THE ASYMPTOTIC LOCATION OF THE ROOTS OF A CERTAIN TRANSCENDENTAL EQUATION*

BY

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The determination of the characteristic values in many boundary problems associated with differential or integral equations depends upon the location of the roots of an equation of the form

$$(1) \qquad | \phi_{ij}(\lambda) + \psi_{ij}(\lambda)e^{\lambda A_j} | = 0.$$

The symbol $|\alpha_{ij}|$ is used here to designate the determinant of order n with general element α_{ij}. In expanded form the equation (1) may be written

$$(2) \qquad \sum \omega_l(\lambda)e^{\lambda B_l} = 0,$$

where each B_l is a combination of the values A_j, and ω_l is expressible in terms of the coefficients ϕ_{ij}, ψ_{ij}.

The following are the hypotheses to be made concerning equation (1):

(i) the exponents A_j are complex constants;

(ii) the functions $\phi_{ij}(\lambda)$, $\psi_{ij}(\lambda)$ are such that the coefficients $\omega_l(\lambda)$ of the equation in the expanded form (2) are analytic for $|\lambda|$ sufficiently large, and are of the form

$$(3) \qquad \omega_l(\lambda) = \lambda^{\nu_l}[a_l],$$

where ν_l is real and a_l is a constant not zero. If ν is not an integer, λ^ν is taken to mean exp ($\nu \log \lambda$), the principal value of the logarithm to be chosen.

The symbol $[a_l]$ in (3) designates a quantity $a_l + \epsilon(\lambda)$, where $\epsilon(\lambda)$ is used as a generic designation for a function which is analytic for $|\lambda|$ sufficiently large and which approaches zero uniformly as $|\lambda| \to \infty$.

The equation of the type (1) either in form (1) or form (2) has been extensively studied under hypotheses of varying degrees of restrictiveness, by Tamarkin,† Wilder,‡ and Pólya and Schwengeler.§ Of these, the two last

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† (1) *Some General Problems of the Theory of Ordinary Linear Differential Equations* etc., Petrograd, 1917 (in Russian), also (2), Mathematische Zeitschrift, vol. 27 (1927), p. 24.

‡ These Transactions, vol. 18 (1917), p. 415.

§ Pólya, Sitzungsberichte der Bayerischen Akademie, München, 1920, p. 285.

Schwengeler, E., *Geometrisches über die Verteilung der Nullstellen* etc., Zürich, 1925.

named focus the attention in large measure upon the geometric determination of the configuration of regions in which the roots of the equation are located. They consider the case in which the coefficients are polynomials; a restriction which would appear from the following discussion to be unessential. The present note is intended as a brief analytic rather than geometric discussion. The final result is more precise than that obtained by Schwengeler.* The deductions are based directly on the work of the other authors cited above. This work thus appears, indeed, to include in large measure the analytic deductions on the location of the roots made subsequently by Schwengeler.

The constants A_j are by hypothesis complex. We designate by $\mu_1, \mu_2, \cdots, \mu_r$ the distinct values of the set $(\arg A_j)$ arranged in the order

$$0 \leqq \mu_1 < \mu_2 < \cdots < \mu_r < \pi.$$

The restriction of the values μ_j to the interval $(0, \pi)$ is unessential, inasmuch as the multiplication of the jth column of equation (1) by $\exp(-\lambda A_j)$ leaves the type of the equation unchanged and substitutes $(-A_j)$ in the rôle of A_j. We redesignate now by A_{hk}, $k=1, 2, \cdots$, those of the quantities A_j with argument μ_h, arranging them in the order

$$|A_{hk}| < |A_{h,k+1}|.$$

The rays

$$\arg \lambda = \frac{\pi}{2} - \mu_h \tag{4}$$

proceed with h in clockwise succession in the λ plane, and are followed in this succession by the rays

$$\arg \lambda = -\frac{\pi}{2} - \mu_h. \tag{4'}$$

We shall confine the attention to a sector S_c enclosing a single one of the rays (4), (4′), i.e.

$$\arg \lambda = \pm \frac{\pi}{2} - \mu_c, \tag{5}$$

and defined by the relations

$$S_c: \ \pm \pi/2 - \mu_c + \eta_c \geqq \arg \lambda > \pm \pi/2 - \mu_{c+1} + \eta_{c+1},$$

where η_c, η_{c+1} are positive constants chosen sufficiently small but otherwise arbitrary. The character of equation (1) in sector S_c is typical of its character

* Cf. Schwengeler, loc. cit., p. 48, and the final theorem of this note.

in any analogous sector associated with another of the rays (4) or (4′). The deductions for sector S_c, therefore, reveal the facts for the entire λ plane.

If the determinant in equation (1) is expanded the equation may be written in the form

$$\sum_{l=0}^{L} e^{\lambda K_l} \sum_{p=1}^{P} \omega_{l_p}(\lambda) e^{\lambda K_p'} = 0, \tag{6}$$

in which the constants K_p' are linear combinations with coefficients 0 and 1 of the quantities A_{hk}, $h \neq c$, and the constants K_l are similar combinations of the quantities A_{ck}. We may choose the subscripts, moreover, so that

$$0 = K_0 < |K_1| < \cdots < |K_L| .$$

Inasmuch as the quantities $R(\lambda A_{hk})$,* with $h \neq c$, maintain their signs throughout the sector S_c, the order by numerical magnitude of the quantities $R(\lambda K_p')$ is fixed for λ in the sector. If we suppose then that the subscript 1 is assigned so that

$$R(\lambda K_1') \geqq R(\lambda K_p') \qquad (p = 2, \cdots, P)$$

and divide equation (6) by $\exp(\lambda K_1')$ we have

$$\sum_{l=0}^{L} e^{\lambda K_l} \left\{ \omega_l(\lambda) + \sum_{p=2}^{P} \omega_{l_p}(\lambda) e^{\lambda H_p} \right\} = 0, \tag{7}$$

where $\omega_l = \omega_{l_1}$, and $H_p = K_p' - K_1'$. In S_c we have then

$$R(\lambda H_p) < 0. \tag{8}$$

Let l' and l'' be defined now by the relations

$$\omega_l \equiv 0 \ \text{ for } \ l < l', \ \text{ and } \ l > l'',$$
$$\omega_{l'} \not\equiv 0, \quad \omega_{l''} \not\equiv 0,\dagger$$

and let the sector S_c be divided into sub-sectors S_{c1} and S_{c2} as follows:

$$S_{c1}: \ \pm \pi/2 - \mu_c + \eta_c \geqq \arg \lambda > \pm \pi/2 - \mu_c - \eta_c,$$
$$S_{c2}: \ \pm \pi/2 - \mu_c - \eta_c \geqq \arg \lambda > \pm \pi/2 - \mu_{c+1} + \eta_{c+1}.$$

We observe then readily that if η_c is chosen sufficiently small we have in S_{c1} because of (8)

$$\{\omega_{l_p} e^{\lambda H_p}\} \exp(\lambda K_l) = \omega_{l_1}[0] \exp(\lambda K_{l_1}) \quad (p = 2, \cdots, P), \tag{9}$$

* The symbol $R(x)$ designates "the real part of x."

† In the simplest case we should have $l' = 0$, $l'' = L$.

for any values of l and l_1. Equation (7) may be written, therefore, in the form

$$\sum_{l=l'}^{l''} e^{\lambda K_l}\omega_l(\lambda)[1] = 0, \text{ in sector } S_{c1}. \tag{10}$$

In the sector S_{c2} on the other hand relations (9) are valid only for $l \leqq l_1$. We have, however, further in this case

$$\omega_l \exp(\lambda K_l) = \omega_{l_1}[0] \exp(\lambda K_{l_1}), \text{ for } l < l_1,$$

and hence we may write the equation after division by $\exp(\lambda K_{l''})$ in the form

$$\omega_{l''}(\lambda)[1] + \sum_{l=l''}^{L} \sum_{p=2}^{P} \omega_{l_p}(\lambda)[1] \exp(\lambda\{K_l - K_{l''} + H_p\}) = 0. \tag{11}$$

If $l''=L$ the left member of this consists of its first term alone, and because of (3) there will be no roots of the equation in S_{c2} for $|\lambda|$ sufficiently large. On the other hand if $l''<L$ the equation (11) is of precisely the original type (6) and we may repeat the foregoing discussion. The sector S_c and the ray (5) must be replaced in their rôles by the sector S_{c2} and the ray

$$\arg \lambda = \pm \pi/2 - \mu_{c2}, \tag{12}$$

where μ_{c2} is the largest value of the set

$$\arg\{K_l - K_{l''} + H_p\}, \quad l > l'',$$

and the quantities λK_l must be replaced by those of the quantities $\lambda\{K_l - K_{l''} + H_p\}$ whose real part vanishes on the ray (12). We find from a repetition of the argument then that in a sector enclosing the ray (12) the equation is again of the general form (10), and that the remaining part of S_{c2} is either devoid of roots for $|\lambda|$ sufficiently large, or else that a further repetition of the procedure leads to a third ray to be considered. In any event the number of rays thus found in S_c and hence in the entire plane is finite. Every such ray is determined by a relation

$$R(\lambda Q) = 0, \tag{13}$$

where Q is formed by linear combination with coefficients 0, 1, and -1, from the values A_j of equation (1), and in an arbitrarily small sector enclosing such a ray the equation is of type (10), i.e. in virtue of (3) of the form

$$\sum_{m=0}^{M} \lambda^{\nu_m}[\alpha_m] e^{\lambda Q_m} = 0. \tag{14}$$

Excepting in these small sectors the equation has no roots for $|\lambda|$ sufficiently large.

The location of the roots of the equation (14) remains to be determined, and to this end we quote the following theorems which deal with the equation in certain specialized forms.

THEOREM A.* *If B and K are real, $B>0$, then for $|\rho|$ sufficiently large the roots ρ_m of the equation*

$$[a_1] + \rho^K[a_2]e^{\rho B} = 0 \tag{15}$$

are asymptotically spaced along the curve

$$\left|\rho^K\left(\frac{a_2}{a_1}\right)e^{\rho B}\right| = 1, \tag{16}$$

and

$$|\rho_m| \sim B^{-1}\left\{ \pm\left(2m - \frac{K}{2}\right)\pi + \arg\left(-\frac{a_2}{a_1}\right)\right\}.$$

Moreover if ρ remains uniformly away from the roots of (15) *then the left hand member of the equation is uniformly bounded from zero.*

We note that for $|\rho|$ sufficiently large one branch of the curve (16) lies entirely within an arbitrarily small sector about the positive axis of imaginaries and approaches parallelism with this axis, while the other branch is similarly located with respect to the negative axis of imaginaries.

THEOREM B.† *If the constants B_j are real and*

$$0 = B_0 < B_1 < \cdots < B_J,$$

then for $|\rho|$ sufficiently large the roots of the equation

$$\sum_{j=0}^{J}[b_j]e^{\rho B_j} = 0, \quad b_0 \neq 0, \quad b_J \neq 0, \tag{17}$$

lie in the strip bounded by the lines

$$R(\rho) = \pm c, \tag{18}$$

where c is a suitably chosen real constant. The number N of roots‡ lying in any interval of this strip of length l satisfies the relation

$$B_Jl/(2\pi) - (J+1) \leqq N \leqq B_Jl/(2\pi) + (J+1).$$

Moreover if ρ remains uniformly away from the zeros of the equation (17) *the left hand member of the equation is uniformly bounded from zero.*

* Wilder, loc. cit., pp. 424, 425. Tamarkin, loc. cit. (1), pp. 284–289. This theorem and Theorem B are proved by the authors cited under the hypotheses that K is an integer and that $[a]=a+E(\rho)/\rho$, where $E(\rho)$ is bounded for $|\rho|$ sufficiently large. Their proofs, however, are still valid with the more general hypotheses of this paper.

† Wilder, loc. cit., p. 421–423. Tamarkin, loc. cit. (1), pp. 168–173, (2), pp. 27–29.

§ Counted with their multiplicities.

We consider now the transformation of this theorem by means of the substitution

$$\rho = \mu + \sigma \log \mu, \tag{19}$$

in which σ is a real constant and the principal value of the logarithm is to be understood. The region into which (18) is transformed is bounded by the curves

$$| \mu^\sigma e^\mu | = e^{\pm c}. \tag{20}$$

It is readily shown to be a strip asymptotically of constant width $2c$, and, like the curve (16), approaching parallelism with the axes of imaginaries.

If we designate by ρ' and ρ'' the real and imaginary parts of ρ, with corresponding notation for μ, the relation (19) may be resolved into the equalities

$$\rho' = \mu' + \sigma \log | \mu | ,$$

$$\rho'' = \mu'' + \sigma \arg \mu .$$

From the first of these we obtain, upon dividing it by $|\mu|$, restricting ρ to the region (18), and allowing $|\mu|$ to increase indefinitely, the result $\mu'/|\mu| \to 0$, i.e. $\arg \mu \to \pm\pi/2$. It follows readily that the portion $T_{R,l}$ of region (20) which lies between the arcs $|\mu| = R$, and $|\mu| = R+l$, is asymptotically congruent, as $R \to \infty$ with l fixed, to the corresponding portion of strip (18), and this in turn approaches conformity with a piece of the strip of length l. Moreover if μ_1 and μ_2 lie in $T_{R,l}$ and correspond to ρ_1 and ρ_2, then clearly

$$(\rho_2 - \rho_1) - (\mu_2 - \mu_1) \to 0, \quad \text{as} \quad R \to \infty .$$

Hence when R is sufficiently large any two points of $T_{R,l}$ at a distance greater than δ from each other correspond to points of strip (18) which are at a distance greater than $\frac{1}{2}\delta$ from each other. Inasmuch as a function $\epsilon(\rho)$ clearly becomes a function $\epsilon(\mu)$ we may state the following theorem in which we have again replaced the μ of the preceding discussion by ρ.

THEOREM C. *If* $0 = B_0 < B_1 < \cdots < B_J$, *and* σ *is real, then for* $|\rho|$ *sufficiently large the roots of the equation*

$$\sum_{j=0}^{J} [b_j] \{ \rho^\sigma e^\rho \}^{B_j} = 0, \quad b_0 \neq 0, \quad b_J \neq 0, \tag{21}$$

lie within the strip bounded by a pair of curves (20). *The number* N *of roots* *lying in a portion of this strip cut out by a pair of arcs* $|\rho|=R$, *and* $|\rho|=R+l$, *where* l *is fixed, satisfies when* R *is sufficiently large the relation*

$$\frac{B_J l}{2\pi}-(J+1)\leqq N\leqq\frac{B_J l}{2\pi}+(J+1).$$

Moreover if ρ *remains uniformly away from the zeros of* (21) *then the left hand member of the equation is uniformly bounded from zero.*

Consider now the equation of type (14), i.e.

$$\sum_{m=0}^{M}\rho^{\nu_m}[a_m]e^{\rho C_m}=0, \tag{22}$$

in which the constants C_m are real and $0=C_0<C_1<\cdots<C_m$. Since multiplication of the equation by any power of ρ is permissible we may suppose $\nu_0=0$. Let σ_1 and m_1 be determined by the relations

$$\nu_m/C_m\begin{cases}\leqq\sigma_1, & \text{for}\quad 1\leqq m<m_1,\\ =\sigma_1, & \text{for}\quad m=m_1,\\ <\sigma_1, & \text{for}\quad m>m_1{}^*,\end{cases}$$

and choose η_1, so that for every $m\geqq 1$

either $\nu_m/C_m=\sigma_1$

or $\nu_m/C_m<\sigma_1-\eta_1.$

Further let S_1 designate the region bounded on the left by any ray through the origin with argument between $\pi/2$ and π, and on the right by the curve

$$|\rho^{\sigma_1}e^{\rho}|=e^{\eta_1}. \tag{23}$$

In the region S_1 we have

$$\rho^{\nu_m}e^{\rho C_m}=\begin{cases}\{\rho^{\sigma_1}e^{\rho}\}^{C_m}, & \text{if}\quad \sigma_1=\nu_m/C_m,\\ [0], & \text{if}\quad \sigma_1>\nu_m/C_m.\end{cases}$$

Hence in S_1 the equation (22) is of the form (15) or (21) and its roots are located as described in Theorem A or Theorem C.

On the other hand in any region S_2 bounded on the left by the curve (23) we have

$$\rho^{\nu_m}e^{\rho C_m}=\{\rho^{\sigma_1}e^{\rho}\}^{C_m}[0],\quad\text{for}\quad m<m_1.$$

Hence equation (22) is of the form

* In the simplest case we should have $m_1=M$.

$$\sum_{m=m_1}^{M} \rho^{\nu_m'}[a_m]e^{\rho C_m'} = 0, \quad \text{in} \quad S_2,$$

where $\nu_m' = \nu_m - \nu_{m_1}$, and $C_m' = C_m - C_{m_1}$. If $m_1 = M$ this latter equation has no roots for $|\rho|$ large. On the other hand if $m_1 < M$ the equation is precisely of the type (22) and the discussion applied above to S_1 may be repeated for S_2. The roots of equation (22) are thus found to be confined to a finite number of strips of the type (20).

Lastly to apply to equation (14) the results obtained for equation (22) it is necessary only to set

$$\lambda = \rho e^{-i \arg Q_m}.$$

In conclusion we state, therefore, the following theorem.

THEOREM. *For* $|\lambda|$ *sufficiently large the roots of equation* (1) *lie within arbitrarily small sectors each containing a ray* (13). *Within each of these sectors the roots are further confined to a finite number of strips which are asymptotically of constant width and approach parallelism with the ray* (13) *enclosed in the sector. The number of roots N, counted with their multiplicities, which lie in any portion of such a strip bounded by a pair of arcs* $|\lambda| = R$, *and* $|\lambda| = R + l$, *satisfies, for R sufficiently large, a relation*

$$\alpha l - \beta \leqq N \leqq \alpha l + \beta,$$

in which α *and* β *are suitably determined constants. Moreover if* λ *remains uniformly away from a root of equation* (1) *the left hand member of the equation is uniformly bounded from zero.*

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