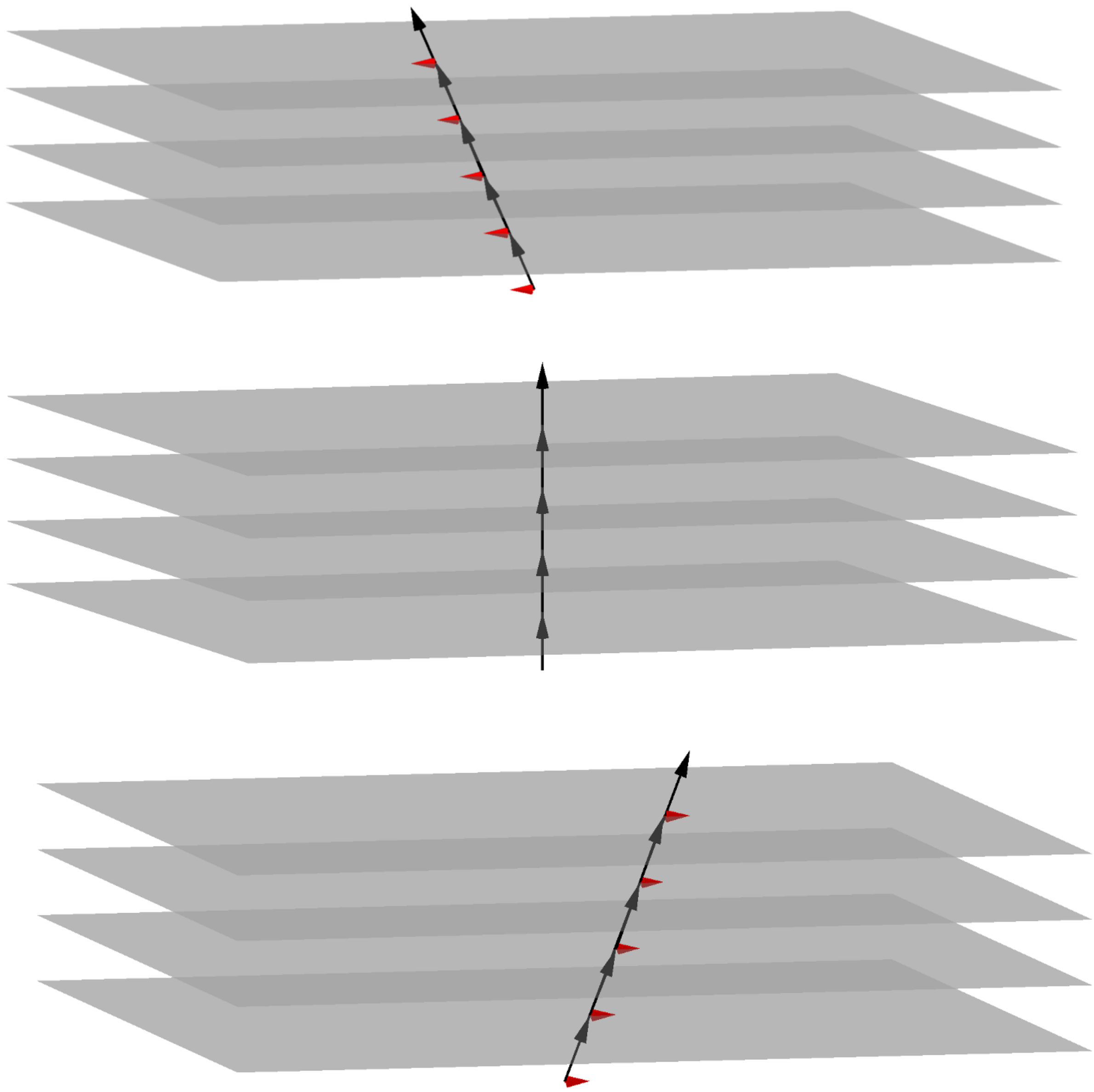

Figure 4: The above three diagrams show three possible inertial world-lines of force-free test particles and their 4-velocities which define autoparallels in a globally flat affine space. The surfaces represent spatial hypersurfaces T = const., compressed to two dimensions, and the red arrows are the spatial components of the 4-velocities. The three diagrams can also be thought of as representing the same inertial world-line and 4-velocities under an inertial change of coordinates

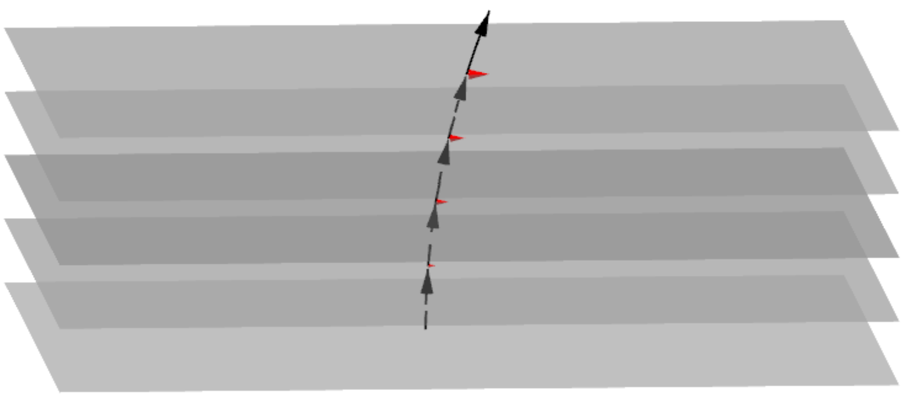

Figure 5: Newton's Second Law, $\vec{a} = \frac{\vec{F}}{m}$, states that a particle's worldline deviates from affine-flat autoparallel transport when subjected to a net force. Fields are entities defined at points of the spacetime that deflect particles at those points of spacetime from autoparallel transport, depending on the nature of the field and particle interacting with that field.

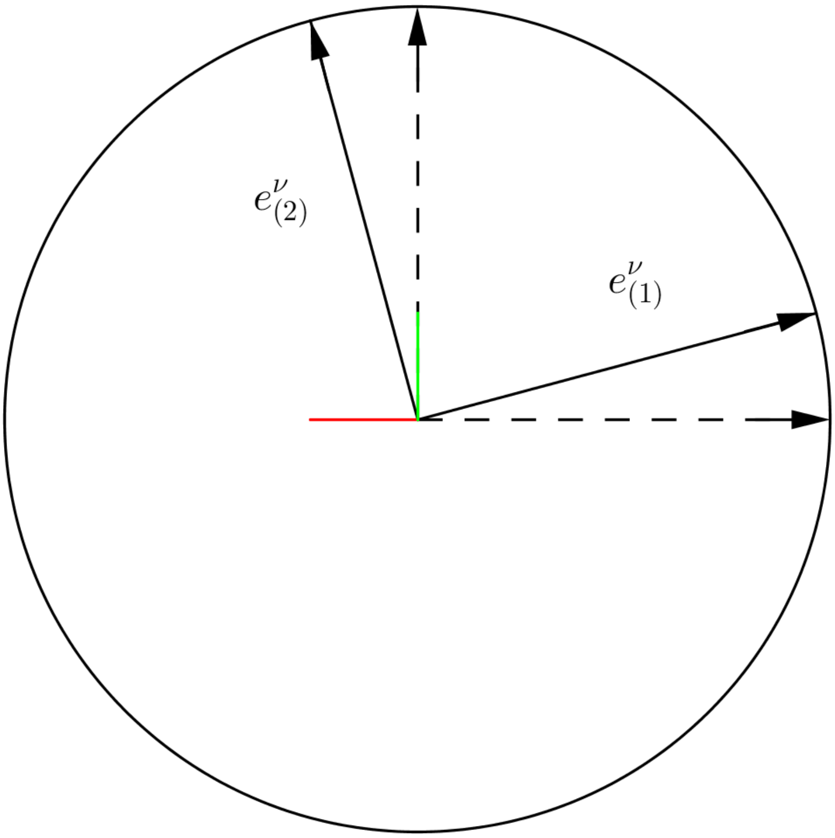

Figure 6: The green segment represents $\Gamma^{(2)}_{(0)(1)}$, while the red segment represents $\Gamma^{(1)}_{(0)(2)}$. Clearly $\Gamma^{(2)}_{(0)(1)} = -\Gamma^{(1)}_{(0)(2)}$ implies a rotation of $e^{\nu}_{(1)}$ and $e^{\nu}_{(2)}$.

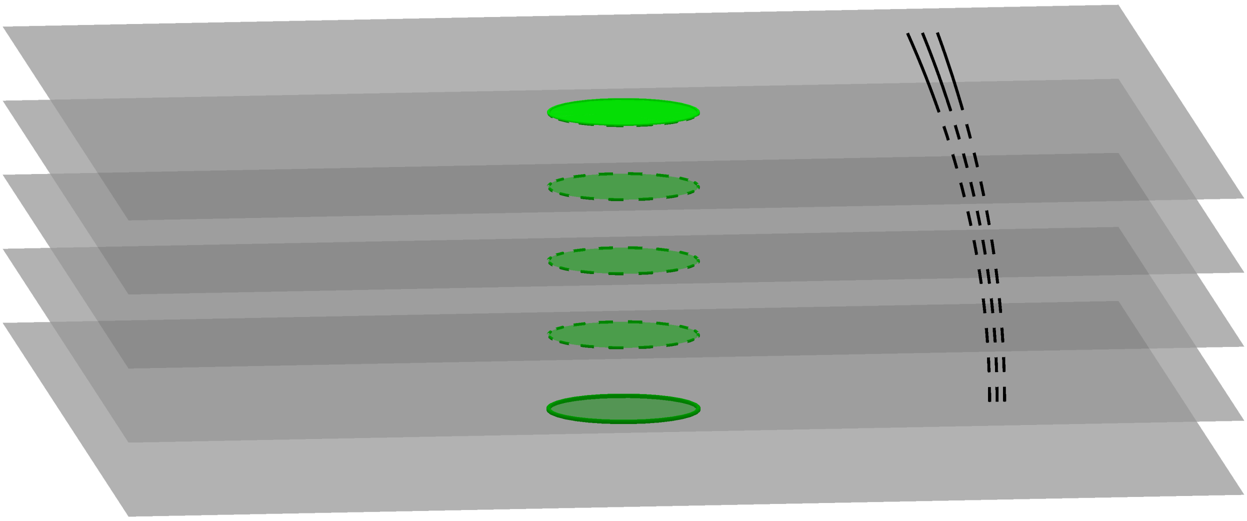

Figure 7: The nature of the gravitational field is such that any entity on which it acts has its four-velocity deflected from autoparallel transport in a globally flat affine spacetime in exactly the same way, owing to the equality of inertial and gravitational masses. This makes a simpler interpretation that the autoparallels of the spacetime are simply modified, and the spacetime is not in fact affine flat. Under this interpretation, there is no need for a gravitational field defined as an entity giving deflection from autoparallel transport in affine-flat spacetime, as gravitation is now a modification of the affine relations of spacetime points.