Adversarial Estimation of Network Formation Models

Consider a nutshell network formation model, the adjacency matrix $\boldsymbol{A}$ is generated as

$$a_{ij} = \mathbb{1}\left\{\theta_{0,i} + \theta_{0,j} > u_{ij}\right\},\, i < j,$$

where θ_i is the fixed effect for each $i = 1, 2, \cdots, n$, and u_{ij} follows i.i.d. standard logistic distribution. Let $a_{ij} = a_{ji}$ for $i < j$ and $a_{ii} = 0$.

Denote $\boldsymbol{\theta_0} = (\theta_{0,1}, \theta_{0,2}, \cdots, \theta_{0,n})$. For given $\boldsymbol{\theta}$, we can generate

$$\widetilde{u}_{ij,k} \sim \text{i.i.d.}\Lambda\left(\cdot\right),\, i < j,\, k = 1, 2, \cdots, K,$$

where $\Lambda\left(u\right) = \left(1 + e^{-u}\right)^{-1}$ is the CDF of the standard logistic distribution. Transform $\widetilde{u}_{ij,k}$ to $\widetilde{d}_{ij,k}$ by

$$a_{ij,k,\boldsymbol{\theta}} = \mathbb{1}\left\{\theta_i + \theta_j > \widetilde{u}_{ij,k}\right\}.$$

When $K = 1$, a single network is generated based on parameter $\boldsymbol{\theta}$.

The oracle discriminator,

$$D_{\boldsymbol{\theta}}\left(a_{ij,k}\right) = \frac{p_0\left(a_{ij,k}\right)}{p_0\left(a_{ij,k}\right) + p_{\boldsymbol{\theta}}\left(a_{ij,k}\right)},$$

where

$$p_0\left(a_{ij,\cdot}\right) = \begin{cases} \Lambda\left(\theta_{0,i} + \theta_{0,j}\right) & \text{if } a_{ij,\cdot} = 1, \\ 1 - \Lambda\left(\theta_{0,i} + \theta_{0,j}\right) & \text{if } a_{ij,\cdot} = 0, \end{cases}$$

and

$$p_{\boldsymbol{\theta}}\left(a_{ij,\cdot}\right) = \begin{cases} \Lambda\left(\theta_i + \theta_j\right) & \text{if } a_{ij,\cdot} = 1, \\ 1 - \Lambda\left(\theta_i + \theta_j\right) & \text{if } a_{ij,\cdot} = 0. \end{cases}$$

The estimation follows Kaji et al. (2022, KMP) with the oracle discriminator,

$$\hat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \frac{1}{n\left(n-1\right)} \sum_{i<j} \log D_{\boldsymbol{\theta}}\left(a_{ij}\right) + \frac{1}{Kn\left(n-1\right)} \sum_{k=1}^{K} \sum_{i<j} \log\left(1 - D_{\boldsymbol{\theta}}\left(a_{ij,k,\boldsymbol{\theta}}\right)\right).$$

Remark 1. *Alternatively, apply the test in Auerbach (2022) for $\boldsymbol{A}$ and $\boldsymbol{A_\theta}$.*

Remark 2. *Alternatively, consider $\theta_{0,i} \sim F\left(\cdot\right)$*

1 Identification

Chatterjee et al. (2011), Gao (2020).

2 Simulation

DGP 1. Let

$$A_{ij} = \mathbb{1}\left\{\gamma_i + \gamma_j > u_{ij}\right\}, \tag{2.1}$$

where $\gamma_i = g\left(X_i\right) + \varepsilon_i$.

DGP 2. Let

$$A_{ij} = \mathbb{1}\left\{\alpha_0 + \beta_0 \|X_i - X_j\|_2 > u_{ij}\right\}. \tag{2.2}$$

DGP 3. Let

$$A_{ij} = \mathbb{1}\left\{\alpha_0 + \beta_0 \|X_i - X_j\|_2 + \gamma_i + \gamma_j > u_{ij}\right\}, \tag{2.3}$$

where $\gamma_i = g\left(X_i\right) + \varepsilon_i$.

References

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Appendix

A.1 Idea

Consider a heterogeneous treatment effect model,

$$Y_i = \tau_i d_i + f(\boldsymbol{X}_i) + U_i,\ i \in \mathcal{N}, \tag{A.1.1}$$

where the treatment d_i is assigned in a random experiment, and $\boldsymbol{X}_i \in \mathbb{R}^p$ is the observed characteristics. Both treated and control units are drawn from the same super population. Denote $\mathcal{T} = \{i \in \mathcal{N} : d_i = 1\}$ and $\mathcal{C} = \mathcal{N} \setminus \mathcal{T}$ as the sets of treated and control units, respectively, and correspondingly $N_1 = |\mathcal{T}|$ and $N_0 = |\mathcal{C}|$.

Suppose $\{\tau_i\}_{i\in\mathcal{T}}$ is known, define

$$\tilde{Y}_i = \begin{cases} Y_i & \text{if } d_i = 0, \\ Y_i - \tau_i & \text{if } d_i = 1, \end{cases} \quad i \in \mathcal{N}, \tag{A.1.2}$$

then

$$\tilde{Y}_i = f(\boldsymbol{X}_i) + U_i,\ \forall i \in \mathcal{N},$$

i.e. $S_{\mathcal{T}} = \left\{\tilde{Y}_i, \boldsymbol{X}_i\right\}_{i\in\mathcal{T}}$ and $S_{\mathcal{C}} = \left\{\tilde{Y}_i, \boldsymbol{X}_i\right\}_{i\in\mathcal{C}}$ follow the same data generating process. In this case, one cannot $S_{\mathcal{T}}$ and $S_{\mathcal{C}}$.

In practice, the heterogeneous treatment effects τ_i are unknown parameter of interests. In Wager and Athey (2018), τ_i is modeled,

$$\tau(\boldsymbol{x}) = \mathbb{E}(\tau_i | \boldsymbol{X}_i = \boldsymbol{x}).$$

Following the intuition from the case where $\{\tau_i\}_{i\in\mathcal{T}}$ is known, we propose to adopt the generative adversarial network (GAN) framework (Goodfellow et al., 2014; Kaji et al., 2022) to estimate $\{\tau_i\}_{i\in\mathcal{T}}$. Consider a minimax game between two components, a generator G and a discriminator D , which can be modeled as deep neural networks. The estimation problem is defined as

$$\min_{G\in\mathcal{G}} \max_{D\in\mathcal{D}} \frac{1}{N_1} \sum_{i\in\mathcal{T}} \log D\left(\tilde{Y}_i(G(\boldsymbol{X}_i)), \boldsymbol{X}_i\right) + \frac{1}{N_0} \sum_{i\in\mathcal{C}} \log(1 - D(Y_i, \boldsymbol{X}_i)). \tag{A.1.3}$$