Haoran Zhang | HW # 9 | STAT 519

prob. 1

$X_1, \ldots, X_n \overset{iid}{\sim} f(x; \theta_1, \theta_2) = \left(\frac{1}{\theta_2}\right) e^{-\frac{x-\theta_1}{\theta_2}}$

$\theta_1 \le x < \infty, \quad -\infty < \theta_2 < \infty$

$L(\theta_1, \theta_2) = \prod_{i=1}^{n} f(x_i; \theta_1, \theta_2) = \prod_{i=1}^{n} \left(\frac{1}{\theta_2}\right) e^{-\frac{x_i-\theta_1}{\theta_2}}$

$= \left(\frac{1}{\theta_2}\right)^n e^{-\frac{\sum_{i=1}^{n}(x_i-\theta_1)}{\theta_2}}$

$l(\theta_1, \theta_2) = -n \ln \theta_2 - \left(\frac{1}{\theta_2}\right) \sum_{i=1}^{n} (X_i - \theta_1)$

$= \underbrace{-n \ln \theta_2 - \frac{1}{\theta_2}\left(\sum_{i=1}^{n} X_i - n\theta_1\right)}$

① $\frac{\partial l(\theta_1, \theta_2)}{\partial \theta_1}$

$\Rightarrow -\frac{1}{\theta_2}(-n) = \frac{n}{\theta_2} > 0$

$\therefore \underbrace{\hat{\theta}_1 = \min(X_1, \ldots, X_n)}$

② $\frac{\partial l(\theta_1, \theta_2)}{\partial \theta_2} = -\frac{n}{\theta_2} + \frac{1}{\theta_2^2}\left(\sum_{i=1}^{n} X_i - n\theta_1\right) = 0$

$\Rightarrow \hat{\theta}_2 = \frac{\sum_{i=1}^{n} X_i - n\theta_1}{n} = \underbrace{\frac{\sum_{i=1}^{n} X_i}{n} - \hat{\theta}_1}$

prob. 2 $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

a) $P(X \le b) = P\left(\frac{X-\mu}{\sigma} \le \frac{b-\mu}{\sigma}\right) = 0.90$

$z_{0.90} = 1.28 \Rightarrow \frac{\hat{b}-\hat{\mu}}{\hat{\sigma}} = 1.28$

$\Rightarrow \hat{b} = \hat{\mu} + 1.28\,\hat{\sigma}$

$L(\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(X_i-\mu)^2}{2\sigma^2}} = \frac{1}{(2\pi)^{\frac{n}{2}}\sigma^n} e^{\frac{-\Sigma(X_i-\mu)^2}{2\sigma^2}}$

$\theta = (\mu, \sigma) \in \mathbb{R}^2$

$$l(\theta) = -\frac{n}{2}\log(2\pi) - n\log\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2$$

$$\begin{cases} \dfrac{\partial l}{\partial \mu} = \dfrac{1}{\sigma^2}\displaystyle\sum_{i=1}^{n}(X_i-\mu) = 0 \\ \dfrac{\partial l}{\partial \sigma} = -\dfrac{n}{\sigma} + \dfrac{1}{\sigma^3}\displaystyle\sum_{i=1}^{n}(X_i-\mu)^2 = 0 \end{cases}$$

$$\Rightarrow \hat{\mu} = \bar{X} \qquad \hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2}$$

we know that $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2$

$$\therefore \hat{\sigma} = \sqrt{\frac{n-1}{n}}\,S$$

$$\therefore \hat{b} = \bar{X} + 1.28\sqrt{\frac{n-1}{n}}\,S$$

b) $P(X \le c) = P\left(\frac{X-\mu}{\sigma} \le \frac{c-\mu}{\sigma}\right)$

$$\therefore \hat{P}(X \le c) = \Phi\left(\frac{c-\hat{\mu}}{\hat{\sigma}}\right) = \Phi\left(\frac{c-\bar{X}}{\sqrt{\frac{n-1}{n}}\,S}\right)$$

prob. 3

$X_1, \dots, X_n \overset{iid}{\sim}$ Poisson(θ), $0<\theta<\infty$

$Y = \sum_{i=1}^{n} X_i$, $\mathcal{L}(\theta, \delta(Y)] = [\theta - \delta(Y)]^2$

$$R(\theta, \delta) = E\,\mathcal{L}(\theta, \delta(Y))$$

$$= E\left\{\left[\theta - \left(b+\frac{Y}{n}\right)\right]^2\right\} = E\left\{\theta^2 - 2\left(b+\frac{Y}{n}\right)\theta + \left(b+\frac{Y}{n}\right)^2\right\}$$

$$= \left\{\theta^2 - 2\theta b - 2\theta E(\bar{X}) + b^2 + 2bE(\bar{X}) + E(\bar{X}^2)\right\}$$

$$E(\bar{X}) = E\left(\frac{X_1+\dots+X_n}{n}\right) = \frac{1}{n}\cdot n\theta = \underline{\underline{\theta}}$$

$$Var(\bar{X}) = Var\left(\frac{X_1+\dots+X_n}{n}\right) = \frac{1}{n^2}\cdot n\theta = \underline{\underline{\frac{\theta}{n}}}$$

$\therefore R(\theta,\delta) = \theta^2 \cancel{-2\theta b} - 2\theta^2 + b^2 \cancel{+2b\theta} + \left(Var(\bar{X}) + [E(\bar{X})]^2\right)$

$= -\theta^2 + b^2 + \left(\frac{\theta}{n} + \theta^2\right) = \boxed{b^2 + \frac{\theta}{n}}$

Select a decision function that minimizes the risk for all $\theta \in \Omega$

$\min_{\delta} \max_{\theta} R(\theta,\delta) \equiv \min_{\delta} \max_{\theta} \left[b^2 + \frac{\theta}{n}\right]$

we can get $b=0$ and $\delta = \frac{Y}{n}$

$R(\theta,\delta) = \frac{\theta}{n}$

however $\max_{\theta} R(\theta,\delta)$ cannot exist

prob. 4

$X_1 \cdots X_n \overset{iid}{\sim} b(1,\theta)$, $0 \le \theta \le 1$

$Y = \sum_{i=1}^{n} X_i$ $\quad \mathcal{L}[\theta, \delta(y)] = [\theta - \delta(y)]^2$ $\quad \delta(y) = by$

$R(\theta,\delta) = E\,\mathcal{L}[\theta, \delta(y)]$

$= E\left\{(\theta - by)^2\right\} = E(\theta^2 - 2\theta by + b^2y^2)$

$= \theta^2 - 2\theta bn\,E(\bar{X}) + b^2n^2\,E(\bar{X}^2)$

$E(\bar{X}) = \frac{1}{n} E(X_1 + \cdots + X_n) = \theta$

$Var(\bar{X}) = \frac{1}{n^2} Var(X_1 + \cdots + X_n) = \frac{1}{n^2} \cdot n\theta(1-\theta) = \frac{\theta(1-\theta)}{n}$

$\therefore R(\theta,\delta) = \theta^2 - 2bn\theta^2 + b^2n^2\left(Var(\bar{X}) + (E(\bar{X}))^2\right)$

$= \theta^2 - 2bn\theta^2 + b^2n^2\left(\frac{\theta(1-\theta)}{n} + \theta^2\right)$

$= \theta^2 - 2bn\theta^2 + b^2n\theta(1-\theta) + \theta^2b^2n^2$

$= b^2n\theta(1-\theta) + \theta^2(bn-1)^2$

$$\frac{\partial R(\theta, \delta)}{\partial \theta} = b^2 n - 2b^2 n \theta + 2(bn-1)^2 \theta = 0$$

$$\Rightarrow \theta = \frac{b^2 n}{2b^2 n - 2(bn-1)^2}$$

$$\therefore \max_{\theta} R[\theta, \delta(y)] = b^2 n \frac{b^2 n [b^2 n - 2(bn-1)^2]}{4[b^2 n - (bn-1)^2]^2} + \frac{b^4 n^2 (nb-1)^2}{4[b^2 n - (bn-1)^2]^2} = \boxed{\frac{b^4 n^2}{4[b^2 n - (bn-1)^2]}}$$

We know that $0 \le \theta \le 1$,

So we need to ensure that $0 \le \frac{b^2 n}{2b^2 n - 2(bn-1)^2} \le 1$

It's provided that $b^2 n > (bn-1)^2$

$\therefore 2b^2 n - 2(bn-1)^2 > 0$ and $b^2 n \ge 0$

besides, we need to ensure $b^2 n \ge 2(bn-1)^2$

$$\min_{\delta} \max_{\theta} R(\theta, \delta) \equiv \min_{\delta} \max_{\theta} \left[b^2 n \theta (1-\theta) + \theta^2 (bn-1)^2 \right]$$

$$\equiv \min_{\delta} \left[\frac{b^4 n^2}{4[b^2 n - (bn-1)^2]} \right] \equiv \min_{b} \left[\frac{b^4 n^2}{4[b^2 n - (bn-1)^2]} \right]$$

$$\frac{\partial R}{\partial b} = \frac{4b^3 n^2 \cdot 4[b^2 n - (bn-1)^2] - b^4 n^2 \cdot 4[2bn - 2(bn-1) \cdot n]}{16(b^2 n - (bn-1)^2)^2} = 0$$

$$\Rightarrow 2b^2 n + bn^2 - 1 = 0$$

when $b = \frac{1}{n}$, we get $2 + n - 1 \neq 0$

$\therefore b = \frac{1}{n}$ does not minimize $\max_{\theta} R(\theta, \delta)$

Prob.5 $X_1 \cdots X_n \overset{iid}{\sim} f(x,\theta) = \frac{1}{\theta}\exp(-\frac{x}{\theta})$, $0 < x < \infty$

$\theta > 0$

$Y_1 < Y_2 < \cdots < Y_r$

a) $L(\theta) = f_{Y_1 \cdots Y_r}(y_1 \cdots y_r) =$

$$\frac{n!}{(1-1)!\,(2-1-1)!\cdots(n-r)!} F(y_1)^{1-1}(F(y_2)-F(y_1))^{2-1-1}\cdots$$

$$\cdot(1-F(y_r))^{n-r}\cdot f(y_1)\cdots f(y_r)$$

$$= \frac{n!}{(n-r)!}(1-F(y_r))^{n-r}\cdot\prod_{i=1}^{r} f(y_i)$$

$f(x,\theta) = \frac{1}{\theta}\exp(-\frac{x}{\theta})$, $0 < x < \infty$

$\int_0^\infty f(x,\theta)\,dx = 1 - e^{-\frac{x}{\theta}}$ $\therefore F_X(x) = 1 - e^{-\frac{x}{\theta}}$

$$L(\theta) = \frac{n!}{(n-r)!}\left[\exp(-\tfrac{1}{\theta}(n-r)y_r)\cdot\right]\frac{1}{\theta^r}\left[\exp\left(-\frac{\sum_{i=1}^{r} y_i}{\theta}\right)\right]$$

b)

$$l(\theta) = -\frac{1}{\theta}(n-r)y_r + \log\left(\frac{n!}{(n-r)!}\right) - r\log(\theta)$$

$$-\frac{1}{\theta}\left(\sum_{i=1}^{r} y_i\right)$$

$$l'(\theta) = \frac{1}{\theta^2}(n-r)y_r - \frac{r}{\theta} + \frac{1}{\theta^2}\left(\sum_{i=1}^{r} y_i\right) = 0$$

$$\Rightarrow \boxed{\hat\theta = \frac{n-r}{r}y_r + \frac{\sum_{i=1}^{r} y_i}{r}}$$

c) MLE: $\hat{\theta} = \frac{n-r}{r} y_r + \frac{\sum_{i=1}^{r} y_i}{r}$

$\Rightarrow r\hat{\theta} = (n-r) y_r + \sum_{i=1}^{r-1} y_i + y_r$

$\Rightarrow r\hat{\theta} = (n-r+1) y_r + \sum_{i=1}^{r-1} y_i$

$\Rightarrow \boxed{y_r = \frac{r\hat{\theta} - \sum_{i=1}^{r-1} y_i}{n-r+1}}$

$$f_{Y_1, \cdots Y_r}(y_1, \cdots y_r) = \frac{n!}{(n-r)!} \exp\left(-\frac{1}{\theta}(n-r) y_r\right)$$

$$\cdot \frac{1}{\theta^r} \exp\left(-\frac{\sum_{i=1}^{r-1} y_i + y_r}{\theta}\right)$$

$$|J| = \begin{vmatrix} \frac{\partial y_1}{\partial y_1} & \cdots & \frac{\partial y_1}{\partial y_r} \\ \frac{\partial y_2}{\partial y_1} & \cdots & \frac{\partial y_2}{\partial y_r} \\ \vdots & & \\ \frac{\partial \hat{\theta}}{\partial y_1} & \cdots & \frac{\partial \hat{\theta}}{\partial y_r} \end{vmatrix} = \begin{vmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & \cdots & & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & 1 & 0 \\ \frac{1}{r} & \frac{1}{r} & \cdots & \frac{1}{r} & \frac{n-r+1}{r} \end{vmatrix} = \boxed{\frac{n-r+1}{r}}$$

$$f(y_1, \cdots, y_{r-1}, \hat{\theta}) = f_{Y_1, \cdots Y_r}(y_1, \cdots y_r) |J|$$

$$\therefore f_{Y_1 \cdots \hat{\theta}}(y_1, \cdots y_{r-1}, \hat{\theta}) = \frac{n!}{(n-r)!} \cdot \frac{1}{\theta^r} \cdot \frac{n-r+1}{r}$$

$$\cdot \exp\left(-\frac{\sum_{i=1}^{r-1} y_i + (n-r+1) y_r}{\theta}\right)$$

$$= \frac{n!}{(n-r)!} \cdot \frac{1}{\theta^r} \cdot \frac{n-r+1}{r} \cdot e^{\frac{-r\hat{\theta}}{\theta}}$$

$$f_{\hat{\theta}}(\hat{\theta}) = \iint \cdots \int f_{Y \cdots \hat{\theta}}(y_1, \cdots y_{r-1}, \hat{\theta})\, dy_1 \cdots dy_{r-1}$$

$$= \frac{n!}{(n-r)!} \frac{1}{\theta^r} \frac{n-r+1}{r} \int_0^{y_{r-1}} \cdots \int_0^{y_2} e^{-\frac{r\hat{\theta}}{\theta}}\, dy_1 \cdots dy_{r-1}$$

$$= \frac{n!}{(n-\gamma)!\,\gamma!} \frac{1}{\theta^\gamma} (n-\gamma+1) \cdot e^{\frac{-\gamma\hat\theta}{\theta}} \cdot y_\gamma^{\gamma-1} \quad \text{pdf}$$

$$M(t) = \int_0^{\frac{n-\gamma}{\gamma} y_\gamma + \frac{\sum_{i=1}^{\gamma} y_i}{\gamma}} y_\gamma^{\gamma-1} \cdot \frac{n!}{(n-\gamma)!\,\gamma!} \frac{1}{\theta^\gamma} (n-\gamma+1) \cdot e^{t\hat\theta - \frac{\gamma\hat\theta}{\theta}} \, d\hat\theta$$

$$= \frac{n!}{(n-\gamma)!\,\gamma!} \frac{1}{\theta^\gamma} (n-\gamma+1) \int_0^{\frac{n-\gamma}{\gamma} y_\gamma + \frac{\sum_{i=1}^{\gamma} y_i}{\gamma}} e^{t\hat\theta - \frac{\gamma\hat\theta}{\theta}} \, d\hat\theta$$

$$= \binom{n}{\gamma} \frac{1}{\theta^\gamma} (n-\gamma+1) \cdot \frac{y_\gamma^{\gamma-1}}{t - \frac{\gamma}{\theta}} \cdot \exp\left(\left(t - \frac{\gamma}{\theta}\right) \cdot \left[\frac{n-\gamma}{\gamma} y_\gamma + \frac{\sum_{i=1}^{\gamma} y_i}{\gamma} \right] \right)$$

d)
$$\frac{f(y_1 \cdots y_\gamma \mid \theta)}{f_{\hat\theta}(\hat\theta)} = \frac{\frac{n!}{(n-\gamma)!} \frac{1}{\theta^\gamma} \exp\left(- \frac{\sum_{i=1}^{\gamma} y_i + (n-\gamma) y_\gamma}{\theta} \right)}{\frac{n!}{(n-\gamma)!\,\gamma!} \frac{1}{\theta^\gamma} (n-\gamma+1) \exp\left(- \frac{\gamma\hat\theta}{\theta} \right) \cdot y_\gamma^{\gamma-1}}$$

$$= \frac{\gamma!}{n-\gamma+1} \cdot \frac{1}{y_\gamma^{\gamma-1}} \cdot \frac{\exp\left(- \frac{\sum_{i=1}^{\gamma} y_i + (n-\gamma) y_\gamma}{\theta} \right)}{\exp\left(- \frac{(n-\gamma) y_\gamma + \sum_{i=1}^{\gamma} y_i}{\theta} \right)}$$

$$= \frac{\gamma!}{n-\gamma+1} \cdot \frac{1}{y_\gamma^{\gamma-1}} \cdot 1 = \boxed{\frac{\gamma!}{(n-\gamma+1)\, y_\gamma^{\gamma-1}}}$$

assume $f_{\hat\theta}(\hat\theta)$ is $k_1[\hat\theta, \theta]$

and $\frac{\gamma!}{(n-\gamma+1) y_\gamma^{\gamma-1}}$ is $k_2[y_1 \cdots y_\gamma]$

where $k_2[y_1 \cdots y_n]$ does not depend on θ

$\therefore$ $\hat\theta$ is sufficient statistic for θ

prob. 6

$X_1, X_2 \overset{iid}{\sim} f(x,\theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}} & 0 < x < \infty \\ & 0 < \theta < \infty \\ 0 & \text{o.w} \end{cases}$

$Y_1 = X_1 + X_2$ and $Y_2 = X_2$

$\begin{cases} Y_1 = X_1 + X_2 \\ Y_2 = X_2 \end{cases} \Rightarrow \begin{cases} X_1 = Y_1 - X_2 \\ X_2 = Y_2 \end{cases}$

$\therefore |J| = \begin{vmatrix} \frac{\partial X_1}{\partial Y_1} & \frac{\partial X_1}{\partial Y_2} \\ \frac{\partial X_2}{\partial Y_1} & \frac{\partial X_2}{\partial Y_2} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$

$\therefore f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(y_1 - y_2, y_2)\, |J|$

$= \frac{1}{\theta} e^{-\frac{y_1 - y_2}{\theta}} \cdot \frac{1}{\theta} e^{-\frac{y_2}{\theta}} = \underbrace{\frac{1}{\theta^2} \cdot e^{-\frac{y_1}{\theta}}}$ $\quad 0 < y_2 < y_1 < \infty$

$E(Y_2) = \int_0^\infty y_2 \cdot \int_{y_2}^\infty \frac{1}{\theta^2} e^{-\frac{y_1}{\theta}} dy_1 dy_2 = \cdot \left[\int_0^\infty e^{-\frac{y_2}{\theta}} dy_2 - y_2 \cdot e^{-\frac{y_2}{\theta}} \Big|_0^\infty \right]$

$= \boxed{\theta}$

$\therefore Y_2$ is an unbiased estimator for θ

$var(Y_2) = \int_0^\infty y_2^2 \cdot \frac{1}{\theta} e^{-\frac{y_2}{\theta}} dy_2 - [E(Y_2)]^2$

$= \int_0^\infty 2y_2 \cdot e^{-\frac{y_2}{\theta}} dy_2 - \left[y_2^2 \cdot e^{-\frac{y_2}{\theta}} \right] \Big|_0^\infty - \theta^2$

$= 2\theta^2 - \theta^2 = \boxed{\theta^2}$

$$f_{Y_1}(y_1) = \int_0^{y_1} f_{Y_1 Y_2}(y_1, y_2)\, dy_2 = \int_0^{y_1} \frac{1}{\theta^2} e^{-\frac{y_1}{\theta}}\, dy_2$$

$$= \frac{y_1}{\theta^2} e^{-\frac{y_1}{\theta}} \qquad \begin{matrix} 0 < y_1 < \infty \\ 0 < \theta < \infty \end{matrix}$$

$$f_{Y_2, Y_1}(y_2 | y_1) = \frac{f_{Y_1 Y_2}(y_1, y_2)}{f_{Y_1}(y_1)} = \frac{\frac{1}{\theta^2} e^{-\frac{y_1}{\theta}}}{\frac{y_1}{\theta^2} e^{-\frac{y_1}{\theta}}} = \boxed{\frac{1}{y_1}}$$

$$E(Y_2 | y_1) = \int_0^{y_1} \frac{y_2}{y_1}\, dy_2 = \frac{y_2^2}{2y_1}\Big|_0^{y_1} = \boxed{\frac{y_1}{2}} = \varphi(y_1)$$

$$\mathrm{var}(\varphi(y_1)) = \frac{1}{4}\mathrm{var}(Y_1) = \frac{1}{4} \cdot (\mathrm{Var}(X_1) + \mathrm{Var}(X_2))$$

$$\mathrm{Var}(X_1) = \mathrm{Var}(X_2) = \theta^2$$

$$\therefore \mathrm{var}(\varphi(y_1)) = \frac{2\theta^2}{4} = \boxed{\frac{\theta^2}{2}}$$