Complex analysis chapter 1.

- Think of $\mathbb{C}$ as a vector space over the real numbers.

- The sequence of complex numbers $\{i^n\}_{n=0}^{\infty}$ is periodic with period 4.

$$i^n = \begin{cases} 1 & \text{if } n = 4k, \\ i & \text{if } n = 4k+1, \\ -1 & \text{if } n = 4k+2, \\ -i & \text{if } n = 4k+3. \end{cases}$$

Similarly $\frac{1}{i^n}$ is also periodic with period 4, $(1, -i, -1, i)$

- For a positive real number r, we call $i\sqrt{r}$ the principle value of the square root of $-r$. The second root of $-r$ is then $-i\sqrt{r}$.

- real numbers : points on a line

complex numbers : points ~~on a~~ in the plane

- $(-1) \times z \to$ reflect about the origin.

$\bar{z}$ ~~and z are~~ reflection of z in real axis

$|z_1 - z_2| \to$ distance between the points z_1 and z_2

- Absolute value identities :

Let $z, z_1, z_2, \ldots$ be complex numbers. We have

$$|z| = \sqrt{z\bar{z}} \quad \text{or} \quad |z|^2 = z\bar{z}.$$

Furthermore, we have

$$z = |\bar{z}|$$

$$|z_1 z_2| = |z_1|\,|z_2|.$$

$$|z_1 z_2 \cdots z_n| = |z_1|\,|z_2| \cdots |z_n|.$$

$$|z^n| = |z|^n \quad (n = 1, 2, \ldots).$$

Moreover, a quotient satisfies :

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \quad (z_2 \neq 0).$$

- Absolute value inequalities :

~~Let z, z~~ $|\operatorname{Re} z| \leq |z|$, $|\operatorname{Im} z| \leq |z|$;

$$|z| \leq |\operatorname{Re} z| + |\operatorname{Im} z|.$$

$|z_1 + z_2| \leq |z_1| + |z_2| \to$ triangle inequality.

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$$

$$|z_1 - z_2| \leq |z_1| + |z_2|.$$

$|z_1 + z_2| \geq ||z_1| - |z_2||$, and $|z_1 - z_2| \geq ||z_1| - |z_2||$.

~~$|z_1 + z_2| \geq ||z_1| - |z_2||$~~

Proof : geometric proof uses the fact that 两边和大于第三边.

• Polar Form: $z = r(\cos\theta + i\sin\theta)$.

where $r = \sqrt{x^2+y^2} > 0$,

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}, \quad \sin\theta = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r} \quad (r \neq 0).$$

• principle value of ~~t~~ the argumene of a complex nuber $z = x + iy$ is $\operatorname{Arg} z$ with the properties:

$$-\pi < \operatorname{Arg} z \leq \pi, \quad \cos(\operatorname{Arg} z) = \frac{x}{|z|}, \quad \sin(\operatorname{Arg} z) = \frac{y}{|z|}.$$

The set of all values of the argument is denoted by

$$\arg z = \{\operatorname{Arg} z + 2k\pi : k = 0, \pm 1, \pm 2, \dots\}.$$

Un like $\operatorname{Arg} z$, which is single-valued, $\arg z$ is multivalued or a set-valued function. For a set S and a number c we write

$$c + S = \{c + s : s \in S\},$$

and

$$cS = \{cs : s \in S\}.$$

• polar form of the product:

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].$$

• polar form of the quotient:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)).$$

• De Moivre's identity:

$$(\cos\theta + i\sin\theta)^n = \cancel{\cos\theta}\ \cos n\theta + i\sin n\theta.$$

(proof is using induction).

• Roots of complex numbers:

Let $w \neq 0$ be a complex number & n a positive integer.

A number z is an n^{th} root of w if ~~to~~ $z^n = w$.

(De Moive's identity worked backwards)

- let $w = \rho(\cos\phi + i\sin\phi) \neq 0$. The nth roots of w are the solutions of the equation $z^n = w$. These are

$$z_{k+1} = \rho^{1/n}\left[\cos\left(\frac{\phi}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\phi}{n} + \frac{2k\pi}{n}\right)\right],$$

$k = 0, 1, \ldots, n-1$.

- The unique number z s.t. $z^n = w$ and $\operatorname{Arg} z = \frac{\operatorname{Arg} w}{n}$ is the principal n^{th} root of w.

~~The~~ taking $\phi = \operatorname{Arg} w$ and $k = 0$.