Notes on Complex Analysis

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Abstract

This document is to serve as a set of notes to fill the gaps in my understanding of Complex Analysis relevant to the project. This is thus not intended to be comprehensive, since only the concepts I am currently unfamiliar with will be included. The reference book is *Complex Analysis with Applications* [1].

1 The Complex Field

Definition 1. A complex number z is an ordered pair (a, b) of real numbers. The set of all complex numbers is denoted by $\mathbb{C}$. We think of $\mathbb{C}$ as a vector space over the real numbers, and we define

$$1 \equiv (0, 0) \text{ and } 0 \equiv (0, 1).$$

Then the set of real numbers $\mathbb{R}$ is contained in $\mathbb{C}$, and for a, b real, we have the identification

$$(a, b) = a(1, 0) + b(0, 1) \equiv a1 + bi = a + bi.$$

This definition is intuitively very similar to the definition of the vector space over $\mathbb{R}^2$, where $1 \equiv (0, 0)$ and $0 \equiv (0, 1)$ are analogous to the idea of "unit vectors," or more generally basis. This prompted me to explore the similarities and differences between the $\mathbb{R}^2$ and $\mathbb{C}$, through which my initial intuition is proven to be useful only to a limited extent and can be potentially misleading.

A comparison between $\mathbb{R}^2$ and $\mathbb{C}$:

1. The most fundamental thing is $\mathbb{R}^2$ is a vector space (over the field of real numbers), whereas $\mathbb{C}$ is a field.

2. In a vector space of $\mathbb{F}^n$, multiplication between vectors is not defined.

   Recall that we define a vector space to be a set V with an addition and a scalar multiplication on V. In contrast, multiplication is defined on the field of complex numbers, and so as the other algebraic properties associated with multiplication (for example, multiplicative identity, multiplicative inverse, commutativity and associativity of multiplication).

   Note that the multiplicative inverse for the field of complex numbers is defined with the concept of the complex conjugate.

3. A lingering question: is the vector space $\mathbb{C}$ isomorphic to the vector space $\mathbb{R}^2$.

   This question is misleading! A vector space isomorphism is only defined between two vector spaces over the same field. $\mathbb{R}^2$ is a two dimensional field over $\mathbb{R}$ and the vector space $\mathbb{C}$ is a one dimensional vector space over field $\mathbb{C}$.

However, the field of complex numbers can be viewed as an extension field of $\mathbb{R}$ which treats $\mathbb{C}$ as a two dimensional vector space over R with basis 1, i.

The clearest relationship between $\mathbb{C}$ and $\mathbb{R}^2$ is to say that: "C is a two dimensional extension field of R." (See Hungerford's Algebra, 1974.)

(The last point was retrieved from this lecture note.)

2 Cartesian Form: $z = x + iy$.

The first way to visualize the complex numbers is to think of them as points in a Cartesian plane. This is achieved by associating each complex number $z = x + iy$ the ordered pair (x, y) and then plot the point $P = (x, y)$. (hence the construction of the real axis, the imaginary axis, and the complex plane)

Some unfamiliar operations associated with the Cartesian form:

- The Absolute Value (also called norm, modulus)

$$|z| = \sqrt{x^2 + y^2}$$
$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

- The Absolute Identity

$$|z| = \sqrt{z\bar{z}} \text{ or } |z|^2 = z\bar{z}$$

  and its corollaries (refer to AG2010a [1]).

- The Absolute Inequalities

$$|\operatorname{Re} z| \leqslant |z|, |\operatorname{Im} z| \leqslant |z|;$$
$$|z| \leqslant |\operatorname{Re} z| + |\operatorname{Im} z|;$$

  and the triangle inequality:

$$||z_1| - |z_2|| \leqslant |z_1 + z_2| \leqslant |z_1| + |z_2|,$$
$$||z_1| - |z_2|| \leqslant |z_1 - z_2| \leqslant |z_1| + |z_2|.$$

3 Polar Form: $z = r(\cos\theta + i\sin\theta)$.

Another useful way to visualize the complex numbers with "rays from the origin." This is achieved by identifying a complex number with the pair (r, θ) :

- r is the distance from P to the origin O, and
- θ is the angle between the x-axis and the ray OP.

We define the polar form formally:

Definition 2. Let $z = x + iy$ be a nonzero complex number. We define a number $r > 0$ by setting $r = \sqrt{x^2 + y^2} > 0$, and let θ be an angle such that

$$\theta = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r}, \sin\theta = \frac{y}{\sqrt{x^2 + y^2}} = \frac{y}{r}.$$

r is called the modulus of z and θ the argument of z.
$z = r(\cos\theta + i\sin\theta)$.

3.1 Arg z and arg z

An important concept that comes with the polar form is the principle vlaue of the argument, defined as follow:

Definition 3. The *principle value of the argument* of a complex number $z = x+iy$ is the unique number Arg z with the properties:

- $-\pi < \text{Arg } z \leqslant \pi$,
- $\cos(\text{Arg } z) = \frac{x}{|z|}$,
- $\sin(\text{Arg } z) = \frac{y}{|z|}$.

With this unique principle value defined, we can then derive the set of all values of argument: $\arg z = \{\text{Arg } z + 2k\pi : k = 0, t_1, t_2, \dots\}$.

3.2 Arithmetics of the Polar Form

- Multiplication.
$$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$
- Division.
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta + 1 - \theta_2) + i\sin(\theta_1 - \theta_2))$$

3.3 Roots of Complex Numbers

Definition 4. Let $w \neq 0$ be a complex number and n a positive integer.
A number z is called an nth roots of w if $z^n = w$.

Proposition 5. *(De Moivre's Identity)* $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

Let $w = \rho(\cos\phi + i\sin\phi) \neq 0$. The nth roots of w are the solutions of the equation $z^n = w$. These are
$$z_{k+1} = \rho^{\frac{1}{n}}(\cos(\frac{\phi}{n} + \frac{2k\pi}{n}) + i\sin(\theta_1 + \theta_2)),$$
where $k = 0, 1, \dots, n-1$.

The unique number z such that
$$z^n = w \text{ and Arg } z = \frac{\text{Arg } w}{n}$$
is called the principal nth root of w.

The principle root is obtained by taking $\phi = \text{Arg } w$ and $k = 0$.

4 Complex Functions

Technically, if we follow the convention for visualizing a real-valued function, then a complex function should be viewed in four-dimensional space. But that is practically impossible and not useful. Thus, we think of a complex function as a mapping from the z-plane to the w-plane, as shown in the figure below.

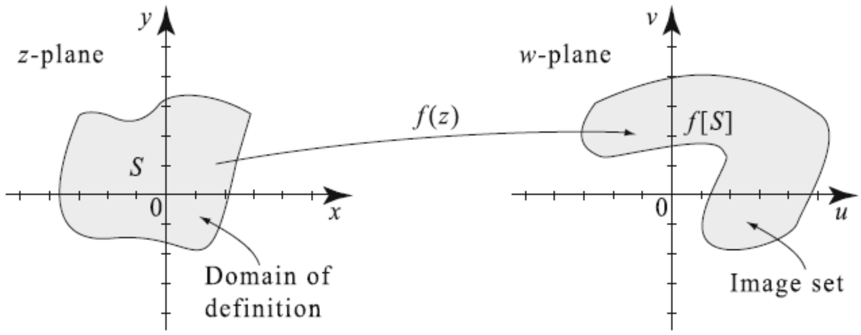

Fig. 1.22 To visualize a mapping by a complex-valued function $w = f(z)$, we use two planes: the z-plane or (x, y)-plane for the domain of definition and the w-plane or (u, v)-plane for the image.

Figure 1: Visualization for A Complex-valued Function

4.1 Transformations

1. Translations (linear).

   Mappings of the form $z \to z + a + bi$ are *translations* by a units up/down, and b units right/left.

   Example 6. Let S denote the disk $S = \{z : |z| \leqslant 1\}$. Find the image of S under the mapping $f(z) = z + 2 + i$.

   This function translates the point z two units to the right and one unit up.

2. Dilations (linear).

   Mappings of the form $z \to rz$, where $r > 0$ are *dilations* by a factor r.

3. Rotations (linear).

   Mappings of the form $z \to (\cos\theta + i\sin\theta)z$ are *rotations* by the angle θ.

4. Identity (linear).

   Mappings of the form $z \to z$.

5. Linear Fractional Transformation or Mobius Transformation (non-linear).

   Mappings of the form

   $$z \to w, w = \frac{az + b}{cz + d}(ad \neq bc).$$

   A special case: inversion.

   Mappings of the form $z \to \frac{1}{z}$.

4.2 Real and Imaginary Parts of Functions

For a complex function f, let $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$. The functions u, v are real-valued functions. With a slight abuse of notation, we can write them as the function for the real part $u(x, y)$ and the function for the imaginary part $v(x, y)$.

The motivation behind splitting the complex function into two separate functions, one for the real part and one for the imaginary part is to help us determine algebraically the image of a set when the answer is not geometrically obvious.

4.3 Mappings in Polar Coordinates

Some complex functions and regions are more naturally suited to polar coordinates, especially in our project (and specifically `issue #40`). It is useful to express $w = f(z)$ in polar form:

$$w = \rho(\cos\phi + i\sin\phi).$$

Using similar logic behind splitting a complex function into $u(x, y)$ and $v(x, y)$, we can identify the polar coordinates of w with a "modulus function" and an "argument function," as defined:

$$\rho(r, \theta) = |f(r\cos\theta + ir\sin\theta)|,$$
$$\phi(r, \theta) = \arg(f(r\cos\theta + ir\sin\theta)).$$

5 The Complex Exponential

5.1 Complex Exponential Function: $\exp(z)$ or e^z

Definition 7. We define the *complex exponential function* $\exp(z)$ or e^z as the convergent series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots, \forall z \in \mathbb{C}.$$

Remark 8. This is exactly analogous to the real exponential function:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots,$$

reminding us again of the idea that $\mathbb{C}$ is an extension field of $\mathbb{R}$.

Similarly, the properties of the complex exponential function follow exactly as those of the real exponential function. (refer to 1.6.3 to 1.6.5 in AG2010a [1])

5.2 Euler's identity and its consequencies

Proposition 9. *(Euler's Identity) If $z = i\theta$, where θ is real, then*

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Using Euler's identity, we can express e^z in terms of the basic functions: $e^x, \cos x, \sin x$, where $x \in \mathbb{R}$.

Corollary 10. *For $z = x + iy$, with x, y real, we have*

$$e^z = e^x(\cos y + i\sin y) = e^x\cos y + ie^x\sin y.$$

Taking real and imaginary parts of the above, we find

$$\mathrm{Re}(e^z) = e^x\cos y \text{ and } \mathrm{Im}(e^z) = e^x\sin y.$$

Following from the previous discussion, we also have the facts:

$$|e^z| = e^x > 0,$$

$$\arg(e^z) = y + 2k\pi, k \in \mathbb{Z}.$$

Another useful consequence of Euler's Identity is to give rise to an exponential representation of a complex number. To realize that the polar form and exponential form are in fact equivalent, refer to the figure below.

Proposition 11. *(Exponential Representation) Let* $z = r(\cos\theta + i\sin\theta)$ *with* $r = |z| > 0, \theta \in \mathbb{R}, \arg z = \theta + 2k\pi$. *Then*

$$z = re^{i\theta},$$

and its corollaries. (see 1.6..17 to 1.6.20 in AG2010a [1])

5.3 The Exponential as a Mapping

The equation $\arg(e^z) = y + 2k\pi, k \in \mathbb{Z}$ discussed in the previous section has a consequence: the argument of e^z is equal to the imaginary part of z. Because of this, we expect the exponential function to map line segments (think the imaginary axis) to circular arcs (think the arc formed due to θ), and by the same logic, rectangular regions to circular regions.

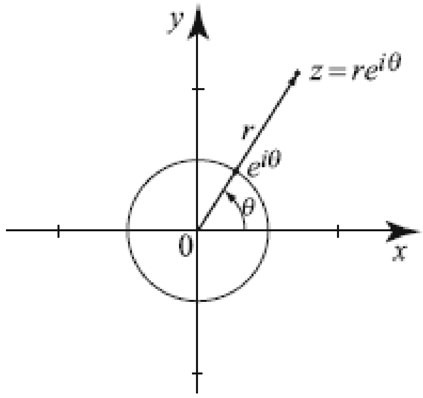

Fig. 1.39 Plotting $z = re^{i\theta}$.

Figure 2: Visualization for Exponential Representation

6 Relating Trigo Functions to the Exponential Function

Definition 12. For a complex number z, we set

$$\cos z = \frac{e^{iz} + e^{-iz}}{2},$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

References

[1] Nakhlé Asmar and Loukas Grafakos, *Complex analysis with applications*, Undergraduate texts in mathematics, Springer, 2010.