linear_regression

January 20, 2020

0.1 Linear regression workbook

This workbook will walk you through a linear regression example. It will provide familiarity with Jupyter Notebook and Python. Please print (to pdf) a completed version of this workbook for submission with HW #1.

ECE C147/C247 Winter Quarter 2020, Prof. J.C. Kao, TAs W. Feng, J. Lee, K. Liang, M. Kleinman, C. Zheng

```
In [1]: import numpy as np
        import matplotlib.pyplot as plt

        #allows matlab plots to be generated in line
        %matplotlib inline
```

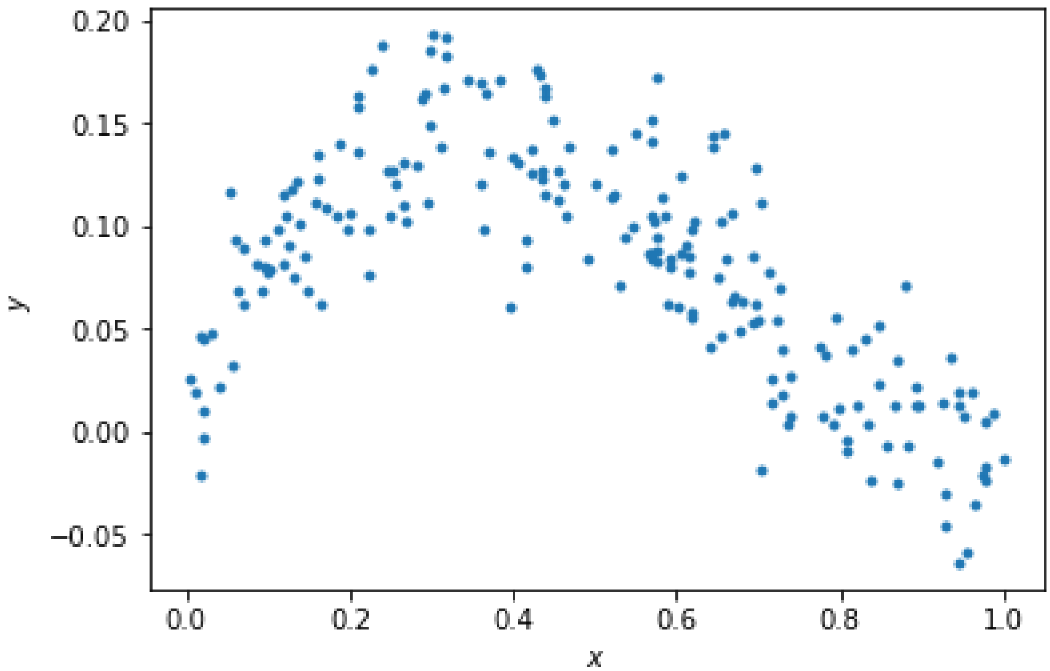
0.1.1 Data generation

For any example, we first have to generate some appropriate data to use. The following cell generates data according to the model: $y = x - 2x^2 + x^3 + \epsilon$

```
In [2]: np.random.seed(0)  # Sets the random seed.
        num_train = 200     # Number of training data points

        # Generate the training data
        x = np.random.uniform(low=0, high=1, size=(num_train,))
        y = x - 2*x**2 + x**3 + np.random.normal(loc=0, scale=0.03, size=(num_train,))
        f = plt.figure()
        ax = f.gca()
        ax.plot(x, y, '.')
        ax.set_xlabel('$x$')
        ax.set_ylabel('$y$')
```

```
Out[2]: Text(0,0.5,'$y$')
```



0.1.2 QUESTIONS:

Write your answers in the markdown cell below this one:

(1) What is the generating distribution of x?

(2) What is the distribution of the additive noise ϵ?

0.1.3 ANSWERS:

(1) x is uniformly distributed with a low of 0 and a high of 1.

(2) ϵ is normally distributed with mean of 0 and standard deviation of 0.03.

0.1.4 Fitting data to the model (5 points)

Here, we'll do linear regression to fit the parameters of a model $y = ax + b$.

In [3]:
```
# xhat = (x, 1)
xhat = np.vstack((x, np.ones_like(x)))

# ==================== #
# START YOUR CODE HERE #
# ==================== #
# GOAL: create a variable theta; theta is a numpy array whose elements are [a, b]
```

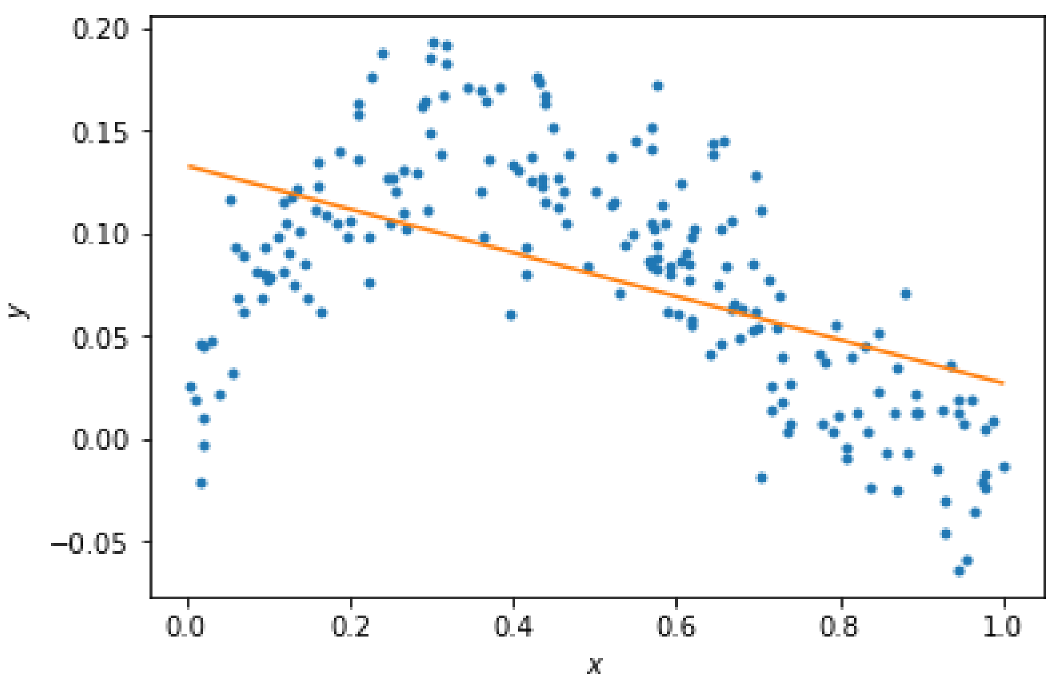
```
theta = np.linalg.inv(xhat.dot(xhat.T)).dot(xhat.dot(y))

# ================== #
# END YOUR CODE HERE #
# ================== #
```

In [4]:
```
# Plot the data and your model fit.
f = plt.figure()
ax = f.gca()
ax.plot(x, y, '.')
ax.set_xlabel('$x$')
ax.set_ylabel('$y$')

# Plot the regression line
xs = np.linspace(min(x), max(x),50)
xs = np.vstack((xs, np.ones_like(xs)))
plt.plot(xs[0,:], theta.dot(xs))
```

Out[4]: [<matplotlib.lines.Line2D at 0x110324bd0>]



0.1.5 QUESTIONS

(1) Does the linear model under- or overfit the data?

(2) How to change the model to improve the fitting?

0.1.6 ANSWERS

(1) The linear model underfits the data.

(2) We can add higher order terms to our model to make it more expressive.

0.1.7 Fitting data to the model (10 points)

Here, we'll now do regression to polynomial models of orders 1 to 5. Note, the order 1 model is the linear model you prior fit.

In [5]:
```
N = 5
xhats = []
thetas = []

# ==================== #
# START YOUR CODE HERE #
# ==================== #

# GOAL: create a variable thetas.
# thetas is a list, where theta[i] are the model parameters for the polynomial fit of 
#   i.e., thetas[0] is equivalent to theta above.
#   i.e., thetas[1] should be a length 3 np.array with the coefficients of the x^2, x,
#   ... etc.

xhats.append(xhat)
thetas.append(theta)

for i in range(1, N):
    xhats.append(np.vstack((x ** (i + 1), xhats[i - 1])))
    thetas.append(np.linalg.inv(xhats[i].dot(xhats[i].T)).dot(xhats[i].dot(y)))

pass

# ================== #
# END YOUR CODE HERE #
# ================== #
```

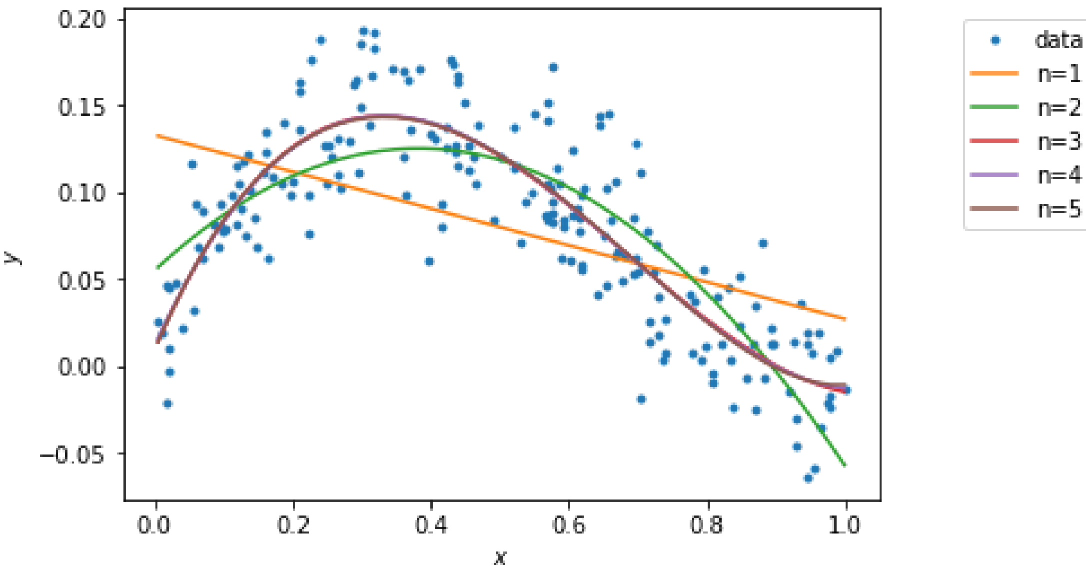
In [6]:
```
# Plot the data
f = plt.figure()
ax = f.gca()
ax.plot(x, y, '.')
ax.set_xlabel('$x$')
ax.set_ylabel('$y$')

# Plot the regression lines
plot_xs = []
for i in np.arange(N):
    if i == 0:
```

```
            plot_x = np.vstack((np.linspace(min(x), max(x),50), np.ones(50)))
        else:
            plot_x = np.vstack((plot_x[-2]**(i+1), plot_x))
        plot_xs.append(plot_x)

    for i in np.arange(N):
        ax.plot(plot_xs[i][-2,:], thetas[i].dot(plot_xs[i]))

    labels = ['data']
    [labels.append('n={}'.format(i+1)) for i in np.arange(N)]
    bbox_to_anchor=(1.3, 1)
    lgd = ax.legend(labels, bbox_to_anchor=bbox_to_anchor)
```



0.1.8 Calculating the training error (10 points)

Here, we'll now calculate the training error of polynomial models of orders 1 to 5:
$L(\theta) = \frac{1}{2} \sum_j (\hat{y}_j - y_j)^2$

```
In [7]: training_errors = []

    # ==================== #
    # START YOUR CODE HERE #
    # ==================== #

    # GOAL: create a variable training_errors, a list of 5 elements,
    # where training_errors[i] are the training loss for the polynomial fit of order i+1.
    for i in range(N):
        yhat = thetas[i].dot(xhats[i])
```

```
            training_errors.append(np.sum((yhat - y)**2)/2)

        # ================== #
        # END YOUR CODE HERE #
        # ================== #

        print ('Training errors are: \n', training_errors)
```

```
Training errors are:
 [0.2379961088362701, 0.1092492220926853, 0.08169603801105374, 0.08165353735296976, 0.08161479
```

0.1.9 QUESTIONS

(1) Which polynomial model has the best training error?

(2) Why is this expected?

0.1.10 ANSWERS

(1) The highest order model has the best training error.

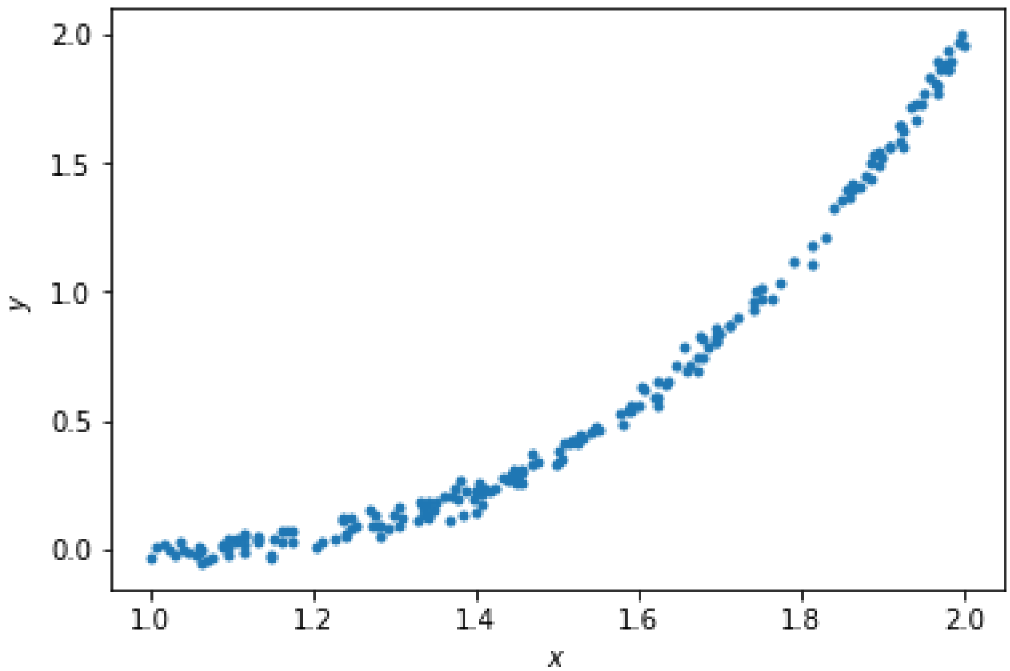
(2) The n-th order model will always do as good as a lower order model because the lower order model can be expressed in terms of the higher order model by setting the coefficient of the higher order terms to 0.

0.1.11 Generating new samples and testing error (5 points)

Here, we'll now generate new samples and calculate the testing error of polynomial models of orders 1 to 5.

```
In [8]: x = np.random.uniform(low=1, high=2, size=(num_train,))
        y = x - 2*x**2 + x**3 + np.random.normal(loc=0, scale=0.03, size=(num_train,))
        f = plt.figure()
        ax = f.gca()
        ax.plot(x, y, '.')
        ax.set_xlabel('$x$')
        ax.set_ylabel('$y$')
```

```
Out[8]: Text(0,0.5,'$y$')
```




```
In [9]: xhats = []
        for i in np.arange(N):
            if i == 0:
                xhat = np.vstack((x, np.ones_like(x)))
                plot_x = np.vstack((np.linspace(min(x), max(x),50), np.ones(50)))
            else:
                xhat = np.vstack((x**(i+1), xhat))
                plot_x = np.vstack((plot_x[-2]**(i+1), plot_x))

            xhats.append(xhat)
```

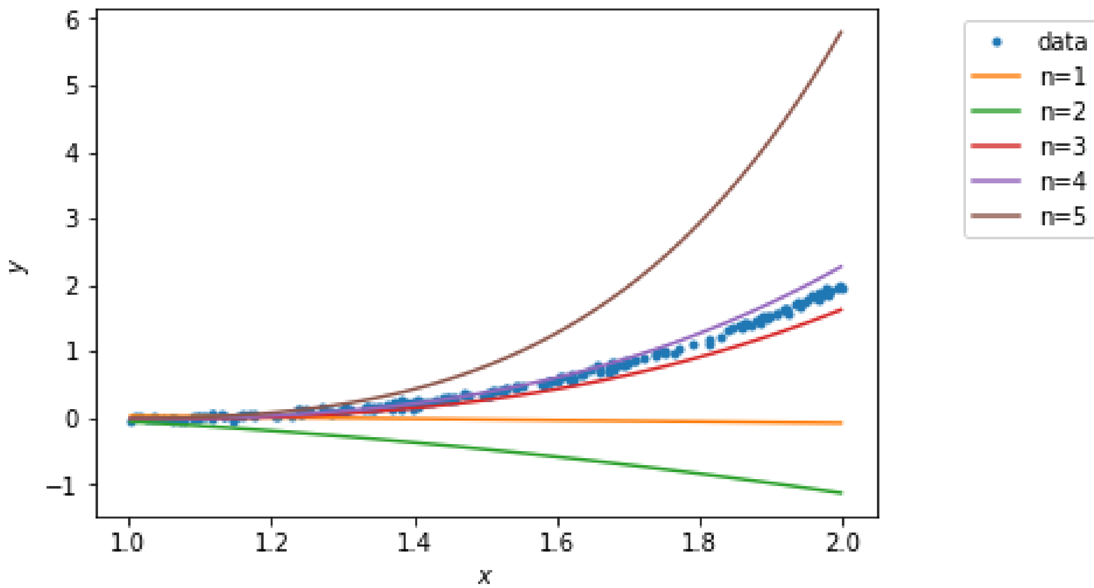
```
In [10]: # Plot the data
         f = plt.figure()
         ax = f.gca()
         ax.plot(x, y, '.')
         ax.set_xlabel('$x$')
         ax.set_ylabel('$y$')

         # Plot the regression lines
         plot_xs = []
         for i in np.arange(N):
             if i == 0:
                 plot_x = np.vstack((np.linspace(min(x), max(x),50), np.ones(50)))
             else:
                 plot_x = np.vstack((plot_x[-2]**(i+1), plot_x))
```

```
    plot_xs.append(plot_x)

for i in np.arange(N):
    ax.plot(plot_xs[i][-2,:], thetas[i].dot(plot_xs[i]))

labels = ['data']
[labels.append('n={}'.format(i+1)) for i in np.arange(N)]
bbox_to_anchor=(1.3, 1)
lgd = ax.legend(labels, bbox_to_anchor=bbox_to_anchor)
```




```
In [11]: testing_errors = []

# ==================== #
# START YOUR CODE HERE #
# ==================== #

# GOAL: create a variable testing_errors, a list of 5 elements,
# where testing_errors[i] are the testing loss for the polynomial fit of order i+1.
for i in range(N):
    yhat = thetas[i].dot(xhats[i])
    testing_errors.append(np.sum((yhat - y)**2)/2)

# ================== #
# END YOUR CODE HERE #
# ================== #

print ('Testing errors are: \n', testing_errors)
```

```
Testing errors are:
 [80.86165184550586, 213.19192445057908, 3.1256971084083736, 1.1870765211496224, 214.910217470
```

0.1.12 QUESTIONS

(1) Which polynomial model has the best testing error?

(2) Why does the order-5 polynomial model not generalize well?

0.1.13 ANSWERS

(1) The 4th order polynomial model has the best testing error.

(2) The order-5 polynomial model overfit the training data, meaning that it did not capture the underlying distribution of the data well.