Notes on Complex Analysis

Juwon Lee

June 22, 2020

Abstract

The goal of this document is to revise Complex Analysis relevant to the project. In addition, I aim to review my skills for LaTeX documentation.

1 Complex Numbers and Functions

1.1 Forms of Complex Numbers

1. *Algebraic form*: $x + iy$ where $x, y \in \mathbb{R}$.
2. *Geometric form*: (x, y).
3. *Polar form*: $z = r(\cos\theta + i\sin\theta)$, where

$$r = \sqrt{x^2 + y^2} > 0, \cos\theta = \frac{x}{r}, \sin\theta = \frac{y}{r}.$$

The argument z is not defined when $z = 0$ or equivalently when $r = 0$.

1.2 Complex Functions

We start by stating the definition of complex-valued function.

Definition 1. Complex-valued function. The complex-valued function f of a complex variable is a relation that assigns to each complex number z in a set S a unique complex number $f(z)$.

Definition 2. Domain. The set S is a subset of the complex numbers and is called the domain of definition of f.

1.3 Visualization

Real-valued functions require two dimensions for visualization, the x-plane and the y-plane. Following this, the visualization of complex-valued functions requires four dimensions: two for variable z and two for the values $w = f(z)$, as both z and w are comprised of real and imaginary parts. In reality, we use two planes, the z-plane and w-plane and view the function as a mapping from a subset of one plane to the other as shown in Figure 1.

1.4 Linear Transformations

Linear transformations can be thought of in terms of a dilation, a rotation, and a translation, which map regions to geometrically similar regions.

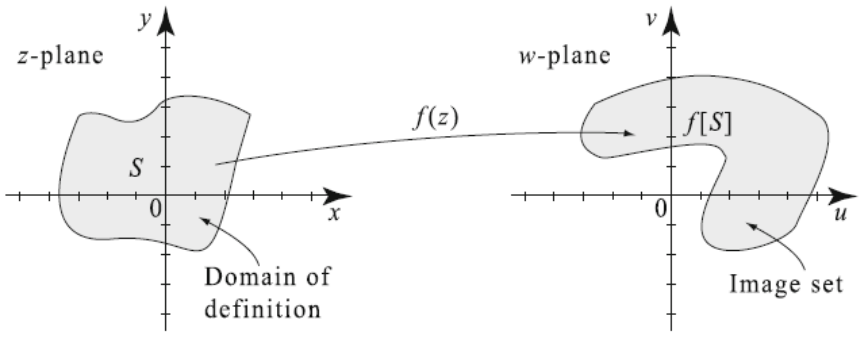


Fig. 1.22 To visualize a mapping by a complex-valued function $w = f(z)$, we use two planes: the z-plane or (x, y)-plane for the domain of definition and the w-plane or (u, v)-plane for the image.

function vis.png

Figure 1: Visualization for A Complex-valued Function

1.5 Complex Exponentials

For $x \in \mathbb{R}$, we recall that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Now, we extend the exponential function to the complex plane by substituting x with a complex number z.

Definition 3. Complex exponential function. For all $z \in \mathbb{C}$,

$$e^z = \sum_{n=0}^{\infty} \frac{|z|^n}{n!} = 1 + \frac{|z|}{1!} + \frac{|z|^2}{2!} + \dots < \infty.$$

Theorem 4. *Euler's identity. If $z = i\theta$ where $\theta \in \mathbb{R}$, then*

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

Corollary 5. *For all $z = x + iy \in CC$ where $x, y \in \mathbb{R}$,*

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y).$$

This is the polar form of e^z, where $|e^z| = e^x$ and $arg(e^z) = y + 2k\pi$ where $k \in \mathbb{Z}$, visualized as the figure below.

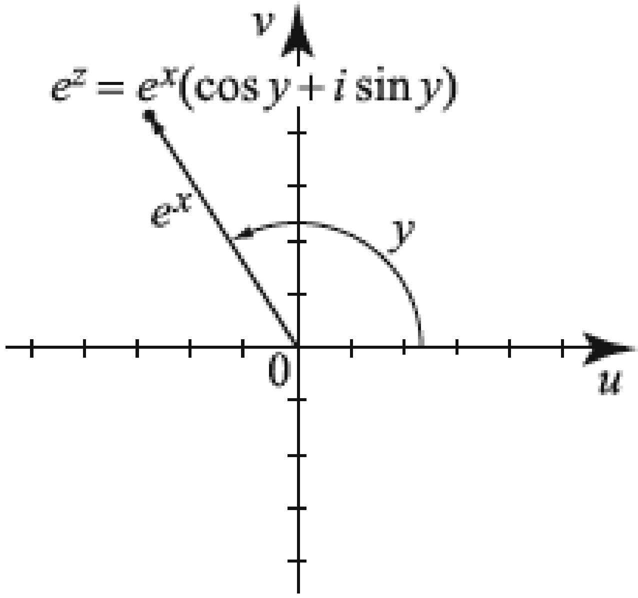
Euler's identity function is proven using Definition 3 and the power series expansions for $\cos\theta$ and $\sin\theta$. Since understanding the theorem is more important for this project than the proofs, I will not write the proof down. However, to better understand the theorem, I will solve the following example using Euler's identity.

Example 6. $e^{2+i\pi} = e^2(\cos\pi + i\sin\pi) = e^2(-1+0) = -e^2$. From this polar form, we can find the argument and absolute of the complex exponential. For $k \in \mathbb{Z}$,

$$\left|e^{2+i\pi}\right| = e^2, \mathrm{arg} e^{2+i\pi} = \pi + 2k\pi.$$

Proposition 7. *Exponential representation. Let $z = r(\cos\theta + i\sin\theta)$ with $r = |z| > 0, \theta \in \mathbb{R}$, $argz = \theta + 2k\pi$. Then,*

$$z = re^{i\theta}.$$



exp vis.png

Figure 2: The modulus and argument of e^z

1.6 Complex Logarithms

The logarithm is defined as the inverse of the exponential function. For $z \in \mathbb{C}\backslash 0$, we define the complex function $w = \text{log} z$, therefore getting

$$w = \text{log} z \iff e^w = z.$$

To determine w in terms of z, we write $w = u + iv$ and $z = re^{i\theta}$, with $|z| = r > 0$ and $\theta = \text{arg} z$. Then,

$$e^{u+iv} = e^u e^{iv} = z = re^{i\theta}$$

and hence $e^u = r$ and $e^{iv} = e^{i\theta}$. This means that $u = \text{ln} r$, and v and θ differ by an integer multiple of 2π because the complex exponential $2\pi i$ is periodic. So $v = \theta + 2k\pi \Rightarrow v = \text{arg} z$ where $k \in \mathbb{Z}$.

Definition 8. Complex logarithm. The formula for the complex logarithm is

$$\text{log} z = \ln |z| + i \text{arg} z.$$

Example 9. We know that $|i| = 1$ and $\text{arg} i = \frac{\pi}{2} + 2k\pi$. From this, we can obtain $\text{log} i$ to be the following.

$$\begin{aligned} \text{log} i &= \ln |i| + i \text{arg} i \\ &= \text{ln} 1 + i(\frac{\pi}{2} + 2k\pi) \\ &= i(\frac{\pi}{2} + 2k\pi) \end{aligned}$$

Definition 10. Principal branch. The principal value or branch of the complex logarithm is defined by

$$\text{Log}z = \ln|z| + i\text{Arg}z.$$

$\text{Log}z$ is the particular value of $\log z$ whose imaginary part is in the interval $(-\pi, \pi]$.

Definition 11. Branch. Let α be a fixed real number. For $z \neq 0$, we call the unique value of $\arg z$ that falls in the interval $(\alpha, \alpha + 2\pi]$ the α-th branch of $\arg z$, denoting it by $\arg_\alpha z$.

$$\log_\alpha z = \ln|z| + i\arg_\alpha z, \text{where}, \alpha < \arg_\alpha z < \alpha + 2\pi$$

1.7 Complex Powers

Definition 12. Complex power. For $z \in \mathbb{C}$,

$$z^a = e^{a\log z}$$

where $\log z$ is the complex logarithm.

Example 13. Evaluating complex numbers

1. Using the principal branch of the logarithm in Definition 6, we know that $\text{Log}(-i) = \ln|-i| + \text{Arg}(-1) = 0 + \frac{-i\pi}{2}$. Using the Definition 8, we obtain

$$\begin{aligned}(-i)^{1+i} &= e^{(1+i)\text{Log}(-i)} \\ &= e^{(1+i)\frac{-i\pi}{2}} \\ &= e^{\frac{-i\pi}{2}+\frac{pi}{2}} = -ie^{\frac{\pi}{2}}\end{aligned}$$

2. Using the logarithm with a branch cut at angle 0 in Definition 7, we obtain

$$(i)^{1+i} = e^{(1+i)\log_0(-i)} = e^{(1+i)\frac{3i\pi}{2}} = -ie^{\frac{-3\pi}{2}}.$$

2 Analytic Functions

2.1 History and Cauchy's Contributions

Most of the theory of analytic functions is due to Augustin-Louis Cauchy.

1. Defined the derivative and integral of complex functions
2. Defined the notion of limit for functions and gave rigorous definitions of continuity and differentiability for real-valued functions
3. Developed groundwork for the theory of definite integrals and series
4. Established theoretical aspects of complex analysis with great attention to rigorous mathematical proof which characterizes pure mathematics

2.2 Open Sets

Definition 14. Neighborhoods. Let $r > 0$ be a positive real number and z_0 a point in the plane. The r-neighborhood of z_0 is the set of all complex numbers z satisfying $|z - z_0| < r$. We denote this set by $B_r(z_0)$.

Definition 15. Deleted Neighborhood.

$$B_r'(z_0) = z : 0 < |z - z_0| < r.$$

Definition 16. Let S be a subset of $\mathbb{C}$.

1. Interior point: z_0 is an interior point of S if we can find a neighborhood of z_0 that is wholly contained in S.
2. Boundary point: z in the complex plane is called boundary point of S if every neighborhood of z contains at least one point in S and at least one point not in S.
3. Boundary: the set of all boundary points of S is called the boundary of S.

Definition 17. Closure. A subset S of the complex numbers is called open if every point in S is an interior point of S. An r-neighborhood, $B_r(z_0)$ is an open disk of radius r centered at z_0. Sets that contain all of their boundary points are called closed. For example, $z : |z - z_0| \leqslant r$ is a closed disk. The smallest closed set that contains a set A is called the closure of A.

Definition 18. Complex Derivative. Let f be defined on an open subset U of $\mathbb{C}$ and let $z_0 \in U$. We say that f has a complex derivative at the point z_0 if the limit

$$\lim z \to z_0 \frac{f(z) - f(z_0)}{z - z_0}$$

exists. This is called the complex derivative of f at z_0 and is denoted by $f'(z_0)$.

We say that f is analytic on U if it has a complex derivative at every point in U.

Definition 19. Entire function. An analytic function defined on the complex plane is said to be entire.

Lemma 20. *If c is a constant, $f(z) = cz \Rightarrow f'(z) = c$.*

Theorem 21. *An analytic function defined on an open subset of the complex plane is continuous.*

Lemma 22. *Discontinuous function at z_0 does not have a complex derivative at z_0.*

Theorem 23. *Properties of Analytic Functions. Suppose that f and g are analytic functions on an open subset U of the complex plane and let c_1, c_2 be complex constants. Then,*

1. *$c_1 f + c_2 g$ and fg are analytic on U and for all $z \in U$.*
2. *The function fg is analytic on U and for all $z \in U$.*
3. *The function $\frac{f}{g}$ is analytic on $W = U \setminus w \in U : g(w) = 0$ and for all $z \in W$,*

$$(\frac{f}{g})'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}$$

Theorem 24. *Cauchy-Riemann Equations Let U be an open subset of $\mathbb{R}^2$ and let u, v be real-valued functions defined on U. Then the complex-valued function $f(x + iy) = u(x, y) + iv(x, y)$ is analytic on U if and only if u, v are differentiable functions on U and satisfy*

$$u_x = v_y, u_y = -v_x$$

for all points in U. If this is the case, then for all $(x, y) \in U$, we have

$$f'(x + iy) = u_x(x, y) + iv_x(x, y) or f'(x_i y) = v_y(x, y) - iu_y(x, y).$$

2.3 Differentiation of Analytic Functions

Theorem 25. *Derivative of the complex exponential function. Let* $f(z) = e^z$ *where* $z \in \mathbb{C}$*. Then* f *is analytic on all of* $\mathbb{C}$ *and* $f'(z) = e^z$*.*

Theorem 26. *Derivative of the complex logarithmic function. Let* $f(z) = Log(z)$ *where* $Log(z) = log|z| + iArg(z)$ *where* $Arg(z)$ *is such that* $0 < Arg(z) < 2\pi$*, i.e., the principal branch of the logarithmic function. Then* f *is analytic on all of* $\mathbb{C} \setminus \{x + yi \in \mathbb{C} : x \geqslant 0, y = 0\}$*, and* $f'(z) = \frac{1}{z}$ *on this set.*

Theorem 27. *Derivative of the complex power function. Let* $f(z) = z^a$ *where* $z \in \mathbb{C}$*. Then* f *is analytic on all of* $\mathbb{C}$ *and* $f'(z) = az^{a-1}$*.*

Analytic functions obey the following rules.

1. Linearity of derivatives: $\frac{d}{dz}(cf(z) + dg(z)) = cf' + dg'$
2. Product rule: $\frac{d}{dz}(f(z)g(z)) = f'g + fg'$
3. Quotient rule: $\frac{d}{dz}\frac{(f(z)}{g(z)} = \frac{g'(f(z)}{g^2}$
4. Chain rule: $\frac{d}{dz}g(f(z)) = g'(f(z))f'(z)$
5. Inverse rule: $\frac{df^{-1}(z)}{dz} = \frac{1}{f'(f^{-1}(z)}$

Therefore, it is sufficient to conclude that analytic function obey the same differentiation rules as real functions.