Haoran Zhang | HW#7 | STAT 517

prob. 1

$$Y \sim b(400, \tfrac{1}{5})$$

$$EY = np = 400 \times \tfrac{1}{5} = 80$$

$$Var(Y) = np(1-p) = 400 \times \tfrac{1}{5} \times \tfrac{4}{5} = 64$$

$$P\left(\frac{Y}{400} > 0.25\right) = P(Y > 100)$$

<u>Continuity correction</u> $\Rightarrow P(Y > 100.5)$

$$= P\left(\frac{Y-80}{\sqrt{64}} > \frac{100.5-80}{\sqrt{64}}\right)$$

$$= 1 - \Phi(2.5625) = \boxed{0.0052}$$

prob. 2

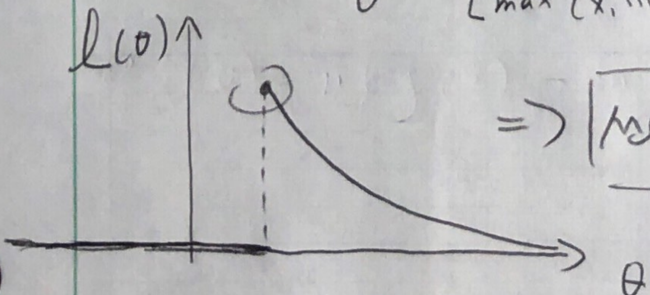
a) $X_1, \cdots X_n \overset{iid}{\sim} f(x,\theta) = \frac{2x}{\theta^2} \quad 0 < x \le \theta$

$$f(x,\theta) = \frac{2}{\theta^2} I_{(0,\theta]}(x) \Rightarrow L(\theta) = \prod_{i=1}^{n} \frac{2}{\theta^2} I_{(0,\theta]}(x_i)$$

$$= \frac{2^n}{\theta^{2n}} \prod_{i=1}^{n} I_{(0,\theta]}(x_i)$$

$$\prod_{i=1}^{n} I_{(0,\theta]}(x_i) = 1 \iff \max(X_1, \cdots X_n) \le \theta$$

$$\Rightarrow L(\theta) = \frac{2^n}{\theta^{2n}} I_{[\max(X_1, \cdots X_n), +\infty)}(\theta)$$



$$\Rightarrow \boxed{\text{MLE: } \hat{\theta} = \max(X_1, \cdots X_n)}$$

b) $E(c\hat{\theta}) = \theta$

$\hat{\theta} = \max(X_1 \cdots X_n) \Rightarrow Y = \max(X_1 \cdots X_n)$

$F_Y(y) = P(X_1 \le y, \cdots X_n \le y)$

$= [P(X_1 \le y)]^n = \left[\int_0^y \frac{2x}{\theta^2} dx\right]^n$

$= \left(\frac{y}{\theta}\right)^{2n}$

$f(y) = \frac{d}{dy}\left[\left(\frac{y}{\theta}\right)^{2n}\right] = \frac{2n}{\theta^{2n}} \cdot y^{2n-1}, \quad 0 < y \le \theta$

$E(Y) = \int_{-\infty}^{+\infty} y f(y)\, dy = \frac{2n}{\theta^{2n}} \int_0^{\theta} y^{2n}\, dy = \frac{2n}{\theta^{2n}} \cdot \left(\frac{y^{2n+1}}{2n+1}\right)\Big|_0^{\theta}$

$= \frac{2n}{\theta^{2n}} \cdot \frac{\theta^{2n+1}}{2n+1} = \frac{2n}{2n+1} \cdot \theta$

$E(Y) = \frac{2n}{2n+1} \cdot \theta \Rightarrow E\left(\frac{2n+1}{2n} Y\right) = \theta$

$\Rightarrow c = \frac{2n+1}{2n}$

c) $P(X_i \le m) = \frac{1}{2} \Rightarrow \int_0^m \frac{2x}{\theta^2} dx = \frac{1}{2}$

$\Rightarrow \left(\frac{m}{\theta}\right)^2 = \frac{1}{2} \Rightarrow \underline{m = \sqrt{\frac{1}{2}\theta^2}}$

$\because \hat{\theta} = \max(X_1 \cdots X_n)$

$\therefore \hat{m} = \sqrt{\frac{1}{2}[\max(X_1 \cdots X_n)]^2}$

prob. 3

$$L(p) = \prod_{i=1}^{50} f(x_i, p) = \prod_{i=1}^{50} \binom{5}{x_i} p^{x_i} (1-p)^{5-x_i}$$

$$\Rightarrow l(p) = \sum_{i=1}^{50} \log \binom{5}{x_i} + \sum_{i=1}^{50} x_i \log p + \sum_{i=1}^{50} (5-x_i) \log(1-p)$$

$$l'(p) = 0 + \frac{\sum_{i=1}^{50} x_i}{p} - \frac{\sum_{i=1}^{50} (5-x_i)}{1-p} = 0$$

$$\Rightarrow (1-p)\sum_{i=1}^{50} x_i - p\sum_{i=1}^{50} (5-x_i) = 0$$

$$\Rightarrow \sum_{i=1}^{50} x_i - \cancel{p\sum_{i=1}^{50} x_i} - p \cdot 50 \cdot 5 + \cancel{p\sum_{i=1}^{50} x_i} = 0$$

$$\Rightarrow \hat{p} = \frac{\sum_{i=1}^{50} x_i}{250} \qquad \text{check } l''(p) < 0 \checkmark$$

$$\Rightarrow \hat{p} = 0.424$$

$$P(X \geq 3) = \sum_{x=3}^{5} \binom{5}{x} \hat{p}^{x} (1-\hat{p})^{5-x}$$

$$= \binom{5}{3}(0.424)^3(1-0.424)^2$$

$$+ \binom{5}{4}(0.424)^4(1-0.424)^1$$

$$+ \binom{5}{5}(0.424)^5(1-0.424)^0$$

$$= 0.253 + 0.093 + 0.014 = \boxed{0.36}$$

prob. 4

$$l(\lambda) = \prod_{i=1}^{55} f(x_i, p) = \prod_{i=1}^{55} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \frac{e^{-55\lambda} \cdot \lambda^{\sum_{i=1}^{55} x_i}}{\prod_{i=1}^{55} x_i!}$$

$$l(\lambda) = -55\lambda + \sum_{i=1}^{55} x_i \cdot \ln(\lambda) - \ln\left(\prod_{i=1}^{55} x_i!\right)$$

$$l'(\lambda) = -55 + \frac{\sum_{i=1}^{55} x_i}{\lambda} = 0 \Rightarrow \hat{\lambda} = \frac{\sum_{i=1}^{55} x_i}{n}$$

$$\therefore \hat{\lambda} = 2.109$$

$$P(X=2) = \frac{e^{-2.109} \cdot 2.109^2}{2!} = \boxed{0.2699}$$

prob. 5

$$X_1 \cdots X_n \underset{}{\overset{iid}{\sim}} \begin{cases} f(x, \theta=1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, -\infty < x < \infty \\ f(x, \theta=2) = \frac{1}{\pi[1+x^2]}, -\infty < x < \infty \end{cases}$$

$$L(\theta=1) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_i^2}{2}} = (2\pi)^{-\frac{n}{2}} \cdot \exp\left(-\frac{1}{2} \cdot \sum_{i=1}^{n} x_i^2\right)$$

$$L(\theta=2) = \prod_{i=1}^{n} \frac{1}{\pi(1+x^2)} = \pi^{-n} \cdot \prod_{i=1}^{n} \frac{1}{1+x_i^2}$$

$$l(\theta=1) = -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\sum_{i=1}^{n} x_i^2$$

$$l(\theta=2) = -n\ln(\pi) - \sum_{i=1}^{n} \ln(1+x_i^2)$$

when $n=1$, $l(\theta=1) - l(\theta=2) = -\frac{1}{2}\ln(2\pi) - \frac{x_i^2}{2} + \ln(\pi) + \ln(1+x_i^2)$

$$\frac{d(l(\theta=1) - l(\theta=2))}{dt} = -\frac{1}{2} + \frac{1}{1+t} = 0, \quad \underline{\text{take } x_i^2 = t, \quad 0 \le t < \infty}$$

$$\Rightarrow t = 1$$

$$\frac{d^2(l(\theta=1) - l(\theta=2))}{dt^2} = -(1+t)^{-2} < 0$$

when $t=1$, $l(\theta=1) - l(\theta=2) = 0.4189$

$\therefore$ There are two cases :

① if $l(\theta=1) - l(\theta=2) > 0$,

$$\sum_{i=1}^{n}\left(\ln(1+x_i^2) - \frac{x_i^2}{2}\right) > \frac{n}{2}\ln(2\lambda) - n\ln(\lambda)$$

$\hat{\theta} = 1$

② if $l(\theta=1) - l(\theta=2) < 0$,

$$\sum_{i=1}^{n}\left(\ln(1+x_i^2) - \frac{x_i^2}{2}\right) < \frac{n}{2}\ln(2\lambda) - n\ln(\lambda)$$

$\hat{\theta} = 2$

③ if $l(\theta=1) = l(\theta=2)$

$\hat{\theta} = 1$ or 2