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Zeros of Sections of Exponential Sums

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We derive the large n asymptotics of zeros of sections of a generic exponential sum. We divide all the zeros of the nth section of the exponential sum into "genuine zeros," which approach, as $n \to \infty$, the zeros of the exponential sum, and "spurious zeros," which go to infinity as $n \to \infty$. We show that the spurious zeros, after scaling down by the factor of n, approach a "rosette," a finite collection of curves on the complex plane, resembling the rosette. We derive also the large n asymptotics of the "transitional zeros," the intermediate zeros between genuine and spurious ones. Our results give an extension to the classical results of Szegö about the large n asymptotics of zeros of sections of the exponential, sine, and cosine functions.

1 Introduction

We will be interested in this paper in the distribution of zeros of sections of exponential sums. We consider the exponential sum

$$f(z) = \sum_{j=1}^{M} c_j e^{\lambda_j z}, \tag{1.1}$$

where $c_j, \lambda_j \in \mathbb{C}$, and its Taylor series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k. \tag{1.2}$$

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The nth section of $f(z)$ is the finite Taylor series,

$$f_n(z) = \sum_{k=0}^{n} a_k z^k. \tag{1.3}$$

The problem is to find the distribution of zeros of f_n, $f_n(z_k) = 0$, as $n \to \infty$. This problem was posed and solved for $f(z) = e^z$ in the classical paper of Szegö [20]. Szegö proved that as $n \to \infty$, the rescaled zeros

$$\zeta_k = \frac{z_k}{n} \tag{1.4}$$

approach the curve

$$\Gamma = \left\{ \zeta : \left| e^{1-\zeta}\zeta \right| = 1,\ |\zeta| \le 1 \right\}, \tag{1.5}$$

on the complex plane, and the limiting distribution of the zeros on Γ is the measure of the maximal entropy, the preimage of the uniform measure on the circle under the Riemann map. Precise asymptotics of the zeros of sections of e^z and the sections themselves were obtained in the works of Buckholtz [3], Newman and Rivlin [15], Carpenter, Varga, and Waldvogel [4], Pritsker and Varga [17]. The absence of zeros in some parabolic domains on the complex plane was established in the works of Newman and Rivlin [15] and Saff and Varga [18]. For connections of zeros of sections of e^z to the Riemann zeta-function see the works of Conrey and Ghosh [5] and Yildrim [24].

Szegö also found the limiting distribution of the sections zeros for $f(z) = \cos z$ and $f(z) = \sin z$. In this case a part of the zeros of f_n approaches the zeros of f as $n \to \infty$, but there is another part of the zeros, the "spurious zeros," which go to infinity as $n \to \infty$. Szegö proved that as $n \to \infty$ the rescaled spurious zeros approach a limiting curve and have a limiting distribution on this curve. Close results were obtained by Dieudonné [6], by a different method. Detailed asymptotics of the zeros of sections of $\cos z$ and $\sin z$ were obtained in the works of Kappert [10] and Varga and Carpenter [22, 23]. See also the review papers of Varga [21], Ostrovskii [16], and Zemyan [25]. The distribution of zeros of analytic functions is a classical area of complex analysis, and many results concerning the distribution of zeros of analytic functions are discussed in the monograph of Levin [14]. The distribution of sections of analytic functions of the Mittag-Leffler type is studied in the work of Edrei, Saff, and Varga [7].

Our main goal in this work is to obtain asymptotics of zeros of sections of exponential sums. First we discuss, in Section 2, the asymptotics of large zeros of exponential sums themselves. The rest of the paper is devoted to the asymptotics of zeros of the

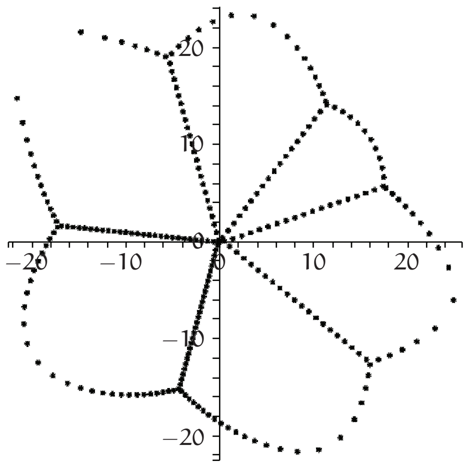

Figure 1.1 The zeros of the $n = 250$ section of exponential sum (1.6).

sections of exponential sums. As an example, let us consider the eight-term exponential sum

$$f(z) = 3e^{(8+2i)z} + (-9+12i)e^{(4+7i)z} + (2+i)e^{(-7+4i)z} - 5e^{(-6-6i)z} + (6-7i)e^{(1-8i)z} + (8-5i)e^{(6-4i)z} + (3-9i)e^{(4+4i)z} + 2ie^{(-2-4i)z}. \tag{1.6}$$

The zeros of the section of this function for $n = 250$ are depicted in Figure 1.1. The zeros form a shape resembling a rosette with six petals. In this paper we obtain the large n asymptotics of the zeros of exponential sums, which provide us with explicit equations for different parts of the rosette.

We divide the zeros of f_n into four classes: (1) finite zeros, (2) zeros of the main series, (3) spurious zeros, and (4) transitional zeros. They are described as follows.

(1) The *finite zeros* are the ones that lie in a finite disk, $D(0, R_0) = \{z \in \mathbb{C} : |z| \le R_0\}$.

(2) The *zeros of the main series* are located in a small neighborhood of the rays, on the intervals $R_0 \le |z| \le nr_c(j, n) - R_1$, $R_1 > 0$, where j is the number of the ray, and $\lim_{n\to\infty} r_c(j, n) = r_c(j) > 0$ is the *critical radius* on the jth ray. We derive the angular coordinate of the jth ray and a transcendental equation, which determines $r_c(j)$ uniquely.

(3) As $n \to \infty$, both the finite zeros and the zeros of the main series converge to the zeros of the exponential sum, $f(z)$. We call them the *genuine zeros* of f_n. In addition to them, there are *spurious zeros* of f_n, which go to infinity as $n \to \infty$. If we scale down the spurious zeros by the factor of n, they approach to some curves $\mathcal{G}_j$. We derive the equations of the curves $\mathcal{G}_j$. As a better approximation to the spurious zeros, we construct

curves $\mathcal{G}_j^n$, which approach $\mathcal{G}_j$ as $n \to \infty$, and such that the scaled down spurious zeros lie in the $O(n^{-2})$-neighborhood of $\mathcal{G}_j^n$.

(4) The *transitional zeros* of f_n are the intermediate ones, located near the triple points on Figure 1.1, where the zeros of the main series and the spurious zeros merge. We derive an equation, which gives the asymptotic location of the transitional zeros. This is determined by zeros of a three-term exponential sum.

We derive the asymptotics of the zeros of f_n, as $n \to \infty$, in Sections 4–9 below. In Appendices A and B, we obtain uniform asymptotics of zeros of the sections of $e^{n\zeta}$, and of the sections themselves, in a fixed neighborhood of the point $\zeta = 1$. These uniform asymptotics are used in the main part of the paper to derive the asymptotics of the spurious zeros of f_n.

We would like to mention here the work of Kuijlaars and McLaughlin [12], where the Riemann-Hilbert approach to distribution of zeros of Laguerre polynomials with nonclassical parameters is developed. The distribution of zeros in [12] has many similarities to the distribution of zeros of exponential sums. Also we would like to mention the work of Bergkvist and Rullgård [2], in which the distribution of zeros of polynomial eigenfunctions of some differential equations of higher order was studied. The distribution of zeros in [2] seems to have similarities to the distribution of zeros of sections of exponential sums as well.

2 Zeros of exponential sums

We consider the exponential sum

$$f(z) = \sum_{j=1}^{M} c_j e^{\lambda_j z}, \quad c_j \neq 0, \tag{2.1}$$

where we assume that the numbers λ_j satisfy the following condition.

Condition 1. The numbers λ_j, $j = 1, \dots, m$, $m \geq 3$, are the vertices of a convex m-gon P_m on the complex plane, and the numbers λ_j, $j = m+1, \dots, M$, lie strictly inside of P_m.

The polygon P_m is the *convex hull* of the numbers λ_j, $j = 1, \dots, m$, on the complex plane, and Condition 1 restricts the remaining numbers λ_j, $j = m+1, \dots, M$, to lie strictly inside of P_m (not on the sides of P_m). For the sake of definiteness, we will assume that the vertices $\lambda_1, \dots, \lambda_m$ are enumerated counterclockwise along P_m. Figure 2.1 shows the convex hull for exponential sum (1.6).

In this section we describe the asymptotics of zeros of $f(z)$ on the complex plane as $|z| \to \infty$. For related results see the paper of Langer [13] and references therein. We

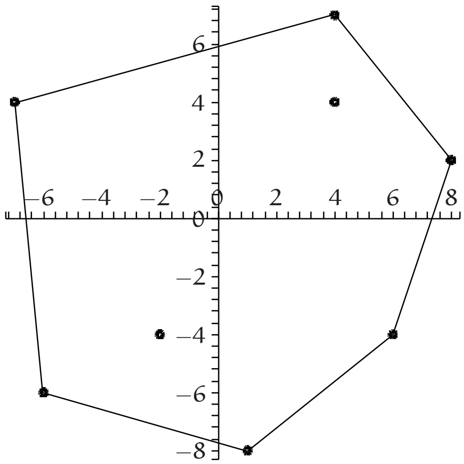

Figure 2.1 The convex hull for exponential sum (1.6), with $\lambda_1 = 8+2i$, $\lambda_2 = 4+7i$, $\lambda_3 = -7+4i$, $\lambda_4 = -6-6i$, $\lambda_5 = 1-8i$, $\lambda_6 = 6-4i$, $\lambda_7 = 4+4i$, $\lambda_8 = -2-4i$.

begin with a description of sectors free of large zeros of f. Define

$$\theta_{jk} = -\arg\left(\lambda_k - \lambda_j\right) + \frac{\pi}{2} \bmod 2\pi. \tag{2.2}$$

Partition the complex plane into the sectors

$$\mathcal{U}_j = \left\{z = re^{i\theta} : \theta_{j,j+1} < \theta < \theta_{j-1,j} \bmod 2\pi,\ r > 0\right\}, \quad j = 1, \ldots, m, \tag{2.3}$$

where we take the convention that

$$\theta_{m,m+1} = \theta_{01} = \theta_{m1}. \tag{2.4}$$

The notation $\theta_{j,j+1} < \theta < \theta_{j-1,j} \bmod 2\pi$ means that θ belongs to the interval from $\theta_{j,j+1}$ to $\theta_{j-1,j}$ on the unit circle in the positive direction. Define also the rays

$$\mathcal{S}_{j,j+1} = \left\{z = re^{i\theta} : \theta = \theta_{j,j+1},\ r \geq 0\right\}, \quad j = 1, \ldots, m, \tag{2.5}$$

so that $\mathcal{U}_j$ is the sector between the rays $\mathcal{S}_{j,j+1}$ and $\mathcal{S}_{j-1,j}$. Observe that the rays $\mathcal{S}_{j,j+1}$ are orthogonal to the sides of the complex conjugate convex hull, $\overline{P_m}$, of the numbers λ_j. Figure 2.2 shows the complex conjugate convex hull and the rays $\mathcal{S}_{j,j+1}$ for exponential sum (1.6).

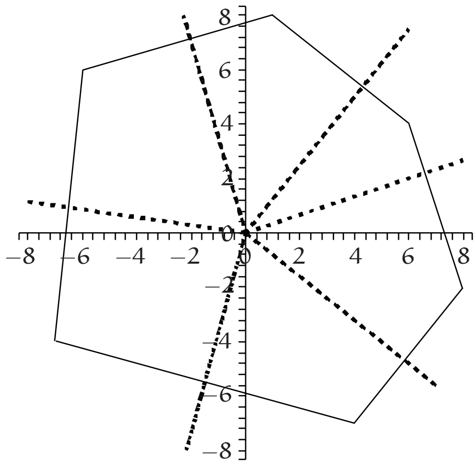

Figure 2.2 The complex conjugate convex hull for exponential sum (1.6), and the corresponding rays $\mathcal{S}_{j,j+1}$, $j = 1, \ldots, 6$.

For a given $j = 1, \ldots, m$, we write

$$\sum_{k=1}^{M} c_k e^{\lambda_k z} = c_j e^{\lambda_j z}\left[1 + \sum_{k:k\neq j} \frac{c_k}{c_j} e^{(\lambda_k-\lambda_j)z}\right]. \tag{2.6}$$

We will describe a region where the sum in the brackets on the right is small and, as a result, $f(z) \neq 0$.

Proposition 2.1. Fix θ in the interval

$$\theta_{j,j+1} < \theta < \theta_{j-1,j} \bmod 2\pi. \tag{2.7}$$

Then for any $k \neq j$,

$$\lim_{r\to\infty} e^{(\lambda_k-\lambda_j)z} = 0, \quad z = re^{i\theta}. \tag{2.8}$$

□

Proof. Since $\lambda_{j-1}, \lambda_j, \lambda_{j+1}$ are the vertices of the convex hull of the numbers λ_k, we have that

$$\arg\left(\lambda_{j+1} - \lambda_j\right) \leq \arg\left(\lambda_k - \lambda_j\right) \leq \arg\left(\lambda_{j-1} - \lambda_j\right) \bmod 2\pi, \tag{2.9}$$

hence

$$-\theta_{j,j+1} + \frac{\pi}{2} \leq \arg\left(\lambda_k - \lambda_j\right) \leq -\theta_{j,j+1} + \frac{3\pi}{2} \bmod 2\pi. \tag{2.10}$$

By adding this inequality and (2.7), we obtain that

$$\frac{\pi}{2} < \arg\left(\lambda_k - \lambda_j\right)z < \frac{3\pi}{2} \bmod 2\pi, \tag{2.11}$$

which implies (2.8). Proposition 2.1 is proved. ■

For the future use, observe that if $k \neq j-1, j, j+1$, then inequality (2.9) is strict and hence there exists $\varepsilon > 0$ such that

$$\frac{\pi}{2} + \varepsilon < \arg\left(\lambda_k - \lambda_j\right)z < \frac{3\pi}{2} - \varepsilon. \tag{2.12}$$

This gives that for some $c > 0$,

$$e^{(\lambda_k-\lambda_j)z} = O\left(e^{-c|z|}\right), \quad |z| \longrightarrow \infty \; k \neq j-1, j, j+1, \tag{2.13}$$

uniformly in the closed sector $\overline{\mathcal{U}_j}$. From (2.3) we obtain that for some $\varepsilon > 0$,

$$\frac{\pi}{2} < \arg\left(\lambda_{j+1} - \lambda_j\right)z < \frac{3\pi}{2} - \varepsilon, \quad z \in \mathcal{U}_j, \tag{2.14}$$

and from (2.5), that

$$\arg\left(\lambda_{j+1} - \lambda_j\right)z = \frac{\pi}{2}, \quad z \in \mathcal{S}_{j,j+1}. \tag{2.15}$$

This gives that for some $c > 0$,

$$e^{(\lambda_{j+1}-\lambda_j)z} = O\left(e^{-c\,d_{j,j+1}(z)}\right), \quad d_{j,j+1}(z) \equiv \operatorname{dist}\left(z, \mathcal{S}_{j,j+1}\right) \longrightarrow \infty; \quad z \in \mathcal{U}_j. \tag{2.16}$$

Similarly,

$$e^{(\lambda_{j-1}-\lambda_j)z} = O\left(e^{-c\,d_{j-1,j}(z)}\right), \quad d_{j-1,j}(z) \longrightarrow \infty; \quad z \in \mathcal{U}_j. \tag{2.17}$$

Estimates (2.13), (2.16), and (2.17) imply that there exist large numbers $r_0, R_0 > 0$ such that for $j = 1, \ldots, m$,

$$\sum_{k:k\neq j} \left|\frac{c_k}{c_j} e^{(\lambda_k-\lambda_j)z}\right| < \frac{1}{2}, \quad z \in \mathcal{U}_j\left(r_0, R_0\right), \tag{2.18}$$

where

$$\mathcal{U}_j\left(r_0, R_0\right) = \left\{z \in \mathcal{U}_j : |z| > R_0,\ \operatorname{dist}\left(z, \mathcal{S}_{j,j+1}\right) > r_0,\ \operatorname{dist}\left(z, \mathcal{S}_{j-1,j}\right) > r_0\right\}. \tag{2.19}$$

We will call $\mathcal{U}_j(r_0, R_0)$ the j*th one-term domination domain*. When $z \in \mathcal{U}_j(r_0, R_0)$, the term $c_j e^{\lambda_j z}$ dominates in $f(z)$ the other terms. Define

$$\mathcal{U}(r_0, R_0) = \bigcup_{j=1}^{m} \mathcal{U}_j(r_0, R_0). \tag{2.20}$$

Define also

$$\mathcal{S}_{j,j+1}(r_0, R_0) = \{z : |z| > R_0,\ \mathrm{dist}\,(z, \mathcal{S}_{j,j+1}) \leq r_0\}, \quad j = 1, \ldots, m. \tag{2.21}$$

We will assume that R_0 is big enough so that

$$\mathcal{S}_{j,j+1}(r_0, R_0) \cap \mathcal{S}_{j-1,j}(r_0, R_0) = \varnothing, \quad j = 1, \ldots, m. \tag{2.22}$$

We will call $\mathcal{S}_{j,j+1}(r_0, R_0)$ the $(j, j+1)$*st two-term domination strip*. Define

$$\mathcal{S}(r_0, R_0) = \bigcup_{j=1}^{m} \mathcal{S}_{j,j+1}(r_0, R_0). \tag{2.23}$$

Proposition 2.2 (absence of zeros of f in the one-term domination domains)**.** There exists $r_0, R_0 > 0$ such that

$$\sum_{k=1}^{M} c_k e^{\lambda_k z} \neq 0, \quad z \in \mathcal{U}(r_0, R_0). \tag{2.24}$$

□

Proof. The proof follows from (2.6) and (2.18). ■

Proposition 2.2 implies that all the large zeros of f are concentrated in the two-term domination strips, $\mathcal{S}_{j,j+1}(r_0, R_0)$. To describe these zeros consider the two-term equation

$$f_0(z) \equiv c_j e^{\lambda_j z} + c_{j+1} e^{\lambda_{j+1} z} = 0. \tag{2.25}$$

By the linear change of variable,

$$u = \frac{(\lambda_{j+1} - \lambda_j)z}{2i} + \frac{1}{2i} \log \frac{c_{j+1}}{c_j}, \tag{2.26}$$

we reduce f_0 to

$$f_0(z) = 2\sqrt{c_j c_{j+1}}\, e^{(\lambda_{j+1} + \lambda_j)z/2} \cos u. \tag{2.27}$$

Therefore, the general solution to (2.25) is $u = \pi/2 + \pi l$, or

$$z = z^0(j, j+1; l) \equiv \alpha_{j,j+1} + l\tau_{j,j+1}, \quad l \in \mathbb{Z}, \tag{2.28}$$

where

$$\alpha_{j,j+1} = \frac{\pi i - \log \dfrac{c_{j+1}}{c_j}}{\lambda_{j+1} - \lambda_j}, \qquad \tau_{j,j+1} = \frac{2\pi i}{\lambda_{j+1} - \lambda_j}. \tag{2.29}$$

Observe that

$$\arg \tau_{j,j+1} = \theta_{j,j+1}. \tag{2.30}$$

Now we can describe the zeros of f in the two-term domination strips. We will use the following general proposition. Denote

$$D\big(z_0, r\big) = \big\{z : \big|z - z_0\big| < r\big\}, \quad r > 0. \tag{2.31}$$

Proposition 2.3. Let $f(z) = f_0(z) + f_1(z)$ where f_0, f_1 are analytic functions in the disk $D(z_0, r)$, $r > 0$. Suppose that

(i) $f_0(z_0) = 0$,

(ii) $|f_0(z)| \geq A|z - z_0|$, for all $z \in D(z_0, r)$, where $A > 0$,

(iii) $|f_1(z)| \leq \varepsilon$, $z \in D(z_0, r)$, $\varepsilon > 0$.

Then if $r_0 \equiv 2\varepsilon/A < r$, then there is a unique simple zero of f in the disk $D(z_0, r_0)$. □

Proof. For $|z - z_0| = r_0$, $|f_0(z)| \geq 2\varepsilon > |f_1(z)|$, hence f has a unique simple zero in $D(z_0, r_0)$ by the Rouché theorem. Proposition 2.3 is proved. ■

With the help of Proposition 2.3 we prove the following result.

Proposition 2.4 (zeros of f in the two-term domination strips)**.** There exist $r_0, R_0 > 0$ such that all zeros z_k of exponential sum (2.1) in $\mathcal{S}_{j,j+1}(r_0, R_0)$ are simple and close to zeros (2.28), so that for some $l = l(k) > 0$,

$$\big|z_k - z^0(j, j+1; l)\big| = O\big(e^{-cl}\big), \quad c > 0, \tag{2.32}$$

and for each $z^0(j, j+1; l) \in \mathcal{S}_{j,j+1}(r_0, R_0)$, there is a zero z_k of f satisfying (2.32). □

Proof. From (2.13) and (2.17) we obtain that if $z \in \mathcal{S}_{j,j+1}(r_0, R_0)$, then for some $c > 0$,

$$e^{(\lambda_k - \lambda_j)z} = O\big(e^{-c|z|}\big), \quad e^{(\lambda_k - \lambda_{j+1})z} = O\big(e^{-c|z|}\big), \quad |z| \longrightarrow \infty;\ k \neq j, j+1. \tag{2.33}$$

Let us write equation $f(z) = 0$ as

$$f_0(z) + f_1(z) = 0, \qquad f_1(z) = \sum_{k \neq j, j+1} c_k e^{\lambda_k z}. \tag{2.34}$$

Then (2.33) implies that if $z \in \mathcal{S}_{j,j+1}(r_0, R_0)$, then

$$e^{-(\lambda_{j+1}+\lambda_j)z/2} f_1(z) = O\left(e^{-c|z|}\right), \quad c > 0, \ |z| \longrightarrow \infty, \tag{2.35}$$

and under transformation (2.26), (2.34) becomes

$$\cos u + g_1(u) = 0, \qquad g_1(u) = O\left(e^{-c_0 \operatorname{Re} u}\right), \quad c_0 > 0; \operatorname{Re} u \longrightarrow \infty. \tag{2.36}$$

Proposition 2.3 implies that for any $a > 0$ there exists $b > 0$ such that all zeros of the latter equation in the region

$$\left\{u : |\operatorname{Im} u| < a, \ \operatorname{Re} u > b\right\} \tag{2.37}$$

are simple and of the form

$$u^0(l) = \frac{\pi}{2} + \pi l + O\left(e^{-c_0 \operatorname{Re} u}\right). \tag{2.38}$$

This implies (2.32). Proposition 2.4 is proved. ■

We will call $z_k \in \mathcal{S}(r_0, R_0)$, the *zeros of the main series*. We summarize the results of this section as follows.

Theorem 2.5 (zeros of the exponential sum)**.** Suppose that the numbers λ_j satisfy Condition 1. Then there exists $r_0, R_0 > 0$ such that all the zeros of f belong to one of the following categories:

(i) $|z_k| \leq R_0$ (finite zeros),
(ii) $z_k \in \mathcal{S}(r_0, R_0)$, described by formula (2.32) (zeros of the main series). □

3 Zeros of sections of exponential sums

Denote by $f_n(z)$ the section of the exponential sum $f(z)$,

$$f_n(z) = \sum_{k=0}^{n} \frac{f^{(k)}(0) z^k}{k!}. \tag{3.1}$$

By (2.1),

$$f_n(z)=\sum_{k=0}^{n} a_k z^k,\quad a_k=\sum_{j=1}^{M}\frac{c_j\lambda_j^k}{k!}. \tag{3.2}$$

Our main goal will be to describe the zeros of the polynomial $f_n(z)$,

$$f_n(z)=0, \tag{3.3}$$

as $n\to\infty$. We expect that as $n\to\infty$ some of the zeros of $f_n(z)$ approach the zeros of $f(z)$. We call them the *genuine zeros* of f_n. We divide the genuine zeros into *finite zeros* and *zeros of the main series*, in accordance with Theorem 2.5. But there is also a family of other zeros, which go to infinity as $n\to\infty$. We call them the *spurious zeros*. In addition, there will be a relatively small number of intermediate zeros. We call them the *transitional zeros*. In the following sections, we will describe all these zeros of f_n.

4 Finite zeros

It will be more convenient for us to consider, instead of (3.3), the equation

$$f_{n-1}(z)=0. \tag{4.1}$$

We rewrite it as

$$f(z)=\sum_{k=n}^{\infty} a_k z^k = a_n z^n\sum_{k=0}^{\infty}\frac{a_{n+k}}{a_n}z^k. \tag{4.2}$$

In addition to Condition 1, we will assume the following condition.

Condition 2. One of $|\lambda_j|$'s, say $|\lambda_1|$, is bigger than the others.

By the change of variables, $\lambda_1 z\to z$, we can reduce λ_1 to 1, so we will assume that

$$\lambda_1=1>|\lambda_j|,\quad j=2,\dots,M. \tag{4.3}$$

Also we can assume that

$$c_1=1. \tag{4.4}$$

In this case, by (3.2), as $n\to\infty$,

$$a_n=\frac{1}{n!}\left(1+O\left(q^n\right)\right),\quad 0<q<1. \tag{4.5}$$

Therefore, (4.2) reads

$$f(z) = \frac{z^n}{n!}\big(1+O\big(q^n\big)\big)\left[1+\sum_{k=1}^{\infty}\frac{z^k\big(1+O\big(q^{n+k}\big)\big)}{(n+1)\cdots(n+k)\big(1+O\big(q^n\big)\big)}\right]. \tag{4.6}$$

By the Stirling formula,

$$n! = \frac{n^n}{e^n}\sqrt{2\pi n}e^{\theta/12n}, \quad 0<\theta<1, \tag{4.7}$$

hence we can rewrite (4.6) as

$$f(z) = \frac{e^n z^n e^{-\theta/12n}}{n^n\sqrt{2\pi n}}\big(1+O\big(q^n\big)\big)\left[1+\sum_{k=1}^{\infty}\frac{z^k\big(1+O\big(q^{n+k}\big)\big)}{(n+1)\cdots(n+k)\big(1+O\big(q^n\big)\big)}\right], \tag{4.8}$$

where the O-terms are independent of z. If z is bounded, $|z|<R_0$, then the right-hand side is $O(e^{-An})$ as $n\to\infty$ for any $A>0$. Hence the zeros, with multiplicities, of f_n are close to those of f. More precisely, the following proposition holds.

Proposition 4.1 (finite zeros of f_{n-1})**.** Let $R_0>0$ be a fixed number such that f has no zeros on the circle $|z|=R_0$. Then for large n, there is a one-to-one correspondence between zeros $z_k\in D(0,R_0)$ of f, and zeros $z_k(n)\in D(0,R_0)$ of f_{n-1} such that

$$z_k(n)-z_k = O\big(e^{-An}\big), \quad n\longrightarrow\infty, \tag{4.9}$$

for any $A>0$. Here any zero of multiplicity p is counted as p zeros. □

5 Zeros of the main series

Consider now zeros of f in the two-term domination strip $S_{j,j+1}(r_0,R_0)$. Let us write f as

$$f(z) = f_0(z)+f_1(z), \quad f_0(z) = c_j e^{\lambda_j z}+c_{j+1}e^{\lambda_{j+1}z}, \tag{5.1}$$

so that f_0 dominates f_1 in $S_{j,j+1}(r_0,R_0)$. With the help of substitution (2.26), we reduce f_0 to form (2.27). In (2.26), (2.27) we choose the branch for $\log(c_{j+1}/c_j)$ and $\sqrt{c_jc_{j+1}}$ as follows: if $c_j=r_je^{i\theta_j}$, $-\pi<\theta_j\le\pi$, $j=1,\dots,m$, then we define

$$\log\frac{c_{j+1}}{c_j} = \ln\frac{r_{j+1}}{r_j}+i\big(\theta_{j+1}-\theta_j\big), \qquad \sqrt{c_jc_{j+1}} = \sqrt{r_jr_{j+1}}e^{i((\theta_j+\theta_{j+1})/2)}. \tag{5.2}$$

Under (2.26), (4.8) reduces to the form

$$\cos u + O\left(e^{-c\,\mathrm{Re}\,u}\right) = \frac{e^n z^n e^{-(\lambda_{j+1}+\lambda_j)z/2} e^{-\theta/12n}}{2\sqrt{c_j c_{j+1}}\, n^n \sqrt{2\pi n}} \left(1 + O\left(q^n\right)\right)$$
$$\times \left[1 + \sum_{k=1}^{\infty} \frac{z^k\left(1 + O\left(q^{n+k}\right)\right)}{(n+1)\cdots(n+k)\left(1 + O\left(q^n\right)\right)}\right]. \tag{5.3}$$

After the rescaling,

$$z = n\zeta, \tag{5.4}$$

we obtain the equation,

$$\cos u + O\left(e^{-c\,\mathrm{Re}\,u}\right) = \frac{e^n \zeta^n e^{-(\lambda_{j+1}+\lambda_j)n\zeta/2} e^{-\theta/12n}}{2\sqrt{c_j c_{j+1}} \sqrt{2\pi n}} \left(1 + O\left(q^n\right)\right)$$
$$\times \left[1 + \sum_{k=1}^{\infty} \frac{n^k \zeta^k\left(1 + O\left(q^{n+k}\right)\right)}{(n+1)\cdots(n+k)\left(1 + O\left(q^n\right)\right)}\right]. \tag{5.5}$$

Let us discuss the condition when the right-hand side in this equation is $o(1)$ as $n \to \infty$. As a first approximation to this, consider the *critical radius* $r_c = r_c(j, j+1) > 0$ on the ray $\{\zeta : \arg \zeta = \theta_{j,j+1}\}$ as a solution of the equation

$$e\left|\zeta e^{-(\lambda_{j+1}+\lambda_j)\zeta/2}\right| = 1, \quad \zeta = r_c e^{i\theta_{j,j+1}}, \tag{5.6}$$

on the interval $0 < r_c < 1$. This equation can be rewritten as

$$r_c e^{1 + r_c x_{j,j+1}} = 1, \tag{5.7}$$

where

$$x_{j,j+1} = \frac{\left|\lambda_{j+1} + \lambda_j\right|}{2} \cos \beta_{j,j+1},$$
$$\beta_{j,j+1} = \arg\left(\lambda_{j+1} + \lambda_j\right) + \theta_{j,j+1} - \pi = -\frac{\pi}{2} + \arg \frac{\lambda_{j+1} + \lambda_j}{\lambda_{j+1} - \lambda_j}. \tag{5.8}$$

Proposition 5.1 (existence of the critical radius)**.** There exists a unique solution of (5.6) on the interval $0 < r_c < 1$. □

Proof. Observe that (4.3) implies that

$$-1 < x_{j,j+1} < 1. \tag{5.9}$$

From this condition we obtain that the function

$$g(r) = re^{1+rx_{j,j+1}} \tag{5.10}$$

is increasing on $[0,1]$. Indeed,

$$g'(r) = \big(1 + rx_{j,j+1}\big)e^{1+rx_{j,j+1}} > 0, \quad 0 \le r \le 1. \tag{5.11}$$

Also, $g(0) = 0$ and $g(1) > 1$, hence (5.7) has a unique solution on the interval $0 < r_c < 1$. ∎

Let $0 < r^* < 1$ be a solution of the equation

$$r^* e^{1+r^*} = 1. \tag{5.12}$$

We have that

$$r^* = 0.27846\ldots. \tag{5.13}$$

From (5.9) we obtain that $g(r^*) < 1$, hence

$$r^* < r_c < 1. \tag{5.14}$$

In the disk $|\zeta| \le r_c < 1$, the function

$$1 + \sum_{k=1}^{\infty} \frac{n^k \zeta^k \big(1 + O\big(q^{n+k}\big)\big)}{(n+1)\cdots(n+k)\big(1 + O\big(q^n\big)\big)} \tag{5.15}$$

is well approximated by

$$1 + \sum_{k=1}^{\infty} \zeta^k = \frac{1}{1-\zeta}, \tag{5.16}$$

so that

$$\left| 1 + \sum_{k=1}^{\infty} \frac{n^k \zeta^k \big(1 + O\big(q^{n+k}\big)\big)}{(n+1)\cdots(n+k)\big(1 + O\big(q^n\big)\big)} - \frac{1}{1-\zeta} \right| = O\big(n^{-1}\big). \tag{5.17}$$

Therefore, for $z \in \mathcal{S}_{j,j+1} \cap \{|z| \le nr_c\}$, (5.5) reduces to

$$\cos u + O\big(e^{-c\,\mathrm{Re}\,u}\big) = \frac{e^n \zeta^n e^{-(\lambda_{j+1}+\lambda_j)n\zeta/2}}{2\sqrt{c_j c_{j+1}}\sqrt{2\pi n}(1-\zeta)}\big(1 + O\big(n^{-1}\big)\big). \tag{5.18}$$

Introduce the nth critical radius, $r_{c,n} = r_{c,n}(j, j+1) > 0$, on the ray $\{\zeta : \arg \zeta = \theta_{j,j+1}\}$, as a solution of the equation

$$\left| \frac{e\zeta e^{-(\lambda_{j+1}+\lambda_j)\zeta/2}}{\left[2\sqrt{c_j c_{j+1}}\sqrt{2\pi n}(1-\zeta)\right]^{1/n}} \right| = 1, \quad \zeta = \zeta_{c,n} \equiv r_{c,n} e^{i\theta_{j,j+1}}, \tag{5.19}$$

on the interval $0 < r_{c,n} < 1$. Observe that $r_{c,n}$ is a small correction to r_c,

$$r_{c,n} = r_c + O\left(n^{-1} \ln n\right). \tag{5.20}$$

Theorem 5.2 (zeros of the main series)**.** There exists a (big) number $R_1 > 0$ such that for any zero z_k of f in the region,

$$\mathcal{S}_{j,j+1}\left(r_0, R_0, R_1; n\right) = \mathcal{S}_{j,j+1}\left(r_0, R_0\right) \cap \left\{z : |z| < n r_{c,n} - R_1\right\}, \quad r_{c,n} = r_{c,n}(j, j+1), \tag{5.21}$$

there exists a unique zero $z_k(n)$ of f_{n-1} such that

$$z_k(n) - z_k = O\left(e^{-\gamma(n r_{c,n} - |z_k|)}\right), \quad n \longrightarrow \infty, \tag{5.22}$$

where $\gamma > 0$ is independent of n. There exists $N > 0$ such that for all $n > N$, the zeros $z_k(n)$, described by (5.22), exhaust all the zeros of f_{n-1} in the region $\mathcal{S}_{j,j+1}(r_0, R_0, R_1; n)$. □

Proof. Define

$$\beta_n(\zeta) = \frac{e\zeta e^{-(\lambda_{j+1}+\lambda_j)\zeta/2}}{\left[2\sqrt{c_j c_{j+1}}\sqrt{2\pi n}(1-\zeta)\right]^{1/n}}, \quad |\zeta| \le r_{c,n}. \tag{5.23}$$

Then

$$\lim_{n\to\infty} \beta_n(\zeta) = \beta(\zeta) \equiv e\zeta e^{-(\lambda_{j+1}+\lambda_j)\zeta/2}, \quad |\zeta| \le r_{c,n}, \tag{5.24}$$

and by (5.19),

$$\left|\beta_n\left(r_{c,n} e^{i\theta_{j,j+1}}\right)\right| = 1. \tag{5.25}$$

The function $g(r) = |\beta(re^{i\theta_{j,j+1}})|$ is strictly increasing on the interval $0 < r < 1$ and this, together with (5.24), (5.25), implies that for large n,

$$\left|\beta_n\left(re^{i\theta_{j,j+1}}\right)\right| \le e^{c(r-r_{c,n})}, \quad c > 0;\ 0 < r \le r_{c,n}. \tag{5.26}$$

Equation (5.18) is of the form

$$\alpha(u) = \gamma_n(z), \tag{5.27}$$

where u and z are related as in (2.26), $\alpha(u)$ is an entire function such that in the region $\{u : |\operatorname{Im} u| < a,\ \operatorname{Re} u > b\}$, $a, b > 0$,

$$\alpha(u) = \cos u + O\big(e^{-c \operatorname{Re} u}\big), \tag{5.28}$$

and $\gamma_n(z)$ is an entire function such that

$$\gamma_n(z) = \beta_n(\zeta)^n \big(1 + O\big(n^{-1}\big)\big), \quad \zeta = \frac{z}{n}. \tag{5.29}$$

From (5.24) and (5.26) we obtain that in the region $\{u : |\operatorname{Im} u| < a, \operatorname{Re} u > b\}$,

$$\big|\gamma_n(z)\big| < C\big|\beta_n(\zeta)\big|^n < Ce^{c(|z| - nr_{c,n})}, \quad C, c > 0;\ |z| < nr_{c,n}. \tag{5.30}$$

By Proposition 2.3 this implies that for a given $a > 0$ there exist $b > 0$ and $R_0 > 0$ such that all the zeros, $u_k(n)$, of (5.27) in the region $\{u : |\operatorname{Im} u| < a, \operatorname{Re} u > b, |z| < nr_{c,n} - R_1\}$ are simple and close to the zeros, u_k, of the function $\alpha(u)$, so that

$$u_k(n) - u_k = O\big(e^{-c(nr_{c,n} - |z_k|)}\big), \quad n \longrightarrow \infty, \tag{5.31}$$

which implies (5.22). Theorem 5.2 is proved. ■

We will call the zeros $z_k(n)$ satisfying (5.22), the *zeros of the main series* of f_{n-1}. As $n \to \infty$, they approach the zeros z_k of f. This is true also for the finite zeros of Proposition 4.1.

6 Spurious zeros

We will construct a sequence of spurious zeros of $f_{n-1}(z)$ in the jth one-term domination sector, $\mathcal{U}_j(r_0, R_0)$. Let us first discuss the construction informally.

Construction of the rosette. In $\mathcal{U}_j(r_0, R_0)$,

$$f(z) = c_j e^{\lambda_j z}\big(1 + O\big(e^{-d_j(z)}\big)\big), \tag{6.1}$$

where

$$d_j(z) = c \min\big\{\operatorname{dist}\big(z, S_{j,j+1}\big), \operatorname{dist}\big(z, S_{j-1,j}\big), |z|\big\}, \quad c > 0, \tag{6.2}$$

hence (4.8), which is equivalent to the equation $f_{n-1}(z) = 0$, reads

$$\begin{aligned} 1 + O\big(e^{-d_j(z)}\big) \\ &= \frac{e^n z^n e^{-\lambda_j z} e^{-\theta/12n}\big(1+O(q^n)\big)}{n^n c_j \sqrt{2\pi n}} \left[1 + \sum_{k=1}^{\infty} \frac{z^k\big(1+O(q^{n+k})\big)}{(n+1)\cdots(n+k)\big(1+O(q^n)\big)}\right], \end{aligned} \tag{6.3}$$

or after the scaling $z = n\zeta$,

$$\begin{aligned} 1 + O\big(e^{-d_j(z)}\big) &= \frac{e^n \zeta^n e^{-\lambda_j n\zeta} e^{-\theta/12n}\big(1+O(q^n)\big)}{c_j \sqrt{2\pi n}} \\ &\quad \times \left[1 + \sum_{k=1}^{\infty} \frac{n^k \zeta^k\big(1+O(q^{n+k})\big)}{(n+1)\cdots(n+k)\big(1+O(q^n)\big)}\right]. \end{aligned} \tag{6.4}$$

By taking the nth root, we obtain the equation

$$\begin{aligned} \omega_q\big(1 + O\big(e^{-d_j(z)}\big)\big)^{1/n} &= \frac{e\zeta e^{-\lambda_j \zeta} e^{-\theta/12n^2}\big(1+O(q^n)\big)^{1/n}}{\big(c_j \sqrt{2\pi n}\big)^{1/n}} \\ &\quad \times \left[1 + \sum_{k=1}^{\infty} \frac{n^k \zeta^k\big(1+O(q^{n+k})\big)}{(n+1)\cdots(n+k)\big(1+O(q^n)\big)}\right]^{1/n}, \end{aligned} \tag{6.5}$$

where $\omega_q = e^{2\pi q i/n}$, $q = 0, 1, \ldots, n-1$. As an approximation to this equation, consider the equation

$$e\zeta e^{-\lambda_j \zeta} = \omega_q. \tag{6.6}$$

By taking the absolute value of both sides, we obtain the equation of the curve on the complex plane,

$$e\left|\zeta e^{-\lambda_j \zeta}\right| = 1. \tag{6.7}$$

For $\lambda_1 = 1$ it reduces to the Szegö equation,

$$e\left|\zeta e^{-\zeta}\right| = 1, \tag{6.8}$$

and for $\lambda_j = 0$, to the equation of the circle,

$$|\zeta| = e^{-1}. \tag{6.9}$$

If $0 < |\lambda_j| < 1$, it can be reduced to the equation

$$e\left|\xi e^{-|\lambda_j|\xi}\right| = 1, \quad \xi = e^{i \arg \lambda_j}\zeta. \tag{6.10}$$

If $0 < c < 1$, the set of solutions of the equation

$$e\left|\xi e^{-c\xi}\right| = 1 \tag{6.11}$$

on the complex plane consists of two analytic curves: $\Gamma = \Gamma(c)$, inside of the unit circle, and $\Gamma_0 = \Gamma_0(c)$, outside of the unit circle. In the polar coordinates, $\xi = re^{i\theta}$, (6.11) reads

$$\cos\theta = g(r), \quad g(r) = \frac{1 + \ln r}{cr}. \tag{6.12}$$

Observe that

$$g'(r) = -\frac{\ln r}{cr^2}, \tag{6.13}$$

and $g(r)$ attains a maximum at $r = 1$, with $g(1) = 1/c > 1$. Also, $g(0) = -\infty$ and $g(r)$ is increasing on $(0, 1]$. Hence (6.12) has a unique solution in the interval $0 < r < 1$ for any θ, and this solution determines the oval $\Gamma(c)$. We will call (6.11) the *generalized Szegö equation* and $\Gamma(c)$ the *generalized Szegö curve*. Figure 6.1 depicts the generalized Szegö curve for $c = 0.9$. In the Cartesian coordinates the equation of $\Gamma(c)$ has the form

$$y = \pm\sqrt{e^{2cx-2} - x^2}. \tag{6.14}$$

The function

$$h(\xi) = e\xi e^{-c\xi} \tag{6.15}$$

is entire and it conformally maps the interior of the curve $\Gamma(c)$ onto the unit disk,

$$h : \operatorname{Int}\Gamma(c) \longrightarrow D(0, 1) = \{z : |z| < 1\}. \tag{6.16}$$

The preimage, with respect to h, of the uniform probability measure on the unit circle, $(2\pi)^{-1}d\theta$, is the measure $d\mu_{max}(\theta)$ of the maximal entropy on $\Gamma(c)$.

We will denote the curve

$$\mathcal{G}(\lambda_j) = \left\{\zeta : e\left|\zeta e^{-\lambda_j\zeta}\right| = 1,\ |\zeta| < 1\right\}, \tag{6.17}$$

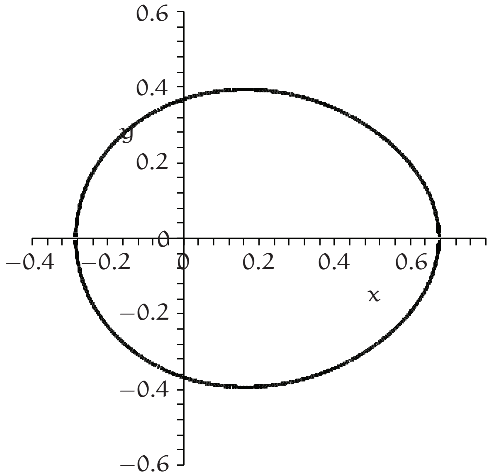

Figure 6.1 The generalized Szegö curve for $c = 0.9$.

so that

$$\mathcal{G}(\lambda_j) = e^{-i \arg \lambda_j} \Gamma\big(|\lambda_j|\big). \tag{6.18}$$

Recall that the sector $\mathcal{U}_j(r_0, R_0)$ is given by the inequalities

$$\theta_{j,j+1} < \arg \zeta < \theta_{j-1,j} \bmod 2\pi, \qquad \theta_{j,j+1} = -\arg\big(\lambda_{j+1} - \lambda_j\big) + \frac{\pi}{2}. \tag{6.19}$$

According to (6.10), this implies that if $\lambda_j \neq 0$, then

$$\begin{aligned} &\alpha_{j,j+1} < \arg \xi < \alpha_{j-1,j} \bmod 2\pi; \\ &\alpha_{j,j+1} = -\arg\big(\lambda_{j+1} - \lambda_j\big) + \frac{\pi}{2} + \arg \lambda_j. \end{aligned} \tag{6.20}$$

This condition holds also for $\lambda_j = 0$, if we take the agreement that $\arg \lambda_j = 0$ for $\lambda_j = 0$. Consider now the arc

$$\Gamma_j = \big\{\xi : \xi \in \Gamma\big(|\lambda_j|\big), \alpha_{j,j+1} < \arg \xi < \alpha_{j-1,j} \bmod 2\pi\big\}, \tag{6.21}$$

on $\Gamma(|\lambda_j|)$, and the arc

$$\mathcal{G}_j = e^{-i \arg \lambda_j} \Gamma_j = \big\{\zeta : \zeta \in e^{-i \arg \lambda_j} \Gamma\big(|\lambda_j|\big),\ \theta_{j,j+1} < \arg \zeta < \theta_{j-1,j} \bmod 2\pi\big\}, \tag{6.22}$$

on $\mathcal{G}(\lambda_j)$. The arc $\mathcal{G}_j \subset \mathcal{G}(\lambda_j)$ goes from one side of the sector $\mathcal{U}_j(r_0, R_0)$ to another. More precisely, we have the following statement. Consider the points

$$\zeta_c(j, j+1) = r_c(j, j+1) e^{i\theta_{j,j+1}}, \quad j = 1, \ldots, m. \tag{6.23}$$

Proposition 6.1. The arc $\mathcal{G}_j$ connects the point $\zeta_c(j-1,j)$ to the point $\zeta_c(j,j+1)$. □

Proof. From (2.26) we have that if $\arg z = \theta_{j,j+1}$, then

$$\lambda_j z = \frac{(\lambda_{j+1}+\lambda_j)z}{2} - iu, \quad u \in \mathbb{R}, \tag{6.24}$$

hence

$$\left|e^{-\lambda_j \zeta}\right| = \left|e^{-(\lambda_{j+1}+\lambda_j)\zeta/2}\right|, \quad \zeta = \frac{z}{n}, \tag{6.25}$$

and (5.6) and (5.7) coincide. This proves that $\zeta_c(j,j+1) \in \mathcal{G}_j$. The relation $\zeta_c(j-1,j) \in \mathcal{G}_j$ is established in the same way. Proposition 6.1 is proved. ■

The *rosette* $\mathcal{H}$ is, by definition, the union of the arcs $\mathcal{G}_j$ and the finite rays,

$$\mathcal{R}_{j,j+1} = \left\{z = re^{i\theta} : \theta = \theta_{j,j+1},\ 0 \le r \le r_c(j,j+1)\right\}, \tag{6.26}$$

$j = 1, \ldots, m$, so that

$$\mathcal{H} = \mathcal{R} \cup \mathcal{G}, \tag{6.27}$$

where

$$\begin{aligned} \mathcal{R} &= \bigcup_{j=1}^{m} \mathcal{R}_{j,j+1}, \\ \mathcal{G} &= \bigcup_{j=1}^{m} \mathcal{G}_j. \end{aligned} \tag{6.28}$$

By definition, the jth *petal*, $\mathcal{P}_j$, of the rosette $\mathcal{H}$ is the region bounded by the rays $\mathcal{R}_{j,j+1}$, $\mathcal{R}_{j-1,j}$, and the arc $\mathcal{G}_j$.

Construction of the nth rosette. As a better, than (6.6), approximation to (6.5), consider the equation

$$\frac{e\zeta e^{-\lambda_j \zeta}}{\left[c_j\sqrt{2\pi n}\,(1-\zeta)\right]^{1/n}} = \omega_q. \tag{6.29}$$

By taking the absolute value of the both sides, we obtain the equation,

$$\frac{e\left|\zeta e^{-\lambda_j \zeta}\right|}{\left[c_j\sqrt{2\pi n}|1-\zeta|\right]^{1/n}} = 1. \tag{6.30}$$

Consider the curve

$$\mathcal{G}^n(\lambda_j) = \left\{ \zeta : \frac{e\left|\zeta e^{-\lambda_j \zeta}\right|}{\left[c_j\sqrt{2\pi n}|1-\zeta|\right]^{1/n}} = 1,\ |\zeta| < 1 \right\} \tag{6.31}$$

and the arc $\mathcal{G}_j^n \subset \mathcal{G}^n(\lambda_j)$, which goes from the point

$$\zeta_{c,n}(j, j+1) = r_{c,n}(j, j+1)e^{i\theta_{j,j+1}} \tag{6.32}$$

to the point $\zeta_{c,n}(j-1, j)$. Observe that if $|\lambda_j| < 1$, then for large n, the curve $\mathcal{G}^n(\lambda_j)$ lies *outside* of the curve $\mathcal{G}(\lambda_j)$, and

$$\operatorname{dist}\left(\mathcal{G}^n(\lambda_j), \mathcal{G}(\lambda_j)\right) = \frac{c\ln n}{n}\left(1 + o(1)\right), \quad n \longrightarrow \infty;\ c > 0. \tag{6.33}$$

Also, if $\delta > 0$ is fixed, then for large n, the curve $\mathcal{G}^n(1) \setminus D(1,\delta)$ lies outside of the curve $\mathcal{G}(1) \setminus D(1,\delta)$, so that the equations of $\mathcal{G}^n(1) \setminus D(1,\delta)$ and $\mathcal{G}(1) \setminus D(1,\delta)$ in the polar coordinates, $\rho = \gamma_n(\theta)$ and $\rho = \gamma(\theta)$, satisfy $\gamma_n(\theta) > \gamma(\theta)$, and

$$\operatorname{dist}\left(\mathcal{G}^n(1) \setminus D(1,\delta), \mathcal{G}(1) \setminus D(1,\delta)\right) = \frac{c(\delta)\ln n}{n}\left(1 + o(1)\right), \quad n \longrightarrow \infty;\ c(\delta) > 0. \tag{6.34}$$

The n*th rosette*, $\mathcal{H}^n$, is, by definition, the union of the arcs $\mathcal{G}_j^n$ and the finite rays,

$$\mathcal{R}_{j,j+1}^n = \left\{ z = re^{i\theta} : \theta = \theta_{j,j+1},\ 0 \le r \le r_{c,n}(j, j+1) \right\}, \tag{6.35}$$

$j = 1, \ldots, m$, so that

$$\mathcal{H}^n = \mathcal{R}^n \cup \mathcal{G}^n, \tag{6.36}$$

where

$$\begin{aligned} \mathcal{R}^n &= \bigcup_{j=1}^m \mathcal{R}_{j,j+1}^n, \\ \mathcal{G}^n &= \bigcup_{j=1}^m \mathcal{G}_j^n. \end{aligned} \tag{6.37}$$

By definition, the jth *petal*, $\mathcal{P}_j^n$, of the rosette $\mathcal{H}^n$ is the region bounded by the rays $\mathcal{R}_{j,j+1}^n$, $\mathcal{R}_{j-1,j}^n$, and the arc $\mathcal{G}_j^n$.

Construction of the spurious zeros. Consider the function

$$h_{j,n}(\zeta) = \frac{e\zeta e^{-\lambda_j \zeta}}{\left[c_j\sqrt{2\pi n}(1-\zeta)\right]^{1/n}}. \tag{6.38}$$

It maps the arc $\mathcal{G}_j^n$ into the unit circle. Define the points $\zeta_q(j,n)$ as the preimages of the points ω_q on the arc $\mathcal{G}_j^n$,

$$\zeta_q(j,n) = \left(h_{j,n}\right)^{-1}\left(\omega_q\right), \quad \zeta_q(j,n) \in \mathcal{G}_j^n. \tag{6.39}$$

Let

$$d_{jn}(\zeta) = n \min\left\{ \operatorname{dist}\left(\zeta, \zeta_{c,n}(j,j+1)\right), \operatorname{dist}\left(\zeta, \zeta_{c,n}(j-1,j)\right)\right\}. \tag{6.40}$$

Theorem 6.2 (spurious zeros, $j \neq 1$)**.** There exists $R_0 > 0$ such that if $j \neq 1$, then for any $\zeta_q(j,n) \in \mathcal{G}_j^n$ such that

$$d_{jn}\left(\zeta_q(j,n)\right) > R_0, \tag{6.41}$$

there exists a unique simple zero $\zeta_k(n)$ of $f_{n-1}(n\zeta)$ such that

$$\zeta_k(n) = \zeta_q(j,n) + O\left(n^{-1}e^{-d_{jn}(\zeta_q(j,n))} + n^{-2}\right), \quad n \longrightarrow \infty. \tag{6.42}$$

□

Proof. By using (6.38), we write (6.5), which is equivalent to the equation $f_{n-1}(z) = 0$, as

$$\omega_q\left(1 + O\left(e^{-d_j(z)}\right)\right)^{1/n} = h_{j,n}(\zeta)\left(1 + O\left(n^{-2}\right)\right), \tag{6.43}$$

or as

$$h_{j,n}(\zeta) = \omega_q + O\left(n^{-1}e^{-d_{jn}(\zeta)} + n^{-2}\right). \tag{6.44}$$

If $j \neq 1$, then the generalized Szegö curve, $\Gamma(|\lambda_j|)$, lies strictly inside of the unit circle, hence the arc $\mathcal{G}_j$ does. Observe that in a neighborhood of $\mathcal{G}_j$,

$$\lim_{n\to\infty} h_{j,n}(\zeta) = h_j(\zeta) \equiv e\zeta e^{-\lambda_j \zeta}, \tag{6.45}$$

and $h_j'(\zeta) \neq 0$, hence $h_{j,n}'(\zeta) \to h_j'(\zeta)$ and $|h_{j,n}'(\zeta)| > \varepsilon > 0$ for large n. Since $|\omega_q - \omega_{q+1}| = 2\pi/n + O(n^{-2})$, we obtain, by Proposition 2.3, that if $d_{jn}(\zeta_q(j,n))$ is big, then there is a simple root $\zeta_k(n)$ of (6.44) such that (6.42) holds. ■

For $j = 1$ the Szegö curve, $\Gamma(1)$, is not strictly inside of the unit disk, because it contains the point $\zeta = 1$. Let us fix some $\rho < 1$ sufficiently close to 1 and partition the zeros $\zeta_k(n)$ of $f_{n-1}(n\zeta)$ into two groups: $|\zeta_k(n)| \le \rho$ and $|\zeta_k(n)| > \rho$. For the first group we will prove formula (6.42) with $j = 1$. For the second group we will consider another approximation to $\zeta_k(n)$. Let $\zeta_k^0(n)$ be the zeros of the section $s_{n-1}(n\zeta)$ of $e^{n\zeta}$, so that

$$s_{n-1}(z) = \sum_{k=0}^{n-1} \frac{z^k}{k!}. \tag{6.46}$$

We will prove for the second group that $\zeta_k(n)$ is well approximated by $\zeta_k^0(n)$. The asymptotics $\zeta_k^0(n)$ is well known for $|\zeta_k^0(n)| < 1-\varepsilon$, $\varepsilon > 0$, and for $|\zeta_k^0(n) - 1| \le C/\sqrt{n}$. In Appendices A and B below we derive uniform asymptotics of $\zeta_k^0(n)$ in the disk $D(1,\delta)$, $\delta > 0$.

Theorem 6.3 (spurious zeros, $j = 1$)**.** There exist $1 > \rho_0 > 0$ and $R_0 > 0$ such that for any $\rho \in (\rho_0, 1)$ the following is true: for any $\zeta_q(1,n) \in \mathcal{G}_1^n$ such that

$$\begin{aligned} &d_{1n}\big(\zeta_q(1,n)\big) > R_0, \\ &\big|\zeta_q(1,n)\big| < \rho, \end{aligned} \tag{6.47}$$

there exists a unique simple zero $\zeta_k(n)$ of $f_{n-1}(n\zeta)$ such that

$$\zeta_k(n) = \zeta_q(1,n) + O\big(n^{-1} e^{-d_{1n}(\zeta_q(1,n))} + n^{-2}\big), \quad n \longrightarrow \infty. \tag{6.48}$$

In addition, for any zero $\zeta_k^0(n)$ of $s_{n-1}(n\zeta)$ such that

$$\big|\zeta_k^0(n)\big| > \rho, \tag{6.49}$$

there exists a unique simple zero $\zeta_k(n)$ of $f_{n-1}(nz)$ such that

$$\zeta_k(n) = \zeta_k^0(n) + O\big(e^{-cn}\big), \quad n \longrightarrow \infty;\ c > 0. \tag{6.50}$$

□

Proof. The first part of the theorem, about (6.48), is proved in the same way as Theorem 6.2, and we omit the proof. For the second part, observe that for any $\delta > 0$ there exists $\rho_0 < 1$ such that for any $\rho \in (\rho_0, 1)$, condition (6.49) implies that

$$\big|\zeta_k^0(n) - 1\big| < \delta. \tag{6.51}$$

Let us write the equation,

$$f_{n-1}(z) = s_{n-1}(z) + f_{n-1}^1(z) = 0, \quad f_{n-1}^1(z) = \sum_{k=0}^{n-1} \frac{(c_2\lambda_2^k + \cdots + c_M\lambda_M^k)z^k}{k!} \tag{6.52}$$

in the form

$$g_n(\zeta) \equiv \frac{s_{n-1}(n\zeta)}{e^{n\zeta}} = -e^{-n\zeta} f^1_{n-1}(n\zeta), \quad z = n\zeta. \tag{6.53}$$

Observe that if $\varepsilon > 0$ is sufficiently small, then there exists $c > 0$ such that for any ζ in the disk $\{|\zeta - 1| \le \varepsilon\}$,

$$e^{-n\zeta} f^1_{n-1}(z) = O\big(e^{-cn}\big), \quad n \longrightarrow \infty. \tag{6.54}$$

The zeros $\zeta^0_k(n)$ solve the equation $g_n(\zeta^0_k(n)) = 0$, and inequality (B.70) in Appendix B below implies that

$$\big|g_n(\zeta)\big| \ge \big|\zeta - \zeta^0_k(n)\big|, \quad \text{if} \, \big|\zeta - \zeta^0_k(n)\big| \le cn^{-1}. \tag{6.55}$$

Therefore, by Proposition 2.3, for any zero $\zeta^0_k(n)$ of $s_{n-1}(n\zeta)$ there exists a unique simple zero $\zeta_k(n)$ of $f_{n-1}(n\zeta)$ such that (6.50) holds. Theorem 6.3 is proved. ■

The zeros $z_k(n)$ described in Theorems 6.2 and 6.3 are called the *spurious zeros* of the section $f_{n-1}(z)$. For these zeros, $\zeta_k(n) = z_k(n)/n$ lies in a small neighborhood of the curve $\mathcal{G}^n$.

7 Transitional zeros

For a given $j = 1, \dots, m$, the jth set, $\mathcal{T}^n_j$, of the transitional zeros of f_{n-1} is located in a neighborhood of the point $n\zeta_{c,n}(j, j+1)$. Let us first describe $\mathcal{T}^n_j$ informally. We set

$$z = n\zeta_{c,n}(j, j+1) + w \tag{7.1}$$

and substitute this into (5.18), which is equivalent to $f_{n-1}(z) = 0$. This gives that

$$\cos u + O\big(e^{-c \,\mathrm{Re}\, u}\big) = \frac{e^n\left(\zeta_{c,n} + \dfrac{w}{n}\right)^n e^{-(\lambda_{j+1}+\lambda_j)(n\zeta_{c,n}+w)/2}}{2\sqrt{c_j c_{j+1}}\sqrt{2\pi n}\left(1 - \zeta_{c,n} + \dfrac{w}{n}\right)} \big(1 + O\big(n^{-1}\big)\big). \tag{7.2}$$

We will assume that $w = O(n^{1/3})$ as $n \to \infty$. Then

$$\left(\zeta_{c,n} + \frac{w}{n}\right)^n = \zeta^n_{c,n} e^{\zeta^{-1}_{c,n} w}\big(1 + O\big(n^{-1/3}\big)\big), \tag{7.3}$$

and (7.2) reduces to the equation

$$\cos u + O\big(e^{-c\,\mathrm{Re}\,u}\big) = A_n e^{(\zeta_c^{-1}-(\lambda_{j+1}+\lambda_j)/2)w}\big(1+O\big(n^{-1/3}\big)\big), \tag{7.4}$$

where

$$A_n = \frac{e^n \zeta_{c,n}^n e^{-(\lambda_{j+1}+\lambda_j)n\zeta_{c,n}/2}}{2\sqrt{c_j c_{j+1}}\sqrt{2\pi n}\big(1-\zeta_{c,n}\big)}. \tag{7.5}$$

By (5.19),

$$\big|A_n\big| = 1, \tag{7.6}$$

and by (2.26),

$$e^{iu} = e^{(\lambda_{j+1}-\lambda_j)z/2}\sqrt{\frac{c_{j+1}}{c_j}}. \tag{7.7}$$

By substituting (7.1), we obtain that

$$e^{iu} = e^{(\lambda_{j+1}-\lambda_j)n\zeta_{c,n}/2} e^{(\lambda_{j+1}-\lambda_j)w/2}\sqrt{\frac{c_{j+1}}{c_j}}. \tag{7.8}$$

Observe that

$$\big(\lambda_{j+1}-\lambda_j\big)\zeta_{c,n} = \big(\lambda_{j+1}-\lambda_j\big)r_{c,n}e^{i\theta_{j,j+1}} = ir_{c,n}\big|\lambda_{j+1}-\lambda_j\big|, \tag{7.9}$$

hence

$$e^{iu} = B_n e^{(\lambda_{j+1}-\lambda_j)w/2}, \tag{7.10}$$

where

$$B_n = e^{inr_{c,n}|\lambda_{j+1}-\lambda_j|/2}\sqrt{\frac{c_{j+1}}{c_j}}. \tag{7.11}$$

We have that

$$\big|B_n\big| = \sqrt{\left|\frac{c_{j+1}}{c_j}\right|}. \tag{7.12}$$

Thus, (7.4) reduces to

$$B_n e^{(\lambda_{j+1}-\lambda_j)w/2} + B_n^{-1}e^{(\lambda_j-\lambda_{j+1})w/2} = 2A_n e^{(\zeta_c^{-1}-(\lambda_{j+1}+\lambda_j)/2)w} + O\big(n^{-1/3}\big), \tag{7.13}$$

or

$$B_n e^{\lambda_{j+1} w} + B_n^{-1} e^{\lambda_j w} - 2A_n e^{\zeta_c^{-1} w} = O\big(n^{-1/3}\big). \tag{7.14}$$

The expression on the left is a three term exponential sum with the exponents λ_{j+1}, λ_j, and ζ_c^{-1}. Observe that

$$\big|\zeta_c^{-1}\big| = r_c^{-1} > 1, \tag{7.15}$$

and the vector ζ_c^{-1} is orthogonal to the one $(\lambda_{j+1} - \lambda_j)$. Since $|\lambda_j|, |\lambda_{j+1}| \le 1$, this implies that the numbers λ_{j+1}, λ_j, and ζ_c^{-1} do not lie on the same line. Now we can formulate the asymptotic formula for the transitional zeros.

Theorem 7.1 (transitional zeros)**.** There is a one-to-one correspondence between the zeros $w_k(n)$ of the exponential sum,

$$g_n(w) = B_n e^{\lambda_{j+1} w} + B_n^{-1} e^{\lambda_j w} - 2A_n e^{\zeta_c^{-1} w}, \tag{7.16}$$

in the disk $D(0, n^{1/3})$ and the zeros $z_k(n)$ of $f_{n-1}(z)$ in the disk $D(n\zeta_{c,n}(j, j+1), n^{1/3})$, such that

$$z_k(n) = n\zeta_{c,n}(j, j+1) + w_k(n) + O\big(n^{-1/6}\big). \tag{7.17}$$

$\square$

Proof. It follows from (7.6) and (7.12) that there exists $R > 0$, independent of n, such that all the zeros of $g_n(w)$ with $|w| > R$ are simple and belong to the main series, see Section 2. For them (7.14) implies (7.17), with even a better error term, $O(n^{-1/3})$. It remains to consider zeros $|w_k(n)| \le R$.

The zeros of $g_n(w)$ are at most double, since the Vandermonde determinant is nonzero. Therefore, there exists $\varepsilon > 0$, independent of n, such that for any three zeros, $w_k(n), w_l(n), w_m(n)$, of $g_n(w)$,

$$\max\big\{\big|w_k(n) - w_l(n)\big|, \big|w_k(n) - w_m(n)\big|, \big|w_m(n) - w_l(n)\big|\big\} > \varepsilon. \tag{7.18}$$

This implies that there exists $\delta > 0$ such that for any $w_k(n) \in D(0, R)$, there exists $0.1\varepsilon < r < 0.2\varepsilon$ such that

$$\big|g_n(w)\big| > \delta, \quad \forall \big|w - w_k(n)\big| = r. \tag{7.19}$$

Equation (7.14) implies now that there exists $N > 0$ such that for all $n > N$, there exists a zero $z_k(n)$ of f_{n-1} which satisfies (7.17). Relation (7.18) enables us to make the correspondence $w_k(n) \to z_k(n)$ one-to-one. Theorem 7.1 is proved. ■

We call the zeros $z_k(n)$ satisfying (7.17) the *transitional zeros in the neighborhood of the point* $n\zeta_{c,n}(j, j+1)$. The set of these zeros is denoted by $\mathcal{T}_j^n$. The set of all the transitional zeros is

$$\mathcal{T}^n = \bigcup_{j=1}^{m} \mathcal{T}_j^n. \tag{7.20}$$

The set of transitional zeros overlaps with both the zeros of the main series and the spurious zeros.

8 Completeness of zeros

Theorem 8.1 (completeness)**.** There exists $N > 0$ such that for any $n > N$, any zero $z_k(n)$ of f_{n-1} belongs to one of the following four categories: finite zeros, zeros of the main series, spurious zeros, and transitional zeros. □

Proof. We will consider zeros in different regions on the complex plane. As usual, $z_k(n) = n\zeta_k(n)$.

Region 1. $z_k(n) \in D(0, R_0)$. The completeness follows from Proposition 4.1.

Region 2. $\zeta_k(n) \in \mathbb{C} \setminus [D(0,\rho) \cup D(1,\delta)]$, $0 < \rho < 1$, $0 < \delta$.

Lemma 8.2 (absence of zeros outside of $D(0,\rho) \cup D(1,\delta)$)**.** For any $\delta > 0$ there exist $1 > \rho > 0$ and $N > 0$ such that for any $n > N$, $f_{n-1}(n\zeta) \neq 0$ if $\zeta \notin D(0,\rho) \cup D(1,\delta)$. □

Proof. Fix any $\delta > 0$. The function $s_{n-1}(n\zeta)$ is a polynomial of degree $(n-1)$ and for large ζ the leading term of this polynomial dominates

$$s_{n-1}(n\zeta) = \frac{n^{n-1}\zeta^{n-1}}{(n-1)!}\big(1 + o(1)\big), \quad n \longrightarrow \infty. \tag{8.1}$$

On the other hand, for small ζ,

$$s_{n-1}(n\zeta) = e^{n\zeta}\big(1 + o(1)\big), \quad n \longrightarrow \infty. \tag{8.2}$$

The transition from one asymptotics to another occurs in a neighborhood of the Szegö curve. Let Γ_ε, $\varepsilon > 0$, be the ε-neighborhood of the Szegö curve Γ, and let $D_\varepsilon^{\text{int}}$ and $D_\varepsilon^{\text{out}}$ be the two connected components of $\mathbb{C} \setminus \Gamma_\varepsilon$, interior and exterior. Then, as shown by Szegö [20], asymptotics (8.1) holds in $D_\varepsilon^{\text{int}}$, while asymptotics (8.2) holds in $D_\varepsilon^{\text{out}}$, with a uniform with respect to z estimate of the error terms $o(1)$. It follows from (8.1) that for any $\varepsilon_0 > 0$ there exists $N > 0$ such that

$$e^{n(1+\varepsilon_0)}|\zeta|^{n-1} \geq \left|s_{n-1}(n\zeta)\right| \geq e^{n(1-\varepsilon_0)}|\zeta|^{n-1}, \quad \zeta \in D_\varepsilon^{\text{out}},\ n > N. \tag{8.3}$$

It is obvious that for all ζ,

$$\left|s_{n-1}(n\zeta)\right| \leq \sum_{k=0}^{\infty} \frac{|n\zeta|^k}{k!} = e^{n|\zeta|}. \tag{8.4}$$

For the given $\delta > 0$, let us choose $\varepsilon > 0$, $\varepsilon_0 > 0$, $\varepsilon_1 > 0$, and $1 > \rho > 0$ such that the following three conditions are satisfied:

(1)

$$\Gamma_\varepsilon \subset D(0,\rho) \cup D(1,\delta); \tag{8.5}$$

(2)

$$\left|\lambda_j\right|\left(1+\varepsilon_1\right) < 1 - \varepsilon_0 - \varepsilon + \ln\rho, \quad j = 2,\dots,M; \tag{8.6}$$

(3)

$$\left|\lambda_j\right| < e^{-2\varepsilon_0-\varepsilon}, \quad j = 2,\dots,M. \tag{8.7}$$

We claim that then there exists $N > 0$ such that for $j = 2,\dots,M$,

$$\left|s_{n-1}\left(n\lambda_j\zeta\right)\right| \leq e^{-\varepsilon n}\left|s_{n-1}(n\zeta)\right|, \quad \zeta \notin D(0,\rho) \cup D(1,\delta),\ n > N. \tag{8.8}$$

Indeed, consider two cases: (1) $\rho \leq |\zeta| \leq 1+\varepsilon_1$, and (2) $|\zeta| > 1+\varepsilon_1$. In case (1), by (8.3), (8.4), and (8.6),

$$\begin{aligned}\left|s_{n-1}\left(n\lambda_j\zeta\right)\right| &\leq e^{n|\lambda_j\zeta|} \leq e^{n(1-\varepsilon_0-\varepsilon)}\rho^{n-1} \leq e^{n(1-\varepsilon_0-\varepsilon)}|\zeta|^{n-1} \\ &\leq e^{-\varepsilon n}\left|s_{n-1}(n\zeta)\right|, \quad n > N,\end{aligned} \tag{8.9}$$

and in case (2), by (8.3) and (8.7),

$$\left|s_{n-1}\left(n\lambda_j\zeta\right)\right| \leq e^{n(1+\varepsilon_0)}\left|\lambda_j\zeta\right|^{n-1} \leq e^{n(1-\varepsilon_0-\varepsilon)}|\zeta|^{n-1} \leq e^{-\varepsilon n}\left|s_{n-1}(n\zeta)\right|, \quad n > N. \tag{8.10}$$

This proves (8.8). From (8.8) we obtain that there exists $N > 0$ such that

$$\left|f^1_{n-1}(n\zeta)\right| \le Me^{-\varepsilon n}\left|s_{n-1}(n\zeta)\right|, \quad \zeta \notin D(0,\rho) \cup D(1,\delta),\ n > N, \tag{8.11}$$

hence $f_{n-1}(n\zeta) = s_{n-1}(n\zeta) + f^1_{n-1}(n\zeta) \neq 0$ if N is big enough. ∎

Region 3. $z_k(n) \in \mathcal{S}_{j,j+1}(r_0, R_0) \cap D(0, n\rho)$. We go back to Section 5. In the disk $\zeta \in D(0,\rho)$, function (5.15) is well approximated by (5.16), hence the equation $f_{n-1}(n\zeta) = 0$ reduces to (5.18). Therefore, if $z_k(n) \in \mathcal{S}_{j,j+1}(r_0, R_0) \cap D(0, n\rho)$ and $|z| > nr_{c,n} + 0.5n^{1/3}$, then equation $f_{n-1}(n\zeta) = 0$ has no zeros for large n, because the absolute value of the right-hand side in (5.18) approaches infinity, as $n \to \infty$, while the left-hand side remains bounded. This proves that the only zeros of f_{n-1} in $\mathcal{S}_{j,j+1}(r_0, R_0) \cap D(0, n\rho)$ are the zeros of the main series and transitional zeros.

Region 4. $z_k(n) \in \mathcal{U}_j(r_0, R_0) \cap D(0, n\rho)$. We go back to Section 6. In the disk $\zeta \in D(0,\rho)$, function (5.15) is well approximated by (5.16), hence the equation $f_{n-1}(n\zeta) = 0$ reduces to (6.44). This implies that all the zeros of f_{n-1} in $\mathcal{S}_{j,j+1}(r_0, R_0) \cap D(0, n\rho)$ are either spurious or transitional.

Region 5. $\zeta_k(n) \in D(1,\delta)$. According to (6.53), (6.54), the equation $f_{n-1} = 0$ reduces to the one

$$\frac{s_{n-1}(n\zeta)}{e^{n\zeta}} = O\left(e^{-cn}\right). \tag{8.12}$$

Asymptotics (B.39), in Appendix B below, proves that all the zeros of the latter equation are located near the zeros of s_{n-1}, which implies that all these zeros are spurious, described by (6.50).

This ends the proof of Theorem 8.1. ∎

9 Limiting distribution of zeros on the rosette

It follows from Theorems 5.2, 6.2, and 6.3 that as $n \to \infty$, the normalized zeros $\zeta_k = z_k/n$ of the section f_{n-1} approach the rosette $\mathcal{H}$, and the δ-function measure of zeros,

$$d\mu_{n-1} = \frac{1}{n-1}\sum_{k=1}^{n-1} \delta\left(\zeta - \zeta_k\right) d\zeta, \tag{9.1}$$

weakly converges to a probability measure on $\mathcal{H}$,

$$\lim_{n\to\infty} \mu_{n-1} = \mu_{\mathcal{H}}, \tag{9.2}$$

such that for any continuous test function $\varphi(\zeta)$,

$$\lim_{n\to\infty}\int\varphi(\zeta)d\mu_{n-1}=\int\varphi(\zeta)d\mu_{\mathcal{H}}=\int_{\mathcal{H}}\varphi(\zeta)p(\zeta)|d\zeta|, \tag{9.3}$$

where $p(\zeta)\geq 0$ is a density function on $\mathcal{H}$. The above theorems give the following description of the density $p(\zeta)$.

Theorem 9.1. On the ray $\mathcal{R}_{j,j+1}$, $p(\zeta)$ is constant,

$$p(\zeta)=\frac{|\lambda_{j+1}-\lambda_j|}{2\pi},\quad \zeta\in\mathcal{R}_{j,j+1}, \tag{9.4}$$

$j=1,\dots,m$. On the curve $\mathcal{G}_j$,

$$p(\zeta)=\frac{|h_j'(\zeta)|}{2\pi},\quad \zeta\in\mathcal{G}_j, \tag{9.5}$$

where

$$h_j(\zeta)=\zeta e^{(1-\lambda_j)\zeta}. \tag{9.6}$$

□

Proof. Ray $\mathcal{R}_{j,j+1}$. By (5.22), the scaled zeros $\zeta_k(n)$ of f_{n-1} are close to the scaled zeros ζ_k of f, so that

$$\zeta_k(n)-\zeta_k=O\big(n^{-1}e^{-\gamma n(r_{c,n}-|\zeta|_k)}\big). \tag{9.7}$$

On the other hand, by (2.32) for $z_k\in S_{j,j+1}(r_0,R_0)$,

$$\zeta_k-n^{-1}\left(\alpha_{j,j+1}+\frac{2\pi li}{\lambda_{j+1}-\lambda_j}\right)=O\big(n^{-1}e^{-cl}\big). \tag{9.8}$$

This implies that $\zeta_k(n)$ are close to the points of the lattice

$$\mathcal{L}_{j,j+1}=\left\{z=n^{-1}\left(\alpha_{j,j+1}+\frac{2\pi li}{\lambda_{j+1}-\lambda_j}\right),\ l\in\mathbb{Z}\right\}, \tag{9.9}$$

hence (9.4) follows.

Curve $\mathcal{G}_j$, $j\neq 1$. From (6.42) we obtain that

$$\begin{aligned}\zeta_k(n)&=\big(h_j^n\big)^{-1}(\omega_q)+O\big(n^{-1}e^{-d_{jn}(\zeta_q(j,n))}+n^{-2}\big)\\&=\big(h_j\big)^{-1}(\omega_q)+O\big(n^{-1}e^{-d_{jn}(\zeta_q(j,n))}+n^{-2}\ln n\big),\end{aligned} \tag{9.10}$$

hence

$$h_j\big(\zeta_k(n)\big) = \omega_q + O\big(n^{-1}e^{-d_{jn}(\zeta_q(j,n))} + n^{-2}\ln n\big). \tag{9.11}$$

If $\zeta = \zeta_k(n)$ and ζ^0 is $\zeta_k(n)$ which corresponds to ω_{q+1}, then

$$\omega_{q+1} - \omega_q = h_j'(\zeta)\big(\zeta^0 - \zeta\big) + O\big(n^{-1}e^{-d_{jn}(\zeta_q(j,n))} + n^{-2}\ln n\big), \tag{9.12}$$

which implies (9.5). This proves Theorem 9.1. ■

10 Beyond conditions 1 and 2

10.1 Beyond Condition 1

Condition 1 means that there is no λ_j, $j = m+1, \dots, M$, on the sides of the polygon P_m. Suppose that this condition does not hold and there are some λ_k's on the side $[\lambda_j, \lambda_{j+1}]$. Denote

$$\sigma_j = \big\{\lambda_k : \lambda_k \in \big[\lambda_j, \lambda_{j+1}\big]\big\}. \tag{10.1}$$

Observe that σ_j includes λ_j and λ_{j+1}. In this case, instead of two-term equation (2.25), we consider the multiterm equation

$$\sum_{k:\lambda_k \in \sigma_j} c_k e^{\lambda_k z} = 0. \tag{10.2}$$

With the help of substitution (2.26) we reduce it to the equation

$$\sum_{k:\ \lambda_k \in \sigma_j} \widetilde{c}_k e^{iy_k u} = 0, \quad -1 \le y_k \le 1. \tag{10.3}$$

The function on the left is quasiperiodic. Its zeros are concentrated in a finite strip $\{u : |\operatorname{Im} u| < A\}$. The distribution of zeros of quasiperiodic exponential sums was studied in the works of Kreĭn and Levin [11], Levin [14], Zhdanovich [26], Soprunova [19], and others. It was shown that the zeros also have a property of quasiperiodicity, and its average number is the same as for the function $\cos u$.

The main results of the present paper, concerning the distribution of zeros of the exponential sum and its sections, can be extended, with proper modifications, to the case when Condition 1 does not hold, but it requires some additional considerations for the zeros of the main series.

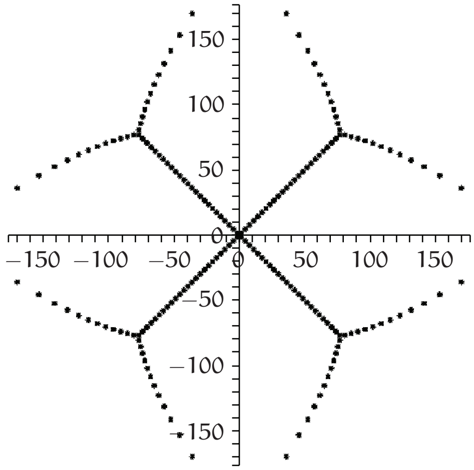

Figure 10.1 The zeros of the $n = 200$ section of exponential sum (10.4) for $m = 4$.

10.2 Beyond Condition 2

If Condition 2 violates, then there are several maximal $|\lambda_j|$'s. As an example, consider the symmetric sum,

$$f(z) = \frac{1}{m}\sum_{j=0}^{m-1} e^{\omega^j z} = 0, \quad \omega = e^{2\pi i/m}. \tag{10.4}$$

In this case,

$$f_n(z) = \sum_{k:0\le mk\le n} \frac{z^{mk}}{(mk)!}. \tag{10.5}$$

The rosette is symmetric and all the results are extended to this case. Figure 10.1 depicts zeros of the $n = 200$ section of exponential sum (10.4) for $m = 4$. The spurious zeros are described in the symmetric case by parts of the original Szegö curve, $\Gamma(1)$.

All the results are extended also to a slightly more general case of

$$f(z) = \sum_{j=0}^{m-1} c_j e^{a\omega^j z} = 0, \quad \omega = e^{2\pi i/m}, \tag{10.6}$$

where $c_j \neq 0$, $j = 0, \ldots, m-1$, and $a \neq 0$. For $m = 2$ this includes the sine and cosine functions.

Consider now the exponential sum

$$f(z) = \sum_{j=1}^{m} e^{\lambda_j z}, \tag{10.7}$$

where

$$\lambda_j = e^{i\varphi_j}, \quad \varphi_j \in \mathbb{R}. \tag{10.8}$$

Then

$$f_n(z) = \sum_{k=0}^{n} a_k z^k, \tag{10.9}$$

where

$$a_k = \frac{e^{ik\varphi_1} + \cdots + e^{ik\varphi_m}}{k!}. \tag{10.10}$$

In this case the asymptotic behavior of the coefficients a_k depends on the arithmetic properties of the numbers φ_j, and the asymptotic behavior of the zeros z_k of f_n can be rather complicated.

Appendices

A Uniform asymptotics of the zeros of $s_{n-1}(n\zeta)$ in the disk $D(1, \delta)$

We write the equation

$$s_{n-1}(n\zeta) \equiv \sum_{k=0}^{n-1} \frac{(n\zeta)^k}{k!} = 0 \tag{A.1}$$

as

$$e^{n\zeta} = \frac{(n\zeta)^n}{n!} \sum_{k=0}^{\infty} \frac{n!(n\zeta)^k}{(n+k)!}, \tag{A.2}$$

or

$$\frac{e^{-n\zeta} n^n \zeta^n}{n!} \sum_{k=0}^{\infty} \frac{n! n^k \zeta^k}{(n+k)!} = 1. \tag{A.3}$$

We will assume that

$$|\zeta| \le 1. \tag{A.4}$$

With the help of the Stirling formula we obtain that

$$\frac{n!n^k}{(n+k)!} = \exp\left[k - \left(n+k+\frac{1}{2}\right)\ln\left(1+\frac{k}{n}\right) + \frac{\theta_n}{12n} - \frac{\theta_{n+k}}{12(n+k)}\right]. \tag{A.5}$$

Since

$$\ln(1+x) = x - \frac{x^2}{2} + O\big(x^3\big), \quad x \longrightarrow 0, \tag{A.6}$$

this gives that as $n \to \infty$,

$$\frac{n!n^k}{(n+k)!} = e^{-k^2/2n}\left(1 + O\left(\frac{k^3}{n^2} + \frac{k}{n}\right)\right), \quad 0 \le k \le n^{0.6}. \tag{A.7}$$

Also,

$$\frac{n!n^k}{(n+k)!} = \begin{cases} O\big(e^{-n^{0.1}}\big), & n^{0.6} \le k \le n, \\ O\big(2^{n-k}e^{-n^{0.1}}\big), & n \le k. \end{cases} \tag{A.8}$$

Therefore,

$$\sum_{k=0}^{\infty} \frac{n!n^k\zeta^k}{(n+k)!} = \sum_{k=0}^{\infty} e^{-k^2/2n}\zeta^k + O(1), \quad |\zeta| \le 1, \tag{A.9}$$

and (A.3) reduces to

$$\frac{e^{-n\zeta}n^n\zeta^n}{n!}\left[\sum_{k=0}^{\infty} e^{-k^2/2n}\zeta^k + O(1)\right] = 1, \quad |\zeta| \le 1. \tag{A.10}$$

Let

$$\zeta = e^{\tau n^{-1/2}}, \quad \operatorname{Re}\tau \le 0. \tag{A.11}$$

Then

$$\sum_{k=0}^{\infty} e^{-k^2/2n}\zeta^k = \sum_{k=0}^{\infty} e^{-k^2/2n+\tau k n^{-1/2}} = n^{1/2}\int_0^{\infty} e^{-x^2/2+\tau x}\,dx + O\big(|\tau|+1\big), \tag{A.12}$$

by the Euler-Maclaurin integration formula. To justify the error term, observe that

$$\begin{aligned}&\left|e^{-k^2/2n+\tau k n^{-1/2}}-\int_k^{k+1}e^{-x^2/2n}e^{\tau x n^{-1/2}}\,dx\right|\\&\qquad\le\int_k^{k+1}\left(\frac{x}{n}+|\tau|n^{-1/2}\right)\left|e^{-x^2/2n+\tau x n^{-1/2}}\right|dx\end{aligned}\tag{A.13}$$

hence

$$\begin{aligned}&\left|\sum_{k=0}^{\infty}e^{-k^2/2n+\tau k n^{-1/2}}-\int_0^{\infty}e^{-x^2/2n+\tau x n^{-1/2}}\,dx\right|\\&\qquad\le\int_0^{\infty}\left(\frac{x}{n}+|\tau|n^{-1/2}\right)\left|e^{-x^2/2n+\tau x n^{-1/2}}\right|dx\le C\big(1+|\tau|\big),\end{aligned}\tag{A.14}$$

which implies (A.12). From (A.9) and (A.12),

$$\sum_{k=0}^{\infty}\frac{n!n^k\zeta^k}{(n+k)!}=n^{1/2}\int_0^{\infty}e^{-x^2/2+\tau x}\,dx+O\big(|\tau|+1\big).\tag{A.15}$$

Equation (A.3) reduces to

$$\frac{e^{-n\zeta}n^n\zeta^n}{n!}\left[n^{1/2}\int_0^{\infty}e^{-x^2/2+\tau x}\,dx+O\big(|\tau|+1\big)\right]=1.\tag{A.16}$$

By applying the Stirling formula, we obtain that

$$e^{n(1-\zeta)}\zeta^n\left[\frac{1}{\sqrt{2\pi}}\int_0^{\infty}e^{-x^2/2+\tau x}\,dx+O\Big(\big(|\tau|+1\big)n^{-1/2}\Big)\right]=1.\tag{A.17}$$

Let us fix any big number $M>0$ and any small number $\varepsilon>0$, and consider the three (partially overlapping) cases:

(1) $0\le|\tau|\le M$,
(2) $M\le|\tau|\le n^{1/6-\varepsilon}$, $\operatorname{Re}\tau<0$,
(3) $\operatorname{Re}\tau\le -n^{\varepsilon}$.

Case 1 ($0\le|\tau|\le M$). We assumed $\operatorname{Re}\tau\le 0$, but if $|\tau|\le M$, (A.10) and (A.12) hold without this restriction. In the case under consideration,

$$\zeta=e^{\tau n^{-1/2}}=1+\tau n^{-1/2}+\frac{\tau^2 n^{-1}}{2}+O\big(n^{-3/2}\big),\tag{A.18}$$

hence

$$e^{n(1-\zeta)}\zeta^n=e^{-\tau^2/2}\big(1+O\big(n^{-1/2}\big)\big),\tag{A.19}$$

hence (A.17) reduces to

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-(x-\tau)^2/2}dx + O\big(n^{-1/2}\big) = 1, \tag{A.20}$$

or

$$\frac{1}{\sqrt{2\pi}}\int_\tau^\infty e^{-x^2/2}dx + O\big(n^{-1/2}\big) = 0, \tag{A.21}$$

which is the Newman-Rivlin equation [15], with an error term of the order of $n^{-1/2}$. All zeros of the function

$$E(\tau) = \frac{1}{\sqrt{2\pi}}\int_\tau^\infty e^{-x^2/2}dx, \qquad E(\tau) = \frac{1}{2}\,\mathrm{erfc}\left(\frac{\tau}{\sqrt{2}}\right) \tag{A.22}$$

are simple and they lie in the left half-plane, $\{z : \mathrm{Re}\, z < 0\}$, see [8]. If we enumerate the zeros τ_q in the second quadrant by $|\tau_q|$, then

$$\tau_1 = -1.915990857\cdots + i2.816359418\cdots, \tag{A.23}$$

$$\tau_q = 2\sqrt{\pi q}e^{3\pi i/4} + \frac{1}{4\sqrt{\pi q}}\ln(8\pi q)e^{\pi i/4} + O\big(q^{-1/2}\big), \quad q \longrightarrow \infty. \tag{A.24}$$

Since $E(\tau)$ is real, it also has zeros $\overline{\tau_k}$ in the third quadrant. From (A.20) we obtain that the zeros $\zeta_q(n)$ of $s_{n-1}(n\zeta)$ such that $|\zeta_q(n) - 1| \le Mn^{-1/2}$ and $\mathrm{Im}\,\zeta_k \ge 0$ are simple and they have the asymptotics

$$\zeta_q(n) = 1 + n^{-1/2}\tau_q + O\big(n^{-1}\big), \quad n \longrightarrow \infty. \tag{A.25}$$

This formula, with the error term $o(n^{-1/2})$, was obtained by Newman and Rivlin.

Case 2 ($M \le |\tau| \le n^{1/6-\varepsilon}$, $\mathrm{Re}\,\tau < 0$). For large τ formula (A.19) is modified as follows:

$$e^{n(1-\zeta)}\zeta^n = e^{-\tau^2/2}\big(1 + O\big(\tau^3 n^{-1/2}\big)\big), \tag{A.26}$$

hence instead of (A.21) we obtain the equation

$$\frac{1}{\sqrt{2\pi}}\int_\tau^\infty e^{-x^2/2}dx + O\big(\tau^3 n^{-1/2}\big) = 0. \tag{A.27}$$

Under the assumption $|\tau| \le n^{1/6-\varepsilon}$ the error term is of the order of $O(n^{-3\varepsilon})$. We can rewrite the equation $E(\tau) = 0$ in the form

$$\widetilde{E}(\tau) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\tau e^{-x^2/2}dx = 1. \tag{A.28}$$

For $\operatorname{Re}\tau \to -\infty$,

$$\widetilde{E}(\tau) \sim -\frac{1}{\sqrt{2\pi}}\tau^{-1}e^{-\tau^2/2}, \tag{A.29}$$

hence

$$\begin{aligned} &-\frac{1}{\sqrt{2\pi}}\tau_q^{-1}e^{-\tau_q^2/2} \sim 1, \\ &\widetilde{E}'(\tau_q) = \frac{1}{\sqrt{2\pi}}e^{-\tau_q^2/2} \sim -\tau_q. \end{aligned} \tag{A.30}$$

If $\delta \equiv |(\tau-\tau_q)\tau_q| \ll |\tau_q|^{-1}$, then

$$\widetilde{E}'(\tau) = \frac{1}{\sqrt{2\pi}}e^{-\tau^2/2} = \widetilde{E}'(\tau_q)\big(1+O(\delta)\big), \tag{A.31}$$

hence

$$\big|\widetilde{E}(\tau)-1\big| \geq c\big|\tau-\tau_q\big|, \quad |\tau-\tau_q| \leq \delta;\ c > 0. \tag{A.32}$$

Equation (A.27) can be rewritten in the form

$$\widetilde{E}(\tau) = 1 + O\big(\tau^3 n^{-1/2}\big). \tag{A.33}$$

From Proposition 2.3 we obtain now that

$$\zeta_q(n) = 1 + n^{-1/2}\tau_q + O\big(n^{-1}q\big), \quad q \leq n^{1/3-\varepsilon},\ n \longrightarrow \infty. \tag{A.34}$$

This asymptotics gives an extension of the asymptotics of Newman and Rivlin to $q \leq n^{1/3-\varepsilon}$.

Case 3 ($\operatorname{Re}\tau \leq -n^{\varepsilon}$, $\varepsilon > 0$). Our calculations in this case are based on the formula,

$$\sum_{k=0}^{\infty} e^{-k^2/2n}\zeta^k = \frac{1}{1-\zeta}\left[1 + O\left(\frac{1}{n\big(1-|\zeta|\big)^2}\right)\right], \quad |\zeta| \leq \exp\big(-n^{-1/2+\varepsilon}\big). \tag{A.35}$$

To prove this formula, observe that

$$\begin{aligned} &\sum_{k=\sqrt{n}}^{\infty} |\zeta|^k = O\big(n\exp\big(-n^{\varepsilon}\big)\big), \quad n \longrightarrow \infty, \\ &\sum_{k=0}^{\sqrt{n}} \big(1-e^{-k^2/2n}\big)|\zeta|^k \leq \frac{C_0}{n}\sum_{k=0}^{\sqrt{n}} k^2|z|^k \leq \frac{C_1}{n\big(1-|\zeta|\big)^3}. \end{aligned} \tag{A.36}$$

This implies (A.35). Similarly, from (A.7) we obtain the estimate

$$\sum_{k=0}^{\infty} \frac{n! n^k \zeta^k}{(n+k)!} - \sum_{k=0}^{\infty} e^{-k^2/2n} \zeta^k = O\left(\frac{1}{n\left(1-|\zeta|\right)^2}\right), \quad |\zeta| \le \exp\left(-n^{-1/2+\varepsilon}\right). \tag{A.37}$$

We reduce now (A.10) to

$$\frac{e^{-n\zeta} n^n \zeta^n}{n!(1-\zeta)} \left[1 + O\left(\frac{1}{n\left(1-|\zeta|\right)^2}\right)\right] = 1, \quad |\zeta| \le \exp\left(-n^{-1/2+\varepsilon}\right). \tag{A.38}$$

By applying the Stirling formula and by taking the nth root, we obtain the equation

$$\frac{e^{1-\zeta}\zeta}{\left[\sqrt{2\pi n}(1-\zeta)\right]^{1/n}} \left[1 + O\left(\frac{1}{n^2\left(1-|\zeta|\right)^2}\right)\right] = \omega_q, \quad |\zeta| \le \exp\left(-n^{-1/2+\varepsilon}\right), \tag{A.39}$$

$\omega_q = e^{2\pi q i/n}$. As an approximation to this equation, consider the equation

$$h(\zeta) \equiv e^{1-\zeta}\zeta = \omega_q, \qquad |\zeta| \le \exp\left(-n^{-1/2+\varepsilon}\right). \tag{A.40}$$

We have that if $\zeta = 1 - t$, then

$$e^{1-\zeta}\zeta = 1 - \frac{t^2}{2}\left(1 + O(t)\right), \quad t \longrightarrow 0, \tag{A.41}$$

hence the solution to the equation $h(\zeta_q) = \omega_q$ has the asymptotics

$$\zeta_q = 1 + \sqrt{\frac{2q}{n}} e^{3\pi i/4}\left(1 + O\left(\frac{q}{n}\right)\right), \quad \frac{q}{n} \longrightarrow 0. \tag{A.42}$$

Also,

$$\left|h(\zeta) - \omega_q\right| \ge c\sqrt{\frac{q}{n}}\left|\zeta - \zeta_q\right|, \quad \left|\zeta - \zeta_q\right| \le \frac{z_q}{2};\ c > 0. \tag{A.43}$$

As a better approximation to (A.39), consider the equation

$$h^n(\zeta) \equiv \frac{e^{1-\zeta}\zeta}{\left[\sqrt{2\pi n}(1-\zeta)\right]^{1/n}} = \omega_q, \qquad |\zeta| \le \exp\left(-n^{-1/2+\varepsilon}\right). \tag{A.44}$$

Observe that

$$\left[\sqrt{2\pi n}(1-\zeta)\right]^{1/n} = 1 + O\left(\frac{\ln n}{n}\right), \quad |\zeta| \le \exp\left(-n^{-1/2+\varepsilon}\right), \tag{A.45}$$

hence by Proposition 2.3, there exists a zero ζ_q^n of h^n such that

$$\zeta_q^n = \zeta_q + O\left(\frac{\ln n}{\sqrt{nq}}\right), \quad q \geq n^{\varepsilon}. \tag{A.46}$$

This in turn implies that there is a simple zero $\zeta_q(n)$ of (A.39) such that

$$\zeta_q(n) = \zeta_q^n + O\left(\frac{1}{n^{1/2} q^{3/2}}\right), \quad q \geq n^{\varepsilon}. \tag{A.47}$$

The following theorem summarizes the results of this appendix.

Theorem A.1. For any $\varepsilon > 0$, the zeros $\zeta_q(n)$ of $s_{n-1}(n\zeta)$ have asymptotics (A.34) in the interval $1 \leq q \leq n^{1/3-\varepsilon}$ and asymptotics (A.47) in the interval $n^{\varepsilon} \leq q \leq n/2$. □

B Uniform asymptotics of the function $s_n(n\zeta)$ in the disk $D(1, \delta)$

The function

$$s_n(z) \equiv \sum_{k=0}^{n} \frac{z^k}{k!} \tag{B.1}$$

solves the equation

$$s_n' = s_n - \frac{z^n}{n!}, \tag{B.2}$$

or

$$\left(s_n e^{-z}\right)' = -\frac{e^{-z} z^n}{n!}. \tag{B.3}$$

In addition,

$$\lim_{z \to +\infty} s_n(z) e^{-z} = 0, \tag{B.4}$$

hence

$$s_n(z) = e^z \int_z^{+\infty} \frac{e^{-u} u^n \, du}{n!}. \tag{B.5}$$

This gives that

$$s_n(n\zeta) = \frac{n^{n+1} e^{n(\zeta-1)}}{n!} \int_{\zeta}^{+\infty} e^{-n\phi(u)} du, \tag{B.6}$$

where

$$\phi(\zeta) = \zeta - \ln\zeta - 1. \tag{B.7}$$

We will assume that $\ln\zeta$ is taken on the principal branch, with a cut on $(-\infty, 0]$. Observe that $\zeta = 1$ is a critical point of $\phi(\zeta)$ and

$$\phi(\zeta) = \frac{(\zeta-1)^2}{2} - \frac{(\zeta-1)^3}{3} + \cdots. \tag{B.8}$$

Therefore, the function

$$\xi(\zeta) = \sqrt{\phi(\zeta)} = \sqrt{\zeta - \ln\zeta - 1} \tag{B.9}$$

is analytic in some disk $D(1,\delta)$, $\delta > 0$, and ξ is the conformal mapping

$$\xi : D(1,\delta) \longrightarrow \Omega, \tag{B.10}$$

where Ω is a domain with analytic boundary, $0 \in \Omega$. It follows from (B.7) that $\xi(\zeta)$ is analytically continued to the half-line $(0,\infty)$ and $\xi(0) = -\infty$, $\xi(+\infty) = +\infty$. From (B.8) we have that

$$\xi(\zeta) = \frac{\zeta-1}{\sqrt{2}} - \frac{(\zeta-1)^2}{6\sqrt{2}} + \frac{(\zeta-1)^3}{36\sqrt{2}} + \cdots. \tag{B.11}$$

For the inverse mapping, $\eta = \xi^{-1} : \Omega \to D(1,\delta)$, we have that

$$\zeta = \eta(\xi) = 1 + \sqrt{2}\xi + \frac{2\xi^2}{3} + \frac{\sqrt{2}\xi^3}{18} + \cdots. \tag{B.12}$$

After the substitution $v = \xi(u)$, (B.6) becomes

$$s_n(n\zeta) = \frac{n^{n+1}e^{n(\zeta-1)}}{n!}\int_{\xi(\zeta)}^{+\infty} e^{-nv^2}\eta'(v)\,dv, \tag{B.13}$$

or if we put $w = \sqrt{n}v$,

$$s_n(n\zeta) = \frac{n^{n+1/2}e^{n(\zeta-1)}}{n!}\int_{\sqrt{n}\xi(\zeta)}^{+\infty} e^{-w^2}\eta'\left(\frac{w}{\sqrt{n}}\right)dw. \tag{B.14}$$

By applying the Stirling formula, we obtain that

$$\frac{s_n(n\zeta)}{e^{n\zeta}} = \frac{e^{-\theta/12n}}{\sqrt{2\pi}}\int_{\sqrt{n}\xi(\zeta)}^{+\infty} e^{-w^2}\eta'\left(\frac{w}{\sqrt{n}}\right)dw. \tag{B.15}$$

The asymptotics of the integral on the right is described in terms of the complementary error function

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{+\infty} e^{-w^2}\, dw. \tag{B.16}$$

For any $\varepsilon > 0$, as $|z| \to \infty$,

$$\operatorname{erfc}(z) = \frac{e^{-z^2}}{z\sqrt{\pi}} \left(1 - \frac{1}{2z^2} + O\big(z^{-4}\big)\right), \quad |\arg z| < \frac{3\pi}{4} - \varepsilon, \tag{B.17}$$

see [1]. Let us fix an arbitrary (big) number $M > 1$ and consider two cases: (1) $|\zeta - 1| \le M/\sqrt{n}$, and (2) $M/\sqrt{n} \le |\zeta - 1| \le \delta$.

Case 1 ($|\zeta - 1| \le M/\sqrt{n}$). Since

$$\eta'\left(\frac{w}{\sqrt{n}}\right) = \eta'(0) + O\big(n^{-1/2}\big) = \sqrt{2} + O\big(n^{-1/2}\big) \tag{B.18}$$

if w is bounded, we obtain from (B.15) that

$$\frac{s_n(n\zeta)}{e^{n\zeta}} = \frac{1}{2} \operatorname{erfc}\left(\sqrt{n}\xi(\zeta)\right) + O\big(n^{-1/2}\big), \quad |\zeta - 1| \le \frac{M}{\sqrt{n}}. \tag{B.19}$$

Case 2 ($M/\sqrt{n} \le |\zeta - 1| \le \delta$). Suppose first that

$$\big|\arg(\zeta - 1)\big| \le \frac{2\pi}{3}. \tag{B.20}$$

Then by (B.11),

$$\big|\arg\left(\xi(\zeta)\right)\big| < 0.7\pi < \frac{3\pi}{4} \tag{B.21}$$

if δ is small enough. Set $a = \sqrt{n}\xi(\zeta)$. We have that

$$\begin{aligned}\int_a^{+\infty} e^{-w^2} \eta'\left(\frac{w}{\sqrt{n}}\right) dw = \eta'\left(\frac{a}{\sqrt{n}}\right) \int_a^{+\infty} e^{-w^2}\, dw \\ + \int_a^{+\infty} e^{-w^2} \left[\eta'\left(\frac{w}{\sqrt{n}}\right) - \eta'\left(\frac{a}{\sqrt{n}}\right)\right] dw,\end{aligned} \tag{B.22}$$

and as $n \to \infty$,

$$\int_a^{+\infty} e^{-w^2} \left[\eta'\left(\frac{w}{\sqrt{n}}\right) - \eta'\left(\frac{a}{\sqrt{n}}\right)\right] dw = O\left(\frac{e^{-a^2}}{a^2\sqrt{n}}\right), \quad |\arg a| < \frac{3\pi}{4} - \varepsilon. \tag{B.23}$$

Indeed, set $w = a + t$. Then the latter integral becomes

$$\int_0^{+\infty} e^{-a^2-2at-t^2}\left[\eta'\left(\frac{a+t}{\sqrt{n}}\right) - \eta'\left(\frac{a}{\sqrt{n}}\right)\right]dt. \tag{B.24}$$

We can choose the contour of integration near $t = 0$ as $t = re^{-i\arg a}$, $r_0 > r > 0$, where $r_0 = 0.1|a|$, and then from $t_0 = r_0 e^{-i\arg a}$ to $+\infty$ in such a way that $|e^{-(a+t)^2}|$ is decreasing to 0. Observe that for $r_0 > r > 0$,

$$\left|\eta'\left(\frac{a+re^{-i\arg a}}{\sqrt{n}}\right) - \eta'\left(\frac{a}{\sqrt{n}}\right)\right| \le \frac{Cr}{\sqrt{n}}, \tag{B.25}$$

hence

$$\left|\int_0^{t_0} e^{-2|a|r-t^2}\left[\eta'\left(\frac{a+t}{\sqrt{n}}\right) - \eta'\left(\frac{a}{\sqrt{n}}\right)\right]dt\right| \le \frac{C}{\sqrt{n}}\int_0^{r_0} e^{-|a|r} r\,dr < \frac{C}{\sqrt{n}|a|^2}. \tag{B.26}$$

This gives (B.23).

Since $\eta(\xi(\zeta)) = \zeta$ and $a = \sqrt{n}\xi(\zeta)$, we have that

$$\eta'\left(\frac{a}{\sqrt{n}}\right) = \eta'(\xi(\zeta)) = \frac{1}{\xi'(\zeta)}. \tag{B.27}$$

We obtain now from (B.17), (B.22), and (B.23) that

$$\begin{aligned}\int_a^{+\infty} e^{-w^2}\eta'\left(\frac{w}{\sqrt{n}}\right)dw &= \frac{1}{\xi'(\zeta)}\int_a^{+\infty} e^{-w^2}dw\left[1 + O\left(\frac{1}{a\sqrt{n}}\right)\right]\\ &= \frac{\sqrt{\pi}}{2\xi'(\zeta)}\operatorname{erfc}(a)\left[1 + O\left(\frac{1}{a\sqrt{n}}\right)\right], \quad |\arg a| < \frac{3\pi}{4} - \varepsilon,\end{aligned} \tag{B.28}$$

hence by (B.15),

$$\frac{s_n(n\zeta)}{e^{n\zeta}} = \frac{1}{2\sqrt{2}\,\xi'(\zeta)}\operatorname{erfc}\left(\sqrt{n}\xi(\zeta)\right)\left[1 + O\left(\frac{1}{(\zeta-1)n}\right)\right], \quad |\arg(\zeta-1)| \le \frac{2\pi}{3}. \tag{B.29}$$

Suppose now that

$$|\arg(\zeta - 1) - \pi| \le \frac{2\pi}{3}. \tag{B.30}$$

Then by (B.11),

$$|\arg\left(\xi(\zeta)\right) - \pi| < 0.7\pi < \frac{3\pi}{4} \tag{B.31}$$

if δ is small enough. Observe that $s_n(0) = 1$, hence from (B.6) we obtain that

$$\int_0^{+\infty} e^{-n\phi(u)} du = \frac{e^n n!}{n^{n+1}}. \tag{B.32}$$

Therefore, (B.6) can be rewritten as

$$s_n(n\zeta) = e^{n\zeta} - \frac{n^{n+1} e^{n(\zeta-1)}}{n!} \int_0^{\zeta} e^{-n\phi(u)} du, \tag{B.33}$$

and (B.15) as

$$\frac{s_n(n\zeta)}{e^{n\zeta}} = 1 - \frac{e^{-\theta/12n}}{\sqrt{2\pi}} \int_{-\infty}^{\sqrt{n}\xi(\zeta)} e^{-w^2} \eta'\left(\frac{w}{\sqrt{n}}\right) dw. \tag{B.34}$$

Observe that

$$\operatorname{erfc}(-z) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{z} e^{-w^2} dw, \tag{B.35}$$

and by (B.17),

$$\operatorname{erfc}(-z) = -\frac{e^{-z^2}}{z\sqrt{\pi}} \left(1 - \frac{1}{2z^2} + O\left(z^{-4}\right)\right), \quad |\arg z - \pi| < \frac{3\pi}{4} - \varepsilon. \tag{B.36}$$

Also,

$$\operatorname{erfc}(-z) + \operatorname{erfc} z = 2. \tag{B.37}$$

Similar to (B.29), we obtain now that

$$\frac{s_n(n\zeta)}{e^{n\zeta}} = 1 - \frac{1}{2\sqrt{2}\xi'(\zeta)} \operatorname{erfc}\left(-\sqrt{n}\xi(\zeta)\right) \left[1 + O\left(\frac{1}{(\zeta-1)n}\right)\right], \quad \left|\arg(\zeta-1) - \pi\right| \leq \frac{2\pi}{3}. \tag{B.38}$$

Let us summarize the results of this appendix.

Theorem B.1. There exists $\delta > 0$ such that for any $M > 1$ as $n \to \infty$,

$$\frac{s_n(n\zeta)}{e^{n\zeta}} = \begin{cases} \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{n}\xi(\zeta)\right) + O\left(n^{-1/2}\right), & \text{if } |\zeta-1| \le \frac{M}{\sqrt{n}}; \\ \frac{1}{2\sqrt{2}\xi'(\zeta)}\,\mathrm{erfc}\left(\sqrt{n}\xi(\zeta)\right)\left[1+O\left(\frac{1}{(\zeta-1)n}\right)\right], & \text{if } \frac{M}{\sqrt{n}} \le |\zeta-1| \le \delta, \\ & \left|\arg(\zeta-1)\right| \le \frac{2\pi}{3}; \\ 1-\frac{1}{2\sqrt{2}\xi'(\zeta)}\,\mathrm{erfc}\left(-\sqrt{n}\xi(\zeta)\right)\left[1+O\left(\frac{1}{(\zeta-1)n}\right)\right], & \text{if } \frac{M}{\sqrt{n}} \le |\zeta-1| \le \delta, \\ & \left|\arg(\zeta-1)-\pi\right| \le \frac{2\pi}{3}. \end{cases} \tag{B.39}$$

□

A similar asymptotics for real $\zeta > 0$ was obtained by Jet Wimp (unpublished), see [9]. Asymptotics (B.39) can be used to locate the zeros of s_n. The zeros σ_q, $\overline{\sigma}_q$ of $\mathrm{erfc}(\sigma)$ are located in the second and the third quadrants, and the ones in the second quadrant have the asymptotics [8]

$$\sigma_q = \sqrt{2\pi q}\,e^{3\pi i/4} + \frac{1}{4\sqrt{2\pi q}}\ln(8\pi q)e^{\pi i/4} + O\left(q^{-1/2}\right), \quad q \longrightarrow \infty, \tag{B.40}$$

(cf. (A.24)) where $\tau_q = \sqrt{2}\sigma_q$. The first zero is

$$\sigma_1 = -1.3548101281\cdots + i1.9914668430\ldots. \tag{B.41}$$

Asymptotics (B.40) can be obtained from the equation

$$2 = \mathrm{erfc}\left(-\sigma_q\right) = -\frac{e^{-\sigma_q^2}}{\sigma_q\sqrt{\pi}}\left(1 - \frac{1}{2\sigma_q^2} + O\left(\sigma_q^{-4}\right)\right), \quad q \longrightarrow \infty. \tag{B.42}$$

It follows from (B.39) that the zeros $\zeta_q(n)$, $\overline{\zeta_q(n)}$ of $s_n(n\zeta)$ are also located in the second and the third quadrants. Let us find the large n asymptotics of $\zeta_q(n)$. First we consider the problem informally. By (B.39), the equation $s_n(n\zeta) = 0$ can be rewritten as

$$\mathrm{erfc}\left(-\sqrt{n}\xi(\zeta)\right) = \frac{2\sqrt{2}}{\eta'\left(\xi(\zeta)\right)}\left[1 + O\left(\frac{1}{(\zeta-1)n}\right)\right]. \tag{B.43}$$

Set

$$\sigma = \sqrt{n}\xi(\zeta); \tag{B.44}$$

then (B.43) reads

$$\operatorname{erfc}(-\sigma) = \frac{2\sqrt{2}}{\eta'\left(\frac{\sigma}{\sqrt{n}}\right)}\left[1 + O\left(\frac{1}{\sigma\sqrt{n}}\right)\right]. \tag{B.45}$$

We are looking for $\sigma = \sigma_q + \tau$, where τ is a small correction to be determined. It is convenient to take logarithm of both sides of (B.45),

$$\log \operatorname{erfc}\big(-\sigma_q - \tau\big) = \frac{3\ln 2}{2} - \log \eta'\left(\frac{\sigma_q + \tau}{\sqrt{n}}\right) + O\left(\frac{1}{\sqrt{nq}}\right). \tag{B.46}$$

(Observe that $\sigma_q^{-1} = O(q^{-1/2})$.) We have the Taylor expansion

$$\begin{aligned}\log \operatorname{erfc}\big(-\sigma_q - \tau\big) &= \log \operatorname{erfc}\big(-\sigma_q\big) - \frac{\operatorname{erfc}'\big(-\sigma_q\big)\tau}{\operatorname{erfc}\big(-\sigma_q\big)} + O\big(\tau^2\big)\\ &= \ln 2 + \frac{2e^{-\sigma_q^2}\tau}{\sqrt{\pi}\operatorname{erfc}\big(-\sigma_q\big)} + O\big(\tau^2\big),\end{aligned} \tag{B.47}$$

where the error term is uniform in σ_q. Indeed, it follows from asymptotics (B.36) that

$$\frac{\operatorname{erfc}'(-\sigma)}{\operatorname{erfc}(-\sigma)} = -\frac{2e^{-\sigma^2}}{\sqrt{\pi}\operatorname{erfc}(-\sigma)} = 2\sigma - \frac{1}{\sigma} + O\big(\sigma^{-3}\big), \tag{B.48}$$

hence

$$\big[\log \operatorname{erfc}(-\sigma)\big]'' = O(1), \quad \sigma \longrightarrow \infty; \quad |\arg \sigma - \pi| < \frac{3\pi}{4}. \tag{B.49}$$

We find τ from the equation

$$\ln 2 + \frac{2e^{-\sigma_q^2}\tau}{\sqrt{\pi}\operatorname{erfc}\big(-\sigma_q\big)} = \frac{3\ln 2}{2} - \log \eta'\left(\frac{\sigma_q}{\sqrt{n}}\right), \tag{B.50}$$

so that

$$\tau = \frac{\sqrt{\pi}}{2}\left[\frac{\ln 2}{2} - \log \eta'\left(\frac{\sigma_q}{\sqrt{n}}\right)\right] e^{\sigma_q^2} \operatorname{erfc}\big(-\sigma_q\big), \tag{B.51}$$

or

$$\tau = \frac{1}{\sqrt{n}} g\left(\frac{\sigma_q}{\sqrt{n}}\right) h\big(\sigma_q\big), \tag{B.52}$$

where the function

$$g(z) = -\frac{1}{2z}\left[\frac{\ln 2}{2} - \log \eta'(z)\right] = \frac{\sqrt{2}}{3} - \frac{5z}{36} + \cdots \tag{B.53}$$

is analytic at $z = 0$, and

$$h(\sigma) = -\sqrt{\pi}\sigma e^{\sigma^2} \operatorname{erfc}(-\sigma) = 1 - \frac{1}{2\sigma^2} + O\big(\sigma^{-4}\big), \quad \sigma \longrightarrow \infty. \tag{B.54}$$

Theorem B.2. There exists $\delta > 0$ such that all the zeros $\zeta_q(n)$ of $s_n(n\zeta)$ in the domain $D(1,\delta) \cap \{\operatorname{Im}\zeta > 0\}$ have the asymptotics

$$\zeta_q(n) = \eta\left(\frac{\sigma_q}{\sqrt{n}} + \frac{\tau_q}{n}\right) + O\left(\frac{1}{n\sqrt{q}}\right), \quad n \to \infty, \tag{B.55}$$

where η is the inverse function of $\xi(\zeta) = \sqrt{\zeta - \ln\zeta - 1}$, see (B.12), $\{\sigma_q, q = 1, 2, \ldots\}$ are the zeros of $\operatorname{erfc}(\sigma)$ in the upper half-plane, and

$$\tau_q = g\left(\frac{\sigma_q}{\sqrt{n}}\right) h(\sigma_q), \tag{B.56}$$

where g and h are defined in (B.53) and (B.54), respectively. □

Proof (existence). Let us write (B.46) as

$$\log \operatorname{erfc}\big(-\sigma_q - \tau\big) = \frac{3\ln 2}{2} - \log \eta'\left(\frac{\sigma_q + \tau}{\sqrt{n}}\right) + \varepsilon(\tau), \quad \varepsilon(\tau) = O\left(\frac{1}{\sqrt{n}}\right), \tag{B.57}$$

or as

$$f(\tau) \equiv \log \operatorname{erfc}\big(-\sigma_q - \tau\big) - \frac{3\ln 2}{2} + \log \eta'\left(\frac{\sigma_q + \tau}{\sqrt{n}}\right) - \varepsilon(\tau) = 0. \tag{B.58}$$

Let

$$\tau^0 = \frac{1}{\sqrt{n}} g\left(\frac{\sigma_q}{\sqrt{n}}\right) h(\sigma_q). \tag{B.59}$$

Then from (B.50) we obtain that

$$f\big(\tau^0\big) = O\left(\frac{1}{\sqrt{nq}}\right), \tag{B.60}$$

and from (B.47), (B.48) that

$$\big|f'\big(\tau^0\big)\big| > c > 0; \qquad f''(\tau) = O(1), \qquad \big|\tau - \tau^0\big| \le n^{-1/4}. \tag{B.61}$$

This implies the existence of a zero τ^1 of $f(\tau)$ such that

$$\tau^1 = \tau^0 + O\left(\frac{1}{\sqrt{nq}}\right). \tag{B.62}$$

By (B.44), this means that there is a zero ζ^1 of $s_n(n\zeta)$ such that

$$\sigma_q + \tau^1 = \sqrt{n}\xi(\zeta^1), \tag{B.63}$$

hence

$$\zeta^1 = \eta\left(\frac{\sigma_q + \tau^0}{\sqrt{n}} + O\left(\frac{1}{n\sqrt{q}}\right)\right), \tag{B.64}$$

which implies (B.55). The existence is proved.

Uniqueness. From (B.36) it follows that any zero σ of (B.45) must be in the disk $D(\sigma_q, n^{-1/3})$, but by (B.61) there is a unique zero in this disk. This proves the uniqueness. Theorem B.2 is proved. ■

It follows from (B.53), (B.54), and (B.56) that τ_q is uniformly bounded, hence (B.55) can be rewritten in the form

$$\zeta_q(n) = \eta\left(\frac{\sigma_q}{\sqrt{n}}\right) + \eta'\left(\frac{\sigma_q}{\sqrt{n}}\right)\frac{\tau_q}{n} + O\left(\frac{1}{n\sqrt{q}}\right), \quad n \longrightarrow \infty. \tag{B.65}$$

Equation (B.55) implies also that

$$\xi(\zeta_q(n)) = \frac{\sigma_q}{\sqrt{n}} + \frac{\tau_q}{n} + O\left(\frac{1}{n\sqrt{q}}\right), \quad n \longrightarrow \infty. \tag{B.66}$$

It follows from Theorems B.1 and B.2 that

$$\left(\frac{s_n(n\zeta)}{e^{n\zeta}}\right)'\Bigg|_{\zeta=\zeta_q(n)} = \sqrt{2n}\sigma_q\left(1 + O(q^{-1})\right). \tag{B.67}$$

Indeed, when we differentiate the last formula in (B.39), we obtain, with the help of (B.42) and (B.66), that

$$\begin{aligned}\left(\frac{s_n(n\zeta)}{e^{n\zeta}}\right)'\Bigg|_{\zeta=\zeta_q(n)} &= \frac{\sqrt{n}}{2\sqrt{2}}\,\mathrm{erfc}'\left(-\sqrt{n}\xi(\zeta)\right)\Big|_{\zeta=\zeta_q(n)} + \mathrm{erfc}\left(-\sqrt{n}\xi(\zeta)\right)\Big|_{\zeta=\zeta_q(n)}O(1)\\ &= -\frac{\sqrt{n}}{\sqrt{2\pi}}e^{-n\xi(\zeta)^2}\Big|_{\zeta=\zeta_q(n)} + O(1) = -\frac{\sqrt{n}}{\sqrt{2\pi}}e^{-\sigma_q^2} + O(1)\end{aligned} \tag{B.68}$$

which implies (B.67), due to (B.42). If we differentiate the last formula in (B.39) twice, we obtain similarly that

$$\left|\left(\frac{s_n(n\zeta)}{e^{n\zeta}}\right)''\right| = O(nq), \quad \text{if } \left|\zeta - \zeta_q(n)\right| \le \frac{1}{\sqrt{nq}}. \tag{B.69}$$

By combining (B.67) and (B.69), we obtain the following result.

Proposition B.3. There exists $c > 0$ and $N > 0$ such that for all $n > N$,

$$\left|\frac{s_n(n\zeta)}{e^{n\zeta}}\right| \ge \sqrt{nq}\left|\zeta - \zeta_q(n)\right|, \quad \text{if } \left|\zeta - \zeta_q(n)\right| \le \frac{c}{\sqrt{nq}}. \tag{B.70}$$

□

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