Haoran Zhang | HW# 5 | STAT 511

prob. 1

$Y_1 < Y_2 < Y_3 < Y_4$ $\quad f(x) = \begin{cases} e^{-x}, & 0 < x < \infty \\ 0 & o.w. \end{cases}$

Find $P(Y_4 \geq 3)$

$X_1, X_2, X_3, X_4 \overset{iid}{\sim} f(x)$

$P(Y_4 \geq 3) = 1 - P(Y_4 < 3) = 1 - P(X_1, X_2, X_3, X_4 < 3)$

$= 1 - P(X_1 < 3) P(X_2 < 3) P(X_3 < 3) P(X_4 < 3)$

$P(X_1 < 3) = \int_0^3 e^{-x} dx = -e^{-x}\Big|_0^3 = -(e^{-3} - e^0)$

$= 1 - e^{-3}$

$\therefore P(Y_4 \geq 3) = 1 - [P(X_1 < 3)]^4 = \boxed{1 - [1 - e^{-3}]^4}$

prob. 2

$Y_1 < Y_2 \qquad X_1, X_2 \sim N(0, \sigma^2)$

a) $E(Y_1) = \iint y_1 g_{12}(y_1, y_2)\, dy_2\, dy_1$

$g_{12}(y_1, y_2) = 2!\, f(y_1) f(y_2) = 2! \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y_1^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y_2^2}{2\sigma^2}}$

$= \frac{1}{\pi\sigma^2} e^{-\frac{y_1^2 + y_2^2}{2\sigma^2}}$

$\therefore E(Y_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{y_2} y_1 \cdot \frac{1}{\pi\sigma^2} e^{-\frac{y_1^2 + y_2^2}{2\sigma^2}} dy_1\, dy_2$

$= \frac{1}{\pi\sigma^2} \int_{-\infty}^{\infty} \exp\left(-\frac{y_2^2}{2\sigma^2}\right) \left[\int_{-\infty}^{y_2} y_1 \exp\left(-\frac{y_1^2}{2\sigma^2}\right) dy_1\right] dy_2$

$= \frac{1}{\pi\sigma^2} \int_{-\infty}^{\infty} \exp\left(-\frac{y_2^2}{2\sigma^2}\right) \cdot -\sigma^2 \exp\left(-\frac{y_2^2}{2\sigma^2}\right) dy_2$

$$= -\frac{1}{\pi}\int_{-\infty}^{\infty} \exp\left(-\frac{y_2^2}{2\sigma^2/2}\right) dy_2$$

$$= -\frac{1}{\pi}\cdot\left(\sqrt{2\pi\sigma^2/2}\right) = \boxed{-\frac{\sigma}{\sqrt{\pi}}}$$

b) $\operatorname{cov}(Y_1, Y_2) = E[Y_1 Y_2] - EY_1 EY_2$

$$E(Y_1Y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{y_2} y_1 y_2 \cdot \frac{1}{\pi\sigma^2} \exp\left(-\frac{y_1^2+y_2^2}{2\sigma^2}\right) dy_1 dy_2$$

$$= -\frac{1}{\pi}\int_{-\infty}^{\infty} y_2 \exp\left(-\frac{y_2^2}{2\sigma^2/2}\right) dy_2 = \underline{\underline{0}}$$

$$EY_2 = \int_{-\infty}^{\infty}\int_{y_1}^{\infty} y_2 \cdot \frac{1}{\pi\sigma^2} \exp\left(-\frac{y_1^2+y_2^2}{2\sigma^2}\right) dy_2 dy_1$$

$$= \int_{-\infty}^{\infty} \frac{1}{\pi\sigma^2} \exp\left(-\frac{y_1^2}{2\sigma^2}\right)\cdot\left[\int_{y_1}^{\infty} y_2 \exp\left(-\frac{y_2^2}{2\sigma^2}\right) dy_2\right] dy_1$$

$$= \int_{-\infty}^{\infty} \frac{1}{\pi\sigma^2} \exp\left(-\frac{y_1^2}{2\sigma^2}\right)\cdot \sigma^2 \exp\left(-\frac{y_1^2}{2\sigma^2}\right) dy_1$$

$$= \frac{\sigma}{\sqrt{\pi}}$$

$$\therefore \operatorname{cov}(Y_1, Y_2) = 0 - \left(-\frac{\sigma}{\sqrt{\pi}}\right)\left(\frac{\sigma}{\sqrt{\pi}}\right) = \boxed{\frac{\sigma^2}{\pi}}$$

prob. 3

$n = 120$

$$Q_{k-1} = \sum_{i=1}^{k} \frac{(X_i - np_{i0})^2}{np_{i0}} \Rightarrow Q_5 = \sum_{i=1}^{k} \frac{(X_i - 20)^2}{20}$$

$$= \frac{(b-20)^2}{20} + \frac{(20-b)^2}{20} = \frac{(b-20)^2}{10}$$

if $Q_5 \sim \chi^2_5$

we would reject H_0, if $Q_5 \geq \chi^2_5$, $\alpha = 0.025$

$\chi^2_{5, \alpha=0.025} = 12.833$ from the table

$\therefore \frac{(b-20)^2}{10} \geq 12.833$

$\Rightarrow (b-20)^2 \geq 128.33$

$\Rightarrow b \geq 31.328$ or $b \leq 8.672$

$\Rightarrow$ b could be in $(0, 8.672]$ or

$[31.328, 120)$

prob. 4

a) $\bar{x} = \frac{0\cdot20 + 1\cdot40 + 2\cdot16 + 3\cdot18 + 4\cdot6}{100}$

$= 1.5$

$X \sim \text{Poisson}(1.5)$

$\Rightarrow P(X=x) = \frac{e^{-1.5} \cdot 1.5^x}{x!}$

x	0	1	2	3	x>3
obs.	20	40	16	18	6
Exp.	22.313	33.470	25.102	12.551	6.564

$\chi^2 = \Sigma \frac{(f_{obs} - f_{exp})^2}{f_{exp}} = \frac{(20-22.313)^2}{22.313} + \frac{(40-33.470)^2}{33.470}$

$+ \frac{(16-25.102)^2}{25.102} + \frac{(18-12.551)^2}{12.551} + \frac{(6-6.564)^2}{6.564}$

$= \boxed{7.228}$

b) $df = k - p - 1 = 5 - 2 = \underline{\underline{3}}$

c) reject H_0 if $Q_3 \geq \chi^2_{3, \alpha = 0.05}$

from table, $\chi^2_{3, \alpha=0.05} = 7.815$

$\because Q_3 = 7.228 < 7.815$

$\therefore$ we fail to reject H_0, and the data is evident to be poisson distributed.