Haoran Zhang | HW#6 | STAT 517

prob. 1

$Y_n \sim b(n, p)$ $\quad E\bar{Y}_n = p$ $\quad var(\bar{Y}_n) = \frac{p(1-p)}{n}$

a)

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

$$\Rightarrow P\left(|\bar{Y}_n - p| \geq \varepsilon\frac{\sqrt{n}}{\sqrt{p(1-p)}} \cdot \sqrt{\frac{p(1-p)}{n}}\right) \leq \frac{p(1-p)}{\varepsilon^2 n}$$

as $n \to \infty$, $P(|\bar{Y}_n - p| \geq \varepsilon) \leq \frac{p(1-p)}{\varepsilon^2 n} \to 0$

$$\therefore \frac{Y_n}{n} \xrightarrow{P} p$$

b)

$$E\left(1 - \frac{Y_n}{n}\right) = 1-p \quad var\left(1 - \frac{Y_n}{n}\right) = \frac{p(1-p)}{n}$$

$$\Rightarrow P\left(\left|1 - \frac{Y_n}{n} - (1-p)\right| \geq \varepsilon\frac{\sqrt{n}}{\sqrt{p(1-p)}} \cdot \sqrt{\frac{p(1-p)}{n}}\right) \geq \frac{p(1-p)}{\varepsilon^2 n}$$

as $n \to \infty$, $P\left(\left|1 - \frac{Y_n}{n} - (1-p)\right| \geq \varepsilon\right) \geq \frac{p(1-p)}{\varepsilon^2 n} \to 0$

$$\therefore 1 - \frac{Y_n}{n} \xrightarrow{P} 1-p$$

c)

$\because \frac{Y_n}{n} \xrightarrow{P} p$ and $1 - \frac{Y_n}{n} \xrightarrow{P} 1-p$

$$\therefore \frac{Y_n}{n}\left(1 - \frac{Y_n}{n}\right) \xrightarrow{P} p(1-p)$$

prob. 2

$$W_n \sim \begin{matrix} \text{mean } \mu \\ \text{var } b/n^p \end{matrix}$$

$$P\left(|W_n - \mu| \geq \varepsilon \sqrt{\frac{n^p}{b}} \cdot \sqrt{\frac{b}{n^p}}\right) \geq \frac{b}{\varepsilon^2 n^p}$$

as $n \to \infty$, $P(|W_n - \mu| \geq \varepsilon) \geq \frac{b}{\varepsilon^2 n^p} \to 0$

$\therefore W_n \xrightarrow{P} \mu$

prob. 3

$$f(x) = \begin{cases} e^{-(x-\theta)} & x > \theta, \ -\infty < \theta < \infty \\ 0 & \text{o.w.} \end{cases}$$

$Y_n = \min(X_1, \cdots, X_n)$

$F_X(x) = 1 - e^{\theta - x}$, $x > \theta$, $-\infty < \theta < \infty$

$F_{Y_n}(y) = P(\min(X_1, \cdots, X_n) \leq y)$

$= 1 - P(X_1 > y)^n = 1 - [1 - (1 - e^{\theta - y})]^n$

$= 1 - (e^{\theta - y})^n$, $y > \theta$, $-\infty < \theta < \infty$

$P(|Y_n - \theta| < \varepsilon) = P(\theta - \varepsilon < Y_n < \theta + \varepsilon)$

$= F_{Y_n}(\theta + \varepsilon) - F_{Y_n}(\theta - \varepsilon)$, $\varepsilon > 0$

$= 1 - e^{-\varepsilon n} - 0$

$= 1 - e^{-\varepsilon n}$

as $n \to \infty$, $P(|Y_n - \theta| < \varepsilon) \to 1$

$\therefore Y_n \xrightarrow{P} \theta$

prob. 4

$$X_1, \cdots, X_n \overset{iid}{\sim} \begin{cases} f(x) = e^{-(x-\theta)}, & \theta < x < \infty \\ f(x) = 0, & o.w. \end{cases}$$

$Y_1 = \min(X_1 \cdots X_n)$ $\quad \underline{Z_n = n(Y_1 - \theta)}$

$F_X(x) = 1 - e^{\theta - x}, \quad \theta < x < \infty$

$F_{Y_1}(y) = P(\min(X_1 \cdots X_n) < y)$

$= 1 - P(X_1 > y, \cdots X_n > y) = 1 - (1 - (1 - e^{\theta - y}))^n$

$= 1 - e^{n(\theta - y)}$

$F_{Z_n}(z) = P(Z_n < z) = P(n(Y_1 - \theta) < z)$

$= P(Y_1 < \frac{z}{n} + \theta) = F_{Y_1}(\frac{z}{n} + \theta) = 1 - e^{n(\theta - \frac{z}{n} - \theta)}$

$= 1 - e^{-z}, \quad z > 0$

$$\boxed{\lim_{n \to \infty} F_{Z_n}(t) = \begin{cases} 1 - e^{-t}, & t > 0 \\ 0, & o.w. \end{cases}}$$

prob. 5

$F_{Y_n}(y) = P(\max(X_1 \cdots X_n) \leq y)$

$= P(X_1 \leq y, \cdots, X_n \leq y) = [F(y)]^n$

$F_{Z_n}(t) = P(n(1 - F(Y_n)) \leq t) = P(1 - F(Y_n) \leq \frac{t}{n})$

$= P(F(Y_n) \geq 1 - \frac{t}{n}) = 1 - P(F(Y_n) \leq 1 - \frac{t}{n})$

$$= 1 - P\left(Y_n \le \bar{F}^{-1}\left(1 - \frac{t}{n}\right)\right)$$

$$= 1 - P\left(\max(X_1 \cdots X_n) \le \bar{F}^{-1}\left(1 - \frac{t}{n}\right)\right)$$

$$= 1 - \left[\bar{F}\left(\bar{F}^{-1}\left(1 - \frac{t}{n}\right)\right)\right]^n$$

$$= 1 - \left(1 - \frac{t}{n}\right)^n = 1 - e^{-t}, \quad t > 0$$

$$\lim_{n \to \infty} F_{Z_n}(t) = \begin{cases} 1 - e^{-t}, & t > 0 \\ 0, & \text{o.w.} \end{cases}$$

prob. 6

$$X_1 \cdots X_n \overset{iid}{\sim} \quad f(x) = \begin{cases} e^{-x}, & 0 < x < \infty \\ 0, & \text{o.w.} \end{cases}$$

a) $$M_{Y_n}(t) = E\left(e^{tY_n}\right) = E\left(e^{t\sqrt{n}(\bar{X}_n - 1)}\right)$$

$$= E\left(e^{t\sqrt{n}\bar{X}_n - t\sqrt{n}}\right) = e^{-t\sqrt{n}} E\left(e^{t\sqrt{n}\bar{X}_n}\right)$$

$$= e^{-t\sqrt{n}} E\left(e^{t\sqrt{n} \cdot \sum_{i=1}^{n} X_i / n}\right)$$

$$M_X(t) = E\left(e^{tX}\right) = \int_0^\infty e^{tx} \cdot e^{-x} \, dx = \frac{1}{t-1} e^{tx - x} \Big|_0^\infty$$

$$= \frac{1}{1-t}, \quad t < 1$$

$$\therefore M_{Y_n}(t) = e^{-t\sqrt{n}} \cdot \left(M_X\left(\frac{t}{\sqrt{n}}\right)\right)^n = e^{-t\sqrt{n}} \cdot \left(\frac{1}{1 - \frac{t}{\sqrt{n}}}\right)^n$$

$$= e^{-t\sqrt{n}} \cdot \left(\frac{\sqrt{n}}{\sqrt{n} - t}\right)^n, \quad \frac{t}{\sqrt{n}} < 1 \Rightarrow t < \sqrt{n}$$

b)

$$\lim_{n\to\infty} M_{Y_n}(t) = \lim_{n\to\infty}\left(e^{-t\sqrt{n}}\cdot\left(\frac{\sqrt{n}}{\sqrt{n}-t}\right)^{n}\right)$$

$$= \lim_{n\to\infty}\left(\left(e^{+\frac{t}{\sqrt{n}}}\right)^{-n}\cdot\left(\frac{\sqrt{n}-t}{\sqrt{n}}\right)^{-n}\right)$$

$$= \lim_{n\to\infty}\left(\left(e^{\frac{t}{\sqrt{n}}}\cdot\left(1-\frac{t}{\sqrt{n}}\right)\right)^{-n}\right)$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{t}{\sqrt{n}}+\frac{t^2}{2!\,n}+\frac{t^3}{3!\,n^{\frac{3}{2}}}+\cdots\right)\cdot\left(1-\frac{t}{\sqrt{n}}\right)\right]^{-n}$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{t}{\sqrt{n}}+\frac{t^2}{2!\,n}+\cdots\right)-\left(\frac{t}{\sqrt{n}}+\frac{t^2}{n}+\frac{t^3}{2!\,n^{\frac{3}{2}}}+\cdots\right)\right]^{-n}$$

$$= \lim_{n\to\infty}\left[1-\frac{t^2}{2n}-\frac{t^3}{3n^{\frac{3}{2}}}-\cdots\right]^{-n}$$

$$\approx \lim_{n\to\infty}\left[1-\frac{t^2}{2n}+\frac{\varphi(n)}{n}\right]^{-n}$$

$\varphi(n) \to 0$ as $n\to\infty$

$$= e^{-\frac{t^2}{2}\cdot(-1)} = \boxed{e^{\frac{t^2}{2}}}$$

$e^{\frac{t^2}{2}}$ ~ mgf of $N(0,1)$

$\Rightarrow Y_n \xrightarrow{D} N(0,1)$