Introduction to Spectral Sequences

A concise approach

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Motivation[1]

We mainly work in the category of abelian groups **Ab**. We aim to generalise the following fact.

For a complex $C_* : C_0 \leftarrow C_1 \leftarrow C_2 \leftarrow \cdots$ with a sub-complex $A_* : A_0 \leftarrow A_1 \leftarrow A_2 \leftarrow \cdots$, we can deduce the homology of C_* from certain information of that of A_* and C_*/A_*.

In fact, recall that the long exact sequence of the pair (C_*, A_*) gives

$$H_{n+1}(C_*/A_*) \xrightarrow{\partial_{n+1}} H_n(A_*) \to H_n(C_*) \to H_n(C_*/A_*) \xrightarrow{\partial_n} H_{n-1}(A_*)$$

Cutting it short, we get

$$0 \to \mathrm{coker}(\partial_{n+1}) \to H_n(C_*) \to \ker(\partial_n) \to 0$$

This helps to determine $H_n(C_*)$. We want to have similar results in a more general context: filtered complexes.

[1] The concept is introduced in: J. Leray. *L'anneau d'homologie d'une représentation.* C. R. Acad. Sci. 222 (1946)

Filtered complexes

For the complex (C_*, ∂) equipped with an *ascending filtration* of each of its terms:

$$F_p C_i \subseteq F_{p+1} C_i \subseteq \cdots \subseteq C_i,$$

compatible with the differential in the sense that
$\partial(F_p C_i) \subseteq F_p C_{i-1}$.
We denote also the graded term by $G_p C_i = F_p C_i / F_{p-1} C_i$.
Observe that the filtration also induces a filtration on the homology complex $H_q(C_*)$. In fact,

$$F_p H_q(C_*) = \{[x] \in H_q(C_*) \mid x \in F_p C_q, \partial x = 0\}, \qquad (1)$$

and it is clear that $F_p H_q \subseteq F_{p+1} H_q$.
The target of the spectral sequences is to calculate
$G_p H_q = F_p H_q / F_{p-1} H_q$ from the groups $G_p C_q$ and certain connecting morphisms.

Homological spectral sequences: 0th page

We denote by $E^0_{p,q} = G_p C_{p+q} = F_p C_{p+q}/F_{p-1} C_{p+q}$. By definition, the differential induces a morphism $\partial_0 : E^0_{p,q} \to E^0_{p,q-1}$ by $x \mapsto [\partial x]$.
Recall that $\partial(F_p C_{p+q}) \subseteq F_p C_{p+q-1}$, $\partial(F_{p-1} C_{p+q}) \subseteq F_{p-1} C_{p+q-1}$.
Clearly $\partial_0^2 = 0$. Thus we can consider the homology of the complex $(E^0_{p,*}, \partial_0)$, denoted by $E^1_{p,*}$, i.e.

$$E^1_{p,q} = \frac{\ker(E^0_{p,q} \xrightarrow{\partial_0} E^0_{p,q-1})}{\operatorname{im}(E^0_{p,q+1} \xrightarrow{\partial_0} E^0_{p,q})} = \frac{\ker(G_p C_{p+q} \to G_p C_{p+q-1})}{\operatorname{im}(G_p C_{p+q+1} \to G_p C_{p+q})} = H_{p+q}(G_p C_*).$$

The 0th page E^0

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \\
\downarrow & & \downarrow & & \\
E^0_{02} & & E^0_{12} & & \cdots \\
\downarrow{\scriptstyle \partial_0} & & \downarrow{\scriptstyle \partial_0} & & \\
E^0_{01} & & E^0_{11} & & \cdots \\
\downarrow{\scriptstyle \partial_0} & & \downarrow{\scriptstyle \partial_0} & & \\
E^0_{00} & & E^0_{10} & & \cdots
\end{array}
$$

1st page

We define a morphism $\partial_1 : E^1_{p,q} \to E^1_{p-1,q}$ as follows.
For $\alpha = [x] \in E^1_{p,q}$, where $x \in E^0_{p,q}$, we know
$0 = \partial_0 x = [\partial x] \in E^0_{p,q-1}$. Thus $\partial x \in F_{p-1}C_{p+q-1}$. Thus we let
$\partial_1(\alpha) = [\partial x] \in E^1_{p-1,q}$.
It is easy to verify that ∂_1 is a well-defined morphism and $\partial_1^2 = 0$.
This allows to consider the homology of the complex $(E^1_{*,q}, \partial_1)$,
which is denoted by $E^2_{*,q}$.

1st page: another perspective

Alternatively, consider the short exact sequence of the triple $(F_pC_*, F_{p-1}C_*, F_{p-2}C_*)$:

$$0 \to G_{p-1}C_* = \frac{F_{p-1}C_*}{F_{p-2}C_*} \to F_pC_*/F_{p-2}C_* \to G_pC_* = \frac{F_pC_*}{F_{p-1}C_*} \to 0$$

It induces a long exact sequence of homology:

$$\cdots \to H_{p+q}(G_pC_*) \xrightarrow{\partial_1} H_{p+q-1}(G_{p-1}C_*) \to \cdots$$

where the connecting morphism is exactly ∂_1 we have defined.

The 1st page E^1

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
E^1_{02} & \xleftarrow{\partial_1} & E^1_{12} & \xleftarrow{\partial_1} & E^1_{22} & \longleftarrow & \cdots \\
E^1_{01} & \xleftarrow{\partial_1} & E^1_{11} & \xleftarrow{\partial_1} & E^1_{21} & \longleftarrow & \cdots \\
E^1_{00} & \xleftarrow{\partial_1} & E^1_{10} & \xleftarrow{\partial_1} & E^1_{20} & \longleftarrow & \cdots
\end{array}$$

Second pages

Rather than continuing by hand, we define directly for $r \in \mathbb{N}$,

$$E^r_{p,q} = \frac{\{x \in F_p C_{p+q} \mid \partial x \in F_{p-r} C_{p+q-1}\}}{F_{p-1} C_{p+q} + \partial(F_{p+r-1} C_{p+q+1})} \qquad (2)$$

Here we omit the intersection in the denominator:
$A/B := A/(A \cap B)$.
Similarly one can show that the differential induces a morphism $\partial_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$, satisfying $\partial_r^2 = 0$.
For easy memory, E^{r+1} is just the homology of the complex (E^r, ∂_r), i.e.

$$E^{r+1}_{p,q} = \frac{\ker(E^r_{p,q} \xrightarrow{\partial_r} E^r_{p-r,q+r-1})}{\operatorname{im}(E^r_{p+r,q-r+1} \xrightarrow{\partial_r} E^r_{p,q})}.$$

The 2nd page E^2

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \\
E^2_{02} & E^2_{12} & E^2_{22} & E^2_{32} & \cdots \\
E^2_{01} & E^2_{11} & E^2_{21} \arrow[llu, "\partial_2"'] & E^2_{31} \arrow[llu, "\partial_2"'] & \cdots \\
E^2_{00} & E^2_{10} & E^2_{20} \arrow[llu, "\partial_2"'] & E^2_{30} \arrow[llu, "\partial_2"'] & \cdots
\end{tikzcd}$$

Limit page

Suppose that for every i, the filtration $F_* C_i$ is bounded, i.e. $F_0 C_i = 0$ and $F_p C_i = C_i$ for $p \gg 0$; then for every p, q, there exists r_0 such that when $r \geq r_0$, we have

$$E^r_{p,q} = G_p H_{p+q}(C_*). \tag{3}$$

Proof.

When $r \gg 0$, $F_{p-r} C_{p+q-1} = 0$, and $F_{p+r-1} C_{p+q+1} = C_{p+q+1}$. Now use (1) and (2). $\square$

In general, we say that the spectral sequence *degenerates* at E^r if $\partial_k = 0$ for all $k \geq r$. In this case, $E^r = E^{r+k} = E^\infty$ for all $k \geq 0$. Then (3) applies.

Case of 1 sub-complex

Let $F_0C_* = A_*$ and $F_1C_* = C_*$. This is exactly the case in the motivation. We denote by $B_* = C_*/A_*$.
Notice that $p = 0, 1$ are the only possible values.
Then $E^0_{0,q} = G_0C_q = A_q$, and
$E^0_{1,q} = G_1C_{q+1} = C_{q+1}/A_{q+1} = B_{q+1}$.
Further, $E^1_{0,q} = H_q(A_*)$ and $E^1_{1,q} = H_{q+1}(B_*)$.
Notice that the morphism ∂_1 is nothing but the connecting morphism of the long exact sequence associated with the short exact sequence $0 \to A_* \to C_* \to B_* \to 0$.

E^0 and E^1 for 1 sub-complex

$$
E^0: \quad
\begin{array}{cc}
\vdots & \vdots \\
A_2 & B_3 \\
\downarrow & \downarrow \\
A_1 & B_2 \\
\downarrow{\scriptstyle \partial_0} & \downarrow \\
\color{orange}{A_0} & B_1 \\
 & \downarrow \\
 & B_0
\end{array}
\qquad
E^1: \quad
\begin{array}{ccc}
\vdots & & \vdots \\
H_2(A) & \longleftarrow & H_3(B) \\
H_1(A) & \longleftarrow & H_2(B) \\
\color{orange}{H_0(A)} & \xleftarrow{\partial_1} & H_1(B) \\
0 & \longleftarrow & H_0(B)
\end{array}
$$

Remark: The orange colour is temporarily used to determine the position $(0, 0)$.

Case of 1 sub-complex: cont.

Recall that E^2 is defined as the homology of (E^1, ∂_1), so $E^2_{0,q} = \mathrm{coker}(\partial_1^{1,q})$ and $E^2_{1,q} = \ker(\partial_1^{1,q})$.
By the shape of the 2nd page E^2, $\partial_2 = 0$, so $E^2 = E^3$. Similarly $\partial_3 = 0$ implies $E^3 = E^4$, and so on. So $E^2 = E^\infty$, and hence the sequence degenerates at E^2.
As a result, $E^2_{p,q} = G_p H_{p+q}$. Recall that $p = 0, 1$, and thus we have a short exact sequence

$$0 \to G_0 H_{p+q} \to H_{p+q} \to G_1 H_{p+q} \to 0,$$

which now reads

$$0 \to \mathrm{coker}(\partial_1^{0,p+q}) \to H_{p+q} \to \ker(\partial_1^{0,p+q-1}) \to 0.$$

We have thus regained the information carried by the long exact sequence.

E^2 for 1 sub-complex

$$
\begin{tikzcd}
 & \vdots & \vdots & \\
0 & \mathrm{coker}(\partial_1^{1,2}) & \ker(\partial_1^{1,2}) & 0 \\
0 & \mathrm{coker}(\partial_1^{1,1}) & \ker(\partial_1^{1,1}) \arrow[ull, "\partial_2"'] & 0 \arrow[ull, "\partial_2"'] \\
0 & \mathrm{coker}(\partial_1^{1,0}) & \ker(\partial_1^{1,0}) \arrow[ull, "\partial_2"] & 0 \arrow[ull, "\partial_2"']
\end{tikzcd}
$$

E^2 :

Case of a double complex

Now we are given a sequence of abelian groups $C_{p,q}$ with $p, q \geq 0$, and 2 differentials $\partial' : C_{p,q} \to C_{p-1,q}$ and $\partial'' : C_{p,q} \to C_{p,q-1}$ satisfying $\partial'^2 = \partial''^2 = \partial'\partial'' + \partial''\partial' = 0$.
We define its *total complex* (C_*, ∂) by $C_n = \bigoplus_{p+q=n} C_{p,q}$ and $\partial = \partial' + \partial''$. Remember that our target is always to calculate the homology of C_*, which we filter in 2 ways:

$$'F_p C_n = \bigoplus_{k+l=n, k \leq p} C_{k,l}, \quad ''F_p C_n = \bigoplus_{k+l=n, l \leq p} C_{k,l}.$$

Now by definition $'E^0_{p,q} = C_{p,q}$, $'E^1_{p,q} = H_q(C_{p,*}, \partial'')$, and also $''E^0_{p,q} = C_{q,p}$, $''E^1_{p,q} = H_p(C_{*,q}, \partial')$. Notice that $'\partial_0$ is induced by ∂'', and $''\partial_0$ is induced by ∂'.

Case of a double complex: cont.

The 2nd page is more interesting. Notice that

$$'\partial_1 : {}'E^1_{p,q} = H_q(C_{p,*}, \partial'') \to {}'E^1_{p-1,q} = H_q(C_{p-1,*}, \partial'')$$

is induced by $\partial' : C_{p,*} \to C_{p-1,*}$. So it is easy to see

$$'E^2_{p,q} = H_p(H_q(C_{p,*}, \partial''), \partial').$$

Dually,

$$''E^2_{p,q} = H_q(H_p(C_{*,q}, \partial'), \partial'').$$

Useful fact: if the 2 spectral sequences converge, they must converge to the same group $H_n(C_*)$, though for 2 different filtrations $'G_p$ and $''G_p$.This is seen by summing over the *diagonal* $p + q = n$.

Diagonal sum

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
E^{\infty}_{0,2} & E^{\infty}_{1,2} & E^{\infty}_{2,2} & \cdots \\
E^{\infty}_{0,1} & E^{\infty}_{1,1} & E^{\infty}_{2,1} & \cdots \\
E^{\infty}_{0,0} & E^{\infty}_{1,0} & E^{\infty}_{2,0} & \cdots
\end{array}$$

We have

$$H_n(C_*) = \bigoplus_{p+q=n} E^{\infty}_{p,q}.$$

Application: universal coefficient theorem

As an example, let M be an abelian group, and P_1, P_0 be free abelian groups such that there is a short exact sequence

$$0 \to P_1 \xrightarrow{\phi} P_0 \to M \to 0.$$

We write $P_i = 0$ if $i \notin \{0, 1\}$.
If X is a topological space, we can form the double complex $C_{p,q} = C_p(X) \otimes P_i$ with natural differentials: $\partial' = d \otimes 1$ and $\partial'' = 1 \otimes \phi$.
Since $C_p(X)$ is free, ${}'E^1_{p,0} = C_p(X) \otimes M$ and ${}'E^1_{p,q} = 0$ if $q \neq 0$.
Thus ${}'E^2$ is concentrated on the row $q = 0$ as is ${}'E^1$, and ${}'E^2_{p,0} = H_p(X, M)$.
By the shape of ${}'E^2$, ${}'\partial_2 = {}'\partial_3 = \cdots = 0$, so ${}'E^\infty = {}'E^2$.

The 1st spectral sequence $'E$

$$
'E^0 : \quad
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
C_0(X) \otimes P_1 \arrow[d, "\partial_0"] & C_1(X) \otimes P_1 \arrow[d] & \cdots \\
C_0(X) \otimes P_0 & C_1(X) \otimes P_0 & \cdots
\end{tikzcd}
$$

$$
'E^1 : \quad
\begin{tikzcd}
0 & 0 \arrow[l] & 0 \arrow[l] & \\
C_0(X) \otimes M & C_1(X) \otimes M \arrow[l] & C_2(X) \otimes M \arrow[l, "\partial_1"'] & \cdots \arrow[l]
\end{tikzcd}
$$

$$
'E^2 : \quad
\begin{tikzcd}
0 & 0 & 0 & \\
H_0(X, M) & H_1(X, M) & H_2(X, M) \arrow[ull, "\partial_2"'] & \cdots \arrow[ull]
\end{tikzcd}
$$

Universal coefficient theorem: cont.

On the other hand, $''E^0_{p,q} = C_{q,p} = C_q(X) \otimes P_p$, so $''E^1_{p,q} = H_q(X) \otimes P_p$. This spectral sequence is concentrated on 2 columns $p = 0, 1$.
It is not hard to continue the calculation: $''E^2_{0,q} = H_q(X) \otimes M$ and $''E^2_{1,q} = \mathrm{Tor}(H_q(X), M)$: use the long exact sequence

$$\cdots \to \mathrm{Tor}(H_q(X), P_0) = 0 \to \mathrm{Tor}(H_q(X), M) \to$$
$$\to H_q(X) \otimes P_1 \to H_q(X) \otimes P_0 \to H_q(X) \otimes M \to 0.$$

By the shape of $''E^2$, $''\partial_2 = {''\partial_3} = \cdots = 0$ and $''E^\infty = {''E^2}$. Recall that the 2 spectral sequences converge to the same homology $H_n(X, M)$ by summing over the diagonal $p + q = n$:

$$H_n(X, M) \cong H_n(X) \otimes M \ \oplus \ \mathrm{Tor}(H_{n-1}(X), M). \qquad (4)$$

The 2nd spectral sequence $''E$

$$''E^0 : \quad \begin{tikzcd} \vdots \arrow[d] & \vdots \arrow[d] \\ C_1(X) \otimes P_0 \arrow[d] & C_1(X) \otimes P_1 \arrow[d] \\ C_0(X) \otimes P_0 & C_0(X) \otimes P_1 \end{tikzcd}$$

$$''E^1 : \quad \begin{tikzcd} \vdots & \vdots \\ H_1(X) \otimes P_0 & H_1(X) \otimes P_1 \arrow[l] \\ H_0(X) \otimes P_0 & H_0(X) \otimes P_1 \arrow[l] \end{tikzcd}$$

$$''E^2 : \quad \begin{tikzcd} \vdots & \vdots & \\ H_1(X) \otimes M & \mathrm{Tor}(H_1(X), M) & 0 \arrow[ull] \\ H_0(X) \otimes M & \mathrm{Tor}(H_0(X), M) & 0 \arrow[ull] \end{tikzcd}$$

Exercise: Künneth formula

Let A_* and B_* be 2 complexes of k-vector spaces (of positive degree). There is an isomorphism

$$H_n(A_* \otimes B_*) = \bigoplus_{p+q=n} H_p(A_*) \otimes H_q(B_*). \tag{5}$$

Remark: the isomorphism is not canonical.

Proof.

To begin with, consider $C_n = \bigoplus_{p+q=n} A_p \otimes B_q$ with differential

$$\partial(x \otimes y) = \partial x \otimes y + (-1)^{\deg x} x \otimes \partial y.$$

□

Leray-Serre spectral sequence

We are given a *(Serre) fibration*[2] $p : X \to B$ with pointed base (B, b_0). Assume that its fibre $F = p^{-1}(b_0)$ has a base point x_0, and F, X, B are all path-connected. Recall that this means that $\pi_1(X, x_0)$ acts on $\pi_k(F)$.
For simplicity, we suppose in addition that B is a CW-complex, with skeletons

$$B^0 \subseteq B^1 \subseteq \cdots \subseteq B,$$

and filter $C_*(X)$ by the preimages $X^k = p^{-1}(B^k)$. In other words, $F_p C_*(X) = C_*(X^p)$.

---

[2] i.e. having RLP w.r.t. the inclusions $D^n \hookrightarrow D^n \times [0,1]$, $n \in \mathbb{N}$.

Serre spectral sequence: cont.

Then we can calculate $E^0_{p,q} = C_{p+q}(X^p, X^{p-1})$ and $E^1_{p,q} = H_{p+q}(X^p, X^{p-1})$. And what makes the sequence interesting is that, we have a natural isomorphism[3]

$$E^2_{p,q} \xrightarrow{\Phi} H_p(B, H_q(F)), \tag{6}$$

where $H_q(F)$ denotes the homology of the fibre at b_0, viewed as a $\mathbb{Z}[\pi_1(B)]$-module. Since (B, B^k) is k-connected, by HLP so is (X, X^k), and thus $X^k \hookrightarrow X$ induces an isomorphism on $H_n(-, M)$ if $n < k$. On the other hand $X^k = \varnothing$ for $k < 0$, and we can deduce that the spectral sequence for homology with coefficients converges to $H_*(X, M)$.

[3]The result with proof is found in: J.-P. Serre. *Homologie singulière des espaces fibrés. Applications.* Ann. of Math. 54 (1951)

Example: complex projective plane $\mathbb{C}P^2$

$S^1 \subseteq \mathbb{C}$ acts on $S^5 \subseteq \mathbb{C}^3$, with quotient space $\mathbb{C}P^2$. We use the fibration $S^1 \to S^5 \to \mathbb{C}P^2$ to compute $H_*(\mathbb{C}P^2)$.
Since $\pi_1(\mathbb{C}P^2) = 0$ and $H_*(S^1)$ are free, the Serre spectral sequence gives

$$E^2_{p,q} = H_p(\mathbb{C}P^2, H_q(S^1)) = H_p(\mathbb{C}P^2) \otimes H_q(S^1) \Rightarrow H_{p+q}(S^5).$$

Using the fact that $\mathbb{C}P^2$ is a 4-dimensional real manifold, we observe that E^2 is concentrated in $0 \leq p \leq 4$ and $q = 0, 1$. This shape implies that $\partial_3 = \partial_4 = \cdots = 0$, hence the sequence degenerates at E^3.

Homological Serre spectral sequence for $\mathbb{C}P^2$, 2nd page

$$
\begin{tikzcd}
E^2: & 0 & 0 & 0 & 0 & 0 & 0 \\
 & \mathbb{Z} & H_1(\mathbb{C}P^2) & H_2(\mathbb{C}P^2) & H_3(\mathbb{C}P^2) & H_4(\mathbb{C}P^2) & 0 \\
 & \color{orange}{\mathbb{Z}} & H_1(\mathbb{C}P^2) & H_2(\mathbb{C}P^2) \arrow[ull, "\partial_2"'] & H_3(\mathbb{C}P^2) \arrow[ull] \arrow[uulll, dashed, bend right, "\partial_3"'] & H_4(\mathbb{C}P^2) \arrow[ull] & 0
\end{tikzcd}
$$

$\mathbb{C}P^2$, cont.

We then use the homology of S^5 to determine $E^3 = E^\infty$.
Since $H_0(S^5) = H_5(S^5) = \mathbb{Z}$ and other homology groups vanish, $E^3_{0,0} = \mathbb{Z}$, and $E^3_{4,1} = \mathbb{Z}$. This is seen by summing over the diagonals and by taking homology of E^2. These 2 positions are the only non-vanishing ones in E^3. For degree reasons this implies that in E^2 we must have $H_3(\mathbb{C}P^2) \simeq H_1(\mathbb{C}P^2) = 0$, and $H_4(\mathbb{C}P^2) \simeq H_2(\mathbb{C}P^2) \simeq \mathbb{Z}$.
As a result, $H_k(\mathbb{C}P^2) = \mathbb{Z}$ iff $k = 0, 2, 4$ and other homology groups vanish.

Homological Serre spectral sequence for $\mathbb{C}P^2$, 3rd page

$$
\begin{array}{cccccc}
0 & & & & & \\
0 & 0 & & & & \\
0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & \mathbb{Z} & \\
\mathbb{Z} & 0 & 0 & 0 & 0 & 0
\end{array}
$$

E^3 :

Exercise: loop space of spheres

Let (X, x_0) be a pointed topological space. Its *path space* is

$$PX = \{\gamma : [0,1] \to X \mid \gamma(0) = x_0\}$$

with compact-open topology, and let $p(\gamma) = \gamma(1)$. We denote the *loop space* of X at x_0 by $\Omega X = p^{-1}(x_0)$. Then it is not hard to see that we have a fibration sequence

$$\Omega X \hookrightarrow PX \xrightarrow{p} X.$$

Try to use it to calculate the homology groups of ΩS^n.
In order to proceed smoothly, observe first that PS^n is contractible.

Cohomological spectral sequences: filtration

Everything is formally dual to its homological counterpart. We are given a complex $C^0 \xrightarrow{d} C^1 \xrightarrow{d} \cdots$ with a *descending filtration* $F_p C^* \supseteq F_{p+1} C^*$ compatible with differential, i.e. $d(F_p C^n) \subseteq F_p C^{n+1}$. Similarly we denote the graded term by $G_p C^n = F_p C^n / F_{p+1} C^n$. Now we again have an induced filtration on $H^*(C)$ by

$$F_p H^n = \{[x] \in H^n(C^*) \mid x \in F_p C^n, dx = 0\}. \tag{7}$$

Once again our target is to calculate the graded terms $G_p H^n = F_p H^n / F_{p+1} H^n$.

Cohomological spectral sequences: pages

We define (*omitting the intersection in the denominator*)

$$E_r^{p,q} = \frac{\{x \in F_p C^{p+q} \mid dx \in F_{p+r} C^{p+q+1}\}}{F_{p+1} C^{p+q} + dF_{p-r+1} C^{p+q-1}}, \tag{8}$$

and we obtain a differential $d^r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ induced by d, so that

$$E_{r+1}^{p,q} = \frac{\ker(E_r^{p,q} \xrightarrow{d^r} E_r^{p+r,q-r+1})}{\operatorname{im}(E_r^{p-r,q+r-1} \xrightarrow{d^r} E_r^{p,q})}$$

For the initial pages, $E_0^{p,q} = G_p C^{p+q}$, and $d^0 : E_0^{p,q} \to E_0^{p,q+1}$; $E_1^{p,q} = H^{p+q}(G_p C^*)$, and $d^1 : E_1^{p,q} \to E_1^{p+1,q}$.

Pages of cohomological spectral sequences

$$
\begin{tikzcd}
\vdots & \vdots & \\
E_0^{0,1} \arrow[u] & E_0^{1,1} \arrow[u] & \cdots \\
E_0^{0,0} \arrow[u, "d^0"] & E_0^{1,0} \arrow[u] & \cdots
\end{tikzcd}
\qquad
\begin{tikzcd}
\vdots & \vdots & \\
E_1^{0,1} \arrow[r] & E_1^{1,1} \arrow[r] & \cdots \\
E_1^{0,0} \arrow[r, "d^1"] & E_1^{1,0} \arrow[r] & \cdots
\end{tikzcd}
$$

$$
\begin{tikzcd}
\vdots \arrow[drr] & \vdots \arrow[drr] & \vdots & \\
E_2^{0,1} \arrow[drr, "d^2"] & E_2^{1,1} \arrow[drr] & E_2^{2,1} & \cdots \\
E_2^{0,0} & E_2^{1,0} & E_2^{2,0} & \cdots
\end{tikzcd}
$$

Convergence and degeneration

Suppose that the filtration is bounded, i.e. $\forall n$, $F_0 C^n = C^n$ and $F_p C^n = 0$ for $p \gg 0$; then the spectral sequence degenerates at some E_r, and converges to

$$E_r^{p,q} = E_\infty^{p,q} = G_p H^{p+q}(C^*). \tag{9}$$

In this case we write $E_r^{p,q} \Rightarrow H^n$. Again in the case of a double complex, the cohomology H^n is calculated by taking the sum over the diagonal $p + q = n$ in the limit page E^∞.
Exercise: Try to get a dual version of the UCT.

Multiplicative structure

Recall that the singular cohomology of spaces has the merit over the homology of possessing a multiplicative structure. This motivates us to suppose that there is a bilinear map

$$* : C^p \times C^q \to C^{p+q}$$

satisfying the Leibniz rule[4]

$$d(\alpha * \beta) = d\alpha * \beta + (-1)^p \alpha * d\beta,$$

and preserving the filtration

$$F_p C^* * F_q C^* \subseteq F_{p+q} C^*.$$

[4] Note that this implies that $* : H^p \times H^q \to H^{p+q}$.

Multiplicative structure: cont.

Now $*$ induces for every $r \in \mathbb{N}$ a bilinear map

$$*_r : E_r^{p,q} \times E_r^{p',q'} \to E_r^{p+p',q+q'}$$

such that

- d^r is a derivation:

$$d^r(\alpha *_r \beta) = d^r\alpha *_r \beta + (-1)^{p+q}\alpha *_r d^r\beta;$$

- $*_{r+1}$ is the induced product on cohomology by $*_r$;
- if the filtration is bounded, $*_r$ stabilises to the product

$$G_pH^{p+q} \times G_{p'}H^{p'+q'} \to G_{p+p'}H^{p+q+p'+q'}.$$

Cohomological Serre spectral sequence

Again we are given a fibration (or rather a *fibre bundle*[5] for simplicity) $p : X \to B$ where the base B is a path-connected CW-complex with base point b_0 and the fibre $F = p^{-1}(b_0)$ is also path-connected.
We filter the singular cohomology of X by

$$F_k C^n(X) = C^n(X, X^{k-1}) = \{\phi : C_n(X) \to \mathbb{Z};\ \phi\restriction_{C_n(X^{k-1})} = 0\},$$

where $X^k = p^{-1}(B^{k-1})$, so that $F_0 C^n(X) = C^n(X)$ and $F_k C^n(X) = 0$ for $k \gg 0$. Now it is easy to calculate $E_0^{p,q} = C^{p+q}(X^p, X^{p-1})$, $E_1^{p,q} = H^{p+q}(X^p, X^{p-1})$, and Serre has also proved that

$$E_2^{p,q} = H^p(B, H^q(F)). \tag{10}$$

We remark that $E_r^{p,q} \Rightarrow H^n(X)$, dual to the homological case.

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[5]i.e. every $b \in B$ has a neighbourhood U such that $p^{-1}(U) \overset{\phi}{\simeq} U \times F$, and $p = \mathrm{proj}_1 \circ \phi$.

Adding the multiplicative structure

We wish to empower the cohomological Serre spectral sequence with a multiplicative structure, and we have a natural choice: the cup product $\smile$: $C^k(X) \times C^l(X) \to C^{k+l}(X)$. Recall that $\phi \smile \psi(\sigma) = \phi(\sigma \restriction_{\Delta\{0,\cdots,k\}})\psi(\sigma \restriction_{\Delta\{k,\cdots,l\}})$ for any chain $\sigma : \Delta^{k+l} \to X$.

However, it does not necessarily preserve our filtration. This means that it does not induce a well-defined product on E_0-level. Instead we turn to E^1-level, i.e. the cohomology level. We consider the cup product as a composition

$$H^*(X) \times H^*(X) \xrightarrow{\times} H^*(X \times X) \xrightarrow{\Delta^*} H^*(X),$$
$$(\alpha, \beta) \mapsto \mathrm{proj}^*\alpha \smile \mathrm{proj}^*\beta \mapsto \alpha \smile \beta,$$

where $\Delta : X \to X \times X$ is the diagonal embedding.

Multiplicative structure on E_1

$X \times X$ is the union of its skeletons $(X \times X)^k = \bigcup_{i+j=k} X^i \times X^j$ which are the preimages of the k-skeletons of $B \times B$. By excision,

$$H^*((X \times X)^k, (X \times X)^{k-1}) = \bigoplus_{i+j=k} H^*(X^i \times X^j, X^{i-1} \times X^j \cup X^i \times X^{j-1})$$

which induces an inclusion of each factor in the direct sum into the term on the left. Now we have the following diagram which gives

$$*_1 : E_1^{p,q} \times E_1^{p',q'} \to E_1^{p+p',q+q'}.$$

And this product is shown to be compatible with d^1.

$*_1$ on E_1, cont.

$$\begin{tikzcd}
H^m(X^k, X^{k-1}) \times H^n(X^l, X^{l-1}) \arrow[r, dashed, "*_1"] \arrow[d, "\times"] & H^{m+n}(X^{k+l}, X^{k+l-1}) \\
H^{n+m}(X^k \times X^l, U) \arrow[r, hook] & H^{n+m}((X \times X)^{k+l}, (X \times X)^{k+l-1}) \arrow[u, "\Delta^*"']
\end{tikzcd}$$

Here $U = X^k \times X^{l-1} \ \cup \ X^{k-1} \times X^l$.

Multiplicative structure on E_2

Since $H^*(F)$ has a graded algebra structure, we have the following diagram

$$\begin{tikzcd}
H^p(B, H^q(F)) \times H^{p'}(B, H^{q'}(F)) \arrow[r, "\smile"] \arrow[dr, "*_2"] & H^{p+p'}(B, H^q(F) \otimes H^{q'}(F)) \arrow[d, "\smile"] \\
 & H^{p+p'}(B, H^{q+q'}(F))
\end{tikzcd}$$

Useful fact: The multiplicative structure on E_2

$$*_2 : E_2^{p,q} \times E_2^{p',q'} \to E_2^{p+p',q+q'}$$

induced by $*_1$ is $(-1)^{q'p}$ times the natural "double" cup product.

Remark: The compatibility is due to the fact that for a ring map $h : R \to R'$ and $\alpha_0, \alpha_1 \in H^\bullet(X; R)$,

$$h^*(\alpha_1 \smile \alpha_2) = h^*(\alpha_1) \smile h^*(\alpha_2).$$

Example: infinite complex projective space[6]

We calculate the cohomology of $\mathbb{C}P^\infty$ with the help of the fibre bundle $S^1 \hookrightarrow S^\infty \to \mathbb{C}P^\infty$ where the total space S^∞ is contractible. So $E_r^{p,q} \Rightarrow H^n(S^\infty) = 0$ $(n > 0)$, and the 2nd page gives

$$E_2^{p,q} = H^p(\mathbb{C}P^\infty, H^q(S^1)).$$

So the 2nd page is concentrated in 2 rows $q = 0, 1$. Let 1 be the generator of $H^0(S^1)$ and x be that of $H^1(S^1)$, we have the following diagram:

$$\begin{tikzcd}
H^0(\mathbb{C}P^\infty)x \arrow[drr, "d^2"] & H^1(\mathbb{C}P^\infty)x \arrow[drr] & H^2(\mathbb{C}P^\infty)x \arrow[drr] & H^3(\mathbb{C}P^\infty)x & \cdots \\
H^0(\mathbb{C}P^\infty) & H^1(\mathbb{C}P^\infty) & H^2(\mathbb{C}P^\infty) & H^3(\mathbb{C}P^\infty) & \cdots
\end{tikzcd}$$

Thus $d^3 = d^4 = \cdots = 0$, and $E_3 = E_\infty$. We deduce that every d^2 is isomorphic, and further $H^{2k}(\mathbb{C}P^\infty) = \mathbb{Z}$, $H^{2k+1}(\mathbb{C}P^\infty) = 0$.

[6]Using the Eilenberg-MacLane notion, $\mathbb{C}P^2 = K(\mathbb{Z}, 2)$.

The ring structure of $H^*(\mathbb{C}P^\infty)$

Consider the isomorphism $d^2 : H^0(\mathbb{C}P^\infty)x \to H^2(\mathbb{C}P^\infty)$. As an isomorphism between $\mathbb{Z}$, it maps the generator to generator, i.e. $\alpha = d^2(x) \in H^2(\mathbb{C}P^\infty)$ is the generator. Note that here we implicitly assume the generator of $H^0(\mathbb{C}P^\infty)$ to be 1. Similarly consider the isomorphism $d^2 : H^2(\mathbb{C}P^\infty)x \to H^4(\mathbb{C}P^\infty)$. This time $d^2(\alpha x)$ becomes the generator of H^4. But we have

$$d^2(\alpha x) = d^2(\alpha)x + (-1)^{2+0}\alpha d^2(x) = d^2(\alpha)x + \alpha^2,$$

and notice that $d^2(\alpha) \in E_2^{4,-1} = 0$. So $d^2(\alpha x) = \alpha^2$ is the generator of H^4. Similarly we can get $H^{2k} = \mathbb{Z}\alpha^k$.
In consequence, $H^*(\mathbb{C}P^\infty) = \mathbb{Z}[\alpha]$, where $\deg \alpha = 2$.

Exercise: Cohomology of higher EM-spaces

Now that we know the cohomology of $K(\mathbb{Z},1) \simeq S^1$ and $K(\mathbb{Z},2) \simeq \mathbb{C}P^2$, it is natural to consider that of $K(\mathbb{Z},3)$ and higher EM-spaces. A good starting point is the homotopy sequence

$$K(G,n) \to * \to K(G,n+1)$$

obtained by using the mapping cylinder. Try to see how many cohomology groups of $K(\mathbb{Z},3)$ with $\mathbb{Z}$ coefficients can be obtained by applying the Serre spectral sequence.
In fact, knowing $H^n(K(\mathbb{Z},3))$ for $n \leq 8$ would allow us to determine the cohomology ring of rational coefficients:

$$H^*(K(\mathbb{Z},3),\mathbb{Q}) = \Lambda_{\mathbb{Q}}[x], \quad \deg x = 3.$$

By induction it is possible to show that $H^*(K(\mathbb{Z},n),\mathbb{Q})$ is of similar form, while we need to distinguish between the case n is odd (exterior algebra) and n is even (polynomial ring).