Haoran Zhang | HW# 4 | STAT 519

prob. 1

$$f(x;\theta) = \begin{cases} \theta x^{\theta-1} & 0<x<1 \\ 0 & \text{o.w.} \end{cases} \quad \text{where } \theta = 1 \text{ or } 2$$

$H_0: \theta = 1$ vs. $H_a: \theta = 2$

power function: $\gamma(\theta) = P_\theta(\text{reject } H_0)$

$= P(x_1 x_2 \geq \frac{3}{4})$

$X_1 \perp X_2 \Rightarrow f(x_1, x_2) = f(x_1) f(x_2) = \theta^2 (x_1 x_2)^{\theta-1}$

when $\frac{3}{4} < x_1 < 1$, and $x_1 x_2 \geq \frac{3}{4}$

$\Rightarrow \frac{3}{4x_1} \leq x_2 < 1$

$$\Rightarrow P(X_1 X_2 \geq \tfrac{3}{4}) = \int_{\frac{3}{4}}^{1} \int_{\frac{3}{4x_1}}^{1} \theta^2 (x_1 x_2)^{\theta-1}\, dx_2\, dx_1$$

$$= \int_{\frac{3}{4}}^{1} \theta x_1^{\theta-1} x_2^{\theta} \Big|_{\frac{3}{4x_1}}^{1} dx_1 = \int_{\frac{3}{4}}^{1} \theta \left(x_1^{\theta-1} - \left(\tfrac{3}{4x_1}\right)^{\theta} x_1^{\theta-1}\right) dx_1$$

$$= \left[x_1^{\theta} - \left(\tfrac{3}{4}\right)^{\theta} \log(\theta) \cdot \theta\right] \Big|_{\frac{3}{4}}^{1}$$

$$= 1 - \left(\tfrac{3}{4}\right)^{\theta} \cdot 0 - \left(\left(\tfrac{3}{4}\right)^{\theta} - \left(\tfrac{3}{4}\right)^{\theta} \log\left(\tfrac{3}{4}\right)\theta\right)$$

$$= 1 - \left(\tfrac{3}{4}\right)^{\theta} + \left(\tfrac{3}{4}\right)^{\theta} \theta \log\left(\tfrac{3}{4}\right) \quad \text{for } \theta = 1, 2$$

prob. 2

$$f(x;\theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}} & \text{when } 0 < x < \infty \\ 0 & \text{o.w.} \end{cases}$$

$H_0: \theta = 2$ vs. $H_1: \theta = 1$

$\Omega = \{\theta : \theta = 1, 2\}$

$$\frac{f(x_1;2)f(x_2;2)}{f(x_1;1)f(x_2;1)} \le \frac{1}{2} \Rightarrow \text{reject } H_0$$

$\because X_1 \perp X_2$

$$\therefore \frac{f(x_1;2)f(x_2;2)}{f(x_1;1)f(x_2;1)} = \frac{1}{4} \cdot e^{(+x_1+x_2)\cdot\frac{1}{2}} \le \frac{1}{2}$$

$$\Rightarrow \exp\left(\frac{x_1+x_2}{2}\right) \le 2 \Rightarrow x_1 + x_2 \le 2\ln(2)$$

$$\gamma(\theta) = P_\theta(\text{reject } H_0) = P_{\theta=1}(X_1 + X_2 \le 2\ln(2))$$

$\because X_1, X_2 \sim \exp(1)$ $\therefore X_1 + X_2 \sim \Gamma(2, 1)$

$$\Rightarrow \gamma(1) = \int_0^{2\ln(2)} x e^{-x}\, dx = -xe^{-x} - e^{-x} \Big|_0^{2\ln(2)}$$

$$= -2\ln(2)\exp(-2\ln(2)) - e^{-2\ln(2)} - (0 - 1)$$

$$= -2\ln(2)\cdot(2)^{-2} - (2)^{-2} + 1$$

$$= \frac{3}{4} - \frac{1}{2}\ln(2) = \underline{\underline{0.4034}}$$

For calculating α, $X_1, X_2 \sim \exp(2)$

$\Rightarrow Z = X_1 + X_2 \sim \Gamma(2, \underline{2})$

$\beta = 2$, $\lambda = \frac{1}{2}$

$$f(z) = \frac{(\frac{1}{2})^2 x^1 e^{-2x}}{1} = \frac{1}{4} x e^{-\frac{1}{2}x} \quad \text{for } x > 0$$

$$\therefore \alpha = \int_0^{2\ln(2)} \frac{1}{4} x e^{-\frac{1}{2}x}\, dx = \frac{1}{4}\left(-2xe^{-\frac{1}{2}x} - 4e^{-\frac{1}{2}x}\right)\Big|_0^{2\ln(2)}$$

$$= \frac{1}{4}\left[-4\ln(2)e^{-\ln(2)} - 4e^{-\ln(2)} - (0-4)\right]$$

$$= \frac{1}{4}\left[-2\ln(2) - 2 + 4\right] = \frac{1}{2} - \frac{1}{2}\ln(2)$$

$$= \underline{\underline{0.1534}}$$

prob. 3

$Y \sim Bin(n, p)$

$H_0: p = \frac{1}{2}$ vs. $H_1: p > \frac{1}{2}$ if $Y \geq C$

$$P(Y = y) = \binom{n}{y} p^y (1-p)^{n-y}$$

$$\gamma(p) = P_y(\text{reject } H_0) = \underline{P_y(Y \geq C)}$$

when n is large enough, apply CLT

$\Rightarrow$ Test Statistic: $\frac{Y - np}{\sqrt{np(1-p)}} \sim N(0,1)$

$$\Rightarrow \gamma(p) = P\left(\frac{Y-np}{\sqrt{np(1-p)}} \geq \frac{C - np}{\sqrt{np(1-p)}}\right)$$

$$\gamma(\tfrac{1}{2}) = P\left(Z \geq \frac{C - \frac{n}{2}}{\sqrt{n \cdot \frac{1}{4}}}\right) = 0.10$$

$$\gamma(\tfrac{2}{3}) = P\left(Z \geq \frac{C - \frac{2}{3}n}{\sqrt{n \cdot \frac{2}{9}}}\right) = 0.95$$

$$\Rightarrow \frac{C - \frac{n}{2}}{\frac{1}{2}\sqrt{n}} = 1.28 \quad \& \quad \frac{C - \frac{2}{3}n}{\frac{1}{3}\sqrt{2n}} = -1.65$$

$① \Rightarrow \frac{1}{6}n = 1.4178\sqrt{n}$

$\Rightarrow \frac{1}{36}n^2 = 2.0102\,n \Rightarrow n \cong 72.37$

$\Rightarrow \boxed{n = 73}$

$② \Rightarrow c - \frac{73}{2} = 1.28 \cdot \frac{1}{2}\sqrt{73} \Rightarrow c \cong 41.968$

$\Rightarrow \boxed{c = 42}$

prob. 4

$Y_1 < Y_2 < Y_3 < Y_4$ $\quad f(x;\theta) = \begin{cases} \frac{1}{\theta}, & 0 < x < \theta \\ 0, & \text{o.w.} \end{cases}$ $\quad \theta > 0$

$H_0: \theta = 1$ vs. $H_1: \theta > 1$

$Y_4 \geq c \Rightarrow H_0$ is rejected

a) $F(y) = P(Y \leq y) = \int^{y} \frac{1}{\theta}\,dy = \frac{y}{\theta}$

accept H_0: $Y_1 < c$, $Y_2 < c$, $Y_3 < c$ and $Y_4 < c$

$\Rightarrow F_{cum}(y) = \left(\frac{y}{\theta}\right)^4$

$\alpha = P_{\theta=1}(Y_4 \geq c) = 1 - \left(\frac{c}{1}\right)^4 = 1 - c^4 = 0.05$

$\Rightarrow c = \underline{0.987}$

b) $\gamma(\theta) = P_\theta(Y_4 \geq c) = 1 - \left(\frac{c}{\theta}\right)^4$

when $c = 0.987$, $\boxed{\gamma(\theta) = 1 - \frac{0.95}{\theta^4}, \quad \theta > 0}$

prob. 5

$p = 0.14$ $\quad y = 104$ out of a random sample of $n = 590 \Rightarrow CLT$

a) $H_0: p = 0.14 \qquad H_a: p > 0.14$

b) Critical region for $\alpha = 0.01$

Test Statistic: $\dfrac{\frac{y}{n} - p}{\sqrt{np(1-p)}} \sim N(0,1)$

$\Rightarrow P\left(\dfrac{\frac{y}{n} - p}{\sqrt{p(1-p)}/\sqrt{n}} > z\right) = 0.01$

$\Rightarrow \dfrac{\frac{y}{590} - 0.14}{0.01429} > z_{0.01} = 2.33$

$\Rightarrow \boxed{y > 103}$

$C = \{Y : y \geq 103\}$

c) p-value: $P\left(z \geq \dfrac{\frac{104}{590} - 0.14}{\sqrt{\frac{p(1-p)}{n}}}\right) = \boxed{0.0055}$

According to the critical region: $C = \{Y : y \geq 103\}$ $y = 104$ is in this region, so the H_0 should be rejected, and H_a should be accepted. Therefore, it's evident that this advertising campaign is successful.