THE ZEROS OF CERTAIN INTEGRAL FUNCTIONS*

J. D. Tamarkin†.

In a very interesting paper which appeared recently in the *Proceedings*‡, Titchmarsh has pointed out a strict analogy existing between the equation

$$\int_a^b e^{zt} f(t)\, dt = 0$$

and the equation

$$P_1(z)\, e^{a_1 z} + \ldots + P_m(z)\, e^{a_m z} = 0, \tag{1}$$

where $P_1(z), \ldots, P_m(z)$ are polynomials. In connexion with equation (1) Titchmarsh mentions a paper by Pólya§. However, the principal results of Pólya (and indeed results in some respects more general) were obtained independently by myself|| and Wilder¶, in connexion with some general problems of the expansion theory for linear differential equations. My paper is not easily available, being published in Russia and in Russian; and this is the reason why I now take the opportunity of indicating briefly the method therein used, which may be of some interest to those working in this field. The equation in question is of the form

$$[M_1]\, e^{m_1 z} + \ldots + [M_\sigma]\, e^{m_\sigma z} = 0, \tag{2}$$

where $$[M_1], \ldots, [M_\sigma]$$

are functions which can be expanded in descending powers of z, so that

$$M_i(z) = M_i + \frac{M_i^{(1)}}{z} + \ldots + \frac{M_i^{(\nu)}}{z^\nu} + \ldots,$$

the series being convergent outside a circle $|z| = R_0$, or only asymptotic on a sector

$$\text{(C)} \qquad \alpha_0 \leqslant \arg z \leqslant \beta_0.$$

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* Extract from a letter addressed to Prof. G. H. Hardy.

† Received 14 December, 1926; read 13 January, 1927.

‡ "The zeros of certain integral functions", *Proc. London Math. Soc.* (2), 25 (1926), 283.

§ "Geometrisches über die Verteilungen der Nullstellen gewisser transzendenter Funktionen", *Münchener Sitzungsberichte* (1920), 285–290.

|| "General problems of the theory of ordinary linear differential equations and expansion of an arbitrary function in series", Petrograd (1917), ch. IV.

¶ "Expansion problems of ordinary linear differential equations with auxiliary conditions at more than two points", *Trans. Amer. Math. Soc.*, 18 (1917), 415–442.

Without loss of generality the discussion may be confined to the case where all the exponents $m_1, m_2, \ldots, m_\sigma$ are real and $m_1 < m_2 < \ldots < m_\sigma$, the general case presenting merely a combination of these special ones. The most interesting case, from the point of view of the expansion problem, is that in which

$$M_1 \neq 0, \quad M_\sigma \neq 0, \tag{3}$$

and this restriction was made in my paper*. Some modifications are necessary in order to adapt the method to the general case where (3) is not satisfied. On the other hand, condition (3) makes it possible to obtain more precise information as to the distribution of the roots of (2) than was obtained by Pólya.

Putting $z = \xi + i\eta$, it is easy to establish the existence of a positive constant h such that all the roots of (2) which are in the sector (C) are situated in the strip

$$\text{(D)} \qquad -h \leqslant \xi \leqslant h.$$

The central point of the method is now to prove that there exist two positive constants A_0 and q, and a sequence of positive numbers

$$\eta_1, \eta_2, \ldots, \eta_\nu, \ldots \quad \big((\nu - 1)\, q < \eta_\nu < \nu q\big)$$

such that

$$|H(z)| \geqslant A_0 \quad \big(H(z) \equiv M_1 e^{m_1 z} + \ldots + M_\sigma e^{m_\sigma z}\big) \tag{4}$$

on the segment (η_ν) of the line $\eta = \eta_\nu$ cut off by the strip (D). If this assertion is true, then an easy discussion of the integral

$$\int d\Phi \quad \big(\Phi = \arg\{[M_1]\, e^{m_1 z} + \ldots + [M_\sigma]\, e^{m_\sigma z}\}\big),$$

taken over a rectangle

$$\xi = \pm h, \quad \eta = \eta_1, \quad \eta = \eta_\nu,$$

will lead to a proof of the asymptotic formula

$$N(r) = r\left\{\frac{m_\sigma - m_1}{2\pi} + O\left(\frac{1}{r}\right)\right\},$$

where $N(r)$ denotes the number of roots of (2) which do not exceed r in absolute value. A simple geometric consideration will then show that

$$r_n = n\left\{\frac{2\pi}{m_\sigma - m_1} + O\left(\frac{1}{n}\right)\right\}, \tag{5}$$

* Wilder has considered the more general case where $M_1 = M_\sigma = 0$, but $M_2 \neq 0$, $M_{\sigma-1} \neq 0$.

if $r_1 \leqslant r_2 \leqslant \ldots \leqslant r_n \leqslant \ldots$ are the absolute values of the roots of (2), arranged in increasing order.

The proof of inequality (4) is based upon a well known lemma of Kronecker and Bohl, which may be stated as follows.

Let $a_1, a_2, \ldots, a_k$ be a set of real numbers such that $1/a_1, \ldots, 1/a_k$ are linearly independent, in other words such that no relation exists of the form

$$\frac{n_1}{a_1}+\ldots+\frac{n_k}{a_k}=0,$$

$n_1, \ldots, n_k$ being integers, not all zero. Let $\beta_1, \ldots, \beta_k$ be any real numbers whatever. Then, if a positive number ϵ_0 is preassigned, it is always possible to find a positive constant q, depending on $a_1, \ldots, a_k, \epsilon_0$ only, such that any interval of length q contains an interval of length ϵ_0 which, in its turn, contains at least one point of each of the k point-sets

$$x_\nu^{(i)}=\beta_i+\nu a_i \quad (i=1, 2, \ldots, k;\ \nu=0, \pm 1, \pm 2 \ \ldots).$$

The exponents $m_1, \ldots, m_\sigma$ may not be linearly independent; but we always can assume the existence of k numbers $\mu_1, \ldots, \mu_k$ which, on the one hand, are linearly independent and, on the other hand, are such that

$$m_i=\sum_{j=1}^{k} l_{ij}\mu_j,$$

where the l_{ij} are integers.

Suppose that $H(z) \neq 0$ on a segment

$$(\eta_0) \qquad \eta=\eta_0, \quad -h \leqslant \xi \leqslant h,$$

which is certainly the case if η_0 is suitably chosen. Then, on (η_0),

$$|H(z)| \geqslant A > 0,$$

where A is a positive constant. We put

$$a_i=2\pi/\mu_i, \quad \beta_i=\eta_0 \quad (i=1, 2, \ldots, k).$$

The expression $H(z)$ is a polynomial in

$$e^{\mu_i \xi}, \quad \cos \mu_i \eta, \quad \sin \mu_i \eta.$$

Hence, if a positive number ϵ is given and $|\xi| \leqslant h$, we always can find a positive number δ such that

$$|H(\xi+i\eta')-H(\xi+i\eta'')|<\epsilon,$$

provided only that

$$|\cos \mu_i \eta'-\cos \mu_i \eta''|<\delta, \quad |\sin \mu_i \eta'-\sin \mu_i \eta''|<\delta \quad (i=1, 2, \ldots, k).$$

Let A_0 be any positive constant less than A. We put $\epsilon = A - A_0$ and take ϵ_0 so small that

$$|\cos \mu_i(\eta_0+\epsilon_i)-\cos \mu_i\eta_0| < \delta, \quad |\sin \mu_i(\eta_0+\epsilon_i)-\sin \mu_i\eta_0| < \delta \quad (|\epsilon_i| \leqslant \epsilon_0).$$

In virtue of the lemma above we can determine such a number q that any interval $(\nu-1)\,q < \eta < \nu q$ contains an interval (i_ν), of length ϵ_0, which contains at least one representative of each of the k point-sets

$$\eta_0+2\pi\nu/\mu_i \quad (i = 1, 2, \ldots, k;\ \nu = 0, \pm 1, \pm 2, \ldots).$$

Let η_ν be any fixed number in (i_ν). It follows immediately from what has been said above that

$$|\cos \mu_i\eta_\nu-\cos \mu_i\eta_0| < \delta, \quad |\sin \mu_i\eta_\nu-\sin \mu_i\eta_0| < \delta.$$

Hence on the segment (η_ν) we have

$$|H(z)| \geqslant |H(\xi+i\eta_0)|-|H(\xi+i\eta_0)-H(\xi+i\eta_\nu)| \geqslant A-\epsilon = A_0,$$

which is (4).

The asymptotic formula (5) for the absolute values of the roots of (2) is not contained explicitly in Wilder's paper. However, this is not an adverse criticism of his method, which in some respects is superior to mine. Wilder's starting point is the following lemma, which may present some interest in itself, and the proof of which is quite elementary :

Given $f(z, x_1, \ldots, x_n)$, continuous in all its arguments and analytic in z when $a_i \leqslant x \leqslant b$ and z lies in a closed finite region S of the complex plane, and such that for no set of values of the x's is the number of zeros of $f(z)$ greater than a given constant N; then if, for any set of values of the x's, z is a point at a distance greater than δ from a zero of $f(z)$ and from the boundary of S, we have

$$|f(z_1, x_1, \ldots, x_n)| \geqslant \lambda,$$

where $\lambda = \lambda(\delta)$ is a real positive constant independent of the x's.

Using this lemma it is fairly easy to show that, if small circles of radius δ are described around all the zeros of the function $H(z)$, then on the remaining part (D_δ) of the strip (D)

$$|H(z)| \geqslant A_\delta > 0,$$

where A_δ denotes a positive constant which depends only on δ. This result is somewhat more precise than that obtained in my paper. Analogous conclusions may be obtained also by using the theory of almost periodic functions of Bohr.