

Introduction to Spectral Sequences

A concise approach

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Table of Contents

Homological spectral sequences

- Case of 1 sub-complex

- Case of a double complex

- Universal coefficient theorem

Homological Serre spectral sequence

- Example: complex projective plane

Cohomological spectral sequences

- Multiplicative structure

Cohomological Serre spectral sequence

- Multiplicative structure

- Example: infinite complex projective space

Motivation¹

We mainly work in the category of abelian groups **Ab**. We aim to generalise the following fact.

For a complex $C_* : C_0 \leftarrow C_1 \leftarrow C_2 \leftarrow \cdots$ with a sub-complex $A_* : A_0 \leftarrow A_1 \leftarrow A_2 \leftarrow \cdots$, we can deduce the homology of C_* from certain information of that of A_* and C_*/A_* .

In fact, recall that the long exact sequence of the pair (C_*, A_*) gives

$$H_{n+1}(C_*/A_*) \xrightarrow{\partial_{n+1}} H_n(A_*) \rightarrow H_n(C_*) \rightarrow H_n(C_*/A_*) \xrightarrow{\partial_n} H_{n-1}(A_*)$$

Cutting it short, we get

$$0 \rightarrow \operatorname{coker}(\partial_{n+1}) \rightarrow H_n(C_*) \rightarrow \ker(\partial_n) \rightarrow 0$$

This helps to determine $H_n(C_*)$. We want to have similar results in a more general context: filtered complexes.

¹The concept is introduced in: J. Leray. *L'anneau d'homologie d'une représentation*. C. R. Acad. Sci. 222 (1946)

Filtered complexes

For the complex (C_*, ∂) equipped with an *ascending filtration* of each of its terms:

$$F_p C_i \subseteq F_{p+1} C_i \subseteq \cdots \subseteq C_i,$$

compatible with the differential in the sense that

$$\partial(F_p C_i) \subseteq F_p C_{i-1}.$$

We denote also the graded term by $G_p C_i = F_p C_i / F_{p-1} C_i$.

Observe that the filtration also induces a filtration on the homology complex $H_q(C_*)$. In fact,

$$F_p H_q(C_*) = \{[x] \in H_q(C_*) \mid x \in F_p C_q, \partial x = 0\}, \quad (1)$$

and it is clear that $F_p H_q \subseteq F_{p+1} H_q$.

The target of the spectral sequences is to calculate

$G_p H_q = F_p H_q / F_{p-1} H_q$ from the groups $G_p C_q$ and certain connecting morphisms.

Homological spectral sequences: 0th page

We denote by $E_{p,q}^0 = G_p C_{p+q} = F_p C_{p+q} / F_{p-1} C_{p+q}$. By definition, the differential induces a morphism $\partial_0 : E_{p,q}^0 \rightarrow E_{p,q-1}^0$ by $x \mapsto [\partial x]$.

Recall that $\partial(F_p C_{p+q}) \subseteq F_p C_{p+q-1}$, $\partial(F_{p-1} C_{p+q}) \subseteq F_{p-1} C_{p+q-1}$. Clearly $\partial_0^2 = 0$. Thus we can consider the homology of the complex $(E_{p,*}^0, \partial_0)$, denoted by $E_{p,*}^1$, i.e.

$$\begin{aligned} E_{p,q}^1 &= \frac{\ker(E_{p,q}^0 \xrightarrow{\partial_0} E_{p,q-1}^0)}{\operatorname{im}(E_{p,q+1}^0 \xrightarrow{\partial_0} E_{p,q}^0)} \\ &= \frac{\ker(G_p C_{p+q} \rightarrow G_p C_{p+q-1})}{\operatorname{im}(G_p C_{p+q+1} \rightarrow G_p C_{p+q})} = H_{p+q}(G_p C_*). \end{aligned}$$

The 0th page E^0

$$\begin{array}{ccc} \vdots & \vdots & \\ \downarrow & \downarrow & \\ E_{02}^0 & E_{12}^0 & \dots \\ \downarrow \partial_0 & \downarrow \partial_0 & \\ E_{01}^0 & E_{11}^0 & \dots \\ \downarrow \partial_0 & \downarrow \partial_0 & \\ E_{00}^0 & E_{10}^0 & \dots \end{array}$$

We define a morphism $\partial_1 : E_{p,q}^1 \rightarrow E_{p-1,q}^1$ as follows.

For $\alpha = [x] \in E_{p,q}^1$, where $x \in E_{p,q}^0$, we know

$0 = \partial_0 x = [\partial x] \in E_{p,q-1}^0$. Thus $\partial x \in F_{p-1} C_{p+q-1}$. Thus we let

$\partial_1(\alpha) = [\partial x] \in E_{p-1,q}^1$.

It is easy to verify that ∂_1 is a well-defined morphism and $\partial_1^2 = 0$.

This allows to consider the homology of the complex $(E_{*,q}^1, \partial_1)$, which is denoted by $E_{*,q}^2$.

1st page: another perspective

Alternatively, consider the short exact sequence of the triple $(F_p C_*, F_{p-1} C_*, F_{p-2} C_*)$:

$$0 \rightarrow G_{p-1} C_* = \frac{F_{p-1} C_*}{F_{p-2} C_*} \rightarrow F_p C_* / F_{p-2} C_* \rightarrow G_p C_* = \frac{F_p C_*}{F_{p-1} C_*} \rightarrow 0$$

It induces a long exact sequence of homology:

$$\cdots \rightarrow H_{p+q}(G_p C_*) \xrightarrow{\partial_1} H_{p+q-1}(G_{p-1} C_*) \rightarrow \cdots$$

where the connecting morphism is exactly ∂_1 we have defined.

The 1st page E^1

$$\vdots$$
$$\vdots$$
$$\vdots$$

$$E_{02}^1 \xleftarrow{\partial_1} E_{12}^1 \xleftarrow{\partial_1} E_{22}^1 \xleftarrow{\quad} \cdots$$

$$E_{01}^1 \xleftarrow{\partial_1} E_{11}^1 \xleftarrow{\partial_1} E_{21}^1 \xleftarrow{\quad} \cdots$$

$$E_{00}^1 \xleftarrow{\partial_1} E_{10}^1 \xleftarrow{\partial_1} E_{20}^1 \xleftarrow{\quad} \cdots$$

Second pages

Rather than continuing by hand, we define directly for $r \in \mathbb{N}$,

$$E_{p,q}^r = \frac{\{x \in F_p C_{p+q} \mid \partial x \in F_{p-r} C_{p+q-1}\}}{F_{p-1} C_{p+q} + \partial(F_{p+r-1} C_{p+q+1})} \quad (2)$$

Here we omit the intersection in the denominator:

$$A/B := A/(A \cap B).$$

Similarly one can show that the differential induces a morphism

$$\partial_r : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r, \text{ satisfying } \partial_r^2 = 0.$$

For easy memory, E^{r+1} is just the homology of the complex (E^r, ∂_r) , i.e.

$$E_{p,q}^{r+1} = \frac{\ker(E_{p,q}^r \xrightarrow{\partial_r} E_{p-r,q+r-1}^r)}{\operatorname{im}(E_{p+r,q-r+1}^r \xrightarrow{\partial_r} E_{p,q}^r)}.$$

The 2nd page E^2

$$\begin{array}{ccccccc}
 \vdots & & \vdots & & \vdots & & \vdots \\
 E_{02}^2 & \leftarrow & E_{12}^2 & \xleftarrow{\partial_2} & E_{22}^2 & \xleftarrow{\partial_2} & E_{32}^2 & \cdots \\
 E_{01}^2 & \leftarrow & E_{11}^2 & \xleftarrow{\partial_2} & E_{21}^2 & \xleftarrow{\partial_2} & E_{31}^2 & \cdots \\
 E_{00}^2 & & E_{10}^2 & & E_{20}^2 & & E_{30}^2 & \cdots
 \end{array}$$

Suppose that for every i , the filtration F_*C_i is bounded, i.e. $F_0C_i = 0$ and $F_pC_i = C_i$ for $p \gg 0$; then for every p, q , there exists r_0 such that when $r \geq r_0$, we have

$$E_{p,q}^r = G_p H_{p+q}(C_*). \quad (3)$$

Proof.

When $r \gg 0$, $F_{p-r}C_{p+q-1} = 0$, and $F_{p+r-1}C_{p+q+1} = C_{p+q+1}$.
Now use (1) and (2). □

In general, we say that the spectral sequence *degenerates* at E^r if $\partial_k = 0$ for all $k \geq r$. In this case, $E^r = E^{r+k} = E^\infty$ for all $k \geq 0$. Then (3) applies.

Case of 1 sub-complex

Let $F_0 C_* = A_*$ and $F_1 C_* = C_*$. This is exactly the case in the motivation. We denote by $B_* = C_*/A_*$.

Notice that $p = 0, 1$ are the only possible values.

Then $E_{0,q}^0 = G_0 C_q = A_q$, and

$$E_{1,q}^0 = G_1 C_{q+1} = C_{q+1}/A_{q+1} = B_{q+1}.$$

Further, $E_{0,q}^1 = H_q(A_*)$ and $E_{1,q}^1 = H_{q+1}(B_*)$.

Notice that the morphism ∂_1 is nothing but the connecting morphism of the long exact sequence associated with the short exact sequence $0 \rightarrow A_* \rightarrow C_* \rightarrow B_* \rightarrow 0$.

E^0 and E^1 for 1 sub-complex

$$\begin{array}{ccc}
 \vdots & \vdots & \vdots \\
 A_2 & B_3 & H_2(A) \longleftarrow H_3(B) \\
 \downarrow & \downarrow & \\
 A_1 & B_2 & H_1(A) \longleftarrow H_2(B) \\
 \downarrow \partial_0 & \downarrow & \\
 \textcolor{orange}{A}_0 & B_1 & \textcolor{orange}{H}_0(A) \xleftarrow{\partial_1} H_1(B) \\
 & \downarrow & \\
 E^0 : & B_0 & E^1 : \quad 0 \longleftarrow H_0(B)
 \end{array}$$

Remark: The orange colour is temporarily used to determine the position $(0,0)$.

Case of 1 sub-complex: cont.

Recall that E^2 is defined as the homology of (E^1, ∂_1) , so $E_{0,q}^2 = \text{coker}(\partial_1^{1,q})$ and $E_{1,q}^2 = \ker(\partial_1^{1,q})$.

By the shape of the 2nd page E^2 , $\partial_2 = 0$, so $E^2 = E^3$. Similarly $\partial_3 = 0$ implies $E^3 = E^4$, and so on. So $E^2 = E^\infty$, and hence the sequence degenerates at E^2 .

As a result, $E_{p,q}^2 = G_p H_{p+q}$. Recall that $p = 0, 1$, and thus we have a short exact sequence

$$0 \rightarrow G_0 H_{p+q} \rightarrow H_{p+q} \rightarrow G_1 H_{p+q} \rightarrow 0,$$

which now reads

$$0 \rightarrow \text{coker}(\partial_1^{0,p+q}) \rightarrow H_{p+q} \rightarrow \ker(\partial_1^{0,p+q-1}) \rightarrow 0.$$

We have thus regained the information carried by the long exact sequence.

E^2 for 1 sub-complex

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & & & & & \\
 0 & \xleftarrow{\partial_2} & \text{coker}(\partial_1^{1,2}) & \xleftarrow{\partial_2} & \text{ker}(\partial_1^{1,2}) & \xleftarrow{\partial_2} & 0 \\
 & & & & & & \\
 0 & \xleftarrow{\partial_2} & \text{coker}(\partial_1^{1,1}) & \xleftarrow{\partial_2} & \text{ker}(\partial_1^{1,1}) & \xleftarrow{\partial_2} & 0 \\
 & & & & & & \\
 0 & \xleftarrow{\partial_2} & \text{coker}(\partial_1^{1,0}) & \xleftarrow{\partial_2} & \text{ker}(\partial_1^{1,0}) & \xleftarrow{\partial_2} & 0
 \end{array}$$

$E^2 :$

Case of a double complex

Now we are given a sequence of abelian groups $C_{p,q}$ with $p, q \geq 0$, and 2 differentials $\partial' : C_{p,q} \rightarrow C_{p-1,q}$ and $\partial'' : C_{p,q} \rightarrow C_{p,q-1}$ satisfying $\partial'^2 = \partial''^2 = \partial'\partial'' + \partial''\partial' = 0$.

We define its *total complex* (C_*, ∂) by $C_n = \bigoplus_{p+q=n} C_{p,q}$ and $\partial = \partial' + \partial''$. Remember that our target is always to calculate the homology of C_* , which we filter in 2 ways:

$${}'F_p C_n = \bigoplus_{k+l=n, k \leq p} C_{k,l}, \quad {}''F_p C_n = \bigoplus_{k+l=n, l \leq p} C_{k,l}.$$

Now by definition ${}'E_{p,q}^0 = C_{p,q}$, ${}'E_{p,q}^1 = H_q(C_{p,*}, \partial'')$, and also ${}''E_{p,q}^0 = C_{q,p}$, ${}''E_{p,q}^1 = H_p(C_{*,q}, \partial')$. Notice that ${}'\partial_0$ is induced by ∂'' , and ${}''\partial_0$ is induced by ∂' .

Case of a double complex: cont.

The 2nd page is more interesting. Notice that

$$' \partial_1 : 'E_{p,q}^1 = H_q(C_{p,*}, \partial'') \rightarrow 'E_{p-1,q}^1 = H_q(C_{p-1,*}, \partial'')$$

is induced by $\partial' : C_{p,*} \rightarrow C_{p-1,*}$. So it is easy to see

$$'E_{p,q}^2 = H_p(H_q(C_{p,*}, \partial''), \partial').$$

Dually,

$$''E_{p,q}^2 = H_q(H_p(C_{*,q}, \partial'), \partial'').$$

Useful fact: if the 2 spectral sequences converge, they must converge to the same group $H_n(C_*)$, though for 2 different filtrations $'G_p$ and $''G_p$. This is seen by summing over the *diagonal* $p + q = n$.

Diagonal sum

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \ddots & & \ddots & & \ddots \\ E_{0,2}^{\infty} & & E_{1,2}^{\infty} & & E_{2,2}^{\infty} & & \cdots \\ & \ddots & & \ddots & & \ddots & \\ E_{0,1}^{\infty} & & E_{1,1}^{\infty} & & E_{2,1}^{\infty} & & \cdots \\ & \ddots & & \ddots & & \ddots & \\ E_{0,0}^{\infty} & & E_{1,0}^{\infty} & & E_{2,0}^{\infty} & & \cdots \end{array}$$

We have

$$H_n(C_*) = \bigoplus_{p+q=n} E_{p,q}^{\infty}.$$

Application: universal coefficient theorem

As an example, let M be an abelian group, and P_1, P_0 be free abelian groups such that there is a short exact sequence

$$0 \rightarrow P_1 \xrightarrow{\phi} P_0 \rightarrow M \rightarrow 0.$$

We write $P_i = 0$ if $i \notin \{0, 1\}$.

If X is a topological space, we can form the double complex $C_{p,q} = C_p(X) \otimes P_q$ with natural differentials: $\partial' = d \otimes 1$ and $\partial'' = 1 \otimes \phi$.

Since $C_p(X)$ is free, $'E_{p,0}^1 = C_p(X) \otimes M$ and $'E_{p,q}^1 = 0$ if $q \neq 0$. Thus $'E^2$ is concentrated on the row $q = 0$ as is $'E^1$, and $'E_{p,0}^2 = H_p(X, M)$.

By the shape of $'E^2$, $'\partial_2 = '\partial_3 = \cdots = 0$, so $'E^\infty = 'E^2$.

The 1st spectral sequence $'E$

$$\begin{array}{ccccc}
 'E^0 : & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
 & C_0(X) \otimes P_1 & & C_1(X) \otimes P_1 & \dots \\
 & \downarrow \partial_0 & & \downarrow & \\
 & C_0(X) \otimes P_0 & & C_1(X) \otimes P_0 & \dots
 \end{array}$$

$$\begin{array}{ccccccc}
 'E^1 : & & 0 & \longleftarrow & 0 & \longleftarrow & 0 \\
 & & & & & & \\
 & C_0(X) \otimes M & \longleftarrow & C_1(X) \otimes M & \xleftarrow{\partial_1} & C_2(X) \otimes M & \longleftarrow \dots
 \end{array}$$

$$\begin{array}{ccccccc}
 'E^2 : & & 0 & & \partial_2^0 & & 0 \\
 & & \swarrow & & \swarrow & & \swarrow \\
 & H_0(X, M) & & H_1(X, M) & & H_2(X, M) & \dots
 \end{array}$$

Universal coefficient theorem: cont.

On the other hand, ${}''E_{p,q}^0 = C_{q,p} = C_q(X) \otimes P_p$, so ${}''E_{p,q}^1 = H_q(X) \otimes P_p$. This spectral sequence is concentrated on 2 columns $p = 0, 1$.

It is not hard to continue the calculation: ${}''E_{0,q}^2 = H_q(X) \otimes M$ and ${}''E_{1,q}^2 = \text{Tor}(H_q(X), M)$: use the long exact sequence

$$\begin{aligned} \cdots \rightarrow \text{Tor}(H_q(X), P_0) = 0 \rightarrow \text{Tor}(H_q(X), M) \rightarrow \\ \rightarrow H_q(X) \otimes P_1 \rightarrow H_q(X) \otimes P_0 \rightarrow H_q(X) \otimes M \rightarrow 0. \end{aligned}$$

By the shape of ${}''E^2$, ${}''\partial_2 = {}''\partial_3 = \cdots = 0$ and ${}''E^\infty = {}''E^2$. Recall that the 2 spectral sequences converge to the same homology $H_n(X, M)$ by summing over the diagonal $p + q = n$:

$$H_n(X, M) \cong H_n(X) \otimes M \oplus \text{Tor}(H_{n-1}(X), M). \quad (4)$$

The 2nd spectral sequence "E

$$\begin{array}{ccc}
 {}''E^0 : & \begin{array}{c} \vdots \\ \downarrow \\ C_1(X) \otimes P_0 \\ \downarrow \\ C_0(X) \otimes P_0 \end{array} & \begin{array}{c} \vdots \\ \downarrow \\ C_1(X) \otimes P_1 \\ \downarrow \\ C_0(X) \otimes P_1 \end{array}
 \end{array}$$

$$\begin{array}{ccc}
 {}''E^1 : & \begin{array}{c} \vdots \\ H_1(X) \otimes P_0 \longleftarrow H_1(X) \otimes P_1 \\ H_0(X) \otimes P_0 \longleftarrow H_0(X) \otimes P_1 \end{array} &
 \end{array}$$

$$\begin{array}{ccc}
 {}''E^2 : & \begin{array}{c} \vdots \\ H_1(X) \otimes M \\ H_0(X) \otimes M \end{array} & \begin{array}{c} \vdots \\ \text{Tor}(H_1(X), M) \longrightarrow 0 \\ \text{Tor}(H_0(X), M) \longrightarrow 0 \end{array}
 \end{array}$$

Exercise: Künneth formula

Let A_* and B_* be 2 complexes of k -vector spaces (of positive degree). There is an isomorphism

$$H_n(A_* \otimes B_*) = \bigoplus_{p+q=n} H_p(A_*) \otimes H_q(B_*). \quad (5)$$

Remark: the isomorphism is not canonical.

Proof.

To begin with, consider $C_n = \bigoplus_{p+q=n} A_p \otimes B_q$ with differential

$$\partial(x \otimes y) = \partial x \otimes y + (-1)^{\deg x} x \otimes \partial y.$$



Leray-Serre spectral sequence

We are given a (*Serre*) *fibration*² $p : X \rightarrow B$ with pointed base (B, b_0) . Assume that its fibre $F = p^{-1}(b_0)$ has a base point x_0 , and F, X, B are all path-connected. Recall that this means that $\pi_1(X, x_0)$ acts on $\pi_k(F)$.

For simplicity, we suppose in addition that B is a CW-complex, with skeletons

$$B^0 \subseteq B^1 \subseteq \cdots \subseteq B,$$

and filter $C_*(X)$ by the preimages $X^k = p^{-1}(B^k)$. In other words, $F_p C_*(X) = C_*(X^p)$.

²i.e. having RLP w.r.t. the inclusions $D^n \hookrightarrow D^n \times [0, 1]$, $n \in \mathbb{N}$.

Serre spectral sequence: cont.

Then we can calculate $E_{p,q}^0 = C_{p+q}(X^p, X^{p-1})$ and $E_{p,q}^1 = H_{p+q}(X^p, X^{p-1})$. And what makes the sequence interesting is that, we have a natural isomorphism³

$$E_{p,q}^2 \xrightarrow{\Phi} H_p(B, H_q(F)), \quad (6)$$

where $H_q(F)$ denotes the homology of the fibre at b_0 , viewed as a $\mathbb{Z}[\pi_1(B)]$ -module. Since (B, B^k) is k -connected, by HLP so is (X, X^k) , and thus $X^k \hookrightarrow X$ induces an isomorphism on $H_n(-, M)$ if $n < k$. On the other hand $X^k = \emptyset$ for $k < 0$, and we can deduce that the spectral sequence for homology with coefficients converges to $H_*(X, M)$.

³The result with proof is found in: J.-P. Serre. *Homologie singulière des espaces fibrés. Applications*. Ann. of Math. 54 (1951)

Example: complex projective plane $\mathbb{C}P^2$

$S^1 \subseteq \mathbb{C}$ acts on $S^5 \subseteq \mathbb{C}^3$, with quotient space $\mathbb{C}P^2$. We use the fibration $S^1 \rightarrow S^5 \rightarrow \mathbb{C}P^2$ to compute $H_*(\mathbb{C}P^2)$.

Since $\pi_1(\mathbb{C}P^2) = 0$ and $H_*(S^1)$ are free, the Serre spectral sequence gives

$$E_{p,q}^2 = H_p(\mathbb{C}P^2, H_q(S^1)) = H_p(\mathbb{C}P^2) \otimes H_q(S^1) \Rightarrow H_{p+q}(S^5).$$

Using the fact that $\mathbb{C}P^2$ is a 4-dimensional real manifold, we observe that E^2 is concentrated in $0 \leq p \leq 4$ and $q = 0, 1$. This shape implies that $\partial_3 = \partial_4 = \cdots = 0$, hence the sequence degenerates at E^3 .

Homological Serre spectral sequence for $\mathbb{C}P^2$, 2nd page

$$\begin{array}{ccccccccc}
 E^2 : & 0 & \leftarrow & 0 & & 0 & & 0 & & 0 \\
 & & & \swarrow \partial_3 & & & & & & \\
 & \mathbb{Z} & \leftarrow & H_1(\mathbb{C}P^2) & \leftarrow & H_2(\mathbb{C}P^2) & \leftarrow & H_3(\mathbb{C}P^2) & \leftarrow & H_4(\mathbb{C}P^2) & \leftarrow & 0 \\
 & & \swarrow \partial_2 & & \swarrow & & \swarrow & & \swarrow & & \\
 & \mathbb{Z} & & H_1(\mathbb{C}P^2) & & H_2(\mathbb{C}P^2) & & H_3(\mathbb{C}P^2) & & H_4(\mathbb{C}P^2) & & 0
 \end{array}$$

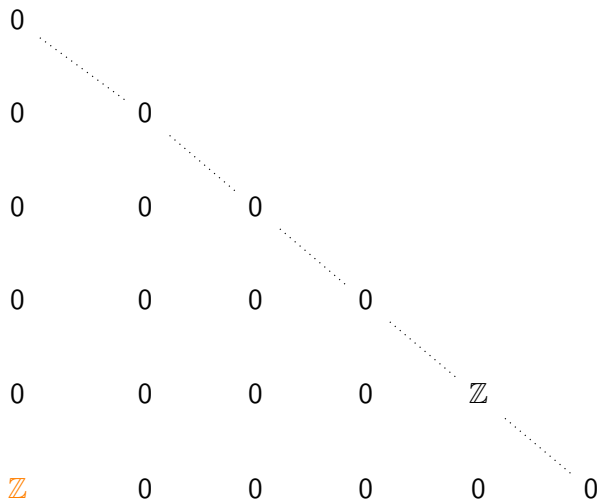
$\mathbb{C}P^2$, cont.

We then use the homology of S^5 to determine $E^3 = E^\infty$.

Since $H_0(S^5) = H_5(S^5) = \mathbb{Z}$ and other homology groups vanish, $E_{0,0}^3 = \mathbb{Z}$, and $E_{4,1}^3 = \mathbb{Z}$. This is seen by summing over the diagonals and by taking homology of E^2 . These 2 positions are the only non-vanishing ones in E^3 . For degree reasons this implies that in E^2 we must have $H_3(\mathbb{C}P^2) \simeq H_1(\mathbb{C}P^2) = 0$, and $H_4(\mathbb{C}P^2) \simeq H_2(\mathbb{C}P^2) \simeq \mathbb{Z}$.

As a result, $H_k(\mathbb{C}P^2) = \mathbb{Z}$ iff $k = 0, 2, 4$ and other homology groups vanish.

Homological Serre spectral sequence for $\mathbb{C}P^2$, 3rd page



$E^3 :$

Exercise: loop space of spheres

Let (X, x_0) be a pointed topological space. Its *path space* is

$$PX = \{\gamma : [0, 1] \rightarrow X \mid \gamma(0) = x_0\}$$

with compact-open topology, and let $p(\gamma) = \gamma(1)$. We denote the *loop space* of X at x_0 by $\Omega X = p^{-1}(x_0)$. Then it is not hard to see that we have a fibration sequence

$$\Omega X \hookrightarrow PX \xrightarrow{p} X.$$

Try to use it to calculate the homology groups of ΩS^n .

In order to proceed smoothly, observe first that PS^n is contractible.

Cohomological spectral sequences: filtration

Everything is formally dual to its homological counterpart. We are given a complex $C^0 \xrightarrow{d} C^1 \xrightarrow{d} \dots$ with a *descending filtration* $F_p C^* \supseteq F_{p+1} C^*$ compatible with differential, i.e. $d(F_p C^n) \subseteq F_p C^{n+1}$. Similarly we denote the graded term by $G_p C^n = F_p C^n / F_{p+1} C^n$. Now we again have an induced filtration on $H^*(C)$ by

$$F_p H^n = \{[x] \in H^n(C^*) \mid x \in F_p C^n, dx = 0\}. \quad (7)$$

Once again our target is to calculate the graded terms $G_p H^n = F_p H^n / F_{p+1} H^n$.

Cohomological spectral sequences: pages

We define (*omitting the intersection in the denominator*)

$$E_r^{p,q} = \frac{\{x \in F_p C^{p+q} \mid dx \in F_{p+r} C^{p+q+1}\}}{F_{p+1} C^{p+q} + dF_{p-r+1} C^{p+q-1}}, \quad (8)$$

and we obtain a differential $d^r : E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$ induced by d , so that

$$E_{r+1}^{p,q} = \frac{\ker(E_r^{p,q} \xrightarrow{d^r} E_r^{p+r, q-r+1})}{\operatorname{im}(E_r^{p-r, q+r-1} \xrightarrow{d^r} E_r^{p,q})}$$

For the initial pages, $E_0^{p,q} = G_p C^{p+q}$, and $d^0 : E_0^{p,q} \rightarrow E_0^{p, q+1}$; $E_1^{p,q} = H^{p+q}(G_p C^*)$, and $d^1 : E_1^{p,q} \rightarrow E_1^{p+1, q}$.

Pages of cohomological spectral sequences

$$\begin{array}{ccccccc}
 \vdots & & \vdots & & \vdots & & \vdots \\
 \uparrow & & \uparrow & & & & \\
 E_0^{0,1} & & E_0^{1,1} & \cdots & E_1^{0,1} & \longrightarrow & E_1^{1,1} \longrightarrow \cdots \\
 \uparrow d^0 & & \uparrow & & & & \\
 E_0^{0,0} & & E_0^{1,0} & \cdots & E_1^{0,0} & \xrightarrow{d^1} & E_1^{1,0} \longrightarrow \cdots
 \end{array}$$

$$\begin{array}{ccccccc}
 \vdots & & \vdots & & \vdots & & \\
 \searrow & & \searrow & & \searrow & & \\
 E_2^{0,1} & & E_2^{1,1} & \longrightarrow & E_2^{2,1} & \longrightarrow & \cdots \\
 \searrow d^2 & & \searrow & & \searrow & & \\
 E_2^{0,0} & & E_2^{1,0} & \longrightarrow & E_2^{2,0} & \longrightarrow & \cdots
 \end{array}$$

Convergence and degeneration

Suppose that the filtration is bounded, i.e. $\forall n, F_0 C^n = C^n$ and $F_p C^n = 0$ for $p \gg 0$; then the spectral sequence degenerates at some E_r , and converges to

$$E_r^{p,q} = E_\infty^{p,q} = G_p H^{p+q}(C^*). \quad (9)$$

In this case we write $E_r^{p,q} \Rightarrow H^n$. Again in the case of a double complex, the cohomology H^n is calculated by taking the sum over the diagonal $p + q = n$ in the limit page E^∞ .

Exercise: Try to get a dual version of the UCT.

Multiplicative structure

Recall that the singular cohomology of spaces has the merit over the homology of possessing a multiplicative structure. This motivates us to **suppose** that there is a bilinear map

$$* : C^p \times C^q \rightarrow C^{p+q}$$

satisfying the Leibniz rule⁴

$$d(\alpha * \beta) = d\alpha * \beta + (-1)^p \alpha * d\beta,$$

and preserving the filtration

$$F_p C^* * F_q C^* \subseteq F_{p+q} C^*.$$

⁴Note that this implies that $* : H^p \times H^q \rightarrow H^{p+q}$.

Multiplicative structure: cont.

Now $*$ induces for every $r \in \mathbb{N}$ a bilinear map

$$*_r : E_r^{p,q} \times E_r^{p',q'} \rightarrow E_r^{p+p',q+q'}$$

such that

- ▶ d^r is a derivation:

$$d^r(\alpha *_r \beta) = d^r \alpha *_r \beta + (-1)^{p+q} \alpha *_r d^r \beta;$$

- ▶ $*_{r+1}$ is the induced product on cohomology by $*_r$;
- ▶ if the filtration is bounded, $*_r$ stabilises to the product

$$G_p H^{p+q} \times G_{p'} H^{p'+q'} \rightarrow G_{p+p'} H^{p+q+p'+q'}.$$

Cohomological Serre spectral sequence

Again we are given a fibration (or rather a *fibre bundle*⁵ for simplicity) $p : X \rightarrow B$ where the base B is a path-connected CW-complex with base point b_0 and the fibre $F = p^{-1}(b_0)$ is also path-connected.

We filter the singular cohomology of X by

$$F_k C^n(X) = C^n(X, X^{k-1}) = \{\phi : C_n(X) \rightarrow \mathbb{Z}; \phi \upharpoonright_{C_n(X^{k-1})} = 0\},$$

where $X^k = p^{-1}(B^{k-1})$, so that $F_0 C^n(X) = C^n(X)$ and $F_k C^n(X) = 0$ for $k \gg 0$. Now it is easy to calculate $E_0^{p,q} = C^{p+q}(X^p, X^{p-1})$, $E_1^{p,q} = H^{p+q}(X^p, X^{p-1})$, and Serre has also proved that

$$E_2^{p,q} = H^p(B, H^q(F)). \quad (10)$$

We remark that $E_r^{p,q} \Rightarrow H^n(X)$, dual to the homological case.

⁵i.e. every $b \in B$ has a neighbourhood U such that $p^{-1}(U) \xrightarrow{\phi} U \times F$, and $p = \text{proj}_1 \circ \phi$.

Adding the multiplicative structure

We wish to empower the cohomological Serre spectral sequence with a multiplicative structure, and we have a natural choice: the cup product $\smile: C^k(X) \times C^l(X) \rightarrow C^{k+l}(X)$. Recall that $\phi \smile \psi(\sigma) = \phi(\sigma \upharpoonright_{\Delta\{0, \dots, k\}}) \psi(\sigma \upharpoonright_{\Delta\{k, \dots, k+l\}})$ for any chain $\sigma: \Delta^{k+l} \rightarrow X$.

However, it does not necessarily preserve our filtration. This means that it does not induce a well-defined product on E_0 -level. Instead we turn to E^1 -level, i.e. the cohomology level. We consider the cup product as a composition

$$\begin{aligned} H^*(X) \times H^*(X) &\xrightarrow{\times} H^*(X \times X) \xrightarrow{\Delta^*} H^*(X), \\ (\alpha, \beta) &\mapsto \text{proj}^* \alpha \smile \text{proj}^* \beta \mapsto \alpha \smile \beta, \end{aligned}$$

where $\Delta: X \rightarrow X \times X$ is the diagonal embedding.

Multiplicative structure on E_1

$X \times X$ is the union of its skeletons $(X \times X)^k = \bigcup_{i+j=k} X^i \times X^j$ which are the preimages of the k -skeleta of $B \times B$. By excision,

$$H^*((X \times X)^k, (X \times X)^{k-1}) = \bigoplus_{i+j=k} H^*(X^i \times X^j, X^{i-1} \times X^j \cup X^i \times X^{j-1})$$

which induces an inclusion of each factor in the direct sum into the term on the left. Now we have the following diagram which gives

$$*_1 : E_1^{p,q} \times E_1^{p',q'} \rightarrow E_1^{p+p',q+q'}.$$

And this product is shown to be compatible with d^1 .

$*_1$ on E_1 , cont.

$$\begin{array}{ccc}
 H^m(X^k, X^{k-1}) \times H^n(X^l, X^{l-1}) & \xrightarrow{\quad *_1 \quad} & H^{m+n}(X^{k+l}, X^{k+l-1}) \\
 \downarrow \times & & \uparrow \Delta^* \\
 H^{n+m}(X^k \times X^l, U) & \hookrightarrow & H^{n+m}((X \times X)^{k+l}, (X \times X)^{k+l-1})
 \end{array}$$

Here $U = X^k \times X^{l-1} \cup X^{k-1} \times X^l$.

Multiplicative structure on E_2

Since $H^*(F)$ has a graded algebra structure, we have the following diagram

$$\begin{array}{ccc} H^p(B, H^q(F)) \times H^{p'}(B, H^{q'}(F)) & \xrightarrow{\smile} & H^{p+p'}(B, H^q(F) \otimes H^{q'}(F)) \\ & \searrow *_2 & \downarrow \smile \\ & & H^{p+p'}(B, H^{q+q'}(F)) \end{array}$$

Useful fact: The multiplicative structure on E_2

$$*_2 : E_2^{p,q} \times E_2^{p',q'} \rightarrow E_2^{p+p',q+q'}$$

induced by $*_1$ is $(-1)^{q'p}$ times the natural "double" cup product.

Remark: The compatibility is due to the fact that for a ring map $h : R \rightarrow R'$ and $\alpha_0, \alpha_1 \in H^\bullet(X; R)$,

$$h^*(\alpha_1 \smile \alpha_2) = h^*(\alpha_1) \smile h^*(\alpha_2).$$

Example: infinite complex projective space⁶

We calculate the cohomology of $\mathbb{C}P^\infty$ with the help of the fibre bundle $S^1 \hookrightarrow S^\infty \rightarrow \mathbb{C}P^\infty$ where the total space S^∞ is contractible. So $E_r^{p,q} \Rightarrow H^n(S^\infty) = 0$ ($n > 0$), and the 2nd page gives

$$E_2^{p,q} = H^p(\mathbb{C}P^\infty, H^q(S^1)).$$

So the 2nd page is concentrated in 2 rows $q = 0, 1$. Let 1 be the generator of $H^0(S^1)$ and x be that of $H^1(S^1)$, we have the following diagram:

$$\begin{array}{ccccccc} H^0(\mathbb{C}P^\infty)_x & H^1(\mathbb{C}P^\infty)_x & H^2(\mathbb{C}P^\infty)_x & H^3(\mathbb{C}P^\infty)_x & \dots \\ & \searrow d^2 & \searrow & \searrow & \\ \textcolor{brown}{H^0(\mathbb{C}P^\infty)} & H^1(\mathbb{C}P^\infty) & H^2(\mathbb{C}P^\infty) & H^3(\mathbb{C}P^\infty) & \dots \end{array}$$

Thus $d^3 = d^4 = \dots = 0$, and $E_3 = E_\infty$. We deduce that every d^2 is isomorphic, and further $H^{2k}(\mathbb{C}P^\infty) = \mathbb{Z}$, $H^{2k+1}(\mathbb{C}P^\infty) = 0$.

⁶Using the Eilenberg-MacLane notion, $\mathbb{C}P^2 = K(\mathbb{Z}, 2)$.

The ring structure of $H^*(\mathbb{C}P^\infty)$

Consider the isomorphism $d^2 : H^0(\mathbb{C}P^\infty)_X \rightarrow H^2(\mathbb{C}P^\infty)$. As an isomorphism between \mathbb{Z} , it maps the generator to generator, i.e. $\alpha = d^2(x) \in H^2(\mathbb{C}P^\infty)$ is the generator. Note that here we implicitly assume the generator of $H^0(\mathbb{C}P^\infty)$ to be 1. Similarly consider the isomorphism $d^2 : H^2(\mathbb{C}P^\infty)_X \rightarrow H^4(\mathbb{C}P^\infty)$. This time $d^2(\alpha x)$ becomes the generator of H^4 . But we have

$$d^2(\alpha x) = d^2(\alpha)x + (-1)^{2+0}\alpha d^2(x) = d^2(\alpha)x + \alpha^2,$$

and notice that $d^2(\alpha) \in E_2^{4,-1} = 0$. So $d^2(\alpha x) = \alpha^2$ is the generator of H^4 . Similarly we can get $H^{2k} = \mathbb{Z}\alpha^k$.

In consequence, $H^*(\mathbb{C}P^\infty) = \mathbb{Z}[\alpha]$, where $\deg \alpha = 2$.

Exercise: Cohomology of higher EM-spaces

Now that we know the cohomology of $K(\mathbb{Z}, 1) \simeq S^1$ and $K(\mathbb{Z}, 2) \simeq \mathbb{C}P^2$, it is natural to consider that of $K(\mathbb{Z}, 3)$ and higher EM-spaces. A good starting point is the homotopy sequence

$$K(G, n) \rightarrow * \rightarrow K(G, n+1)$$

obtained by using the mapping cylinder. Try to see how many cohomology groups of $K(\mathbb{Z}, 3)$ with \mathbb{Z} coefficients can be obtained by applying the Serre spectral sequence.

In fact, knowing $H^n(K(\mathbb{Z}, 3))$ for $n \leq 8$ would allow us to determine the cohomology ring of rational coefficients:

$$H^*(K(\mathbb{Z}, 3), \mathbb{Q}) = \Lambda_{\mathbb{Q}}[x], \quad \deg x = 3.$$

By induction it is possible to show that $H^*(K(\mathbb{Z}, n), \mathbb{Q})$ is of similar form, while we need to distinguish between the case n is odd (exterior algebra) and n is even (polynomial ring).