Grothendieck Spectral Sequences

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Extending an additive functor

Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories. It is immediately extend component-wise to a functor $C(\mathcal{A}) \to C(\mathcal{B})$ between categories of (cochain) complexes. Moreover we know that we can naturally extend F to a functor

$$K(F) : K(\mathcal{A}) \to K(\mathcal{B}).$$

Proof.

Let $f, g : A^\bullet \to B^\bullet$ be homotopic morphisms of complexes in $C(\mathcal{A})$. Thus $\exists h = (h^n : A^n \to B^{n-1})$ such that

$$f^n - g^n = d_B^{n-1} h^n + h^{n+1} d_A^n.$$

By additivity of F we then have

$$F(f)^n - F(g)^n = d_{F(B)}^{n-1} F(h^n) + F(h^{n+1}) d_{F(A)}^n,$$

and thus $F(f), F(g) : F(A^\bullet) \to F(B^\bullet)$ are homotopic by $F(h) := (F(h^n) : F(A^n) \to F(B^{n-1}))$. $\square$

Extending to homotopy categories

We see that $K(F) : K(\mathcal{A}) \to K(\mathcal{B})$ transforms cones to cones, and hence it defines an functor of triangulated categories, i.e. additive, essentially commuting with [1] and sending Δ to Δ. This is because the distinguished triangles in $K(\mathcal{A})$ are precisely those isomorphic to

$$A^\bullet \xrightarrow{f^\bullet} B^\bullet \hookrightarrow \mathrm{Cone}(f^\bullet) \twoheadrightarrow A^\bullet[1],$$

where the cone is the complex defined by

$$\mathrm{Cone}(f) = A^\bullet[1] \oplus B^\bullet,\ d = \begin{pmatrix} d_{A^\bullet[1]} & 0 \\ f[1] & d_{B^\bullet} \end{pmatrix}.$$

Extending to derived categories?

Now we wish to extend $K(F)$ further to a functor $D(F) : D(\mathcal{A}) \to D(\mathcal{B})$ making the following diagram commute:

$$\begin{array}{ccc} K(\mathcal{A}) & \xrightarrow{K(F)} & K(\mathcal{B}) \\ \downarrow{\scriptstyle \pi_{\mathcal{A}}} & & \downarrow{\scriptstyle \pi_{\mathcal{B}}} \\ D(\mathcal{A}) & \xrightarrow{D(F)} & D(\mathcal{B}) \end{array}$$

By the universal property of localisations, it suffices to check that $K(F)$ preserves qis, i.e. for any quasi-isomorphism u in $K(\mathcal{A})$, $K(F)(u)$ is a quasi-isomorphism in $K(\mathcal{B})$.
It is easy to see that if F is exact then this holds, and vice versa.
In fact, a functor between abelian categories is exact iff it preserves Ker, Coker and $\oplus$. But this is too special a case. So we usually focus on extending F to (left or right) bounded derived categories.

Preserving acyclic complexes

Now let $F : K(\mathcal{A}) \to K(\mathcal{B})$ be any functor of triangulated categories. As we have seen, F extends to derived categories if it transforms qis to qis, and in particular, acyclic complexes (i.e. with 0 cohomology) to acyclic complexes. Indeed, if $A^\bullet \in K(\mathcal{A})$ is acyclic, $(0 \to A^\bullet) \in \mathrm{qis}$. So $F(0 \to A^\bullet) = 0 \to F(A^\bullet) \in \mathrm{qis}$, and thus $H^i(F(A^\bullet)) = H^i(0) = 0$, $\forall i$.
Conversely, let $f : X^\bullet \to Y^\bullet$ be a qis in $K(\mathcal{A})$. By (TR2) we can extend it to a distinguished triangule
$X^\bullet \xrightarrow{f} Y^\bullet \to \mathrm{Cone}(f) \to X^\bullet[1]$, which gives an acyclic complex $\mathrm{Cone}(f)$. The reason is that any distinguished triangle of complexes defines a long exact sequence of cohomology:

$$\cdots \to H^i(X^\bullet) \xrightarrow{\sim} H^i(Y^\bullet) \to H^i(C(f)) \to H^i(X^\bullet[1]) \xrightarrow{\sim} H^i(Y^\bullet[1]) \to \cdots$$

here $H^i(X^\bullet[1]) = H^{i+1}(X^\bullet)$ and $H^i(Y^\bullet[1]) = H^{i+1}(Y^\bullet)$.

Preserving acyclic complexes, cont.

Consider the distinguished triangle

$$F(X^\bullet) \xrightarrow{F(f)} F(Y^\bullet) \to F(\mathrm{Cone}(f)) \to F(X^\bullet[1]) \cong F(X^\bullet)[1].$$

If we assume that F transforms acyclic complexes to acyclic complexes, then $F(\mathrm{Cone}(f))$ is acyclic and $F(f)$ is a qis.
The idea of defining the derived functor: find a sufficiently large subcategory of $K(\mathcal{A})$ such that the restriction of F to it preserves acyclic complexes.

Adapted subcategories

In the following, $C^*(\mathcal{A})$ denotes one of $C(\mathcal{A})$, $C^\pm(\mathcal{A})$ and $C^b(A)$. Similarly for $K^*(\mathcal{A})$ and $D^*(\mathcal{A})$.
Recall that a subcat of a triangulated cat is called triangulated subcat, if (TR2) holds in that subcat. A full triangulated subcat $K^*(\mathcal{A})'$ of $K^*(\mathcal{A})$ is called left (resp. right) adapted for a left (resp. right) exact functor $F : K(\mathcal{A}) \to K(\mathcal{B})$ if

A1. $F(X^\bullet)$ is acyclic for any acyclic complex $X^\bullet \in K^*(\mathcal{A})'$;

A2. $\forall X^\bullet \in K^*(\mathcal{A})$, $\exists X^\bullet \to R^\bullet \in \mathrm{qis}$, (resp. $\exists R^\bullet \to X^\bullet \in \mathrm{qis}$), where $R^\bullet \in K^*(\mathcal{A})'$;

A3. the inclusion $\iota : K^*(\mathcal{A})' \to K^*(\mathcal{A})$ defines an equivalence of triangulated cats $\Psi : K^*(A)'_{\mathrm{qis}} \to D^*(\mathcal{A})$.

Deriving the functor

By (A1), $F \circ \iota$ preserves acyclic complexes. By the universal property of localisations this defines a functor $\tilde{F} : K^*(\mathcal{A})'_{\mathrm{qis}} \to D^*(\mathcal{B})$ such that $\pi_{\mathcal{B}} \circ F \circ \iota = \tilde{F} \circ \pi'_{\mathcal{A}}$. Let $\Phi : D^*(\mathcal{A}) \to K^*(\mathcal{A})'_{\mathrm{qis}}$ be a quasi-inverse functor. We set $D^*(F)' := \tilde{F} \circ \Phi : D^*(\mathcal{A}) \to D^*(\mathcal{B})$. By (A3) the functor $D^*(F)'$ is a functor of triangulated cats. See the following diagram:

$$\begin{tikzcd}
K^*(\mathcal{A})' \arrow[r, "\iota"] \arrow[dd, "\pi'_{\mathcal{A}}"'] & K^*(\mathcal{A}) \arrow[r, "F"] \arrow[d, "\pi_{\mathcal{A}}"] & K^*(\mathcal{B}) \arrow[dd, "\pi_{\mathcal{B}}"] \\
 & D^*(\mathcal{A}) \arrow[dl, bend right, "\Phi"'] \arrow[dr, "D^*(F)'"] & \\
K^*(\mathcal{A})'_{\mathrm{qis}} \arrow[ur, "\Psi"'] \arrow[rr, "\tilde{F}"] & & D^*(\mathcal{B})
\end{tikzcd}$$

Deriving the functor, cont.

Take $X^\bullet \in K^*(\mathcal{A})$ and consider it in $D^*(\mathcal{A})$. Using the iso of functors $1_{D^*(A)} \to \Psi \circ \Phi$ we can find a functorial iso $X^\bullet \to \Psi(Y^\bullet)$ in $D^*(\mathcal{A})$, where $Y^\bullet = \Phi(X^\bullet) \in K^*(\mathcal{A})'$. Write this iso in the form of a roof, say $X^\bullet \xrightarrow{s} Z^\bullet \xleftarrow{t} Y^\bullet$, where $s, t \in \mathrm{qis}$. By (A2) we can find $a : Z^\bullet \to W^\bullet \in \mathrm{qis}$, where $W^\bullet \in K^*(\mathcal{A})'$. Replacing s with $a \circ s$ and t with $a \circ t$, we may assume that $Z^\bullet \in K^*(\mathcal{A})'$. Applying F we get a morphism in $K^*(\mathcal{B})$, namely $F(X^\bullet) \xrightarrow{F(s)} F(Z^\bullet) \xleftarrow{F(t)} F(Y^\bullet)$. Since $t \in \mathrm{qis}$, $F(t) \in \mathrm{qis}$. Applying $\pi_{\mathcal{B}}$, we get a morphism in $D^*(\mathcal{B})$:

$$\pi_{\mathcal{B}} F(X^\bullet) \to \pi_{\mathcal{B}} F \iota(Y^\bullet) = \tilde{F}\pi'_{\mathcal{A}}(Y^\bullet) = \tilde{F}\Phi\pi_{\mathcal{A}}(X^\bullet) = D^*(F)'\pi_{\mathcal{A}}(X^\bullet).$$

This defines a natural transformation $\epsilon_F : \pi_{\mathcal{B}} \circ F \to D^*(F)' \circ \pi_{\mathcal{A}}$. By definition $\tilde{F}\Phi\Psi = D^*(F)'\Psi$, thus we have iso of functors $\tilde{F} \to D^*(F)'\Psi$. Composing with $\pi'_{\mathcal{A}}$ we get an iso of functors

$$\pi_{\mathcal{B}} F \iota = \tilde{F}\pi'_{\mathcal{A}} \xrightarrow{\sim} D^*(F)'\Psi\pi'_{\mathcal{A}} = D^*(F)'\pi_{\mathcal{A}}\iota.$$

Right derived functors

This shows that, restricted to $K^*(\mathcal{A})'$, ϵ_F defines an iso of functors. Thus the pair $(D^*(F)', \epsilon_F)$ satisfies the following definition:

- Let $F : K(\mathcal{A}) \to K(\mathcal{B})$ be a left exact additive functor of abelian cats. A right derived functor of F is a pair $(D^+(F), \epsilon_F)$, where $D^+(F) : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is an additive exact functor, and $\epsilon_F : \pi_{\mathcal{B}} \circ K^+(F) \to D^+(F) \circ \pi_{\mathcal{A}}$ is a natural transformation, where $\pi_{\mathcal{A}} : K^+(A) \to D^+(A)$ and $\pi_{\mathcal{B}} : K^+(B) \to D^+(B)$ are the natural morphisms of the localisations; moreover, this pair is required to satisfy the following universal property: for any exact functor $G : D^+(A) \to D^+(B)$ and a natural transformation $\epsilon : \pi_{\mathcal{B}} \circ K^+(F) \to G \circ \pi_{\mathcal{A}}$ there exists a unique natural transformation $\eta : D^+(F) \to G$ making the following diagram commutative:

$$\begin{tikzcd}
 & \pi_{\mathcal{B}} \circ K^+(F) \arrow[dl, "\epsilon_F"'] \arrow[dr, "\epsilon"] & \\
D^+(F) \circ \pi_{\mathcal{A}} \arrow[rr, "\eta \circ \pi_{\mathcal{A}}"] & & G \circ \pi_{\mathcal{A}}
\end{tikzcd}$$

Left derived functors

Similarly we have the following definition:

- Let $F : K(\mathcal{A}) \to K(\mathcal{B})$ be a right exact additive functor of abelian cats. A left derived functor of F is a pair $(D^-(F), \epsilon_F)$, where $D^-(F) : D^-(\mathcal{A}) \to D^-(\mathcal{B})$ is an additive exact functor, and $\epsilon_F : D^-(F) \circ \pi_{\mathcal{A}} \to \pi_{\mathcal{B}} \circ K^-(F)$ is a natural transformation, satisfy the following universal property:
- for any exact functor $G : D^-(A) \to D^-(B)$ and a natural transformation $\epsilon : G \circ \pi_{\mathcal{A}} \to \pi_{\mathcal{B}} \circ K^-(F)$ there exists a unique natural transformation $\eta : G \to D^-(F)$ making the following diagram commutative:

$$\begin{tikzcd}
& \pi_{\mathcal{B}} \circ K^-(F) & \\
D^-(F) \circ \pi_{\mathcal{A}} \arrow[ur, "\epsilon_F"] & & G \circ \pi_{\mathcal{A}} \arrow[ul, "\epsilon"'] \arrow[ll, "\eta \circ \pi_{\mathcal{A}}"']
\end{tikzcd}$$

Existence

The following result is stated without proof.

- Assume that a left exact functor $F : K(\mathcal{A}) \to K(\mathcal{B})$ has a left adapted subcat $K(\mathcal{A})'$ of $K(\mathcal{A})$. Then $(D^+(F)', \epsilon_F)$ is a right derived functor of F. In particular, it does not depend, up to natural isomorphism, on the choice of adapted subcats.
- A similar result holds for any right exact functor with a right adapted subcat.

To define an a left (resp. right) adapted subcat $K^{\pm}(\mathcal{A})'$ for a left (resp. right) exact additive functor $F : K^{\pm}(\mathcal{A}) \to K^{\pm}(\mathcal{B})$, we choose it to be the full subcat $K^{\pm}(\mathcal{R})$ of $K^{\pm}(\mathcal{A})$ formed by complexes of objects belong to a left (resp. right) adapted subset $\mathcal{R} \subset ob(\mathcal{A})$, in the following sense:

Adapted subsets

A set $\mathcal{R}$ of objects in $\mathcal{A}$ is left (resp. right) adapted for a left (resp. right) additive functor $F : K^{\pm}(\mathcal{A}) \to K^{\pm}(\mathcal{B})$, if

1A. for any acyclic complex $X^{\bullet} \in C^{\pm}(\mathcal{R})$, $F(X^{\bullet})$ is acyclic;

2A. any object $A \in \mathcal{A}$ admits a monomorphism $A \hookrightarrow R$ (resp. epimorphism $R \twoheadrightarrow A$), where $R \in \mathcal{R}$;

3A. $\mathcal{R}$ is closed under finite direct sums.

We state, without proof, that for $\mathcal{R}$ defined above, $K^{\pm}(\mathcal{R})$ is an adapted subcat for F.

Object-complexes

Recall that we have canonical fully faithful functor $\mathcal{A} \to C^b(\mathcal{A})$ by taking object-complexes. The composition $\mathcal{A} \to C^b(\mathcal{A}) \to K^b(\mathcal{A})$ is indeed also fully faithful, since a morphism $A \to B$ of object-complexes is null-homotopic iff it is the 0 morphism. Moreover, notice that an object-complex is stable under taking cohomology, i.e. $H^\bullet(A) = A$, so any qis of object-complexes is an iso. Thus the composition

$$\mathcal{A} \to C^b(\mathcal{A}) \to K^b(\mathcal{A}) \to D^b(\mathcal{A})$$

is again fully faithful, which identifies $\mathcal{A}$ with a full subcat of $D^b(\mathcal{A})$.

n-th derived functor

- Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact additive functor, extended to a functor of triangulated cats $F : K(\mathcal{A}) \to K(\mathcal{B})$. We denote by $\mathbf{R}F$ the right derived functor $D^+(F) : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ defined by some choice of left adapted subset. This defines a functor
$$R^nF := H^n \circ \mathbf{R}F : \mathcal{A} \to \mathcal{B},$$
called the n-th right derived functor of F.
- If F is right exact, we can similarly define the left derived functor $\mathbf{L}F$: $D^-(\mathcal{A}) \to D^-(\mathcal{B})$ and the n-th left derived functor
$$L_nF := H^{-n} \circ \mathbf{L}F : \mathcal{A} \to \mathcal{B}.$$

It follows from the construction that for object-complexs,

$$R^nF(A) = D^+(K^+(F))(A[n]),\ L_nF(A) = D^-(K^-(F))(A[-n]),$$

where $K^{\pm}(F) : K^{\pm}(\mathcal{A}) \to K^{\pm}(\mathcal{B})$ is the canonical extension of F to a functor of triangulated categories.

Long exact sequences

- In fact, the derived functors $\mathbf{R}F$ and $\mathbf{L}F$ are triangulated. One way to see this (in the global version) is by using the expression $\mathbf{R}F(A^\bullet) \simeq \operatorname{colim}_{B^\bullet \to A^\bullet} K(F)(B^\bullet)$ to show that $\mathbf{R}F$ commutes with finite products, where $B^\bullet \to A^\bullet$ runs in $D(\mathcal{A})$. Then it also commutes with cones and is thus triangulated. See the book *Categories and Sheaves* by Kashiwara and Shapira for details.
- Now for any distinguished triangle $X^\bullet \to Y^\bullet \to Z^\bullet \to X^\bullet[1]$ we have a distinguished triangle $\mathbf{R}F(X^\bullet) \to \mathbf{R}F(Y^\bullet) \to \mathbf{R}F(Z^\bullet) \to \mathbf{R}F(X^\bullet)[1]$ that defines a long exact sequence of cohomology

$$\cdots \to H^n\mathbf{R}F(X^\bullet) \to H^n\mathbf{R}F(Y^\bullet) \to H^n\mathbf{R}F(Z^\bullet) \to H^{n+1}\mathbf{R}F(X^\bullet) \to \cdots$$

Long exact sequences, cont.

In particular, for a short exact sequence in $\mathcal{A}$, $0 \to A \xrightarrow{u} B \to C \to 0$, we can consider it, up to a qis, as a distinguished triangle

$$A \to \mathrm{Cyl}(u) \to \mathrm{Cone}(u) \to A[1],$$

where the cylinder of a map $f : X^\bullet \to Y^\bullet$ is defined by

$$\mathrm{Cyl}(f) = X^\bullet \oplus X^\bullet[1] \oplus Y^\bullet,\ d = \begin{pmatrix} d_{X^\bullet} & -1 & 0 \\ 0 & d_{X^\bullet[1]} & 0 \\ 0 & f[1] & d_{Y^\bullet} \end{pmatrix}.$$

Now we have long exact sequences

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to R^1F(B) \to R^1F(C) \to \cdots$$

and

$$\cdots \to L_1F(A) \to L_1F(B) \to L_1F(C) \to F(A) \to F(B) \to F(C) \to 0.$$

Case of enough injectives

- Very often we choose $\mathcal{R}$ to be the set of injectives in $\mathcal{A}$.
- Recall that for any cat $\mathcal{C}$, $I \in \mathcal{C}$ is an injective object, if $h^I : \mathcal{C} \to \mathbf{Set}$ transforms any monomorphism to an epimorphism, where $h^I(X) = Hom_{\mathcal{C}}(X, I)$. When $\mathcal{C}$ is additive, an injective object I is characterised by the property that h^I is exact. We say that $\mathcal{C}$ has enough injectives if any object admits a monomorphism to an injective object.
- We state the following result without proof: in an abelian cat $\mathcal{A}$ with enough injectives, the set $\mathcal{I}$ of injectives becomes a left adapted set for any left exact additive functor $F : \mathcal{A} \to \mathcal{B}$ between abelian cats.
- Moreover, if $\mathcal{A}$ has enough injectives, we have

$$K^+(\mathcal{I}) \simeq D^+(\mathcal{A}).$$

Co-resolutions

- Let $A \in \mathcal{A}$, $\mathcal{R} \subset ob(\mathcal{A})$ be a set of objects. Then an $\mathcal{R}$-resolution of A is a complex $X^\bullet \in C^+(\mathcal{R})$ with an augmentation map $A \xrightarrow{\alpha} X^0$ such that $X^i = 0$ for $i < 0$, $H^i(X^\bullet) = 0$ for $i > 0$, and that α is isomorphic onto $\mathrm{Ker}(X^0 \xrightarrow{d^0} X^1)$.
- This implies that the object-complex A is isomorphic to $H^0(X^\bullet)$, in other words, $A \to X^\bullet$ is a qis; a choice of such a qis gives an acyclic complex

$$0 \to A \to X^0 \to X^1 \to \cdots$$

- When $\mathcal{R} := \mathcal{I}$, we get the definition of injective resolutions. If $\mathcal{A}$ has enough injectives, each object $A \in \mathcal{A}$ has an injective resolution, and any morphism $A \xrightarrow{f} B$ can be extended to a morphism of injective resolutions uniquely up to homotopy.

Case of enough injectives, cont.

- Therefore, for $\mathcal{A}$ with enough injectives, $\mathcal{A}$ is equivalent to the full subcat of $K^+(\mathcal{I})$ that consists of injective resolutions.
- Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact additive functor of abelian cats. Suppose $\mathcal{A}$ has enough injectives, then there is a natural iso $F \cong R^0 F$.

Proof.

Let $\mathcal{I}$ be an adapted set of F and derive it accordingly. Let $A \to I^\bullet$ be an injective resolution of A, then $R^0 F(A) = H^0(F(I^\bullet))$. Since F is left exact, $F(A) \to F(I^0)$ is a isomorphic onto $\mathrm{Ker}(F(X^0) \to F(X^1))$, so $R^0 F(A) \cong F(A)$. To make it functorial, we use that any morphism $A \to A'$ in $\mathcal{A}$ defines a unique morphism in $K(\mathcal{A})$ of their injective resolutions. Since $H^0 : K(\mathcal{A}) \to \mathcal{B}$ is functorial, we see that $R^0 F \cong F$. $\square$

- Notice that in general, if the derived functor is defined by using an arbitrary adapted set of objects, the isomorphism may not be functorial.

Example: Hom and Ext functors

- Assume $\mathcal{A}$ has enough injectives. Consider the left exact functor $\mathrm{Hom}_{\mathcal{A}}(A,-) : \mathcal{A} \to \mathbf{Ab}$. We denote by $\mathbf{R}\mathrm{Hom}(A,-) : D^+(\mathcal{A}) \to D^+(\mathbf{Ab})$ its right derived functor. And we let $\mathrm{Ext}^i_{\mathcal{A}}(A,-) := R^i\mathrm{Hom}_{\mathcal{A}}(A,-) : \mathcal{A} \to \mathbf{Ab}$.
- Explicitly, we first extend $\mathrm{Hom}(A,-)$ to $K\mathrm{Hom}(A,-) : K^+(\mathcal{A}) \to K^+(\mathbf{Ab})$. By definition $K\mathrm{Hom}(A,X^\bullet) = \mathrm{Hom}^\bullet(A,X^\bullet)$, where $\mathrm{Hom}^i(A,X^\bullet) = \mathrm{Hom}(A,X^i) = \mathrm{Hom}_{K(\mathcal{A})}(A,X^\bullet[i])$.
- To extend to derived functor, we replace $X^\bullet$ with a quasi-isomorphic complex $I^\bullet$ of injectives and apply the extended functor to it, to get

$$\begin{aligned}\mathrm{Ext}^i(A,X^\bullet) := H^i\mathbf{R}\mathrm{Hom}(A,I^\bullet) &\cong \mathrm{Hom}_{K(\mathcal{A})}(A,I^\bullet[i]) \\ &\cong \mathrm{Hom}_{D^+(\mathcal{A})}(A,X^\bullet[i]).\end{aligned}$$

- When $X^\bullet = B$ is an object-complex, $I^\bullet$ is an injective resolution of B, and $\mathrm{Hom}^i(A,B) = \mathrm{Ext}^i(A,B)$.

Generalisation: Hom and Ext bifunctors

- More generally, let $A^\bullet$ and $B^\bullet$ be 2 complexes in $\mathcal{A}$, we define a complex of abelian groups $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$ by

$$\mathrm{Hom}^n(A^\bullet, B^\bullet) = \prod_{i \in \mathbb{Z}} \mathrm{Hom}(A^i, b^{i+n}),$$
$$d^n(f) = d_{B^\bullet} \circ f - (-1)^n f \circ d_{A^\bullet}, \;\; f \in \mathrm{Hom}^n(A^\bullet, B^\bullet).$$

- Note that $\mathrm{Ker}(d^n) = \mathrm{Hom}_{C(\mathcal{A})}(A^\bullet, B^\bullet[n])$, and $\mathrm{Im}(d^n)$ consists of null-homotopic maps. Thus $H^n(\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)) \cong \mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet[n]) \cong \mathrm{Hom}_{K(\mathcal{A})}(A^\bullet[-n], B^\bullet)$.

$$\begin{tikzcd}
\cdots \arrow[r] & A^i \arrow[r, "d^i_{A^\bullet}"] \arrow[d, "f_i"] \arrow[dr, dashed, "g_i"] & A^{i+1} \arrow[r] \arrow[d, "f_{i+1}"] & \cdots \\
\cdots \arrow[r] & B[n]^i \arrow[r, "(-1)^n d^{i+n}_{B^\bullet}"'] & B[n]^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

Hom and Ext bifunctors, cont.

- In other words, the bifunctor $\mathrm{Hom}^\bullet(-,-) : C(\mathcal{A})^{\mathrm{op}} \times C(\mathcal{A}) \to C(\mathbf{Ab})$ can be extended to a bitriangulated bifunctor $\mathrm{Hom}^\bullet(-,-) : K(\mathcal{A})^{\mathrm{op}} \times K(\mathcal{A}) \to K(\mathbf{Ab})$.
- If $\mathcal{A}$ has a right adapted subset for the 1st partial functor[1] $\mathrm{Hom}^\bullet(-,B^\bullet)$, e.g. $\mathcal{A}$ has enough projectives, then we can extend it to a bifunctor $\mathbf{R}\mathrm{Hom}^\bullet(-,-) : D^-(\mathcal{A})^{\mathrm{op}} \times D^+(\mathcal{A}) \to D^b(\mathbf{Ab})$.
- Notice that the compositions of both partial derived functors with $H^i : D^b(\mathbf{Ab}) \to \mathbf{Ab}$ are isomorphic functors, so we may choose the either that exists and set $\mathrm{Hom}^i(A^\bullet, B^\bullet) := H^i(\mathrm{Hom}^\bullet(A^\bullet, B^\bullet))$.
- If the 1st (resp. 2nd) partial derived functor exists, then this equals $\mathrm{Hom}_{D^-(\mathcal{A})}(A^\bullet[-i].B^\bullet)$ (resp. $\mathrm{Hom}_{D^+(\mathcal{A})}(A^\bullet, B^\bullet[i])$).
- Restricted to $\mathcal{A}^{\mathrm{op}} \times \mathcal{A}$, we get the familiar $\mathrm{Ext}^i(-,-)$.

[1] Notice that it is contravariant "left" exact.

Excursion: sheaves

- Let $\mathcal{C} \in \{\mathbf{Set}, \mathbf{Ab}, \mathbf{Gp}, \mathbf{CRing}, {}_A\mathbf{Mod}, {}_A\mathbf{Alg}, \cdots\}$, and X a topological space. Let $O(X)$ be the category of open subsets of X with inclusion morphisms.
- A $\mathcal{C}$-valued presheaf on X is a functor $\mathscr{F} : O(X)^{\mathrm{op}} \to \mathcal{C}$. $\forall U \in O(X)$, $s \in \mathscr{F}(U)$ is a section of $\mathscr{F}$ on U. The stalk of $\mathscr{F}$ at $x \in X$ is defined as $\mathscr{F}_x := \mathrm{colim}_{U \ni x} \mathscr{F}(U)$. The image of s in $\mathscr{F}_x$, denoted by s_x, is the germ of the section s at x.
- A presheaf $\mathscr{F}$ on X is a sheaf if [2]for any $U \in O(X)$, any open covering $(U_i)_{i \in I}$ of U, and any $(s_i)_i \in \prod_{i \in I} \mathscr{F}(U_i)$ satisfying

$$\forall (i, j) \quad s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j},$$

  we have $\exists! s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i$, $\forall i$.
- $\mathbf{Sh}_X$ is a full subcat of $\mathbf{Presh}_X$. The inclusion admits a left adjoint $\mathscr{F} \mapsto \hat{\mathscr{F}}$, called the sheafification which keeps the stalks, i.e. $\mathrm{Hom}_{\mathbf{Presh}_X}(\mathscr{F}, \mathscr{G}) \simeq \mathrm{Hom}_{\mathbf{Sh}_X}(\hat{\mathscr{F}}, \mathscr{G})$.

[2]the so-called gluing condition; it requires $\mathscr{F}(\varnothing) = *$.

Funtoriality

- For continuous $f : X \to Y$ and $\mathscr{F} \in \mathbf{Presh}_X$, the formula $U \mapsto \mathscr{F}(f^{-1}(U))$ defines $f_*\mathscr{F} \in \mathbf{Presh}_Y$; we can verify $f_*(\mathbf{Sh}_X) \subset \mathbf{Sh}_Y$.
- Conversely, for $\mathscr{G} \in \mathbf{Presh}_Y$, $V \mapsto \mathrm{colim}_{U \supset f(V)} \mathscr{G}(U)$ defines $f^{-1}\mathscr{G} \in \mathbf{Presh}_X$; if $\mathscr{G} \in \mathbf{Sh}_Y$, we now denote by $f^{-1}\mathscr{G}$ the sheafification of the presheaf $f^{-1}\mathscr{G}$.
- f^{-1} preserves the stalks: $f^{-1}\mathscr{G}_x \simeq \mathscr{G}_{f(y)}$; similar result does not hold in general for f_*.
- We have adjunctions $(f^{-1}, f_*) : \mathbf{Presh}_Y \to \mathbf{Presh}_X$ and $(f^{-1}, f_*) : \mathbf{Sh}_Y \to \mathbf{Sh}_X$. Moreover, $(f \circ g)_* \simeq f_* \circ g_*$ and $(f \circ g)^{-1} \simeq g^{-1} \circ f^{-1}$ in both cases.
- Let $i : \{x\} \hookrightarrow X$ be the inclusion, $\mathscr{F} \in \mathbf{Presh}_X$, then $i^{-1}\mathscr{F}(\{x\}) = \mathscr{F}_x$; if $\mathscr{F} \in \mathbf{Sh}_X$, then moreover $i^{-1}\mathscr{F}(\varnothing) = *$.
- Let $j : U \hookrightarrow X$ be inclusion of open subspace, $\mathscr{F} \in \mathbf{Presh}_X$, then $j^{-1}\mathscr{F} = \mathscr{F}|_U$; same for sheaves upon sheafification.

Changing the context: ringed spaces

- A ringed space is a pair $(X, \mathscr{O}_X)$ of a topological space and a (structural) sheaf of rings on it. An $\mathscr{O}_X$-module is a sheaf $\mathscr{F}$ of abelian groups such that $\forall U \in O(X)$, $\mathscr{F}(U) \in {}_{\mathscr{O}_X(U)}\mathbf{Mod}$, and for any open subsets $V \subset U$, $\forall s \in \mathscr{F}(U)$, $\forall f \in \mathscr{O}_X(U)$, $(fs)|_V = f|_V \cdot s|_V$.Note that in this case $\mathscr{F}_x \in {}_{\mathscr{O}_{X,x}}\mathbf{Mod}$.
- A morphism of ringed spaces $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a continuous map $\phi : X \to Y$ together with a morphism of sheaves $\mathscr{O}_Y \to \phi_*\mathscr{O}_X$, or equivalently a morphism of sheaves $\phi^{-1}\mathscr{O}_Y \to \mathscr{O}_X$.
- For $\mathscr{F} \in {}_{\mathscr{O}_X}\mathbf{Mod}$, $\phi_*\mathscr{F} \in {}_{\phi_*\mathscr{O}_X}\mathbf{Mod}$. Via the morphism of sheaves $\mathscr{O}_Y \to \phi_*\mathscr{O}_X$, $\phi_*\mathscr{F} \in {}_{\mathscr{O}_Y}\mathbf{Mod}$. Conversely, for $\mathscr{G} \in {}_{\mathscr{O}_Y}\mathbf{Mod}$, $\phi^{-1}\mathscr{G} \in {}_{\phi^{-1}\mathscr{O}_Y}\mathbf{Mod}$, and we let $\phi^*\mathscr{M} := \mathscr{O}_X \otimes_{\phi^{-1}\mathscr{O}_Y} \phi^{-1}\mathscr{G} \in {}_{\mathscr{O}_X}\mathbf{Mod}$, where the $\otimes$ is sheafified. One can show $(\phi^*\mathscr{G})_x = \mathscr{O}_{X,x} \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{G}_{f(x)}$. Moreover, $(\phi^*, \phi_*) : {}_{\mathscr{O}_Y}\mathbf{Mod} \to {}_{\mathscr{O}_X}\mathbf{Mod}$ is an adjunction.
- $(\phi \circ \psi)^* \simeq \psi^* \circ \phi^*$, $\phi^*\mathscr{M} \otimes_{\mathscr{O}_X} \phi^*\mathscr{N} \simeq \phi^*(\mathscr{M} \otimes_{\mathscr{O}_Y} \mathscr{N})$.

$\mathscr{H}om$ and $\mathscr{E}xt$ functors

- Let $(X, \mathscr{O}_X)$ be a ringed space with $\mathscr{O}_X$ a sheaf of commutative rings. Consider the left exact functor $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{M}, -) : {}_{\mathscr{O}_X}\mathbf{Mod} \to {}_{\mathscr{O}_X}\mathbf{Mod}$, where $\mathscr{H}om(\mathscr{M}, \mathscr{N})$ is the $\mathscr{O}_X$-module defined by $U \mapsto \mathrm{Hom}_{\mathscr{O}_X|_U}(\mathscr{M}|_U, \mathscr{N}|_U)$.
- An injective object in ${}_{\mathscr{O}_X}\mathbf{Mod}$ is some $\mathscr{I} \in {}_{\mathscr{O}_X}\mathbf{Mod}$ whose stalks $\mathscr{I}_x$ are injective $\mathscr{O}_{X,x}$-modules. ${}_{\mathscr{O}_X}\mathbf{Mod}$ admits enough injectives. See Hartshorne's *Algebraic Geometry*.
- Now we see that $\mathscr{H}om(\mathscr{M}, -)$ admits the right derived functor[3] $\mathbf{R}\mathscr{H}om(\mathscr{M}^\bullet, -) : D^+({}_{\mathscr{O}_X}\mathbf{Mod}) \to D^+(\mathbf{Sh}_X^{ab})$.
- Restricted to object-complexes, we define $\mathscr{E}xt^n_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N}) = R^n\mathscr{H}om(\mathscr{M}, \mathscr{N})$.
- In particular, $\mathscr{H}om(\mathscr{O}_X, -)$ is the identity functor, so $\mathscr{E}xt^0(\mathscr{O}_X, \mathscr{G}) = \mathscr{G}$, and $\mathscr{E}xt^n(\mathscr{O}_X, G) = \underline{0}$ for $n > 0$.

[3]Here $\mathbf{Sh}_X^{ab} = {}_{\underline{\mathbb{Z}}}\mathbf{Mod}$; notice that constant presheaf $\underline{\mathbb{Z}}$ is not a sheaf!

$\otimes$ and Tor functors

- Let $(X, \mathscr{O}_X)$ be a ringed space, $\mathscr{M} \in \mathbf{Mod}_{\mathscr{O}_X}$, $\mathscr{N} \in {}_{\mathscr{O}_X}\mathbf{Mod}$. We can define their tensor product $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N} \in \mathbf{Sh}_X^{ab}$ via sheafification. If moreover $\mathscr{M}$ (resp. $\mathscr{N}$) $\in {}_{\mathscr{O}_X}\mathbf{Mod}_{\mathscr{O}_X}$, then the tensor product is a left (resp. right) $\mathscr{O}_X$-module[4].
- Now consider the right exact functors $\mathscr{M} \otimes_{\mathscr{O}_X} - : {}_{\mathscr{O}_X}\mathbf{Mod} \to \mathbf{Sh}_X^{ab}$ and $- \otimes_{\mathscr{O}_X} \mathscr{N} : \mathbf{Mod}_{\mathscr{O}_x} \to \mathbf{Sh}_X^{ab}$. They can be extended to a triangulated functor between homotopy categories of complexes.
- $\mathscr{M} \in \mathbf{Mod}_{\mathscr{O}_X}$ is flat if $\mathscr{M} \otimes_{\mathscr{O}_X} -$ is exact. For example, a locally free sheaf $\mathscr{F}$ is flat. In fact, $\forall x \in X$, $\exists x \in U \in O(X)$ such that $\mathscr{F}|_U \cong \mathscr{O}_X|_U^{(I_x)}$ for some set I_x.
- Any $\mathscr{O}_X$-module admits a flat resolution. This is because $\mathbf{Mod}_{\mathscr{O}_X}$ has a projective generator, namely $\bigoplus_{U \in O(X)} \mathscr{O}_U$, since there is a natural bijection $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_U, \mathscr{F}) \to \mathscr{F}(U)$, where the sheaves $\mathscr{O}_U$ are defined by $(\mathscr{O}_U)_x = \mathscr{O}_{X,x}$ for $x \in U$ and 0 otherwise.

[4] This is the case when the $\mathscr{O}_X$ is **CRing**-valued.

$\otimes$ and Tor functors, cont.

- It is not hard to show that the set of flat sheaves is a right adapted set for $\mathscr{M} \otimes_{\mathscr{O}_X} -$, and one defines the left derived functor $\mathscr{M} \overset{\mathbf{L}}{\otimes} - : D^-({}_{\mathscr{O}_X}\mathbf{Mod}) \to D^-(\mathbf{Sh}_X^{ab})$, and $\mathrm{Tor}_n^{\mathscr{O}_X}(\mathscr{M}, -) = H^{-n}(\mathscr{M} \overset{\mathbf{L}}{\otimes} -)$.
- Replacing $\mathscr{N}$ by its flat resolution $\mathscr{P}_\bullet \to \mathscr{N} \to 0$, we obtain $\mathrm{Tor}_n^{\mathscr{O}_X}(\mathscr{M}, \mathscr{N}) = H^{-n}(\cdots \to \mathscr{M} \otimes \mathscr{P}_1 \to \mathscr{M} \otimes \mathscr{P}_0 \to 0)$.
- In particular, when $X = *$ and $\mathscr{O}_X$ is essentially just a commutative ring R, and the definition becomes familiar.

Generalisation: $\otimes$ of complexes

- For $\mathscr{M}^\bullet \in C^*(\mathbf{Mod}_{\mathscr{O}_X})$, $\mathscr{N}^\bullet \in C^*({}_{\mathscr{O}_X}\mathbf{Mod})$, one defines their tensor product $\mathscr{M}^\bullet \otimes \mathscr{N}^\bullet \in C^*(\mathbf{Sh}_X^{ab})$ by $(\mathscr{M}^\bullet \otimes \mathscr{N}^\bullet)^n = \bigoplus_{i+j=n} \mathscr{M}^i \otimes \mathscr{N}^j$, and $d^n(x^i \otimes y^j) = d_{\mathscr{M}^\bullet}(x^i) \otimes y^j + (-1)^n x^i \otimes d_{\mathscr{N}^\bullet}(y^j)$.
- If $\mathbf{Mod}_{\mathscr{O}_X}$ has enough flat objects, we can extend this definition to some $\mathscr{M}^\bullet \overset{\mathbf{L}}{\otimes} \mathscr{N}^\bullet \in D^-(\mathbf{Sh}_X^{ab})$.
- If $\mathscr{M}^\bullet = \mathscr{M}$, $\mathscr{N}^\bullet = \mathscr{N}$, we let $\mathrm{Tor}_i^{\mathscr{O}_X}(\mathscr{M}, \mathscr{N}) = H^{-i}(\mathscr{M} \overset{\mathbf{L}}{\otimes} \mathscr{N})$. It follows from the definitions that $\mathscr{M} \overset{\mathbf{L}}{\otimes} -$ coincides with the one defined before.
- Notice that one can compute $\mathrm{Tor}_i^{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})$ either by using the flat resolution of $\mathscr{M}$ or by using that of $\mathscr{N}$. The result is the same.

Derived functors of a direct image

Let $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ and $f_* : \mathbf{Mod}_{\mathscr{O}_X} \to \mathbf{Mod}_{\mathscr{O}_Y}$ be the direct image functor which is left exact.

- We have mentioned that $\mathbf{Mod}_{\mathscr{O}_X}$ has enough injectives, which form a left adapted set for f_*. Moreover, injective sheaves are flabby, i.e. the restriction maps $\mathscr{I}(U) \to \mathscr{I}(V)$ are surjective, $\forall V \subset U$.
- By definition, f_* is exact on the subcat of flabby sheaves, thus we can define the right global derived functor $\mathbf{R}f_* : D(\mathbf{Mod}_{\mathscr{O}_X}) \to D(\mathbf{Mod}_{\mathscr{O}_Y})$.
- Specialising to object-complexes we define $R^i f_* : \mathbf{Mod}_{\mathscr{O}_X} \to \mathbf{Mod}_{\mathscr{O}_Y}, \quad \mathscr{F} \mapsto H^i(\mathbf{R}f_* \mathscr{F})$.[5]
- In particular, let $(Y, \mathscr{O}_Y) = (*, R)$, $f_*\mathscr{F}(*) = \mathscr{F}(f^{-1}(*)) = \mathscr{F}(X) =: \Gamma(X, \mathscr{F})$ is the R-module of global sections. Also $R^i f_* \mathscr{F} =: H^i(X, \mathscr{F})$ is the i-th sheaf cohomology R-module.

[5] One can show that $R^i f_* \mathscr{F}$ is the sheafification of $U \mapsto H^i(f^{-1}(U), \mathscr{F})$.

Deriving an inverse image?

Recall that we also have $f^* : \mathbf{Mod}_{\mathscr{O}_Y} \to \mathbf{Mod}_{\mathscr{O}_X}$, $\mathscr{M} \mapsto \mathscr{M} \otimes_{\mathscr{O}_Y} \mathscr{O}_X$, which is right exact.
Since we can compute Tor's by using either argument, we check that flat sheaves is a right adapted set for f^*. This allows us to define the left derived functor

$$\mathbf{L}f^* : D^-(\mathbf{Mod}_{\mathscr{O}_X}) \to D^-(\mathbf{Mod}_{\mathscr{O}_X}).$$

Compatibility with composition

Suppose we have 2 left exact additive functor $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{C}$ between abelian cats. We would like to compare $\mathbf{R}(G \circ F)$ with $\mathbf{R}(G) \circ \mathbf{R}(F)$, provided that both are defined.

- Suppose $\mathcal{A}$ has a left adapted subset $\mathcal{R}$ for F, $\mathcal{B}$ has a left adapted subset $\mathcal{S}$ for G, and that $F(\mathcal{R}) \subset \mathcal{S}$. Then there is a natural isomorphism $\mathbf{R}(G \circ F) \simeq \mathbf{R}G \circ \mathbf{R}F$.

Proof.

By definition $\mathcal{R}$ is adapted for $G \circ F$, thus $\mathbf{R}(G \circ F)$ exists. Consider the natural transformation $\pi_{\mathcal{A}} \circ K(G \circ F) = \pi_{\mathcal{A}} \circ K(G) \circ K(F) \to \mathbf{R}G \circ \pi_{\mathcal{B}} \circ K(F) \to \mathbf{R}G \circ \mathbf{R}F \circ \pi_{\mathcal{A}}$. Moreover we have a canonical natural transformation $\pi_{\mathcal{A}} \circ K(G \circ F) \to \mathbf{R}(G \circ F) \circ \pi_{\mathcal{A}}$. By the universal property we get some $\mathbf{R}(G \circ F) \to \mathbf{R}G \circ \mathbf{R}F$. Now use the conditions to show that this map is isomorphic on complexes. $\square$

- To compute explicitly $R^n(G \circ F)$ in terms of $R^q G \circ R^p F$ one uses spectral sequences.

Spectral sequences

Let $\mathcal{A}$ be an abelian cat. A spectral sequence in $\mathcal{A}$ is a collection of complexes $(E_r^\bullet, d_r)_{r\in\mathbb{N}}$ and a collection of limit objects $\{H^n\}_{n\in\mathbb{Z}}$, with filtration of subobjects $F^\bullet = (\cdots \to F^i(H^n) \xrightarrow{u_i} F^{i-1}(H^n) \to \cdots)$, where u_i are monomorphisms, such that

SS1. each $E_r^n = \bigoplus_{p+q=n} E_r^{p,q}$;

SS2. the composition of $E_r^{p,q} \to E_r$ with d_r defines a morphism $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$;

SS3. there are isomorphisms $\alpha_r^{p,q} : \mathrm{Ker}(\mathrm{Coker}(d_r^{p-r,q+r-1}) \to E_r^{p+r,q-r+1}) \to E_{r+1}^{p,q}$;

SS4. $\exists r_0$ such that $d_r^{p,q} = d_r^{p-r,q+r-1} = 0$ for $r \geq r_0$, thus $E_r^{p,q} \cong E_{r_0}^{p,q}$ for $r \geq r_0$;[6]

SS5. $\forall p, q \in \mathbb{Z}$, there is an isomorphism

$$\beta_{p,q} : E_\infty^{p,q} \to Gr^p(H^{p+q}) := \mathrm{Coker}(F^{p+1}(H^{p+q}) \xrightarrow{u_{p+1}} F^p(H^{p+q})).$$

One denotes the spectral sequence by $E_r^{p,q} \Rightarrow H^n$.

[6] So the SS degenerates at E_{r_0} and $E_{r_0} = E_\infty$.

Case of a double complex $X^{\bullet,\bullet}$

We review this very important example. Consider differentials ${}'d^{i,j} : X^{i,j} \to X^{i+1,j}$ and ${}''d^{i,j} : X^{i,j} \to X^{i,j+1}$, with ${}'d \circ {}'d = 0$ and ${}''d \circ {}''d = 0$. Suppose moreover that it is commutative: ${}''d^{i+1,j} \circ {}'d^{i,j} =' d^{i,j+1} \circ {}''d^{i,j}$.
The total complex is defined by $tot(X^{\bullet,\bullet}) = (X^n, d^n)$, $X^n = \bigoplus_{p+q=n} X^{p,q}$, and
$d^n(x^{p,q}) = {}'d^{p,q}(x^{p,q}) + (-1)^p \, {}''d^{p,q}(x^{p,q})$, so that $d \circ d = 0$.
It should now be easy to get the 2 spectral sequences of the double complex:

$$
\begin{aligned}
{}'E_1^{p,q} &= {}''H^{p,q}(X^{\bullet,\bullet}),\ {}''E_1^{p,q} = {}'H^{p,q}(X^{\bullet,\bullet});\\
{}'E_2^{p,q} &= {}'H^p({}''H^q(X^{\bullet,\bullet})),\ {}'E_2^{p,q} = {}''H^p({}'H^q(X^{\bullet,\bullet})).
\end{aligned}
$$

Grothendieck spectral sequence

As before, let $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{C}$ be left exact additive functors between abelian cats, and suppose there exists $\mathcal{R} \subset \mathcal{A}$ adapted for F, $\mathcal{S} \subset \mathcal{B}$ adapted for G, and that $F(\mathcal{R}) \subset \mathcal{S}$. Then for any complex $A \in \mathcal{A}$, there exists a spectral sequence[7], functorial in A, such that

$$E_2^{p,q} = R^p G(R^q F(A)) \Rightarrow R^n(G \circ F)(A).$$

[7] See Tôhoku paper.

Idea of proof

- For simplicity, let $\mathcal{R}$ be the set of injectives, and suppose $\mathcal{A}$ has enough of them.
- We first replace A by its injective resolution $E^\bullet$ and consider $F(E^\bullet) \in C(\mathcal{S})$. On the one hand, $R^n(G \circ F)(X) = H^n G(F(E^\bullet))$. On the other hand, we want to replace $F(E^\bullet)$ with a quasi-isomorphic injective complex. This is done by taking its Cartan-Eilenberg resolution.
- For any complex $(X^\bullet, d)$, $d : X^\bullet \to X^\bullet[1]$, recall that we have complexes of boundaries, cycles and cohomology, given by

$$B(X^\bullet) = \mathrm{Im}(d), \; Z(X^\bullet) = \mathrm{Ker}(d),$$
$$H(X^\bullet) = \mathrm{Coker}(B(X^\bullet) \to Z(X^\bullet)).$$

Cartan-Eilenberg resolutions

Let $\mathcal{A}$ be abelian cat with enough injectives, $K^\bullet \in C(\mathcal{A})$, and $I^{\bullet,\bullet}$ a double complex in $\mathcal{A}$. We assume $I^{p,q} = 0$ if $p \le 0$ or $q < 0$. A resolution of $K^\bullet$ in $C(\mathcal{A})$,

$$K^\bullet \xrightarrow{\epsilon} I^{\bullet,0} \to I^{\bullet,1} \to I^{\bullet,2} \to \cdots$$

is a Cartan-Eilenberg resolution of $K^\bullet$, if

CE1. $K^i \to I^{i,\bullet}$ is an injective resolution of K^i, $\forall i \ge 0$;

CE2. it induces the following resolutions:

$$\begin{aligned} B(K^\bullet) &\to {}'B(I^{\bullet,0}) \to {}'B(I^{\bullet,1}) \to \cdots \\ Z(K^\bullet) &\to {}'Z(I^{\bullet,0}) \to {}'Z(I^{\bullet,1}) \to \cdots \\ H(K^\bullet) &\to {}'H(I^{\bullet,0}) \to {}'H(I^{\bullet,1}) \to \cdots \end{aligned}$$

CE3. the following exact sequences split:

$$\begin{aligned} 0 &\to {}'B(I^{\bullet,q}) \to {}'Z(I^{\bullet,q}) \to {}'H(I^{\bullet,q}) \to 0, \\ 0 &\to {}'Z(I^{\bullet,q}) \to I^{\bullet,q} \to {}'B(I^{\bullet,q+1}) \to 0. \end{aligned}$$

Observations

- Since direct summands of injectives are still injective, (CE3) implies that the resolutions in (CE2) are injective resolutions. So for a left exact G for which injectives are adapted, one can compute $R^pG(H^q(K^\bullet))$ by the injective resolution $H^q(K^\bullet) \to H^q(I^{\bullet,0}) \to H^q(I^{\bullet,1}) \to \cdots$
- $\mathcal{A}$ has enough injectives, so $C^+(\mathcal{A})$ has enough injectives, and any injective resolution of a complex in $C^+(\mathcal{A})$ is a CE resolution.
- Let $\epsilon : K^\bullet \to L^{\bullet,\bullet}$ be a resolution in $K(\mathcal{A})$, then the natural morphism $K^\bullet \to tot(L^{\bullet,\bullet})$ is a quasi-isomorphism. This is because ${}'H^p(L^{\bullet,\bullet}) = 0$ for $p > 0$ and ${}'H^0(L^{\bullet,\bullet}) = K^\bullet$, so the 2nd spectral sequence of the double complex $L^{\bullet,\bullet}$ is concentrated in 1 row on 2nd page, i.e. $E_2^{p,q} = 0$ for $p > 0$ and $E_2^{0,q} = H^q(K^\bullet)$. Hence it degenerates there. We thus have, by (SS5), an isomorphism $E_2^{0,q} \cong Gr^0(H^q) = H^q(tot(L^{\bullet,\bullet}))$.

Proof of theorem

- We take $K^\bullet = F(E^\bullet)$, where $E^\bullet$ is an injective resolution of A. We consider its Cartan-Eilenberg resolution, which gives a double complex $I^{\bullet,\bullet}$ of injectives such that all cohomology ${}'H^{p,q}(I^{\bullet,\bullet})$ are injective.
- In particular, we have an injective resolution $R^qF(A) = H^q(K^\bullet) \to {}'H^{q,0}(I^{\bullet,\bullet}) \to {}'H^{q,1}(I^{\bullet,\bullet}) \to \cdots$ so $R^pG(R^qF(A)) = H^p(G({}'H^{q,\bullet}))$.
- By (CE3), the 1st exact sequence is reversible, and ${}'H^{q,\bullet}$ can be expressed by kernels; since G is left exact, we can change the order: $H^p(G({}'H^{q,\bullet})) = {}''H^p\,{}'H^q(G(I^{\bullet,\bullet}))$.
- Notice that this is exactly the term of 2nd page from the 2nd spectral sequence for the double complex $G(I^{\bullet,\bullet})$, which converges to $H^nG(tot(L^{\bullet,\bullet})) = H^nG(K^\bullet) = R^n(G \circ F)(A)$, as we have seen.

Remarks

- For an additive functor $F : \mathcal{A} \to \mathcal{B}$ and a double complex $L^{\bullet,\bullet}$ in $\mathcal{A}$, the cohomology of $tot(F(L^{\bullet,\bullet})$ are called the hypercohomology of $L^{\bullet,\bullet}$ with respect to F.
- More generally, for left (resp. right) exact functors F, G as in the theorem (resp. its dual version), there is a spectral sequence $E_2^{p,q} = R^{\pm p} G \circ R^{\pm q} F(A^\bullet) \Rightarrow R^{p+q}(G \circ F)(A^\bullet)$, functorial in any complex $A^\bullet \in K^{\pm(\mathcal{A})}$, where $R^{>0}$ means right and $R^{<0}$ means left derived functors.

Example: Global section functors

- Let $(X, \mathscr{O}_X)$ be a ringed space, $R = \mathscr{O}_X(X)$. Recall the left exact global section functor $\Gamma_X : \mathbf{Sh}_X^{ab} \to \mathbf{Mod}_R$ given by $\mathscr{M} \mapsto \mathscr{M}(X) \cong \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X, \mathscr{M})$.
- The set of injective $\mathscr{O}_X$-modules form an adapted set for Γ_X, which defines the right derived functor $\mathbf{R}\Gamma_X : D^+(\mathbf{Mod}_{\mathscr{O}_X}) \to D^+(\mathbf{Mod}_R)$, and we have defined the sheaf cohomology by $H^n(X, \mathscr{M}) = R^n\Gamma_X(\mathscr{M})$.
- This is clearly a special case of the direct image functor when $(Y, \mathscr{O}_Y) = (*, R)$. In general, let $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$, we have $\Gamma_Y \circ f_* = \Gamma_X, \quad \mathbf{R}\Gamma_X = \mathbf{R}\Gamma_Y \circ \mathbf{R}f_*$.
- The spectral sequence of the composition of functors $E_2^{p,q} = H^p(Y, R^q f_* \mathscr{M}) \Rightarrow H^n(X, \mathscr{M})$ is the Leray spectral sequence.

Example: $\mathscr{H}om$ and $\mathscr{E}xt$

- Let $(X, \mathscr{O}_X)$ be a ringed space with the structural sheaf **CRing**-valued. The functor $\mathscr{H}om(\mathscr{M}, -) : \mathbf{Mod}_{\mathscr{O}_X} \to \mathbf{Mod}_{\mathscr{O}_X}$ admits the right derived functor $\mathbf{R}\mathscr{H}om_{\mathscr{O}_X}(\mathscr{M}^\bullet, -) : D^+(\mathbf{Mod}_{\mathscr{O}_X}) \to D^+(\mathbf{Sh}_X^{ab})$, and we have defined $\mathscr{E}xt^n(\mathscr{M}, \mathscr{N}) = R^n\mathscr{H}om(\mathscr{M}, \mathscr{N})$.
- Consider the global section functor $\Gamma_X : \mathbf{Mod}_{\mathscr{O}_X} \to \mathbf{Mod}_{\mathscr{O}_X(X)}$, and we have $\Gamma_X \circ \mathscr{H}om(\mathscr{M}, -) = \mathrm{Hom}(\mathscr{M}, -)$, thus there is a local-to-global Ext spectral sequence $E_2^{p,q} = H^p(X, \mathscr{E}xt^q(\mathscr{M}, \mathscr{N})) \Rightarrow \mathrm{Ext}^n(\mathscr{M}, \mathscr{N})$.
- As a reminder, the condition of the theorem is satisfied, since $\mathscr{H}om(\mathscr{M}, -)$ sends injective sheaves to flabby sheaves. See Godement's *Topologie algébrique et théorie des faisceaux*.

Exercise: $\otimes$ and Tor

Keeping the notations, consider $\mathscr{H}om(-, \mathscr{O}_X) : \mathbf{Mod}_{\mathscr{O}_X} \to \mathbf{Mod}_{\mathscr{O}_X}$ and $- \otimes \mathscr{N} : \mathbf{Mod}_{\mathscr{O}_X} \to \mathbf{Mod}_{\mathscr{O}_X}$.
As an exercise, try to show that there is an isomorphism in $D^b(\mathbf{Mod}_{\mathscr{O}_X})$ functorial in $\mathscr{M}^\bullet$ and $\mathscr{N}^\bullet$:

$$\mathbf{R}\mathscr{H}om_{\mathscr{O}_X}(\mathscr{M}^\bullet, \mathscr{N}^\bullet) \cong \mathbf{R}\mathscr{H}om_{\mathscr{O}_X}(\mathscr{M}^\bullet, \mathscr{O}_X) \overset{\mathbf{L}}{\otimes} \mathscr{N}^\bullet,$$

so that when specialising to object-complexes, there is a spectral sequence

$$E_2^{p,q} = \mathrm{Tor}_p^{\mathscr{O}_X}(\mathscr{E}xt^q_{\mathscr{O}_X}(\mathscr{M}, \mathscr{O}_X)) \cong \mathscr{E}xt^n_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N}).$$

How to apply the Grothendieck SS?

- In fact, in algebraic geometry, Grothendieck spectral sequences serve as a useful tool for the study of the derived category of coherent sheaves. And usually we use them to deduce certain properties of the limit objects, knowing those of the terms in pages.
- However, to give a concrete introduction on that, we need to polish our space X first. Usually we assume X to be a noetherian scheme of finite Krull dimension, and to satisfy:
- Each coherent sheaf on X is a quotient of a locally free $\mathscr{O}_X$-module.
- Then $D^b(\mathbf{Coh}(X)) \to D^b(\mathbf{Qcoh}(X))$ is fully faithful triangulated. For definitions, consider the locally exact sequence:

$$\mathscr{O}_U^{(I)} \to \mathscr{O}_U^{(J)} \to \mathscr{F}|_U \to 0.$$

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