Haoran zhang | HW # 8 | STAT 517 | 1/

prob. 1

a) $X \sim Gamma(\alpha = 4, \ \beta = \theta > 0)$

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}$$

$$f(x,\theta) = \frac{1}{6\theta^4} x^3 e^{-\frac{x}{\theta}}$$

$$\log f(x,\theta) = -\log(6\theta^4) + 3\log x - \frac{x}{\theta}$$

$$\frac{\partial \log f(x,\theta)}{\partial \theta} = -\frac{24\theta^3}{6\theta^4} + \frac{x}{\theta^2} = -\frac{4}{\theta} + \frac{x}{\theta^2}$$

$$\frac{\partial^2 \log f(x,\theta)}{\partial \theta^2} = \frac{4}{\theta^2} - \frac{2x}{\theta^3}$$

$$I(\theta) = -E\left(\frac{\partial^2 \log f(x,\theta)}{\partial\theta^2}\right) = -E\left(\frac{4}{\theta^2} - \frac{2x}{\theta^3}\right)$$

$$= -\frac{4}{\theta^2} + \frac{2}{\theta^3}E(X) = -\frac{4}{\theta^2} + \frac{2}{\theta^3}\cdot\alpha\beta = -\frac{4}{\theta^2} + \frac{2}{\theta^3}\cdot 4\theta$$

$$= -\frac{4}{\theta^2} + \frac{8}{\theta^2} = \boxed{\frac{4}{\theta^2}}$$

b) $X_1 \cdots X_n \ \underset{\sim}{iid} \ f(x_i,\theta)$

$$L(\theta) = \prod_{i=1}^n f(x_i,\theta) = \prod_{i=1}^n \left(\frac{1}{6\theta^4} x_i^3 e^{-\frac{x_i}{\theta}}\right) = (6\theta^4)^{-n} \cdot \prod_{i=1}^n x_i^3 \cdot e^{-\frac{\sum x_i}{\theta}}$$

$$l(\theta) = -n\log(6\theta^4) + \log\left(\prod_{i=1}^n x_i^3\right) - \frac{\sum_{i=1}^n x_i}{\theta}$$

$$l'(\theta) = -n\cdot\frac{24\theta^3}{6\theta^4} + \frac{\sum_{i=1}^n x_i}{\theta^2} = 0$$

$$\Rightarrow \frac{-4n}{\theta} + \frac{\sum_{i=1}^n x_i}{\theta^2} = 0 \Rightarrow \hat{\theta} = \frac{1}{4}\cdot\frac{\sum_{i=1}^n x_i}{n} = \frac{1}{4}\bar{X}$$

R-C lower bound: $\frac{1}{nI(\theta)} = \frac{1}{n\cdot\frac{4}{\theta^2}} = \frac{\theta^2}{4n}$

$$var(\hat{\theta}) = var\left(\frac{\bar{X}}{4}\right) = \left(\frac{1}{4}\right)^2 var\left(\frac{\sum_{i=1}^n X_i}{n}\right) = \frac{1}{16}\cdot\frac{1}{n}\cdot var(X_i)$$

$$= \frac{\alpha\beta^2}{16n} = \frac{4\theta^2}{16n} = \frac{\theta^2}{4n}$$

$$\text{Eff.} = \frac{\text{R-C lower bound}}{Var(\hat{\theta})} = \boxed{1} \Rightarrow \text{efficient} ✓$$

c) $\sqrt{n}(\hat{\theta}-\theta) \dot{\sim} N\left(0, \frac{1}{I(\theta_0)}\right)$ when n is large

$\therefore \underline{\sqrt{n}(\hat{\theta}-\theta) \dot{\sim} N\left(0, \frac{\theta^2}{4}\right)}$

prob. 2

$X \sim N(0, \theta)$, $0<\theta<\infty$

a) $f(x,\theta) = \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{x^2}{2\theta}\right)$

$\log f(x,\theta) = -\frac{1}{2}\log(2\pi\theta) + \left(\frac{-x^2}{2\theta}\right)$

$\frac{\partial \log f(x,\theta)}{\partial\theta} = -\frac{1}{2\theta} + \frac{x^2}{2\theta^2}$

$I(\theta) = -E\left(\frac{\partial^2 \log f(x,\theta)}{\partial\theta^2}\right) = E\left(\frac{\partial \log f(x,\theta)}{\partial\theta}\right)^2$

$= E\left(\frac{1}{4\theta^2} + \frac{x^4}{4\theta^4} - \frac{x^2}{2\theta^3}\right)$

$= \frac{1}{4\theta^2} + \frac{1}{4\theta^4}E[x^4] - \frac{1}{2\theta^3}E[x^2]$

$Var(x) = E[x^2] - (Ex)^2 = E[x^2] - 0 = \theta$

$\therefore \underline{E[x^2] = \theta}$

$\because \frac{x-0}{\sqrt{\theta}} \sim N(0,1) \quad \therefore \frac{x^2}{\theta} \sim \chi^2(1)$

$\Rightarrow Var\left(\frac{x^2}{\theta}\right) = 2 \Rightarrow Var(x^2) = 2\theta^2$

$E(x^4) = Var(x^2) + (E(x^2))^2 = 2\theta^2 + \theta^2 = 3\theta^2$

$\therefore I(\theta) = \frac{1}{4\theta^2} + \frac{1}{4\theta^4}\cdot 3\theta^2 - \frac{1}{2\theta^3}\cdot\theta = \frac{1}{4\theta^2} + \frac{3}{4\theta^2} - \frac{1}{2\theta^2}$

$= \boxed{\frac{1}{2\theta^2}}$

b) $X_1 \cdots X_n \overset{iid}{\sim} f(x_i, \theta)$

$$L(\theta) = \prod_{i=1}^{n} f(x_i, \theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{x_i^2}{2\theta}\right) = (2\pi\theta)^{-\frac{n}{2}} \exp\left(-\frac{\sum_{i=1}^{n} x_i^2}{2\theta}\right)$$

$$l(\theta) = -\frac{n}{2}\log(2\pi\theta) + \left(-\frac{\sum_{i=1}^{n} x_i^2}{2\theta}\right)$$

$$l'(\theta) = -\frac{n}{2} \cdot 2\pi \cdot \frac{1}{2\pi\theta} + \frac{\sum_{i=1}^{n} x_i^2}{2\theta^2} = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i^2 = \frac{n}{2\theta} \cdot 2\theta^2 = \theta \cdot n \Rightarrow \boxed{\hat{\theta} = \frac{\sum_{i=1}^{n} X_i^2}{n}}$$

$$\text{R-C lower bound} = \frac{1}{nI(\theta)} = \frac{1}{n \cdot \frac{1}{2\theta^2}} = \frac{2\theta^2}{n}$$

$$Var(\hat{\theta}) = Var\left(\frac{\sum_{i=1}^{n} X_i^2}{n}\right) = \frac{1}{n^2} \cdot Var\left(\sum_{i=1}^{n} X_i^2\right)$$

$$= \frac{1}{n^2} \cdot (n \cdot 2\theta^2) = \frac{2\theta^2}{n}$$

$$\text{Eff.} = \frac{\text{R-C lower bound}}{Var(\hat{\theta})} = \boxed{1} \Rightarrow \text{Efficient} \checkmark$$

c) $\sqrt{n}(\hat{\theta} - \theta) \overset{\cdot}{\sim} N\left(0, \frac{1}{I(\theta_0)}\right)$ when n is large

$\therefore \sqrt{n}(\hat{\theta} - \theta) \overset{\cdot}{\sim} N(0, 2\theta^2)$

prob. 3

a) $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 \Rightarrow \bar{X} \perp S^2$

$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$

$$\Rightarrow \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1) \Rightarrow \frac{(n-1)S^2}{\theta} \sim \chi^2(n-1)$$

$$\Rightarrow Var\left(\frac{(n-1)S^2}{\theta}\right) = 2(n-1) \Rightarrow Var(S^2) = \frac{2(n-1)\theta^2}{(n-1)^2} = \frac{2\theta^2}{n-1}$$

$$f(x, \theta) = \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{(x-\mu)^2}{2\theta}\right)$$

$$\log f(x, \theta) = -\frac{1}{2}\log(2\pi\theta) - \frac{(x-\mu)^2}{2\theta}$$

$$\frac{\partial \log f(x,\theta)}{\partial \theta} = -\frac{1}{2}\cdot 2\pi \cdot \frac{1}{2\pi\theta} + \frac{(x-\mu)^2}{2\theta^2} = -\frac{1}{2\theta} + \frac{(x-\mu)^2}{2\theta^2}$$

$$I(\theta) = E\left(\frac{\partial \log f(x,\theta)}{\partial\theta}\right)^2 = E\left(\frac{1}{4\theta^2} + \frac{(x-\mu)^4}{4\theta^4} - \frac{(x-\mu)^2}{2\theta^3}\right)$$

$$= \frac{1}{4\theta^2} + \frac{1}{4\theta^4} E\left[(x-\mu)^4\right] - \frac{1}{2\theta^3} E\left[(x-\mu)^2\right]$$

$\because$ $x-\mu \sim N(0,\theta)$

$\therefore$ From prob. 2, we can get $I(\theta) = \frac{1}{2\theta^2}$

R-C lower bound $= \frac{[k'(\theta)]^2}{nI(\theta)}$

$k(\theta) = ES^2 = \theta \Rightarrow$ R-C lower bound $= \frac{1}{nI(\theta)} = \frac{2\theta^2}{n}$

Eff. $= \frac{2\theta^2/n}{2\theta^2/(n-1)} = \boxed{\frac{n-1}{n}}$

b) $X_1 \cdots X_n \overset{iid}{\sim} f(x_i,\theta)$

$$L(\theta) = \prod_{i=1}^n f(x_i,\theta) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{(X_i-\mu)^2}{2\theta}\right)$$

$$= (2\pi\theta)^{-\frac{n}{2}} \exp\left(-\frac{\sum_{i=1}^n (X_i-\mu)^2}{2\theta}\right)$$

$$l(\theta) = -\frac{n}{2}\log(2\pi\theta) - \frac{\sum_{i=1}^n (X_i-\mu)^2}{2\theta}$$

$$l'(\theta) = -\frac{n}{2}\cdot 2\pi\cdot\frac{1}{2\pi\theta} + \frac{\sum_{i=1}^n (X_i-\mu)^2}{2\theta^2} = 0$$

$$\Rightarrow \boxed{\hat\theta = \frac{\sum_{i=1}^n (X_i-\mu)^2}{n}}$$

c) $\sqrt{n}(\hat\theta-\theta) \,\dot\sim\, N\left(0, \frac{1}{I(\theta_0)}\right)$ when n is large

$\therefore \sqrt{n}(\hat\theta-\theta) \,\dot\sim\, N(0, 2\theta^2)$

prob. 4

$X_1, \dots, X_n \underbrace{\quad}_{} \text{iid}$ Poisson(θ)

$$f(x,\theta) = \frac{e^{-\theta}\theta^{x}}{x!} \quad x = 0, 1, 2, 3, \dots, \quad \theta > 0$$

a) $H_0: \theta = \theta_0 \qquad H_1: \theta \neq \theta_0$

$$L(\theta) = \prod_{i=1}^{n} f(x_i, \theta) = \prod_{i=1}^{n} \frac{e^{-\theta}\theta^{x_i}}{x_i!} = \frac{e^{-\theta n}\theta^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}$$

$$l(\theta) = -\theta n + \sum_{i=1}^{n} x_i \log(\theta) - \log\left(\prod_{i=1}^{n} x_i!\right)$$

$$l'(\theta) = -n + \frac{\sum_{i=1}^{n} x_i}{\theta} = 0 \Rightarrow \hat{\theta} = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta})} = \frac{e^{-n\theta_0}\theta_0^{\sum_{i=1}^{n} x_i} \Big/ \prod_{i=1}^{n} x_i!}{e^{-\sum_{i=1}^{n} x_i}\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^{\sum_{i=1}^{n} x_i} \Big/ \prod_{i=1}^{n} x_i!}$$

$$= \frac{e^{-n\theta_0}\theta_0^{\sum_{i=1}^{n} x_i}}{e^{-\sum_{i=1}^{n} x_i}\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^{\sum_{i=1}^{n} x_i}}$$

if $Y = \sum_{i=1}^{n} x_i$, $\Lambda = \dfrac{e^{-n\theta_0}\theta_0^{Y}}{e^{-Y}(Y/n)^{Y}} = e^{Y - n\theta_0} \cdot \left(\dfrac{\theta_0}{Y/n}\right)^{Y} \leq 1$

$$\underbrace{Y \sim \text{Poisson}(n\theta) \text{ since } \bar{Y} = n\theta}$$

b) $\theta_0 = 2$, $n = 5$

$$\alpha = P_{\theta=\theta_0}(\Lambda \leq C) = P_{\theta=\theta_0}(Y \leq 4 \text{ or } Y \geq 17)$$

$$= P(Y \leq 4 \text{ or } Y \geq 17 \mid \theta = 2)$$

$$= P(Y \leq 4 \mid \theta = 2) + P(Y \geq 17 \mid \theta = 2)$$

$$Y \sim \text{Poisson}(10) \Rightarrow f(y, \theta_0 = 10) = \frac{e^{-10}10^{y}}{y!}$$

$$\Rightarrow \alpha = \sum_{y=0}^{4} \frac{e^{-10}10^{y}}{y!} + \sum_{17}^{\infty} \frac{e^{-10}10^{y}}{y!}$$

$$= 1 - \sum_{y=5}^{16} \frac{e^{-10}10^{y}}{y!} = 1 - 0.9437 = \boxed{0.0563}$$

Prob. 5

$X_1, \dots X_n \sim Beta(\alpha = \beta = \theta)$ and $\Omega = \{\theta : \theta = 1, 2\}$

$f(x, \theta) = \dfrac{\Gamma(2\theta)}{\Gamma(\theta)\Gamma(\theta)} x^{\theta-1}(1-x)^{\theta-1}$, $0 < x < 1$

$L(\theta) = \prod_{i=1}^{n} f(x_i, \theta) = \prod_{i=1}^{n} C \cdot x_i^{\theta-1}(1-x_i)^{\theta-1}$, $C = \dfrac{\Gamma(2\theta)}{\Gamma(\theta)\Gamma(\theta)}$

$= C^n \prod_{i=1}^{n} x_i^{\theta-1}(1-x_i)^{\theta-1}$

$L(\theta=1) = \left(\dfrac{\Gamma(2)}{\Gamma(1)\Gamma(1)}\right)^n = \underline{1}$

$L(\theta=2) = \left(\dfrac{\Gamma(4)}{\Gamma(2)\Gamma(2)}\right)^n \prod_{i=1}^{n} x_i^1(1-x_i)^1 = \underline{6^n \prod_{i=1}^{n} X_i(1-X_i)}$

$\because 0 < X_i < 1 \quad \therefore \max(X_i(1-X_i)) = \dfrac{1}{4}$

① if $L(\theta=2) \geq L(\theta=1)$

then $L(\hat{\theta}) = L(\theta=2)$

$\Lambda = \dfrac{L(\theta_0)}{L(\hat{\theta})} = \dfrac{L(\theta=1)}{L(\theta=2)} = \dfrac{1}{6^n \prod_{i=1}^{n} X_i(1-X_i)}$

$\log \Lambda = -\left(\sum_{i=1}^{n} \log X_i + \sum_{i=1}^{n} \log(1-X_i) + n\log 6\right)$

$W = \sum_{i=1}^{n} \log X_i + \sum_{i=1}^{n} \log(1-X_i) \Rightarrow \log \Lambda = -(W + C)$

$\Rightarrow \underline{\Lambda = e^{-(W+C)}}$

② if $L(\theta=2) < L(\theta=1)$

$\Lambda = 1 \Rightarrow$ efficient

prob. 6 $X_1 \cdots X_n \overset{iid}{\sim} p(x,\theta) = \theta^x (1-\theta)^{1-x}$

$x = 0, 1$ where $0<\theta<1$

$H_0: \theta = \frac{1}{3}$ vs. $H_1 \neq \frac{1}{3}$

a) $L(\theta) = \prod_{i=1}^{n} f(x_i, \theta) = \prod_{i=1}^{n} \theta^{x_i} (1-\theta)^{1-x_i}$

$= \theta^{\sum_{i=1}^n x_i} (1-\theta)^{n-\sum_{i=1}^n x_i}$

$l(\theta) = \sum_{i=1}^n x_i \cdot \log(\theta) + (n - \sum_{i=1}^n x_i) \log(1-\theta)$

$l'(\theta) = \frac{\sum_{i=1}^n x_i}{\theta} - \frac{n-\sum_{i=1}^n x_i}{1-\theta} = 0 \Rightarrow \hat{\theta} = \frac{\sum_{i=1}^n x_i}{n} = \bar{X}$

$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta})} = \frac{\left(\frac{1}{3}\right)^{\sum_{i=1}^n x_i} \left(\frac{2}{3}\right)^{n-\sum_{i=1}^n x_i}}{(\bar{x})^{\sum_{i=1}^n x_i} (1-\bar{x})^{n-\sum_{i=1}^n x_i}}$$

$$= \underline{\left(\frac{1}{3\bar{x}}\right)^{\sum_{i=1}^n x_i} \left(\frac{2}{3(1-\bar{x})}\right)^{n-\sum_{i=1}^n x_i}}$$

$\log \Lambda = \sum_{i=1}^n x_i \log\left(\frac{1}{3\bar{x}}\right) + (n - \sum_{i=1}^n x_i) \log\left(\frac{2}{3(1-\bar{x})}\right)$

$\underline{-2\log\Lambda = +2\sum_{i=1}^n \log(3\bar{x}) + 2(n - \sum_{i=1}^n x_i) \log\left(\frac{3(1-\bar{x})}{2}\right)}$

b) $f(x,\theta) = \theta^x (1-\theta)^{1-x}$

$\log f(x,\theta) = x\log\theta + (1-x)\log(1-\theta)$

$\frac{\partial}{\partial\theta} \log f(x,\theta) = \frac{x}{\theta} - \frac{1-x}{1-\theta}$

$\frac{\partial^2}{\partial\theta^2} \log f(x,\theta) = -\frac{x}{\theta^2} - \frac{1-x}{(1-\theta)^2}$

$I(\theta) = -E\left(-\frac{x}{\theta^2} - \frac{(1-x)}{(1-\theta)^2}\right) = \frac{EX}{\theta^2} + \frac{E(1-X)}{(1-\theta)^2}$

$EX = p \qquad E(1-X) = 1-p$

$\quad = \theta \qquad\qquad\quad = 1-\theta$

$\therefore I(\theta) = \frac{1}{\theta} + \frac{1}{1-\theta} = \boxed{\frac{1}{\theta(1-\theta)}}$

$\chi^2_w = \left\{\sqrt{nI(\hat{\theta})}\,(\hat{\theta} - \theta_0)\right\}^2$ under $H_0 : \chi^2_w \sim \chi^2_{(1)}$

$= \boxed{\left\{\sqrt{\frac{n}{\bar{x}(1-\bar{x})}}\,\left(\bar{x} - \frac{1}{3}\right)\right\}^2}$

reject H_0 if $\chi^2_w \geq \chi^2_{1,\alpha}$

c) $\chi^2_R = \left(\frac{l'(\theta_0)}{\sqrt{nI(\theta_0)}}\right)^2$

$l'(\theta) = \frac{\sum_{i=1}^n x_i}{\theta} - \frac{n - \sum_{i=1}^n x_i}{1-\theta} = \frac{n\bar{x}}{\theta} - \frac{n - n\bar{x}}{1-\theta}$

$l'(\theta_0) = \frac{n\bar{x}}{\theta_0} - \frac{n-n\bar{x}}{1-\theta_0} = \frac{n(\bar{x}-\theta_0)}{\theta_0(1-\theta_0)}$

$\chi^2_R = \left(\frac{n^2(\bar{x}-\theta_0)^2}{\theta_0^2(1-\theta_0)^2} \cdot \frac{\theta_0(1-\theta_0)}{n}\right) = \boxed{\frac{n(\bar{x}-\theta_0)^2}{\theta_0(1-\theta_0)}}$

reject H_0 if $\chi^2_R \geq \chi^2_{1,\alpha}$