[Simplicial Complex].

Def 1: An abstract simplicial complex is a family Δ consisting of finite subsets of a given set X s.t. the following condition holds:

$$\text{If } \tau \in \Delta \text{ and } \sigma \subseteq \tau, \text{ then } \sigma \in \Delta$$

$\tau \in \Delta$ is face of Δ. The dimension of a face τ is $|\tau| - 1$.

Def 2: A face τ is a maximal face of Δ if there is no face σ of Δ s.t. $\cancel{\tau \subsetneq \sigma}$. $\tau \subsetneq \sigma$.

- 0-dimensional faces : vertices
- 1-dimensional faces : edges
- A simplicial complex of dimension ≤ 1 : (simple & loopless) graph.

Def 3: A simplicial complex Δ_0 is a subcomplex of Δ if $\Delta_0 \subseteq \Delta$.

For $k \geq -1$, the k-skeleton $\Delta^{(k)}$ of Δ is the subcomplex of Δ obtained by removing all faces of dimension higher than k.

Def 4: (f-vector) For each $n \geq -1$, ~~we~~ let $f_n = f_n(\Delta)$ be the # of faces of Δ of dimension n.

The f-vector of Δ is the vector $(f_{-1}, f_0, f_1, \dots, f_d)$ where d is the dimension of Δ.

eg 1 The sets $\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}$ form a simplicial complex G_1.

Then the f-vector of G_1 is $(\underset{\substack{\uparrow \\ \emptyset}}{1}, 4, 5, 1)$.

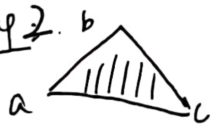
Pause ~~&~~ for some intuitions:

- a simplicial complex in $\mathbb{R}^n$ is a collection of simplices ~~of~~ in $\mathbb{R}^n$. (of possibly varying dimensions) s.t.

① Every face of a ~~complex~~ simplex of K is in K.

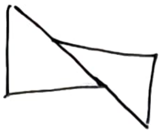
② The intersection of any two simplices of K is a face of each.

eg 2.



This 2-simplex together w all its faces is a simplicial complex.

$\{\{a,b,c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a\}, \{b\}, \{c\}\}$.

eg. 3.  ← this is NOT a simplicial complex

eg. 4.  ← this is a simplicial complex.

~~Bad~~

Def 5: (Geometric realizations of simplicial complexes).

~~Def 5~~: ~~The~~ For any $d \geq 0$, the standard d-simplex is the set

- $X_d = \{(\lambda_0, \dots, \lambda_d) : \lambda_i \geq 0 \text{ for } 0 \leq i \leq d,\ \lambda_0 + \dots + \lambda_d = 1\} \subset \mathbb{R}^{d+1}$

equ. X_d is the convex hull of $(1,0,\dots,0), (0,1,0,0\dots0), \dots, (0,0,\dots,0,1))$

- Let Δ be abstract simplicial complex with vertex set V,
  let $f : V \to \mathbb{R}^n$ be any map.
  For any non-empty d-face $\sigma = \{a_0, \dots, a_d\}$ of Δ,
  f induces a map: ~~f_x~~ $f_\sigma : X_d \to \mathbb{R}^n$

  $(\lambda_0, \dots, \lambda_d) \mapsto \lambda_0 f(a_0) + \dots + \lambda_d f(a_d)$.

- f induces a geometric realization of Δ if:
  ① The map f_σ is injective for each $\sigma \in \Delta \setminus \{\emptyset\}$.
  ② For any nonempty $\sigma, \tau \in \Delta$, we have:
  $\operatorname{im} f_\sigma \cap \operatorname{im} f_\tau = \operatorname{im} f_{\sigma \cap \tau}$.

i.e.. ① = " there is no ~~trivi~~ non-trivial linear combination $\sum_i \lambda_i f(a_i) = 0$ s.t. $\sum_i \lambda_i = 0$.

② = " the intersection between the realizations of any two faces contains nothing more than the realization of the greatest common face of the two simplicies

- The actual geometric realization:

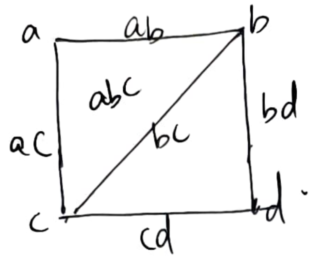
$$|\Delta| = \bigcup_{\sigma \in \Delta \setminus \{\emptyset\}} \operatorname{im} f_\sigma .$$

Back to eg. 1 :

$E1 = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{c,d\}, \{a,c\}, \{b,d\}, \{a,b,c\}\}$.

$\downarrow$

Geom realization of E1:



Def 6 : Oriented complex

Def 6 : (intuition) An oriented simplex is a simplex σ together with an orientation of σ.

If $\{a_0, a_1, \dots, a_p\}$ spans a p-simplex σ, then we denote the oriented ~~complex~~ simplex with $[a_0, a_1, \dots, a_p]$.

(formal) . Let $\mathbb{F}$ be a commutative ring, Δ be a simplicial complex.
For each $n \geq -1$, we form a free $\mathbb{F}$-module $\tilde{C}_n(\Delta; \mathbb{F})$ with a basis indexed by the n-dimensional faces of Δ.
For each face $a_0 a_1 \dots a_n$, (n-dimensional) we have a basis element $\vec{e}_{a_0, a_1, \dots, a_n}$.
we refer to a basis element as an oriented simplex.
(analogous to the idea of unit vectors) give the direction!

Def 7 : From the above context, we refer to $\tilde{C}_n(\Delta; \mathbb{F})$ as the chain group of degree n.
(note that the rank of $\tilde{C}_n(\Delta; \mathbb{F})$ is the n-th value $f_n(\Delta)$ in the f-vector of Δ.

Back to eg. 1 :

For $E1 = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}, \dots\}$.

$\tilde{C}_{-1}(E1) = \{\lambda e_\phi : \lambda \in \mathbb{F}\} \cong \mathbb{F}$

$\tilde{C}_0(E1) = \{\lambda_a e_a + \lambda_b e_b + \lambda_c e_c + \lambda_d e_d : \lambda_a, \lambda_b, \lambda_c, \lambda_d \in \mathbb{F}\} \cong \mathbb{F}^4$

$\tilde{C}_1(E1) = \{\lambda_{ab} e_{a,b} + \dots + \lambda_{cd} e_{c,d} : \lambda_{ab}, \dots, \lambda_{cd} \in \mathbb{F}\} \cong \mathbb{F}^5$.

$\tilde{C}_2(E1) = \{\lambda e_{a,b,c} : \lambda \in \mathbb{F}\} \cong \mathbb{F}$.

The exterior product notation for oriented complex:

$a_0 \wedge a_1 \wedge \cdots \wedge a_n = e_{a_0, a_1, \ldots, a_n}$.

e.g. $a \wedge b$ represents the oriented simplex $e_{a,b}$.
(in degree -1, we use $e_\emptyset$).

$\wedge$, exterior product : ① $b \wedge a = -a \wedge b$
② $a \wedge a = 0$.

Def 8 : (Boundary Maps, ∂_n).

~~Define~~ The boundary map ∂_n is a homomorphism

$$\partial_n : \tilde{C}_n(\Delta) \longrightarrow \tilde{C}_{n-1}(\Delta).$$

$$(\star)\ (a_0 \wedge a_1 \wedge \cdots \wedge a_n) \longmapsto \sum_{r=0}^{n} (-1)^r a_0 \wedge \cdots \wedge a_{r-1} \wedge \hat{a}_r \wedge a_{r+1} \wedge \cdots \wedge a_n$$

for each n, where $\hat{a}_r$ denotes removing the element a_r.

(special case $n=0$, let $\partial_0(a) = e_\emptyset$ for each vertex a).

Proposition 9: (The double boundary condition).

Boundary maps satisfy the double boundary condition, i.e., $\partial_n \circ \partial_{n+1} = 0 \ \forall n$.

(The go-to eg.: the boundary of a disk is a circle, which is a closed curve. And a closed curve has vanishing boundary. Taking the boundary of boundary of a disk leaves us with nothing!)

pf : Suppose $\partial_{n+1}(a_0 \wedge a_1 \wedge \cdots \wedge a_{n+1}) = \sum_{r=0}^{n+1} \lambda_r \cdot a_0 \wedge \cdots \wedge a_{r-1} \wedge \hat{a}_r \wedge a_{r+1} \wedge \cdots \wedge a_{n+1}$

for some constants $\lambda_0, \ldots, \lambda_{n+1}$, where $\lambda_0 = 1$.

For $r<k$, 2 appearances in ~~A~~ $\partial_n \circ \partial_{n+1}(a_0 \wedge \cdots \wedge a_{n+1})$:

① removing a_k first then a_r : coefficient $= \lambda_k \cdot (-1)^r$.

② removing a_r first then a_k : coefficient $= \lambda_r \cdot (-1)^{k-1}$ ← since a_k ends up in position $k-1$ after ~~a_r~~ is removed

Thus,

$\lambda_k \cdot (-1)^r + \lambda_r \cdot (-1)^{k-1} = 0$

$\Leftrightarrow \qquad \lambda_k \cdot (-1)^r = \lambda_r (-1)^k$.

since $\lambda_0 = 1$, $\lambda_k = (-1)^k \ \forall k$.

Def 10: (simplicial chain complex).

$$C(\Delta): \cdots \xrightarrow{\partial_{n+2}} \tilde{C}_{n+1}(\Delta) \xrightarrow{\partial_{n+1}} \tilde{C}_n(\Delta) \xrightarrow{\partial_n} \tilde{C}_{n-1}(\Delta) \xrightarrow{\partial_{n-1}} \cdots$$

Back to eg. 1:

$$\begin{array}{ccccccccccc} C(E_1): 0 & \longrightarrow & \tilde{C}_2(E_1) & \xrightarrow{\partial_2} & \tilde{C}_1(E_1) & \xrightarrow{\partial_1} & \tilde{C}_0(E_1) & \xrightarrow{\partial_0} & \tilde{C}_{-1}(E_1) & \to & 0. \\ & & \cong & & \cong & & \cong & & \cong & & \\ 0 & \longrightarrow & \mathbb{F}^{1} & \longrightarrow & \mathbb{F}^{5} & \longrightarrow & \mathbb{F}^{4} & \longrightarrow & \mathbb{F}^{1} & \longrightarrow & 0 \end{array}$$

↳ Recall that E_1 has f-vector $(1, 4, 5, 1)$!

Rule for arranging vertices in an oriented complex:

① 1-dimension case:

$a \wedge b$ and $b \wedge a$ have opposite orientations.

i.e. $a \wedge b = -b \wedge a$.

② 2-dimension case:

check: $a \wedge b \wedge c = -b \wedge a \wedge c = b \wedge c \wedge a = -c \wedge b \wedge a = c \wedge a \wedge b = -a \wedge c \wedge b$ ✓

check: $\partial_2(b \wedge a \wedge c) = b \wedge a + a \wedge c + c \wedge b$

$= -a \wedge b - b \wedge c - c \wedge a$

$= -\partial_2(a \wedge b \wedge c)$ → aligns with $b \wedge a \wedge c = -a \wedge b \wedge c$ ✓

③ General rule:

- Fix a total order on the vertex set of Δ.
- For any vertices $x_0, x_1, \dots, x_n$ forming an n-dimensional face, define a pair (x_i, x_j) to be an inversion in $\vec{x} = x_0 \wedge x_1 \wedge \cdots \wedge x_n$ if $i < j$ and $x_i > x_j$.

eg! If $a < b < c < d$, then $b \wedge d \wedge a \wedge c$ contains 3 inversions, namely (b, a), (d, a), (d, c).

- Define $\text{inv}(x)$ to be the number of inversions of x.

- Consider a face $a_0 a_1 \dots a_n$, assuming $a_0 < a_1 < \cdots < a_n$, and let π be a permutation of $\{0, \dots, n\}$. Write $b_r = a_{\pi(r)}$, $\vec{a} = a_0 \wedge a_1 \wedge \cdots \wedge a_n$, $\vec{b} = b_0 \wedge b_1 \wedge \cdots \wedge b_n$.

Define $\vec{b} = (-1)^{\text{inv}(\vec{b})} \cdot \vec{a}$. (*)

(Def 11).

Lemma 12: The assignment in Def 11 (*) aligns w the boundary map ∂_n. i.e. when computing $\partial_r(\vec{b})$ according to rule in def 8 (☆), the result is $(-1)^{inv(\vec{b})} \cdot \partial_n(\vec{a})$.

Def 13: Two oriented ~~so~~ simplices $\vec{a} = a_0 \wedge \cdots \wedge a_r$ and $\vec{b} = b_0 \wedge \cdots \wedge b_k$ are compatible if $\{a_0, \ldots, a_r\} \cup \{b_0, \ldots, b_k\} \in \Delta$.

If $\vec{a}$ and $\vec{b}$ are compatible we define the (exterior) product between $\vec{a}$ and $\vec{b}$:

$$\vec{a} \wedge \vec{b} = a_0 \wedge \cdots \wedge a_r \wedge b_0 \wedge \cdots \wedge b_k .$$

Generalization of the exterior product:

Suppose that $c_1 = \sum_a \lambda_a \cdot \vec{a} \in \tilde{C}_r(\Delta)$

$$c_2 = \sum_b \mu_b \cdot \vec{b} \in \tilde{C}_k(\Delta)$$

are two linear combinations of oriented simplizies s.t. $\vec{a}$ and $\vec{b}$ are compatible whenever $\lambda_a \neq 0$ and $\lambda_b \neq 0$. Then we can form the product:

$$c_1 \wedge c_2 = \sum_{a,b} \lambda_a \mu_b \cdot \vec{a} \wedge \vec{b} \in \tilde{C}_{r+k-1}(\Delta).$$

Def 14: (Homology of a simplicial complex Δ).

(Intuition) The homology gives an algebraic measure on the amount of cycles that are not boundaries.

(formal) Define the $\mathbb{F}$-module $Z_n(\Delta;\mathbb{F})$ of cycles and the $\mathbb{F}$-module $B_n(\Delta;\mathbb{F})$ of boundaries by the following formulas:

- $Z_n(\Delta;\mathbb{F}) = \ker \partial_n = \{ z \in \tilde{C}_n(\Delta;\mathbb{F}) : \partial_n(z) = 0 \}$,
- $B_n(\Delta;\mathbb{F}) = \operatorname{im} \partial_{n+1} = \{ z \in \tilde{C}_n(\Delta;\mathbb{F}) : z = \partial_{n+1}(x)$ for some $x \in \tilde{C}_{n+1}(\Delta;\mathbb{F}) \}$.

By Prop 9, $B_n(\Delta;\mathbb{F})$ is a submodule of $Z_n(\Delta;\mathbb{F})$.

We define the simplicial homology in degree n of Δ to be the quotient:

$$\tilde{H}_n(\Delta;\mathbb{F}) = Z_n(\Delta;\mathbb{F}) / B_n(\Delta;\mathbb{F}).$$

Def 15: (homology class).
Each member of $\tilde{H}_n(\Delta;\mathbb{F})$ is a <u>homology class</u>. Each such member is an equivalent class under the relation $z \sim z' \Leftrightarrow z - z' \in B_n(\Delta;\mathbb{F})$.
Let $[z]$ denote the homology class containing the cycle z.

Def 16: (reduced vs unreduced homology).

- $\tilde{H}_n(\Delta;\mathbb{F})$ is reduced homology.
  - $\tilde{C}_{-1}(\Delta;\mathbb{F}) = \mathbb{F}\cdot e_\emptyset$.
  - $\partial_0(a) = e_\emptyset$ for each vertex a.

- $H_n(\Delta;\mathbb{F})$ is unreduced homology.
  - $C_{-1}(\Delta;\mathbb{F}) = 0$.
  - ∂_0 is the zero map.

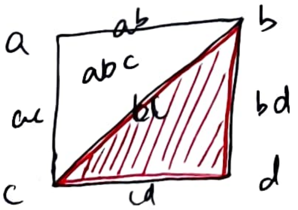
- For degree $n \geq 1$, $\tilde{H}_n(\Delta;\mathbb{F}) = H_n(\Delta;\mathbb{F})$.

<u>Homology as amount of "holes" in Δ:</u>

- In certain well-behaved cases, homology can be interpreted as an algebraic measure on the amount of "holes" in Δ.

"hole": Given $X \subseteq \mathbb{R}^n$, a <u>hole</u> of X is a bounded connected component of $\mathbb{R}^n \setminus X$.
For a given $k \geq 0$, suppose $(k+1)$-skeleton $\Delta^{(k+1)}$ of Δ has a geometric realization X in $\mathbb{R}^{k+1}$. Then $\tilde{H}_k(\Delta;\mathbb{F})$ is a free $\mathbb{F}$-module of rank the # of holes of X.

<u>Back to eg. 1</u>:
Geometric realization of E_1:



- The shaded region is bounded & separated by $|E_1|$ from the unbounded region outside $|E_1|$.
- $\tilde{H}_1(E_1, \mathbb{F})$ has rank 1
- The homology class of the cycle $(b\wedge c + c\wedge d + d\wedge b)$ is a generator of $\tilde{H}_1(E_1, \mathbb{F})$.

- In the general case, however, the interpretation (that homology = amount of holes) does NOT hold.

Methods for Computing homology (overview).

1. Combinatorial techniques.
   - splitting chain complexes
   - collapsing
   - Discrete Morse theory.
   - Joins, links, and deletions

2. Algebraic techniques.
   - exact sequences
   - chain maps & chain isomorphisms.
   - relative homology.
   - mayer - Vietoris sequences.