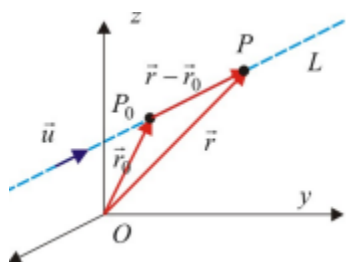


Lesson 4: Unit 3 – Equations of Lines and Planes

Vector Equation of a Line

In both 2-d and 3-d cases we can determine the equation of a line using its direction and any fixed point on the line.

Suppose a line passes through a fixed point $P_0(x_0, y_0, z_0)$ with position vector \vec{r}_0 , and that the line is parallel to the vector \vec{u} . Consider $P(x, y, z)$ be an arbitrary point on the line so that $\overrightarrow{OP} = \vec{r}$.



Using the triangle law of vector addition,

$$\overrightarrow{OP} = \overrightarrow{OP_0} + \overrightarrow{P_0P}$$

$$(x, y, z) = (x_0, y_0, z_0) + t\vec{u}$$

or

$$\vec{r} = \vec{r}_0 + t\vec{u}, t \in \mathbb{R}$$

Where $\vec{u} = (u_x, u_y, u_z)$ is a direction vector for the line.

This is the **vector equation of a line** in \mathbb{R}^3 .

Ex. Find two vector equations of the line L that passes through the points $A(1, 2, 3)$ and $B(2, -1, 0)$.

Solution

If we use the direction vector $\vec{d} = \overrightarrow{AB} = (1, -3, -3)$ and the point $A(1, 2, 3) \in L$, then the vector equation of the line L is:

$$L : (x, y, z) = (1, 2, 3) + t(1, -3, -3), t \in \mathbb{R}.$$

Ex. Find the vector equation of a line L_2 that passes through the origin and is parallel to the line

$$L_1 : \vec{r} = (-2, 0, 3) + t(-1, 0, 2), t \in \mathbb{R}.$$

Solution

$$L_2 : \vec{r} = (0, 0, 0) + s(-1, 0, 2), s \in \mathbb{R}.$$

$$L_2 : \vec{r} = s(-1, 0, 2), s \in \mathbb{R}.$$

Ex. Find the vector equation of a line that:

a) passes through $A(3, -2, 0)$ and is parallel to the y-axis

b) passes through $M(-1, 0, 4)$ and is perpendicular to the yz-plane

Solution

$$a) \vec{r} = (3, -2, 0) + t(0, 1, 0), t \in \mathbb{R}$$

$$b) \vec{r} = (-1, 0, 4) + t(1, 0, 0), t \in \mathbb{R}$$

Parametric Equations of a Line in \mathbb{R}^3

Let's rewrite the vector equation of a line:

$$\vec{r} = \vec{r}_0 + t\vec{u}, t \in \mathbb{R}$$

as

$$(x, y, z) = (x_0, y_0, z_0) + t(u_x, u_y, u_z), t \in \mathbb{R} \text{ where } \vec{u} = (u_x, u_y, u_z).$$

The **parametric equations of a line** in \mathbb{R}^3 are:

$$\begin{cases} x = x_0 + tu_x \\ y = y_0 + tu_y \\ z = z_0 + tu_z \end{cases}, t \in \mathbb{R}$$

Ex. Find the parametric equations of the line L that passes through the points $A(0, -1, 2)$ and $B(1, -1, 3)$. Describe the line.

Solution

$$\vec{u} = \overrightarrow{AB} = (1, 0, 1); \quad A(0, -1, 2) \in L$$

$$L: \vec{r} = (0, -1, 2) + t(1, 0, 1), \quad t \in R$$

$$(x, y, z) = (0, -1, 2) + t(1, 0, 1), \quad t \in R$$

$$L: \begin{cases} x = t \\ y = -1 \\ z = 2 + t \end{cases}, \quad t \in R$$

Symmetric Equation of a line

The parametric equations of a line may be written as:

$$\begin{cases} x = x_0 + tu_x \\ y = y_0 + tu_y \\ z = z_0 + tu_z \end{cases}, \quad t \in R$$

From here, **the symmetric equations of the line** are:

$$\frac{x-x_0}{u_x} = \frac{y-y_0}{u_y} = \frac{z-z_0}{u_z}$$

$$u_x \neq 0, u_y \neq 0, u_z \neq 0$$

Ex. Convert the vector equation of the line $L: \vec{r} = (0, 1, -3) + t(-1, 2, 0), t \in R$ to the parametric and symmetric equations.

Solution

$$(x, y, z) = (0, 1, -3) + t(-1, 2, 0), \quad t \in R$$

$$\therefore \begin{cases} x = -t \\ y = 1 + 2t \\ z = -3 \end{cases}, \quad t \in R$$

$$\frac{x}{-1} = \frac{y-1}{2} = \frac{z+3}{0}$$

$$\therefore \frac{x}{-1} = \frac{y-1}{2}, \quad z = -3$$

Ex. Convert the symmetric equations for a line: $\frac{x-2}{3} = \frac{y+1}{-2} = \frac{z}{4}$ to the parametric and vector equations.

Solution

$$\frac{x-2}{3} = \frac{y+1}{-2} = \frac{z}{4} = t \rightarrow \begin{cases} x-2 = 3t \\ y+1 = -2t \\ z = 4t \end{cases}, t \in R$$

$$\therefore \begin{cases} x = 2 + 3t \\ y = -1 - 2t \\ z = 4t \end{cases}, t \in R$$

$$\therefore \vec{r} = (2, -1, 0) + t(3, -2, 4), t \in R$$

Ex. For each case, find if the given point lies on the given line.

a) L: $\vec{r} = (1, 2, -3) + t(0, 1, -2)$; P(1, 4, -7)

b) L: $\begin{cases} x = -2 + 3t \\ y = -t \\ z = 5 \end{cases}$; P(0, 1, 5)

c) L: $\frac{x+1}{-2} = \frac{y-2}{1} = \frac{z}{-3}$; P(-2, 3, -3)

Solution

a) $(1, 4, -7) = (1, 2, -3) + t(0, 1, -2)$

$$(1, 4, -7) - (1, 2, -3) = t(0, 1, -2)$$

$$(0, 2, -4) = t(0, 1, -2) \Rightarrow t = 2 \therefore P \in L$$

b) $\begin{cases} 0 = -2 + 3t \Rightarrow t = \frac{2}{3} \\ 1 = -t \Rightarrow t = -1 \\ 5 = 5 \Rightarrow \text{true} \end{cases} \therefore P \notin L$

c) $\frac{-3+1}{-2} = \frac{3-2}{1} = \frac{-3}{-3}$

$$1 = 1 = 1 \therefore P \in L$$

Ex. Consider the line L: $\vec{r} = (3, -2, 3) + t(-1, 2, -3)$, $t \in R$. Find the intersection points between this line and the coordinates axes and planes.

Solution

$$L: \begin{cases} x = 3 - t \\ y = -2 + 2t \\ z = 3 - 3t \end{cases}$$

$$x = 0 \rightarrow t = 3 \rightarrow y = -2 + 2(3) = 4, z = 3 - 3(3) = -6$$

$$\therefore \text{yz-int is } A(0, 4, -6) = L \cap \text{yz-plane}$$

$$y=0 \rightarrow t=1 \rightarrow x=2, z=0$$

$$\therefore \text{xz-int is } B(2,0,0) = L \cap \text{xz-plane}$$

$$z=0 \rightarrow t=1 \rightarrow x=2, y=0$$

$$\therefore \text{xy-int is } B(2,0,0) = L \cap \text{xy-plane}$$

$$\therefore \text{x-int is } B(2,0,0) = L \cap \text{x-axis}$$

Note that y-intercept and z-intercept do not exist.

Now let's recap

Vector and Parametric Equations of a Line in R^2 in R^3

$$\vec{r} = \vec{r}_0 + t\vec{d}, t \in \mathbb{R}$$

$$\text{where } \vec{d} = (d_x, d_y).$$

$$\begin{cases} x = x_0 + td_x \\ y = y_0 + td_y \end{cases} \quad t \in \mathbb{R}$$

$$\vec{r} = \vec{r}_0 + t\vec{d}, t \in \mathbb{R}$$

$$\text{where } \vec{d} = (d_x, d_y, d_z)$$

$$\begin{cases} x = x_0 + td_x \\ y = y_0 + td_y \\ z = z_0 + td_z \end{cases} \quad t \in \mathbb{R}$$

Note. The vector equation of a line is not unique. It depends on the specific point P_0 and on the direction vector u or r that are used.

Parallel lines

Two lines L_1 and L_2 with direction vectors \vec{u}_1 and \vec{u}_2 are parallel if $\vec{u}_1 \parallel \vec{u}_2$ (collinear) or, there exist $k \in \mathbb{R}$ such that: $\vec{u}_2 = k\vec{u}_1$

$$\text{or: } \vec{u}_1 \times \vec{u}_2 = \vec{0}$$

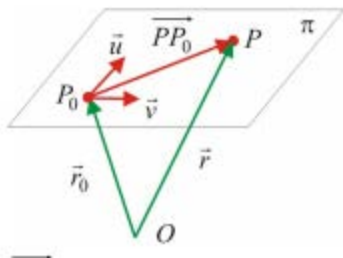
Perpendicular lines

Two lines L_1 and L_2 with direction vectors \vec{u}_1 and \vec{u}_2 are perpendicular if \vec{u}_1 and \vec{u}_2 .

$$L_1 \perp L_2 \quad \text{if} \quad \vec{u}_1 \perp \vec{u}_2 \quad \text{or} \quad \vec{u}_1 \cdot \vec{u}_2 = 0.$$

Vector Equations of a Plane

Let consider a plane π . Two vectors \vec{u} and \vec{v} , parallel to the plane π but not parallel between them (not collinear), are called *direction vectors* of the plane π .



The vector $\overrightarrow{P_0P}$ from a specific point $P_0(x_0, y_0, z_0)$ to a generic point $P(x, y, z)$ of the plane is a linear combination of direction vectors \vec{u} and \vec{v} :

$$\overrightarrow{P_0P} = s\vec{u} + t\vec{v}; \quad s, t \in \mathbb{R}$$

The **vector equation of the plane** is:

$$: \vec{r} = \vec{r}_0 + s\vec{u} + t\vec{v}; \quad s, t \in \mathbb{R}$$

Ex. A plane π is given by the following vector equation:

$$: \vec{r} = (-1, 0, 2) + s(0, 0, 1) + t(1, 0, -1); \quad s, t \in \mathbb{R}$$

a) Find two points on this plane.

b) Find one line on this plane.

c) Find the vector equation of a line L_\perp that passes through the origin and is perpendicular to this plane.

Solution

a) If $s=0$ and $t=0$, then $\vec{r} = (-1, 0, 2) \Rightarrow P_0(-1, 0, 2) \in \pi$

If $s=1$ and $t=2$, then $\vec{r} = (-1, 0, 2) + 1(0, 0, 1) + 2(1, 0, -1) = (1, 0, 1) \Rightarrow A(1, 0, 1) \in \pi$

b) Let $L: \vec{r} = (-1, 0, 2) + s(0, 0, 1); \quad s \in \mathbb{R} \Rightarrow L \subset \pi$

c) A director vector for the line L_\perp is:

$$\vec{u} \times \vec{v} = (0, 1, 0)$$

$$L_\perp : \vec{r} = q(0, 1, 0); \quad q \in \mathbb{R}.$$

Parametric Equations of a Plane

The vector equation of a plane given by

$$: \vec{r} = \vec{r}_0 + s\vec{u} + t\vec{v}; \quad s, t \in \mathbb{R}$$

$$(x, y, z) = (x_0, y_0, z_0) + t(u_x, u_y, u_z) + s(u_x, u_y, u_z)$$

can be written also as

$$\begin{cases} x = x_0 + su_x + tv_x \\ y = y_0 + su_y + tv_y \\ z = z_0 + su_z + tv_z \end{cases} ; \quad s, t \in \mathbb{R}$$

These are **the parametric equations of a line**.

Ex. Convert the vector equation to the parametric equations.

$$: \vec{r} = (-1, 0, 2) + s(0, 1, -1) + t(1, -2, 0); \quad s, t \in \mathbb{R}$$

Solution

$$(x, y, z) = (-1, 0, 2) + s(0, 1, -1) + t(1, -2, 0); \quad s, t \in \mathbb{R}$$

$$\begin{cases} x = -1 + t \\ y = s - 2t \\ z = 2 - s \end{cases} ; \quad s, t \in \mathbb{R}$$

Ex. Convert the parametric equations to the vector equation.

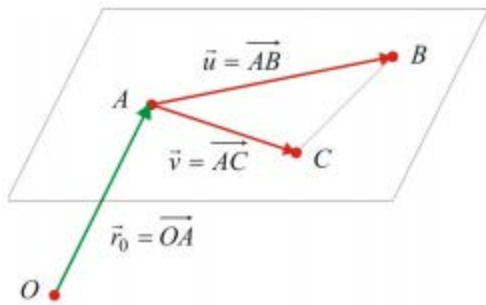
$$\begin{cases} x = 1 + s - 2t \\ y = 3t \\ z = 4 - s \end{cases} ; \quad s, t \in \mathbb{R}$$

Solution

$$: \vec{r} = (1, 0, 4) + s(1, 0, -1) + t(-2, 3, 0); \quad s, t \in \mathbb{R}$$

Ex. Find the vector equation of the plane π that passes through the points $A(0, 1, -1)$, $B(2, -1, 0)$, and $C(0, 0, 1)$.

Solution



Let $\vec{r}_0 = \vec{OA} = (0,1,-1)$, $\vec{u} = \vec{AB} = (2,-2,1)$, and $\vec{v} = \vec{AC} = (0,-1,2)$.

Then:

$$: \vec{r} = (0,1,-1) + s(2,-2,1) + t(0,-1,2); \quad s, t \in \mathbb{R}$$

Ex. Find the vector and parametric equations of the plane π that contains the following parallel and distinct lines:

$$L_1 : \vec{r} = (1,2,1) + s(0,-1,-2); \quad s \in \mathbb{R}$$

$$L_2 : \vec{r} = (3,4,0) + t(0,1,2); \quad t \in \mathbb{R}$$

Solution

$$A(1,2,2), B(3,4,0) \Rightarrow \vec{AB} = (3-1,4-2,0-2) = (2,2,-2)$$

$$\text{Take } \vec{u} = \vec{AB} \text{ and } \vec{v} = (0,1,2) \text{ and } P_0 = A \Rightarrow \pi : \vec{r} = (1,2,2) + s(0,1,2) + t(2,2,-2); \quad s, t \in \mathbb{R}$$

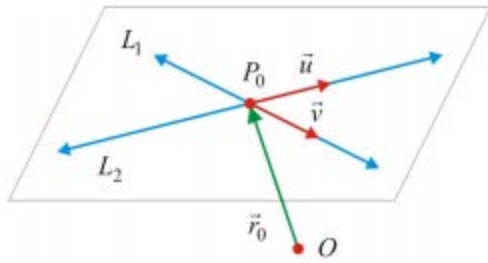
$$: \begin{cases} x = 1 + 2t \\ y = 2 + s + 2t \\ z = 2 + 2s - 2t \end{cases}; \quad s, t \in \mathbb{R}$$

Ex. Find the vector equation of the plane π determined by the following intersecting lines:

$$L_1 : \vec{r} = (0,0,1) + s(-1,0,0); \quad s \in \mathbb{R}$$

$$L_2 : \vec{r} = (-3,0,1) + t(0,0,2); \quad t \in \mathbb{R}$$

Solution



First let's find the point of intersection.

$$\begin{cases} -s = -3 \\ 0 = 0 \\ 1 = 1 + 2t \end{cases} \Rightarrow s=3 \text{ and } t=0$$

$$P_0 = L_1 \cap L_2 \Rightarrow P_0(-3,0,1)$$

$$\text{Let } \vec{r}_0 = \overrightarrow{OP_0} = (-3,0,1), \vec{u} = \vec{u}_2 = (0,0,2), \text{ and } \vec{v} = \vec{u}_1 = (-1,0,0).$$

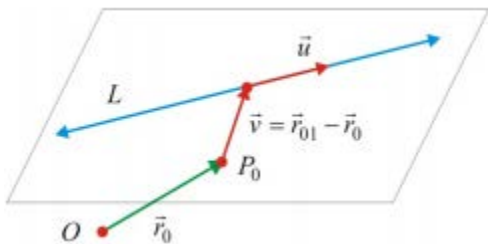
Then:

$$\pi : \vec{r} = (-3,0,1) + s(0,0,2) + t(-1,0,0); \quad s, t \in \mathbb{R}$$

Ex. Find the vector equation of the plane π that passes through the origin and contains the line

$$L : \vec{r} = (0,1,2) + t(-1,0,3); \quad t \in \mathbb{R}$$

Solution



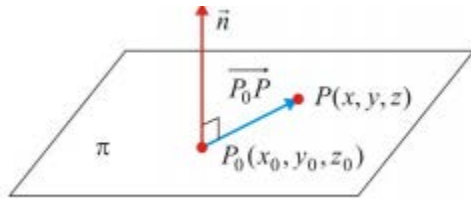
Let $\vec{r}_0 = (0,0,0)$, $\vec{u} = (-1,0,3)$, and $\vec{v} = (0,1,2) - (0,0,0) = (0,1,2)$. Then the vector equation of the plane is:

$$\pi : \vec{r} = (0,0,0) + s(-1,0,3) + t(0,1,2); \quad s, t \in \mathbb{R}$$

$$\therefore : \vec{r} = s(-1,0,3) + t(0,1,2); \quad s, t \in \mathbb{R}$$

Normal Equation of a Plane

A plane may be determined by a point $P_0(x_0, y_0, z_0)$ and a vector perpendicular to the plane \vec{n} called the normal vector.



If $P(x, y, z)$ is a generic point on the plane, then:

$$\overrightarrow{P_0P} \perp \vec{n} \text{ and}$$

$$\overrightarrow{P_0P} \cdot \vec{n} = 0$$

This is **the normal equation of a plane**.

10. Cartesian Equation of a Plane

Let's write the normal vector of a plane in the form:

$$\vec{n} = (A, B, C)$$

Then, the normal equation may be written as:

$$(x - x_0, y - y_0, z - z_0) \cdot (A, B, C) = 0$$

$$Ax + By + Cz - Ax_0 - By_0 - Cz_0 = 0$$

or

$$Ax + By + Cz + D = 0$$

equation which is called **the Cartesian equation of a plane**.

Note. A normal vector to the plane is:

$$\vec{n} = \vec{u} \times \vec{v}$$

where \vec{u} and \vec{v} are the direction vectors of the plane.

Ex. Consider the plane π defined the Cartesian equation $\pi : 2x - 3y + 6z + 12 = 0$.

- a) Find a normal vector to this plane.
- b) Find two points on this plane
- c) Find if the point $P(1,2,3)$ is a point on this plane.

Solution

- a) $\vec{n} = (2, -3, 6)$
- b) If $x=0$ and $y=0$ then $z=-2$. So $(0,0,-2) \in \pi$.
If $x=0$ and $z=0$ then $y=4$. So $(0,4,0) \in \pi$.
- c) $2(1) - 3(2) + 6(3) + 12 = 26 \neq 0 \Rightarrow P \notin \pi$.

Ex. Find the Cartesian equation of a plane π that passes through the points $A(1,-1,0)$, $B(0,0,1)$, and $C(0,-2,1)$.

Solution

$$\vec{u} = \overrightarrow{AB} = (-1, 1, 1); \quad \vec{v} = \overrightarrow{AC} = (-1, -1, 1)$$

$$\vec{n} = \vec{u} \times \vec{v} = (2, 0, 2) = (A, B, C)$$

$$\pi: 2x + 2z + D = 0$$

$$A \in \pi \Rightarrow 2(1) + 2(0) + D = 0 \Rightarrow D = -2$$

$$\therefore \pi: 2x + 2z - 2 = 0 \text{ or } x + z - 1 = 0$$

Ex. Find parametric and vector equations for the plane: $\pi: x - 2y + 3z - 6 = 0$.

Solution

Take $y=s$ and $z=t$ then $x=6+2s-3t$.

$$: \begin{cases} x = 6 + 2s - 3t \\ y = s \\ z = t \end{cases} ; \quad s, t \in \mathbb{R}$$

$$\pi: \vec{r} = (6, 0, 0) + s(2, 1, 0) + t(-3, 0, 1); \quad s, t \in \mathbb{R}$$

Ex. Find the intersections with the coordinate axes for the plane $\pi : 3x + 2y + z - 6 = 0$.
Represent the plane graphically.

Solution

Let $A = \pi \cap x\text{-axis} \Rightarrow y_A = z_A = 0 \Rightarrow x_A = 2$.

x-int is $A(2,0,0)$

Similarly,

y-int is $B(0,3,0)$

z-int is $C(0,0,6)$

