Notes on Langer 1973

Zhang Liu

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1 Introduction

An exponential sum is a type of function in the form

$$\Phi(z) = \sum_{j=0}^{n} A_j(z) e^{c_j z}, \tag{1}$$

where $A_j(z)$ and c_j are constants, $c_j \in \mathbb{R}$.

The function (1) can be expressed in a form

$$\Phi(z - z_0) = z \int_{c_0}^{c_n} \phi(t) e^{tz} dt. \tag{2}$$

The integral of the type here involved with less specialized function $\phi(t)$ represents a generalization of certain sums of type (1).

2 Constant Coefficients and Real Commensurable Exponents.

Theorem 1. *If in the exponential sum (1) the coefficients are constants and the exponents are real and commensurable, then the distribution of zeros is given explicitly by the formula:*

$$z = \frac{1}{\alpha}\{2m\pi i + \log \xi_j\},$$
$$(j = 1, 2, \cdots, p_n, m = 0, \pm 1, \pm 2, \cdots).$$

In this distribution the number of zeros which lie between two lines $y = y_1$ and $y = y_2$, is restricted by the relation (7), i.e.,

$$-n + \frac{c_n}{2\pi}(y_2 - y_1) \leqslant n(R) \leqslant n + \frac{c_n}{2\pi}(y_2 - y_1).$$

Theorem 2. *If the coefficients a_j are real and the zeros of the polynomial*

$$P(\xi) = \sum_{j=0}^{n} a_j \xi^j$$

all lie within the unit circle about $\xi = 0$, then the zeros of the corresponding trigonometric sums are all real and simple, where the trigonometric sums are:

$$\Phi_c(z) = \sum_{j=0}^{n} a_j \cos jz$$

$$\Phi_s(z) = \sum_{j=1}^{n} a_j \sin jz.$$

Each of these sumes has precisely $2n$ zeros on the interval $0 \leqslant z < 2\pi$ and the zeros of either sum alternate with those of the other. (By a theorem of Kakya the hypothesis is fulfilled if $0 \leqslant 1_0 < a_1 < \cdots < a_n$.)

3 Constant Coefficients and General Real Exponents.

Under the case of "Constant Coefficients and General Real Exponents," the sum $\Phi(z)$ is expressed by the formula:

$$\Phi(z) = \sum_{j=0}^{n} a_j e^{c_j z}, c_0 = 0. \tag{10}$$

Theorem 3. *If in the exponential sum (10) the coefficients are constants and the exponents are real, then the zeros of the sum all lie within a strip (5), i.e.,*

$$|(| x) < K(z = x + iy),$$

and in any portion of this strip the number of zeros is limited by relation (7), i.e.,

$$-n + \frac{c_n}{2\pi}(y_2 - y_1) \leqslant n(R) \leqslant n + \frac{c_n}{2\pi}(y_2 - y_1).$$

When z is uniformly bounded from the zeros of $\Phi(z)$, then $|\Phi|(z)$ is uniformly bounded from zero.

4 Coefficients Asymptotically Constant.

Under the case of "Coefficients Asymptotically Constant," the form assumed for the sum (1) is:

$$\Phi(z) = \sum_{j=0}^{n} a_{j+\varepsilon(z)} e^{c_j z}, a_0 a_n \neq 0. \tag{14}$$

Theorem 4. *If the function $\Phi(z)$ (or a determination of it) is of the form (14), then in the region $|(|z) > M$ the distribution of zeros of $\Phi(z)$ (or of the branch of $\Phi(z)$ in question) may be described as in Theorem 3. The zeros are asymptotically represented by those of the related sum (15), i.e.,*

$$\Phi_1(z) = \sum_{j=0}^{n} a_j e^{c_j z}.$$

5 Coefficients which are Asymptotically Power Functions.

6 The Values v_j and c_j Proportional.

Theorem 5. *If in the exponential sum (1) the coefficients are of the form (16) with values v_j proportional to the exponents c_j, and all terms are ordinary terms, then the zeros of the sum are asymptotically located within a logarithmic curvilinear strip bounded by curves of the form (18), i.e.,*