# 6. Special Relativity

## Grade 12 Physics

Olympiads School

Summer 2018

# Introduction

- By now, we've covered most fundamental topics in classical mechanics, plus a few other things.
- But physics is much more than that!
- Our next two units are definitely going to be. . . weird
  - Special Relativity–"the very fast"
  - Quantum Mechanics–"the very small"

# What Everyone Knows

$$\boxed{E = mc^2}$$

Most people know about *this* about the theory of relativity, but what does it actually mean?

# Starting with The Story

- We start with how Einstein came up with this theory
- It’s a very interesting story
- Shows us many aspects about how physics is done
- We’ll learn a lot more about relativity than most schools can teach

# Frame of Reference

## A Quick Review

A **frame of reference** (or “reference frame”, or just “frame”) is a hypothetical mobile “laboratory” an observer uses to make measurements (e.g. mass, lengths, time). At a minimum, it includes:

- A ruler to measure lengths
- A clock to measure the passage of time
- A scale to compare forces
- A balance to measure masses

High-school textbooks often refer to the frame of reference as a “coordinate system”. While it certainly includes that, this definition often makes it difficult to understand special relativity.

# Frame of Reference

## A Quick Review

- We assume that the hypothetical laboratory is *perfect*—the hypothetical "instruments" have zero errors
- What matters is the *motion* (at rest, uniform motion, acceleration etc) of your laboratory, and how it affects the measurement that you make
- "From the point of view of…"

# Newtonian (Classical) Relativity

In Newtonian physics, space and time are *absolute*:

- $1\,\text{m}$ is $1\,\text{m}$ no matter where you are in the universe
- $1\,\text{s}$ is $1\,\text{s}$ no matter where you are in the universe
- Measurements of space and time do not depend on motion

If space and time are absolute, then *all* velocities are relative

- Measured velocities depend on the motion of the observer

An **inertial frame of reference** is moving in uniform motion (constant velocity, zero acceleration)

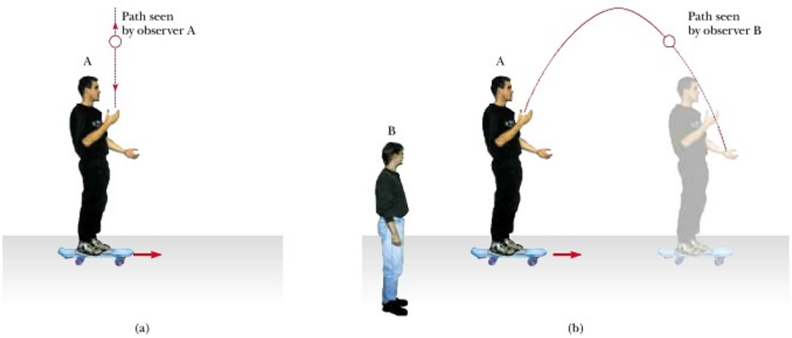
## The Principle of Relativity

All laws of motion must apply equally in all inertial frames of reference.

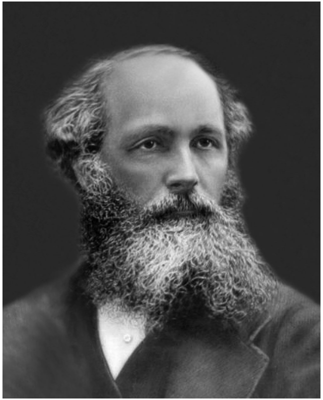
# Inertial Frame of Reference

## A Quick Review

- Observer A is moving with the skateboard at a constant velocity
- Observer B is standing on the side of the road
- Both observers see different motion, but they agree on the equations that govern the motion



# New Physics: Maxwell's Equations



James Clerk Maxwell

- For centuries, when experimental results would always agree with this Newtonian framework. . .
- Until the middle of 19th Century, when instruments were accurate enough to measure behaviours that were a bit different from prediction
- Then came Maxwell's equations in 1861 and 1862
- Classical laws of electrodynamics that explains the relationship between
    - Electricity
    - Electric Circuits
    - Magnetism
    - Optics

# Maxwell's Equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = -\mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

That's a lot of symbols that you won't recognize. Solving them require *a lot* of difficult calculus that even most science students in university don't need to learn. (**i.e. you don't need to learn this**)

# Speed of Light

If you are able to solve those equations (an exercise in AP Physics), they show that all electromagnetic waves (including light) travel with the same speed ("speed of light"):

$$\boxed{c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 299\,792\,458\,\mathrm{m/s}}$$

- **Problem:** relative to *what*?
- The waves that physicists knew have to travel in some medium
  - Ocean wave $\longrightarrow$ water
  - Sound wave $\longrightarrow$ air

# Peculiar features of Maxwell's equation

- Maxwell's equations do not mention the *medium* in which EM waves travels
- When applying *Galilean transformation* (classical equation for relative motion) to Maxwell's equations, the law for magnetism break down: magnetic field lines appear to have beginnings/ends
- In *some* inertial frames of reference, Maxwell's equations are simple and elegant, but in another inertial frame of reference, they are ugly and complex
- Physicists at the time theorized that—perhaps—there is/are actually *preferred* inertial frame(s) of references
- This violate the *principle of relativity*

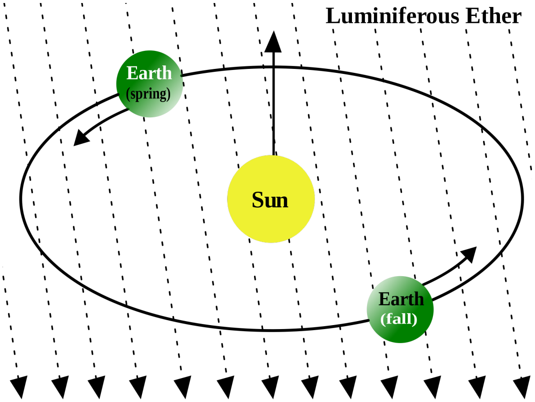
# The Illusive Aether

- Maxwell's hypothesis: the speed of light $c_0$ is relative to a hypothetical "luminiferous aether"
- In order for this "aether" (or "ether") to exist, it must have some fantastic (as in, a fantasy, too good to be true!) properties:
  - *All* space is filled with aether
  - Massless
  - Zero viscosity
  - Non-dispersive
  - Incompressible
  - Continuous at a very small (sub-atomic) scale

**So, it sounds fantastic, but how do we find it?**

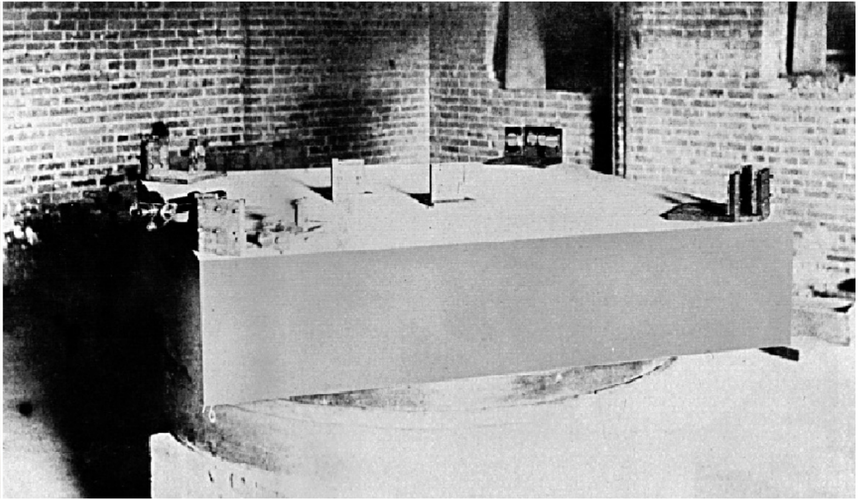
# The Michelson-Morley Experiment

If aether exists, then at different times of the year, then Earth will have a different relative velocity with respect to it:



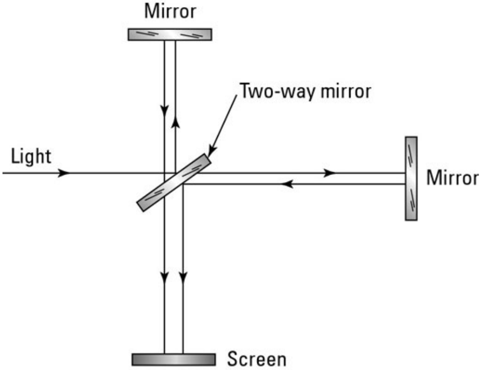

And this aether will cause the light to either speed up, or slow down.

# The Michelson Interferometer



The experiment is ingenious but very difficult. . .

# The Michelson Interferometer



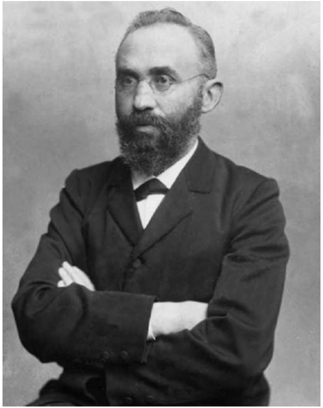

- A beam of light is split into two using a two-way (half-silvered) mirror
- The two beams are reflected and finally arriving at the screen
- The path are the same length, so if the *speed* of the light changes, we should see an interference pattern.
- **Except none were ever found!**

# What To Do with “Null Result”

The Michelson-Morley experiment failed to detect the illusive ether, even after many refinements. What does this mean?

- Majority view
  - **The experiment was flawed!**
  - Keep improving the experiment (or design a better experiment) and the ether will eventually be found
- Minority view:
  - **The hypothesis is wrong!**
  - The experiment showed it for what it is: ether cannot be found
- A few physicists: The must be **another explanation** that saves both experiment and theory

# Hendrik Lorentz



Hendrik Antoon Lorentz

- Considered the Michelson-Morley experiment to be significant
- Objects travelling in the direction of ether contracts in length, nullifying the experimental results
- Lorentz Factor:

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

- *No known physical phenomenon* can cause anything to contract; this makes no sense!
- Lorentz was on to something, but his thinking was wrong

# Making Maxwell's Equations Work

## Albert Einstein in 1905, Age 26

Albert Einstein

- Einstein was 26 years old working as a patent clerk in Switzerland
    - believed in the principle of relativity, and therefore
    - rejected the concept of a preferred frame of reference
- The failure of the Michelson-Morley experiment to find the flow of ether proves that it does not exist
- In order to make Maxwell's equations to work again, Einstein revisited two most fundamental concepts in physics: *space* and *time*

# Special Relativity

Published in the journal *Annalen der Physik* on September 26, 1905 in the article *On the Electrodynamics of Moving Bodies*

- Submitted on June 30, 1905 and passed for publication
- Einstein's third paper that year
- Mentions the names of only five other scientists: Newton, Maxwell, Hertz, Doppler and Lorentz
- Does not have any references to any publications
- Ignored by most physicists at first, until Max Planck took interests
- Called "special relativity" because it describes a "special case" without effects of gravity
- Later published theory of "general relativity" (much more complicated)

# Postulates of Special Relativity

## The Principle of Relativity

All laws of physics must apply equally in all inertial frames of reference.

- Reaffirms the principle in which all physics is based on
- Extend the principle to include electrodynamics

## The Principle of Invariant Light Speed

As measured in any inertial frame of reference, light is always propagated in empty space with a definite velocity $c$ that is independent of the state of motion of the emitting body.

- Reaffirms the results from Michelson-Morley experiment
- The hypothetical aether does not exist!

# Postulates of Special Relativity

The two postulates are unremarkable by themselves, but Einstein is able to show that when combined, the consequences are profound

# What's so Special About Special Relativity?

**Classical (Newtonian) relativity:**

- Space and time are absolute—speed of light must be relative
- When I catch a baseball, its speed depends on whether the person who threw it was moving towards me, or away from me. We can apply the equation of relative motion

$$\mathbf{v}_{AC} = \mathbf{v}_{AB} + \mathbf{v}_{BC}$$

- Similarly, the speed of light from any light source should depend on the motion of the source, and whether it is moving towards, or away from me!

**Einstein says: Impossible! This would violate Maxwell's equations and the principle of relativity.**

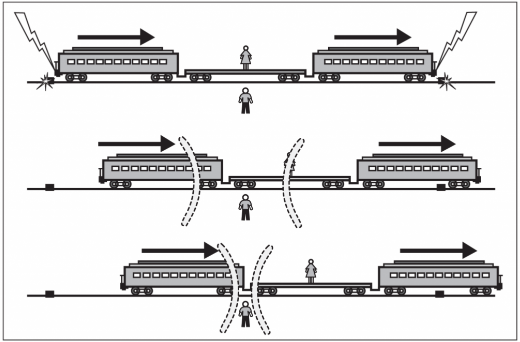
# What's so Special About Special Relativity?

**Einstein's special relativity:**

- Speed of light is absolute
- Space and time must be relative
- We must modify our traditional concepts:
  - Measurement of space (our ruler in the $x$-, $y$- and $z$-directions)
  - Measurement of time (our clock)
  - Concept of simultaneity (whether or not two events happens at the same time)

# Simultaneity: A Thought Experiment

Lightning bolt strikes the ends of a moving train



- The man on the ground sees the lightning bolt striking at the same time
- The woman on the moving train sees the lightning bolt on the right first

# Simultaneity: A Thought Experiment

From the man's perspective:

- He is stationary, but the train is moving
- When the lightnings strike, he is at an equal distance from the front and the back of the train
- The flash from the two lightning bolts arrive at his eyes at the same time

Therefore, his conclusions are:

- The two lightnings must have happened at the same time
- The woman in the train made the wrong observation: she only thinks that the lightning struck the front first because she's moving towards the light from the front

# Simultaneity: A Thought Experiment

Now, from the woman's perspective:

- *She* and the train are stationary, but the man and the rest of the world are moving
- When the lightnings strike, she is at an equal distance from the two ends of the train
- The flash from the front arrive first, then the back

Therefore, her conclusions are:

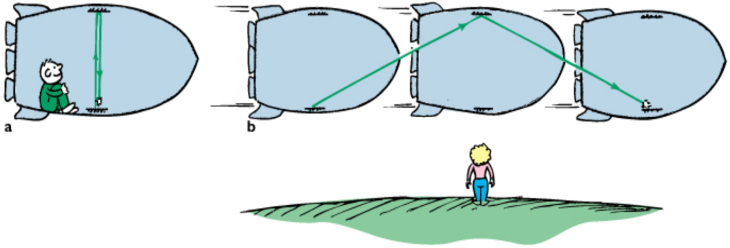
- Lightnings must have struck the front first
- The man on the road made the wrong observation: he only thinks that the lightning struck at the same time because he's moving towards the light from the back

# Simultaneity: A Thought Experiment

- The two observers disagree on the result, but
  - Neither person is wrong
  - Neither person is misinformed
- Both observers are valid *inertial* frames of reference
- This means that simultaneity depends on your motion

**Events that are simultaneous in one inertial frame of reference are not simultaneous in another.**

# Time Dilation: A Thought Experiment




I'm on a spaceship travelling in deep space, and I shine a light from $A$ to $B$. The distance between $A$ and $B$ is really just:

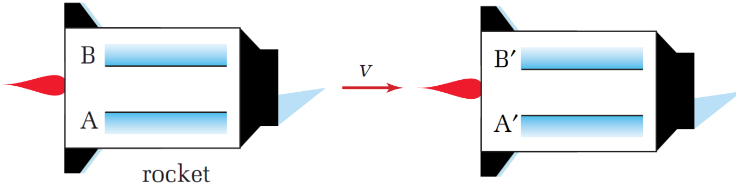
$$|AB| = c\Delta t_0$$

I know the speed of light $c$, and I know how long it took for the light pulse to reach $B$. (The reason I used $\Delta t_0$ will be obvious later.)

# Abandoning Concept of Absolute Time: Time Dilation

A “thought experiment”

**B**

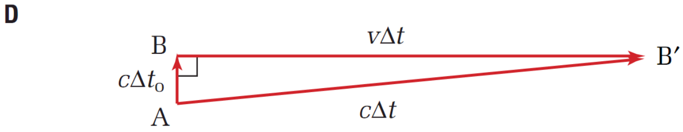



You are in space station watching my spaceship go past you at speed $v$. You would see that same beam of light travel from $A$ to $B'$ instead.

**C**

# Abandoning Concept of Absolute Time: Time Dilation

A “thought experiment”




$$c^2\Delta t^2 = v^2\Delta t^2 + c^2\Delta t_0^2$$

$$\left(c^2 - v^2\right)\Delta t^2 = c^2\Delta t_0^2$$

$$\left(1 - \frac{v^2}{c^2}\right)\Delta t^2 = \Delta t_0^2$$

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

# Abandoning Concept of Absolute Time: Time Dilation

$$\boxed{\Delta t = \frac{\Delta t_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}}$$

- $\Delta t_0$ is called the the **ordinary time**, **expanded time**, or **proper time**. It is the time measured by a person at rest relative to the object or event.
- $\Delta t$ is called the expanded time or **dilated time**. It is the time measured by a moving observer in another inertial frame of reference. Since $\sqrt{1 - \left(\frac{v}{c}\right)^2}$ is always smaller than 1, $\Delta t$ is always greater than $\Delta t_0$.

# Example Problem (A Simple One)

**Example 1a:** A rocket speeds past an asteroid at $0.800c$. If an observer in the rocket sees $10.0\,\mathrm{s}$ pass on her watch, how long would that time interval be as seen by an observer on the asteroid?

**Example 1b:** A rocket speeds past an asteroid at $0.800c$. If an observer in the *asteroid* sees $10.0\,\mathrm{s}$ pass on his watch, how long would that time interval be as seen by an observer on the *rocket*?

How can that be?!

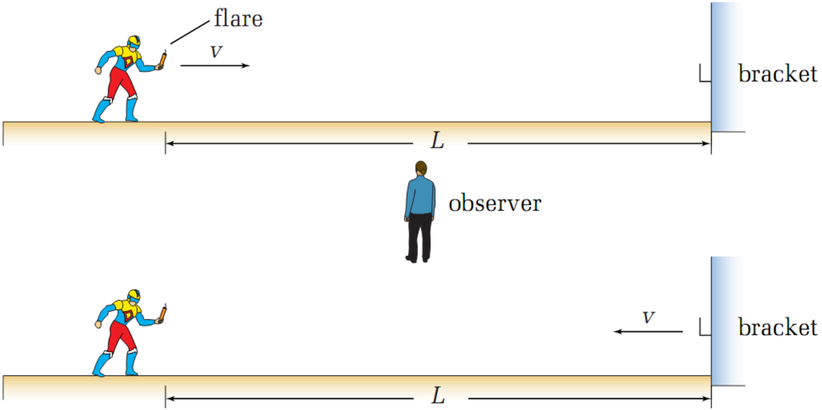
# Abandoning Concept of Absolute Space: Length Contraction

## Another Example

Captain Quick is a comic book hero who can run at nearly the speed of light. In his hand, he is carrying a flare with a lit fuse set to explode in $1.5\,\mu s$. The flare must be placed into its bracket before this happens. The distance $(L)$ between the flare and the bracket is $402\,m$.

# Abandoning Concept of Absolute Space: Length Contraction

## Another Example



# Abandoning Concept of Absolute Space: Length Contraction

## Another Example

If Captain Quick runs at $2.00 \times 10^8\,\mathrm{m/s}$, according to classical mechanics, he will not make it in time:

$$\Delta t = \frac{L}{v} = \frac{402\,\mathrm{m}}{2.00 \times 10^8\,\mathrm{m/s}} = 2.01 \times 10^{-6}\,\mathrm{s} = 2.01\,\mu\mathrm{s}$$

But according to relativistic mechanics, he makes it just in time...

# Abandoning Concept of Absolute Space: Length Contraction

## Another Example

To a stationary observer, the time on the flare is slowed:

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{1.5 \times 10^{-6}\,\mathrm{s}}{\sqrt{1-\left(\frac{2}{3}\right)^2}} = \frac{1.5 \times 10^{-6}\,\mathrm{s}}{0.7454} = 2.01 \times 10^{-6}\,\mathrm{s}$$

The stationary observer sees a passage of time of $\Delta t = 2.01\,\mu\mathrm{s}$, but Captain Quick, who is in the same reference frame as the flare, experiences a passage of time of $\Delta t_0 = 1.50\,\mu\mathrm{s}$, precisely the time for the flare to explode.

# Abandoning Concept of Absolute Space: Length Contraction

## Another Example

- So, if Captain Quick sees only $\Delta t_0 = 1.50\,\mu\text{s}$, then how far did he travel?
- He isn't travelling any faster, so he only other possibility is that **the distance actually got shorter** (in his frame of reference).
- How much did the distance contract?

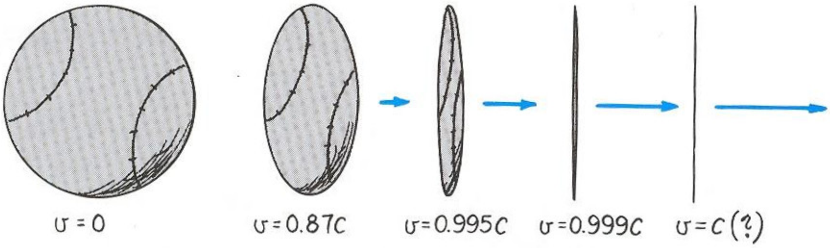
$$\boxed{L = L_0\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

For this example:

$$L = L_0\sqrt{1 - \left(\frac{v}{c}\right)^2} = (402\,\text{m}) \cdot \sqrt{1 - \left(\frac{2}{3}\right)^2} = 300\,\text{m}$$

# Length Contraction

Length contraction only occurs in the direction of motion



# Lorentz Factor

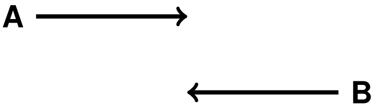
The **Lorentz factor** $\gamma$ is a short-hand for writing length contraction, time dilation and relativistic mass:

$$\boxed{\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}}$$

Then time dilation and length contraction can be written simply as:

$$\boxed{\Delta t = \gamma \Delta t_o} \qquad \boxed{L = \frac{L_o}{\gamma}}$$

# Let’s Summarize




If Person A and Person B are moving at constant speed with respect to one another (doesn’t matter if they’re moving towards, or away from each other)

- They cannot agree whether any events happens at the same time or not
- Each sees the other’s clock running slow
- Each sees the other “contracted” in length along the direction of motion

# Example Problem

**Example 2:** A spacecraft passes Earth at a speed of $2.00 \times 10^8\ \mathrm{m/s}$. If observers on Earth measure the length of the spacecraft to be $554\ \mathrm{m}$, how long would it be according to its passengers?

# But What About…

We've gone quite far with relativity already, but what about this very famous equation?

$$\boxed{E = mc^2}$$

We seem to be no closer to learning about it!

# Relativistic Momentum

In Unit 2, you were taught that momentum is mass times velocity. And back in Physics 11, you were taught that velocity is displacement over time:

$$\mathbf{p} = m\mathbf{v} = m\frac{\Delta\mathbf{x}}{\Delta t}$$

Now that you know $\Delta t$ depends on the motion, we can find the "relativistic version" of momentum for when $\mathbf{v}$ is high compared to $c$):

$$\mathbf{p} = m_0\frac{\Delta\mathbf{x}}{\Delta t_0} = \frac{m_0\Delta\mathbf{x}}{\Delta t\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{m_0\mathbf{v}}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

# Relativistic Mass

From the relativistic momentum expression, we can see the mass is also relativistic. The apparent mass $m$ as measured by a moving observer is related to its rest mass (intrinsic mass, invariant mass) $m_0$ by the Lorentz factor:

$$\boxed{m = \frac{m_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \gamma m_0}$$

The intrinsic mass has not increased, but a moving observer will note that the object behaves as if it is more massive. As $v \to c$, $m \to \infty$.

# Force and Work

Also in Unit 2, you were taught that force is the rate of change in momentum:

$$\mathbf{F} = \frac{\Delta \mathbf{p}}{\Delta t} = \frac{d\mathbf{p}}{dt}$$

and of course you remember that work is force times displacement:

$$W = \mathbf{F} \cdot \Delta \mathbf{d} = \int \mathbf{F} \cdot d\mathbf{x}$$

We can substitute the impulse expression for force, then substitute expression for relativistic momentum, and after some pretty difficult calculus (okay, difficult for some of you, may be)...

# Work and Kinetic Energy

$$W = \frac{m_0 c^2}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} - m_o c^2 = K$$

We know from the work-kinetic-energy theorem that the work $W$ done by a force $\mathbf{F}$ is equal to the change in kinetic energy $K$

# Relativistic Energy

$$\boxed{K = mc^2 - m_0c^2}$$

| Variable | Symbol | SI Unit |
|---|---|---|
| Kinetic energy of an object | $K$ | J |
| Relativistic mass (measured in moving frame) | $m$ | kg |
| Rest mass (measured in stationary frame) | $m_0$ | kg |
| Speed of light | $c$ | m/s |

# Relativistic Energy

## What This All Means

$$\boxed{K = mc^2 - m_0c^2 = (\gamma - 1)m_0c^2}$$

The minimal energy that any object has, regardless of it's motion (or lack of) is its **rest energy**:

$$E_0 = m_0c^2$$

The **total energy** of an object has is

$$E_T = mc^2 = \gamma m_0c^2$$

The difference between total energy and rest energy is the kinetic energy:

$$K = E_T - E_0$$

# Kinetic Energy–Classical vs. Relativistic

**Relativistic:**

$$K = (\gamma - 1)m_0c^2$$

**Newtonian:**

$$K = \frac{1}{2}mv^2$$

But are they really that different? If we do a series expansion of the square-root term term, we get:

$$K = m_0c^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \cdots\right) - m_0c^2 \approx \frac{1}{2}mv^2 + \cdots$$

When $v$ is small compared to $c$

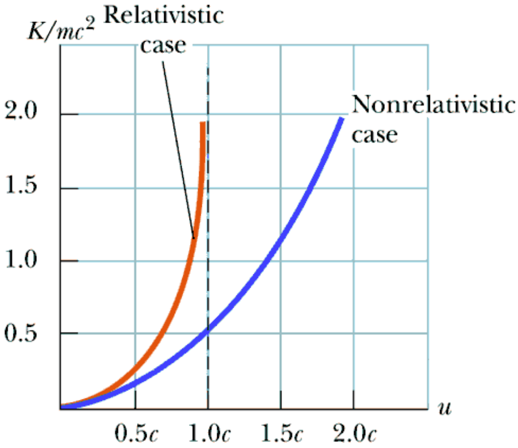
# Comparing Classical and Relativistic Energy

In classical mechanics:

$$K = \frac{1}{2}mv^2$$

In relativistic mechanics:

$$K = mc^2 - m_o c^2$$



# Example Problem

**Example 4:** A rocket car with a mass of $2.00 \times 10^3$ kg is accelerated from rest to $1.00 \times 10^8$ m/s. Calculate its kinetic energy:

1. Using the classical equation
2. Using the relativistic equation