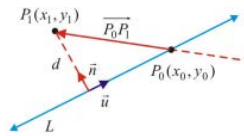
Lesson 6: Unit 4 – Relationships Between Points, Lines and Planes (2)

Next we will investigate the distance from a point to a line in both 2 and 3 dimensions, starting with 2-d.

Distance from a Point to a Line in R^2

Let L : Ax + By +C = 0 be a line in R^2, $P_1(x_1,y_1)$ be a generic point on the xy-plane and $P_0(x_0,y_0)$ be a specific point on this line, so: Ax_0 + By_0 +C = 0 . The distance d between the point P_1 to the line L is given by (scalar projection of $\overrightarrow{P_0P_1}$ onto the normal vector $\vec{n}$):



d = $\frac{|\overrightarrow{P_0P_1}\cdot\vec{n}\,|}{|\vec{n}|}$

Using $\vec{n}$ = (A,B), $|\vec{n}\,|$ = $\sqrt{A^2 + B^2}$ and

$\overrightarrow{P_0P_1}\cdot\vec{n}$ = $(x_1 - x_0, y_1 - y_0)$ · (A,B)

= A$(x_1 - x_0)$ + B$(y_1 - y_0)$

= Ax_1 + By_1 - Ax_0- By_0

= Ax_1 + By_1 + C

We get

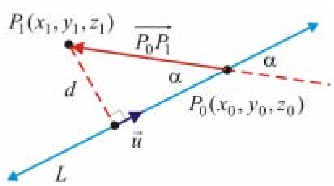
d = $\frac{|\mathbf{A}x_1 + \mathbf{B}y_1 + \mathbf{C}|}{\sqrt{A^2+B^2}}$

Ex. Find the distance between the point P_1 (3, 1) and the line L: −2x + 3y + 6 = 0.

d = $\frac{|-2(3) + 3(1) + 6|}{\sqrt{(-2)^2+(3)^2}} = \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13}$

Distance from a Point to a Line in R^3

Let L: $\vec{r} = \vec{r_0} + t\vec{u}$, t ∈ R be a line defined by its vector equation and $P_0(x_0, y_0, z_0)$ be a specific point on this line.



The distance d from a point $P_1(x_1, y_1, z_1)$ to the line L may be found using:

d =| $\overrightarrow{P_0P_1}$|sinα.

Because $|\overrightarrow{P_0P_1} \times \vec{u}| = |\overrightarrow{P_0P_1}|\ |\vec{u}|\sin\alpha$, the distance formula can be written

d = $\dfrac{|\overrightarrow{\boldsymbol{P_0P_1}} \times \vec{\boldsymbol{u}}\,|}{|\vec{\boldsymbol{u}}|}$

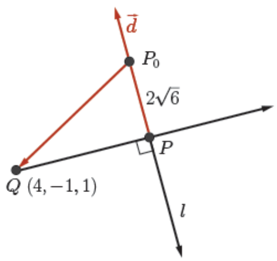
Note: You don't need to memorize this formula, just understand how was derived.

Ex. Find the distance from the point $Q(4, -1, 1)$ to the line

l: $\begin{cases} x = 1 + 2t \\ y = 3 - t \\ z = -1 + t \end{cases}$, t∈R

Solution

Method 1



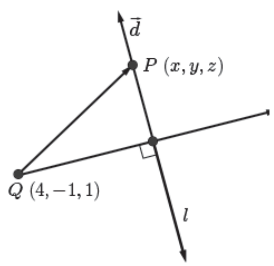
$P_0\text{P} = \dfrac{|\overrightarrow{P_0Q}\cdot\vec{d}\,|}{|\vec{d}|} = \dfrac{|(3,-4,2)\cdot(2,-1,1)\,|}{\sqrt{(2)^2+(-1)^2+(1)^2}} = 2\sqrt{6}.$

$P_0\text{Q} = \sqrt{29}$

$P_0\text{Q}^2 = P_0\text{P}^2 + PQ^2$

$\text{PQ} = \sqrt{5}$

Method 2



We require that QP ⊥ *l*.

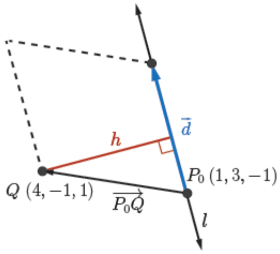
$\overrightarrow{QP} \cdot \vec{d} = 0$

(-3+2t , 4-t , -2+t) · (2, -1, 1) = 0

t = 2

$|\overrightarrow{QP}\,| = \sqrt{5}$

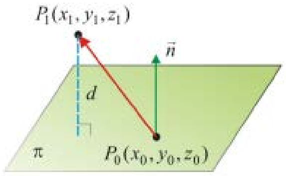
Method 3



d = $\frac{|\overrightarrow{P_0Q}\times\vec{d}\,|}{|\vec{d}|}$ = $\sqrt{5}$

Distance from a Point to a Plane

Let **π**: Ax + By +Cz + D = 0 be a plane, $P_1(x_1,y_1,z_1)$ be a generic point on the xy-plane and $P_0(x_0,y_0,z_0)$ be a specific point on this plane, so: Ax_0 + By_0 +Cz_0 = 0 . The distance d between the point P_1 to the plane π is given by (scalar projection of $\overrightarrow{P_0P_1}$ onto the normal vector $\vec{n}$):



d = $\frac{|\overrightarrow{P_0P_1}\cdot\vec{n}\,|}{|\vec{n}|}$

And we derive a formula for the distance from a point to a plane:

d = $\boldsymbol{\frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2+B^2+C^2}}}$

Ex. Find the distance between the point R(−2, 0, 3) and the plane π : 2x − 3y + z − 6 = 0 .

Solution

d = $\frac{|2(-2)-3(0)+3-6|}{\sqrt{2^2+(-3)^2+1^2}} = \frac{\sqrt{14}}{2}$

Ex. Find the distance between the parallel planes. π_1: 3x + 6y − 9z − 3 = 0 , π_2: 2x + 4y − 6z − 4 = 0

Solution

$P_1(1,0,0) \in \pi_1$

d = $\frac{|2(1)+4(0)-6(0)-4|}{\sqrt{2^2+4^2+(-6)^2}} = \frac{\sqrt{14}}{14}$