Introduction to Spectral Sequences

A concise approach

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Motivation¹

We mainly work in the category of abelian groups **Ab**. We aim to generalise the following fact.

For a complex $C_*: C_0 \leftarrow C_1 \leftarrow C_2 \leftarrow \cdots$ with a sub-complex $A_*: A_0 \leftarrow A_1 \leftarrow A_2 \leftarrow \cdots$, we can deduce the homology of C_* from certain information of that of A_* and C_*/A_* . In fact, recall that the long exact sequence of the pair (C_*, A_*) gives

$$H_{n+1}(C_*/A_*) \xrightarrow{\partial_{n+1}} H_n(A_*) \to H_n(C_*) \to H_n(C_*/A_*) \xrightarrow{\partial_n} H_{n-1}(A_*)$$

Cutting it short, we get

$$0 \to \operatorname{coker}(\partial_{n+1}) \to H_n(C_*) \to \ker(\partial_n) \to 0$$

This helps to determine $H_n(C_*)$. We want to have similar results in a more general context: filtered complexes.

¹The concept is introduced in: J. Leray. L'anneau d'homologie d'une représentation. C. R. Acad. Sci. 222 (1946)

Filtered complexes

For the complex (C_*, ∂) equipped with an ascending filtration of each of its terms:

$$F_pC_i \subseteq F_{p+1}C_i \subseteq \cdots \subseteq C_i$$
,

compatible with the differential in the sense that $\partial(F_pC_i) \subseteq F_pC_{i-1}$.

We denote also the graded term by $G_pC_i = F_pC_i/F_{p-1}C_i$. Observe that the filtration also induces a filtration on the homology complex $H_q(C_*)$. In fact,

$$F_p H_q(C_*) = \{ [x] \in H_q(C_*) \mid x \in F_p C_q, \partial x = 0 \},$$
 (1)

and it is clear that $F_pH_q\subseteq F_{p+1}H_q$.

The target of the spectral sequences is to calculate $G_pH_q=F_pH_q/F_{p-1}H_q$ from the groups G_pC_q and certain connecting morphisms.

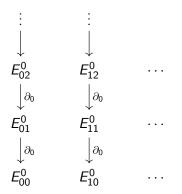
Homological spectral sequences: 0th page

We denote by $E_{p,q}^0 = G_p C_{p+q} = F_p C_{p+q} / F_{p-1} C_{p+q}$. By definition, the differential induces a morphism $\partial_0 : E_{p,q}^0 \to E_{p,q-1}^0$ by $x \mapsto [\partial x]$.

Recall that $\partial(F_pC_{p+q})\subseteq F_pC_{p+q-1}$, $\partial(F_{p-1}C_{p+q})\subseteq F_{p-1}C_{p+q-1}$. Clearly $\partial_0^2=0$. Thus we can consider the homology of the complex $(E_{p,*}^0,\partial_0)$, denoted by $E_{p,*}^1$, i.e.

$$\begin{split} E_{p,q}^1 &= \frac{\ker(E_{p,q}^0 \xrightarrow{\partial_0} E_{p,q-1}^0)}{\operatorname{im}(E_{p,q+1}^0 \xrightarrow{\partial_0} E_{p,q}^0)} \\ &= \frac{\ker(G_p C_{p+q} \to G_p C_{p+q-1})}{\operatorname{im}(G_p C_{p+q+1} \to G_p C_{p+q})} = H_{p+q}(G_p C_*). \end{split}$$

The 0th page E^0



1st page

We define a morphism $\partial_1: E^1_{p,q} \to E^1_{p-1,q}$ as follows.

For $\alpha = [x] \in E_{p,q}^1$, where $x \in E_{p,q}^0$, we know

$$0 = \partial_0 x = [\partial x] \in E^0_{p,q-1}$$
. Thus $\partial x \in F_{p-1}C_{p+q-1}$. Thus we let

$$\partial_1(\alpha) = [\partial x] \in E^1_{p-1,a}$$
.

It is easy to verify that ∂_1 is a well-defined morphism and $\partial_1^2=0$. This allows to consider the homology of the complex $(E_{*.a}^1, \partial_1)$, which is denoted by $E_{*,a}^2$.

1st page: another perspective

Alternatively, consider the short exact sequence of the triple $(F_pC_*, F_{p-1}C_*, F_{p-2}C_*)$:

$$0 \to G_{p-1}C_* = \frac{F_{p-1}C_*}{F_{p-2}C_*} \to F_pC_*/F_{p-2}C_* \to G_pC_* = \frac{F_pC_*}{F_{p-1}C_*} \to 0$$

It induces a long exact sequence of homology:

$$\cdots \rightarrow H_{p+q}(G_pC_*) \xrightarrow{\partial_1} H_{p+q-1}(G_{p-1}C_*) \rightarrow \cdots$$

where the connecting morphism is exactly ∂_1 we have defined.

The 1st page E^1

$$\vdots \qquad \vdots \qquad \vdots$$

$$E_{02}^{1} \leftarrow \xrightarrow{\partial_{1}} E_{12}^{1} \leftarrow \xrightarrow{\partial_{1}} E_{22}^{1} \leftarrow \cdots$$

$$E_{01}^{1} \leftarrow \xrightarrow{\partial_{1}} E_{11}^{1} \leftarrow \xrightarrow{\partial_{1}} E_{21}^{1} \leftarrow \cdots$$

$$E_{00}^{1} \leftarrow \xrightarrow{\partial_{1}} E_{10}^{1} \leftarrow \xrightarrow{\partial_{1}} E_{20}^{1} \leftarrow \cdots$$

Second pages

Rather than continuing by hand, we define directly for $r \in \mathbb{N}$,

$$E_{p,q}^{r} = \frac{\{x \in F_{p}C_{p+q} \mid \partial x \in F_{p-r}C_{p+q-1}\}}{F_{p-1}C_{p+q} + \partial(F_{p+r-1}C_{p+q+1})}$$
(2)

Here we omit the intersection in the denominator:

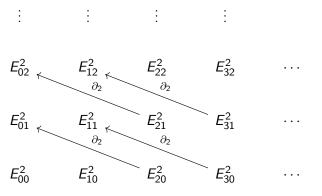
 $A/B:=A/(A\cap B).$

Similarly one can show that the differential induces a morphism $\partial_r: E^r_{p,q} \to E^r_{p-r,q+r-1}$, satisfying $\partial_r^2 = 0$.

For easy memory, E^{r+1} is just the homology of the complex (E^r, ∂_r) , i.e.

$$E_{p,q}^{r+1} = \frac{\ker(E_{p,q}^r \xrightarrow{\partial_r} E_{p-r,q+r-1}^r)}{\operatorname{im}(E_{p+r,q-r+1}^r \xrightarrow{\partial_r} E_{p,q}^r)}.$$

The 2nd page E^2



Limit page

Suppose that for every i, the filtration F_*C_i is bounded, i.e. $F_0C_i=0$ and $F_pC_i=C_i$ for $p\gg 0$; then for every p,q, there exists r_0 such that when $r\geq r_0$, we have

$$E_{p,q}^r = G_p H_{p+q}(C_*).$$
 (3)

Proof.

When $r \gg 0$, $F_{p-r}C_{p+q-1} = 0$, and $F_{p+r-1}C_{p+q+1} = C_{p+q+1}$. Now use (1) and (2).

In general, we say that the spectral sequence degenerates at E^r if $\partial_k = 0$ for all $k \ge r$. In this case, $E^r = E^{r+k} = E^{\infty}$ for all $k \ge 0$. Then (3) applies.

Case of 1 sub-complex

Let $F_0C_*=A_*$ and $F_1C_*=C_*$. This is exactly the case in the motivation. We denote by $B_*=C_*/A_*$.

Notice that p = 0, 1 are the only possible values.

Then $E_{0,q}^0=G_0C_q=A_q$, and

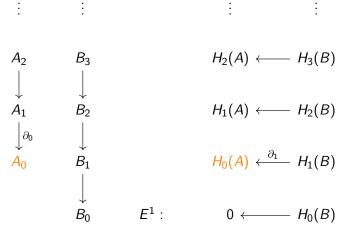
$$E_{1,q}^0 = G_1 C_{q+1} = C_{q+1} / A_{q+1} = B_{q+1}.$$

Further, $E_{0,q}^1 = H_q(A_*)$ and $E_{1,q}^1 = H_{q+1}(B_*)$.

Notice that the morphism ∂_1 is nothing but the connecting morphism of the long exact sequence associated with the short exact sequence $0 \to A_* \to C_* \to B_* \to 0$.

E^0 and E^1 for 1 sub-complex

 F^0 :



Remark: The orange colour is temporarily used to determine the position (0,0).

Case of 1 sub-complex: cont.

Recall that E^2 is defined as the homology of (E^1, ∂_1) , so $E^2_{0,q} = \operatorname{coker}(\partial_1^{1,q})$ and $E^2_{1,q} = \ker(\partial_1^{1,q})$.

By the shape of the 2nd page E^2 , $\partial_2=0$, so $E^2=E^3$. Similarly $\partial_3=0$ implies $E^3=E^4$, and so on. So $E^2=E^\infty$, and hence the sequence degenerates at E^2 .

As a result, $E_{p,q}^2=G_pH_{p+q}$. Recall that p=0,1, and thus we have a short exact sequence

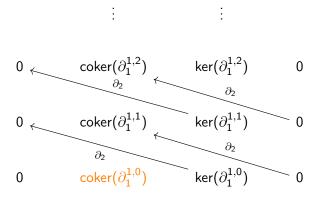
$$0 \rightarrow G_0 H_{p+q} \rightarrow H_{p+q} \rightarrow G_1 H_{p+q} \rightarrow 0,$$

which now reads

$$0 \to \operatorname{\mathsf{coker}}(\partial_1^{0,p+q}) \to H_{p+q} \to \ker(\partial_1^{0,p+q-1}) \to 0.$$

We have thus regained the information carried by the long exact sequence.

E^2 for 1 sub-complex



 E^2 :

Case of a double complex

Now we are given a sequence of abelian groups $C_{p,q}$ with $p,q \geq 0$, and 2 differentials $\partial': C_{p,q} \to C_{p-1,q}$ and $\partial'': C_{p,q} \to C_{p,q-1}$ satisfying $\partial'^2 = \partial''^2 = \partial'\partial'' + \partial''\partial' = 0$.

We define its total complex (C_*, ∂) by $C_n = \bigoplus_{p+q=n} C_{p,q}$ and $\partial = \partial' + \partial''$. Remember that our target is always to calculate the homology of C_* , which we filter in 2 ways:

$${}^{\prime}F_{p}C_{n}=\bigoplus_{k+l=n,k\leq p}C_{k,l},\quad {}^{\prime\prime}F_{p}C_{n}=\bigoplus_{k+l=n,l\leq p}C_{k,l}.$$

Now by definition ${}'E^0_{p,q} = C_{p,q}$, ${}'E^1_{p,q} = H_q(C_{p,*},\partial'')$, and also ${}''E^0_{p,q} = C_{q,p}$, ${}''E^1_{p,q} = H_p(C_{*,q},\partial')$. Notice that ${}'\partial_0$ is induced by ∂'' , and ${}''\partial_0$ is induced by ∂' .

Case of a double complex: cont.

The 2nd page is more interesting. Notice that

$$'\partial_1: 'E^1_{p,q} = H_q(C_{p,*}, \partial'') \to 'E^1_{p-1,q} = H_q(C_{p-1,*}, \partial'')$$

is induced by $\partial': C_{p,*} \to C_{p-1,*}$. So it is easy to see

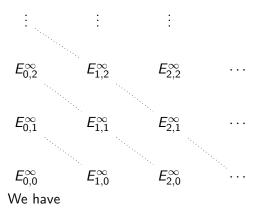
$${}^{\prime}E_{p,q}^2=H_p(H_q(C_{p,*},\partial''),\partial').$$

Dually,

$$^{\prime\prime}E_{p,q}^2=H_q(H_p(C_{*,q},\partial'),\partial'').$$

Useful fact: if the 2 spectral sequences converge, they must converge to the same group $H_n(C_*)$, though for 2 different filtrations ${}'G_p$ and ${}''G_p$. This is seen by summing over the diagonal p+q=n.

Diagonal sum



$$H_n(C_*) = \bigoplus_{n+q=n} E_{p,q}^{\infty}.$$

Application: universal coefficient theorem

As an example, let M be an abelian group, and P_1, P_0 be free abelian groups such that there is a short exact sequence

$$0 \to P_1 \xrightarrow{\phi} P_0 \to M \to 0.$$

We write $P_i = 0$ if $i \notin \{0, 1\}$.

If X is a topological space, we can form the double complex $C_{p,q}=C_p(X)\otimes P_i$ with natural differentials: $\partial'=d\otimes 1$ and $\partial''=1\otimes \phi$.

Since $C_p(X)$ is free, ${}'E^1_{p,0}=C_p(X)\otimes M$ and ${}'E^1_{p,q}=0$ if $q\neq 0$. Thus ${}'E^2$ is concentrated on the row q=0 as is ${}'E^1$, and ${}'E^2_{p,0}=H_p(X,M)$.

By the shape of $'E^2$, $'\partial_2 = '\partial_3 = \cdots = 0$, so $'E^\infty = 'E^2$.

The 1st spectral sequence 'E

Universal coefficient theorem: cont.

On the other hand, ${}''E^0_{p,q}=C_{q,p}=C_q(X)\otimes P_p$, so ${}''E^1_{p,q}=H_q(X)\otimes P_p$. This spectral sequence is concentrated on 2 columns p=0,1.

It is not hard to continue the calculation: ${}''E_{0,q}^2 = H_q(X) \otimes M$ and ${}''E_{1,q}^2 = \operatorname{Tor}(H_q(X), M)$: use the long exact sequence

$$\cdots \to \operatorname{\mathsf{Tor}}(H_q(X),P_0) = 0 \to \operatorname{\mathsf{Tor}}(H_q(X),M) \to \\ \to H_q(X) \otimes P_1 \to H_q(X) \otimes P_0 \to H_q(X) \otimes M \to 0.$$

By the shape of ${}''E^2$, ${}''\partial_2 = {}''\partial_3 = \cdots = 0$ and ${}''E^\infty = {}''E^2$. Recall that the 2 spectral sequences converge to the same homology $H_n(X,M)$ by summing over the diagonal p+q=n:

$$H_n(X,M) \cong H_n(X) \otimes M \oplus \operatorname{Tor}(H_{n-1}(X),M).$$
 (4)

The 2nd spectral sequence "E

Exercise: Künneth formula

Let A_* and B_* be 2 complexes of k-vector spaces (of positive degree). There is an isomorphism

$$H_n(A_* \otimes B_*) = \bigoplus_{p+q=n} H_p(A_*) \otimes H_q(B_*). \tag{5}$$

Remark: the isomorphism is not canonical.

Proof.

To begin with, consider $C_n = \bigoplus_{p+q=n} A_p \otimes B_q$ with differential

$$\partial(x\otimes y)=\partial x\otimes y+(-1)^{\deg x}x\otimes\partial y.$$

Leray-Serre spectral sequence

We are given a (Serre) fibration² $p: X \to B$ with pointed base (B, b_0) . Assume that its fibre $F = p^{-1}(b_0)$ has a base point x_0 , and F, X, B are all path-connected. Recall that this means that $\pi_1(X, x_0)$ acts on $\pi_k(F)$.

For simplicity, we suppose in addition that B is a CW-complex, with skeletons

$$B^0 \subseteq B^1 \subseteq \cdots \subseteq B$$
,

and filter $C_*(X)$ by the preimages $X^k = p^{-1}(B^k)$. In other words, $F_pC_*(X) = C_*(X^p)$.

²i.e. having RLP w.r.t. the inclusions $D^n \hookrightarrow D^n \times [0,1], n \in \mathbb{N}$.

Serre spectral sequence: cont.

Then we can calculate $E_{p,q}^0 = C_{p+q}(X^p, X^{p-1})$ and $E_{p,q}^1 = H_{p+q}(X^p, X^{p-1})$. And what makes the sequence interesting is that, we have a natural isomorphism³

$$E_{p,q}^2 \xrightarrow{\Phi} H_p(B, H_q(F)),$$
 (6)

where $H_q(F)$ denotes the homology of the fibre at b_0 , viewed as a $\mathbb{Z}[\pi_1(B)]$ -module. Since (B, B^k) is k-connected, by HLP so is (X, X^k) , and thus $X^k \hookrightarrow X$ induces an isomorphism on $H_n(-, M)$ if n < k. On the other hand $X^k = \emptyset$ for k < 0, and we can deduce that the spectral sequence for homology with coefficients converges to $H_*(X, M)$.

³The result with proof is found in: J.-P. Serre. *Homologie singulière des espaces fibrés. Applications.* Ann. of Math. 54 (1951)

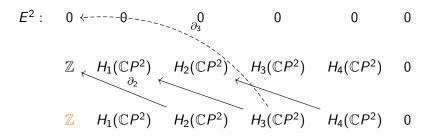
Example: complex projective plane $\mathbb{C}P^2$

 $S^1\subseteq\mathbb{C}$ acts on $S^5\subseteq\mathbb{C}^3$, with quotient space $\mathbb{C}P^2$. We use the fibration $S^1\to S^5\to\mathbb{C}P^2$ to compute $H_*(\mathbb{C}P^2)$. Since $\pi_1(\mathbb{C}P^2)=0$ and $H_*(S^1)$ are free, the Serre spectral sequence gives

$$E_{p,q}^2 = H_p(\mathbb{C}P^2, H_q(S^1)) = H_p(\mathbb{C}P^2) \otimes H_q(S^1) \Rightarrow H_{p+q}(S^5).$$

Using the fact that $\mathbb{C}P^2$ is a 4-dimensional real manifold, we observe that E^2 is concentrated in $0 \le p \le 4$ and q = 0, 1. This shape implies that $\partial_3 = \partial_4 = \cdots = 0$, hence the sequence degenerates at E^3 .

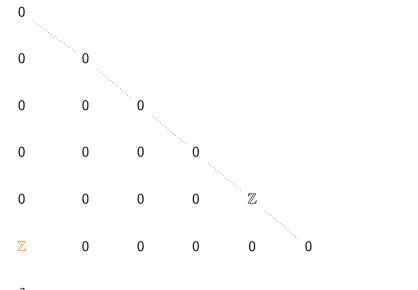
Homological Serre spectral sequence for $\mathbb{C}P^2$, 2nd page



$\mathbb{C}P^2$, cont.

We then use the homology of S^5 to determine $E^3=E^\infty$. Since $H_0(S^5)=H_5(S^5)=\mathbb{Z}$ and other homology groups vanish, $E_{0,0}^3=\mathbb{Z}$, and $E_{4,1}^3=\mathbb{Z}$. This is seen by summing over the diagonals and by taking homology of E^2 . These 2 positions are the only non-vanishing ones in E^3 . For degree reasons this implies that in E^2 we must have $H_3(\mathbb{C}P^2)\simeq H_1(\mathbb{C}P^2)=0$, and $H_4(\mathbb{C}P^2)\simeq H_2(\mathbb{C}P^2)\simeq \mathbb{Z}$. As a result, $H_k(\mathbb{C}P^2)=\mathbb{Z}$ iff k=0,2,4 and other homology groups vanish.

Homological Serre spectral sequence for $\mathbb{C}P^2$, 3rd page



E³ :

Exercise: loop space of spheres

Let (X, x_0) be a pointed topological space. Its *path space* is

$$PX = \{ \gamma : [0,1] \to X \mid \gamma(0) = x_0 \}$$

with compact-open topology, and let $p(\gamma) = \gamma(1)$. We denote the *loop space* of X at x_0 by $\Omega X = p^{-1}(x_0)$. Then it is not hard to see that we have a fibration sequence

$$\Omega X \hookrightarrow PX \xrightarrow{p} X$$
.

Try to use it to calculate the homology groups of ΩS^n . In order to proceed smoothly, observe first that PS^n is contractible.

Cohomological spectral sequences: filtration

Everything is formally dual to its homological counterpart. We are given a complex $C^0 \xrightarrow{d} C^1 \xrightarrow{d} \cdots$ with a descending filtration $F_pC^* \supseteq F_{p+1}C^*$ compatible with differential, i.e. $d(F_pC^n) \subseteq F_pC^{n+1}$. Similarly we denote the graded term by $G_pC^n = F_pC^n/F_{p+1}C^n$. Now we again have an induced filtration on $H^*(C)$ by

$$F_pH^n = \{ [x] \in H^n(C^*) \mid x \in F_pC^n, dx = 0 \}.$$
 (7)

Once again our target is to calculate the graded terms $G_pH^n=F_pH^n/F_{p+1}H^n$.

Cohomological spectral sequences: pages

We define (omitting the intersection in the denominator)

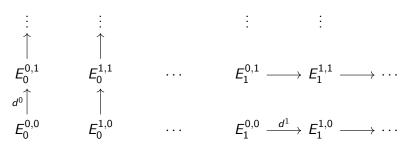
$$E_r^{p,q} = \frac{\{x \in F_p C^{p+q} \mid dx \in F_{p+r} C^{p+q+1}\}}{F_{p+1} C^{p+q} + dF_{p-r+1} C^{p+q-1}},$$
 (8)

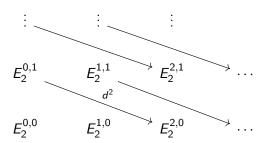
and we obtain a differential $d^r: E_r^{p,q} \to E_r^{p+r,q-r+1}$ induced by d, so that

$$E_{r+1}^{p,q} = \frac{\ker(E_r^{p,q} \xrightarrow{d'} E_r^{p+r,q-r+1})}{\operatorname{im}(E_r^{p-r,q+r-1} \xrightarrow{d'} E_r^{p,q})}$$

For the initial pages, $E_0^{p,q} = G_p C^{p+q}$, and $d^0 : E_0^{p,q} \to E_0^{p,q+1}$; $E_1^{p,q} = H^{p+q}(G_p C^*)$, and $d^1 : E_1^{p,q} \to E_1^{p+1,q}$.

Pages of cohomological spectral sequences





Convergence and degeneration

Suppose that the filtration is bounded, i.e. $\forall n, F_0C^n = C^n$ and $F_pC^n = 0$ for $p \gg 0$; then the spectral sequence degenerates at some E_r , and converges to

$$E_r^{p,q} = E_{\infty}^{p,q} = G_p H^{p+q}(C^*).$$
 (9)

In this case we write $E_r^{p,q} \Rightarrow H^n$. Again in the case of a double complex, the cohomology H^n is calculated by taking the sum over the diagonal p+q=n in the limit page E^{∞} .

Exercise: Try to get a dual version of the UCT.

Multiplicative structure

Recall that the singular cohomology of spaces has the merit over the homology of possessing a multiplicative structure. This motivates us to suppose that there is a bilinear map

$$*: C^p \times C^q \to C^{p+q}$$

satisfying the Leibniz rule⁴

$$d(\alpha * \beta) = d\alpha * \beta + (-1)^p \alpha * d\beta,$$

and preserving the filtration

$$F_pC^**F_qC^*\subseteq F_{p+q}C^*$$
.

⁴Note that this implies that $*: H^p \times H^q \to H^{p+q}$.

Multiplicative structure: cont.

Now * induces for every $r \in \mathbb{N}$ a bilinear map

$$*_r: E_r^{p,q} \times E_r^{p',q'} \to E_r^{p+p',q+q'}$$

such that

 $ightharpoonup d^r$ is a derivation:

$$d^{r}(\alpha *_{r} \beta) = d^{r}\alpha *_{r} \beta + (-1)^{p+q}\alpha *_{r} d^{r}\beta;$$

- $\triangleright *_{r+1}$ is the induced product on cohomology by $*_r$;
- ▶ if the filtration is bounded, *r stabilises to the product

$$G_pH^{p+q}\times G_{p'}H^{p'+q'}\to G_{p+p'}H^{p+q+p'+q'}.$$

Cohomological Serre spectral sequence

Again we are given a fibration (or rather a fibre bundle⁵ for simplicity) $p: X \to B$ where the base B is a path-connected CW-complex with base point b_0 and the fibre $F = p^{-1}(b_0)$ is also path-connected.

We filter the singular cohomology of X by

$$F_kC^n(X)=C^n(X,X^{k-1})=\{\phi:C_n(X)\to\mathbb{Z};\ \phi\upharpoonright_{C_n(X^{k-1})}=0\},$$

where $X^k = p^{-1}(B^{k-1})$, so that $F_0C^n(X) = C^n(X)$ and $F_kC^n(X) = 0$ for $k \gg 0$. Now it is easy to calculate $E_0^{p,q} = C^{p+q}(X^p, X^{p-1})$, $E_1^{p,q} = H^{p+q}(X^p, X^{p-1})$, and Serre has also proved that

$$E_2^{p,q} = H^p(B, H^q(F)).$$
 (10)

We remark that $E_r^{p,q} \Rightarrow H^n(X)$, dual to the homological case.

⁵i.e. every $b \in B$ has a neighbourhood U such that $p^{-1}(U) \stackrel{\phi}{\simeq} U \times F$, and $p = \operatorname{proj}_1 \circ \phi$.

Adding the multiplicative structure

We wish to empower the cohomological Serre spectral sequence with a multiplicative structure, and we have a natural choice: the cup product \smile : $C^k(X) \times C^l(X) \to C^{k+l}(X)$. Recall that $\phi \smile \psi(\sigma) = \phi(\sigma \upharpoonright_{\Delta^{\{0,\cdots,k\}}}) \psi(\sigma \upharpoonright_{\Delta^{\{k,\cdots,l\}}})$ for any chain $\sigma : \Delta^{k+l} \to X$.

However, it does not necessarily preserve our filtration. This means that it does not induce a well-defined product on E_0 -level. Instead we turn to E^1 -level, i.e. the cohomology level. We consider the cup product as a composition

$$H^*(X) \times H^*(X) \xrightarrow{\times} H^*(X \times X) \xrightarrow{\Delta^*} H^*(X),$$

 $(\alpha, \beta) \mapsto \operatorname{proj}^* \alpha \smile \operatorname{proj}^* \beta \mapsto \alpha \smile \beta,$

where $\Delta: X \to X \times X$ is the diagonal embedding.

Multiplicative structure on E_1

 $X \times X$ is the union of its skeletons $(X \times X)^k = \bigcup_{i+j=k} X^i \times X^j$ which are the preimages of the k-skeletons of $B \times B$. By excision,

$$H^*((X\times X)^k,(X\times X)^{k-1})=\bigoplus_{i+j=k}H^*(X^i\times X^j,X^{i-1}\times X^j\cup X^i\times X^{j-1})$$

which induces an inclusion of each factor in the direct sum into the term on the left. Now we have the following diagram which gives

$$*_1: E_1^{p,q} \times E_1^{p',q'} \to E_1^{p+p',q+q'}.$$

And this product is shown to be compatible with d^1 .

 $*_1$ on E_1 , cont.

$$H^{m}(X^{k}, X^{k-1}) \times H^{n}(X^{l}, X^{l-1}) \xrightarrow{--^{*1}} H^{m+n}(X^{k+l}, X^{k+l-1})$$

$$\downarrow^{\times} \qquad \qquad \Delta^{*} \uparrow$$

$$H^{n+m}(X^{k} \times X^{l}, U) \longrightarrow H^{n+m}((X \times X)^{k+l}, (X \times X)^{k+l-1})$$
Here $U = X^{k} \times X^{l-1} \cup X^{k-1} \times X^{l}$.

Multiplicative structure on E_2

Since $H^*(F)$ has a graded algebra structure, we have the following diagram

$$H^{p}(B, H^{q}(F)) \times H^{p'}(B, H^{q'}(F)) \xrightarrow{*_{2}} H^{p+p'}(B, H^{q}(F) \otimes H^{q'}(F))$$
 $H^{p+p'}(B, H^{q+q'}(F))$

Useful fact: The multiplicative structure on E_2

$$*_2: E_2^{p,q} \times E_2^{p',q'} \to E_2^{p+p',q+q'}$$

induced by $*_1$ is $(-1)^{q'p}$ times the natural "double" cup product. Remark: The compatibility is due to the fact that for a ring map $h: R \to R'$ and $\alpha_0, \alpha_1 \in H^{\bullet}(X; R)$,

$$h^*(\alpha_1 \smile \alpha_2) = h^*(\alpha_1) \smile h^*(\alpha_2).$$

Example: infinite complex projective space⁶

We calculate the cohomology of $\mathbb{C}P^{\infty}$ with the help of the fibre bundle $S^1 \hookrightarrow S^{\infty} \to \mathbb{C}P^{\infty}$ where the total space S^{∞} is contractible. So $E_r^{p,q} \Rightarrow H^n(S^{\infty}) = 0$ (n > 0), and the 2nd page gives

$$E_2^{p,q}=H^p(\mathbb{C}P^\infty,H^q(S^1)).$$

So the 2nd page is concentrated in 2 rows q=0,1. Let 1 be the generator of $H^0(S^1)$ and x be that of $H^1(S^1)$, we have the following diagram:

$$H^0(\mathbb{C}P^{\infty})x$$
 $H^1(\mathbb{C}P^{\infty})x$ $H^2(\mathbb{C}P^{\infty})x$ $H^3(\mathbb{C}P^{\infty})x$ \cdots
 $H^0(\mathbb{C}P^{\infty})$ $H^1(\mathbb{C}P^{\infty})$ $H^2(\mathbb{C}P^{\infty})$ $H^3(\mathbb{C}P^{\infty})$ \cdots

Thus $d^3=d^4=\cdots=0$, and $E_3=E_\infty$. We deduce that every d^2 is isomorphic, and further $H^{2k}(\mathbb{C}P^\infty)=\mathbb{Z}$, $H^{2k+1}(\mathbb{C}P^\infty)=0$.

⁶Using the Eilenberg-MacLane notion, $\mathbb{C}P^2 = K(\mathbb{Z}, 2)$.

The ring structure of $H^*(\mathbb{C}P^{\infty})$

Consider the isomorphism $d^2: H^0(\mathbb{C}P^\infty)x \to H^2(\mathbb{C}P^\infty)$. As an isomorphism between \mathbb{Z} , it maps the generator to generator, i.e. $\alpha = d^2(x) \in H^2(\mathbb{C}P^\infty)$ is the generator. Note that here we implicitly assume the generator of $H^0(\mathbb{C}P^\infty)$ to be 1. Similarly consider the isomorphism $d^2: H^2(\mathbb{C}P^\infty)x \to H^4(\mathbb{C}P^\infty)$. This time $d^2(\alpha x)$ becomes the generator of H^4 . But we have

$$d^{2}(\alpha x) = d^{2}(\alpha)x + (-1)^{2+0}\alpha d^{2}(x) = d^{2}(\alpha)x + \alpha^{2},$$

and notice that $d^2(\alpha) \in E_2^{4,-1} = 0$. So $d^2(\alpha x) = \alpha^2$ is the generator of H^4 . Similarly we can get $H^{2k} = \mathbb{Z}\alpha^k$. In consequence, $H^*(\mathbb{C}P^\infty) = \mathbb{Z}[\alpha]$, where $\deg \alpha = 2$.

Exercise: Cohomology of higher EM-spaces

Now that we know the cohomology of $K(\mathbb{Z},1)\simeq S^1$ and $K(\mathbb{Z},2)\simeq \mathbb{C}P^2$, it is natural to consider that of $K(\mathbb{Z},3)$ and higher EM-spaces. A good starting point is the homotopy sequence

$$K(G, n) \rightarrow * \rightarrow K(G, n + 1)$$

obtained by using the mapping cylinder. Try to see how many cohomology groups of $K(\mathbb{Z},3)$ with \mathbb{Z} coefficients can be obtained by applying the Serre spectral sequence.

In fact, knowing $H^n(K(\mathbb{Z},3))$ for $n \leq 8$ would allow us to determine the cohomology ring of rational coefficients:

$$H^*(K(\mathbb{Z},3),\mathbb{Q}) = \Lambda_{\mathbb{Q}}[x], \quad \deg x = 3.$$

By induction it is possible to show that $H^*(K(\mathbb{Z}, n), \mathbb{Q})$ is of similar form, while we need to distinguish between the case n is odd (exterior algebra) and n is even (polynomial ring).