~~[illegible]~~

~~[illegible]~~

$\boxed{\text{Langer, 1973}}$

- Define exponential polynomial / exponential sum:

(1) $\Phi(z) = \sum_{j=0}^{n} A_j(z)\, e^{c_j z}$, where $A_j(z)$ and c_j are constants, $c_j \in \mathbb{R}$.

- (1) can be expresses in a form with less specialized functions $\phi(t)$:

(2) $\Phi(z - z_0) = z \int_{c_0}^{c_n} \phi(t)\, e^{tz}\, dt$

$\phi(t)$ ~~can be~~ represents a generalization of certain sums of type (1).

Constant coefficients + real, commensurable exponents.

- Theoretically the simplest ~~[illegible]~~ type of sum is one in which the problem of distribution of zeros is essentially an ~~[illegible]~~ algebraic one.

This occurs in particular when $c_j = \alpha p_j$, $j = 1, 2, 3, \dots n$, $\alpha \in \mathbb{R}$, $p_j \in \mathbb{Z}$.

The sum is of the form:

(3) $\Phi(z) = \sum_{j=0}^{n} a_j \left(e^{az}\right)^{p_j}$, $p_0 = 0$.

- If the polynomial alone in (3) ~~[illegible]~~ ~~gives~~ admits zeros the values $\xi_1, \xi_2, \dots, \xi_{p_n}$

the function (3) vanishes for such values of z, and for only such, as satisfy a relation

$$e^{az} = \xi_j.$$

- The zeros of $\Phi(z)$ are, therefore, given by the formula

(4) $z = \frac{1}{\alpha}\{2m\pi i + \log \xi_j\}$, $(j = 1, 2, \dots, p_n)$, $(m = 0, \pm 1, \pm 2, \dots)$.

→ countably finite set (bijection to $\mathbb{N}$ or $\mathbb{Z}$)

→ distributed in the complex plane at regular intervals of length $2\pi/\alpha$, along p_n lines which are normal to the axis of ~~#~~ reals.

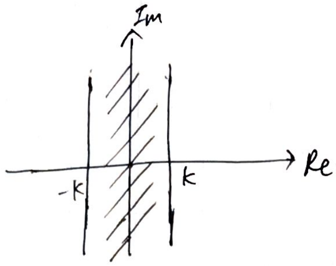
~~This formula dete~~

If the explicit solution of the ~~[illegible]~~ polynomial equation involved is feasible, the determination of the zeros of $\Phi(z)$ is completed by formula 4.

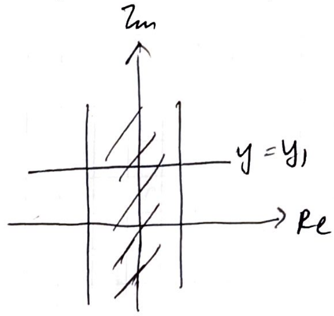
→ i.e. formula 4 determines which terms are non-significant / negligible.

~~(3)~~ • From (4), with any specifically given function $\Phi(z)$, the choice of a constant K is possible so that the zeros of $\Phi(z)$ all lie within the rectilinear strip of the z plane given by the relation

(5) $|x| < K \quad (z = x + iy)$.



~~(6)~~ • Any line $y = y_1$, parallel to the axis of reals, cuts the strip, and on such a line is found by

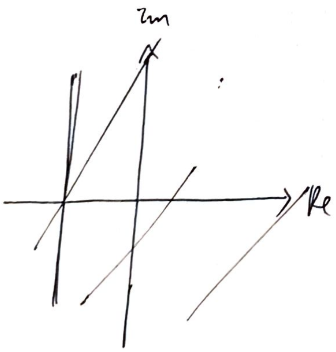


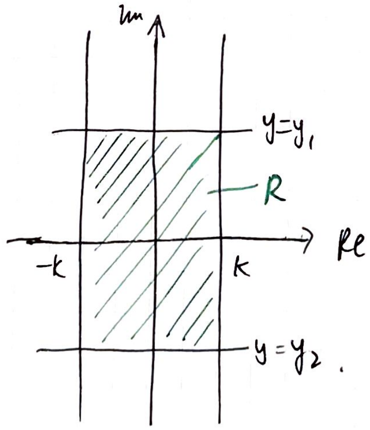
$$(6) \quad I\left\{\frac{1}{a_0}\Phi(z)\right\} = \sum_{j=1}^{n} b_j(y_1)(e^{ax})^{p_j},$$

$b_j \in \mathbb{R}$ and its value depends on y_1.

• By Descartes' rule, the expression (6) can vanish at most as many times as there are changes of sign in the sequence of b_j, which $\le n-1$ since the number of terms in (6) is n.

•





$y = y_1$ and $y = y_2$ are chosen s.t. on neither of them there lies a zero of $\Phi(z)$

- The number $n(R)$ of zeros of $\Phi(z)$ within the rectangle R is subject to the bounds

$$(7) \quad -n + \frac{c_n}{2\pi}(y_2 - y_1) \le n(R) \le n + \frac{c_n}{2\pi}(y_2 - y_1).$$

- This limits, both above & below, the density of zeros in any portion of the strip (5)
- This shows that no sums of the form (3) with $(n+1)$ terms can have a zero of multiplicity greater than n.

$\boxed{\text{Theorem 1}}$.

If in the exponential sum (1) the coefficients are constants ~~and exponentia~~ and the exponents are real and commensurable, then the distribution of zeros ~~are~~ is given explicitly by the formula (4).

i.e. $z = \frac{1}{\alpha}\{2m\pi i + \log \xi_j\}$, $(j = 1, 2, \dots, p_n)$, $(m = 0, \pm 1, \pm 2 \dots)$.

In this distribution the # of zeros which lie between two lines $y = y_1$ and $y = y_2$, is restricted by the relation (7).

i.e. $-n + \frac{c_n}{2\pi}(y_2 - y_1) \le n(R) \le n + \frac{c_n}{2\pi}(y_2 - y_1)$

~~$\Phi(z) =$~~ (8) *

$\boxed{\text{Theorem 2}}$

If the coefficients a_j are real and the zeros of the polynomial

$$(9) \quad P(\xi) = \sum_{j=0}^{n} a_j \xi^j$$

all lie within the unit circle about $\xi = 0$, then the zeros of the corresponding trigonomic sums (8) are all real and simple.

Each of these sums has precisely $2n$ zeros on the interval $0 \le z < 2\pi$ and the zeros of either sum alternate with those of the other.

(By a theorem of Kakeya the hypothesis is fulfilled if $0 \le a_0 < a_1 < \dots < a_n$.)

*(8) ~~$\Phi(z$~~ Trigonometric sums:

$$\Phi_c(z) = \sum_{j=0}^{n} a_j \cos jz$$

$$\Phi_s(z) = \sum_{j=1}^{n} a_j \sin jz$$

Constant coefficients and general real exponents

- When the exponents are not commensurable the determination of the distribution of zeros of $\Phi(z)$ is not in general of an algebraic character ~~the~~. In this case, the sum $\Phi(z)$ ~~can~~ is expressed by the formula

$$(10)\quad \Phi(z) = \sum_{j=0}^{n} a_j e^{c_j z}, \quad c_0 = 0.$$

~~Fall the Fowing this case,~~

- The establishment of the relation (7) depends only on the consideration of the quantity

$$(11)\quad I\left\{\frac{1}{a_0}\Phi(z)\right\} = \sum_{j=1}^{n} b_j(y) e^{c_j x},$$

on a line for which y is constant.

- An important note: in the case of an exponential sum, the value $|\Phi(z)|$ is uniformly bounded from zero when the variable z is uniformly bounded from the zeros of $\Phi(z)$.

This means that given a sufficiently small positive δ there exists a constant H depending only on δ and such that

$$(12)\quad |\Phi(z)| > H, \text{ for } |z - z_m| > \delta,$$

where z_m designates the set of zeros of $\Phi(z)$.

$\boxed{\text{Theorem 3}}$

i.e. $\Phi(z) = \sum_{j=0}^{n} g_j e^{c_j z}, c_0 = 0$.

If in the exponential sum (10), the coefficients are constants and the ~~expectation~~ exponents are real, then the zeros of the sum all lie within a strip (5), i.e. $|x| < K, (z = x + iy)$, ~~and~~

and in any portion of this strip the # of zeros is limited by relation (7)

i.e. $-n + \frac{c_n}{2\pi}(y_2 - y_1) \leq n(R) \leq n + \frac{c_n}{2\pi}(y_2 - y_1)$.

When z is uniformly bounded ~~fro~~ from the zeros of $\Phi(z)$, then $|\Phi(z)|$ is uniformly bounded from zero.

Note: ~~The~~ In the general case of incommensurable exponents, the distance between distinct zeros of $\Phi(z)$ admits of no positive lower bound.

4. Coefficients asymptotically constant.

- Suppose now that either the coefficients $A_j(z)$ in the sum (1) are ~~sifu~~ single valued and of the form (13) $A_j(z) = a_j + \varepsilon(z)$, in the region $|z| > M$;

  or ~~they are multi~~ $A_j(z)$ are multivalued but in the region $|z| > M$, $-\pi < \arg z \leq \pi$, their various branches are each of the form (13).

- Hence the form assumed for the sum (1) is ~~therefore~~,

$$(14) \quad \Phi(z) = \sum_{j=0}^{n} \{a_j + \varepsilon(z)\} e^{c_j z}, \quad a_0 a_n \neq 0.$$

- Note: $\varepsilon(z)$: in a region R of the z plane including the point $z = \infty$, ~~a function~~ an epsilon function, $\varepsilon(z)$, is a function that is analytic in every finite portion of R and that approaches zero uniformly in R as $|z| \to \infty$.

- The zeros of the sum (14) are asymptotically represented by those of the related sum

$$(15) \quad \Phi_1(z) = \sum_{j=0}^{n} a_j e^{c_j z}.$$

$\boxed{\text{Theorem 4}}$

If the function $\Phi(z)$ (or a determination of it) is of the form (14)

i.e. $\Phi(z) = \sum_{j=0}^{n} \{a_j + \varepsilon(z)\} e^{c_j z}$, $a_0 a_n \neq 0$,

then in the region $|z| > M$ the distribution of zeros of $\Phi(z)$ (or of the branch of $\Phi(z)$ in question) may be described as in Theorem 3.

The zeros are asymptotically represented by those of the related sum (15).

5. Coefficients which are Asymptotically Power Functions.

- Suppose now that in the form (1) the coefficients $A_j(z)$, or chosen branches of them, are of the form

(16) $A_j(z) = z^{v_j}\{a_j + \varepsilon(z)\}$,

for z in the region $|z| > M$, $-\pi < \arg z \leq \pi$. $v_j \in \mathbb{R}$

6. The values v_j and c_j proportional.

- If the real constant β is defined by the relation

$$v_j = \beta c_j, \quad (j = 1, 2, \dots, n),$$

the formula

(17) $\zeta = z + \beta \log z$.

defines a single-valued analytic map of the portion of the z-plane to which z was restricted above upon a complex ζ plane, the point $z = \infty$ corresponding to $\zeta = \infty$.

- The rectilinear strip $|\xi| < k$ corresponds to the curvilinear strip bounded by the logarithmic curves.

(18) $x + \beta \log |z| = \pm k$.

$\boxed{\text{Theorem 5}}$

If in the exponential sum (1) i.e. $\Phi(z) = \sum_{j=0}^{n} A_j(z) e^{c_j z}$, the coefficients are of the form (16) i.e. $A_j(z) = z^{v_j}\{a_j + \varepsilon(z)\}$, with values v_j proportional to the exponents c_j, and all terms are ordinary terms, ~~and all~~ then the zeros of the sum are asymptotically located within a logarithmic curvilinear strip bounded by curves of the form (18), i.e. $x + \beta \log|z| = \pm k$,

and the # of zeros lying between any two lines parallel to the axis of reals is asymptotically subject to the relations (7), i.e.

$$-n + \frac{c_n}{2\pi}(y_2 - y_1) \leq n(R) \leq n + \frac{c_n}{2\pi}(y_2 - y_1).$$

7. Numerical Real Values v_j.

- with each ordinary ~~form~~ term of the sum

$$(19)\quad \Phi(z) = \sum_{j=0}^{n} z^{v_j}\{a_j + \varepsilon(z)\} e^{c_j z},$$

there can be associated in the z-plane the corresponding point P_j with Cartesian coordinates (c_j, v_j).

- If the sum (19) contains exceptional terms it must be made a matter of hypothesis that for every such term a choice of the value v_j is possible under the form (16) s.t. the corresponding point (c_j, v_j) lies below the broken line L.
- Consider any intermediate segment L_r of the line L, and let its slope be m_r, the slope of the ~~proceeding seg~~ preceding segment m_{r-1}. Then the real parameter k varies over the range

$$m_{r-1} - \varepsilon \geq k > m_r - \varepsilon,$$

$\varepsilon > 0$ and sufficiently small but otherwise ~~arbitrary~~ arbitrary, the curve

$$(20)\quad x = -k \log|z|$$

sweeps out the region of the z plane bounded by the curves.

$$(21)\quad x = -(m_{r-1} - \varepsilon)\log|z|,$$
$$x = -(m_r - \varepsilon)\log|z|.$$

省略很多步. but it follows that

- Sum (19) can be written in the form

$$\cancel{(22)\quad \Phi(z) = \sum_{h=1}^{n_r} z^{v_{rh}}}$$

$$(22)\quad \Phi(z) = \sum_{h=1}^{n_r} z^{v_{rh}}\{a_{rh} + \varepsilon(z)\} e^{c_{rh} z},$$

for all z of the region bounded by curves (21).

- It may then be concluded:

the zeros of $\Phi(z)$ in the region (21) are asymptotically confined to the logarithmic strip bounded by the curves.

$$(23)\quad x + m_r \log|z| = \pm K.$$

• and that ~~the~~ the # of zeros in this strip and between two lines $y=y_1$ and $y=y_2$ is restricted by the relation obtained from (7) by replacing n and C_r by n_r and ~~C_r,~~ $(C_{r,n_r} - C_{r_1})$.

$\boxed{\text{Theorem 6}}$.

If $\Phi(z)$ is an exponential sum with coefficient of the form (16)
i.e. $A_j(z) = z^{v_j}\{a_j + \varepsilon(z)\}$, (the v_j for exceptional terms satisfying the hypothesis of the text),

then the zeros of the sum are asymptotically confined to a finite number of logarithmic strips (23), i.e., $x + m_r \log|z| = \pm k$,

the # of zeros in any strip between two lines parallel to the axis of reals being asymptotically subject to a relation similar to (7).

i.e. $\left[-n_r + \frac{C_{n,n_r} - C_{r_1}}{2\pi}(y_2 - y_1) \le n_r(R) \le n_r + \frac{C_{n,n_r} - C_{r_1}}{2\pi}(y_2 - y_1)\right]$.

not sure tho...
" replace C_r by $(C_{r,n_r} - C_{r_1})$??

8. Colinear Complex Exponents.

• Previously, we've assumed the exponents to be real (the various cases of sum (1) concern primarily with the structure of the coefficient function $A_j(z)$).
~~How~~ But this assumption is dispensable, & the distribution of zeros of the sum having complex exponents is also determinable.

• Let l designate the line on which the points C_j are located, and let the subscripts be assigned to these points in the order of their location on l.
Then if θ designates the inclination angle of the line l w.r.t. the axis of the reals (the positive sense of l being from C_0 to C_n), it follows that the relations

$$C_j = C_0 + \gamma_j e^{i\theta}, \quad (j = 0, 1, 2, \ldots, n),$$

are satisfied by a set of real values ~~$0 \le \gamma_0 < \gamma$~~ ~~$0 < \gamma$~~ $0 = \gamma_0 < \gamma_1 < \ldots < \gamma_n$.

- In terms of the variable ξ defined by the relation

$$(15) \quad \xi = z e^{i\theta},$$

the sum (1), therefore, takes the form

$$(16) \quad \Phi = e^{c_0 z} \sum_{j=0}^{n} B_j(\xi) e^{\gamma_j \xi},$$

where $B_j(\xi) \equiv A_j(z)$ under the substitution (15)

- The sum in (16) is one in which the exponents γ_j are real, and the coefficients $B_j(\xi)$ have the essential structural characteristics of $A_j(z)$.

↳ transformed into previous form!

- Hence, the distribution of zeros of the sum Φ in the ξ plane is as described in the theorem of the preceding sections.

i.e. zeros are asymptotically ~~not~~ confined to one or more strips which are parallel or approach parallelism with the axis $\arg \xi = \pm \pi/2$.

| Theorem 7 |

If the exponents c_j in the exponential sum (1) are collinear complex constants, the distribution of ~~zero~~ zeros of $\Phi(z)$ is obtainable from the theorem previously enunciated by substituting in the role of the axis of the reals the line containing $\bar{c}_j$ conjugate to the exponents c_j.

9. General Complex Constants

- The most general type of exponential sum in this paper: exponents may be any set of complex constants.

-

Theorem 8

If in the sum (1) the exponents are any complex constants, the zeros of $\Phi(z)$ are confined for $|z| > M$ to a finite number of strips each of asymptotically constant width. These strips are associated in groups with the exterior normals to the sides of the polygon described in the text, and approach parallelism with the respective normals. Within each group of strips the distribution of zeros may be described as in the previously stated theorems, the role of the axis of reals being ~~transformed~~ transferred to the respective side of the polygon.