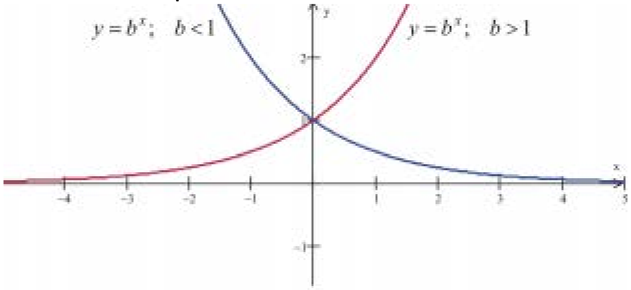
Lesson 5 (on Calculus)

Unit 6 – Derivatives (3)

Derivatives of Exponential and Trigonometric Functions

Review of Exponential Functions

The exponential function is defined as: y = f (x) = b^x ; b > 0,b ≠ 1. The graph of the exponential function is represented below:



The x-axis (y = 0) is a horizontal asymptote.

Number e

The number e is defined by:

e = $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ (1)

which can be written also as:

e = $\lim\limits_{u\to 0}(1+u)^{\frac{1}{u}}$ (2)

Exponential Function

The exponential function e^x may be evaluate using the limit:

e^x = $\lim\limits_{n\to\infty}\left(1+\frac{x}{n}\right)^n$ (3)

Derivative of e^x

$(e^x)'$ = e^x (4)

Proof

$\lim\limits_{h\to 0}\frac{e^{x+h}-e^x}{h}$ = $\lim\limits_{h\to 0}\frac{e^x(e^h-1)}{h}$ = $(\lim\limits_{h\to 0} e^x)(\lim\limits_{h\to 0}\frac{e^h-1}{h})$ = e^x

We used that $\lim\limits_{h\to 0}\frac{e^h-1}{h}$ = 1(substitute h by ± 0.1, ± 0.01, ± 0.001 to estimate $\frac{e^h-1}{h}$).

Derivative of $e^{f(x)}$

$(e^{f(x)})' = e^{f(x)}f'(x)$.

Ex. Differentiate and simplify.

a. x^3e^x

$(x^3e^x)' = 3x^2e^x + x^3e^x = (3x^2 + x^3)e^x$

b. e^{x^2+x+2}

$(e^{x^2+x+2})' = (2x+1)\, e^{x^2+x+2}$

Derivative of b^x *, b > 0, b ≠ 1*

$(b^x)' = b^x \ln b$

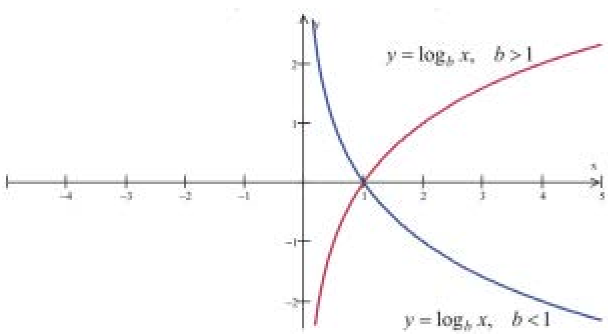
Proof

$(b^x)' = (e^{x\ln b})' = e^{x\ln b}\ln b = b^x \ln b$

Review of Logarithmic Function

$y = b^x \iff x = \log_b y$

y = f(x) = $\log_b y$ **, b > 0, b ≠ 1, x > 0**



Derivative of ln x

$(\ln x)' = \frac{1}{x}$

Proof

$y = \ln x \Rightarrow x = e^y$

$x' = (e^y)' \Rightarrow 1 = e^y y' \Rightarrow y' = \frac{1}{e^y} \Rightarrow y' = \frac{1}{x}$

Derivative of $\log_b x$

$(\log_b x)' = \frac{1}{x \ln b}$

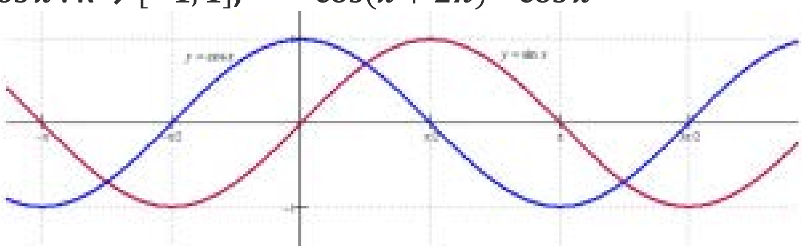
(Hint: $\log_b x = \frac{\ln x}{\ln b}$)

Derivative of Trigonometric Functions

Review of Trigonometric Functions

$\sin x$: R → $[-1, 1]$, **$\sin(x + 2\pi) = \sin x$**

$\cos x$: R → $[-1, 1]$, **$\cos(x + 2\pi) = \cos x$**



Derivative of $\sin x$ *and* $\cos x$

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

Proof

$$\lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h} = (\lim_{h\to 0} \sin x)(\lim_{h\to 0} \frac{(\cos h - 1)}{h} + (\lim_{h\to 0} \cos x)\,(\lim_{h\to 0} \frac{\sin h}{h})$$

$$= \cos x$$

Ex. Differentiate.

a. $x^2 \tan x$
b. $\cot x$
c. $\tan \sqrt{x^2 + 1}$

Solution

a. $(x^2 \tan x)' = 2x\tan x + x^2(\tan x)'$

$= 2x\tan x + x^2(\frac{\sin x}{\cos x})'$

$= 2x\tan x + x^2(\frac{sin^2 x + cos^2 x}{cos^2 x})$

$= 2x\tan x + \frac{x^2}{cos^2 x}$

b. $(\cot x)' = -\frac{1}{sin^2 x}$

c. $(\tan\sqrt{x^2+1})' = \frac{1}{cos^2\sqrt{x^2+1}}(\sqrt{x^2+1})'$

$= \frac{x}{\sqrt{x^2+1}cos^2\sqrt{x^2+1}}$.