Asymptotic zero locus

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1 Hypothesis

For $x > 0$, as $x \to \infty$, $f(x) = g(x)$ approximately when $f(x) = 0$, ie when x is an integer multiple of π.

2 Qualitative asymptotics

As x gets larger, the approximation gets better. This is because $g(x)$ tends to zero.

3 Quantitative asymptotics

First, we recall the Maclaurin series representations for $\cos, \sin$, and exponential functions.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Hence,

$$\begin{gathered} e^{-x} = \sin x \iff \\ 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \iff \\ 1 - 2x + \frac{x^2}{2!} + \frac{x^4}{4!} - 2\frac{x^5}{5!} + \frac{x^6}{6!} + \cdots = 0 \iff \\ (1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots) - 2(x + \frac{x^5}{5!} + \frac{x^9}{9!} + \frac{x^{13}}{13!} + \cdots) = 0. \end{gathered}$$

At the kth positive solution, the bound on the error is