Lesson 4 (on Calculus)

Unit 6 – Derivatives (2)

The Chain Rule

Next we want to differentiate composite functions.

Why? We need to be able to differentiate more complicated functions.

We can often write complicated functions as the composite of two or more simpler functions.

Composite functions are formed by combining one function with another.
To form a composite, one function is substituted into the input of a second function.
In function notation, if $g(x)$ and $h(x)$ are two functions, then the composite of g and h is the function $f(x)$ defined by

$$f(x)=(g\circ h)(x)=g(h(x))$$

where $g(x)$ is the outer function and $h(x)$ is the inner function. It is read either as g composed with h or as g of $h(x)$.
For example, the function $f(x) = (5x^2 + 2x)^7$ may be written as the composition of two other functions.
It can be decomposed into two separate component functions: $g(x) = x^7$ and $h(x) = 5x^2 + 2x$.

Note: The order of the composition is important.

Now, how to differentiate $f(x) = (5x^2 + 2x)^7$?
It would be very time consuming to expand, simplify, and then differentiate this expression.
Instead, we will develop a method called the chain rule to differentiate composite functions.

If we let $g(u) = u^7$ and we let $u = h(x) = 5x^2 + 2x$, then $f(x) = g(h(x))$.
Since we know how to differentiate $g(u)$ and $h(x)$ individually, the chain rule allows us to differentiate $y = g(h(x))$.
It turns out that the derivative of the composite function, $f(x) = g(h(x))$, is the product of the derivatives of $g(u)$ and $h(x)$. That is, $\boldsymbol{f'(x) = g'(u)h'(x)}$.

Consider the following derivatives as rates of change:

$\frac{dy}{du}$ is the rate of change of y with respect to u.

$\frac{du}{dx}$ is the rate of change of *u* with respect to *x*.
$\frac{dy}{dx}$ is the rate of change of *y* with respect to *x*.

Suppose *y* changes 3 times as fast as *u*, and *u* changes 4 times as fast as *x*. Then it makes sense that *y* changes 3 × 4 = 12 times as fast as *x*.
That is, the rate of change in *y* with respect to *x* is equal to the product of the other two rates.

The Chain Rule

The Chain Rule

If *h*(*x*) is differentiable at *x* and *g*(*x*) is differentiable at *h*(*x*), then the composite function, *f*(*x*) = *g*(*h*(*x*)) or *f*(*x*) = (*g*∘*h*)(*x*), is differentiable at *x* and $f'(x)$ is given by the product

$$f'(x) = g'(h(x)) \cdot h'(x)$$

In Leibniz notation, if *y*=*g*(*u*) and *u*=*h*(*x*) are both differentiable functions, then

$$\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right)$$

Note: Before using the chain rule, it is very important to first identify the inner and outer functions that make up the composition.

Ex: Differentiate *y* = $(5x^2 + 2x)^7$ using the chain rule.

Solution

Start by identifying the inner and outer functions.
Since *y* = $(5x^2 + 2x)^7$, let the inner function be *u* = $5x^2 + 2x$.
Then, in terms of *u*, the outer function is *y* = u^7.

Now, differentiate these two equations separately.

u = $5x^2 + 2x$ implies $\frac{du}{dx}$ = 10x + 2
y = u^7, implies $\frac{dy}{du}$ = $7u^6$ ⇨ $\frac{dy}{du}$ = $7(5x^2 + 2x)^6$

Recall the formula for the chain rule: $\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right)$.

Therefore, the derivative of *y* = $(5x^2 + 2x)^7$ is
$\frac{dy}{dx} = 7(5x^2 + 2x)^6(10x + 2) = 14(5x^2 + 2x)^6(5x + 1)$

Differentiating Rational Functions with the Chain Rule

A rational expression, $\frac{p(x)}{q(x)}$, can be written as

$$\frac{p(x)}{q(x)} = p(x)\cdot[q(x)]^{-1}$$

Once written in this form, we can apply the product rule and chain rule to differentiate.

$\left(\frac{p(x)}{q(x)}\right)' = \frac{p'(x)q(x)-p(x)q'(x)}{q^2(x)}$

Ex. Differentiate. Simplify the answer.

a. f(x) = $\frac{x^2+1}{x^3-1}$
b. f(x) = $\frac{x^3}{(1+x)^2}$

Solution

a. f'(x) = $\frac{(x^2+1)'(x^3-1)-(x^2+1)(x^3-1)'}{(x^3-1)^2} = \frac{2x(x^3-1)-3x^2(x^2+1)}{(x^3-1)^2}$
$= \frac{x(2x^3-2-3x^3-3x)}{(x^3-1)^2} = \frac{-x(x^3+3x+2)}{(x^3-1)^2}$
$= \frac{-x(x^3+3x+2)}{(x^3-1)^2}$

b. f'(x) = $\frac{x^2(x+3)}{(x+1)^3}$

Higher Order Derivatives

Let consider the function y = f (x) . The first derivative of f or " f prime" is:

f'(x) = y' = $\frac{dy}{dx}$

The second derivative of f or " f double prime" is:

f''(x) = y'' = $\frac{d^2y}{dx^2}$

The third derivative of f or " f triple prime" is:

f'''(x) = y''' = $\frac{d^3y}{dx^3}$

⋮

The n-th derivative of f is:

$f^{(n)}$(x) = $y^{(n)}$ = $\frac{d^ny}{dx^n}$

Ex. Show that y = x^3+3x+1 satisfies y''' + xy'' − 2y' = 0 .

Solution

y' = $3x^2$ + 3

$y'' = 6x$

$y''' = 6$

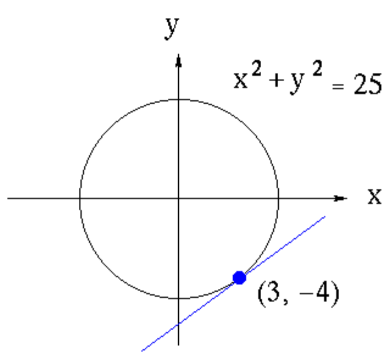
$y''' + xy'' - 2y' = 6 + x(6x) - 2(3x^2 + 3) = 0$

Implicit differentiation

The following problems require the use of implicit differentiation. Implicit differentiation is nothing more than a special case of the well-known chain rule for derivatives. The majority of differentiation problems in first-year calculus involve functions *y* written EXPLICITLY as functions of *x*. For example, if

$$x^2 + y^2 = 25 ,$$

which represents a circle of radius five centered at the origin. Suppose that we wish to find the slope of the line tangent to the graph of this equation at the point (3, -4).



How could we find the derivative of *y* in this instance? One way is to first write *y* explicitly as a function of *x*. Thus,

$x^2 + y^2 = 25$,

$y^2 = 25 - x^2$,

and

$y = \pm\sqrt{25 - x^2}$

where the positive square root represents the top semi-circle and the negative square root represents the bottom semi-circle. Since the point (3, -4) lies on the bottom semi-circle given by

$y = -\sqrt{25 - x^2}$

the derivative of *y* is

$$y' = -(1/2)\left(25 - x^2\right)^{-1/2}(-2x) = \frac{x}{\sqrt{25 - x^2}}$$

i.e.

$$y' = \frac{x}{\sqrt{25 - x^2}}$$

Thus, the slope of the line tangent to the graph at the point (3, -4) is

$$m = y' = \frac{(3)}{\sqrt{25 - (3)^2}} = \frac{3}{4}$$

Unfortunately, not every equation involving *x* and *y* can be solved explicitly for *y*.

For the sake of illustration we will find the derivative of *y* WITHOUT writing *y* explicitly as a function of *x*.

$[(f(x))^2]' = 2f(x) \cdot f'(x)$

Now begin with $x^2 + y^2 = 25$.

Differentiate both sides of the equation, getting

$(x^2 + y^2)' = (25)'$

$2x + 2yy' = 0$,

So that

$$y' = \frac{-x}{y}.$$

Thus, the slope of the line tangent to the graph at the point (3, -4) is

$$m = y' = \frac{-(3)}{(-4)} = \frac{3}{4}.$$

This second method illustrates the process of ***implicit differentiation***. It is important to note that the derivative expression for explicit differentiation involves *x* only, while the derivative expression for implicit differentiation may involve BOTH *x* AND *y*.