THE ZEROS OF CERTAIN INTEGRAL FUNCTIONS

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1.1. The functions with which we are concerned are of the form

$$F(z) = \int_a^b e^{zt} f(t)\, dt \tag{1.11}$$

where $f(t)$ is any real integrable function, or, more generally,

$$f(t) = f_1(t) + i f_2(t),$$

where $f_1(t)$ and $f_2(t)$ are real integrable functions. The limits a, b, are finite. The related forms

$$\int_a^b \cos zt\, f(t)\, dt, \qquad \int_a^b \sin zt\, f(t)\, dt$$

may of course be reduced to (1.11) by simple transformations. The type includes many well known functions, such as Bessel functions.

Some striking results as to the distribution of the zeros when $f(t)$ satisfies simple special conditions have been obtained by Pólya*. Here however, the problem is to determine properties which are common to all the functions $F(z)$, without further restriction on $f(t)$ than that of integrability. The results obtained are much more precise than is the case for the general integral function of exponential type, *i.e.* such that

$$|F(z)| < Ae^{K|z|}.$$

They are more analogous to those obtained by Pólya† for functions of the form

$$P_1(z)\, e^{a_1 z} + P_2(z)\, e^{a_2 z} + \ldots + P_m(z)\, e^{a_m z} \tag{1.12}$$

* G. Pólya, *Math. Zeitschrift*, 2 (1918), 352–383.

† G. Pólya, *Bayer. Akademie der Wissenschaften* (1920), 285–290.

where $P_1(z) \ldots$ are polynomials. The function (1.11) may be regarded as a limiting case of (1.12) when the polynomials reduce to constants, the exponents $a_1 \ldots$ are all real, and $m \to \infty$. However, the properties of (1.11) are rather different from those of (1.12), and they are in fact obtained quite independently.

1.2. It is supposed that a and b are the effective lower and upper limits of the integral; *i.e.* there is no number $\alpha > a$ such that

$$\int_a^{\alpha} |f(t)|\,dt = 0,$$

and no number $\beta < b$ such that

$$\int_{\beta}^{b} |f(t)|\,dt = 0.$$

Let $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$, ... be the zeros of $F(z)$, arranged so that r_n is a non-decreasing function of n. It is supposed throughout that $F(0)$ is not zero. We may do this without loss of generality; for if $F(0) = 0$, then

$$F(z) = -z\int_a^b e^{zt} g(t)\,dt \quad \left(g(t) = \int_a^t f(u)\,du\right),$$

so that on dividing by z we obtain a function of the same type, with a zero of smaller order. So the zero may be removed altogether.

The first theorem is one already obtained by Pólya* :—

THEOREM I.—*The function $F(z)$ has an infinity of zeros such that the series*

(1.21) $$\sum_1^{\infty} \frac{1}{r_n}$$

is divergent.

The proof given here is simpler than the original one.

THEOREM II.—*The series*

(1.22) $$\sum_1^{\infty} \frac{\cos\theta_n}{r_n}$$

is absolutely convergent.

It follows from these two theorems that, in general, $\cos\theta_n$ is small when n is large; and it might be supposed that $\cos\theta_n \to 0$, or that any

* G. Pólya, *Math. Zeitschrift*, 2 (1918), 380.

angular region including the imaginary axis would contain all zeros of large modulus. This would be the case if we could replace $F(z)$ by a function of the form (1.12). Actually it is not necessarily so, as is shown by

THEOREM III.—*There is a function of the given class with an infinity of real zeros.*

The principal result of the paper is

THEOREM IV.—*If $n(r)$ is the number of zeros of $F(z)$ for which $|z| \leqslant r$, then*

$$n(r) \sim \frac{b-a}{\pi} r.$$

This, of course, includes Theorem I.

THEOREM V.—*The series*

$$\sum_{1}^{\infty} \frac{\sin \theta_n}{r_n} \tag{1.23}$$

is conditionally convergent.

It follows from Theorems I and II that it is not absolutely convergent.

THEOREM VI.—*We have*

$$F(z) = F(0)\, e^{\frac{1}{2}(a+b)z} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right), \tag{1.24}$$

the product being conditionally convergent.

Incidentally we obtain the sums of the series (1.22) and (1.23). We have

$$\sum_{1}^{\infty} \frac{\cos \theta_n}{r_n} = \tfrac{1}{2}(a+b) - R\left\{\frac{F'(0)}{F(0)}\right\}, \tag{1.25}$$

$$\sum_{1}^{\infty} \frac{\sin \theta_n}{r_n} = I\left\{\frac{F'(0)}{F(0)}\right\}. \tag{1.26}$$

In the course of the proofs we obtain a good deal of information about the modes of variation of $|F(z)|$, which sometimes goes further than is actually required for application to the problem of the zeros. It is proved, for instance, that, if $M(r) = \text{Max } |F(z)|\ (|z| = r)$, then

$$\log M(r) \sim r \,\text{Max}\,(|a|, |b|). \tag{1.27}$$

The main theorems may be extended so as to apply to the zeros of $F(z)-k$, where k is any constant, real or complex. The proofs only require slight modifications.

As an application of Theorem IV we have

THEOREM VII.—*If $\phi(t)$ and $\psi(t)$ are integrable functions, such that*

$$\int_0^x \phi(t)\,\psi(x-t)\,dt = 0$$

almost everywhere in the interval $0 < x < \kappa$, then $\phi(t) = 0$ almost everywhere in $(0, \lambda)$, and $\psi(t) = 0$ almost everywhere in $(0, \mu)$, where $\lambda+\mu \geqslant \kappa$.

1.3. The proofs are based on the following known theorems :—

THEOREM OF PHRAGMÉN AND LINDELÖF*.—*If $\phi(z)$ is regular and of order ρ in an angle less than π/ρ, i.e. if for every positive ϵ*

$$|\phi(re^{i\theta})| < e^{r^{\rho+\epsilon}} \quad (a \leqslant \theta \leqslant \beta,\ \beta-a < \pi/\rho,\ r > r_0),$$

and $\phi(z)$ is bounded for $\theta = a$, $\theta = \beta$, then it is bounded for $a \leqslant \theta \leqslant \beta$, and if $\phi(z) \to 0$ as $r \to \infty$ for $\theta = a$, $\theta = \beta$, then it tends uniformly to zero for $a \leqslant \theta \leqslant \beta$.

THEOREM OF JENSEN.—*If $\phi(z)$ is an integral function $\big(\phi(0) \neq 0\big)$, and $n(r)$ is the number of its zeros for $|z| \leqslant r$, then*

$$\int_0^r \frac{n(x)}{x}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|\phi(re^{i\theta})|\,d\theta - \log|\phi(0)|.$$

THEOREM OF CARLEMAN†.—*If $\phi(z)$ is regular for $R(z) \geqslant 0$, and $r_1 e^{i\theta_1}$, $r_2 e^{i\theta_2}$, ... are its zeros in the right half-plane, then*

$$\sum_{l<r_n<R} \left(\frac{1}{r_n}-\frac{r_n}{R^2}\right)\cos\theta_n = \frac{1}{\pi R}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \log|\phi(Re^{i\theta})|\cos\theta\,d\theta$$

$$+\frac{1}{2\pi}\int_l^R \left(\frac{1}{r^2}-\frac{1}{R^2}\right)\{\log|\phi(ir)|+\log|\phi(-ir)|\}\,dr+\psi(R)$$

where $\psi(R)$ is bounded as $R \to \infty$.

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* E. Phragmén and E. Lindelöf, *Acta Math.*, 31 (1908), 381–406.

† T. Carleman, *Arkiv för Mat. Astr. o. Fys.*, 17 (1922), Memoir 9.

THEOREM OF VALIRON*.—*If $\phi(z)$ is an integral function, and $M(r) = \operatorname{Max} |\phi(z)| \, (|z| = r)$, then, given $k > 1$, there is a number $H = H(k)$ and a sequence of circles $|z| = R_n$ $(R_n < kR_{n-1})$ on each of which*

$$|\phi(z)| > \{M(r)\}^{-H}.$$

1 . 4. It will be supposed throughout the proof that $a = -1$, $b = 1$. The general case can be reduced to this. For

$$\int_a^b e^{zt} f(t)\, dt = \tfrac{1}{2}(b-a)\, e^{\frac{1}{2}(a+b)z} \int_{-1}^{1} e^{\zeta t} g(t)\, dt$$

where $$\zeta = \tfrac{1}{2}(b-a)z, \qquad g(t) = f\{\tfrac{1}{2}(b+a) + \tfrac{1}{2}(b-a)\, t\}.$$

2. *Preliminary Lemmas.*

2 . 1. LEMMA 2 . 1.—*If $F(z) = 0$ for all values of z, then $f(t) = 0$ for almost all values of t.*

We have

$$F(iy) + F(-iy) = 2\int_0^1 \cos yt\, \{f(t) + f(-t)\}\, dt,$$

and hence

$$\int_0^\lambda \frac{\sin \xi y}{y} \{F(iy) + F(-iy)\}\, dy = 2\int_0^\lambda \frac{\sin \xi y}{y}\, dy \int_0^1 \cos yt\, \{f(t)+f(-t)\}\, dt$$

$$= 2\int_0^1 \{f(t)+f(-t)\}\, dt \int_0^\lambda \frac{\sin \xi y \cos yt}{y}\, dy.$$

Now if ξ and t are real

$$\left| \int_0^\lambda \frac{\sin \xi y \cos yt}{y}\, dy \right| \leqslant \int_0^\pi \frac{\sin u}{u}\, du.$$

Hence by Lebesgue's convergence theorem

$$\lim_{\lambda \to \infty} \int_0^1 \{f(t)+f(-t)\}\, dt \int_0^\lambda \frac{\sin \xi y \cos yt}{y}\, dy$$

$$= \int_0^1 \{f(t)+f(-t)\}\, dt \int_0^\infty \frac{\sin \xi y \cos yt}{y}\, dy$$

$$= \tfrac{1}{2}\pi \int_0^\xi \{f(t)+f(-t)\}\, dt,$$

* G. Valiron, *Theory of Integral Functions*, Toulouse (1923), 80. The theorem is stated by Valiron for functions of finite order, but his proof applies to any integral function, or indeed to any function which is regular in an appropriate region.

if $0<\xi<1$. Hence

$$(2.11)\quad \int_0^\xi \{f(t)+f(-t)\}\,dt=\frac{1}{\pi}\int_0^\infty \frac{\sin \xi y}{y}\{F(iy)+F(-iy)\}\,dy.$$

Similarly

$$(2.12)\quad \int_0^\xi \{f(t)-f(-t)\}\,dt=\frac{1}{\pi}\int_0^\infty \frac{1-\cos \xi y}{y}\{F(iy)-F(-iy)\}\,dy.$$

It follows that, if $F(z)=0$, then

$$\int_0^\xi f(t)\,dt=0,\qquad \int_0^\xi f(-t)\,dt=0\quad (0<\xi<1),$$

and the result stated follows.

2.2. LEMMA 2.2.—*As* $r\to\infty$,

$$|F(re^{i\theta})|=o(e^{r|\cos\theta|})$$

uniformly with respect to θ.

If x is real and positive

$$|F(x)|=\left|\int_{-1}^{1-\delta} e^{xt}f(t)\,dt+\int_{1-\delta}^{1} e^{xt}f(t)\,dt\right|$$

$$\leqslant e^{(1-\delta)x}\int_{-1}^{1-\delta}|f(t)|\,dt+e^{x}\int_{1-\delta}^{1}|f(t)|\,dt.$$

Now, having given ϵ, we can choose δ so small that

$$\int_{1-\delta}^{1}|f(t)|\,dt<\tfrac{1}{2}\epsilon,$$

and then x_0 so large that

$$e^{-\delta x}\int_{-1}^{1-\delta}|f(t)|\,dt<\tfrac{1}{2}\epsilon\quad (x>x_0).$$

Then $|F(x)|<\epsilon e^{x}$ $(x>x_0)$, *i.e.* $F(x)=o(e^{x})$. Again

$$F(iy)=\int_{-1}^{1}\cos yt\,f(t)\,dt+i\int_{-1}^{1}\sin yt\,f(t)\,dt,$$

which tends to zero as $y\to\infty$, by the "Riemann-Lebesgue lemma."

Hence, by the theorem of Phragmén and Lindelöf, the function

$$G(z) = e^{-z} F(z)$$

tends uniformly to zero for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. This gives the desired result in this range, and the remaining values of θ may be dealt with similarly.

2.3. LEMMA 2.3.—*When* $r \to \infty$.

(2.31) $$|F(re^{i\theta})| \neq O\{e^{r(|\cos\theta|-\delta)}\} \quad (\delta > 0)$$

for any value of θ.

Suppose that, for a value α of θ $(-\frac{1}{2}\pi < \alpha < \frac{1}{2}\pi)$ and a positive value of δ,

(2.32) $$|F(re^{i\alpha})| = O\{e^{r(\cos\alpha - \delta)}\}.$$

Let $$G(z) = e^{-z(1-\delta\sec\alpha)} \int_{1-\delta\sec\alpha}^{1} e^{zt} f(t)\, dt,$$

when we suppose, as obviously we may, that $\delta < 2\cos\alpha$. Then it follows from Lemma 2.2 that $G(z)$ is bounded for $R(z) \leqslant 0$. But also, for $\theta = \alpha$,

$$\int_{1-\delta\sec\alpha}^{1} e^{zt} f(t)\, dt = F(z) - \int_{-1}^{1-\delta\sec\alpha} e^{zt} f(t)\, dt$$

$$= O\{e^{r(\cos\alpha-\delta)}\} + O\{e^{r\cos\alpha(1-\delta\sec\alpha)}\}$$

$$= O\{e^{r(\cos\alpha-\delta)}\},$$

so that $G(z)$ is bounded for $\theta = \alpha$. Hence, by the theorem of Phragmén and Lindelöf, $G(z)$ is bounded everywhere, and so reduces to a constant, which is zero, since $G(-x) \to 0$. Hence, by Lemma 2.1, $f(t) = 0$ almost everywhere in the interval $(1-\delta\sec\alpha < t < 1)$, contrary to hypothesis.

The theorem may be proved in a similar way if $\frac{1}{2}\pi < \alpha < \frac{3}{2}\pi$. The proof fails if $\alpha = \pm\frac{1}{2}\pi$. But if

(2.33) $$F(iy) = O(e^{-\delta y}),$$

then the function $$e^{-(1+i\delta)z} F(z)$$

is bounded for $z = x$, $z = iy$, and so for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. Hence

$$|F(re^{i\theta})| = O\{e^{r(\cos\theta - \delta\sin\theta)}\} \quad (0 < \theta < \tfrac{1}{2}\pi),$$

which contradicts what has been proved already.

3.1. *Proof of Theorem I.*—If the series (1.21) were convergent, we should have

$$F(z) = F(0)\, e^{(a+i\beta)z} \prod_1^\infty \left(1 - \frac{z}{z_n}\right) = e^{(a+i\beta)z} G(z),$$

say, where, for every positive ϵ,

$$|G(z)| = O(e^{\epsilon|z|}).$$

But by Lemma 2.3

$$|G(x)| = e^{-ax}|F(x)| \neq O\{e^{(1-a-\delta)x}\},$$

$$|G(-x)| = e^{ax}|F(-x)| \neq O\{e^{(1+a-\delta)x}\}.$$

Hence $$1 - a - \delta \leqslant \epsilon, \quad 1 + a - \delta \leqslant \epsilon,$$

for all positive values of δ and ϵ; and this is impossible, whatever a is.

3.2. *Proof of Theorem II.*—If a function $\phi(z)$ is regular and bounded for $R(z) \geqslant 0$, and has zeros at $r_1 e^{i\theta_1}$, $r_2 e^{i\theta_2}$, ..., in the right half-plane, then

$$\sum \frac{\cos\theta_n}{r_n}$$

is convergent. A simple direct proof of this is given by Pólya and Szegö*. It also follows from the formula of Carleman. For we have

$$\sum_{l < r_n < R} \left(\frac{1}{r_n} - \frac{r_n}{R^2}\right) \cos\theta_n < K,$$

and *a fortiori*

$$\sum_{l < r_n < \frac{1}{2}R} \frac{\cos\theta_n}{r_n} < \tfrac{4}{3} \sum_{l < r_n < \frac{1}{2}R} \left(\frac{1}{r_n} - \frac{r_n}{R^2}\right) \cos\theta_n < \tfrac{4}{3}K,$$

which gives the result.

Now, by Lemma 2.2, $e^{-z}F(z)$ is bounded for $R(z) \geqslant 0$, and $e^{z}F(z)$ is bounded for $R(z) \leqslant 0$. So Theorem II follows.

* G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis I*, Berlin (1924), 142, problem 298.

3.3. *Proof of Theorem III.*—Let $0 < \theta < 1$, $0 < \delta < \theta(1-\theta)$, and

$$f(t) = (-1)^n \mu_n \quad (\theta^n - \delta^n \leqslant t \leqslant \theta^n),$$

for $n = 1, 2, \ldots$, and $f(t) = 0$ elsewhere. The numbers μ_n are supposed positive and such that the series

$$\Sigma \mu_n \delta^n$$

is convergent, so that $f(t)$ is integrable. Then

$$F(-x) = \sum_{\nu=1}^{\infty} (-1)^\nu \mu_\nu \int_{\theta^\nu - \delta^\nu}^{\theta^\nu} e^{-xt}\,dt.$$

Now
$$\left| \sum_{\nu=1}^{n-1} (-1)^\nu \mu_\nu \int_{\theta^\nu-\delta^\nu}^{\theta^\nu} e^{-xt}\,dt \right| < e^{-x(\theta^{n-1}-\delta^{n-1})} \sum_{\nu=1}^{n-1} \mu_\nu \delta^\nu,$$

$$\left| \sum_{\nu=n+1}^{\infty} (-1)^\nu \mu_\nu \int_{\theta^\nu-\delta^\nu}^{\theta^\nu} e^{-xt}\,dt \right| < \sum_{\nu=n+1}^{\infty} \mu_\nu \delta^\nu,$$

$$\int_{\theta^n-\delta^n}^{\theta^n} e^{-xt}\,dt > \delta^n e^{-x\theta^n}.$$

Hence
$$(-1)^n F(x) > \mu_n \delta^n e^{-x\theta^n} - e^{-x(\theta^{n-1}-\delta^{n-1})} \sum_{\nu=1}^{n-1} \mu_\nu \delta^\nu - \sum_{\nu=n+1}^{\infty} \mu_\nu \delta^\nu.$$

Taking $x = a^n$, $\mu_n = \delta^{-n} e^{-c^n}$ $(c > 1)$, we have

$$(-1)^n F(-a^n) > e^{-c^n - a^n\theta^n} - e^{-a^n(\theta^{n-1}-\delta^{n-1})} \sum_{\nu=1}^{n-1} e^{-c^\nu} - \sum_{\nu=n+1}^{\infty} e^{-c^\nu}$$

$$> e^{-c^n - a^n\theta^n} - A e^{-a^n\theta^{n-1}} - A e^{-c^{n+1}}$$

provided that $\delta < 1/a$. Let $c = a\theta$. Then

$$(-1)^n F(-a^n) > e^{-2a^n\theta^n} - A e^{-a^n\theta^{n-1}} - A e^{-a^{n+1}\theta^{n+1}}$$

$$= e^{-2a^n\theta^n}\{1 - A e^{a^n\theta^n(2-1/\theta)} - A e^{a^n\theta^n(2-a\theta)}\},$$

which is positive for sufficiently large values of n if $\theta < \frac{1}{2}$, $a > 2/\theta$. So with these conditions the function $F(z)$ has a real zero in every interval $(-a^{n+1}, -a^n)$ for $n > n_1$.

4. *Further Lemmas.*

4.1. LEMMA 4.1.—*Let $n(r)$ denote the number of zeros of $F(z)$ for which $|z| \leqslant r$, and let*

$$N(r) = \int_0^r \frac{n(x)}{x}\, dx.$$

Then $$N(r) < 2r/\pi \quad (r > r_0).$$

Jensen's formula is

$$N(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |F(re^{i\theta})|\, d\theta - \log |F(0)|.$$

By Lemma 2.2 $$|F(re^{i\theta})| < \delta e^{r|\cos\theta|} \quad (r > r_0).$$

Hence $$\log |F(re^{i\theta})| < \log \delta + r |\cos\theta|,$$

and $$N(r) < \log\delta + \frac{r}{2\pi} \int_{-\pi}^{\pi} |\cos\theta|\, d\theta - \log |F(0)|$$

$$= 2r/\pi - \log \{|F(0)|/\delta\},$$

so that the result holds if $\delta < |F(0)|$.

Corollary.—$n(r) < Kr$.

For $$n(r) \log k \leqslant \int_r^{kr} \frac{n(x)}{x}\, dx \leqslant N(kr) < Kr.$$

4.2. LEMMA 4.2.—*Let Π_δ denote a product taken over terms involving zeros for which $|\cos\theta_n| < \delta$. Then, given $\delta > 0$, $\epsilon > 0$, we have*

$$\Pi_\delta \left| 1 - \frac{x}{r_n} e^{-i\theta_n} \right|^2 > e^{-(A\delta^2+\epsilon)x} \Pi_\delta \left(1 + \frac{x^2}{r_n^2}\right)$$

for $x > x_0(\delta, \epsilon)$, where A is an absolute constant.*

We have

$$1 - 2\frac{x}{r_n} \cos\theta_n + \frac{x^2}{r_n^2} \geqslant \left(1 - 2\frac{x}{r_n} |\cos\theta_n|\right)\left(1 + \frac{x^2}{r_n^2}\right)$$

$$> e^{-4(x/r_n)|\cos\theta_n|} \left(1 + \frac{x^2}{r_n^2}\right),$$

* The following work would be much simpler if all the zeros were purely imaginary—for instance, this lemma would be trivial. We use the fact (expressed by Theorem II) that $\cos\theta_n$, if not always zero, is at any rate in general small.

if $x\,|\cos\theta_n|/r_n < \frac{1}{4}\log 2$, which is certainly so if $r_n > 8x\delta$. Let

$$\Pi_\delta = \prod_{r_n \leqslant 8x\delta} . \prod_{r_n > 8x\delta} = \Pi_1 . \Pi_2.$$

Then

$$\Pi_2\left(1-2\frac{x}{r_n}\cos\theta_n+\frac{x^2}{r_n^2}\right) > \exp\left(-4x\sum_{r_n>8x\delta}\frac{\cos\theta_n}{r_n}\right)\Pi_2\left(1+\frac{x^2}{r_n^2}\right)$$

$$> e^{-\epsilon x}\,\Pi_2\left(1+\frac{x^2}{r_n^2}\right)$$

for $x > x_0(\epsilon, \delta)$; and

$$\Pi_1\left(1-2\frac{x}{r_n}\cos\theta_n+\frac{x^2}{r_n^2}\right)\Big/\Pi_1\left(1+\frac{x^2}{r_n^2}\right) > \Pi_1\left\{\frac{(x-r_n)^2}{x^2+r_n^2}\right\}$$

$$> \Pi_1\left\{\frac{(x-8x\delta)^2}{x^2+8^2x^2\delta^2}\right\}$$

$$= \left\{\frac{(1-8\delta)^2}{1+8^2\delta^2}\right\}^{n(8x\delta)}$$

$$> \left\{\frac{(1-8\delta)^2}{1+8^2\delta^2}\right\}^{Kx\delta}$$

$$= e^{Kx\delta\log[(1-8\delta)^2/(1+8^2\delta^2)]}$$

$$> e^{-Ax\delta^2}.$$

Combining these two inequalities, we have the result stated.

4.3. LEMMA 4.3.—*Let* Π'_δ *denote a product taken over terms involving zeros for which* $|\cos\theta_n| \geqslant \delta$. *Let*

$$\phi(z) = \Pi'_\delta\left(1-\frac{z}{z_n}\right), \quad G(z) = \frac{F(z)}{\phi(z)}.$$

Then as $x \to \infty$,

$$G(x) = O\{e^{(1+\epsilon)x}\}, \quad \neq O\{e^{(1-\epsilon)x}\},$$

for every positive ϵ.

The series $\Sigma'_\delta 1/|z_n|$ is convergent, since no term is greater than the corresponding term of $\Sigma'_\delta|\cos\theta_n|/\delta r_n$. Hence the product for $\phi(z)$ is absolutely convergent, and

$$|\phi(z)| = O(e^{\epsilon|z|})$$

for every positive ϵ. Hence, by the Theorem of Valiron, there is a sequence of circles $|z| = R_1, R_2, \ldots$ ($R_n < k R_{n-1}$) on each of which

$$|\phi(z)| > e^{-\epsilon|z|}.$$

It follows from this and Lemma 2 . 2 that

$$|G(z)| < K e^{(1+\epsilon)R_n}$$

for $|z| = R_n$, and so, since $G(z)$ is an integral function, for $|z| \leqslant R_n$. In particular, if $R_{n-1} < x \leqslant R_n$, then

$$|G(x)| < K e^{(1+\epsilon)R_n} < K e^{(1+\epsilon)kx}$$

and the first result follows, since ϵ and $k-1$ may be as small as we please.

The second part follows at once from Lemma 2 . 3, since

$$F(x) \neq O\{e^{(1-\epsilon)x}\}, \quad \phi(x) = O(e^{\epsilon x}).$$

Similar results, of course, hold for $G(-x)$.

4 . 4. LEMMA 4 . 4.—*If x is real*

$$|F(x)| = |F(0)| \prod_1^\infty \left|1 - \frac{x}{r_n} e^{-i\theta_n}\right|.$$

We have

$$F(z) = F(0)\, e^{(a+i\beta)z} \prod_1^\infty \left(1 - \frac{z}{z_n}\right) e^{z/z_n}, \tag{4.41}$$

so that if x is real

$$|F(x)| = |F(0)|\, e^{ax} \prod_1^\infty \left|1 - \frac{x}{r_n} e^{-i\theta_n}\right| e^{(x\cos\theta_n)/r_n}.$$

But it follows from Theorem II that the product

$$\prod e^{(x\cos\theta_n)/r_n}$$

is convergent. Hence

$$|F(x)| = |F(0)|\, e^{\gamma x} \prod_1^\infty \left|1 - \frac{x}{r_n} e^{-i\theta_n}\right|,$$

where

$$\gamma = a + \sum_1^\infty \frac{\cos\theta_n}{r_n}. \tag{4.42}$$

We have to prove that $\gamma = 0$. Define $G(z)$, as in **Lemma 4.3**. Then, by Lemma 4.2,

$$|G(x)|^2 = |F(0)|^2 e^{2\gamma x} \Pi_\delta \left|1 - \frac{x}{r_n} e^{-i\theta_n}\right|^2$$

(4.43) $$> K e^{(2\gamma - A\delta^2 - \epsilon)x} \Pi_\delta \left(1 + \frac{x^2}{r_n^2}\right).$$

Also

$$|G(-x)|^2 = |F(0)|^2 e^{-2\gamma x} \Pi_\delta \left|1 + \frac{x}{r_n} e^{-i\theta_n}\right|^2$$

$$< K e^{-2\gamma x} \Pi_\delta \left(1 + \frac{x^2}{r_n^2}\right) \Pi_\delta \left(1 + 2\frac{x}{r_n} |\cos\theta_n|\right)$$

(4.44) $$< K e^{(\epsilon - 2\gamma)x} \Pi_\delta \left(1 + \frac{x^2}{r_n^2}\right).$$

Hence

(4.45) $$\left|\frac{G(x)}{G(-x)}\right|^2 > K e^{(4\gamma - A\delta^2 - 2\epsilon)x} \quad [x > x_0(\epsilon, \delta)].$$

Now it follows from Lemma 4.3 that there is a sequence of values of x, tending to infinity, in which

(4.46) $$|G(x)|^2 < e^{\epsilon x} |G(-x)|^2.$$

Comparing (4.44), (4.45), we have

$$4\gamma - A\delta^2 - 2\epsilon \leqslant \epsilon$$

for all positive values of δ and ϵ. Hence $\gamma \leqslant 0$. Applying the same argument to $F(-z)$, we obtain $\gamma \geqslant 0$. Hence $\gamma = 0$.

This proves the lemma. The relation (1.25) also follows. For, equating the coefficients of z on the two sides of (4.41), we obtain

(4.47) $$\alpha + i\beta = F'(0)/F(0),$$

and (1.25), in the case $a = -1$, $b = 1$, follows from (4.42) and (4.47).

4.5. LEMMA 4.5.—*Given $k > 1$, $\epsilon > 0$, there is a sequence of numbers $x_1, x_2, \ldots$ ($x_n < k x_{n-1}$) such that for every n*

$$|F(x_n)| > e^{(1-\epsilon)x_n}.$$

Let $-1<\xi<1$, and let

$$h(t)=1 \quad (-1<t<\xi), \qquad = f(t) \quad (\xi<t<1).$$

Let $$H(z)=\int_{-1}^{1} e^{zt}\, h(t)\, dt.$$

Let $\zeta_1, \zeta_2, \ldots$ be the zeros of $H(z)$, and let

$$\psi(z)=\Pi_\delta'\left(1-\frac{z}{\zeta_n}\right), \quad K(z)=\frac{H(z)}{\psi(z)}.$$

Applying (4 . 45) to $K(z)$, we have, since $\gamma=0$,

$$|K(x)/K(-x)|^2 > Ae^{(-A\delta^2-2\epsilon)x} \quad [x>x_0(\epsilon, \delta)],$$

or $$|H(x)|^2 > Ae^{(-A\delta^2-2\epsilon)x}\,|H(-x)\,\psi(x)/\psi(-x)|^2.$$

Now, as in the case of $\phi(z)$,

$$|\psi(-x)| = O(e^{\epsilon x})$$

and so there is a sequence of numbers $x_1, x_2, \ldots$ $(x_n<kx_{n-1})$ in which

$$|\psi(x)|>e^{-\epsilon x}.$$

Hence for this sequence

$$|H(x)|>Ae^{(-A\delta^2-2\epsilon)x}|H(-x)|.$$

But $$H(x)=F(x)+O(e^{\xi x}),$$

$$H(-x)=e^x/x+O(e^{-\xi x}).$$

Hence $$|F(x)|>Ae^{(1-A\delta^2-2\epsilon)x}/x-O(e^{\xi x})-O(e^{-\xi x})$$

for the given sequence. This proves the lemma.

4 . 6.—LEMMA 4 . 6.—*For every positive* ϵ

$$\prod_1^\infty\left(1+\frac{x^2}{r_n^2}\right)>e^{(2-\epsilon)x} \quad [x>x_0(\epsilon)].$$

Applying the previous lemma to $F(-x)$, we obtain a sequence of values x_m of x for which

$$|F(-x_m)|>e^{(1-\epsilon)x_m}.$$

Since $\phi(-x_m)=O(e^{\epsilon x_m})$, it follows that

$$|G(-x_m)|>e^{(1-2\epsilon)x_m}.$$

Hence, by (4 . 44), $$\Pi_\delta\left(1+\frac{x_m^2}{r_n^2}\right) > e^{(2-5\epsilon)x_m} \quad [x > x_0(\epsilon)].$$

Hence, if $x_m \leqslant x < x_{m+1}$,

$$\prod_1^\infty\left(1+\frac{x^2}{r_n^2}\right) \geqslant \prod_1^\infty\left(1+\frac{x_m^2}{r_n^2}\right) \geqslant \Pi_\delta\left(1+\frac{x_m^2}{r_n^2}\right)$$

$$> e^{(2-5\epsilon)x_m} > e^{(2-5\epsilon)x/k},$$

and the result follows, since $k-1$ and ϵ may be as small as we please.

4 . 7. LEMMA 4 . 7.—$N(r) \sim 2r/\pi$.

In view of Lemma 4 . 1, we have only to prove that

(4 . 71) $$N(r) > (2/\pi-\epsilon)\, r \quad (r > r_0).$$

By Lemma 4 . 6

(4 . 72) $$\sum_1^\infty \log\left(1+\frac{x^2}{r_n^2}\right) > (2-\epsilon)x \quad (x > x_0).$$

Now $$\sum_1^\infty \log\left(1+\frac{x^2}{r_n^2}\right) = \sum_1^\infty n\left\{\log\left(1+\frac{x^2}{r_n^2}\right)-\log\left(1+\frac{x^2}{r_{n+1}^2}\right)\right\}$$

$$= \sum_1^\infty n\int_{r_n}^{r_{n+1}} \frac{2x^2\,dr}{r(x^2+r^2)}$$

$$= 2x^2\int_0^\infty \frac{n(r)\,dr}{r(x^2+r^2)}$$

(4 . 73) $$= 4x^2\int_0^\infty \frac{r\,N(r)}{(x^2+r^2)^2}\,dr,$$

the last form being obtained by a partial integration. If the lemma is not true, there is a number λ less than unity such that

$$N(r) < 2\lambda r/\pi$$

for indefinitely great values of r, say $r = R_1,\ R_2,\ \ldots$. Since $N(r)$ is a non-decreasing function of r, we have

$$\int_{\lambda R_n}^{R_n} \frac{r\,N(r)}{(x^2+r^2)^2}\,dr < \frac{2\lambda R_n}{\pi}\int_{\lambda R_n}^{R_n}\frac{r\,dr}{(x^2+r^2)^2}.$$

From this and Lemma 4 . 1 it follows that

$$\int_0^\infty \frac{rN(r)}{(x^2+r^2)^2}\,dr < \frac{2}{\pi}\int_0^\infty \frac{r^2\,dr}{(x^2+r^2)^2} - \frac{2}{\pi}\int_{\lambda R_n}^{R_n} \frac{(r-\lambda R_n)\,r\,dr}{(x^2+r^2)^2} + \frac{K}{x^4}$$

$$= \frac{1}{2x} - \frac{2}{\pi}\int_{\lambda R_n}^{R_n} \frac{(r-\lambda R_n)\,r\,dr}{(x^2+r^2)^2} + \frac{K}{x^4}.$$

Hence by (4 . 72), (4 . 73),

$$\frac{2}{\pi}\int_{\lambda R_n}^{R_n} \frac{(r-\lambda R_n)\,r\,dr}{(x^2+r^2)^2} < \frac{\epsilon}{4x} + \frac{K}{x^4} < \frac{\epsilon}{2x} \quad (x > x_1).$$

Taking $x = R_n$, this gives

$$\frac{2}{\pi R_n}\int_\lambda^1 \frac{(u-\lambda)\,u\,du}{(1+u^2)^2} < \frac{\epsilon}{2R_n},$$

which is impossible if ϵ is sufficiently small. This proves the lemma.

4 . 8. LEMMA 4 . 8*.—*If $N(r)/r$ tends to a limit, then $n(r)/r$ tends to the same limit.*

Suppose that $N(r)/r \to \kappa$. Then

$$(\kappa-\epsilon)\,r < N(r) < (\kappa+\epsilon)\,r \quad (r > r_0).$$

Hence
$$\int_r^{r(1+\delta)} \frac{n(x)}{x}\,dx = N\{r(1+\delta)\} - N(r)$$

$$< (\kappa+\epsilon)(1+\delta)\,r - (\kappa-\epsilon)\,r$$

$$= \kappa\delta r + \epsilon(2+\delta)\,r.$$

But
$$\int_r^{r(1+\delta)} \frac{n(x)}{x}\,dx \geqslant n(r)\int_r^{r(1+\delta)} \frac{dx}{x} > \frac{\delta n(r)}{1+\delta}.$$

Hence
$$n(r) < \kappa(1+\delta)\,r + \epsilon(1+\delta)(2+\delta)\,r/\delta.$$

Taking *e.g.* $\delta = \sqrt{\epsilon}$, it follows that

$$\overline{\lim}\, n(r)/r \leqslant \kappa.$$

* This is a form of a well known theorem of Landau. See E. Landau, *Rend. di Palermo*, 26 (1908), 218.

Similarly $$n(r) > \kappa(1-\delta)r-\epsilon(1-\delta)(2-\delta)r/\delta,$$

so that $$\underline{\lim}\, n(r)/r \geqslant \kappa.$$

4 . 9. *Proof of Theorem IV and* (1 . 27).—Theorem IV follows at once from Lemmas 4 . 7 and 4 . 8. Also it is clear from Lemma 2 . 2 that

$$\log M(r) < r \quad (r > r_0). \tag{4.91}$$

Let $$\mu_1(r) = \operatorname*{Max}_{0 \leqslant x \leqslant r} |F(x)|, \quad \mu_2(r) = \operatorname*{Max}_{0 \leqslant x \leqslant r} |F(-x)|.$$

By Lemma 4 . 5 $$\mu_1(r) > e^{(1-\epsilon)r} \quad (r = x_1, x_2, \ldots),$$

where $x_n < kx_{n-1}$. Hence, if $x_n \leqslant r < x_{n+1}$,

$$\mu_1(r) \geqslant \mu_1(x_n) > e^{(1-\epsilon)x_n} > e^{(1-\epsilon)r/k},$$

so that

$$\log \mu_1(r) > (1-\epsilon)r \quad (r > r_0). \tag{4.92}$$

Similarly $$\log \mu_2(r) > (1-\epsilon)r \quad (r > r_0).$$

The general result (1 . 27) easily follows from these equations.

5. *Proof of Theorem V.*—Applying Carleman's formula to $F(iz)$, $F(-iz)$, and making $l \to 0$, we have*

$$\sum_{r_n < R} \left(\frac{1}{r_n} - \frac{r_n}{R^2}\right) \sin\theta_n = \frac{1}{\pi R}\int_{-\pi}^{\pi} \log|F(Re^{i\theta})| \sin\theta\, d\theta + I\left\{\frac{F'(0)}{F(0)}\right\}. \tag{5.1}$$

Now it follows from Jensen's formula and Lemma 4 . 7 that

$$\lim_{R\to\infty} \frac{1}{R}\int_{-\pi}^{\pi} \log|F(Re^{i\theta})|\, d\theta = 4,$$

or $$\lim_{R\to\infty} \int_{-\pi}^{\pi} \left\{|\cos\theta| - \frac{1}{R}\log|F(Re^{i\theta})|\right\} d\theta = 0.$$

But the integrand is positive if R is sufficiently large, by Lemma 2 . 2.

* See Pólya and Szegö, *loc. cit.*, 1, 310 (solution 240).

Hence we have also

$$\lim_{R\to\infty}\int_{-\pi}^{\pi}\left\{|\cos\theta|-\frac{1}{R}\log|F(Re^{i\theta})|\right\}\sin\theta\,d\theta=0,$$

or

(5 . 2) $$\lim_{R\to\infty}\frac{1}{R}\int_{-\pi}^{\pi}\log|F(Re^{i\theta})|\sin\theta\,d\theta=0.$$

Hence*

(5 . 3) $$\lim_{R\to\infty}\sum_{r_n<R}\left(\frac{1}{r_n}-\frac{r_n}{R^2}\right)\sin\theta_n=I\left\{\frac{F'(0)}{F(0)}\right\}.$$

Put $R=\frac{1}{2}\pi m$, where m is an integer. Then

(5 . 31) $$\lim_{m\to\infty}\sum_{n<m}\frac{\sin\theta_n}{r_n}\left(1-\frac{4r_n^2}{\pi^2m^2}\right)=I\left(\frac{F'(0)}{F(0)}\right),$$

for the number of terms which occur in one of these sums but not in the other is

$$|m-n(\tfrac{1}{2}\pi m)|=o(m)$$

and the modulus of each of these terms is $O(1/m)$. Let

$$\sigma_m=\sum_{n<m}\frac{\sin\theta_n}{r_n}\left(1-\frac{4r_n^2}{\pi^2m^2}\right).$$

Then

$$(m+1)^2\sigma_{m+1}-m^2\sigma_m=\sum_{n<m}\frac{\sin\theta_n}{r_n}(2m+1)+\frac{\sin\theta_m}{r_m}\left\{(m+1)^2-\frac{4r_m^2}{\pi^2}\right\},$$

so that $$\sum_{n<m}\frac{\sin\theta_n}{r_n}=\frac{(m+1)^2\sigma_{m+1}-m^2\sigma_m}{2m+1}+o(1).$$

Hence

$$\frac{1}{p}\sum_{m=1}^{p}\sum_{n<m}\frac{\sin\theta_n}{r_n}=\frac{1}{p}\sum_{m=1}^{p}\frac{(m+1)^2\sigma_{m+1}-m^2\sigma_m}{2m+1}+o(1)$$

$$=\frac{1}{p}\sum_{m=1}^{p}\frac{2m^2}{4m^2-1}\sigma_m+\frac{(p+1)^2}{p(2p+1)}\sigma_{p+1}+o(1),$$

which, when $p\to\infty$, tends to the same limit as σ_p. Hence the series (1 . 23) is summable $(C,1)$ to the sum given by (1 . 26); and since its

* This limit is a "Rieszian mean", which we reduce to an ordinary arithmetic mean.

n-th term is $O(1/n)$, it must actually be convergent. This proves Theorem V and (1 . 26).

The result is, of course, obvious if $f(t)$ is real, since then the zeros of $F(z)$ are conjugate complex numbers, and the terms of the series (1 . 23) cancel in pairs. It may be of interest to verify (1 . 26) in a less trivial case. Let

$$f(t) = 1 \quad (-1 < t \leqslant 0), \qquad i \quad (0 < t < 1).$$

Then
$$F(z) = (e^z - 1)(e^{-z} + i)/z.$$

The zeros of $F(z)$ are at

$$\pm 2i\pi,\ \pm 4i\pi,\ \ldots \quad \text{and} \quad i\pi/2,\ -3i\pi/2,\ 5i\pi/2,\ -7i\pi/2,\ \ldots.$$

Hence
$$\sum_1^\infty \frac{\sin\theta_n}{r_n} = \frac{2}{\pi}\,(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\ldots) = \tfrac{1}{2}.$$

Also $F(0) = i+1$, $F'(0) = (i-1)/2$, so that $F'(0)/F(0) = \tfrac{1}{2}i$.

6. *Proof of Theorem VI.*—In view of Theorems II and V, we may write the Weierstrassian product (4 . 41) in the form

$$F(z) = F(0)\, e^{(\gamma+i\delta)z} \prod_1^\infty \left(1 - \frac{z}{z_n}\right),$$

where
$$\gamma = \alpha + \sum_1^\infty \frac{\cos\theta_n}{r_n}, \qquad \delta = \beta - \sum_1^\infty \frac{\sin\theta_n}{r_n}.$$

We have already proved that $\gamma = 0$, and the fact that $\delta = 0$ follows from (1 . 26) and (4 . 47).

7. *Proof of Theorem VII.*—Let

$$f(t) = \phi(t) \quad (0 < t < \kappa), \qquad 1 \quad (\kappa < t < \nu),$$

$$g(t) = \psi(t) \quad (0 < t < \kappa), \qquad 1 \quad (\kappa < t < \nu),$$

$$F(z) = \int_0^\nu e^{zt} f(t)\, dt, \qquad G(z) = \int_0^\nu e^{zt} g(t)\, dt.$$

Then
$$F(z)\, G(z) = \int_0^\nu \int_0^\nu e^{z(s+t)} f(s)\, g(t)\, ds\, dt,$$

and putting $s = \frac{1}{2}(u+v)$, $t = \frac{1}{2}(u-v)$,

$$F(z)\,G(z) = \tfrac{1}{2}\iint e^{zu} f(\tfrac{1}{2}u+\tfrac{1}{2}v)\,g(\tfrac{1}{2}u-\tfrac{1}{2}v)\,du\,dv$$

$$= \int_0^{2\nu} e^{zu}\,h(u)\,du,$$

where

$$h(u) = \tfrac{1}{2}\int_{-u}^{u} f(\tfrac{1}{2}u+\tfrac{1}{2}v)\,g(\tfrac{1}{2}u-\tfrac{1}{2}v)\,dv \quad (0 < u < \nu)$$

$$= \tfrac{1}{2}\int_{u-2\nu}^{2\nu-u} f(\tfrac{1}{2}u+\tfrac{1}{2}v)\,g(\tfrac{1}{2}u-\tfrac{1}{2}v)\,dv \quad (\nu < u < 2\nu).$$

In particular, for $0 < u < \kappa$ (putting $v = 2t-u$),

$$h(u) = \int_0^u f(t)\,g(u-t)\,dt = \int_0^u \phi(t)\,\psi(u-t)\,dt,$$

which is zero for $0 < u < \kappa$, by hypothesis. So applying Theorem IV to $F(z)\ G(z)$, and denoting by $n(r)$ the number of its zeros for $|z| \leqslant r$, we obtain

$$n(r) \sim (2\nu-\rho)\,r/\pi \quad (\rho \geqslant \kappa).$$

Let $n_1(r)$, $n_2(r)$ refer to $F(z)$, $G(z)$, respectively, and suppose that $\phi(t) = 0$ $(0 < t < \lambda)$, $\psi(t) = 0$ $(0 < t < \mu)$. Then

$$n_1(r) \sim (\nu-\lambda)\,r/\pi, \quad n_2(r) \sim (\nu-\mu)\,r/\pi.$$

But

$$n_1(r)+n_2(r) = n(r).$$

Hence $\lambda+\mu = \rho$, and the result follows.