week 7~8

Recap:

called "simplices"

Def 1: Simplicial complex is a family Δ consisting of $\boxed{\text{finite subsets}}$ of a given set X such that

- $\forall \tau \in \Delta$ and $\sigma \in \tau$, $\sigma \in \Delta$.

$\tau \in \Delta$ is a face of Δ. The dimension of a face τ is $|\tau| - 1$.

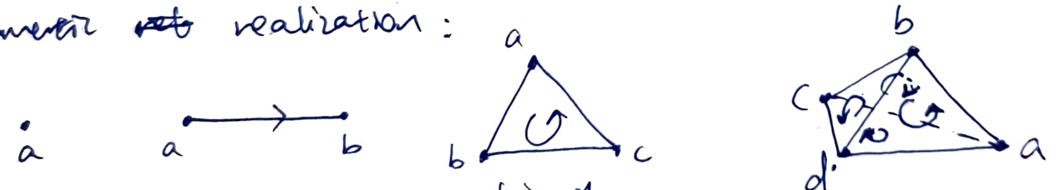
- $\forall v \in X$, $\{v\} \in \Delta$.

sets $\{v\}$ are the vertices of X.

Def 2: $\sigma \in \Delta$ is a k-simplex of dimension k if $|\sigma| = k+1$.
If $\tau \subseteq \sigma$, τ is a face of σ and σ a coface of τ.

An orientation of k-simplex σ, $\sigma = \{v_0, \dots, v_k\}$ is an equivalence class of orderings of the vertices of σ, where $(v_0, \dots, v_k) \sim (v_{\tau(0)}, \dots, v_{\tau(k)})$ if the sign τ is 1.
oriented simplex is denoted as $[\sigma]$.

geometric realization:

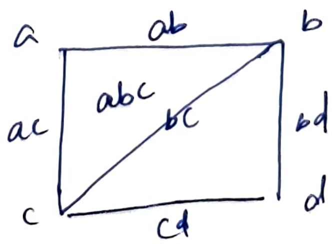


vertex a | edge $[a,b]$ | triangle $[a,b,c]$ | $[a,b,c,d]$

~~Exact~~

Computing homology

Using example 1: $E1 =$



$Z_2(E1; \mathbb{F}) = 0$ $\because$ the only face with dimension $=2$ is abc.
$\cdot\ \partial_2(\lambda \cdot a \wedge b \wedge c) = 0 \iff \lambda = 0$.

$\Rightarrow \tilde{H}_2(E1; \mathbb{F}) = 0$.

Consider $\tilde{H}_1(E_1;\mathbb{F})$.

Known: $B_1(E_1;\mathbb{F})$ is generated by $\partial_2(a\wedge b\wedge c) = a\wedge b - a\wedge c + b\wedge c = z_1$.

Suppose $z \in Z_1(E_1;\mathbb{F})$. Then $\partial_1(z)=0$ and z is a linear combination

$$z = \lambda_{ab}\cdot a\wedge b + \lambda_{ac}\cdot a\wedge c + \lambda_{bd}\cdot b\wedge d + \lambda_{cd}\cdot c\wedge d,$$

(where $\lambda_{ab}, \lambda_{ac}, \lambda_{bc}, \lambda_{bd}, \lambda_{cd} \in \mathbb{F}$)

$$\Rightarrow \partial_1(z) = \partial_1(\lambda_{ab}\cdot a\wedge b + \lambda_{ac}\cdot a\wedge c + \lambda_{bd}\cdot b\wedge d + \lambda_{cd}\cdot c\wedge d)$$

$$= (-\lambda_{ab} - \lambda_{ac})\, a + (\lambda_{ab} - \lambda_{bc} - \lambda_{bd})\, b + (\lambda_{ac} + \lambda_{bc} - \lambda_{cd})\, c + (\lambda_{bd} + \lambda_{cd})\, d = 0.$$

Let $\lambda_{ab} = t$, $\lambda_{cd} = u$.

Then $\lambda_{ac} = -t$, $\lambda_{bd} = -u$, $\lambda_{bc} = t+u$.

$Z_1(E_1;\mathbb{F})$ is generated by:

$z_1 = a\wedge b - a\wedge c + b\wedge c \quad (t=1, u=0)$

$z_2 = b\wedge c - b\wedge d + c\wedge d. \quad (t=0, u=1)$

x and y are in the same homology class $\Leftrightarrow$ $y-x$ is a multiple of z_1.

$+$ z_1 generates $B_1(E_1,\mathbb{F})$

~~$\Rightarrow$ the ...~~

$$\Rightarrow \tilde{H}_1(E_1;\mathbb{F}) = \frac{Z_1(E_1;\mathbb{F})}{B_1(E_1,\mathbb{F})}$$

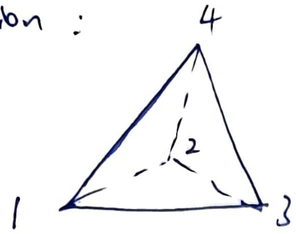
$\forall x \in \tilde{H}_1(E_1;\mathbb{F})$, x is of the form $[u z_2] = \{t z_1 + u z_2 : t \in \mathbb{F}\}$.

$\Rightarrow$ $\tilde{H}_1(E_1,\mathbb{F})$ has dimension 1 and is generated by $[z_2]$.

$\Rightarrow$ To sum up, $\tilde{H}_n(E_1;\mathbb{F}) \cong \begin{cases} \mathbb{F} & \text{if } n=1 \\ 0 & \text{if } n\neq 1 \end{cases}$

~~Exam~~ Example 2 : Homology group of a sphere S^2.

① Triangulation :



$\{1\}, \{2\}, \{3\}, \{4\}$.
$\{1,2\}, \{1,3\}, \{2,3\}, \{1,4\}, \{2,4\}, \{3,4\}$
$\{1,2,3\}, \{1,2,4\}, \{2,3,4\}, \{1,3,4\}$.

② Computing H_0 :
by computing the rank of the matrix :

$$\partial_1 = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Note : $(1,1,0,0)$, $(1,0,1,0)$, $(0,1,1,0)$, $(1,0,0,1)$, $(0,1,0,1)$, $(0,0,1,1)$ are coordinates of $\partial_1(x), x \in C_1$ in terms of the base $\{1\}, \{2\}, \{3\}, \{4\}$.

$\text{rank}(\partial_1) = 3$.
$\Rightarrow \beta_0 = 4 - 3 = 1$.
$\Rightarrow H_0(S^2; \frac{\mathbb{Z}}{2\mathbb{Z}}) \cong (\frac{\mathbb{Z}}{2\mathbb{Z}})$.

③ Computing H_1 :
by computing the rank of the matrix :

$$\partial_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

note : $(1,1,1,0,0,0)$, $(1,0,0,1,1,0)$, $(0,0,1,0,1,1)$, $(0,1,0,1,0,1)$ are coordinates of $\partial_2(x), x \in C_2$ in terms of the base $\{1,2\}, \{1,3\}, \{2,3\}, \{1,4\}, \{2,4\}, \{3,4\}$. (boundaries of 2-simplices written on the base of ~~1-simpl~~ 1-simplices).

· $\text{rank}(\partial_2) = 3$
~~##~~ since $\text{rank}(\partial_1) = 3$. $\Rightarrow \dim(\ker(\partial_1)) = 6 - 3 = 3$.
$\Rightarrow \beta_1 = \dim(\ker(\partial_1)) - \text{rank}(\partial_2) = 3 - 3 = 0$.
$\Rightarrow H_1(S^2; \frac{\mathbb{Z}}{2\mathbb{Z}}) = \{0\}$.

③ Computing H_2 :

~~we~~ There is one 2-cycle $(\{1,2,3\} + \{1,3,4\} + \{1,2,4\} + \{2,3,4\})$ and no 3-simplex.

$$\Rightarrow H_2(S^2; \frac{\mathbb{Z}}{2\mathbb{Z}}) \cong (\frac{\mathbb{Z}}{2\mathbb{Z}}).$$