Comparing Neural Population Responses Based on Pairwise p-Wasserstein Distance between Topological Signatures

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Main Contribution

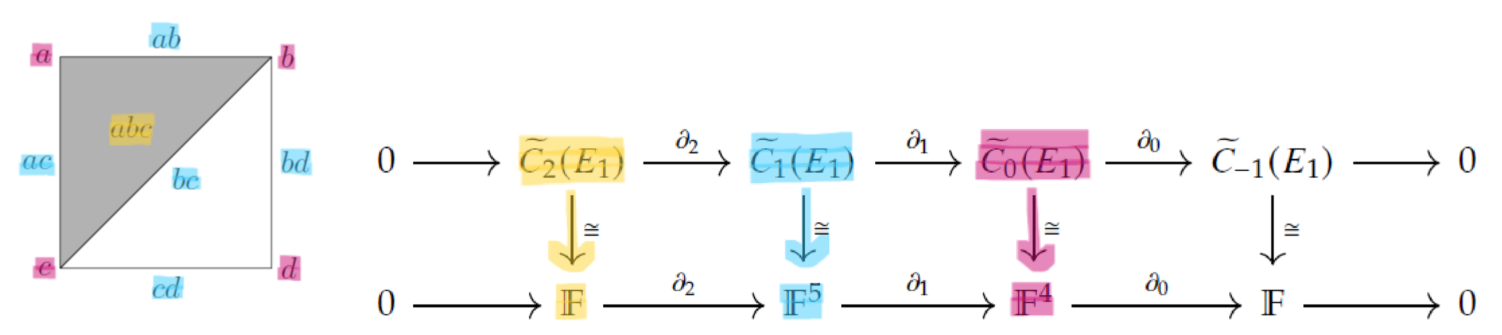
We develop and evaluate a topology-based approach to compare neural population activities as high-dimensional point-clouds. As a demonstration, we apply the approach to compare neural population responses in the mouse retina to different visual stimuli. With the proposed approach, one can

- quantitatively compare between neural population responses arising from artificial and biological neural networks, and
- perform statistical inference on a distribution of topological signatures for the respective neural population responses.

Preliminaries

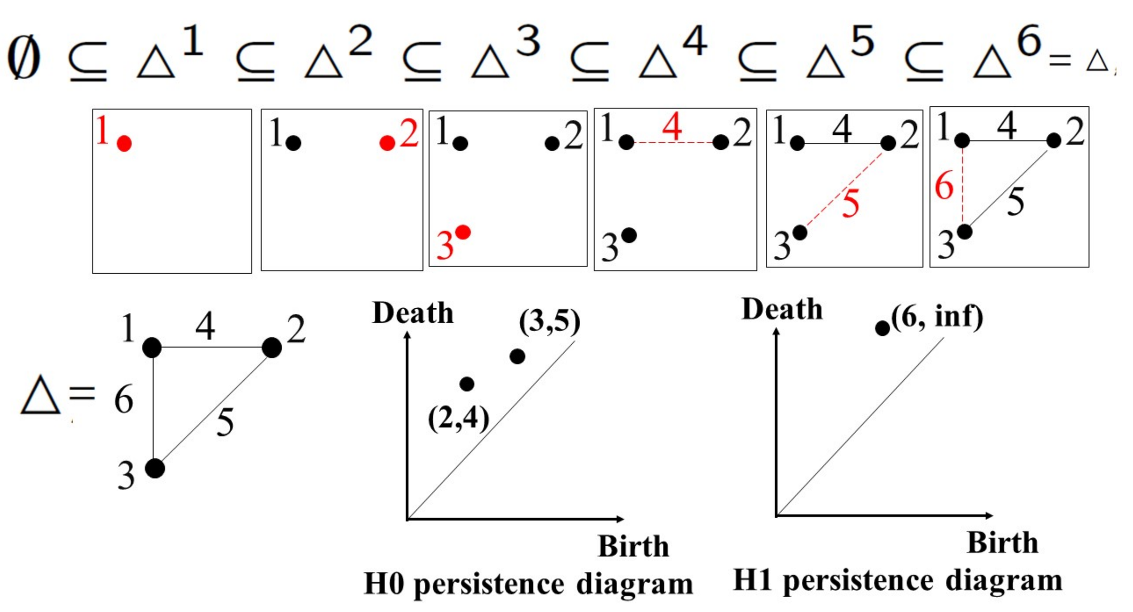
The topological and statistical constructions in the proposed approach build on the following mathematical concepts.

- **Simplicial complex:** a pair $(V, \triangle)$, where V is a finite set, and $\triangle$ is a family of non-empty subsets of V such that $\tau \in \triangle$ and $\sigma \subseteq \tau \implies \sigma \in \triangle$, where $\tau \in \triangle$ is face of $\triangle$.



(a) Simplicial complex. (b) Simplicial chain complex.

- **Simplicial homology in degree k of $\triangle$** is the quotient group $H_k(\triangle;\mathbb{F}) = ker(\partial_k)/im(\partial_{k+1}) = Z_k(\triangle;\mathbb{F})/B_k(\triangle;\mathbb{F})$, where
  - $Z_k(\triangle;\mathbb{F}) = ker(\partial_k) = \{Z \in \widetilde{C}_k(\triangle;\mathbb{F}) : \partial_k(Z) = 0\}$ is the $\mathbb{F}$-module of the cycle group,
  - $B_k(\triangle;\mathbb{F}) = im(\partial_{k+1}) = \{Z \in \widetilde{C}_k(\triangle;\mathbb{F}) : \partial_{k+1}(x), \quad x \in \widetilde{C}_{k+1}(\triangle;\mathbb{F})\}$ is the $\mathbb{F}$-module of the boundary group.
- **Filtered simplicial complex:** A subcomplex of $\triangle$ is a subset $\triangle^i \subseteq \triangle$ that is also a simplicial complex. Let $\triangle$ be a finite simplicial complex and let $\triangle^1 \subset \triangle^2 \subset \cdots \subset \triangle^m = \triangle$ be a finite sequence of nested subcomplexes of $\triangle$. The simplicial complex $\triangle$ with such a sequence of subcomplexes, $\emptyset \subseteq \triangle^1 \subseteq \triangle^2 \subseteq \cdots \subseteq \triangle^m = \triangle$, is called filtered simplicial complex.
- **p-persistent k-th homology group:** Given a filtered complex, for the i-th subcomplex $\triangle^i$ we compute the associated boundary maps ∂_k^i for all dimensions k, boundary matrices M_k^i for all dimensions k, C_k^i, Z_k^i (cycle group), B_k^i (boundary group), and H_k^i (homology group). Then the p-persistent k-th homology group $H_k^{i,p}$ of $\triangle^i$ is $Z_k^i/(B_k^{i+p} \cap Z_k^i)$.



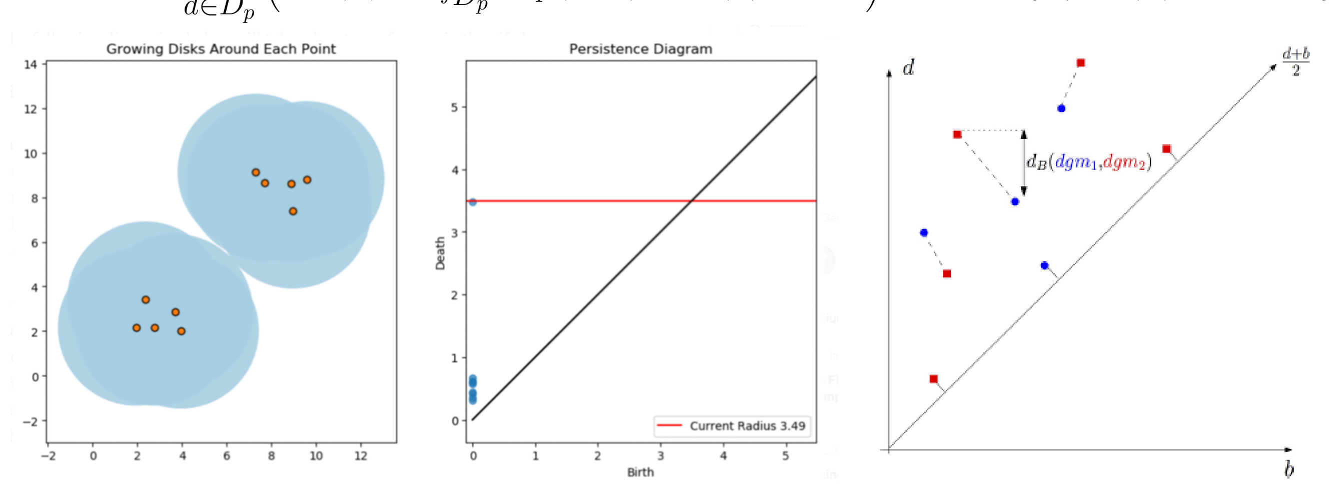
(c) Persistence homology.

- **Barcode and persistent diagrams:** The rank of $H_k^{i\to j}(C;\mathbb{F})$ gives the number of intervals in the barcode of $H_k^{i\to j}(C;\mathbb{F})$ spanning the parameter interval $[i,j]$ (i,j represent the "birth" and "death" of a feature respectively). Persistent diagram is an equivalent representation of barcode, with x-coordinate and y-coordinate representing the "birth" and "death" of a feature respectively.
- **p-Wasserstein distance between persistent diagrams:** Given $p \geq 1$, the p-Wasserstein distance between a pair of persistence diagrams dgm_1 and dgm_2 is defined by

$$W_p(\mathrm{dgm}_1, \mathrm{dgm}_2) = \left(\inf_M \Sigma_{(x,y)\in M} \|x-y\|_\infty^p\right)^{1/p},$$

where the infimum is taken over all possible matchings M. A matching M between dgm_1 and dgm_2 is a subset $M \subseteq \mathrm{dgm}_1 \times \mathrm{dgm}_2$ such that every point in dgm_1 and dgm_2 appears exactly once in M.
- **Statistical inference on the space of persistent diagrams:** The space of persistent diagrams is defined as $D_p = \{d | W_p(d, d') < \infty\} = \{d | \mathrm{Pers}_p(d) < \infty\}$. Given a probability space $(D_p, \mathcal{B}(D_p), \mathcal{P})$, the Fréchet variance and Fréchet expectation are defined as

$$\mathrm{Var}_\mathcal{P} = \inf_{d\in D_p}\left(F_\mathcal{P}(d) = \int_{D_p} W_p(d,e)^2 d\mathcal{P}(e) < \infty\right), \mathbb{E}_\mathcal{P} = \{d | F_\mathcal{P}(d) = \mathrm{Var}_\mathcal{P}\}.$$



(d) Persistence diagrams (credit: Gary Koplik). (e) Matching.

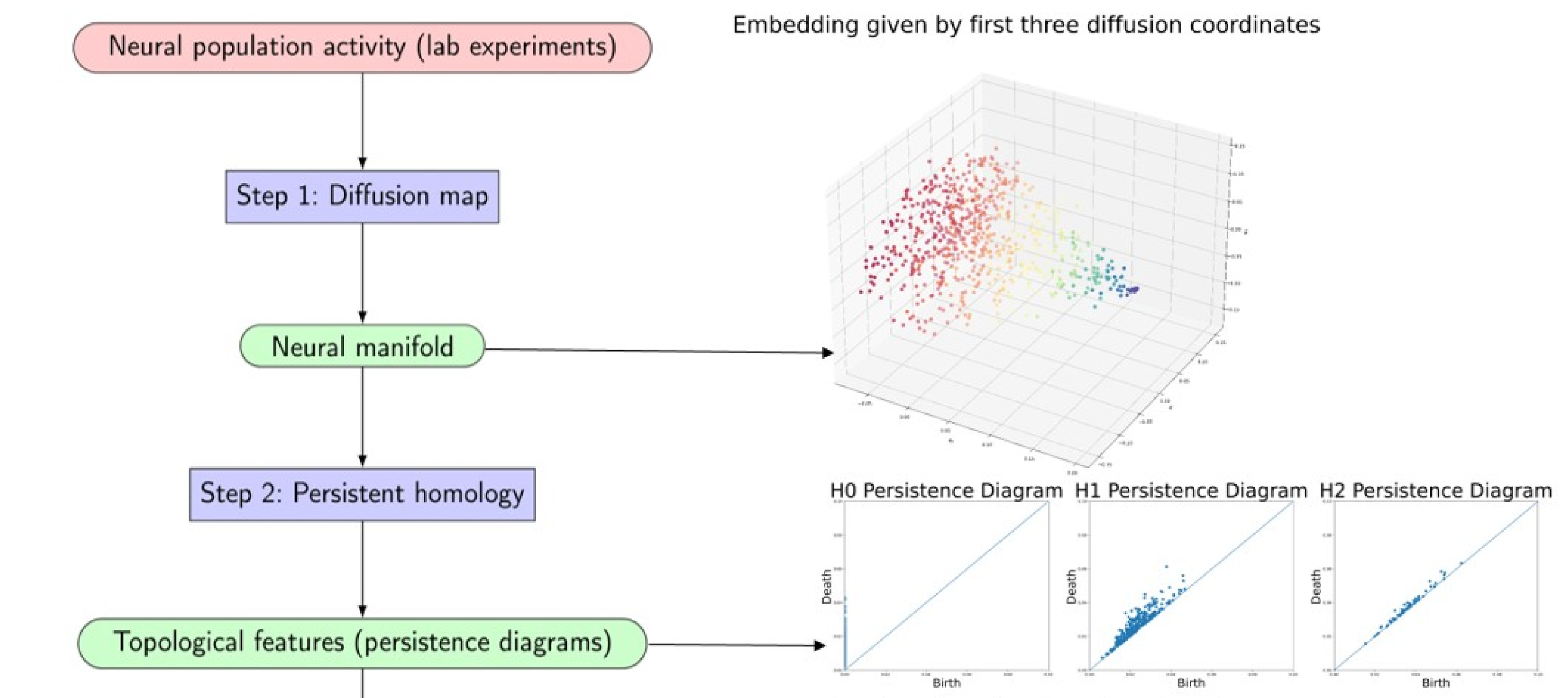
Introduction

Real-world data are often encoded in high-dimensional representations. Moreover, it is often unclear which coordinates and metrics can be meaningfully justified.

- **Topological properties are well-suited:** they are generalized to high-dimensional surfaces and are invariant under different coordinates and metrics.
- **Aim:** compare data point-clouds in terms of their topological properties.
- **Motivation:** analyze the high-dimensional output of a population of neurons in response to some stimulus (neural population response).
- **A crucial gap:** prior works have not considered how these neural population responses can be appropriately compared.

Methods and Results

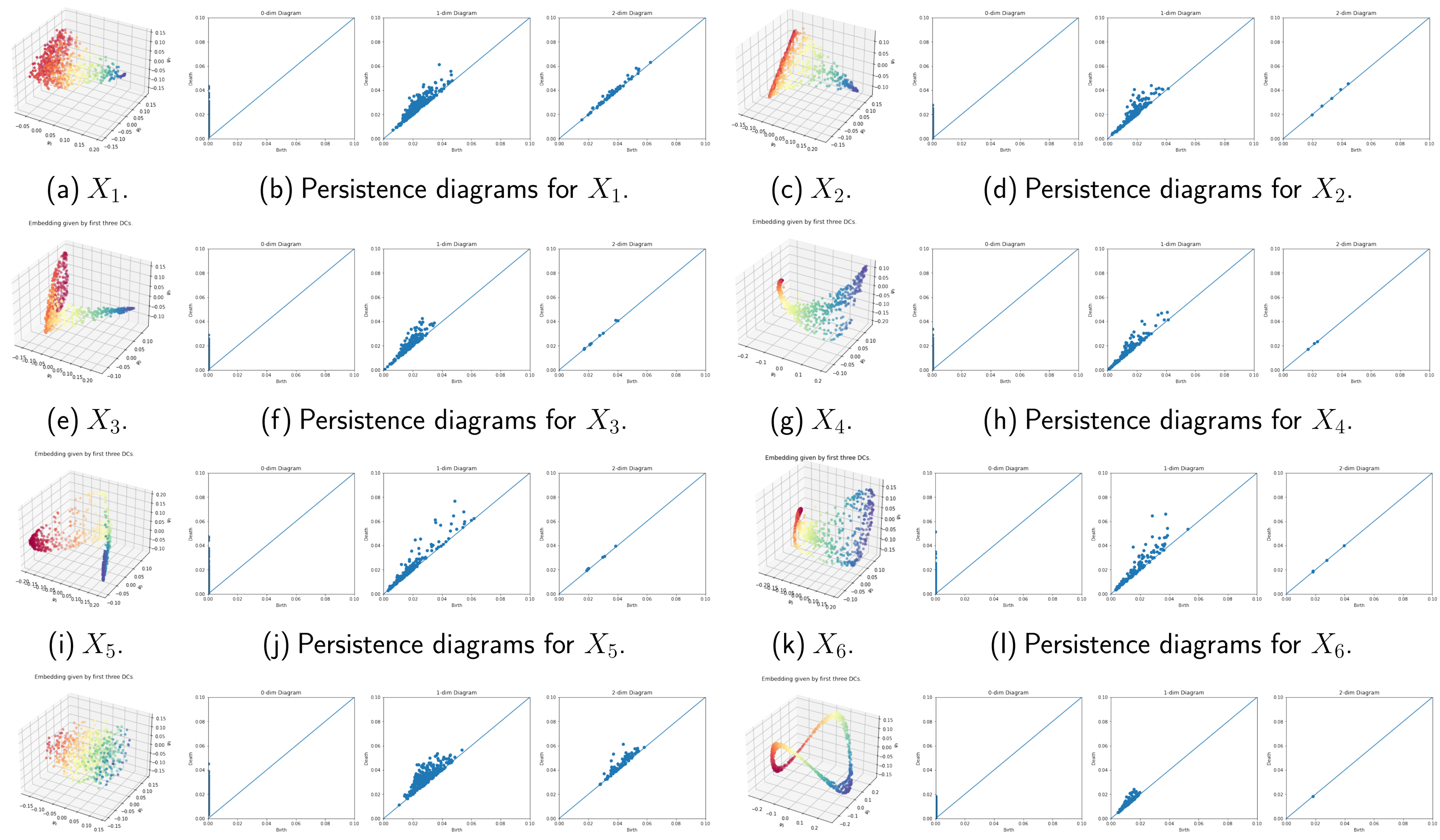
The steps in our approach are summarized in the flowchart.



H_0	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0.0	**2.77**	**3.05**	**3.95**	**3.08**	**3.42**
X_2	**2.77**	0.0	0.43	1.35	1.0	0.84
X_3	**3.05**	0.43	0.0	1.03	0.94	0.66
X_4	**3.95**	1.35	1.03	0.0	1.11	0.72
X_5	**3.08**	1.0	0.94	1.11	0.0	0.51
X_6	**3.42**	0.84	0.66	0.72	0.51	0.0

Figure: Summary of the proposed approach.

- Visual stimuli are flashed in front of the mouse. Neuron output is recorded with electrodes and encoded in peristimulus (PSTH) diagrams: 698 neurons, 6 types of visual stimuli, 264 number of pixels in the PSTH diagram
- Six point clouds each corresponds to the neural population response towards one type of stimuli, which we denote as $X_1, X_2, \ldots, X_6$. Each point cloud X_i consists of 698 points in $\mathbb{R}^{264}$.



(a) X_1. (b) Persistence diagrams for X_1. (c) X_2. (d) Persistence diagrams for X_2.
(e) X_3. (f) Persistence diagrams for X_3. (g) X_4. (h) Persistence diagrams for X_4.
(i) X_5. (j) Persistence diagrams for X_5. (k) X_6. (l) Persistence diagrams for X_6.
(m) S^2 (n) Persistence diagrams for 2-sphere. (o) T^2 (p) Persistence diagrams for a torus.

Based on the results, we have the following hypotheses:

1. The neural population response evoked by the low-frequency stimulus (X_1) is significantly different from the other stimulus types in terms of topological signatures. It might be interesting to conduct further lab experiments to investigate this selective preference.
2. The pairwise p-Wasserstein distances between persistence diagrams for H_2 are nearly negligible. This suggests that the intrinsic dimensionality of this neural data might even be lower than three-dimensional. It would be interesting to compare the results with the hypothesis that if we are given an oriented stimulus, and if the orientation is a circular variable, then the neural population response must be topologically equivalent to a circle.

Conclusion and Discussion

- **Advantage 1:** When there is little knowledge about the underlying coordinates/metrics, topology-based methods are more suitable than geometric methods.
- **Advantage 2:** If one wishes to analyze the probability distributions of neural population responses, this approach allows for standard statistical analysis.
- **Future Direction 1:** To extend this approach analogously to analyze the neural population responses arising from artificial neural networks under numerical simulations.
- **Future Direction 2:** To apply this approach to compare neural population responses across different brain regions.
- **Future Direction 3:** To consider whether it is possible without the dimensionality reduction step - still a challenge in Topological Data Analysis!