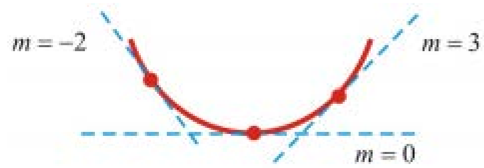
Lesson 7 (on Calculus) –

Unit 7 – Derivatives and Their Applications (2)

Concavity

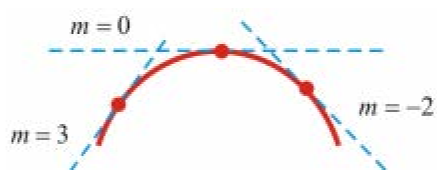
The graph of a function has a *concavity upward* if:

- Graph lies above all its tangents
- Tangents rotate counter-clockwise
- Slope of tangent lines increases
- f '(x) increases or f ''(x) > 0

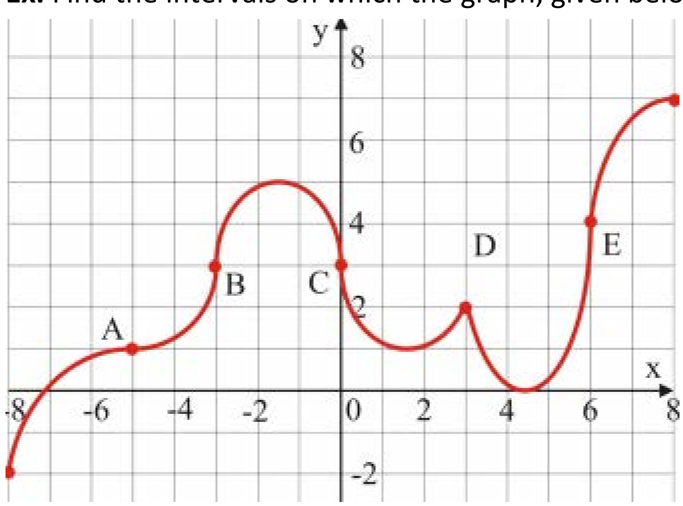


The graph of a function has a concavity downward if:

- Graph lies below all its tangents
- Tangents rotate clockwise
- Slope of tangent lines decreases
- f '(x) decreases or f ''(x) < 0



Ex. Find the intervals on which the graph, given below, is concave upward or downward.



Ans:

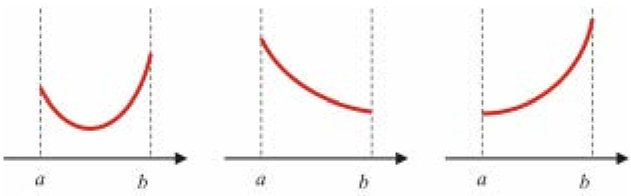
The graph is concave upward over (−5,−3) , (0,3), and (3,6).

The graph is concave downward over (−8,−5) , (−3,0) , and (6,8) .

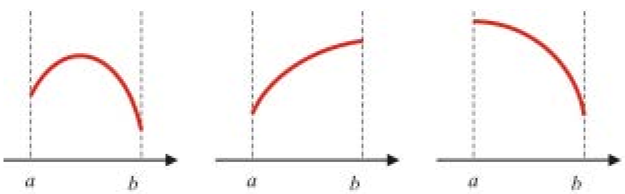
Test for Concavity

Let f be a function twice differentiable (*f''(x) exist*s) over (a, b) .

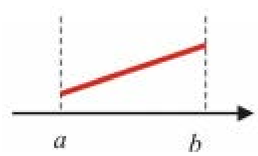
1. If f''(x) > 0 for all x∈(a, b) , then the graph of f is *concave upward* (has a concavity upward) over (a, b) .



2. If f''(x) < 0 for all x∈(a, b) , then the graph of f is *concave downward* (has a concavity downward) over (a, b) .



3. If f''(x) = 0 for all x∈(a, b) , then the graph of f has no concavity over (a, b) (f'(x) = const ; the graph is a straight line).



Ex. Find the intervals of concavity for f (x) = $x^4 - 2x^3$.

Solution

*f'(x)=4*x^3*- 6*x^2

*f''(x)= 12*x^2*- 12x*

f''(x) = 0 ⇨ *x = 0 or x = 1*

f(0) = 0 and f(1) = -1

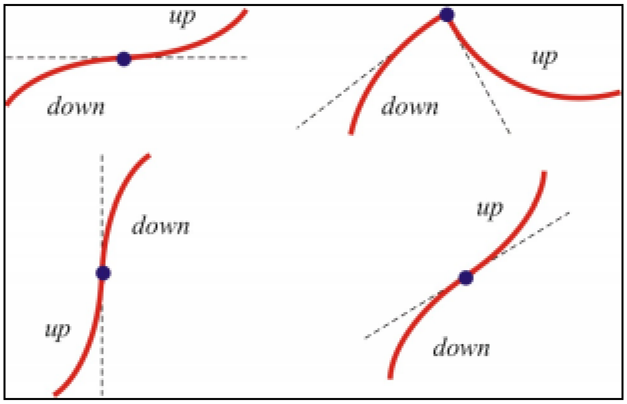
x		0		1	
$f(x)$	∪	0	∩	-1	∪
$f''(x)$	+	0	-	0	+

The graph is concave upward over (−∞, 0) and (1, ∞) .

The graph is concave downward over (0, 1) .

Point of Inflection

A point P(i, f (i)) on the graph of y = f (x) is called *point of inflection* if the concavity of the graph changes at P (from concave upward to concave downward or from concave downward to concave upward).



Second Derivative Test

Let f be a twice differentiable function over an open interval containing the critical number c and f '(c) = 0

((c, f (c)) is a stationary point).

1. If f ''(c) > 0 then f has a local minimum at x = c.

2. If f ''(c) < 0 then f has a local maximum at x = c.

Ex. Identify the points of inflection for f (x) = $x^4 - 2x^3$.

Ans: The points of inflection are (0, 0) , and (1, −1) .

Algorithm for Solving Optimization Problems

1. Read and understand the problem's text.
2. Draw a diagram (if necessary).
3. Assign variables to the quantities involved and state restrictions according to the situation.
4. Write relations between these variables.
5. Identify the variable that is minimized or maximized. This is the dependant variable.
6. Use the other relations (called constraints) to express the dependent variable (the one which is minimized or maximized) as a function of one single variable (the independent variable).
7. Find extrema (maximum or minimum) for the dependant variable (using global extrema algorithm, first derivative test or the second derivative test).
8. Check if extrema satisfy the conditions of the application.
9. Find the value of other variables at extrema (if necessary).
10. Write the conclusion statement.

Ex. Find two positive numbers with a product equal to 200 such that the sum of one number and twice the other number is as small as possible. What is the minimum value of the sum?

Solution

Let x and y be the two numbers.

$x, y \in R$, $x, y > 0$ (restrictions)

$xy = 200$ (constraint)

Let s be the sum of first number x and twice the other number y: $s = x + 2y$

The task is to minimize s .

$y = \frac{200}{x}$

$s(x) = x + \frac{400}{x}$

$s'(x) = 1 - \frac{400}{x^2}$

$s'(x) = 0 \quad \Rightarrow \quad x = \pm 20$

Because $x > 0$, $x = 20$ and $y = 10$.

$s''(x) = \frac{800}{x^3} > 0$ for all $x > 0$.

So s has a minimum value when $s' = 0$. The minimum value of the s is: $s - \min = 20 + 2(10) = 40$

∴ The minimum value of the sum is 40. This minimum is achieved when the first number is 20 and the second number is 10.