Lesson 2 (on Calculus)

Unit 5 – An Introduction to Calculus (2)

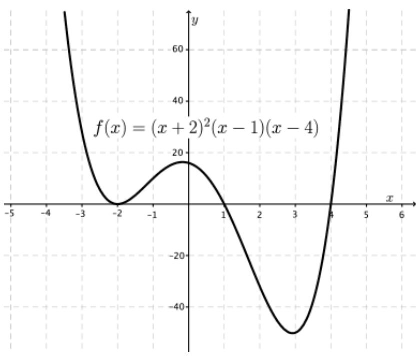
Continuity

In your explorations of functions and their graphs, you will have noticed that the graphs of some functions, from start to end, are made of one unbroken curve, whereas others include breaks within their domain.

The following graphs demonstrate this idea.

Functions Having an Unbroken Curve

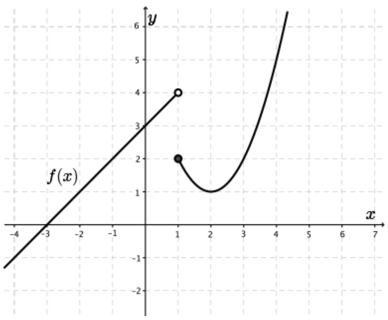
f(x) = $(x + 2)^2$(x-1)(x-4)



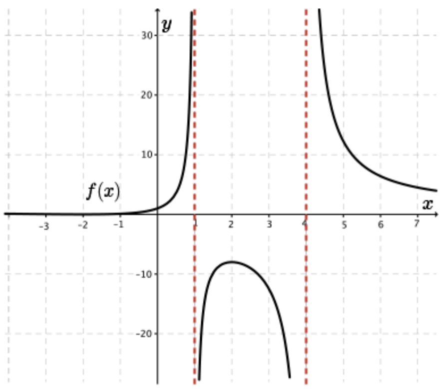
Functions that contain **no breaks** along their entire domain are **continuous**.
As the *x*-values of a continuous function vary, the *y*-values vary continuously and do not jump from one value to another.
If a function is continuous, then it is possible to draw the function's graph from one end of its domain to the other without removing your pencil from the paper.
If this is not possible, then the graph must contain breaks.

Functions with Breaks

f(x) = $\begin{cases} x+3 & if\ x < 1 \\ (x-2)^2+1 & if\ x \geq 1 \end{cases}$



f(x) = $\frac{(x+2)^2}{(x-1)(x-4)}$



A function that is not continuous at *x*=*a* is referred to as **discontinuous** at *a*.

The point, *a*, is known as a point of discontinuity.

Definition

A function, *f*(*x*), is **continuous** at *x*=*a* if $\lim_{x \to a} f(x)$ = *f*(*a*).

For this definition to be satisfied, the following three conditions must be true.

f(a) must exist.

$\lim_{x \to a} f(x)$ must exist.

$\lim_{x \to a} f(x)$ = f(a).

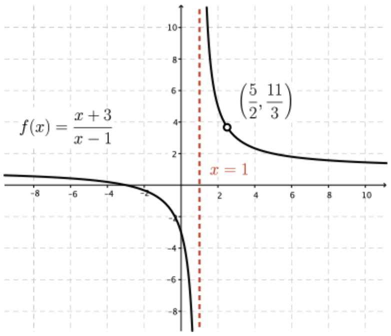
Types of Discontinuity

Removable Discontinuities

Some functions may contain **removable discontinuities** (sometimes commonly referred to as holes or missing points) in their graphs.

For example, holes occur when a rational function has an equal factor in its numerator and denominator that can be cancelled.

Consider the function f(x) =$\frac{(2x-5)(x+3)}{(2x-5)(x-1)}$, which has a hole at x = $\frac{5}{2}$.



On its graph, we see a missing point within the domain of $f(x)$ at x=5/2.
Note that $\lim_{x \to \frac{5}{2}} f(x)$ exists, since we can approach the value x=$\frac{5}{2}$ from the left and right sides.
Here, $\lim_{x \to \frac{5}{2}} f(x)$ =11/3.
However, f(5/2) does not exist because when x=5/2, $f(x)$ has the indeterminate form $f(x)$=0/0.

Infinite discontinuities

The function, $f(x)$ = $\frac{(2x-5)(x+3)}{(2x-5)(x-1)}$, can be simplified, for x≠5/2.
We can cancel the equal factor (2x−5) from the numerator and the denominator of $f(x)$ to give

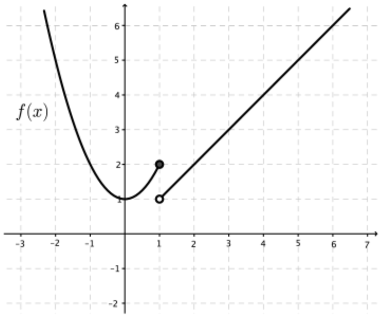
$$f(x)=\frac{x+3}{x-1}$$

where x≠5/2.
Once $f(x)$ has been completely simplified, any factor of $x-a$ that remains in the denominator will produce a vertical asymptote at $x=a$.
Therefore, the simplified function, $f(x)$ =$\frac{x+3}{x-1}$, has a vertical asymptote at x=1.
We say that $f(x)$ has an **infinite discontinuity** at x=1.

Jump Discontinuities

Piecewise functions often have jump discontinuities.

Consider the function, f(x) = $\begin{cases} x^2 + 1 & if\ x \leq 1 \\ x & if\ x > 1 \end{cases}$



Tracing *f*(*x*) from the left side to the right side, we see $\lim_{x \to 1^-} f(x) = 2$.

However, for *x* slightly greater than 1, the value of *f*(*x*) jumps to approximately 1.

We use an open circle to denote a point that is not a part of the function.

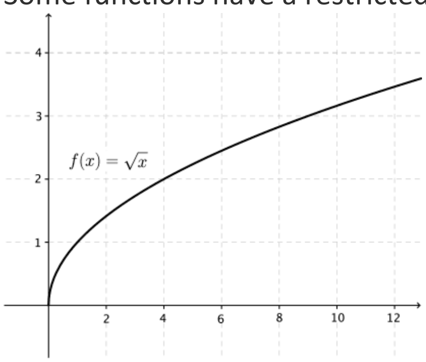
Beginning on the left side of the grid, *f*(*x*) is continuous until the point where *x*=1 and *f*(1)=2.

We say that there is a **jump discontinuity** at *x*=1.

After this point of discontinuity, *f*(*x*) is continuous for *x*>1.

Restricted Domain

Some functions have a restricted domain.

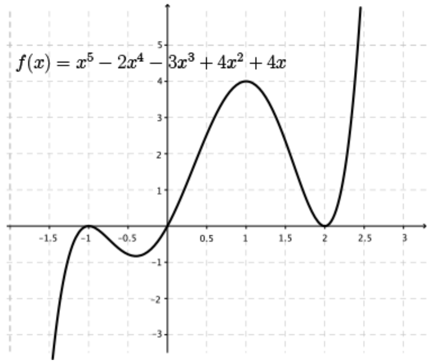


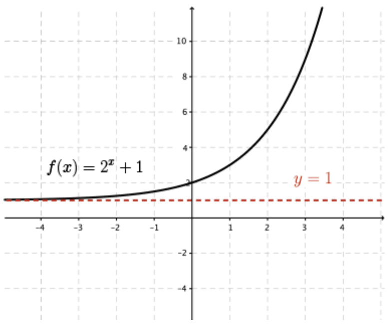
For example, consider the function $f(x)=\sqrt{x}$. Its domain is $\{x \in R | x \geq 0\}$.

This function does not exist for *x*<0; however, it is continuous for *x*>0.
We say that *f*(*x*) is continuous from the right at *x*=0 since $\lim_{x \to 0^+} f(x)$ exists and equals *f*(0).

Functions That Are Continuous Everywhere

Polynomial and exponential functions are continuous for all $x \in R$.

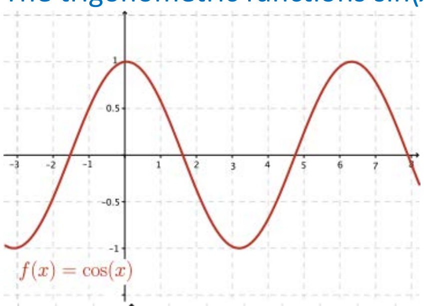


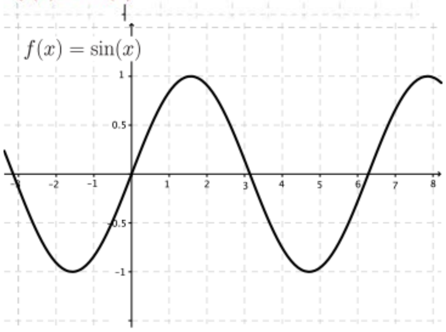


Notice there is a horizontal asymptote at *y*=1, but this does not affect the continuity of the function.

Functions That Are Continuous Everywhere

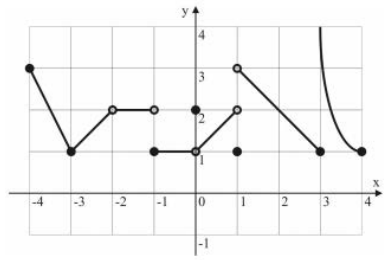
The trigonometric functions sin(*x*) and cos(*x*) are also continuous everywhere for all *x*∈R.





On your own, explore the graphs of tan(*x*) and the reciprocal trigonometric functions to determine their continuity.

Ex. The function y = f (x) is represented graphically in the figure below.



Analyze the continuity of this function at:

a) x = −3 (continuous)

b) x = −2 (discontinuous, removable discontinuity)

c) x = −1 (discontinuous, jump discontinuity)

d) x = 0 (discontinuous, removable discontinuity)

e) x = 3 (discontinuous, infinite discontinuity)

Ex. Analyze the continuity of the function:

f(x) = $\begin{cases} x, & x < 0 \\ x^2, & 0 \le x \le 1 \\ x^3 + 1, & x > 1 \end{cases}$

Solution

The function f (x) is continuous over (−∞,0) , (0,1) and (1,∞) (because y = x , y = x^2 , and y = $x^3 + 1$ are polynomial functions).

Let analyze the continuity at x = 0 :

$\lim_{x \to 0^-} f(x)$ = $\lim_{x \to 0^-} x$ = 0

$\lim_{x \to 0^+} f(x)$ = $\lim_{x \to 0^+} x^2$ = 0

f(0) = 0

∴ f(x) is continuous at x = 0.

Let analyze the continuity at x = 1 :

$\lim_{x \to 1^-} f(x)$ = $\lim_{x \to 1^-} x^2$ = 1

$\lim_{x \to 1^+} f(x)$ = $\lim_{x \to 0^+} (x^3 + 1)$ = 2

f(1) = 1

∴ f(x) is discontinuous at x = 1(jump discontinuity).

Therefore, the function f (x) is discontinuous at x =1 and continuous elsewhere.

Ex. For what value of the constant c is the function

f(x) = $\begin{cases} x^2 + c & if\ x < 0 \\ 2x + c^2 - 2c + 2 & if\ x \geq 0 \end{cases}$

continuous at any number (everywhere)?

Solution

The function f (x) is continuous over (−∞,0) and (0,∞) (because y = x^2 + c and y = 2x +$c^2 - 2c + 2$ are polynomial functions).

Let analyze the continuity at x = 0 :

$\lim_{x \to 0^-} f(x)$ = $\lim_{x \to 0^-} (x^2 + c)$ = c

$\lim_{x \to 0^+} f(x)$ = $\lim_{x \to 0^+} (2x + c^2 - 2c + 2\ )$ = $c^2 - 2c + 2$

f(0) = $c^2 - 2c + 2$

For f(x) to be continuous everywhere then f(x) must be continuous also at x = 0.

So, $c^2 - 2c + 2$ = c ⇨ $c^2 - 3c + 2$ = 0 ⇨ (c-1)(c-2) = 0 ⇨ c=2 or c=1.

Therefore, f (x) is continuous everywhere if c = 2 or c = 1.

Limits to Infinity

If n ≥ 1, then:

$\lim_{x \to \pm\infty} (x^n)$ = $(\pm\infty)^n$

$\lim_{x \to \pm\infty} \frac{1}{x^n}$ = 0

Ex. Compute each limit.

a. $\lim_{x \to \infty} x$ = ∞

b. $\lim_{x \to -\infty} 3x^4$ = ∞

c. $\lim_{x \to \infty} (-3x^2 + 2x + 10)$ = $\lim_{x \to \infty} x^2(-3 + \frac{2}{x} + \frac{10}{x^2})$ = ($\lim_{x \to \infty} x^2$)($\lim_{x \to \infty} (-3 + \frac{2}{x} + \frac{10}{x^2})$)

= -∞

Notice that $\lim_{x \to \infty} (-3x^2 + 2x + 10)$ = $\lim_{x \to \infty} 3x^2$.

d. $\lim_{x \to \infty} \frac{2x-7}{x+1}$ = $\lim_{x \to \infty} \frac{x(2-\frac{7}{x})}{x(1+\frac{1}{x})}$ = 2

e. $\lim_{x \to -\infty} \frac{2x-7}{x^2+1}$ = 0

f. $\lim_{x \to -\infty} \frac{-2x^3-7}{3x^2+1}$ = $\lim_{x \to -\infty} \frac{-2x^3}{3x^2}$ = -∞