STA465 Homework 1

1.1 $Pr(\mu|Y) = \frac{Pr(Y,\mu)}{Pr(Y)} = \frac{P(Y|\mu,\sigma)\,P(\mu)}{P(Y)} \propto P(Y|\mu,\sigma)\,P(\mu)$

likelihood: $P(Y|\mu,\sigma) = \prod_{i=1}^{T} P(Y_i|\mu,\sigma)$

$$= (2\pi\sigma^2)^{-T/2} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{T}(Y_i-\mu)^2\right]$$

$$\propto \sigma^{-T} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{T}(Y_i-\mu)^2\right]$$

prior: $P(\mu) = (2\pi v^2)^{-1/2} \exp\left[-\frac{1}{2v^2}(\mu-s)^2\right]$

Posterior: $P(\mu|Y) = (2\pi\sigma^2)^{-T/2}(2\pi v^2)^{-1/2} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{T}(Y_i-\mu)^2\right]\exp\left[-\frac{1}{2v^2}(\mu-s)^2\right]$

$$= (2\pi\sigma^2)^{-T/2}(2\pi v^2)^{-1/2} \exp\left[-\frac{1}{2}\left(\frac{1}{\sigma^2}\sum_{i=1}^{T}(Y_i-\mu)^2 + \frac{1}{v^2}(\mu-s)^2\right)\right]$$

$$= (2\pi\sigma^2)^{-T/2}(2\pi v^2)^{-1/2} \exp\left[-\frac{1}{2V_T^2}(\mu-S_T)^2\right]$$ by (2.11, 2.12) in Bayesian Data Analysis 3rd ed.

where $S_T = \frac{\frac{1}{v^2}s + \frac{T}{\sigma^2}\bar{Y}}{\frac{1}{v^2}+\frac{T}{\sigma^2}}$

by slide 69, Lec1 and (2.12) in BDA. 3rd ed.

$V_T = \left(\frac{1}{v^2}+\frac{T}{\sigma^2}\right)^{-1}$

so $\mu|Y \sim N(S_T, V_T)$

1.2 $Y_i \sim \text{Binomial}(N, p)$ $\qquad P(Y|p) = \prod_{i=1}^{T} \text{Bin}(y_i|N,p) = \prod_{i=1}^{T} \binom{N}{y_i} p^{y_i}(1-p)^{N-y_i}$

$p \sim \text{Beta}(\alpha, \beta)$

$$P(p) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1}$$

$$P(p|y) = \frac{P(y|p)P(p)}{P(y)} \propto P(Y|P)P(p)$$

$$= \left[\prod_{i=1}^{T} \binom{N}{Y_i} p^{Y_i}(1-p)^{N-Y_i}\right]\left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1}\right]$$

$$\propto p^{T\bar{Y}}(1-p)^{T(N-\bar{Y})} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1}$$

- $\binom{N}{Y_i}$ does not depend on p and can thus be treated as a constant
- similarly, $\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$

$$\propto p^{T\bar{Y}}(1-p)^{T(N-\bar{Y})} p^{\alpha-1}(1-p)^{\beta-1}$$

$$= p^{T\bar{Y}+\alpha-1}(1-p)^{T(N-\bar{Y})+\beta-1}$$

so $p|y \sim \text{Beta}(\alpha^*, \beta^*)$ where $\alpha^* = T\bar{Y}+\alpha$

$\beta^* = T(N-\bar{Y})+\beta$