Introduction to simplicial homology
(work in progress)

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Contents

Chapter 0

Algebraic preliminaries

The study of simplicial homology requires basic knowledge of some fundamental concepts from abstract algebra. The purpose of this introductory chapter is to introduce these concepts. For a more detailed treatment of the subject, we refer the reader to a textbook on groups, rings and modules.

0.1 Groups, rings, and fields

0.1.1 Groups

An *abelian group* $(G, +)$ is a set G together with a binary operation $+ : G \times G \to G$ satisfying the following axioms:

G1 $(a + b) + c = a + (b + c)$ for all $a, b, c \in G$.

G2 There is an identity element 0 satisfying $a + 0 = 0 + a = a$ for each $a \in G$.

G3 For each $a \in G$, there is an element $-a$, the *inverse* of a, such that $a + (-a) = (-a) + a = 0$.

G4 $a + b = b + a$ for all $a, b \in G$.

The first three axioms define a group, whereas the fourth axiom makes the group abelian. For the purposes of this document, only abelian groups are of importance. We typically let G denote the group $(G, +)$, thus assuming that the binary operation $+$ is clear from context.

Some important abelian groups are the set of integers $\mathbb{Z}$, the set of rationals $\mathbb{Q}$, the set of reals $\mathbb{R}$, and the set of complex numbers $\mathbb{C}$. In all cases, the operation $+$ is just ordinary addition. Another important example is $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ equipped with addition modulo n.

We also have a group structure on each of the sets $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$, $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. In this case, the group operation is multiplication, and the identity element is 1. To avoid confusion, let us restate the axioms in the language of multiplicative groups:

G1' $(ab)c = a(bc)$ for all $a, b, c \in G$.

G2' There is an identity element 1 satisfying $a \cdot 1 = 1 \cdot a = a$ for each $a \in G$.

G3' For each $a \in G$, there is an element a^{-1}, the *inverse* of a, such that $aa^{-1} = a^{-1}a = 1$.

G4' $ab = ba$ for all $a, b \in G$.

Here, we write $ab = a \cdot b$.

Note that $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$ is not a group under multiplication. Namely, given $a \in \mathbb{Z}^*$, there is no $b \in \mathbb{Z}^*$ such that $ab = 1$ unless $a \in \{1, -1\}$. For any prime p, the set $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus \{0\}$ turns out to be a group under multiplication modulo p. If n is the product of two integers a and b, both at least 2, then $\mathbb{Z}_n^*$ is not a group. Namely, suppose that there exists an integer x such that $ax = 1$ in $\mathbb{Z}_n$. Then

$$b = b \cdot 1 = b(ax) = (ba)x = nx = 0$$

in $\mathbb{Z}_n$. This is a contradiction; hence a is not invertible in $\mathbb{Z}_n$.

0.1.2 Rings

A *commutative ring* $(R, +, \cdot)$ is a set R together with two binary operations $+ : R \times R \to R$ and $\cdot : R \times R \to R$ satisfying the following axioms:

R1 R forms an abelian group under addition.

R2 There is an element $1 \in R$ satisfying $1 \cdot a = a \cdot 1 = a$ for all $a \in R$.

R3 $a(bc) = (ab)c$ for all $a, b, c \in R$.

R4 $(a + b)c = ac + bc$ for all $a, b, c \in R$.

R5 $ab = ba$ for all $a, b \in R$.

The first four axioms define a ring, whereas the fifth axiom makes the ring commutative. For the purposes of this document, only commutative rings are of importance. As for groups, we let R denote the ring $(R, +, \cdot)$.

We have that $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Z}_n$ are all commutative rings. In this document, we will state most definitions and results in terms of an arbitrary commutative ring, but the reader may always think of this ring as being one of the rings just mentioned. The most important ring for our purposes is $\mathbb{Z}$.

By the way, note the disctinction between R and $\mathbb{R}$. The former will typically denote any commutative ring, whereras the latter denotes the particular ring of real numbers.

0.1.3 Fields

A *field* is a commutative ring $\mathbb{F}$ such that $\mathbb{F} \setminus \{0\}$ forms an abelian group under multiplication. The *characteristic* of a field is the smallest positive integer p such that the sum $1 + \cdots + 1$ of p ones is zero. If no such p exists, then we define the characteristic to be zero. The characteristic of a field is either a prime number or zero.

By the examples in Section 0.1.1, we deduce that $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are infinite fields. Moreover, $\mathbb{Z}_p$ is a finite field for each prime p. The former fields have characteristic zero, whereas $\mathbb{Z}_p$ has characteristic p. We may also deduce that $\mathbb{Z}$ is not a field, and neither is $\mathbb{Z}_n$ when n is composite.

0.2 Modules

0.2.1 Definition of vector spaces and modules

Let $\mathbb{F}$ be a field. A *vector space* $(V, \mathbb{F}, +, \cdot)$ over the field $\mathbb{F}$ is a set V, together with an addition operation $+ : V \times V \to V$ and a scalar multiplication operation $\cdot : \mathbb{F} \times V \to V$, satisfying the following axioms:

1. V forms an abelian group under addition.
2. $(ab)v = a(bv)$ for all $a, b \in \mathbb{F}$ and $v \in V$.
3. $a(u + v) = au + av$ and $(a + b)v = av + bv$ for all $a, b \in \mathbb{F}$ and $u, v \in V$.
4. $1 \cdot v = v$ for all $v \in V$.

We let V denote the vector space $(V, \mathbb{F}, +, \cdot)$.

In Section 0.1.3, we noted that the set of integers $\mathbb{Z}$ does not form a field. In particular, we are not allowed to define vector spaces over $\mathbb{Z}$. Yet, axioms 1-4 above still make sense for $\mathbb{F} = \mathbb{Z}$. Given any commutative ring R, we say that $V = (V, R, +, \cdot)$ is an *R-module* if axioms 1-4 are satisfied with $\mathbb{F} = R$. Note that $\mathbb{F}$-modules and vector spaces over $\mathbb{F}$ coincide when $\mathbb{F}$ is a field.

Example. For a given ring $(R, +, \cdot)$, let $R' = (R, +)$ denote the underlying abelian group. The group R' becomes an R-module if we define the product of $a \in R$ and $x \in R'$ in the obvious way, i.e., as the product ax in R viewed as an element in R'. One typically uses the same symbol R to denote both the ring and the corresponding R-module. Indeed, the formal description of R' is $(R, R, +, \cdot)$. Compare to the convention that $\mathbb{R}$ denotes both the field of reals and the one-dimensional vector space over this field.

Example. Let $n \geq 1$ be an integer. Consider the additive group $\mathbb{Z}_n$. We obtain a $\mathbb{Z}$-module structure on $\mathbb{Z}_n$ by defining the product between $x \in \mathbb{Z}$ and $a \in \mathbb{Z}_n$ to be $(xa) \bmod n$. It is straightforward to check that $\mathbb{Z}_n$ satisfies the axioms for a $\mathbb{Z}$-module.

A subset S of an R-module M is a *submodule* of M if S itself has the structure of an R-module with addition and scalar multiplication being the restrictions of the corresponding operations in M. If R is a field, then submodules are the same as subspaces.

Example. Let $k\mathbb{Z}$ denote the $\mathbb{Z}$-module of integer multiples of k. Then $k\mathbb{Z}$ is a submodule of the $\mathbb{Z}$-module $\mathbb{Z}$.

0.2.2 Homomorphisms and isomorphisms

Let M and N be two R-modules. An *R-module homomorphism* is a map $f : M \to N$ such that

$$f(am + bn) = af(m) + bf(n)$$

for any $a, b \in R$ and $m, n \in M$. In the case that R is a field, one typically refers to module homomorphisms as *linear maps* or *linear transformations*.

An *isomorphism* is a bijective homomorphism. Two modules M and N are *isomorphic*, written $M \cong N$, if there exists an isomorphism $f : M \to N$.

The relation of being isomorphic is an equivalence relation. For example, the relation is symmetric, because if $f : M \to N$ is an isomorphism, then its inverse $f^{-1} : N \to M$ is a homomorphism and hence an isomorphism.

Example. For each $k \geq 2$, we have that the map $\varphi : \mathbb{Z} \to \mathbb{Z}$ given by

$$\varphi(x) = kx$$

is a homomorphism but not an isomorphism. For example, there is no x such that $\varphi(x) = 1$. Now, let $k\mathbb{Z}$ denote the $\mathbb{Z}$-module of integer multiples of k. Then φ defines an isomorphism from $\mathbb{Z}$ to $k\mathbb{Z}$, and the inverse is given by mapping y to y/k for each $y \in k\mathbb{Z}$.

A homomorphism f is injective if and only if the only solution to the equation $f(x) = 0$ is $x = 0$. Namely, $f(x) = f(y)$ if and only if $f(x - y) = 0$.

0.2.3 Sums and direct sums

Given R-modules $M_1, M_2, \ldots, M_k$, we define the *direct sum*

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_k$$

to be the set of all vectors $(m_1, m_2, \ldots, m_k)$ such that $m_i \in M_i$ for $1 \leq i \leq k$.[1] We obtain an R-module structure on M by defining vector addition and scalar multiplication in the obvious manner:

$$\begin{aligned} (m_1, \ldots, m_k) + (m_1', \ldots, m_k') &= (m_1 + m_1', \ldots, m_k + m_k'), \\ r(m_1, \ldots, m_k) &= (rm_1, \ldots, rm_k). \end{aligned}$$

Given two submodules A and B of the same R-module M, we may form the *sum*

$$A + B = \{a + b : a \in A, b \in B\}.$$

The reader may check that $A + B$ is an R-module. We have a surjective homomorphism $\varphi : A \oplus B \to A + B$ given by

$$\varphi(a, b) = a + b.$$

The map φ is an isomorphism if and only if every $x \in A + B$ can be expressed in only one way as a sum $a + b$ such that $a \in A$ and $b \in B$. In such a situation, we will identify the sum $A + B$ with the direct sum $A \oplus B$. A useful criterion is that a given sum $A + B$ is direct if and only if

$$a + b = 0 \Longleftrightarrow a = b = 0.$$

This is equivalent to saying that $A \cap B = \{0\}$.

[1] One may also consider direct sums of infinitely many R-modules, but we will not make any use of this construction in this document.

0.2.4 Free modules

An important R-module is R^n, the direct sum of n copies of R. We refer to such a module as a *free R-module of rank n*. When R is a field, this is the same as an n-dimensional vector space over R. For arbitary commutative rings, the situation is more complicated. For example, we already saw that $\mathbb{Z}_n$ is a $\mathbb{Z}$-module, and this module is not free. Indeed, free $\mathbb{Z}$-modules are either infinite or zero, whereas $\mathbb{Z}_n$ is finite and nonzero for $n \geq 2$.

An R-module M is *finitely generated* if there is a finite set $\{m_1, \ldots, m_k\} \subseteq M$ such that every element in M can be written as $r_1 m_1 + \cdots + r_k m_k$ for some $r_1, \ldots, r_k \in R$. We write

$$M = \langle m_1, \ldots, m_k \rangle. \tag{1}$$

In this document, we will focus almost entirely on finitely generated R-modules.

Let M be a free and finitely generated R-module. A subset $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ in M forms a *basis* for M if every element in M can be written as a linear combination of $\mathbf{b}_1, \ldots, \mathbf{b}_n$ in a unique manner. Specifically, for any $x \in M$, there are unique elements $\lambda_1, \ldots, \lambda_n \in R$ such that

$$x = \lambda_1 \mathbf{b}_1 + \cdots + \lambda_n \mathbf{b}_n, \tag{2}$$

and any such linear combination is an element in M. Just as for vector spaces, every basis of a free R-module M has the same number of elements, the *rank* of M.

Given any set $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, we may define the *free R-module with basis B*. By definition, this module consists of all formal linear combinations of the form (2). In particular, the module is isomorphic to R^n.

0.2.5 Finitely generated abelian groups

By a famous result known as the *fundamental theorem of finitely generated abelian groups*, there is a representation of every finitely generated $\mathbb{Z}$-module M as a direct sum of a free $\mathbb{Z}$-module and a finite $\mathbb{Z}$-module;

$$M = \mathbb{Z}^n \oplus G,$$

where G is finite. Moreover, this representation is unique up to isomorphism. Specifically, if M admits another decomposition as $M = \mathbb{Z}^{n'} \oplus G'$, where G' is finite, then $n = n'$ and $G = G'$. The value n is the *rank* of M. In particular, any finite $\mathbb{Z}$-module has rank zero.

In turn, by the *fundamental theorem of finite abelian groups*, any finite $\mathbb{Z}$-module G admits a decomposition as a direct sum

$$G = \mathbb{Z}_{q_1} \oplus \mathbb{Z}_{q_2} \oplus \cdots \oplus \mathbb{Z}_{q_k},$$

where $q_1 \leq q_2 \leq \cdots \leq q_k$, and q_i is a power of a prime for each i. Again, the decomposition is unique up to isomorphism.

Note that the stated theorems refer to abelian groups, not $\mathbb{Z}$-modules. Yet, for any abelian group M, there is one and only one way to define a $\mathbb{Z}$-module structure on M. We thus have a bijective correspondence between $\mathbb{Z}$-modules and abelian groups, which means that we may suppress the module structure from notation. For example, we will often refer to the free $\mathbb{Z}$-module $\mathbb{Z}^n$ as a *free (abelian) group of rank n*.

To obtain the bijective correspondence, let $G = (G, +)$ be an abelian group. We want to impose a $\mathbb{Z}$-module structure $(G, +, \cdot)$ on G, which amounts to defining scalar multiplication. By axiom 4 in Section 0.2.1, we must define $1 \cdot m = m$ for all $m \in G$. Moreover, letting $k \geq 1$, repeated application of axiom 3 yields that $k \cdot m$ must be the sum of k copies of m, whereas another application of axiom 3 yields that we must define $(-k) \cdot m = -(k \cdot m)$. Namely,

$$m = (0 + 1) \cdot m = 0 \cdot m + 1 \cdot m = 0 \cdot m + m \Longrightarrow 0 \cdot m = 0$$

and

$$0 = 0 \cdot m = (k + (-k)) \cdot m = k \cdot m + (-k) \cdot m.$$

As a consequence, there is indeed only one way to define scalar multiplication. Conversely, one easily checks that the given procedure indeed yields a valid scalar multiplication and hence a well-defined $\mathbb{Z}$-module.

An important fact about abelian groups is that any subgroup of a free abelian group is again free. This is not true for general modules. More precisely, there exist rings R with the property that a submodule of a free R-module is not necessarily a free R-module.

0.2.6 Quotients

Let Z be an R-module, and let B be a submodule of Z. For $x, y \in Z$, define $x \sim y$ to mean that $y - x \in B$. This is an equivalence relation and gives rise to equivalence classes. We define Z/B to be the set of such equivalence classes. Specficially, each element in Z/B is of the form

$$[x] = \{x + b : b \in B\} = \{y : y - x \in B\}.$$

Sometimes, we will find it convenient to write

$$[x] = x + B.$$

We obtain an R-module structure on Z/B by defining

$$\lambda[x] + \mu[y] = [\lambda x + \mu y].$$

Using the definition, one may prove that this is well-defined.

There is a homomorphism $p : Z \to Z/B$ defined by $p(x) = x + B$. We refer to p as the *projection map*.

Example. Let $Z = \mathbb{Z}$ and $B = k\mathbb{Z}$. Then $x \sim y$ if and only if $x \equiv y \pmod{k}$. In particular, the elements of $\mathbb{Z}/(k\mathbb{Z})$ are the congruence classes modulo k, which means that $\mathbb{Z}/(k\mathbb{Z})$ is isomorphic to $\mathbb{Z}_k$. The projection map $p : \mathbb{Z} \to \mathbb{Z}/(k\mathbb{Z})$ maps an integer to its congruence class modulo k.

In the case that R is a field, we have the important identity

$$\dim Z/B = \dim Z - \dim B. \tag{3}$$

To see this, fix a basis $e_1, \ldots, e_r$ for B, and extend the basis with elements $e_{r+1}, \ldots, e_{r+s}$ to a basis for Z. Then $[e_{r+1}], \ldots, [e_{r+s}]$ form a basis for Z/B.

Example. Let $Z = \mathbb{Q}^2$, and let B be the subspace generated by $(1,1)$. Then $(x,y) \sim (x',y')$ if and only if

$$(x,y) = (x',y') + t(1,1) \text{ for some } t \in Q,$$

which is equivalent to saying that $x - x' = y - y'$. This means that the equivalence classes are

$$[(t,0)] = \{(t+y,y) : y \in \mathbb{Q}\}.$$

In particular, Z/B is a one-dimensional space generated by $[(1,0)]$. Indeed, the two vectors $(1,1)$ and $(1,0)$ form a basis for $\mathbb{Q}^2$.

When $R = \mathbb{Z}$, we have an identity similar to (3):

$$\operatorname{rank} Z/B = \operatorname{rank} Z - \operatorname{rank} B. \tag{4}$$

For a complete proof, we refer the interested reader to a textbook on modules. It is important to note that Z/B is not necessarily free even if Z and B are both free. For example, we already noted that $\mathbb{Z}/(k\mathbb{Z}) \cong \mathbb{Z}_k$. This group is not free, whereas both $\mathbb{Z}$ and $k\mathbb{Z}$ are free of rank one.

Quotients commute with direct sums in the following sense.

Proposition 0.2.1 *Let Z_1 and Z_2 be R-modules, and let B_1 and B_2 be submodules of Z_1 and Z_2, respectively. Then*

$$\frac{Z_1}{B_1} \oplus \frac{Z_2}{B_2} \cong \frac{Z_1 \oplus Z_2}{B_1 \oplus B_2}.$$

Proof. Define $\varphi : \frac{Z_1}{B_1} \oplus \frac{Z_2}{B_2} \to \frac{Z_1 \oplus Z_2}{B_1 \oplus B_2}$ by

$$\varphi(z_1 + B_1, z_2 + B_2) = (z_1, z_2) + B_1 \oplus B_2.$$

This is a well-defined homomorphism, because if $z_i' = z_i + b_i$ for some $b_i \in B_i$ and $i \in \{1,2\}$, then

$$(z_1', z_2') + B_1 \oplus B_2 = (z_1, z_2) + (b_1, b_2) + B_1 \oplus B_2 = (z_1, z_2) + B_1 \oplus B_2.$$

The reader may check that φ is a bijection, which concludes the proof. □

0.2.7 Isomorphism theorems

There are three important results in abstract algebra known as the three *isomorphism theorems*. We state and prove them for R-modules.

One may associate two particularly important R-modules to any given homomorphism $f : M \to N$. The *kernel* of f is the submodule of M consisting of all $z \in M$ that are mapped to zero under f;

$$\ker f = \{z \in M : f(z) = 0\}.$$

The *image* under f is the submodule of N defined by

$$\operatorname{im} f = \{b \in N : b = f(x) \text{ for some } x \in M\}.$$

Theorem 0.2.2 (First Isomorphism Theorem) *Let $f : M \to N$ be a homomorphism. Then there exists an isomorphism*

$$g : \frac{M}{\ker f} \to \operatorname{im} f.$$

Proof. For $[x] \in M/\ker f$, we define $g([x]) = f(x)$. To see that this is well-defined, note that any element in $[x]$ is of the form $x + y$ for some $y \in \ker f$, and $f(x + y) = f(x) + f(y) = f(x)$. Now, g is injective, because

$$g([x]) = 0 \Longleftrightarrow f(x) = 0 \Longleftrightarrow x \in \ker f \Longleftrightarrow [x] = [0].$$

Moreover, g is surjective, because we have for any $y \in \operatorname{im} f$ that

$$y = f(x) = g([x]).$$

As a consequence, g is an isomorphism. □

As indicated by the discussion in Section 0.2.3, there is a close relationship between $A+B$ and $A \cap B$. The following theorem makes the connection precise.

Theorem 0.2.3 (Second Isomorphism Theorem) *Let A and B be two submodules of an R-module M. Then there is an isomorphism*

$$g : \frac{A}{A \cap B} \to \frac{A + B}{B}.$$

Proof. Define a map $f : A \to (A + B)/B$ by

$$f(a) = [a] = \{a + b : b \in B\}.$$

We have that f is surjective. Namely, suppose that $[x] \in (A + B)/B$. Then $x = a + b$ for some $a \in A$ and $b \in B$. We obtain that

$$[x] = [a] = f(a).$$

As a consequence, $\operatorname{im} f = (A+B)/B$. By the First Isomorphism Theorem 0.2.2, there exists an isomorphism $g : A/\ker f \to (A + B)/B$. Observing that $\ker f = A \cap B$, we are done. □

The following reformulation of the Second Isomorphism Theorem is very useful for our purposes.

Corollary 0.2.4 *Let B be a submodule of an R-module Z. Suppose that $B_0 \subseteq B$ and $Z_0 \subseteq Z$ are R-modules satisfying the equations*

$$\begin{cases} Z &= Z_0 + B, \\ B_0 &= Z_0 \cap B. \end{cases}$$

Then

$$\frac{Z}{B} \cong \frac{Z_0}{B_0}.$$

Proof. By the assumptions and the Second Isomorphism Theorem 0.2.3, we have that

$$\frac{Z}{B} = \frac{Z_0 + B}{B} \cong \frac{Z_0}{Z_0 \cap B} \cong \frac{Z_0}{B_0}.$$

□

Theorem 0.2.5 (Third Isomorphism Theorem) *Let A, B, and C be three R-modules such that C is a submodule of B and B is a submodule of A. Then there exists an isomorphism*

$$g : \frac{A/C}{B/C} \to \frac{A}{B}.$$

Proof. We define a map $f : A/C \to A/B$ by

$$f(a + C) = a + B,$$

where we let $a + C$ denote the element $\{a + c : c \in C\}$ in A/C, and analogously for $a + B$ in A/B. This is well-defined, because if $a' \in a + C$, then

$$a' - a \in C \subseteq B \Longrightarrow a' + B = a + B.$$

It is clear that f is surjective. By the First Isomorphism Theorem 0.2.2, we deduce that

$$\frac{A/C}{\ker f} \cong \frac{A}{B}.$$

Now, $a + C$ belonging to $\ker f$ is equivalent to $f(a + C) = B$, because B constitutes the zero element in A/B. This is true if and only if $a \in B$, which in turn is equivalent to saying that $a + C \in B/C$. Hence $\ker f = B/C$, which concludes the proof. $\square$

Chapter 1

Simplicial homology

1.1 Simplicial complexes

The term *simplicial complex* may refer to either of two seemingly unrelated concepts. The first concept is that of an *abstract* simplicial complex, which is a family of sets that is closed under deletion of elements. The second concept is that of a *geometric* simplicial complex, which is a geometric object in Euclidean space consisting of simplices[1] of various dimensions (points, line segments, triangles, tetrahedra, and so on), glued together according to certain rules. As we will see in a moment, the two concepts are in fact closely related: For every geometric simplicial complex, there is an underlying abstract simplicial complex describing its combinatorial structure. Conversely, one may realize any abstract complex as a geometric complex.

1.1.1 Abstract simplicial complexes

An *abstract simplicial complex* is a family Δ consisting of finite subsets of a given set X such that the following condition holds:

- If $\tau \in \Delta$ and $\sigma \subseteq \tau$, then $\sigma \in \Delta$.

Members of a simplicial complex Δ are called *faces* of Δ, and the *dimension* of a face τ is $|\tau| - 1$. The dimension of Δ is the maximum among all dimensions of faces of Δ. If $\sigma \subseteq \tau$, then we say that σ is a face of τ. A face τ is a *maximal face* of Δ if there is no face σ of Δ such that $\tau \subsetneqq \sigma$. We refer to 0-dimensional faces as *vertices* and to 1-dimensional faces as *edges*. A simplicial complex of dimension at most 1 is a (simple and loopless) *graph*. A simplicial complex Δ_0 is a *subcomplex* of Δ if $\Delta_0 \subseteq \Delta$. For $k \geq -1$, the *k-skeleton* $\Delta^{(k)}$ of Δ is the subcomplex of Δ obtained by removing all faces of dimension greater than k. For example, the 1-skeleton of Δ is a graph.

In this document, we will only consider finite simplicial complexes. Equivalently, our simplicial complexes will have a finite number of vertices. For each $n \geq -1$, we let $f_n = f_n(\Delta)$ be the number of faces of Δ of dimension n. The *f-vector* of Δ is the vector $(f_{-1}, f_0, f_1, \ldots, f_d)$, where d is the dimension of Δ.

[1] One *simplex*, several *simplices* or *simplexes*.

Running example 1. The sets

$$\emptyset, a, b, c, d, ab, ac, bc, bd, cd, abc$$

form a simplicial complex E_1. Here, we write $a = \{a\}$, $ac = \{a, c\}$, and so on. The f-vector of E_1 is $(1, 4, 5, 1)$.

1.1.2 Geometric realizations of simplicial complexes

One may realize a simplicial complex as a geometric object in $\mathbb{R}^n$, and the procedure is roughly the following. Identify each vertex with a point. For each edge ab, draw a line segment between the points realizing the vertices a and b. Next, for each 2-dimensional face abc, fill the triangle with sides given by the line segments realizing ab, ac, and bc. Continue in this manner in higher dimensions. For example, realize each 3-dimensional face $abcd$ as the tetrahedron with sides given by the four filled triangles realizing the 2-dimensional faces contained in $abcd$. Note that the full realization of an abstract simplicial complex is determined by how we realize the vertices of the complex.

Running example 1. Figure 1.1 gives a geometric realization in the plane of the simplicial complex E_1. Since the label of each face is given by the vertices it

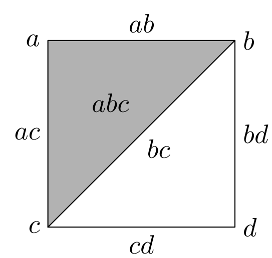

Figure 1.1: Geometric realization of E_1, our first running example.

contains, it suffices to label the vertices in the realization. From now on, we will typically not label any other faces.

For completeness, we give a formal definition of a geometric realization of a simplicial complex. For any $d \geq 0$, define the *standard d-simplex* to be the set

$$X_d = \{(\lambda_0, \ldots, \lambda_d) : \lambda_i \geq 0 \text{ for } 0 \leq i \leq d, \lambda_0 + \cdots + \lambda_d = 1\} \subset \mathbb{R}^{d+1}.$$

Equivalently, X_d is the convex hull of $(1, 0, \cdots, 0), (0, 1, \cdots, 0), \ldots, (0, 0, \cdots, 1)$. Let Δ be an abstract simplicial complex with vertex set V, and let $f : V \to \mathbb{R}^n$ be any map. For any nonempty d-face $\sigma = \{a_0, \ldots, a_d\}$ of Δ, we have that f induces a map $f_\sigma : X_d \to \mathbb{R}^n$ by

$$f_\sigma(\lambda_0, \ldots, \lambda_d) = \lambda_0 f(a_0) + \cdots + \lambda_d f(a_d).$$

In Figure 1.2, we illustrate a possible scenario in the case $d = 2$.

We say that f induces a *geometric realization* of Δ if the following hold:

- The map f_σ is injective for each $\sigma \in \Delta \setminus \{\emptyset\}$.

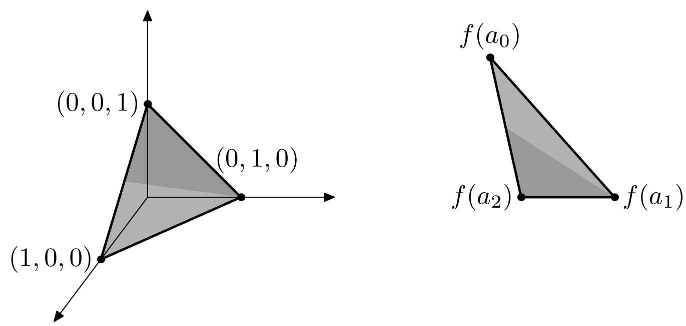

Figure 1.2: The map f_σ transforms the standard 2-simplex on the left into the filled triangle on the right, mapping $(1,0,0)$ to $f(a_0)$, $(0,1,0)$ to $f(a_1)$, and $(0,0,1)$ to $f(a_2)$.

- For any nonempty $\sigma, \tau \in \Delta$, we have that

$$\operatorname{im} f_\sigma \cap \operatorname{im} f_\tau = \operatorname{im} f_{\sigma\cap\tau},$$

  where $\operatorname{im} f$ denotes the image under f. Here, we adopt the convention that $\operatorname{im} f_\emptyset = \emptyset$.

The first condition is equivalent to saying that the point set $\{f(a) : a \in \sigma\}$ is in general position. This means that there is no nontrivial linear combination $\sum_i \lambda_i f(a_i) = 0$ such that $\sum_i \lambda_i = 0$. The second condition is equivalent to saying that

$$\operatorname{im} f_\sigma \cap \operatorname{im} f_\tau \subseteq \operatorname{im} f_{\sigma\cap\tau},$$

because we always have the other direction. In words, the intersection between the realizations of any two faces contains nothing more than the realization of the greatest common face of the two simplices. The actual geometric realization is the union

$$|\Delta| = \bigcup_{\sigma\in\Delta\setminus\{\emptyset\}} \operatorname{im} f_\sigma.$$

Note that $|\Delta|$ depends on the choice of vertex map f.

If Δ has k vertices $a_1, \ldots, a_k$, then we may realize Δ in $\mathbb{R}^k$ by mapping a_i to $\mathbf{e}_i$, where $\mathbf{e}_i$ denotes the ith unit vector in $\mathbb{R}^k$. This is the *standard* geometric realization of Δ, which is unique up to the order of the vertices.

Running example 2. Let E_2 be the simplicial complex with vertices a^+, a^-, b^+, b^-, c^+, and c^- such that the maximal faces are all sets consisting of exactly one element from each of $\{a^+, a^-\}$, $\{b^+, b^-\}$, and $\{c^+, c^-\}$. Counting faces, one obtains that E_2 contains 6 vertices, 12 edges, and 8 triangles, which yields the f-vector $(1, 6, 12, 8)$. On the left in Figure 1.3 is a geometric realization of $E_2 \setminus \{a^-b^-c^-\}$ in $\mathbb{R}^2$. On the right is a geometric realization of E_2 in $\mathbb{R}^3$. This realization is the boundary of an octahedron, one of the five Platonic solids.

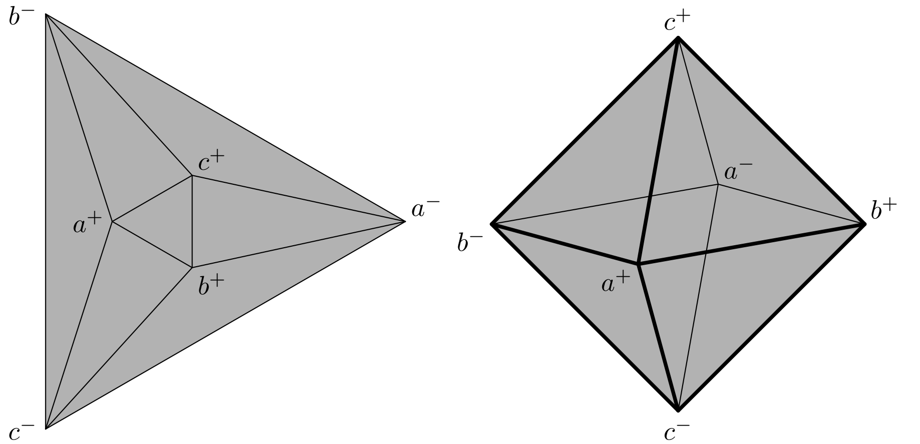

Figure 1.3: Geometric realizations of $E_2 \setminus \{a^-b^-c^-\}$ (on the left) and E_2 (on the right), where E_2 is our second running example.

Running example 3. Let E_3 be the simplicial complex with vertex set $\{a, b, c, d, e, f\}$ and with the following maximal faces:

$$abd, bce, acf, aef, bdf, cde, abe, bcf, acd, def.$$

Clearly, E_3 is not a very big family; the f-vector is $(1, 6, 15, 10)$. Nonetheless, this complex is tricky to realize geometrically. In fact, it is impossible to realize it in $\mathbb{R}^3$ without some intersection of faces containing more than it should.[2] Allowing some cheating, one may still illustrate E_3 in $\mathbb{R}^2$ as a triangulated hexagon, as illustrated in Figure 1.4. Note that the boundary of this hexagon consists of two copies of each of the vertices a, b, and c and also two copies of each of the edges ab, bc, and ac. To obtain a proper realization of E_3, we need to identify all these pairs of copies. This cannot be done in $\mathbb{R}^2$ or even $\mathbb{R}^3$; we need a Euclidean space of dimension at least four.

1.2 Oriented simplices

Let $\mathbb{F}$ be a commutative ring; the reader may think of $\mathbb{F}$ as the ring of integers $\mathbb{Z}$ or a field. Let Δ be a simplicial complex. For each $n \geq -1$, we form a free $\mathbb{F}$-module $\tilde{C}_n(\Delta; \mathbb{F})$ with a basis indexed by the n-dimensional faces of Δ. Specifically, for each face $a_0a_1 \cdots a_n$, we have a basis element $\mathbf{e}_{a_0,a_1,\ldots,a_n}$. We refer to a basis element as an *oriented simplex*. Note that the rank of $\tilde{C}_n(\Delta; \mathbb{F})$ is the nth value $f_n(\Delta)$ in the f-vector of Δ. We refer to $\tilde{C}_n(\Delta; \mathbb{F})$ is the *chain group of degree n*. This terminology stems from the fact that the modules $\tilde{C}_n(\Delta; \mathbb{F})$ form a "chain" via certain maps to be discussed in Section 1.3. When $\mathbb{F}$ is clear from context, we will often write $\tilde{C}_n(\Delta) = \tilde{C}_n(\Delta; \mathbb{F})$.

[2]The reason is that any geometric realization of E_3 is homeomorphic to the real projective plane $\mathbb{RP}^2$, and there is no embedding of this topological space in $\mathbb{R}^3$.

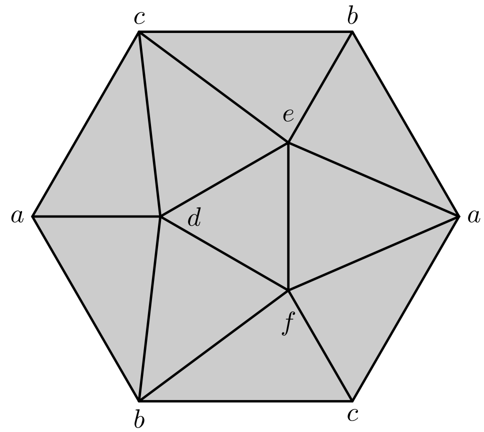

Figure 1.4: Halfway to a geometric realization of E_3, our third running example. To make this a true realization, we need to glue together opposite points on the hexagon.

Running examples. For $E_1 = \{\emptyset, a, b, c, d, ab, ac, bc, bd, cd, abc\}$, we get that

$$\begin{aligned}
\tilde{C}_{-1}(E_1) &= \{\lambda \mathbf{e}_\emptyset : \lambda \in \mathbb{F}\} \cong \mathbb{F}, \\
\tilde{C}_0(E_1) &= \{\lambda_a \mathbf{e}_a + \lambda_b \mathbf{e}_b + \lambda_c \mathbf{e}_c + \lambda_d \mathbf{e}_d : \lambda_a, \lambda_b, \lambda_c, \lambda_d \in \mathbb{F}\} \cong \mathbb{F}^4, \\
\tilde{C}_1(E_1) &= \{\lambda_{ab} \mathbf{e}_{a,b} + \cdots + \lambda_{cd} \mathbf{e}_{c,d} : \lambda_{ab}, \ldots, \lambda_{cd} \in \mathbb{F}\} \cong \mathbb{F}^5, \\
\tilde{C}_2(E_1) &= \{\lambda \mathbf{e}_{a,b,c} : \lambda \in \mathbb{F}\} \cong \mathbb{F}.
\end{aligned}$$

The chain groups for E_2 and E_3 are equally simple to describe.

The tilde symbol over C is a somewhat awkward convention among mathematicians. Its only purpose is to indicate that we define the chain group of degree -1 to be $\mathbb{F}$. Authors who adopt a strictly geometric viewpoint prefer to define this group to be zero, because there is no sensible geometric interpretation of the empty set. Specifically, they define

$$C_n(\Delta; \mathbb{F}) = \begin{cases} \tilde{C}_n(\Delta; \mathbb{F}) & \text{if } n \neq -1, \\ 0 & \text{if } n = -1. \end{cases}$$

For $n \geq 0$, it turns out to be inconvenient to use the notation $\mathbf{e}_{a_0,a_1,\ldots,a_n}$ to denote oriented simplices. Instead, we will write

$$a_0 \wedge a_1 \wedge \cdots \wedge a_n = \mathbf{e}_{a_0,a_1,\ldots,a_n}.$$

For example, $a \wedge b$ represents the oriented simplex $\mathbf{e}_{a,b}$. In degree -1, we stick to the notation $\mathbf{e}_\emptyset$.

The symbol $\wedge$ denotes so-called *exterior product*. It satisfies the rules

$$\begin{aligned}
b \wedge a &= -a \wedge b \\
a \wedge a &= 0
\end{aligned}$$

for any oriented 0-simplices a and b. We may apply this rule to oriented simplices in higher dimensions. For example,

$$b \wedge d \wedge a \wedge c = -b \wedge a \wedge d \wedge c = a \wedge b \wedge d \wedge c = -a \wedge b \wedge c \wedge d.$$

Here, each change of sign corresponds to interchanging two adjacent elements. Throughout Section 1.3, we will assume that this rule makes sense, is well-defined, and does not cause any contradictions. We postpone the formalities to Section 1.4.

1.3 Boundary maps

We define *boundary maps* ∂_n, which are algebraic counterparts to the geometric notion of oriented boundaries. The boundary map ∂_n is a homomorphism that takes an element x in $\tilde{C}_n(\Delta)$ and associates to it an element in $\tilde{C}_{n-1}(\Delta)$, the *boundary* of x.

Formally, we define ∂_n on a given oriented simplex $a_0 \wedge a_1 \wedge \cdots \wedge a_n$ by

$$\partial_n(a_0 \wedge a_1 \wedge \cdots \wedge a_n) = \sum_{r=0}^{n} (-1)^r a_0 \wedge \cdots \wedge a_{r-1} \wedge \hat{a_r} \wedge a_{r+1} \wedge \cdots \wedge a_n \tag{1.1}$$

for each n, where $\hat{a_r}$ denotes removal of the element a_r. In the special case $n = 0$, we let $\partial_0(a) = \mathbf{e}_\emptyset$ for each vertex a. To obtain a homomorphism, we extend ∂_n linearly to the whole of $\tilde{C}_n(\Delta)$.

The boundary maps satisfy the equation $\partial_n \circ \partial_{n+1} = 0$; we always get zero when taking the boundary twice. Before proving this, we give some heuristic motivation for why this is a desirable property of the boundary maps. Consider the geometric notion of boundary in small dimensions. The boundary of a curve consists of the two endpoints of the curve. If the curve is closed, then the two endpoints coincide, which means that the curve does not have any boundary. Now, consider the unit circular disk. The boundary of this disk is a circle, which is a closed curve. In particular, when taking the boundary of the boundary of the disk, we are left with nothing. The same remains true for any "well-behaved" closed and bounded set in $\mathbb{R}^2$; the boundary of the boundary is empty, because the boundary is a closed curve, or the disjoint union of several closed curves.[3]

Interpreting a given edge ab as a line segment oriented from a to b, we define

$$\partial_1(a \wedge b) = b - a$$

and extend ∂_1 linearly to the whole of $\tilde{C}_1(\Delta)$. The counterpart of a closed curve in our setting is the graph-theoretical concept of a set of edges forming a cycle, say a square $\{ab, bc, cd, da\}$. Now,

$$\partial_1(a \wedge b + b \wedge c + c \wedge d + d \wedge a) = (b - a) + (c - b) + (d - c) + (a - d) = 0.$$

Thus the given definition indeed yields an algebraic counterpart of the fact that a closed curve has vanishing boundary.

[3]By the boundary of a d-dimensional manifold, we mean the set of points on the manifold with no open neighborhood homeomorphic to the open $(d-1)$-ball.

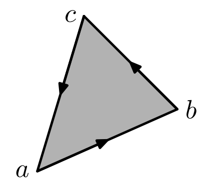

Figure 1.5: Triangle with oriented boundary.

Proceeding to the next dimension, consider a triangle abc as illustrated in Figure 1.5. The boundary of this triangle consists of the three edges ab, bc, and ac. Again aligning with the geometric notion of oriented boundary, we define

$$\partial_2(a \wedge b \wedge c) = b \wedge c - a \wedge c + a \wedge b$$

and extend ∂_2 linearly to the whole of $\tilde{C}_2(\Delta)$. The sign of $a \wedge c$ is -1, because the edge is directed from c to a. We write the boundary the way we do (rather than writing $b \wedge c + c \wedge a + a \wedge b$), because it turns out to be convenient to preserve the order of the vertices in each simplex when computing the boundary.

We note that

$$\partial_1 \circ \partial_2(a \wedge b \wedge c) = (c - b) - (c - a) + (b - a) = 0.$$

By linearity, this implies that $\partial_1 \circ \partial_2(x) = 0$ for any $x \in \tilde{C}_2(\Delta)$. Conversely, given that we want this equation to hold, we do not have much choice when defining ∂_2. Specifically, assigning

$$\partial_2(a \wedge b \wedge c) = \lambda_0 b \wedge c + \lambda_1 a \wedge c + \lambda_2 a \wedge b,$$

we obtain that

$$\begin{aligned} \partial_1 \circ \partial_2(a \wedge b \wedge c) &= \lambda_0(c - b) + \lambda_1(c - a) + \lambda_2(b - a) \\ &= (-\lambda_1 - \lambda_2)a + (-\lambda_0 + \lambda_2)b + (\lambda_0 + \lambda_1)c. \end{aligned}$$

For this to be zero, we must define $\lambda_1 = -\lambda_0$ and $\lambda_2 = \lambda_0$.

Something similar turns out to be true also in higher dimensions. More precisely, adopt the convention that the coefficient of $a_1 \wedge \cdots \wedge a_n$ should be one in $\partial_n(a_0 \wedge a_1 \wedge \cdots \wedge a_n)$ for each n. For $\partial_n \circ \partial_{n+1}$ to be zero for each n, we must then define $\partial_n(a_0 \wedge a_1 \wedge \cdots \wedge a_n)$ as in (1.1) for each n. Namely, suppose that ∂_n is indeed defined like this, and assume that

$$\partial_{n+1}(a_0 \wedge a_1 \wedge \cdots \wedge a_{n+1}) = \sum_{r=0}^{n+1} \lambda_r \cdot a_0 \wedge \cdots \wedge a_{r-1} \wedge \hat{a_r} \wedge a_{r+1} \wedge \cdots \wedge a_{n+1}$$

for some constants $\lambda_0, \ldots, \lambda_{n+1}$, where $\lambda_0 = 1$. For $r < k$, there are two appearances in $\partial_n \circ \partial_{n+1}(a_0 \wedge a_1 \wedge \cdots \wedge a_n \wedge a_{n+1})$ of the oriented simplex obtained from $a_0 \wedge a_1 \wedge \cdots \wedge a_n \wedge a_{n+1}$ by removing a_r and a_k. The first appearance is obtained by removing first a_k and then a_r, and the corresponding coefficient is $\lambda_k \cdot (-1)^r$. The second appearance is obtained by removing first a_r and then a_k. This time, the coefficient is $\lambda_r \cdot (-1)^{k-1}$, because a_k ends up on position $k-1$ after we have removed a_r. Summing, we get that

$$\lambda_k \cdot (-1)^r + \lambda_r \cdot (-1)^{k-1} = 0 \Longleftrightarrow (-1)^r \lambda_k = (-1)^k \lambda_r.$$

Since $\lambda_0 = 1$, this yields that $\lambda_k = (-1)^k$ for all k.

By the above discussion, we obtain the desired double boundary condition:

Proposition 1.3.1 *We have that $\partial_n \circ \partial_{n+1} = 0$ for every n.*

An equivalent way of expressing Proposition 1.3.1 is to say that the sequence

$$\mathsf{C}(\Delta) : \cdots \xrightarrow{\partial_{n+2}} \tilde{C}_{n+1}(\Delta) \xrightarrow{\partial_{n+1}} \tilde{C}_n(\Delta) \xrightarrow{\partial_n} \tilde{C}_{n-1}(\Delta) \xrightarrow{\partial_{n-1}} \cdots$$

defines a *chain complex*. We will discuss general chain complexes in Section 2.1. We refer to $\mathsf{C}(\Delta)$ as a *simplicial* chain complex.

Running example 1. The complex E_1 has dimension 2, which means that we get the following simplicial chain complex:

$$\mathsf{C}(E_1) : 0 \longrightarrow \tilde{C}_2(E_1) \xrightarrow{\partial_2} \tilde{C}_1(E_1) \xrightarrow{\partial_1} \tilde{C}_0(E_1) \xrightarrow{\partial_0} \tilde{C}_{-1}(E_1) \longrightarrow 0.$$

All other chain groups are zero. Remembering that E_1 has f-vector $(1, 4, 5, 1)$, we obtain that $\mathsf{C}(E_1)$ has the following schematic structure:

$$0 \longrightarrow \mathbb{F} \longrightarrow \mathbb{F}^5 \longrightarrow \mathbb{F}^4 \longrightarrow \mathbb{F} \longrightarrow 0.$$

1.4 Orientations of oriented simplices

We still have to settle the issue that there are different ways to arrange the vertices in an oriented simplex.

Start with the 1-dimensional case. There are two ways we may represent an edge ab as an oriented simplex: $a \wedge b$ and $b \wedge a$. Yet, we want just one basis element for each edge, not two. As already discussed in Section 1.2, we solve this problem by making the identification

$$b \wedge a = -a \wedge b.$$

We think of $a \wedge b$ and $b \wedge a$ as having opposite *orientations*.

Proceeding to dimension two, we may arrange the vertices a, b, and c of a face abc in six different ways, which gives rise to six oriented simplices representing the same face:

$$a \wedge b \wedge c \quad a \wedge c \wedge b \quad b \wedge a \wedge c \quad b \wedge c \wedge a \quad c \wedge a \wedge b \quad c \wedge b \wedge a.$$

Again we want just one basis element, not six. We already gave an identification rule in Section 1.2, which yields that

$$a \wedge b \wedge c = -b \wedge a \wedge c = b \wedge c \wedge a = -c \wedge b \wedge a = c \wedge a \wedge b = -a \wedge c \wedge b$$

In each step, we interchange two adjacent elements, and the sign changes accordingly. One may check by hand that we obtain no contradictions.

Yet, we also need to check that the assignment of signs aligns with the boundary map. Again, this can be done by hand. For example,

$$\partial_2(b \wedge a \wedge c) = b \wedge a + a \wedge c + c \wedge b = -a \wedge b - b \wedge c + a \wedge c = -\partial_2(a \wedge b \wedge c),$$

which aligns with the assignment $b \wedge a \wedge c = -a \wedge b \wedge c$.

To explain the general rule, fix a total order on the vertex set of Δ. For any vertices $x_0, x_1, \ldots, x_n$ forming an n-dimensional face, we define a pair (x_i, x_j) to be an *inversion* in $\mathbf{x} = x_0 \wedge x_1 \wedge \cdots \wedge x_n$ if $i < j$ and $x_i > x_j$. For example, if $a < b < c < d$, then $b \wedge d \wedge a \wedge c$ contains the three inversions (b, a), (d, a), and (d, c). We define $\mathrm{inv}(\mathbf{x})$ to be the number of inversions of $\mathbf{x}$. For example, $\mathrm{inv}(b \wedge d \wedge a \wedge c) = 3$.

Consider a face $a_0 a_1 \cdots a_n$, assuming that $a_0 < a_1 < \cdots < a_n$, and let π be a permutation of $\{0, \ldots, n\}$. Write $b_r = a_{\pi(r)}$, $\mathbf{a} = a_0 \wedge a_1 \wedge \cdots \wedge a_n$, and $\mathbf{b} = b_0 \wedge b_1 \wedge \cdots \wedge b_n$. We define

$$\mathbf{b} = (-1)^{\mathrm{inv}(\mathbf{b})} \cdot \mathbf{a}. \tag{1.2}$$

Lemma 1.4.1 *The assignment* (1.2) *aligns with the boundary map ∂_n. Equivalently, when computing $\partial_n(\mathbf{b})$ according to the rule* (1.1)*, the result is $(-1)^{\mathrm{inv}(\mathbf{b})} \cdot \partial_n(\mathbf{a})$.*

Proof. Let $\mathbf{a}_r$ be the oriented $(n-1)$-dimensional simplex obtained from $\mathbf{a}$ by removing a_r, and define $\mathbf{b}_r$ analogously in terms of $\mathbf{b}$ and r. We obtain that

$$\partial_n(\mathbf{b}) = \sum_{r=0}^{n} (-1)^r \cdot \mathbf{b}_r = \sum_{r=0}^{n} (-1)^{r+\mathrm{inv}(\mathbf{b}_r)} \cdot \mathbf{a}_{\pi(r)}.$$

For the last step, we use the fact that $b_r = a_{\pi(r)}$.

It remains to prove that

$$\mathrm{inv}(\mathbf{b}_r) \equiv \mathrm{inv}(\mathbf{b}) + \pi(r) + r \pmod 2. \tag{1.3}$$

for each r. Namely, this will imply that

$$\begin{aligned}
\sum_{r=0}^{n} (-1)^{r+\mathrm{inv}(\mathbf{b}_r)} \cdot \mathbf{a}_{\pi(r)} &= \sum_{r=0}^{n} (-1)^{\pi(r)+\mathrm{inv}(\mathbf{b})} \cdot \mathbf{a}_{\pi(r)} \\
&= (-1)^{\mathrm{inv}(\mathbf{b})} \cdot \sum_{r=0}^{n} (-1)^{\pi(r)} \cdot \mathbf{a}_{\pi(r)} \\
&= (-1)^{\mathrm{inv}(\mathbf{b})} \cdot \sum_{k=0}^{n} (-1)^{k} \cdot \mathbf{a}_{k} \\
&= (-1)^{\mathrm{inv}(\mathbf{b})} \cdot \partial_n(\mathbf{a}).
\end{aligned}$$

Now, let α_r be the number of inversions of the form $(b_i, b_r) = (a_{\pi(i)}, a_{\pi(r)})$ in $\mathbf{b}$, and let β_r be the number of inversions of the form (b_r, b_i). Note that

$$\mathrm{inv}(\mathbf{b}) = \mathrm{inv}(\mathbf{b}_r) + \alpha_r + \beta_r.$$

In particular, (1.3) yields that it suffices to prove that

$$\alpha_r + \beta_r \equiv \pi(r) + r \pmod 2. \tag{1.4}$$

We have that α_r is the number of elements among $\pi(0), \ldots, \pi(r-1)$ that are greater than $\pi(r)$, whereas β_r is the number of elements among $\pi(r+1), \ldots, \pi(n)$

that are less than $\pi(r)$. Since there are $r-\alpha_r$ elements among $\pi(0),\ldots,\pi(r-1)$ that are less than $\pi(r)$, we conclude that $\beta_r = \pi(r)-(r-\alpha_r)$. As a consequence,

$$\alpha_r + \beta_r = \alpha_r + \pi(r) - (r - \alpha_r) = 2\alpha_r + \pi(r) - r \equiv \pi(r) + r \pmod 2,$$

which yields (1.4). □

In some situations, it might be convenient to extend the definition of $a_0 \wedge a_1 \wedge \cdots \wedge a_n$ to the situation that $a_i = a_j$ for some i and j. As indicated already in Section 1.2, we then define $a_0 \wedge a_1 \wedge \cdots \wedge a_n$ to be zero. This is indeed compatible with the boundary map. For example, if $a_0 = a_1$, then

$$\partial_n(a_1 \wedge a_1 \wedge \cdots \wedge a_n) = a_1 \wedge a_2 \wedge \cdots \wedge a_n - a_1 \wedge a_2 \wedge \cdots \wedge a_n + w = w,$$

where w is a linear combination of elements all starting $a_1 \wedge a_1$, hence zero. We leave the general case to the reader.

1.5 Products of chain group elements

It is possible to define some kind of product between chain group elements. Yet, for the product to be compatible with the boundary map (in a manner to be described below), we must leave the product undefined for certain combinations of elements. As a consequence, what we will consider is not a product in the proper sense of the word. In this context, it is worth mentioning the theory of simplicial *cohomology*, a theory dual to the theory of simplicial homology under development in this document. Within that theory, it is indeed possible to define a proper product.

For the purposes of this section, say that two oriented simplices $\mathbf{a} = a_0 \wedge \cdots \wedge a_r$ and $\mathbf{b} = b_0 \wedge \cdots \wedge b_k$ are *compatible* if $\{a_0,\ldots,a_r\} \cup \{b_0,\ldots,b_k\} \in \Delta$. If $\mathbf{a}$ and $\mathbf{b}$ are compatible, we define the (exterior) product between $\mathbf{a}$ and $\mathbf{b}$ to be

$$\mathbf{a} \wedge \mathbf{b} = a_0 \wedge \cdots \wedge a_r \wedge b_0 \wedge \cdots \wedge b_k.$$

By compatibility, this is either an oriented simplex in $\tilde{C}_{r+k-1}(\Delta)$ or zero, the latter being the case if $a_i = b_j$ for some i and j.

More generally, suppose that $c_1 = \sum_{\mathbf{a}} \lambda_{\mathbf{a}} \cdot \mathbf{a} \in \tilde{C}_r(\Delta)$ and $\sum_{\mathbf{a}} \mu_{\mathbf{b}} \cdot \mathbf{b} \in \tilde{C}_k(\Delta)$ are two linear combinations of oriented simplices such that $\mathbf{a}$ and $\mathbf{b}$ are compatible whenever $\lambda_{\mathbf{a}} \neq 0$ and $\lambda_{\mathbf{b}} \neq 0$. Then we may form the product

$$c_1 \wedge c_2 = \sum_{\mathbf{a},\mathbf{b}} \lambda_{\mathbf{a}}\mu_{\mathbf{b}} \cdot \mathbf{a} \wedge \mathbf{b} \in \tilde{C}_{r+k-1}(\Delta).$$

We leave it to the reader to check that

$$c_2 \wedge c_1 = (-1)^{(r+1)(k+1)} c_1 \wedge c_2$$

and

$$\partial_{r+k-1}(c_1 \wedge c_2) = \partial_r(c_1) \wedge c_2 + (-1)^{r+1} c_1 \wedge \partial_k(c_2).$$

In particular, if c_1 and c_2 are both cycles, then $c_1 \wedge c_2$ is again a cycle.

Example. We have that

$$\begin{aligned}(a-b)\wedge(c-d) &= a\wedge c - a\wedge d - b\wedge c + b\wedge d,\\ (a-b)\wedge(a-c) &= a\wedge a - a\wedge c - b\wedge a + b\wedge c\\ &= c\wedge a + a\wedge b + b\wedge c.\end{aligned}$$

Also, our running example E_3 has a cycle in degree 2 that we might write as

$$(a^+ - a^-)\wedge(b^+ - b^-)\wedge(c^+ - c^-).$$

This is indeed an element in $\tilde{C}_2(\Delta;\mathbb{F})$; every selection of one element from each of the three parentheses yields a face of Δ.

We do not allow products between oriented simplices that are not compatible. The problem is that such a product would not be an oriented simplex corresponding to a face of Δ, meaning that we ought to define the product to be zero. Yet, such a definition would conflict with the boundary map. For example, if $\Delta = \{\emptyset, a, b\}$, then we cannot define $a\wedge b$ to be zero, because $\partial_1(a\wedge b) = b - a \neq 0$.

1.6 Definition of simplicial homology

Now, we define the homology of a simplicial complex Δ. Loosely speaking, the homology gives an algebraic measure on the amount of cycles that are not boundaries.

Formally, we define the $\mathbb{F}$-module $Z_n(\Delta;\mathbb{F})$ of *cycles* and the $\mathbb{F}$-module $B_n(\Delta;\mathbb{F})$ of *boundaries* by the following formulas:

$$\begin{aligned}Z_n(\Delta;\mathbb{F}) &= \ker\partial_n\\ &= \{z\in\tilde{C}_n(\Delta;\mathbb{F}) : \partial_n(z) = 0\},\\ B_n(\Delta;\mathbb{F}) &= \operatorname{im}\partial_{n+1}\\ &= \{z\in\tilde{C}_n(\Delta;\mathbb{F}) : z = \partial_{n+1}(x) \text{ for some } x\in\tilde{C}_{n+1}(\Delta;\mathbb{F})\}.\end{aligned}$$

By Proposition 1.3.1, we have that $B_n(\Delta;\mathbb{F})$ is a submodule of $Z_n(\Delta;\mathbb{F})$. We define the simplicial homology in degree n of Δ to be the quotient

$$\tilde{H}_n(\Delta;\mathbb{F}) = Z_n(\Delta;\mathbb{F})/B_n(\Delta;\mathbb{F});$$

see Section 0.2.6 for more information about quotients. We refer to members of $\tilde{H}_n(\Delta;\mathbb{F})$ as *homology classes*; each such member is an equivalence class under the relation $z\sim z' \iff z - z'\in B_n(\Delta;\mathbb{F})$. We let $[z]$ denote the homology class containing the cycle z.

The tilde symbol over H indicates that we consider *reduced* homology. This means that the underlying chain group in degree -1 is $\tilde{C}_{-1}(\Delta;\mathbb{F}) = \mathbb{F}\cdot\mathbf{e}_\emptyset$ and that the boundary map ∂_0 is defined by $\partial_0(a) = \mathbf{e}_\emptyset$ for each vertex a. We obtain *unreduced* homology $H_n(\Delta;\mathbb{F})$ by setting $C_{-1}(\Delta;\mathbb{F}) = 0$ and defining ∂_0 to be the zero map. In all degrees $n\geq 1$, we always have that $\tilde{H}_n(\Delta;\mathbb{F}) = H_n(\Delta;\mathbb{F})$.

In certain particularly well-behaved cases, one may view the homology as an algebraic measure on the amount of "holes" in Δ. Specifically, given a subset X of Euclidean space $\mathbb{R}^n$, we define a *hole* of X to be a bounded connected component of $\mathbb{R}^n \setminus X$, the complement of X in $\mathbb{R}^n$. For a given $k \geq 0$, assume that the $(k+1)$-skeleton $\Delta^{(k+1)}$ of Δ admits a geometric realization X in $\mathbb{R}^{k+1}$. Then it is possible to prove that $\tilde{H}_k(\Delta; \mathbb{F})$ is a free $\mathbb{F}$-module of rank the number of holes of X.

Running examples. The geometric realization of E_1 in Figure 1.1 consists of one hole. Namely, the region in $\mathbb{R}^2$ surrounded by the three edges bc, cd, db is bounded and is separated by $|E_1|$ from the unbounded region outside $|E_1|$. In Section 1.7.1, we show that $\tilde{H}_1(E_1; \mathbb{F})$ indeed has rank one, and the homology class of the cycle $b \wedge c + c \wedge d + d \wedge b$ is a generator of $\tilde{H}_1(E_1; \mathbb{F})$.

The geometric realization X of E_2 in Figure 1.3 also consists of one hole. This is because X divides $\mathbb{R}^3$ into one bounded region inside X and one unbounded region outside X. Indeed, we will see that $\tilde{H}_2(E_2; \mathbb{F})$ has rank one, and a generator is given by the homology class of the cycle

$$(a^+ - a^-) \wedge (b^+ - b^-) \wedge (c^+ - c^-). \tag{1.5}$$

In the general case, the interpretation of homology as a measure on holes breaks down. We just have to look at our third running example E_3 to arrive at a significantly more intricate situation. Let the underlying ring $\mathbb{F}$ be $\mathbb{Z}$. As is indicated in Figure 1.4, we have a cycle $a \wedge b + b \wedge c + c \wedge a$ with the property that *twice* that cycle is a boundary. Yet, the cycle itself is not a boundary. In Section 1.7.3, we show that $\tilde{H}_1(E_3; \mathbb{Z})$ is a finite group with two elements, the nonzero element being the homology class of $a \wedge b + b \wedge c + c \wedge a$.

1.7 Explicit homology computations

In this section, we use explicit methods to compute the homology of our running examples. The main purpose is to give the reader the opportunity to study the technical details of a homology computation. As we will see in Chapters 2 and 3, there exist more efficient methods for computing homology. In particular, the reader should not be discouraged by the technical complexity of some of the computations below.

We will only look at homology in degree one and higher. In Section 1.8, we give general formulas for the homology in degrees -1 and 0. As it turns out, the homology in those degrees is zero for all three running examples.

1.7.1 Running example 1

We have that $Z_2(E_1; \mathbb{F}) = 0$, because the only face of dimension 2 is abc, and $\partial_2(\lambda \cdot a \wedge b \wedge c) = 0$ if and only if $\lambda = 0$. As a consequence, $\tilde{H}_2(E_1; \mathbb{F}) = 0$.

Next, we consider $\tilde{H}_1(E_1; \mathbb{F})$. First, we note that $B_1(E_1; \mathbb{F})$ is generated by

$$\partial_2(a \wedge b \wedge c) = a \wedge b - a \wedge c + b \wedge c = z_1.$$

Next, let z be an element in $Z_1(E_1; \mathbb{F})$. This means that $\partial_1(z) = 0$ and that z is a linear combination

$$z = \lambda_{ab} \cdot a \wedge b + \lambda_{ac} \cdot a \wedge c + \lambda_{bc} \cdot b \wedge c + \lambda_{bd} \cdot b \wedge d + \lambda_{cd} \cdot c \wedge d,$$

where $\lambda_{ab}, \lambda_{ac}, \lambda_{bc}, \lambda_{bd}, \lambda_{cd} \in \mathbb{F}$. Note that

$$\begin{aligned}\partial_1(z) &= \partial_1(\lambda_{ab} \cdot a \wedge b + \lambda_{ac} \cdot a \wedge c + \lambda_{bc} \cdot b \wedge c + \lambda_{bd} \cdot b \wedge d + \lambda_{cd} \cdot c \wedge d) \\ &= (-\lambda_{ab} - \lambda_{ac})a + (\lambda_{ab} - \lambda_{bc} - \lambda_{bd})b \\ &\quad + (\lambda_{ac} + \lambda_{bc} - \lambda_{cd})c + (\lambda_{bd} + \lambda_{cd})d.\end{aligned}$$

Setting $\lambda_{ab} = t$ and $\lambda_{cd} = u$, this yields that $\lambda_{ac} = -t$, $\lambda_{bd} = -u$, and $\lambda_{bc} = t+u$. In particular, $Z_1(E_1; \mathbb{F})$ is generated by

$$\begin{aligned} z_1 &= a \wedge b - a \wedge c + b \wedge c \qquad (t = 1, u = 0), \\ z_2 &= b \wedge c - b \wedge d + c \wedge d \qquad (t = 0, u = 1). \end{aligned}$$

To compute $\tilde{H}_1(E_1; \mathbb{F})$, we note that x and y belong to the same homology class if and only if $y - x$ is a multiple of z_1, the generator of $B_1(E_1; \mathbb{F})$. This means that the members of $\tilde{H}_1(E_1; \mathbb{F}) = Z_1(E_1; \mathbb{F})/B_1(E_1; \mathbb{F})$ are of the form

$$[uz_2] = \{tz_1 + uz_2 : t \in \mathbb{F}\}.$$

In particular, $\tilde{H}_1(E_1; \mathbb{F})$ has dimension 1 and is generated by $[z_2]$.

To summarize, we have the following result.

Proposition 1.7.1 *We have that*

$$\tilde{H}_n(E_1; \mathbb{F}) \cong \begin{cases} \mathbb{F} & \text{if } n = 1, \\ 0 & \text{if } n \neq 1. \end{cases}$$

1.7.2 Running example 2

For convenience, we represent each face σ of E_2 as a sequence rst, where $r, s, t \in \{0, 1, \times\}$. Specifically,

$$r = \begin{cases} 0 & \text{if } a^+ \in \sigma, \\ 1 & \text{if } a^- \in \sigma, \\ \times & \text{if } a^+, a^- \notin \sigma. \end{cases}$$

The symbols s and t are defined analogously in terms of $b^\pm$ and $c^\pm$, respectively. For example, $0{\times}1$ represents the face a^+c^-. We use the same sequences to represent the corresponding oriented simplices.

Computing $Z_2(E_2; \mathbb{F})$ is equivalent to finding all vanishing linear combinations of the column vectors in the matrix in Figure 1.6; in this matrix, we only indicate nonzero elements. Each column in the matrix is the boundary of the face labelling the column, expressed in the basis given by the labels of the rows. Suppose that $z \in Z_2(E_2; \mathbb{F})$, and let λ_{rst} be the coefficient of rst in the expansion of z. By the first four rows in the matrix, we obtain that $\lambda_{1st} = -\lambda_{0st}$ for any $s, t \in \{0, 1\}$. By the next two rows, we get that $\lambda_{010} = -\lambda_{000}$ and $\lambda_{011} = -\lambda_{001}$. Finally, the ninth row yields that $\lambda_{001} = -\lambda_{000}$. To summarize, z is a cycle if and only if

$$z = \lambda \cdot (000 - 001 - 010 + 011 - 100 + 101 + 110 - 111)$$

for some $\lambda \in \mathbb{F}$, meaning that $Z_2(E_2; \mathbb{F}) \cong \mathbb{F}$. Since $B_2(E_2; \mathbb{F}) = 0$, we deduce that $\tilde{H}_2(E_2; \mathbb{F}) \cong \mathbb{F}$. Going back to original notation and setting $\lambda = 1$, we note that we may express the above cycle in the form (1.5).

	000	001	010	011	100	101	110	111
×00	1				1			
×01		1				1		
×10			1				1	
×11				1				1
0×0	−1		−1					
0×1		−1		−1				
1×0					−1		−1	
1×1						−1		−1
00×	1	1						
01×			1	1				
10×					1	1		
11×							1	1

.

Figure 1.6: Matrix describing the map $\partial_2 : \tilde{C}_2(E_2;\mathbb{F}) \to \tilde{C}_1(E_2;\mathbb{F})$.

×00	1						
×01		1					
×10			1				
×11				1			
0×0	−1		−1		1		
0×1		−1		−1		1	
1×0					−1		
1×1						−1	
00×	1	1			−1	−1	1
01×			1	1			−1
10×					1	1	−1
11×							1

.

Figure 1.7: The result after applying a prudent choice of column operations on the matrix in Figure 1.6.

Next, consider $B_1(E_2;\mathbb{F})$. This is the module spanned by the columns of the matrix in Figure 1.6. To obtain a handier description of the module, we perform elementary column operations on this matrix. We leave it to the reader to check that $B_1(E_2;\mathbb{F})$ coincides with the module spanned by the columns in the matrix in Figure 1.7.

To compute $\tilde{H}_1(E_2;\mathbb{F})$, let $z \in Z_1(E_2;\mathbb{F})$. We have that z belongs to the same homology class as some element such that the coefficient of ×00 is zero. Namely, if the coefficient of ×00 is λ, then we may transform z by subtracting λ times the first column of the matrix in (1.7). The resulting element remains in the same homology class as z, because the columns of the matrix all belong to $B_1(E_2;\mathbb{F})$. Proceeding with the other columns, we conclude that z belongs to the same homology class as an element z' with the property that we have zero coefficients in front of ×00, ×01, ×10, ×11, 0×0, 0×1, and 00×. Equivalently, z' is a linear combination of the remaining oriented simplices. Finding all possible z' is equivalent to finding all vanishing linear combinations of the column vectors in the matrix in Figure 1.8. One easily checks that the only such linear combination

	$1{\times}0$	$1{\times}1$	$01{\times}$	$10{\times}$	$11{\times}$
$\times\times 0$	1				
$\times\times 1$		1			
$\times 0 \times$				1	
$\times 1 \times$			1		1
$0 \times\times$			−1		
$1 \times\times$	−1	−1		−1	−1

Figure 1.8: Matrix describing relevant parts of the map $\partial_1 : \tilde{C}_1(E_2;\mathbb{F}) \to \tilde{C}_0(E_2;\mathbb{F})$.

is the trivial one with all zeros. As a consequence, $z' = 0$. The conclusion is that there is just one single homology class, which is equivalent to saying that $\tilde{H}_1(E_2;\mathbb{F})$ is 0.

We summarize our results in a proposition.

Proposition 1.7.2 *We have that*

$$\tilde{H}_n(E_2;\mathbb{F}) \cong \begin{cases} \mathbb{F} & \text{if } n = 2, \\ 0 & \text{if } n \neq 2. \end{cases}$$

1.7.3 Running example 3

Finally, we consider E_3. The module $Z_2(E_3;\mathbb{F})$ is given by all vanishing linear combinations of the column vectors in the matrix in Figure 1.6.

	abd	bce	caf	aef	bfd	cde	abe	bcf	cad	def
ab	1						1			
bc		1						1		
ca			1						1	
ef				1						1
fd					1					1
de						1				1
ad	−1								1	
ae				1			−1			
af			1	−1						
be		−1					1			
bf					1			−1		
bd	1				−1					
cf			−1					1		
cd						1			−1	
ce		1				−1				

Figure 1.9: Matrix describing the map $\partial_2 : \tilde{C}_2(E_3;\mathbb{F}) \to \tilde{C}_1(E_3;\mathbb{F})$.

Performing row operations, we arrive at the matrix in Figure 1.10. Note that we have the element 2 in the lower right corner. This element could be either nonzero or zero depending on the underlying ring $\mathbb{F}$. Specifically, if $\mathbb{F}$ is $\mathbb{Z}$ or a field of characteristic different from 2, then the matrix has full rank, which means that $\tilde{H}_2(E_3;\mathbb{F}) = Z_2(E_3;\mathbb{F}) = 0$. If $\mathbb{F}$ is a field of characteristic two, then the last row of the matrix is zero. In this case, $\tilde{H}_2(E_3;\mathbb{F}) = Z_2(E_3;\mathbb{F}) \cong \mathbb{F}$, and a generator is given by the sum of all ten oriented simplices.

abd	bce	caf	aef	bfd	cde	abe	bcf	cad	def
1						1			
	1						1		
		1						1	
			1						1
				1					1
					1				1
						1		1	
							1	1	
								1	1
									2

.

Figure 1.10: The matrix in Figure 1.9 after a prudent choice of row operations.

Proceeding to degree 1, we obtain a nice description of $B_1(E_3;\mathbb{F})$ by performing column operations on the matrix in Figure 1.9; see Figure 1.11 for the resulting matrix, whose columns span $B_1(E_3;\mathbb{F})$. Note that the rightmost column is equal to twice the cycle $\gamma = -bf + bd + cf - cd = bd + dc + cf + fb$. This column will be zero if $\mathbb{F}$ is a field of characteristic two.

ab	1									
bc		1								
ca			1							
ef				1						
fd					1					
de						1				
ad	-1						1			
ae				1			-1		1	
af			1	-1					-1	
be		-1					1	1		
bf					1			-1	-1	-2
bd	1				-1		-1		1	2
cf			-1					1	2	2
cd						1			-1	-2
ce		1				-1		-1	-1	

.

Figure 1.11: The matrix in Figure 1.9 after a prudent choice of column operations.

To compute $\tilde{H}_1(E_3;\mathbb{F})$, we proceed as with E_2. Precisely, let $z \in Z_1(E_2;\mathbb{F})$. We may use the first nine columns of the matrix in Figure 1.11 to get a cycle z' in the same homology class as z such that the coefficients are zero in front of $ab, bc, ca, de, ef, fd, ad, ae, be$. It remains to determine all cycles that are linear combinations of the remaining six basis elements af, bf, bd, cf, cd, ce. Two such cycles belong to the same homology class if and only if they differ by a multiple of the cycle 2γ; this cycle is the rightmost column of the matrix in Figure 1.11.

$$\begin{array}{c|cccccc|} & af & bf & bd & cf & cd & ce \\ \hline a & -1 & & & & & \\ b & & -1 & -1 & & & \\ c & & & & -1 & -1 & -1 \\ d & & & 1 & & 1 & \\ e & & & & & & 1 \\ f & 1 & 1 & & 1 & & \end{array}.$$

Figure 1.12: Matrix describing relevant parts of the map $\partial_1 : \tilde{C}_1(E_3;\mathbb{F}) \to \tilde{C}_0(E_3;\mathbb{F})$.

Looking at Figure 1.12, we draw the conclusion that all cycles are of the form $\lambda \cdot \gamma$ for some $\lambda \in \mathbb{F}$. Two such cycles $\lambda_1\gamma$ and $\lambda_2\gamma$ belong to the same homology class if and only if $\lambda_1 - \lambda_2 = 2x$ for some $x \in \mathbb{F}$.

To present a general formula for the homology of E_3, we need the concept of *annihilator*. For $x \in \mathbb{F}$, we define $\mathrm{Ann}_{\mathbb{F}}(x) = \{y \in \mathbb{F} : xy = 0\}$.

Proposition 1.7.3 *We have that*

$$\tilde{H}_n(E_3;\mathbb{F}) \cong \begin{cases} \mathbb{F}/(2\mathbb{F}) & \text{if } n = 1, \\ \mathrm{Ann}_{\mathbb{F}}(2) & \text{if } n = 2, \\ 0 & \text{if } n \notin \{1,2\}. \end{cases}$$

For example,

$$\begin{aligned} \tilde{H}_n(E_3;\mathbb{Q}) &\cong \begin{cases} \mathbb{Q}/(2\mathbb{Q}) = 0 & \text{if } n = 1, \\ \mathrm{Ann}_{\mathbb{Q}}(2) = 0 & \text{if } n = 2, \end{cases} \\ \tilde{H}_n(E_3;\mathbb{Z}) &\cong \begin{cases} \mathbb{Z}/(2\mathbb{Z}) \cong \mathbb{Z}_2 & \text{if } n = 1, \\ \mathrm{Ann}_{\mathbb{Z}}(2) = 0 & \text{if } n = 2, \end{cases} \\ \tilde{H}_n(E_3;\mathbb{Z}_2) &\cong \begin{cases} \mathbb{Z}_2/(2\mathbb{Z}_2) = \mathbb{Z}_2 & \text{if } n = 1, \\ \mathrm{Ann}_{\mathbb{Z}_2}(2) = \mathbb{Z}_2 & \text{if } n = 2. \end{cases} \end{aligned}$$

1.8 Homology in low degrees

As promised, we give general formulas for the homology in degrees -1 and 0 of a simplicial complex Δ. We also say a few words about the degree 1 in the case that Δ is a graph.

1.8.1 Homology in degree -1

Unless $\Delta = \emptyset$, we always have that $Z_{-1}(\Delta;\mathbb{F}) = \mathbb{F} \cdot \mathbf{e}_{\emptyset}$. If $\Delta = \{\emptyset\}$, then $B_{-1}(\Delta;\mathbb{F}) = 0$, which means that $\tilde{H}_{-1}(\Delta;\mathbb{F}) \cong \mathbb{F}$. Otherwise, there is some vertex a in Δ. Since $\partial_0(a) = \mathbf{e}_{\emptyset}$, we get that $B_{-1}(\Delta;\mathbb{F}) = Z_{-1}(\Delta;\mathbb{F})$, which means that $\tilde{H}_{-1}(\Delta;\mathbb{F}) = 0$. To summarize,

$$\tilde{H}_{-1}(\Delta;\mathbb{F}) \cong \begin{cases} \mathbb{F} & \text{if } \Delta = \{\emptyset\}, \\ 0 & \text{otherwise.} \end{cases}$$

Note the distinction between the *void* simplicial complex $\emptyset$ containing nothing and the *empty* simplicial complex $\{\emptyset\}$ containing the empty set and nothing else.

1.8.2 Homology in degree 0

Let V be the vertex set of Δ. We note that a linear combination $\sum_{a \in V} \lambda_a a$ belongs to $Z_0(\Delta; \mathbb{F})$ if and only if $\sum_{a \in V} \lambda_a = 0$. Namely, the boundary of $\lambda_a a$ is $\lambda_a \mathbf{e}_\emptyset$. The reader may check that $Z_0(\Delta; \mathbb{F})$ is a free $\mathbb{F}$-module of rank $|V| - 1$.

To determine $B_0(\Delta; \mathbb{F})$, define a relation $\sim$ on the vertex set of Δ by letting $a \sim b$ if and only if there is a edge path $(a_0a_1, a_1a_2, \ldots, a_{m-1}a_m)$ such that $a = a_0$ and $b = a_m$. It is a simple exercise to show that $\sim$ defines an equivalence relation. Let $V_1, V_2, \ldots, V_k$ be the equivalence classes, and fix a vertex r_i in each V_i. For each $a \in V_i \setminus \{r_i\}$, define $\tilde{a} = a - r_i$. We claim that $B_0(\Delta; \mathbb{F})$ coincides with the free submodule M of $\tilde{C}_0(\Delta; \mathbb{F})$ with basis $\mathcal{X} = \{\tilde{a} : a \in V_i \setminus \{r_i\}, 1 \leq i \leq k\}$. In particular, $B_0(\Delta; \mathbb{F})$ has rank $|V| - k$.

To prove the claim, first note that $B_0(\Delta; \mathbb{F})$ is generated by all boundaries $\partial_1(ab) = b - a$ such that ab is an edge of Δ. By definition, $a \sim b$ whenever ab is an edge in Δ, which implies that a and b belong to the same equivalence class V_i. Since

$$\partial_1(ab) = b - a = (b - r_i) - (a - r_i) = \tilde{b} - \tilde{a},$$

we conclude that $B_0(\Delta; \mathbb{F})$ is a submodule of M. For the reverse inclusion, it suffices to show that $\tilde{a}$ is a boundary for each $a \in V_i \setminus \{r_i\}$ and for each i such that $1 \leq i \leq k$. Now $a \in V_i$ if and only if there is an edge path $(a_0a_1, a_1a_2, \ldots, a_{m-1}a_m)$ such that $r_i = a_0$ and $a = a_m$. Observing that

$$\partial_1(a_0a_1 + a_1a_2 + \cdots + a_{m-1}a_m) = a_m - a_0 = a - r_i = \tilde{a},$$

we obtain the desired boundary.

Now, the reader may check that we obtain a basis for $Z_0(\Delta; \mathbb{F})$ by adding the element $\tilde{r}_i = r_i - r_k$ to the set $\mathcal{X}$ for $1 \leq i \leq k - 1$. To conclude, $\tilde{H}_0(\Delta; \mathbb{F})$ is a free $\mathbb{F}$-module of rank $(|V| - 1) - (|V| - k) = k - 1$, and a basis is given by the homology classes of the elements $\tilde{r}_1, \ldots, \tilde{r}_{k-1}$.

We refer to the equivalence classes $V_1, \ldots, V_k$ as the *connected components* of Δ. By the above discussion, we may conclude the following.

Proposition 1.8.1 *For any simplicial complex Δ, we have that*

$$\tilde{H}_0(\Delta; \mathbb{F}) \cong \mathbb{F}^{k-1},$$

where k is the number of connected components of Δ.

Using unreduced homology instead of reduced homology, we obtain the formula $H_0(\Delta; \mathbb{F}) \cong \mathbb{F}^k$. This is one instance for which we get a nicer formula for unreduced homology than for reduced homology. As will be clear later on, there are other instances for which quite the opposite holds.

1.8.3 Homology in degree 1 for 1-dimensional complexes

As suggested by our treatment of our running example E_3 in Section 1.7.3, the situation in degree 1 is too complicated to allow for a simple formula like the one in Proposition 1.8.1. In the case that Δ has no faces of dimension greater than one – i.e., Δ is a graph – there is indeed a nice formula. Specifically, we then have that

$$\tilde{H}_1(\Delta; \mathbb{F}) \cong \mathbb{F}^{e-v+k},$$

where e is the number of edges, v the number of vertices, and k the number of connected components of Δ. We leave the proof to the interested reader. The module $\tilde{H}_1(\Delta; \mathbb{F})$ is the *cycle space* of the graph Δ.

Chapter 2

Combinatorial techniques

So far, our computations have been straightforward applications of linear algebra. Such a brute force approach works fine as long as the underlying simplicial complex is small enough, but once the complex grows too large, we need to find other means for computing the homology. In this chapter, we consider some methods that are mainly combinatorial in nature. In Chapter 3, we proceed with more algebraic methods.

2.1 General chain complexes

We will discuss methods that apply to a more general setting involving arbitrary chain complexes, not just simplicial chain complexes. Moreover, we will frequently need to transform simplicial chain complexes into new chain complexes that are not necessarily simplicial. For this reason, we introduce a more general notion of chain complexes.

Let $\mathbb{F}$ be a commutative ring. A *chain complex* C over $\mathbb{F}$ is a sequence $(C_n : n \in \mathbb{Z})$ of $\mathbb{F}$-modules, called *chain groups*, along with $\mathbb{F}$-module homomorphisms

$$d_n : C_n \to C_{n-1}$$

such that

$$d_n \circ d_{n+1} = 0 \text{ for } n \in \mathbb{Z}. \tag{2.1}$$

We refer to d_n as a *boundary map*. We always denote the boundary map by d_n in the general case, thus reserving the notation ∂_n for boundary maps in simplicial chain complexes.

As we did already in Section 1.3, one typically illustrates a chain complex as a sequence with arrows between the groups in the following manner:

$$\mathsf{C} : \cdots \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

In this document, we will focus on finite chain complexes. In such complexes, only finitely many chain groups are nonzero. The simplicial chain complex associated to a finite simplicial complex is always finite.

As in the case of simplicial complexes, we may define submodules of C_n consisting of cycles and boundaries. More precisely, we define the $\mathbb{F}$-module

$Z_n(\mathsf{C})$ of *cycles* and the $\mathbb{F}$-module $B_n(\mathsf{C})$ of *boundaries* by the following formulas:

$$\begin{aligned} Z_n(\mathsf{C}) &= \ker \partial_n = \{z \in C_n : \partial_n(z) = 0\}, \\ B_n(\mathsf{C}) &= \operatorname{im} \partial_{n+1} = \{z \in C_n : z = \partial_{n+1}(x) \text{ for some } x \in C_{n+1}\}. \end{aligned}$$

In any chain complex, we have that

$$B_n(\mathsf{C}) \subseteq Z_n(\mathsf{C}).$$

Indeed, (2.1) is equivalent to saying that all boundaries are cycles. We define the *homology in degree* n of C to be the quotient

$$H_n(\mathsf{C}) = \frac{Z_n(\mathsf{C})}{B_n(\mathsf{C})} = \frac{\ker d_n}{\operatorname{im} d_{n+1}}.$$

What we refer to as the simplicial homology $\tilde{H}_n(\Delta; \mathbb{F})$ of a simplicial complex Δ is really the homology $H_n(\mathsf{C}(\Delta; \mathbb{F}))$ of the simplicial chain complex $\mathsf{C}(\Delta; \mathbb{F})$ associated to Δ.

2.2 Splitting chain complexes

An important technique for computing the homology of a chain complex is to split it into smaller chain complexes. One may then compute the homology of each piece separately and finally add the pieces together to get the full homology.

To formalize this idea, we need a few concepts. Given a chain complex

$$\mathsf{C} : \cdots \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

a *subcomplex* of C is a chain complex

$$\mathsf{C}' : \cdots \xrightarrow{d_{n+2}} C'_{n+1} \xrightarrow{d_{n+1}} C'_n \xrightarrow{d_n} C'_{n-1} \xrightarrow{d_{n-1}} \cdots$$

such that the following hold:

- C'_n is a submodule of C_n for each n.
- The restriction of d_n to C'_n defines a homomorphism to C'_{n-1} for each n. Equivalently, $d_n(C'_n) \subseteq C'_{n-1}$ for each n.

The most important subcomplexes of a simplicial complex Δ are the ones arising from subcomplexes Δ_0 of Δ.

Proposition 2.2.1 *If Δ is a simplicial complex and Δ_0 is a subcomplex of Δ, then $\mathsf{C}(\Delta_0)$ is a subcomplex of $\mathsf{C}(\Delta)$.*

We leave the proof to the reader.

A chain complex C *splits* into two chain complexes

$$\begin{aligned} \mathsf{C}' &: \cdots \xrightarrow{d_{n+2}} C'_{n+1} \xrightarrow{d_{n+1}} C'_n \xrightarrow{d_n} C'_{n-1} \xrightarrow{d_{n-1}} \cdots \\ \mathsf{C}'' &: \cdots \xrightarrow{d_{n+2}} C''_{n+1} \xrightarrow{d_{n+1}} C''_n \xrightarrow{d_n} C''_{n-1} \xrightarrow{d_{n-1}} \cdots \end{aligned}$$

if the following hold:

- Each of C' and C'' is a subcomplex of C.
- C_n is the direct sum of C'_n and C''_n for each n.

We write $\mathsf{C} = \mathsf{C}' \oplus \mathsf{C}''$.

Theorem 2.2.2 *If $\mathsf{C} = \mathsf{C}' \oplus \mathsf{C}''$, then*

$$H_n(\mathsf{C}) \cong H_n(\mathsf{C}') \oplus H_n(\mathsf{C}'')$$

for every n.

Proof. We may write each element $c \in C_n$ uniquely as a sum $c' + c''$, where $c' \in C'_n$ and $c'' \in C''_n$. To indicate that the sum is direct, we write $c = c' \oplus c''$. The boundary map has the property that

$$d_n(c' \oplus c'') = d_n(c') \oplus d_n(c'').$$

In particular, $d_n(c' \oplus c'') = 0$ if and only if $d_n(c') = d_n(c'') = 0$, which means that

$$\ker d_n = (C'_n \cap \ker d_n) \oplus (C''_n \cap \ker d_n).$$

For similar reasons, we have that

$$\operatorname{im} d_n = (C'_n \cap \operatorname{im} d_n) \oplus (C''_n \cap \operatorname{im} d_n)$$

By Proposition 0.2.1, we get that

$$\begin{aligned} H_n(\mathsf{C}) &= \frac{(C'_n \cap \ker d_n) \oplus (C''_n \cap \ker d_n)}{(C'_n \cap \operatorname{im} d_n) \oplus (C''_n \cap \operatorname{im} d_n)} \\ &\cong \frac{C'_n \cap \ker d_n}{C'_n \cap \operatorname{im} d_n} \oplus \frac{C''_n \cap \ker d_n}{C''_n \cap \operatorname{im} d_n}. \\ &= H_n(\mathsf{C}') \oplus H_n(\mathsf{C}''), \end{aligned}$$

which concludes the proof. □

Running example 1. Let Δ_0 be the subcomplex of E_1 obtained by removing the faces a, ab, ac, and abc. We claim that we may write $\mathsf{C}(E_1) = \mathsf{C}(E_1; \mathbb{F})$ as the direct sum of $\mathsf{C}(\Delta_0)$ and the chain complex C' with the following chain groups:

$$\begin{aligned} C'_0 &= \langle a - b \rangle, \\ C'_1 &= \langle ab, ac + cb \rangle, \\ C'_2 &= \langle abc \rangle; \end{aligned}$$

$C'_n = 0$ for $n \notin \{0, 1, 2\}$. Here, we use the notation (1) from Section 0.2.4, and we write abc instead of $a \wedge b \wedge c$.

To prove the claim, we first note that

$$\begin{aligned} \partial(abc) &= ab - (ac + cb) \in C'_1, \\ \partial(ab) = \partial(ac + cb) &= b - a \in C'_0, \\ \partial(a - b) &= 0 \in C'_{-1}, \end{aligned}$$

which yields that C' is a subcomplex of $\mathsf{C}(E_1)$. By Proposition 2.2.1, $\mathsf{C}(\Delta_0)$ is also a subcomplex. It remains to show that

$$\tilde{C}_n(E_1) = C'_n \oplus \tilde{C}_n(\Delta_0)$$

for each n. Now,

$$\begin{array}{rcrcccc} C'_{-1} & \oplus & \tilde{C}_{-1}(\Delta_0) & = & 0 & \oplus & \langle \mathbf{e}_\emptyset \rangle, \\ C'_0 & \oplus & \tilde{C}_0(\Delta_0) & = & \langle a-b \rangle & \oplus & \langle b,c,d \rangle, \\ C'_1 & \oplus & \tilde{C}_1(\Delta_0) & = & \langle ab, ac+cb \rangle & \oplus & \langle bc, bd, cd \rangle, \\ C'_2 & \oplus & \tilde{C}_2(\Delta_0) & = & \langle abc \rangle & \oplus & 0. \end{array}$$

One easily checks that we indeed get $\tilde{C}_n(E_1)$ in each case, which concludes the proof of the claim. As a consequence,

$$\tilde{H}_n(E_1) \cong H_n(\mathsf{C}') \oplus \tilde{H}_n(\Delta_0).$$

We leave it to the reader to verify that C' has zero homology and that $\tilde{H}_n(\Delta_0)$ is zero unless $n = 1$, in which case $\tilde{H}_1(\Delta_0) \cong \mathbb{F}$. Using Theorem 2.2.2, we hence reestablish the result from Section 1.7.1.

One may generalize Theorem 2.2.2 to the situation where C is the direct sum of k subcomplexes $\mathsf{C}^{(1)}, \ldots, \mathsf{C}^{(k)}$. By an induction argument, we get that

$$H_n(\mathsf{C}) \cong H_n(\mathsf{C}^{(1)}) \oplus \cdots \oplus H_n(\mathsf{C}^{(k)}). \tag{2.2}$$

2.3 Collapses

Let σ and τ be faces of a simplicial complex Δ such that the following hold:

- τ is maximal in Δ.
- $\sigma = \tau \setminus \{x\}$ for some $x \in \tau$.
- τ is the *only* maximal face of Δ containing σ.

The above conditions mean that the family $\Delta \setminus \{\sigma, \tau\}$ is a simplicial complex. We refer to the procedure of removing $\{\sigma, \tau\}$ from Δ as an *elementary collapse.*

Proposition 2.3.1 *For any elementary collapse $\Delta \to \Delta \setminus \{\sigma, \tau\} = \Delta_0$, we have that*

$$\tilde{H}_n(\Delta; \mathbb{F}) \cong \tilde{H}_n(\Delta_0; \mathbb{F})$$

for all n.

Proof. Write $k = \dim \sigma = \dim \tau - 1$. Let C' be the subcomplex of $\mathsf{C}(\Delta) = \mathsf{C}(\Delta; \mathbb{F})$ with the property that the nonzero chain groups are

$$\begin{array}{rcl} C'_{k+1} & = & \langle \tau \rangle, \\ C'_k & = & \langle \partial_{k+1}(\tau) \rangle. \end{array}$$

This is indeed a chain complex, because $\partial_{k+1}(\tau) \in C'_k$ and $\partial_k(\partial_{k+1}(\tau)) = 0 \in C'_{k-1}$. The reader may check that all homology groups of C' are zero.

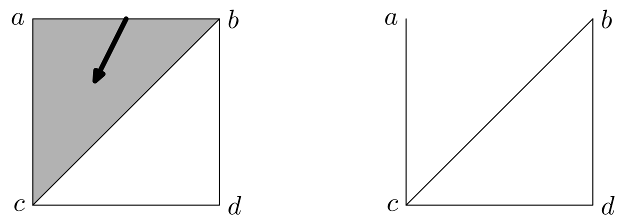

Figure 2.1: Elementary collapse with respect to the pair (ab, abc).

We claim that the chain complex $\mathsf{C}(\Delta)$ splits as

$$\mathsf{C}(\Delta) = \mathsf{C}' \oplus \mathsf{C}(\Delta_0).$$

By Theorem 2.2.2 and the fact that C' has zero homology, this yields the proposition.

Now, it is clear that $C_{k+1}(\Delta)$ is the direct sum of $\langle\tau\rangle$ and $C_{k+1}(\Delta_0)$. Moreover, we may write

$$C_k(\Delta) = \langle\partial(\tau)\rangle + C_k(\Delta_0), \tag{2.3}$$

and this is again a direct sum. Namely, let x be an element in $\tilde{C}_k(\Delta)$. The coefficient of σ in $\partial(\tau)$ is ± 1. For simplicity, let us assume that the coefficient is 1. We may write $x = \lambda\sigma + x_0$ for some $\lambda \in \mathbb{F}$ and $x_0 \in \tilde{C}_k(\Delta_0)$. Noting that

$$x = \lambda\sigma + x_0 = \lambda\partial(\tau) + \lambda(\sigma - \partial(\tau)) + x_0,$$

we obtain (2.3), because $\lambda(\sigma - \partial(\tau)) + x_0 \in C_k(\Delta_0)$. In addition, the sum is direct. Namely, suppose that $\lambda \cdot \partial(\tau) + x_0 = 0$. Then the coefficient of σ in x_0 must be $-\lambda$, because the coefficient is $+\lambda$ in $\partial(\tau)$. Yet, there is no occurrence of σ in x_0, which means that λ must be 0. As a consequence, x_0 is also zero; hence the sum is direct. This concludes the proof. □

Running example 1. Look at the elementary collapse $E_1 \to E_1 \setminus \{ab, abc\} = E_1'$; see Figure 2.1 for a geometric illustration. We obtain that

$$\begin{array}{ccccc}
\tilde{C}_{-1}(E_1) & = & 0 & \oplus & \langle \mathbf{e}_\emptyset \rangle, \\
\tilde{C}_0(E_1) & = & 0 & \oplus & \langle a, b, c, d \rangle, \\
\tilde{C}_1(E_1) & = & \langle ab - ac + bc \rangle & \oplus & \langle ac, bc, bd, cd \rangle, \\
\tilde{C}_2(E_1) & = & \langle abc \rangle & \oplus & 0.
\end{array}$$

Proceeding with the elementary collapse $E_1' \to E_1' \setminus \{a, ac\} = \Delta_0$, we may split the chain complex further as

$$\begin{array}{ccccccc}
\tilde{C}_{-1}(E_1) & = & 0 & \oplus & 0 & \oplus & \langle \mathbf{e}_\emptyset \rangle, \\
\tilde{C}_0(E_1) & = & 0 & \oplus & \langle b - a \rangle & \oplus & \langle b, c, d \rangle, \\
\tilde{C}_1(E_1) & = & \langle ab - ac + bc \rangle & \oplus & \langle ac \rangle & \oplus & \langle bc, bd, cd \rangle, \\
\tilde{C}_2(E_1) & = & \langle abc \rangle & \oplus & 0 & \oplus & 0.
\end{array}$$

The first two components of this direct sum form the subcomplex considered in the example in Section 2.2.

We refer to a sequence of elementary collapses

$$\Delta = \Sigma^0 \to \Sigma^1 \to \Sigma^2 \to \cdots \to \Sigma^r = \Delta_0 \tag{2.4}$$

as a *collapse* from Δ to Δ_0. Applying Proposition 2.3.1 r times, we obtain the following important result.

Proposition 2.3.2 *If there is a collapse from Δ to Δ_0, then*

$$\tilde{H}_n(\Delta;\mathbb{F}) \cong \tilde{H}_n(\Delta_0;\mathbb{F})$$

for every n.

In the example, we applied Proposition 2.3.1 twice to conclude that $\tilde{H}_n(E_1;\mathbb{F}) \cong \tilde{H}_n(\Delta_0;\mathbb{F})$.

Let Δ_0 be a simplicial complex, and let a be a vertex not in Δ_0. The *cone* $\mathrm{Cone}(\Delta_0) = \mathrm{Cone}_a(\Delta_0)$ over Δ_0 with *apex* a is the simplicial complex obtained from Δ_0 by adding $\sigma \cup \{a\}$ for each $\sigma \in \Delta_0$. Equivalently, Δ is a cone with apex a if $\sigma \cup \{a\}$ is a face of Δ whenever σ is a face of Δ.

Proposition 2.3.3 *If Δ is a cone with apex a, then $\tilde{H}_n(\Delta;\mathbb{F}) = 0$ for every n.*

Proof. We use induction on the number of faces of Δ. If Δ is the void complex $\emptyset$, then $\tilde{C}_n(\Delta) = 0$ and hence $\tilde{H}_n(\Delta) = 0$ for all n. Suppose that Δ is nonempty, and let τ be a maximal face of Δ. We must have that τ is of the form $\sigma \cup \{a\}$ for some σ not containing a. Moreover, σ is not contained in any other faces, because if $\sigma \cup \{x\}$ is in Δ for some $x \neq a$, then we also have that $\sigma \cup \{x, a\} \in \Delta$, contradicting the assumption that τ is maximal. We conclude that $\Delta \to \Delta \setminus \{\sigma, \tau\} = \Delta'$ defines an elementary collapse. Clearly, Δ' is again a cone with apex a. By induction, $\tilde{H}_n(\Delta') = 0$ for all n; hence Proposition 2.3.1 yields that $\tilde{H}_n(\Delta) = 0$ for all n. □

One may also prove Proposition 2.3.3 directly. Specifically, let z be a cycle in $\tilde{C}_n(\Delta;\mathbb{F})$. We may write $z = c_0 + a \wedge c_1$, where c_0, c_1 are elements in $\tilde{C}_{n-1}(\Delta_0;\mathbb{F})$ and $\Delta = \mathrm{Cone}_a(\Delta_0)$. We have that

$$0 = \partial_n(z) = \partial_n(c_0) + c_1 - a \wedge \partial_{n-1}(c_1).$$

For this to be zero, we need $c_1 = -\partial(c_0)$ and $\partial_{n-1}(c_1) = 0$. Yet, this means that

$$\partial_{n+1}(a \wedge c_0) = c_0 - a \wedge \partial_n(c_0) = c_0 + a \wedge c_1 = z;$$

hence z is a boundary. Since this is true for every cycle z, we conclude that the homology is zero. The expression $a \wedge c_0$ being a valid chain group element is because Δ is a cone with apex a.

The *full simplex* on a vertex set V is the simplicial complex 2^V of all subsets of V. Writing $d = |V| - 1$, we refer to 2^V as a *d-simplex*.

Corollary 2.3.4 *If Δ is a d-simplex for some $d \geq 0$, then $\tilde{H}_n(\Delta;\mathbb{F}) = 0$ for all $n \geq 0$.*

Proof. For any v in the vertex set of Δ, we have that Δ is a cone with apex v. By Proposition 2.3.3, we are done. □

A complex Δ is *collapsible* if there is a collapse from Δ to the void complex $\emptyset$ (not to be confused with $\{\emptyset\}$). By the proof of Proposition 2.3.3, cones are collapsible.

Corollary 2.3.5 *If a complex Δ is collapsible, then $\tilde{H}_n(\Delta) = 0$ for every n.*

Proof. This is an immediate consequence of Proposition 2.3.2. □

There exist simplicial complexes Δ that are not collapsible but still satisfy $\tilde{H}_n(\Delta; \mathbb{F}) = 0$ for all n.

2.4 Collapsing in practice

Suppose that we are given a simplicial complex Δ and a subcomplex Δ_0. We want to find out what it means for Δ to admit a collapse to Δ_0. By definition, there is then a sequence of elementary collapses as in (2.4). Let σ_i, τ_i be such that $\Sigma^{i-1} \setminus \Sigma^i = \{\sigma_i, \tau_i\}$ and $\sigma_i \subset \tau_i$. Define M to be the set of pairs (σ_i, τ_i) for $1 \le i \le r$. This means that M is a matching of faces of Δ such that every face not in Δ_0 appears in exactly one pair. Equivalently, M is a perfect matching on $\Delta \setminus \Delta_0$.

Here, a *matching* on a family Δ is a set M of pairs (σ, τ), where σ and τ are distinct members of Δ, such that no member of Δ appears in more than one pair. The matching is *perfect* if every member appears in exactly one pair. Members of Δ that appear in some pair are *matched*, whereas other members are *unmatched*.

We say that a matching M on a family Δ of sets is an *element matching* if every pair in M is of the form $(\sigma \setminus \{x\}, \sigma \cup \{x\})$ for some $x \in X$ and $\sigma \in \Delta$.[1] For simplicity, we will often write $\sigma \setminus x = \sigma \setminus \{x\}$ and $\sigma \cup x = \sigma \cup \{x\}$. All matchings considered in this document are element matchings. By the above discussion, we conclude the following.

- For a simplicial complex Δ to admit a collapse to a subcomplex Δ_0, it is necessary that there exists a perfect element matching on $\Delta \setminus \Delta_0$.

Yet, this condition is not sufficient. For example, $\Delta = \{\emptyset, a, ab, b, bc, c, ac\}$ cannot be collapsed to $\Delta_0 = \{\emptyset\}$, but there is a perfect element matching on $\Delta \setminus \Delta_0$ given by the pairs

$$(a, ab), (b, bc), (c, ac). \tag{2.5}$$

The problem with this matching is that none of these pairs can be the pair used in the first elementary collapse; each of a, b, and c belongs to two maximal faces, not just one.

As it turns out, there is a combinatorial description of the condition that a perfect element matching M on $\Delta \setminus \Delta_0$ corresponds to a collapse. For generality, let Δ be an arbitrary family of sets, and let M be an arbitrary element matching on Δ. Form a directed graph $D(\Delta, M)$ with one vertex for each member of Δ and with edges defined according to the following rules.

[1] Note that the pair is $(\sigma, \sigma \cup \{x\})$ if $x \notin \sigma$ and $(\sigma \setminus \{x\}, \sigma)$ if $x \in \sigma$.

- There is a directed edge from $\sigma \setminus x$ to $\sigma \cup x$ whenever $(\sigma \setminus x, \sigma \cup x)$ belongs to the matching M.
- There is a directed edge from $\sigma \cup x$ to $\sigma \setminus x$ whenever $(\sigma \setminus x, \sigma \cup x)$ does not belong to the matching M.

We say that M is an *acyclic matching* if $D(\Delta, M)$ does not contain any directed cycles.

Example. If M is the matching in (2.5), then $D(\Delta, M)$ contains the directed cycle

$$a \to ab \to b \to bc \to c \to ac \to a.$$

In particular, M is not an acyclic matching.

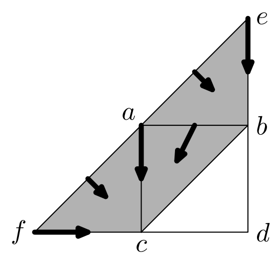

Figure 2.2: A complex Δ along with a matching M on the family of faces outside the subcomplex Δ_0 with maximal faces bc, bd, cd.

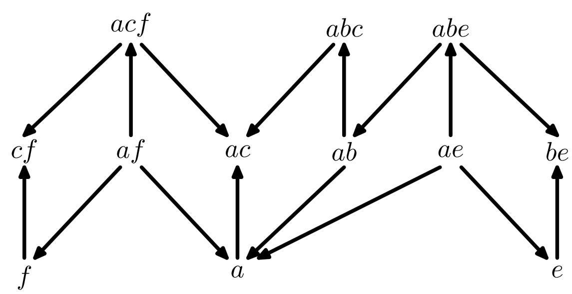

Figure 2.3: The directed graph $D(\Delta, M)$ associated to the matching M illustrated in Figure 2.2. We exclude unmatched faces.

Example. Let Δ be the complex in Figure 2.2; the maximal faces of this complex are abc, abe, acf, bd, cd. Let Δ_0 be the subcomplex with maximal faces bc, bd, cd. In the figure, we have indicated a perfect matching M on $\Delta \setminus \Delta_0$. This matching consists of the following pairs:

$$(e, be), (a, ac), (f, cf), (ae, abe), (ab, abc), (af, acf).$$

We illustrate the associated digraph $D(\Delta, M)$ in Figure 2.3. One may check that $D(\Delta, M)$ does not contain any directed cycles. In particular, M is an acyclic matching.

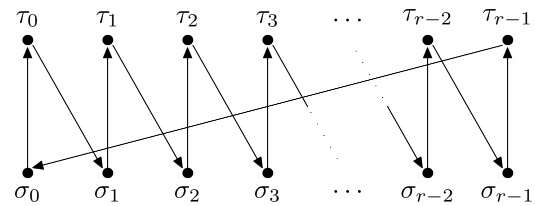

Figure 2.4: Cycle in a digraph corresponding to a non-acyclic matching.

Proposition 2.4.1 *Every directed cycle in $D(\Delta, M)$ is of the form*

$$(\sigma_0, \tau_0, \sigma_1, \tau_1, \ldots, \sigma_{r-1}, \tau_{r-1})$$

such that

$$\sigma_i, \sigma_{(i+1) \bmod r} \subset \tau_i \text{ and } (\sigma_i, \tau_i) \in M, \tag{2.6}$$

and $r \geq 3$; see Figure 2.4 for an illustration.

Proof. In a directed cycle in $D(\Delta, M)$, the number of up-steps (steps of the form $\sigma \to \sigma \cup x$) is equal to the number of down-steps (steps of the form $\sigma \cup x \to \sigma$). There is no directed path (ρ, σ, τ) consisting of two consecutive up-steps, as this would imply that σ is matched with both ρ and τ. As a consequence, at most every other step can be an up-step. A straightforward counting argument yields that exactly every other step is an up-step. □

Proposition 2.4.2 *There is a collapse from Δ to Δ_0 if and only if there exists a perfect acyclic matching M on $\Delta \setminus \Delta_0$.*

Proof. Suppose that M is a perfect acyclic matching on $\Delta \setminus \Delta_0$. Since $D(\Delta, M)$ contains no directed cycles, there must be a face σ in $\Delta \setminus \Delta_0$ such that there are no arrows directed to σ. Let τ be the face matched with σ. Since no arrows are directed to σ, we must have that σ is the smaller face. Let a be such that $\tau = \sigma \cup a$. Now, σ cannot be contained in any face $\tau' \neq \tau$ of the same dimension as τ, because then there would be an arrow from τ' to σ. Moreover, τ must be a maximal face of Δ, because if $\tau \cup b$ belongs to Δ for some $b \notin \tau$, then so does

$$(\tau \cup b) \setminus a = \sigma \cup b,$$

which implies that there is an arrow from $\sigma \cup b$ to σ, a contradiction. As a consequence,

$$\Delta \to \Delta \setminus \{\sigma, \tau\}$$

is an elementary collapse. Proceeding inductively, we obtain a sequence of elementary collapses from Δ to Δ_0.

The other direction is left to the reader. □

Example. With Δ, Δ_0, and M defined as in Figure 2.2, we obtain a sequence of elementary collapses by collapsing with the matched pairs in the following order:

$$(af, acf), (f, cf), (ae, abe), (e, be), (ab, abc), (a, ac).$$

For example, we may start with (af, acf), because there is no arrow directed to af in $D(\Delta, M)$; see Figure 2.3. As a consequence, Δ can be collapsed to Δ_0, which means that $\tilde{H}_n(\Delta) = \tilde{H}_n(\Delta_0)$ for all n.

More generally, we have the following characterization of acyclic matchings.

Proposition 2.4.3 *A matching M on a family Δ is acyclic if and only if the matched pairs can be labelled $(\sigma_1, \tau_1), \dots, (\sigma_k, \tau_k)$ such that the following conditions hold:*

(i) *For $1 \le i < j \le k$, we have that $\dim \sigma_i \le \dim \sigma_j$.*

(ii) *For $1 \le i < j \le k$, we have that σ_j is not contained in τ_i.*

Proof. A matching M satisfying (i)-(ii) is acyclic. Namely, assume the opposite, and let i be minimal such that τ_i appears in a cycle in $D(\Delta, M)$. By Proposition 2.4.1, τ_i is followed in the cycle by σ_j and τ_j for some $j \neq i$ (note that the indices i and j do not have the same meaning here as in that proposition). By (ii), we must have that $j < i$, which is a contradiction to the minimality of i.

Conversely, suppose that M is acyclic. For any labeling $(\sigma_1, \tau_1), \dots, (\sigma_k, \tau_k)$ of the matched pairs such that (i) is satisfied, we have that (ii) can only be violated if $\dim \sigma_i = \dim \sigma_j$. In particular, we may restrict our attention to all matched pairs (σ, τ) such that $\dim \sigma$ is equal to a fixed value. Since $D(\Delta, M)$ does not contain any directed cycles, there must be some matched pair (σ_1, τ_1) with the property that τ_1 does not contain σ for $(\sigma, \tau) \in M \setminus \{(\sigma_1, \tau_1)\}$. By induction on the size of M, the pairs in $D(\Delta, M) \setminus \{(\sigma_1, \tau_1)\}$ can be labelled as $(\sigma_2, \tau_2), \dots, (\sigma_k, \tau_k)$ such that (ii) is satisfied. Adding $(\sigma_1, \tau_1\}$ at the beginning of the list, we observe that (ii) is still satisfied. This concludes the proof. $\square$

Example. Again, consider Δ, Δ_0, and M defined as in Figure 2.2. The following arrangement of the pairs in M satisfies the conditions in Proposition 2.4.3:

$$(a, ac), (e, be), (f, cf), (ab, abc), (ae, abe), (af, acf).$$

We obtain a collapse from Δ to Δ_0 by starting with the pair on the far right and then going left. There are many other possible arrangements, but the pair (ab, abc) must appear before the pair (ae, abe), because abe contains ab.

We conclude the section with some useful results. See Section 2.7 for examples.

A finite *partially ordered set* or *poset* is a pair $P = (X, \le)$, where X is a finite set and $\le$ is a binary relation on X satisfying the following conditions for all $x, y, z \in X$:

- $x \le x$.
- If $x \le y$ and $y \le x$, then $x = y$.
- If $x \le y$ and $y \le z$, then $x \le z$.

An (order-preserving) *poset map* between two posets $P = (X, \leq_P)$ and $Q = (Y, \leq_Q)$ is a function $f : X \to Y$ such that $f(x) \leq_Q f(y)$ whenever $x \leq_P y$. We will often write $f : P \to Q$. We obtain a poset structure on a family Δ of sets by definining $\sigma \leq \tau$ whenever $\sigma \subseteq \tau$.

Lemma 2.4.4 ([1, 2]) *(Cluster Lemma) Let Δ be a family of sets, and let $f : \Delta \to Q$ be a poset map, where Q is an arbitrary poset. For $q \in Q$, let M_q be an acyclic matching on*

$$f^{-1}(q) = \{\sigma \in \Delta : f(\sigma) = q\}.$$

Let

$$M = \bigcup_{q \in Q} M_q.$$

Then M is an acyclic matching on Δ.

Proof. Assume the opposite, and let $(\sigma_0, \tau_0, \ldots, \sigma_{r-1}, \tau_{r-1})$ be a cycle in $D(\Delta, M)$ satisfying (2.6). Let $q_0, \ldots, q_{r-1}$ be such that $\sigma_k, \tau_k \in f^{-1}(q_k)$ for $0 \leq k \leq r-1$. Since $\sigma_{(k+1) \bmod r} \subset \tau_k$, we get that $q_{(k+1) \bmod r} = f(\sigma_{(k+1) \bmod r}) \leq f(\sigma_k) = q_k$. Via a simple induction argument, this implies that $q_{k'} \leq q_k$ for any pair k, k'. Swapping k and k', we obtain $q_k \leq q_{k'}$, which implies that $q_k = q_{k'}$; Q is a poset. Hence all sets in the cycle are contained in one single family $f^{-1}(q)$, which is a contradiction. $\square$

A very common situation is described in the following corollary.

Corollary 2.4.5 *Let Δ be a family of subsets of a set X, and let Y be a subset of X. For $\rho \subseteq Y$, let M_ρ be an acyclic matching on the family*

$$\Delta_\rho = \{\sigma \in \Delta : \sigma \cap Y = \rho\}.$$

Define

$$M = \bigcup_{\rho \subseteq Y} M_\rho.$$

Then M is an acyclic matching on Δ.

Proof. Let 2^Y denote the poset of all subsets of Y ordered by inclusion. A poset map from Δ to 2^Y is given by mapping a face σ to $\sigma \cap Y$. Namely, if $\sigma \subseteq \tau$, then $\sigma \cap Y \subseteq \tau \cup Y$. By Lemma 2.4.4, we are done. $\square$

Our final lemma gives another approach to constructing acyclic matchings. The idea is to pick an element x and match $\sigma \setminus x$ with $\sigma \cup x$ whenever possible.

Lemma 2.4.6 *Let Δ be a family of subsets of a set X, and let $x \in X$. Define*

$$\begin{aligned} M_x &= \{(\sigma \setminus x, \sigma \cup x) : \sigma \setminus x, \sigma \cup x \in \Delta\}; \\ \Delta_x &= \{\sigma : \sigma \setminus x, \sigma \cup x \in \Delta\}. \end{aligned}$$

Let M' be an acyclic matching on $\Delta' = \Delta \setminus \Delta_x$. Then $M = M_x \cup M'$ is an acyclic matching on Δ.

Proof. Assume that $(\sigma_0, \tau_0, \ldots, \sigma_{r-1}, \tau_{r-1})$ is a cycle in $D(\Delta, M)$ satisfying (2.6). Since M' is an acyclic matching on Δ', there must be some pair $\{\sigma_i, \tau_i\}$ that is included in M_x rather than in M'; by construction, we then have that $\tau_i = \sigma_i \cup x$. For simplicity, assume that $i = 0$. Since τ_{r-1} is not matched with σ_0, we must have that $x \notin \tau_{r-1}$. This means that there is some $j \in \{1, \ldots, r-1\}$ such that $x \in \tau_{j-1}$ and $x \notin \tau_j$. However, this implies that $\tau_{j-1} = \sigma_j \cup x$, which is a contradiction, because we would then have that $(\sigma_j, \tau_{j-1}) \in M_x$ by construction. □

2.5 Basics of discrete Morse theory

Now, we generalize the situation from Section 2.3. The topics discussed in this section form the basis of *discrete Morse theory*, a method introduced by Robin Forman.

Let

$$\mathsf{C} : \cdots \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

be an arbitary chain complex C of free and finitely generated $\mathbb{F}$-modules. We assume that we have a fixed basis $\mathcal{E}_n$ of each C_n. For example, if C is the simplicial chain complex associated to a simplicial complex, then we may choose $\mathcal{E}_n$ to be the usual set of oriented simplices of dimension n. For simplicity, we write $d = d_n$ for each n.

Any element c in C_n admits a unique representation

$$c = \sum_{\sigma \in \mathcal{E}_n} \lambda_\sigma \sigma$$

as a linear combination of the basis elements, where $\lambda_\sigma \in \mathbb{F}$ for each σ. We refer to λ_σ as the *coefficient* of σ in c. For $x \in C_{n+1}$, we let $(x : \sigma)$ denote the coefficient of σ_i in $d(x)$.

Let k be an integer, and let $\sigma \in \mathcal{E}_k$ and $\tau \in \mathcal{E}_{k+1}$. We want to define the concept of a *generalized* elementary collapse involving the pair (σ, τ). It turns out to be convenient to drop the maximality requirements on τ. In the simplicial case, this leaves us with one single requirement:

- $\sigma = \tau \setminus x$ for some $x \in \tau$.

In the general case, we require the following:

- The coefficient $(\tau : \sigma)$ of σ in $d(\tau)$ is invertible in $\mathbb{F}$.

In the case that $\mathbb{F}$ is a field, this means that the given coefficient is a nonzero element in $\mathbb{F}$. If $\mathbb{F} = \mathbb{Z}$, then the coefficient must be 1 or -1.

Given an elementary collapse from a simplicial complex Δ to a subcomplex Δ_0, we deduced in Section 2.3 that $\mathsf{C}(\Delta)$ can be written as a direct sum $\mathsf{C}' \oplus \mathsf{C}(\Delta_0)$, where C' has zero homology. In our more general setting, we want to obtain a similar decomposition $\mathsf{C} = \mathsf{C}' \oplus \widehat{\mathsf{C}}$, where again C' has zero homology. As we will see, the procedure is a little bit more complicated than in Section 2.3.

To start with, let

$$\begin{aligned} \widehat{\mathcal{E}}_{k+1} &= \mathcal{E}_{k+1} \setminus \{\tau\} \\ \widehat{\mathcal{E}}_k &= \mathcal{E}_k \setminus \{\sigma\}. \end{aligned}$$

Defining $\widehat{\mathcal{E}}_n = \mathcal{E}_n$ for all other choices of n, we obtain a new family of $\mathbb{F}$-modules $\widehat{C}_n$, where $\widehat{C}_n$ is the free $\mathbb{F}$-module with basis $\widehat{\mathcal{E}}_n$. Yet, we get two problems; for simplicity we write $d = d_n$ for all n.

1. It is not necessarily true that $d(x) \in \widehat{C}_k$ for all $x \in \widehat{C}_{k+1}$, because we might have that $(x : \sigma) \neq 0$.

2. It is not necessarily true that $d(x) \in \widehat{C}_{k+1}$ for all $x \in \widehat{C}_{k+2}$, because we might have that $(x : \tau) \neq 0$.

These problems are a consequence of our dropping the maximality requirements on τ. We start by taking care of the first problem. As it turns out, our solution to this problem will also solve the second problem.

For any basis element $\epsilon \in \mathcal{E}_{k+1} \setminus \{\tau\}$, define

$$\lambda_\epsilon = \frac{(\epsilon : \sigma)}{(\tau : \sigma)}$$

and

$$\widehat{\epsilon} = \epsilon - \lambda_\epsilon \tau = \epsilon - \frac{(\epsilon : \sigma)}{(\tau : \sigma)} \cdot \tau.$$

Note that

$$(\widehat{\epsilon} : \sigma) = (\epsilon : \sigma) - \lambda_\epsilon (\tau : \sigma) = (\epsilon : \sigma) - \frac{(\epsilon : \sigma)}{(\tau : \sigma)} \cdot (\tau : \sigma) = 0.$$

In particular, $d(\widehat{\epsilon}) \in \widehat{C}_k$. Redefine $\widehat{\mathcal{E}}_{k+1}$ as

$$\widehat{\mathcal{E}}_{k+1} = \{\widehat{\epsilon} : \epsilon \in \mathcal{E}_{k+1} \setminus \{\tau\}\}.$$

Moreover, redefine $\widehat{C}_{k+1}$ to be the free $\mathbb{F}$-module with basis $\widehat{\mathcal{E}}_{k+1}$.

Example. Consider the chain complex corresponding to the simplicial complex consisting of all subsets of the set $\{r, s, t, u\}$. Consider the pair (σ, τ), where $\sigma = rt$ and $\tau = rst$. In this case, $k = 1$. Note that

$$(rst : rt) = -1,$$

because the coefficient of rt in $\partial(rst) = rs + st - rt$ is -1.

Now, look at the basis elements $\epsilon \neq rst$ in degree 2. We have that

$$\lambda_\epsilon = \frac{(\epsilon : rt)}{(rst : rt)} = -(\epsilon : rt),$$

which yields that

$$\begin{array}{rcrcr} \lambda_{rsu} & = & -(rsu : rt) & = & 0, \\ \lambda_{rtu} & = & -(rtu : rt) & = & -1, \\ \lambda_{stu} & = & -(stu : rt) & = & 0. \end{array}$$

This means that we should define

$$\begin{array}{rcl l l} \widehat{rsu} & = & rsu - 0 & = & rsu, \\ \widehat{rtu} & = & rtu - (-rst) & = & rtu + rst, \\ \widehat{stu} & = & stu - 0 & = & stu. \end{array}$$

One easily checks that rt does not appear in the boundary of any of these elements. For example,

$$\begin{aligned}\partial(\widehat{rtu}) = \partial(rtu + rst) &= rt + tu - ru + rs + st - rt \\ &= rs - ru + st + tu.\end{aligned}$$

After having solved the first problem, it remains to consider the second problem: We want $d(x)$ to lie in $\widehat{C}_{k+1}$ for every $x \in \widehat{C}_{k+2} = C_{k+2}$. Now, $\widehat{\mathcal{E}}_{k+1} \cup \{\tau\}$ is a basis for C_{k+1}. In particular, there are numbers μ_ϵ such that

$$d(x) = \sum_{\epsilon \in \mathcal{E}_{k+1} \setminus \{\tau\}} \mu_\epsilon \widehat{\epsilon} + \mu_\tau \tau.$$

Since $d^2(x) = 0$, we get that

$$0 = (d(x) : \sigma) = \sum_{\epsilon \in \mathcal{E}_{k+1} \setminus \{\tau\}} \mu_\epsilon (\widehat{\epsilon} : \sigma) + \mu_\tau (\tau : \sigma) = \mu_\tau (\tau : \sigma).$$

The last equality is a consequence of the fact that $(\widehat{\epsilon} : \sigma) = 0$ for all $\epsilon \in \mathcal{E}_{k+1} \setminus \{\tau\}$. Since $(\tau : \sigma)$ is invertible, we obtain that μ_τ must be zero. As a consequence, $d(x) \in \widehat{C}_{k+1}$.

Example. Let us continue with the same example as before. Since $k = 1$, we want to check that $d(x) \in \widehat{C}_2$ for each $x \in C_3$. Now, $C_3 = \langle rstu \rangle$, and we get that

$$\partial(rstu) = stu - rtu + rsu - rst = stu - (rtu + rst) + rsu = \widehat{stu} - \widehat{rtu} + \widehat{rsu}.$$

This indeed belongs to $\widehat{C}_2$.

We may conclude the following.

Proposition 2.5.1 *Let $\sigma \in \mathcal{E}_k$ and $\tau \in \mathcal{E}_{k+1}$ be such that $(\tau : \sigma)$ is invertible in $\mathbb{F}$. Then*

$$\widehat{\mathsf{C}} : \cdots \xrightarrow{d_{n+2}} \widehat{C}_{n+1} \xrightarrow{d_{n+1}} \widehat{C}_n \xrightarrow{d_n} \widehat{C}_{n-1} \xrightarrow{d_{n-1}} \cdots$$

defines a chain complex, where $\widehat{C}_n$ is the submodule of C_n with basis

$$\widehat{\mathcal{E}}_n = \begin{cases} \{\epsilon - \frac{(\epsilon:\sigma)}{(\tau:\sigma)} \cdot \tau : \epsilon \in \mathcal{E}_{k+1}\} & \text{if } n = k+1, \\ \mathcal{E}_k \setminus \{\sigma\} & \text{if } n = k, \\ \mathcal{E}_n & \text{otherwise.} \end{cases}$$

Let C' be the subcomplex of C with the property that the nonzero chain groups are $C'_{k+1} = \langle \tau \rangle$ and $C'_k = \langle d(\tau) \rangle$. We obtain the following.

Proposition 2.5.2 *Let $\widehat{\mathsf{C}}$ be defined as in Proposition 2.5.1. Then C is the direct sum of C' and $\widehat{\mathsf{C}}$. In particular, $H_n(\mathsf{C}) \cong H_n(\widehat{\mathsf{C}})$ for each n.*

Proof. We already concluded that $\widehat{\mathsf{C}}$ is a subcomplex of C. As a consequence, it suffices to show that C_n is the direct sum of C'_n and $\widehat{C}_n$ for each n. This is immediate unless $n \in \{k, k+1\}$. For $n = k$, the proof is identical to that of Proposition 2.3.1. For $n = k+1$, it is easy to check that we obtain a basis for C_{k+1} by adding τ to $\widehat{\mathcal{E}}_{k+1}$. To prove the final statement in the proposition, apply Theorem 2.2.2 and use the fact that the homology of C' is zero. □

After having considered the situation for one single matched pair, we now consider a general matching

$$M = \{(\sigma_i, \tau_i) : 1 \leq i \leq r\},$$

where σ_i and τ_i are basis elements such that $\langle \partial(\tau_i), \sigma_i \rangle$ is invertible for each i. For any $x \in C_d$, we write $\deg x = d$.

Theorem 2.5.3 *Suppose that the following hold:*

(i) *For $1 \leq i < j \leq r$, we have that $\dim \sigma_i \leq \dim \sigma_j$.*

(ii) *For $1 \leq i < j \leq r$, we have that $\langle \partial(\tau_i), \sigma_j \rangle$ is zero.*

Then it is possible to form a new chain complex $\widehat{\mathsf{C}}$ such that $H_n(\widehat{\mathsf{C}}) \cong H_n(\mathsf{C})$ for all n and such that the rank of $\widehat{C}_n$ is equal to the rank of C_n minus the number of basis elements $\mathbf{e} \in \{\sigma_i, \tau_i, : 1 \leq i \leq r\}$ such that $\deg \mathbf{e} = d$.

For simplicial chain complexes, conditions (i)-(ii) are equivalent to saying that the matching is acyclic; apply Proposition 2.4.3.

Let us sketch a proof. Starting with the very last pair (σ_k, τ_k), we may use the above procedure to transform C into a new chain complex $\widehat{\mathsf{C}}$ with the same homology as C. If $k = 1$, then we are done. Otherwise, we observe the folloing: When moving from C to $\widehat{\mathsf{C}}$, we only modify basis elements $\mathbf{e}$ such that $\langle \partial(\mathbf{e}), \sigma_k \rangle \neq 0$. Among the elements $\sigma_1, \tau_1, \ldots, \sigma_{k-1}, \tau_{k-1}$, there is no such element by assumption. In particular, conditions (i)-(ii) still hold for the pairs $(\sigma_1, \tau_1), \ldots, (\sigma_{k-1}, \tau_{k-1})$ in $\widehat{\mathsf{C}}$. Using an induction argument, we may hence transform $\widehat{\mathsf{C}}$ into a chain complex with properties as in the theorem.

2.6 Joins, deletions, and links

Some important constructions in the theory of simplicial complexes are those of joins, deletions, and links.

2.6.1 Joins

Let X and Y be two sets such that $X \cap Y$ is empty. Let Δ be a family of subsets of X, and let Γ be a family of subsets of Y. The *join* of Δ and Γ is the family

$$\Delta * \Gamma = \{\delta \cup \gamma : \delta \in \Delta, \gamma \in \Gamma\}.$$

Observe that $\Delta * \emptyset = \emptyset$ and $\Delta * \{\emptyset\} = \Delta$. If both Δ and Γ are simplicial complexes, then $\Delta * \Gamma$ is also a simplicial complex.

For example, any cone is a join, because

$$\mathrm{Cone}_a(\Gamma) = \{\emptyset, a\} * \Gamma.$$

See Section 2.3 for more discussion on cones.

For another important special case, let Γ be a simplicial complex, and let a and b be two vertices not in Γ. The *suspension* $\mathrm{Susp}(\Gamma) = \mathrm{Susp}_{a,b}(\Gamma)$ is the join $\{\emptyset, a, b\} * \Gamma$. Equivalently,

$$\mathrm{Susp}(\Gamma) = \{\sigma, \sigma \cup \{a\}, \sigma \cup \{b\} : \sigma \in \Gamma\}.$$

We examine suspensions in Section 2.6.2.

Given a matching M_Δ on Δ and a subfamily Γ_0 of Γ, we define

$$M_\Delta * \Gamma_0 = \{(\sigma \cup \gamma, \tau \cup \gamma) : (\sigma, \tau) \in M_\Delta, \gamma \in \Gamma_0\}. \tag{2.7}$$

If M_Δ is a perfect matching on Δ, then $M_\Delta * \Gamma_0$ is a perfect matching on $\Delta * \Gamma_0$. We define $\Delta_0 * M_\Gamma$ analogously for any subfamily Δ_0 of Δ and any matching M_Γ on Γ.

In the following, δ always denotes a member of Δ, whereas γ denotes a member of Γ.

Proposition 2.6.1 *Let X and Y be two sets such that $X \cap Y$ is empty. Let Δ be a family of subsets of X, and let Γ be a family of subsets of Y. Suppose that we have acyclic matchings M_Δ and M_Γ on Δ and Γ, respectively. Then there is an acyclic matching on $\Delta * \Gamma$ such that $\delta \cup \gamma$ is unmatched if and only if δ is unmatched with respect to M_Δ and γ is unmatched with respect to M_Γ.*

Proof. Define a matching M on $\Delta * \Gamma$ as the union of the following two matchings.

- $M_\Delta * \Gamma$.
- $\mathcal{C} * M_\Gamma$, where $\mathcal{C}$ is the family of members of Δ that are unmatched with respect to M_Δ.

Note the asymmetry in the construction. It is clear that $\delta \cup \gamma$ is unmatched if and only if δ is unmatched with respect to M_Δ and γ is unmatched with respect to M_Γ.

It remains to prove that M is an acyclic matching. Assume the opposite, and let

$$(\delta_0 \cup \gamma_0, \delta_1 \cup \gamma_1, \ldots, \delta_{2r-1} \cup \gamma_{2r-1})$$

be a cycle in $D(\Delta * \Gamma, M)$. Consider the sequence

$$(\delta_0, \delta_1, \ldots, \delta_{2r-1}).$$

By construction, for each i, we have that either $\delta_i = \delta_{i+1}$ or there is a directed edge from δ_i to δ_{i+1} in $D(\Delta, M_\Delta)$ (indices are computed modulo $2r$). Since $D(\Delta, M_\Delta)$ is acyclic, we must have that $\delta_i = \delta_{i+1}$ for all i, which yields that all δ_i are equal to some fixed δ.

If δ is matched in M_Δ, then there cannot be a directed edge from $\delta \cup \gamma$ to $\delta \cup \gamma'$ unless $\gamma' \subset \gamma$. This rules out the existence of a directed cycle in this case.

Suppose that δ is unmatched in M_Δ, and consider the sequence

$$(\gamma_0, \gamma_1, \ldots, \gamma_{2r-1}).$$

Again by construction, this sequence forms a directed cycle in $D(\Gamma, M_\Gamma)$, a contradiction. $\square$

Corollary 2.6.2 *Suppose that Δ admits a collapse to Δ_0 and that Γ admits a collapse to Γ_0. Then $\Delta * \Gamma$ admits a collapse to $\Delta_0 * \Gamma_0$.*

Proof. The given conditions are equivalent to $\Delta \setminus \Delta_0$ and $\Gamma \setminus \Gamma_0$ admitting acyclic matchings. Proposition 2.6.1 yields that $\Delta * \Gamma \setminus (\Delta_0 * \Gamma_0)$ admits a perfect acyclic matching. By Proposition 2.4.2, this is equivalent to saying that $\Delta * \Gamma$ admits a collapse to $\Delta_0 * \Gamma_0$. $\square$

In particular, the following is true.

Corollary 2.6.3 *Let Δ and Γ be simplicial complexes. If Δ is collapsible, then $\Delta * \Gamma$ is collapsible.*

2.6.2 Links and deletions

Let Δ be a simplicial complex, and let a be a vertex in Δ. The *deletion* of Δ with respect to a is the subcomplex

$$\mathrm{del}_\Delta(a) = \{\sigma \in \Delta : a \notin \sigma\}.$$

The *link* of Δ with respect to a is the subcomplex

$$\mathrm{link}_\Delta(a) = \{\sigma \in \mathrm{del}_\Delta(a) : \sigma \cup \{a\} \in \Delta\}.$$

Note that Δ is the disjoint union of $\mathrm{del}_\Delta(a)$ and $\{a\} * \mathrm{link}_\Delta(a)$.

The following proposition is a special case of a more general result discussed in Section 3.3. Since we will need the proposition already in Section 2.7, we give a separate proof here.

Proposition 2.6.4 *Let n be integer, let Δ be a simplicial complex, and let a be a vertex such that $\tilde{H}_n(\mathrm{del}_\Delta(a);\mathbb{F}) = \tilde{H}_{n-1}(\mathrm{del}_\Delta(a);\mathbb{F}) = 0$. Then*

$$\tilde{H}_n(\Delta;\mathbb{F}) \cong \tilde{H}_{n-1}(\mathrm{link}_\Delta(a);\mathbb{F}).$$

Proof. Write $D = \mathrm{del}_\Delta(a)$ and $L = \mathrm{link}_\Delta(a)$. We may write any element z in $\tilde{C}_n(\Delta)$ as a sum $z = x + a \wedge y$, where $x \in \tilde{C}_n(D)$ and $y \in \tilde{C}_{n-1}(L)$. Note that z is a cycle if and only if $\partial_n(x) + y = 0$ and $\partial_{n-1}(y) = 0$. In particular, we may write any element z in $Z_n(\Delta)$ as a sum $z = x - a \wedge \partial_n(x)$, where $x \in \tilde{C}_n(D)$ and $\partial_n(x) \in \tilde{C}_{n-1}(L)$. Also note that we may write any element z in $B_n(\Delta)$ as

$$z = \partial_{n+1}(x + a \wedge y) = \partial_{n+1}(x) + y - a \wedge \partial_n(y)$$

for some $x \in \tilde{C}_{n+1}(D)$ and $y \in \tilde{C}_n(L)$.

Define a map $\varphi : \tilde{C}_n(\Delta) \to \tilde{C}_{n-1}(L)$ by

$$\varphi(x + a \wedge y) = y.$$

By the above discussion, φ induces a map $\varphi^* : \tilde{H}_n(\Delta) \to \tilde{H}_{n-1}(L)$. Namely, if $z \in Z_n(\Delta)$, then $\varphi(z)$ is of the form $-\partial_n(x)$ for some $x \in \tilde{C}_n(D)$ such that $\partial_n(x) \in \tilde{C}_{n-1}(L)$, hence an element in $Z_{n-1}(L)$. If $z \in B_n(\Delta)$, then $\varphi(z)$ is of the form $\partial_n(y)$ for some $y \in \tilde{C}_n(L)$, hence an element in $B_{n-1}(L)$.

It remains to prove that φ^* is an isomorphism. First, assume that z is an element in $Z_n(\Delta)$ such that $\varphi(z)$ is an element in $B_{n-1}(L)$. Writing $z = x - a \wedge$

$\partial_n(x)$, where $x \in \tilde{C}_n(D)$ and $\partial_n(x) \in \tilde{C}_{n-1}(L)$, we obtain that $\varphi(z) = -\partial_n(x)$. This belonging to $B_{n-1}(L)$ means that $\partial_n(x) = \partial_n(x')$ for some $x' \in \tilde{C}_n(L)$. Now,

$$\begin{aligned} z - \partial_{n+1}(a \wedge x') &= x - a \wedge \partial_n(x) - x' + a \wedge \partial_n(x') \\ &= x - x' \in Z_n(D) = B_n(D); \end{aligned}$$

the last equality follows from the fact that $\tilde{H}_n(D) = 0$. As a consequence, $z \in B_n(\Delta)$, which yields that φ^* is injective.

Next, let w be any element in $Z_{n-1}(L)$. Since $Z_{n-1}(D) = B_{n-1}(D)$, we have that $w = \partial_n(x)$ for some $x \in \tilde{C}_n(D)$. This yields that $z = -x + a \wedge \partial_n(x)$ is an element in $Z_n(\Delta)$ such that $\varphi(z) = w$. In particular, φ^* is surjective. $\square$

Corollary 2.6.5 *We have that $\tilde{H}_n(\mathrm{Susp}_{a,b}(\Gamma); \mathbb{F}) = \tilde{H}_{n-1}(\Gamma; \mathbb{F})$ for every n.*

Proof. Write $\Delta = \mathrm{Susp}_{a,b}(\Gamma)$. We observe that

$$\begin{aligned} \mathrm{del}_\Delta(a) &= \mathrm{Cone}_b(\Gamma), \\ \mathrm{link}_\Delta(a) &= \Gamma. \end{aligned}$$

Using Proposition 2.3.3, we deduce that $\tilde{H}_n(\mathrm{del}_\Delta(a); \mathbb{F}) = 0$ for all n. As a consequence, Proposition 2.6.4 yields that

$$\tilde{H}_n(\Delta; \mathbb{F}) \cong \tilde{H}_{n-1}(\mathrm{link}_\Delta(a); \mathbb{F}) = \tilde{H}_{n-1}(\Gamma; \mathbb{F}),$$

which concludes the proof. $\square$

Running example 2. We have that E_2 is a triple suspension of $\{\emptyset\}$. Specifically, define $\Sigma_0 = \{\emptyset\}$,

$$\begin{aligned} \Sigma_1 &= \mathrm{Susp}_{a^+,a^-}(\Sigma_0) &&= \{\emptyset, a^+, a^-\}, \\ \Sigma_2 &= \mathrm{Susp}_{b^+,b^-}(\Sigma_1) &&= \{\emptyset, a^+, a^-, b^+, b^-, a^+b^+, a^+b^-, a^-b^+, a^-b^-\}, \\ \Sigma_3 &= \mathrm{Susp}_{c^+,c^-}(\Sigma_2). \end{aligned}$$

Then $E_2 = \Sigma_3$. Since $\tilde{H}_n(\Sigma_0; \mathbb{F}) \cong \mathbb{F}$ if $n = -1$ and $\tilde{H}_n(\Sigma_0; \mathbb{F}) \cong 0$ otherwise, Corollary 2.6.5 applied three times yields that $\tilde{H}_n(E_2; \mathbb{F}) \cong \mathbb{F}$ if $n = -1 + 3 = 2$ and $\tilde{H}_n(E_2; \mathbb{F}) \cong 0$ otherwise. Hence we reestablish the result from Section 1.7.2.

Corollary 2.6.5 is also a special case of a general result about joins, which we state for completeness.

Theorem 2.6.6 *Let Δ be a simplicial complex such that all homology groups of Δ are free $\mathbb{F}$-modules. Let Γ be any simplicial complex. Then*

$$\tilde{H}_n(\Delta * \Gamma; \mathbb{F}) \cong \bigoplus_{i+j=n-1} \tilde{H}_i(\Delta; \mathbb{F}) \otimes_{\mathbb{F}} \tilde{H}_j(\Gamma; \mathbb{F}).$$

Here, $A \otimes_{\mathbb{F}} B$ is the tensor product over $\mathbb{F}$ of the two $\mathbb{F}$-modules A and B. In the case that A and B are free $\mathbb{F}$-modules with bases $\{\mathbf{e}_1, \ldots, \mathbf{e}_r\}$ and $\{\mathbf{e}'_1, \ldots, \mathbf{e}'_s\}$, respectively, the tensor product $A \otimes_{\mathbb{F}} B$ is the free $\mathbb{F}$-module of rank rs with basis $\{\mathbf{e}_i \otimes \mathbf{e}'_j : 1 \leq i \leq r, 1 \leq j \leq s\}$. We refer the interested reader to a textbook in algebra for the general definition and to a textbook in algebraic topology for a proof of Theorem 2.6.6.

2.7 Independence complexes

We look at independence complexes, which are simplicial complexes defined in terms of graphs. Such complexes Δ are characterized by the fact that a face σ belongs to Δ if and only if every subset of σ of size two belongs to Δ.

Let $G = (V, E)$ be a simple and loopless graph. Equivalently, G is a 1-dimensional simplicial complex. A vertex set A in a graph G is *independent* (or *stable*) if no two vertices in A form an edge. We define $\mathsf{Ind}(G)$ to be the family of independent sets in G. The family $\mathsf{Ind}(G)$ has the property that if $B \in \mathsf{Ind}(G)$ and $A \subset B$, then $A \in \mathsf{Ind}(G)$. In particular, $\mathsf{Ind}(G)$ is a simplicial complex. We refer to $\mathsf{Ind}(G)$ as the *independence complex* of G.

For a vertex v, we define $N_G(v)$ to be the set of vertices w such that vw belongs to E. Equivalently, $N_G(v)$ is the set of vertices *adjacent* to v. Note that v itself does not belong to $N_G(v)$. Define $N_G[v] = N_G(v) \cup \{v\}$. We write $N(v) = N_G(v)$ and $N[v] = N_G[v]$ when the graph G is clear from context.

For a vertex set U in a graph G, let $G - U$ be the graph obtained from G by removing the vertex set U and all edges with at least one endpoint in U. We write $G - v = G - \{v\}$. For an edge set K, let $G \setminus K$ be the graph obtained from G by removing all edges in K (but not the vertices contained in the edges), and let $G \cup K$ be the graph obtained from G by adding all edges in K. We write $G \setminus e = G \setminus \{e\}$ and $G \cup e = G \cup \{e\}$.

The following observation is sometimes useful.

Proposition 2.7.1 *Let G be a graph, and let a be a vertex in G. Then*

$$\begin{aligned} \mathrm{del}_{\mathsf{Ind}(G)}(a) &= \mathsf{Ind}(G - a), \\ \mathrm{link}_{\mathsf{Ind}(G)}(a) &= \mathsf{Ind}(G - N[a]). \end{aligned}$$

We leave the proof to the reader.

Let K_n denote the complete graph on n vertices; the vertex set of K_n is a set V of size n, and the edge set consists of all subsets of V of size two. For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ defined on disjoint vertex sets V_1 and V_2, the disjoint sum $G_1 + G_2$ is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$.

We are interested in the homology of $\mathsf{Ind}(G)$. For simplicity, we write $\tilde{H}_n[G] = \tilde{H}_n(\mathsf{Ind}(G); \mathbb{F})$. We start with some simple observations.

Proposition 2.7.2 *If $G' = K_1 + G$, then $\mathsf{Ind}(G')$ is the cone over $\mathsf{Ind}(G)$. In particular, $\tilde{H}_n[G'] = 0$ for all n.*

For the second statement, apply Proposition 2.3.3.

Proposition 2.7.3 *If $G' = K_2 + G$, then $\mathsf{Ind}(G')$ is the suspension over $\mathsf{Ind}(G)$. In particular, $\tilde{H}_n[G'] \cong \tilde{H}_{n-1}[G]$ for all n.*

For the second statement, apply Corollary 2.6.5.

The following two propositions have nearly identical proofs. They are special cases of a more general result discussed in Section 3.3; see Section 3.5 for more details.

Proposition 2.7.4 *Let G be a graph, and let a be a vertex in G. Suppose that $\mathsf{Ind}(G - N[a])$ is collapsible. Then $\mathsf{Ind}(G)$ admits a collapse to $\mathsf{Ind}(G - a)$. In particular, $\tilde{H}_n[G] \cong \tilde{H}_n[G - a]$.*

Proof. The members of $\mathsf{Ind}(G) \setminus \mathsf{Ind}(G-a)$ are all $\tau \in \mathsf{Ind}(G-a)$ such that a belongs to τ. Equivalently, $\tau \setminus a$ is an independent set in $G - N[a]$. To summarize,

$$\mathsf{Ind}(G) \setminus \mathsf{Ind}(G-a) = \{a\} * \mathsf{Ind}(G - N[a]).$$

By assumption and Proposition 2.4.2, $\mathsf{Ind}(G - N[a])$ admits a perfect acyclic matching M. We transform M into a perfect matching M' on $\mathsf{Ind}(G)\setminus\mathsf{Ind}(G-a)$ by replacing each pair (σ, τ) in M with the pair $(\sigma \cup a, \tau \cup a)$. Again applying Proposition 2.4.2, we get that $\mathsf{Ind}(G)$ admits a collapse to $\mathsf{Ind}(G-a)$. $\square$

For vertices a and b, write $N[ab] = N[a] \cup N[b]$.

Proposition 2.7.5 *Let G be a graph, and let a and b be nonadjacent vertices in G. Suppose that $\mathsf{Ind}(G - N[ab])$ is collapsible. Then $\mathsf{Ind}(G)$ admits a collapse to $\mathsf{Ind}(G \cup e)$, where e denotes the edge ab. In particular, $\tilde{H}_n[G] \cong \tilde{H}_n[G \cup e]$.*

Proof. The members of $\mathsf{Ind}(G) \setminus \mathsf{Ind}(G \cup e)$ are all $\tau \in \mathsf{Ind}(G)$ such that both a and b belong to τ. Equivalently, $\tau \setminus ab$ is an independent set in $G - N[ab]$. To summarize,

$$\mathsf{Ind}(G) \setminus \mathsf{Ind}(G \cup e) = \{ab\} * \mathsf{Ind}(G - N[ab]).$$

By assumption and Proposition 2.4.2, $\mathsf{Ind}(G - N[ab])$ admits a perfect acyclic matching M. We transform M into a perfect matching M' on $\mathsf{Ind}(G)\setminus\mathsf{Ind}(G\cup e)$ by replacing each pair (σ, τ) in M with the pair $(\sigma \cup ab, \tau \cup ab)$. Again applying Proposition 2.4.2, we get that $\mathsf{Ind}(G)$ admits a collapse to $\mathsf{Ind}(G \cup e)$. $\square$

Corollary 2.7.6 *Let G be a graph, and let a and b be adjacent vertices in G. Suppose that $\mathsf{Ind}(G - N[ab])$ is collapsible. Then $\mathsf{Ind}(G \setminus e)$ admits a collapse to $\mathsf{Ind}(G)$, where e denotes the edge ab. In particular, $\tilde{H}_n[G] \cong \tilde{H}_n[G \setminus e]$.*

2.7.1 Example

Let G' be a graph with the schematic structure illustrated on the left in Figure 2.5. More precisely, there are vertices a, b, r, s, t in G' such that $N(r) = \{a, s\}$, $N(s) = \{r, t\}$, and $N(t) = \{s, b\}$ and such that a and b are nonadjacent.

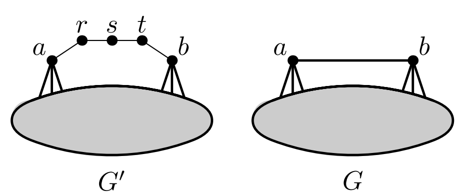

Figure 2.5: Example graphs.

Letting G_1, G_2, G_3, G_4 be the graphs illustrated in Figure 2.6, we note the following:

* The vertex s is isolated in $G' - N[ab]$. In particular, $\mathsf{Ind}(G')$ is collapsible to $\mathsf{Ind}(G_1)$ by Proposition 2.7.5.
* The vertex t is isolated in $G_1 - N[ar]$. In particular, $\mathsf{Ind}(G_2)$ is collapsible to $\mathsf{Ind}(G_1)$ by Proposition 2.7.5.

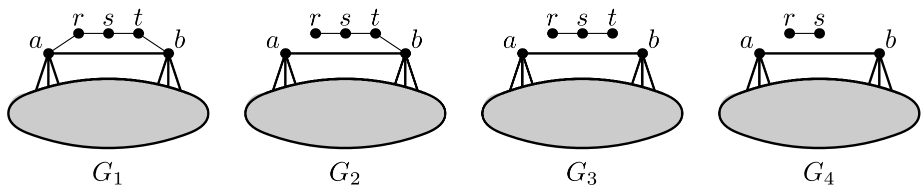

Figure 2.6: More example graphs.

* The vertex r is isolated in $G_2 - N[tb]$. In particular, $\mathsf{Ind}(G_3)$ is collapsible to $\mathsf{Ind}(G_2)$ by Proposition 2.7.5.

* The vertex r is also isolated in $G_3 - N[t]$. In particular, $\mathsf{Ind}(G_3)$ is collapsible to $\mathsf{Ind}(G_4)$ by Proposition 2.7.4.

We conclude that

$$\tilde{H}_n[G'] \cong \tilde{H}_n[G_1] \cong \tilde{H}_n[G_2] \cong \tilde{H}_n[G_3] \cong \tilde{H}_n[G_4]$$

for all n.

Now, G_4 is the disjoint sum of K_2 and G, where G is the graph illustrated on the right in Figure 2.5. In particular, Proposition 2.7.3 yields that

$$\tilde{H}_n[G_4] \cong \tilde{H}_{n-1}[G]$$

for all n. To summarize, we have that

$$\tilde{H}_n[G'] \cong \tilde{H}_{n-1}[G] \tag{2.8}$$

for all n.

For example, let G' be the cycle graph O_k. This is the graph with vertex set $\{1, \dots, k\}$ and edge set $\{12, 23, 34, \dots, (k-1)k, 1k\}$. Assume that $k \geq 6$. Choosing $(a, r, s, t, b) = (1, k, k-1, k-2, k-3)$ and writing $e = ab$, we note that

$$G = (O_k - \{r, s, t\}) \cup e = O_{k-3}.$$

In particular,

$$\tilde{H}_n[O_k] \cong \tilde{H}_{n-1}[O_{k-3}].$$

Examining $\mathsf{Ind}(O_k)$ by hand for $k \leq 5$ and using induction on k, we obtain that $\tilde{H}_n[O_k]$ is zero for all n and k, except that

$$\begin{array}{rcl} \tilde{H}_{k-1}[O_{3m}] & \cong & \mathbb{F}^2, \\ \tilde{H}_k[O_{3m+1}] & \cong & \mathbb{F}, \\ \tilde{H}_k[O_{3m+2}] & \cong & \mathbb{F}, \end{array}$$

for all $m \geq 1$.

Let us give a second method for deriving the identity (2.8). We may divide $\mathsf{Ind}(G')$ into the following families:

* The family $\mathcal{F}_\emptyset$ of graphs not containing a and b;

$$\mathcal{F}_\emptyset = \{\emptyset, r, s, t, rt\} * \mathsf{Ind}(G - \{a, b\}).$$

* The family $\mathcal{F}_a$ of graphs containing a but not b;

$$\mathcal{F}_a = \{a, as, at\} * \mathsf{Ind}(G - N[a] \cup \{b\}).$$

* The family $\mathcal{F}_b$ of graphs containing b but not a;

$$\mathcal{F}_b = \{b, br, bs\} * \mathsf{Ind}(G - \{a\} \cup N[b]).$$

* The family $\mathcal{F}_{ab}$ of graphs containing both a and b;

$$\mathcal{F}_{ab} = \{ab, abs\} * \mathsf{Ind}(G - N[ab]) = \mathsf{Ind}(G') \setminus \mathsf{Ind}(G_1),$$

  where G_1 is the graph on the left in Figure 2.6.

We obtain a collapse from $\mathsf{Ind}(G')$ to $\mathsf{Ind}(G_1)$ by forming the matching

$$M_{ab} = \{(ab, abs)\} * \mathsf{Ind}(G - N[ab]);$$

we use the notation in (2.7). Namely, this is a perfect matching on $\mathcal{F}_{ab}$, and the matching is acyclic by Proposition 2.6.1.

By Corollary 2.4.5, we obtain an acyclic matching on $\mathsf{Ind}(G_1)$ by taking the union of any acyclic matchings on the families $\mathcal{F}_\emptyset$, $\mathcal{F}_a$, and $\mathcal{F}_b$. Let us pick the matchings

$$\begin{aligned} M_\emptyset &= \{(\emptyset, r), (t, rt)\} * \mathsf{Ind}(G - \{a, b\}), \\ M_a &= \{(a, at)\} * \mathsf{Ind}(G - N[a] \cup \{b\}), \\ M_b &= \{(b, br)\} * \mathsf{Ind}(G - \{a\} \cup N[b]). \end{aligned}$$

By Proposition 2.6.1, each of these matchings is acyclic. In particular, the union $M = M_\emptyset \cup M_a \cup M_b$ is acyclic.

Note that M is a perfect matching on $\mathrm{del}_{\mathsf{Ind}(G_1)}(s) = \mathsf{Ind}(G_1 - s)$ and that the family of unmatched faces is

$$\{s\} * \mathrm{link}_{\mathsf{Ind}(G_1)}(s) = \{s\} * \mathsf{Ind}(G_1 - N[s]) = \{s\} * \mathsf{Ind}(G).$$

Since $\tilde{H}_n[G_1 - s] = 0$ for all n, we may apply Proposition 2.6.4 to obtain (2.8).

Chapter 3

Algebraic techniques

3.1 Exact sequences

One of the most important concepts in mathematics is that of an *exact* chain complex. In such a complex

$$\mathsf{C} : \cdots \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots,$$

every cycle is a boundary. Equivalently,

$$Z_n(\mathsf{C}) = B_n(\mathsf{C})$$

for every n. One typically refers to exact chain complexes as *(long) exact sequences*. The word "long" means that the sequence extends indefinitely in both directions; we have a chain group C_n defined for every $n \in \mathbb{Z}$. There is also the concept of a *short exact sequence*. This is a sequence involving only five groups, the very first and the very last being zero;

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{s} C \longrightarrow 0.$$

The sequence being exact means that $i : A \to B$ is injective, $s : B \to C$ is surjective, and $\operatorname{im} i = \ker s$. More generally, we have the following result.

Proposition 3.1.1 *Suppose we have an exact sequence of $\mathbb{F}$-modules*

$$\mathsf{A} : \cdots \xrightarrow{d_{n+2}} A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} \cdots.$$

Then the following hold.

- *If $d_{n+1} = 0$, then $d_n : A_n \to A_{n-1}$ is injective.*
- *If $d_{n-1} = 0$, then $d_n : A_n \to A_{n-1}$ is surjective.*

In particular, $d_n : A_n \to A_{n-1}$ is an isomorphism if d_{n+1} and d_{n-1} are both zero maps.

Proof. The homomorphism d_n is injective if and only if $\ker d_n = 0$, and d_n is surjective if and only if $\operatorname{im} d_n = A_{n-1}$. Now, if $d_{n+1} = 0$, then $0 = \operatorname{im} d_{n+1} = \ker d_n$. Moreover, if $d_{n-1} = 0$, then $A_{n-1} = \ker d_{n-1} = \operatorname{im} d_n$. For the final statement, a homomorphism is an isomorphism if and only if it is injective and surjective. □

There is an entire area of mathematics – *homological algebra* – devoted to the study of exact sequences and their relatives. Since a chain complex C is exact if and only if $H_n(\mathsf{C})$ is zero for every n, one may view the homology of C as a measure on how far C is from being exact.

3.2 Chain maps and chain isomorphisms

Another important concept is that of isomorphic chain complexes. Intuitively, two chain complexes are isomorphic if it is possible to identify the chain groups of the two complexes in a way compatible with the boundary maps.

Let

$$\begin{aligned} \mathsf{X}: \cdots &\xrightarrow{f_{n+2}} X_{n+1} \xrightarrow{f_{n+1}} X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \\ \mathsf{Y}: \cdots &\xrightarrow{g_{n+2}} Y_{n+1} \xrightarrow{g_{n+1}} Y_n \xrightarrow{g_n} Y_{n-1} \xrightarrow{g_{n-1}} \cdots \end{aligned}$$

be two chain complexes. A *chain map* $\varphi : \mathsf{X} \to \mathsf{Y}$ is a sequence $(\varphi_n : n \in \mathbb{Z})$ of $\mathbb{F}$-module homomorphisms $\varphi_n : X_n \to Y_n$ such that

$$\varphi_{n-1} \circ f_n = g_{n-1} \circ \varphi_n$$

for all n. This means that the following diagram *commutes*:

$$\begin{tikzcd} \cdots \arrow[r, "f_{n+2}"] & X_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "\varphi_{n+1}"'] & X_n \arrow[r, "f_n"] \arrow[d, "\varphi_n"'] & X_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "\varphi_{n-1}"'] & \cdots \\ \cdots \arrow[r, "g_{n+2}"] & Y_{n+1} \arrow[r, "g_{n+1}"] & Y_n \arrow[r, "g_n"] & Y_{n-1} \arrow[r, "g_{n-1}"] & \cdots \end{tikzcd}$$

Equivalently, the result when going from X_n to Y_{n-1} does not depend on whether we go first right and then down or first down and then right. Note that φ_n induces a homomorphism

$$\varphi_n^* : H_n(\mathsf{X}) \to H_n(\mathsf{Y}).$$

Namely, if $x \in \ker f_n$, then

$$0 = \varphi_{n-1} f_n(x) = g_n \varphi_n(x) \Longrightarrow \varphi_n(x) \in \ker g_n.$$

Moreover, if $x \in \operatorname{im} f_{n+1}$, meaning that $x = f_{n+1}(x')$ for some $x' \in X_{n+1}$, then

$$\varphi_n(x) = \varphi_n f_{n+1}(x') = g_{n+1}(\varphi_{n+1}(x')) \Longrightarrow \varphi_n(x) \in \operatorname{im} g_{n+1}.$$

We say that a chain map φ is a *chain isomorphism* if every φ_n is an isomorphism.

Proposition 3.2.1 *With notation as above, suppose that φ is a chain isomorphism, and let $n \in \mathbb{Z}$. Then φ_n induces an isomorphism*

$$\varphi_n^* : H_n(\mathsf{X}) \to H_n(\mathsf{Y})$$

for each n.

Proof. We have that

$$\varphi_n(\ker f_n) = \varphi_n(\ker \varphi_{n-1} f_n) = \varphi_n(\ker g_n \varphi_n) = \ker g_n,$$

because φ_{n-1} is injective and φ_n is surjective. As a consequence, φ_n^* is surjective; for every $y \in \ker g_n$, there is an $x \in \ker f_n$ such that $\varphi_n^*(x + \operatorname{im} f_{n+1}) = y + \operatorname{im} g_{n+1}$. Moreover,

$$\varphi_n(\operatorname{im} f_{n+1}) = g_{n+1}(\varphi_{n+1}(X_{n+1})) = g_{n+1}(Y_{n+1}) = \operatorname{im} g_{n+1},$$

because φ_{n+1} is surjective. In particular, $x \in \operatorname{im} f_n$ if and only if $\varphi_n(x) \in \operatorname{im} g_n$, because φ_n is injective. We deduce that φ_n^* is injective; $\varphi_n^*(x + \operatorname{im} f_{n+1})$ is zero if and only $x \in \operatorname{im} f_{n+1}$. $\square$

3.3 Relative homology

Let C be a chain complex, and let $\mathsf{C}^{(0)}$ be a subcomplex of C. We define the *relative* chain complex

$$\mathsf{Q} : \cdots \xrightarrow{\hat{d}_{n+2}} Q_{n+1} \xrightarrow{\hat{d}_{n+1}} Q_n \xrightarrow{\hat{d}_n} Q_{n-1} \xrightarrow{\hat{d}_{n-1}} \cdots$$

of the pair $(\mathsf{C}, \mathsf{C}^{(0)})$ in the following manner. The nth chain group of Q is defined to be the quotient

$$Q_n = \frac{C_n}{C_n^{(0)}}.$$

The boundary map $\hat{d}_n : Q_n \to Q_{n-1}$ is the one induced by $d_n : C_n \to C_{n-1}$. Specifically,

$$\hat{d}_n(x + C_n^{(0)}) = d_n(x) + C_{n-1}^{(0)}.$$

This is well-defined, because $d_n(C_n^{(0)}) \subseteq C_{n-1}^{(0)}$. We let $H_n(\mathsf{C}, \mathsf{C}^{(0)})$ denote the homology in degree n of Q and refer to it as the *relative homology* of the pair $(\mathsf{C}, \mathsf{C}^{(0)})$.

For a simplicial complex Δ and a subcomplex Δ_0, let $\mathsf{C}(\Delta, \Delta_0; \mathbb{F})$ denote the relative chain complex of the pair $(\mathsf{C}(\Delta; \mathbb{F}), \mathsf{C}(\Delta_0; \mathbb{F}))$. For convenience, we write

$$\tilde{H}_n(\Delta, \Delta_0; \mathbb{F}) = H_n(\mathsf{C}(\Delta; \mathbb{F}), \mathsf{C}(\Delta_0; \mathbb{F})).$$

We refer to this as the relative homology of the pair (Δ, Δ_0).

The situation is particularly nice for simplicial complexes. Specifically, we may identify the quotient $C_n(\Delta; \mathbb{F})/C_n(\Delta_0; \mathbb{F})$ with the submodule of $C_n(\Delta; \mathbb{F})$ generated by all oriented simplices corresponding to faces of dimension n in $\Delta \setminus \Delta_0$. We obtain the boundary $\hat{\partial}_n(c)$ by computing the ordinary boundary $\partial_n(c)$ and removing all oriented simplices corresponding to faces of Δ_0.

Running example 1. Let 2^V be the full simplex on the vertex set $V = \{a, b, c, d\}$. Then $2^V \setminus E_1$ is the family $\{ad, abd, acd, bcd, abcd\}$. In particular,

$$\begin{aligned}
\tilde{C}_3(2^V, E_1; \mathbb{F}) &\cong \langle abcd \rangle \cong \mathbb{F}, \\
\tilde{C}_2(2^V, E_1; \mathbb{F}) &\cong \langle abd, acd, bcd \rangle \cong \mathbb{F}^3, \\
\tilde{C}_1(2^V, E_1; \mathbb{F}) &\cong \langle ad \rangle \cong \mathbb{F}.
\end{aligned}$$

For $n \notin \{1,2,3\}$, we have that $C_n(2^V, E_1; \mathbb{F}) = 0$. The boundary maps are defined by

$$\hat{\partial}_3(abcd) = bcd - acd + abd, \quad \begin{array}{rcl} \hat{\partial}_2(abd) & = & -ad, \\ \hat{\partial}_2(acd) & = & -ad, \\ \hat{\partial}_2(bcd) & = & 0, \end{array} \quad \hat{\partial}_1(ad) = 0.$$

For example, abc does not appear in the boundary of $abcd$, because $abc \in E_1$.

Formally, write $T = \Delta \backslash \Delta_0$, and let P_n be the submodule of $\tilde{C}_n(\Delta)$ generated by oriented simplices in T of dimension n. We may write $\tilde{C}_n(\Delta) = \tilde{C}_n(\Delta_0) \oplus P_n$. In particular,

$$\frac{\tilde{C}_n(\Delta)}{\tilde{C}_n(\Delta_0)} = \frac{\tilde{C}_n(\Delta_0) \oplus P_n}{\tilde{C}_n(\Delta_0) \oplus 0} \cong \frac{\tilde{C}_n(\Delta_0)}{\tilde{C}_n(\Delta_0)} \oplus \frac{P_n}{0} \cong P_n;$$

here, we use Proposition 0.2.1. The reader may check that an isomorphism $\varphi_n : P_n \to \tilde{C}_n(\Delta)/\tilde{C}_n(\Delta_0)$ is given by

$$\varphi_n(p) = p + \tilde{C}_n(\Delta_0).$$

Define $\hat{\partial}_n : P_n \to P_{n-1}$ by $\hat{\partial}_n(p) = p_0$, where p_0 is the unique element in P_{n-1} such that $p_0 - \partial_n(p) \in \tilde{C}_{n-1}(\Delta_0)$. Then

$$\begin{array}{rcl} \varphi_{n-1}\hat{\partial}_n(p) & = & p_0 + \tilde{C}_{n-1}(\Delta_0) \\ & = & \partial_n(p) + (p_0 - \partial_n(p)) + \tilde{C}_{n-1}(\Delta_0) \\ & = & \partial_n(p) + \tilde{C}_{n-1}(\Delta_0) = \hat{\partial}_n(p + \tilde{C}_n(\Delta_0)) = \hat{\partial}_n\varphi_n(p). \end{array}$$

In particular, $\mathsf{C}(\Delta, \Delta_0; \mathbb{F})$ is isomorphic to the chain complex

$$\mathsf{P}(\Delta, \Delta_0) : \cdots \xrightarrow{\hat{\partial}_{n+2}} P_{n+1} \xrightarrow{\hat{\partial}_{n+1}} P_n \xrightarrow{\hat{\partial}_n} P_{n-1} \xrightarrow{\hat{\partial}_{n-1}} \cdots$$

It is clear from the above procedure that the chain complex $\mathsf{P}(\Delta, \Delta_0)$ only depends on $\Delta \setminus \Delta_0$, not on the particular choice of Δ and Δ_0. Specifically, if $\Delta \backslash \Delta_0 = \Gamma \backslash \Gamma_0$, then $\mathsf{P}(\Delta, \Delta_0) = \mathsf{P}(\Gamma, \Gamma_0)$. To summarize, we have the following result.

Proposition 3.3.1 *Let Γ and Δ be simplicial complexes, let Γ_0 be a subcomplex of Γ, and let Δ_0 be a subcomplex of Δ. If $\Gamma \setminus \Gamma_0 = \Delta \setminus \Delta_0$, then $\mathsf{C}(\Gamma, \Gamma_0; \mathbb{F})$ and $\mathsf{C}(\Delta, \Delta_0; \mathbb{F})$ are isomorphic. In particular,*

$$\tilde{H}_n(\Gamma, \Gamma_0; \mathbb{F}) \cong \tilde{H}_n(\Delta, \Delta_0; \mathbb{F})$$

for all n.

For the final statement, apply Proposition 3.2.1.

Return to the general case. In what follows, a crucial observation is that there exists a map

$$d_n^* : H_n(\mathsf{C}, \mathsf{C}^{(0)}) \to H_n(\mathsf{C}^{(0)}).$$

This map sends the homology class of an element $x + C_n^{(0)} \in Z_n(\mathsf{C}, \mathsf{C}^{(0)})$ to the homology class of the element $d_n(x) \in Z_{n-1}(\mathsf{C}^{(0)})$. This is well-defined, because if $c \in C_n^{(0)}$, then

$$d_n(x + c) - d_n(x) = d_n(c) \in B_{n-1}(\mathsf{C}^{(0)});$$

hence $d_n(x)$ and $d_n(x+c)$ belong to the same homology class.

Note that elements in $\operatorname{im} d_n^*$ are not necessarily zero. Specifically, an element $y \in C_{n-1}^{(0)}$ might be a boundary of some element in C_n without being a boundary of any element in $C_n^{(0)}$.

The following theorem is one of the most important results in homological algebra. The long exact sequence in the theorem is known as the *long exact sequence for the pair* $(\mathsf{C}, \mathsf{C}^{(0)})$.

Theorem 3.3.2 *For any chain complex* C *and any subcomplex* $\mathsf{C}^{(0)}$ *of* C*, we have the following long exact sequence.*

$$\begin{array}{ccccccc}
 & \cdots & \xrightarrow{i_{n+1}^*} & H_{n+1}(\mathsf{C}) & \xrightarrow{s_{n+1}^*} & H_{n+1}(\mathsf{C}, \mathsf{C}^{(0)}) & \xrightarrow{d_{n+1}^*} \\
\xrightarrow{d_{n+1}^*} & H_n(\mathsf{C}^{(0)}) & \xrightarrow{i_n^*} & H_n(\mathsf{C}) & \xrightarrow{s_n^*} & H_n(\mathsf{C}, \mathsf{C}^{(0)}) & \xrightarrow{d_n^*} \\
\xrightarrow{d_n^*} & H_{n-1}(\mathsf{C}^{(0)}) & \xrightarrow{i_{n-1}^*} & H_{n-1}(\mathsf{C}) & \xrightarrow{s_{n-1}^*} & \cdots &
\end{array}$$

The map i_n^* *is induced by the inclusion map* $i_n : C_n^{(0)} \to C_n$*, the map* s_n^* *is induced by the projection map* $s_n : C_n \to C_n/C_n^{(0)}$*, and the map* d_n^* *is defined above.*

Proof. Write $\mathsf{C}^{(1)} = \mathsf{C}$. For $x \in Z_n(\mathsf{C}^{(i)})$ and $i \in \{0, 1\}$, let $[x]_i$ denote the homology class of x in $H_n(\mathsf{C}^{(i)})$. For $x + C_n^{(0)} \in Z_n(\mathsf{C}^{(1)}, \mathsf{C}^{(0)})$, let $[x]_2$ denote the homology class of $x + C_n^{(0)}$ in $H_n(\mathsf{C}^{(1)}, \mathsf{C}^{(0)})$.

First, we prove that $\operatorname{im} i_n^* = \ker s_n^*$. We note that $[x]_1$ belongs to $\operatorname{im} i_n^*$ if and only if there is an element $x' \in Z_n(\mathsf{C}^{(0)})$ such that $x - x' \in B_n(\mathsf{C}^{(1)})$. As a consequence, $s_n^* i_n^*$ is zero, because s_n^* maps $[x]_1$ to

$$[x]_2 = [x - x']_2 + [x']_2 = [0]_2.$$

Conversely, if $[x]_1 \in \ker s_n^*$, then $x - d(y) \in C_n^{(0)}$ for some $y \in C_{n+1}^{(1)}$. Writing $x' = x - d(y)$, we note that $x' \in Z_n(\mathsf{C}^{(0)})$ and $x - x' \in B_n(\mathsf{C}^{(1)})$; hence $[x]_1 = [x']_1 \in \operatorname{im} i_n^*$.

Second, we prove that $\operatorname{im} s_n^* = \ker d_n^*$. We note that $[x]_2$ belongs to $\operatorname{im} s_n^*$ if and only if there is an element $x' \in Z_n(\mathsf{C}^{(1)})$ such that $x - x' \in \mathsf{C}^{(0)}$. As a consequence, $d_n^* s_n^*$ is zero, because d_n^* maps $[x]_2$ to

$$[d_n(x)]_0 = [d_n(x - x')]_0 + [d_n(x')]_0 = [0]_0.$$

Conversely, if $[x]_2 \in \ker d_n^*$, then $d_n(x) = d_n(x^{(0)})$ for some $x^{(0)} \in C_n^{(0)}$. Writing $x' = x - x^{(0)}$, we note that $x' \in Z_n(\mathsf{C}^{(1)})$ and $x - x' \in \mathsf{C}^{(0)}$; hence $[x]_2 = [x']_2 \in \operatorname{im} s_n^*$.

Third, we prove that $\operatorname{im} d_n^* = \ker i_{n-1}^*$. We note that $[x]_0$ belongs to $\operatorname{im} d_n^*$ if and only if there is an element $y \in C_n(\mathsf{C}^{(1)})$ such that $x - d_n(y) \in B_{n-1}(\mathsf{C}^{(0)})$. As a consequence, $i_{n-1}^* d_n^*$ is zero, because i_{n-1}^* maps $[x]_0$ to

$$[x]_1 = [x - d_n(y)]_1 + [d_n(y)]_1 = [0]_1.$$

Conversely, if $[x]_0 \in \ker i_n^*$, then $x = d_n(y)$ for some $y \in C_n^{(1)}$; hence $[x]_2 \in \operatorname{im} s_n^*$.
□

Corollary 3.3.3 *The following hold for any integer n.*

(i) *If $H_n(\mathsf{C}^{(0)}) = H_{n-1}(\mathsf{C}^{(0)}) = 0$, then $H_n(\mathsf{C}) \cong H_n(\mathsf{C}, \mathsf{C}^{(0)})$.*

(ii) *If $H_n(\mathsf{C}) = H_{n-1}(\mathsf{C}) = 0$, then $H_{n-1}(\mathsf{C}^{(0)}) \cong H_n(\mathsf{C}, \mathsf{C}^{(0)})$.*

(iii) *If $H_{n+1}(\mathsf{C}, \mathsf{C}^{(0)}) = H_n(\mathsf{C}, \mathsf{C}^{(0)}) = 0$, then $H_n(\mathsf{C}) \cong H_n(\mathsf{C}^{(0)})$.*

Proof.

(i) Suppose that $H_n(\mathsf{C}^{(0)}) = H_{n-1}(\mathsf{C}^{(0)}) = 0$. Then we get the exact sequence

$$0 \xrightarrow{i_n^*} H_n(\mathsf{C}) \xrightarrow{s_n^*} H_n(\mathsf{C}, \mathsf{C}^{(0)}) \xrightarrow{d_n^*} 0.$$

In particular, $H_n(\mathsf{C}) \cong H_n(\mathsf{C}, \mathsf{C}^{(0)})$.

(ii) Suppose that $H_n(\mathsf{C}) = H_{n-1}(\mathsf{C}) = 0$. Then we get the exact sequence

$$0 \xrightarrow{s_n^*} H_n(\mathsf{C}, \mathsf{C}^{(0)}) \xrightarrow{d_n^*} H_{n-1}(\mathsf{C}^{(0)}) \xrightarrow{i_{n-1}^*} 0.$$

In particular, $H_{n-1}(\mathsf{C}^{(0)}) \cong H_n(\mathsf{C}, \mathsf{C}^{(0)})$.

(iii) Suppose that $H_{n+1}(\mathsf{C}, \mathsf{C}^{(0)}) = H_n(\mathsf{C}, \mathsf{C}^{(0)}) = 0$. Then we get the exact sequence

$$0 \xrightarrow{d_{n+1}^*} H_n(\mathsf{C}^{(0)}) \xrightarrow{i_n^*} H_n(\mathsf{C}) \xrightarrow{s_n^*} 0.$$

In particular, $H_n(\mathsf{C}) \cong H_n(\mathsf{C}^{(0)})$.

□

Corollary 3.3.4 *For any simplicial complex Δ and any subcomplex Δ_0 of Δ, we have the following long exact sequence.*

$$\begin{array}{ccccccc}
 & \cdots & \xrightarrow{i_{n+1}^*} & \tilde{H}_{n+1}(\Delta) & \xrightarrow{s_{n+1}^*} & \tilde{H}_{n+1}(\Delta, \Delta_0) & \xrightarrow{\partial_{n+1}^*} \\
\xrightarrow{\partial_{n+1}^*} & \tilde{H}_n(\Delta_0) & \xrightarrow{i_n^*} & \tilde{H}_n(\Delta) & \xrightarrow{s_n^*} & \tilde{H}_n(\Delta, \Delta_0) & \xrightarrow{\partial_n^*} \\
\xrightarrow{\partial_n^*} & \tilde{H}_{n-1}(\Delta_0) & \xrightarrow{i_{n-1}^*} & \tilde{H}_{n-1}(\Delta) & \xrightarrow{s_{n-1}^*} & \cdots &
\end{array}$$

The map i_n^* is induced by the inclusion map $i_n : \tilde{C}_n(\Delta_0) \to \tilde{C}_n(\Delta)$, the map s_n^* is induced by the projection map $s_n : \tilde{C}_n(\Delta) \to \tilde{C}_n(\Delta)/\tilde{C}_n(\Delta_0)$, and the map ∂_n^* is defined in the same manner as d_n^* in Theorem 3.3.2.

Corollary 3.3.5 *The following hold for any integer n.*

(i) *If $\tilde{H}_n(\Delta_0; \mathbb{F}) = \tilde{H}_{n-1}(\Delta_0; \mathbb{F}) = 0$, then $\tilde{H}_n(\Delta; \mathbb{F}) \cong \tilde{H}_n(\Delta, \Delta_0; \mathbb{F})$.*

(ii) *If $\tilde{H}_n(\Delta; \mathbb{F}) = \tilde{H}_{n-1}(\Delta; \mathbb{F}) = 0$, then $\tilde{H}_{n-1}(\Delta_0; \mathbb{F}) \cong \tilde{H}_n(\Delta, \Delta_0; \mathbb{F})$.*

(iii) *If $\tilde{H}_{n+1}(\Delta, \Delta_0; \mathbb{F}) = \tilde{H}_n(\Delta, \Delta_0; \mathbb{F}) = 0$, then $\tilde{H}_n(\Delta; \mathbb{F}) \cong \tilde{H}_n(\Delta_0; \mathbb{F})$.*

Let 2^V be a k-simplex; $k = |V| - 1$. We obtain the *boundary of a k-simplex* by removing the maximal face V from 2^V.

Corollary 3.3.6 *If Σ is the boundary of a k-simplex, then*

$$\tilde{H}_n(\Sigma;\mathbb{F}) \cong \begin{cases} \mathbb{F} & \text{if } n = k-1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let Δ be the full k-simplex 2^V; we have that V has size $k+1$. Then $\Delta \setminus \Sigma$ consists of one single simplex of dimension k. As a consequence,

$$\tilde{H}_n(\Delta,\Sigma;\mathbb{F}) \cong \begin{cases} \mathbb{F} & \text{if } n = k, \\ 0 & \text{otherwise.} \end{cases}$$

Now, $\tilde{H}_n(\Delta;\mathbb{F}) = 0$ for all n by Corollary 2.3.4. As a consequence, $\tilde{H}_{n-1}(\Sigma;\mathbb{F}) \cong \tilde{H}_n(\Delta,\Sigma;\mathbb{F})$ by Corollary 3.3.5 (ii), which yields the desired result. ☐

For any simplicial complex Δ_1 and any subcomplex Δ_0, it is easy to construct a simplicial complex Δ_2 with the same homology as the pair (Δ_1,Δ_0).

Corollary 3.3.7 *Let Δ_1 be a simplicial complex, and let Δ_0 be a subcomplex of Δ_1. Let a be a vertex not in Δ_1. Define*

$$\Delta_2 = \mathrm{Cone}_a(\Delta_0) \cup \Delta;$$

this is the simplicial complex obtained from Δ_1 by adding $\sigma \cup a$ for each $\sigma \in \Delta_0$. Then

$$\tilde{H}_n(\Delta_2;\mathbb{F}) \cong \tilde{H}_n(\Delta_1,\Delta_0;\mathbb{F}).$$

Proof. By Proposition 2.3.3, we have that $\mathrm{Cone}_a(\Delta_0)$ is collapsible. As a consequence,

$$\tilde{H}_n(\Delta_2;\mathbb{F}) \cong \tilde{H}_n(\Delta_2,\mathrm{Cone}_a(\Delta_0);\mathbb{F}).$$

Yet, $\Delta_2 \setminus \mathrm{Cone}_a(\Delta_0) = \Delta_1 \setminus \Delta_0$, which implies that

$$\tilde{H}_n(\Delta_2,\mathrm{Cone}_a(\Delta_0)) \cong \tilde{H}_n(\Delta_1,\Delta_0).$$

As a consequence, we obtain the desired result. ☐

Note that $\Delta_1 = \mathrm{del}_{\Delta_2}(a)$ and $\Delta_0 = \mathrm{link}_{\Delta_2}(a)$. In the case that $\tilde{H}_n(\Delta_1) = 0$ for all n, we recover Proposition 2.6.4. Namely,

$$\tilde{H}_n(\Delta_2) \cong \tilde{H}_n(\Delta_1,\Delta_0) \cong \tilde{H}_{n-1}(\Delta_0) = \tilde{H}_{n-1}(\mathrm{link}_{\Delta_2}(a))$$

by Corollary 3.3.7 and Corollary 3.3.5 (ii).

3.4 Mayer-Vietoris sequences

In Section 3.3, we related the homology of a complex C to the homology of a subcomplex $\mathsf{C}^{(0)}$ via a long exact sequence involving the relative homology of the pair $(\mathsf{C},\mathsf{C}^{(0)})$. In this section, we discuss another long exact sequence relating the homology of two subcomplexes of the same complex to the homology of their sum and intersection.

Let

$$\mathsf{X} : \cdots \xrightarrow{d_{n+2}} X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} \cdots$$

$$\mathsf{Y} : \cdots \xrightarrow{d_{n+2}} Y_{n+1} \xrightarrow{d_{n+1}} Y_n \xrightarrow{d_n} Y_{n-1} \xrightarrow{d_{n-1}} \cdots$$

be subcomplexes of the same complex C. Let $\mathsf{X}+\mathsf{Y}$ be the subcomplex of C with chain groups X_n+Y_n. Moreover, let $\mathsf{X}\cap\mathsf{Y}$ be the subcomplex with chain groups $X_n \cap Y_n$. One easily checks that these are indeed chain complexes. The long exact sequence in the following theorem is known as a *Mayer-Vietoris sequence.*

Theorem 3.4.1 *With notation as above, we have a long exact sequence*

$$\begin{array}{ccccccc} & \cdots & \xrightarrow{f^*_{n+1}} & H_{n+1}(\mathsf{X}\oplus\mathsf{Y}) & \xrightarrow{g^*_{n+1}} & H_{n+1}(\mathsf{X}+\mathsf{Y}) & \xrightarrow{h^*_{n+1}} \\ \xrightarrow{h^*_{n+1}} & H_n(\mathsf{X}\cap\mathsf{Y}) & \xrightarrow{f^*_n} & H_n(\mathsf{X}\oplus\mathsf{Y}) & \xrightarrow{g^*_n} & H_n(\mathsf{X}+\mathsf{Y}) & \xrightarrow{h^*_n} \\ \xrightarrow{h^*_n} & H_{n-1}(\mathsf{X}\cap\mathsf{Y}) & \xrightarrow{f^*_{n-1}} & H_{n-1}(\mathsf{X}\oplus\mathsf{Y}) & \xrightarrow{g^*_{n-1}} & \cdots & \end{array}$$

The map f^*_n is induced by the map $f_n : C_n(\mathsf{X}\cap\mathsf{Y}) \to C_n(\mathsf{X}\oplus\mathsf{Y})$ given by $f_n(s) = (s,-s)$. The map g^*_n is induced by the map $g_n : C_n(\mathsf{X}\oplus\mathsf{Y}) \to C_n(\mathsf{X}+\mathsf{Y})$ given by $g_n(x,y) = x+y$. The map h^*_n is given by mapping the homology class of z to that of $d_n(x)$, where x is any element in X_n such that $z-x \in Y_n$.

Remark. Recall from Theorem 2.2.2 that

$$H_n(\mathsf{X}\oplus\mathsf{Y}) \cong H_n(\mathsf{X}) \oplus H_n(\mathsf{Y}).$$

Proof. We may identify $\mathsf{X}\cap\mathsf{Y}$ with a certain subcomplex of $\mathsf{X}\oplus\mathsf{Y}$. Namely, let W be the subcomplex with chain groups

$$W_n = \{(s,-s) : s \in X_n \cap Y_n\}.$$

We leave it to the reader to check that we obtain a chain isomorphism $\varphi : \mathsf{X}\cap\mathsf{Y} \to \mathsf{W}$ defined by $\varphi_n(s) = (s,-s)$ for all $s \in X_n \cap Y_n$. Theorem 3.3.2 yields a long exact sequence

$$\begin{array}{ccccccc} & \cdots & \xrightarrow{i^*_{n+1}} & H_{n+1}(\mathsf{X}\oplus\mathsf{Y}) & \xrightarrow{s^*_{n+1}} & H_{n+1}(\mathsf{X}\oplus\mathsf{Y},\mathsf{W}) & \xrightarrow{d^*_{n+1}} \\ \xrightarrow{d^*_{n+1}} & H_{n+1}(\mathsf{W}) & \xrightarrow{i^*_n} & H_n(\mathsf{X}\oplus\mathsf{Y}) & \xrightarrow{s^*_n} & H_n(\mathsf{X}\oplus\mathsf{Y},\mathsf{W}) & \xrightarrow{d^*_n} \\ \xrightarrow{d^*_n} & H_{n-1}(\mathsf{W}) & \xrightarrow{i^*_{n-1}} & H_{n-1}(\mathsf{X}\oplus\mathsf{Y}) & \xrightarrow{s^*_{n-1}} & \cdots & \end{array}$$

Now, let Q denote the relative chain complex of the pair $(\mathsf{X}\oplus\mathsf{Y},\mathsf{W})$. We claim that we obtain a chain isomorphism

$$\psi : \mathsf{Q} \to \mathsf{X}+\mathsf{Y}$$

by

$$\psi_n((x,y)+W_n) = x+y$$

for any $x \in X$ and $y \in Y$. The map ψ_n is well-defined, because if $(x',y') - (x,y) \in W_n$, then $x'-x+y'-y = 0$, which yields that

$$x'+y' = x+y+(x'-x+y'-y) = x+y.$$

Moreover, ψ_n is injective, because $\psi_n((x,y)+W_n) = 0$ if and only if $y = -x$, which is equivalent to saying that $(x,y) \in W_n$. Finally, ψ_n is surjective, because any element in X_n+Y_n is of the form $x+y = \psi_n((x,y)+W_n)$. As a consequence, we get a long exact sequence as in the theorem, and it only remains to show that f_n^*, g_n^*, and h_n^* are as specified there.

Now, f_n^* is induced by $i_n\varphi_n$, and

$$i_n\varphi_n(s) = i_n(s,-s) = (s,-s) = f_n(s).$$

Moreover, g_n^* is induced by $\psi_n s_n$, and

$$\psi_n s_n(x,y) = \psi_n((x,y)+W_n) = x+y = g_n(x,y).$$

Finally, h_n^* maps the homology class of an element $x+y = \psi_n((x,y)+W_n) \in \ker d_n$ to the homology class of

$$\varphi_n^{-1}(d_n(x), d_n(y)) = \varphi_n^{-1}(d_n(x), -d_n(x)) = d_n(x).$$

This concludes the proof. $\square$

Let Δ and Γ be simplicial complexes. We note that

$$\begin{aligned} \mathsf{C}(\Delta;\mathbb{F}) \cap \mathsf{C}(\Gamma;\mathbb{F}) &= \mathsf{C}(\Delta\cap\Gamma;\mathbb{F}), \\ \mathsf{C}(\Delta;\mathbb{F}) + \mathsf{C}(\Gamma;\mathbb{F}) &= \mathsf{C}(\Delta\cup\Gamma;\mathbb{F}). \end{aligned}$$

In particular, we have the following result.

Corollary 3.4.2 *With notation as above, we have a long exact sequence*

$$\begin{array}{ccccccc} & \cdots & \xrightarrow{f_{n+1}^*} & \tilde{H}_{n+1}(\Delta)\oplus\tilde{H}_{n+1}(\Gamma) & \xrightarrow{g_{n+1}^*} & \tilde{H}_{n+1}(\Delta\cup\Gamma) \\ \xrightarrow{h_{n+1}^*} & \tilde{H}_n(\Delta\cap\Gamma) & \xrightarrow{f_n^*} & \tilde{H}_n(\Delta)\oplus\tilde{H}_n(\Gamma) & \xrightarrow{g_n^*} & \tilde{H}_n(\Delta\cup\Gamma) \\ \xrightarrow{h_n^*} & \tilde{H}_{n-1}(\Delta\cap\Gamma) & \xrightarrow{f_{n-1}^*} & \tilde{H}_{n-1}(\Delta)\cup\tilde{H}_{n-1}(\Gamma) & \xrightarrow{g_{n-1}^*} & \cdots \end{array}$$

The maps f_n^*, g_n^*, and h_n^* are defined as in Theorem 3.4.1.

3.5 Independence complexes revisited

Recall definitions and concepts from Section 2.7. As before, we write $\tilde{H}_n[G] = \tilde{H}_n(\mathsf{Ind}(G);\mathbb{F})$, and we also write $\tilde{C}_n[G] = \tilde{C}_n(\mathsf{Ind}(G);\mathbb{F})$. One may view Theorems 3.5.1 and 3.5.4 as algebraic generalizations of Propositions 2.7.4 and 2.7.5.

Theorem 3.5.1 *Let G be a graph, and let a be a vertex in G. Then we have the following long exact sequence.*

$$\begin{array}{ccccccc} & \cdots & \xrightarrow{i_{n+1}^*} & \tilde{H}_{n+1}[G] & \xrightarrow{t_{n+1}^*} & \tilde{H}_n[G-N[a]] & \xrightarrow{j_n^*} \\ \xrightarrow{j_n^*} & \tilde{H}_n[G-a] & \xrightarrow{i_n^*} & \tilde{H}_n[G] & \xrightarrow{t_n^*} & \tilde{H}_{n-1}[G-N[a]] & \xrightarrow{j_{n-1}^*} \\ \xrightarrow{j_n^*} & \tilde{H}_{n-1}[G-a] & \xrightarrow{i_{n-1}^*} & \tilde{H}_{n-1}[G] & \xrightarrow{t_{n-1}^*} & \cdots & \end{array}$$

The maps i_n^* and j_n^* are induced by the inclusion maps $i_n : \tilde{C}_n[G-a] \to \tilde{C}_n[G]$ and $j_n : \tilde{C}_n[G-N[a]] \to \tilde{C}_n[G-a]$. The map t_n^* is induced by the map $t_n : \tilde{C}_n[G] \to \tilde{C}_{n-1}[G-N[a]]$ given by $t_n(x + a\wedge y) = y$ for each $x \in \tilde{C}_n[G-a]$ and $y \in \tilde{C}_{n-1}[G-N[a]]$.

Proof. Consider the long exact sequence for the pair $(\mathsf{Ind}(G), \mathsf{Ind}(G-a))$. As concluded in the proof of Proposition 2.7.4,

$$\mathsf{Ind}(G) \setminus \mathsf{Ind}(G-a) = \{a\} * \mathsf{Ind}(G - N[a]).$$

Now, this is equal to

$$\mathrm{Cone}_a(\mathsf{Ind}(G-N[a])) \setminus \mathsf{Ind}(G-N[a]).$$

As a consequence, Proposition 3.3.1 and Corollary 2.6.5 yield that

$$\begin{aligned} \tilde{H}_n(\mathsf{Ind}(G), \mathsf{Ind}(G-a)) &\cong \tilde{H}_n(\mathrm{Cone}_a(\mathsf{Ind}(G-N[a])), \mathsf{Ind}(G-N[a])) \\ &\cong \tilde{H}_{n-1}(\mathsf{Ind}(G-N[a])) \\ &= \tilde{H}_{n-1}[G-N[a]]. \end{aligned}$$

By the long exact sequence for the pair $(\mathrm{Cone}_a(\mathsf{Ind}(G-N[a])), \mathsf{Ind}(G-N[a]))$, we have an isomorphism φ_n from $\tilde{H}_n(\mathsf{Ind}(G), \mathsf{Ind}(G-a))$ to $\tilde{H}_{n-1}[G-N[a]]$ given by mapping the homology class of $a \wedge y + \tilde{C}_n[G-a]$ to the homology class of $\partial_n(a \wedge y) = y$. As a consequence, the map $t_n^* = \varphi_n s_n^*$ sends the homology class of $x + a \wedge y$ to the homology class of y, and the map $j_n^* = \partial_{n+1}^* \varphi_n^{-1}$ sends the homology class of y to the homology class of $\partial_{n+1}(a \wedge y) = y$. □

Corollary 3.5.2 *Let G be a graph, and let a be a vertex in G.*

(i) *If $\tilde{H}_n[G-a] = \tilde{H}_{n-1}[G-a] = 0$, then $\tilde{H}_n[G] \cong \tilde{H}_{n-1}[G-N[a]]$.*

(ii) *If $\tilde{H}_{n+1}[G] = \tilde{H}_n[G] = 0$, then $\tilde{H}_n[G-a] \cong \tilde{H}_n[G-N[a]]$.*

(iii) *If $\tilde{H}_n[G-N[a]] = \tilde{H}_{n-1}[G-N[a]] = 0$, then $\tilde{H}_n[G] \cong \tilde{H}_n[G-a]$.*

Corollary 3.5.3 *Let G be a graph, and assume that a and b are vertices such that a is the only neighbor of b. Then*

$$\tilde{H}_n[G] \cong \tilde{H}_{n-1}[G-N[a]]$$

for all n.

Proof. We have that b is isolated in $G-a$, which implies that $\tilde{H}_n[G-a] = 0$ for all n. Applying Corollary 3.5.2 (i), we obtain the desired result. □

Example. Let P_k be the path of vertex length k. This is the graph with vertex set $\{1, \dots, k\}$ and edge set $\{12, 23, 34, \dots, (k-1)k\}$. Let $k \geq 3$, and choose $a = k-1$ and $b = k$. Then the conditions of Corollary 3.5.3 are satisfied, which yields that

$$\tilde{H}_n[P_k] \cong \tilde{H}_{n-1}[P_k - N[a]] = \tilde{H}_{n-1}[P_{k-3}].$$

Observing that $P_0 = \{\emptyset\}$, $P_1 = \{\emptyset, 1\}$, and $P_2 = \{\emptyset, 1, 2\}$, a simple induction argument yields that $\tilde{H}_n[P_k]$ is zero for all n and k, except that

$$\tilde{H}_{m-1}[P_{3m}] \cong \tilde{H}_m[P_{3m+2}] \cong \mathbb{F}$$

for $m \geq 0$.

One may also choose $a = k-2$, in which case we get that $\tilde{H}_n[P_k - N[a]] = 0$ for all n. By Corollary 3.5.2 (iii) and Proposition 2.7.3, we conclude that

$$\tilde{H}_n[P_k] \cong \tilde{H}_n[P_k - a] = \tilde{H}_n[K_2 + P_{k-3}] \cong \tilde{H}_{n-1}[P_{k-3}].$$

Theorem 3.5.4 *Let G be a graph, and let a and b be nonadjacent vertices in G. Then we have the following long exact sequence.*

$$\begin{array}{ccccccc}
 & \cdots & \xrightarrow{i^*_{n+1}} & \tilde{H}_{n+1}[G] & \xrightarrow{u^*_{n+1}} & \tilde{H}_{n-1}[G-N[ab]] & \xrightarrow{\ell^*_{n-1}} \\
\xrightarrow{\ell^*_{n-1}} & \tilde{H}_n[G\cup e] & \xrightarrow{i^*_n} & \tilde{H}_n[G] & \xrightarrow{u^*_n} & \tilde{H}_{n-2}[G-N[ab]] & \xrightarrow{\ell^*_{n-2}} \\
\xrightarrow{\ell^*_{n-2}} & \tilde{H}_{n-1}[G\cup e] & \xrightarrow{i^*_{n-1}} & \tilde{H}_{n-1}[G] & \xrightarrow{u^*_{n-1}} & \cdots &
\end{array}$$

Proof. This time, consider the long exact sequence for the pair $(\mathsf{Ind}(G), \mathsf{Ind}(G \cup e))$. As we concluded in the proof of Proposition 2.7.5,

$$\mathsf{Ind}(G) \setminus \mathsf{Ind}(G \cup e) = \{ab\} * \mathsf{Ind}(G - N[ab]).$$

Now, this is equal to

$$\mathrm{Cone}_{a,b}(\mathsf{Ind}(G - N[ab])) \setminus \mathrm{Susp}_{a,b}(\mathsf{Ind}(G - N[ab])),$$

where $\mathrm{Cone}_{a,b}(\Delta) = \mathrm{Cone}_a(\mathrm{Cone}_b(\Delta))$ As a consequence, Proposition 3.3.1, Corollary 3.3.5 (ii), and Corollary 2.6.5 yield that

$$\begin{aligned}
& \tilde{H}_n(\mathsf{Ind}(G), \mathsf{Ind}(G \cup a)) \\
\cong\ & \tilde{H}_n(\mathrm{Cone}_{a,b}(\mathsf{Ind}(G - N[ab])), \mathrm{Susp}_{a,b}(\mathsf{Ind}(G - N[ab]))) \\
\cong\ & \tilde{H}_{n-1}(\mathrm{Susp}_{a,b}(\mathsf{Ind}(G - N[ab]))) \\
\cong\ & \tilde{H}_{n-2}(\mathsf{Ind}(G - N[ab])) \\
=\ & \tilde{H}_{n-2}[G - N[ab]].
\end{aligned}$$

This concludes the proof. $\square$

The interested reader may check the following:

- The map i^*_n is induced by the inclusion map $i_n : \tilde{C}_n[G \cup e] \to \tilde{C}_n[G]$.
- The map ℓ^*_n is induced by the map $\ell_n : \tilde{C}_n[G - N[ab]] \to \tilde{C}_{n+1}[G \cup e]$ given by $\ell_n(x) = (b - a) \wedge x$.
- The map u^*_n is induced by the map $u_n : \tilde{C}_n[G] \to \tilde{C}_{n-2}[G - N[ab]]$ given by $u_n(x + a \wedge b \wedge y) = y$ for each $x \in \tilde{C}_n[G \cup e]$ and $y \in \tilde{C}_{n-2}[G - N[ab]]$.

Corollary 3.5.5 *Let G be a graph, and let $e = ab$ be an edge not in G.*

(i) *If $\tilde{H}_n[G \cup e] = \tilde{H}_{n-1}[G \cup e] = 0$, then $\tilde{H}_n[G] \cong \tilde{H}_{n-2}[G - N[ab]]$.*

(ii) *If $\tilde{H}_{n+1}[G] = \tilde{H}_n[G] = 0$, then $\tilde{H}_n[G \cup e] \cong \tilde{H}_{n-1}[G - N[ab]]$.*

(iii) *If $\tilde{H}_{n-1}[G - N[ab]] = \tilde{H}_{n-2}[G - N[ab]] = 0$, then $\tilde{H}_n[G \cup e] \cong \tilde{H}_n[G]$.*

Example. Again, consider P_k, and assume that $k \geq 3$. Choose $a = k - 1$ and $b = k$; write $e = ab$. We note that k is an isolated vertex in $P_k \setminus e$. In particular, $\tilde{H}_n[P_k \setminus e] = 0$ for all n. Picking $G = P_k \setminus e$ in Corollary 3.5.5 (ii), we get that

$$\tilde{H}_n[P_k] \cong \tilde{H}_{n-1}[P_k - N[ab]] = \tilde{H}_{n-1}[P_{k-3}].$$

Finally, let G be a graph, and let a and b be adjacent vertices in G. Note that

$$\begin{aligned}\mathsf{Ind}(G-a)\cap\mathsf{Ind}(G-b) &= \mathsf{Ind}(G-\{a,b\}),\\ \mathsf{Ind}(G-a)\cup\mathsf{Ind}(G-b) &= \mathsf{Ind}(G).\end{aligned}$$

Applying Corollary 3.4.2, we obtain the following result.

Corollary 3.5.6 *Let G be a graph, and let a and b be adjacent vertices in G. Then we have the following long exact sequence.*

$$\begin{array}{ccccccc} & \cdots & \xrightarrow{f^*_{n+1}} & \tilde{H}_{n+1}[G-a]\oplus\tilde{H}_{n+1}[G-b] & \xrightarrow{g^*_{n+1}} & \tilde{H}_{n+1}[G] \\ \xrightarrow{h^*_{n+1}} & \tilde{H}_n[G-\{a,b\}] & \xrightarrow{f^*_n} & \tilde{H}_n[G-a]\oplus\tilde{H}_n[G-b] & \xrightarrow{g^*_n} & \tilde{H}_n[G] \\ \xrightarrow{h^*_n} & \tilde{H}_{n-1}[G-\{a,b\}] & \xrightarrow{f^*_{n-1}} & \tilde{H}_{n-1}[G-a]\oplus\tilde{H}_{n-1}[G-b] & \xrightarrow{g^*_{n-1}} & \cdots \end{array}$$

The maps f^*_n, g^*_n, and h^*_n are defined as in Theorem 3.4.1.

Corollary 3.5.6 yields identities similar to the ones in Corollaries 3.5.2 and 3.5.5. We leave the details to the reader.

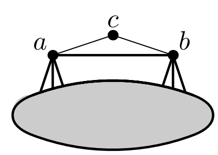

Figure 3.1: The situation in Corollary 3.5.7.

Corollary 3.5.7 *Let G be a graph, and let a, b, and c be vertices forming a triangle such that $N(c)=\{a,b\}$. See Figure 3.1 for an illustration. Then*

$$\tilde{H}_n[G]\cong\tilde{H}_{n-1}[G-N[a]]\oplus\tilde{H}_{n-1}[G-N[b]]$$

for all n.

Proof. Note that c is isolated in $G-\{a,b\}$. In particular, Corollary 3.5.6 yields that

$$\tilde{H}_n[G]\cong\tilde{H}_n[G-a]\oplus\tilde{H}_n[G-b].$$

Now, b is the only neighbor of c in $G-a$, which yields that

$$\tilde{H}_n[G-a]\cong\tilde{H}_{n-1}[(G-a)-N_{G-a}[b]]=\tilde{H}_{n-1}[G-N_G[b]]$$

by Corollary 3.5.3. Similarly, $\tilde{H}_n[G-b]\cong\tilde{H}_{n-1}[G-N_G[a]]$. As a consequence, we obtain the desired result. $\square$

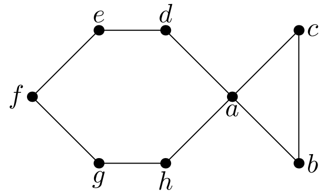

Figure 3.2: Example graph.

Example. Let G be the graph in Figure 3.2. We note that the conditions of Corollary 3.5.7 are satisfied. Now, $G - N[a]$ is isomorphic to P_3, whereas $G - N[b]$ is isomorphic to P_5. As a consequence, we get that

$$\tilde{H}_n[G] \cong \tilde{H}_{n-1}[P_3] \oplus \tilde{H}_{n-1}[P_5] \cong \begin{cases} \mathbb{F} & \text{if } n = 1, \\ \mathbb{F} & \text{if } n = 2, \\ 0 & \text{otherwise.} \end{cases}$$

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