Template Matching

Aim : To match image with template.

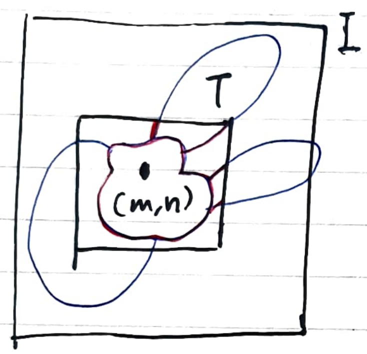
need: ① space of images.

② distance of images : "norm" / "metric".

→ to measure how similar an given image is ~~compare~~ compared to the template.

Image template : $T(i,j)$, the brightness distribution around pixel (i,j)

Given image : $I(i,j)$



"Distance" between 2 images:

~~Euclide~~ • Euclidean norm : L2-norm

$$\text{Match}_{L_2}(m,n) = \Big[\sum_{i,j \text{ s.t. } (i-m,j-n)\in \text{Dom}(T)} (I(i,j) - T(i-m,j-n))^2 \Big]^{\frac{1}{2}}.$$

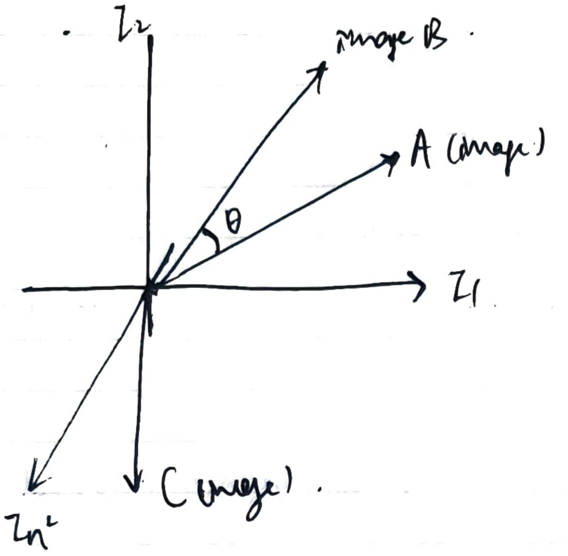
• $\text{Match}^2_{L^2}(m,n)$ ~~norm~~ (MSE).

$$= \sum_{i,j \text{ s.t. } (i-m,j-n)\in \text{Dom}(T)} \Big(\underbrace{I^2(i,j)}_{\text{Image energy}} - 2I(i,j)T(i-m,j-n) + \underbrace{T^2(i-m,j-n)}_{\text{template energy}} \Big).$$

• Cross Correlation $= \sum_{i,j \text{ st. } (i-m,j-n)\in \text{Dom}(T)} I(i,j)\,T(i-m,j-n).$

• Normalized Cross Correlation $= \dfrac{\text{Cross Correlation}}{\Big[\sum_{i,j \text{ st. } (i-m,j-n)\in \text{Dom}(T)} I(i,j)^2\Big]^{\frac{1}{2}}}$

~~correlation as inner product.~~



The more similar the two images
the smaller the angle,
i.e. the smaller the inner product

→ This idea gives rise to K-means clustering !

Lecture 5-6 Limulus Linearity in Frequency

The physical domain

$$I(x) = \sum_i (I \cdot \phi_i)\, \phi_i$$

project onto ϕ_i

ϕ_i is the i^{th} basis vector

The frequency domain

$$a_n = \frac{2}{T} \int_0^T I(t) \cos(n\omega t)\, dt$$

$$b_n = \frac{2}{T} \int_0^T I(t) \sin(n\omega t)\, dt$$

project onto sin/cos

$$a_0 = \frac{1}{T} \int_0^T I(t)\, dt$$

Sinusoids as basis for the frequency domain:

$$\frac{1}{\sqrt{2\pi}},\ \frac{\cos x}{\sqrt{\pi}},\ \frac{\sin x}{\sqrt{\pi}},\ \frac{\cos 2x}{\sqrt{\pi}},\ \frac{\sin 2x}{\sqrt{\pi}},\ \ldots$$

orthogonal: $\int_{-\pi}^{\pi} \phi_i \phi_j \, dx = 0 \quad \forall\, i \neq j$

unit length: $\int_{-\pi}^{\pi} \phi_i^2 = 1 \quad \forall\, i$

linking sin & cos: $(\cos\phi + i \sin\phi) = e^{i\phi}$

Hilbert space:

- ∞ dim
- spanned by complex exponentials
- inner product & norm

Fourier Transform: change in basis

$$\mathcal{F}(\underbrace{f(t)}_{\text{physical domain}}) = \underbrace{F(\omega)}_{\text{freq domain}} = \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t}\, dt$$

$$\mathcal{F}^{-1}(F(\omega)) = f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega)\, e^{i\omega t}\, d\omega$$

(OH MY Prof. is hilarious 00:58:00)

Action of linear system with sin input:

$$L e^{iwt} = \int_{-\infty}^{+\infty} h(u) e^{iw(t-u)} du.$$

Linear operator ↑ a sinusoid ↑

$$= e^{iwt} \int_{-\infty}^{+\infty} h(u) e^{iw(u)} du.$$

$$= H(w) e^{iwt}.$$

a scalar. !! a sinusoid.

Significance of this: output of a linear operator on sinusoid gives a scalar times sinusoid.

⇒ ① Sinusoids are eigenfunctions of a linear system !!

↓

Link back to REU.

② The fourier transform of the input response, $H(w)$ are the eigenvalues associated with the eigenfunc s

Filtering in the frequency domain

Convolution in the physical domain is multiplication in the frequency domain.

image

$$\mathcal{F}(\underbrace{I * h}_{\text{filtered image}}) = \int_{-\infty}^{+\infty} e^{-itw} \left(\int_{-\infty}^{+\infty} I(t-u) h(u) du \right) dt$$

$(t, u) \mapsto (v = t-u, u)$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-i(u+v)w} I(v) h(u) du dv.$$

$$= \left(\int_{-\infty}^{+\infty} e^{-ivw} I(v) dv\right) \cdot \left(\int_{-\infty}^{+\infty} e^{-iuw} h(u) du\right)$$

$$= I(w) \cdot H(w)$$ modulation transfer function

Fourier's duality theorem:

$$I(x) * h(x) \underset{\text{inv. FT}}{\overset{\text{FT}}{\longleftrightarrow}} I(w) \cdot H(w)$$

$$I(x) \cdot h(x) \longleftrightarrow I(w) * H(w)$$

low-pass filter = blur operator
high-pass filter = sharpening operator
band-pass filter

" ~~undext~~ "learning how to think in the frequency domain is really important for understanding perception & life in general!"

Fourier Transform in 1D

$$\begin{cases} F(\xi) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i \xi t}\, dt. \\ f(t) = \int_{-\infty}^{\infty} F(\xi)\, e^{2\pi i \xi t}\, d\xi. \end{cases}$$

Property	$f(t)$ physical domain	$F(\xi)$ frequency domain.
Linearity	$a f_1(t) + b f_2(t)$	$a F_1(\xi) + b F_2(\xi)$
Duality	$F(t)$	$f(-\xi)$
Convolution	$(f * g)(t)$	$F(\xi)\, G(\xi)$.
Product	$f(t)\, g(t)$	$(F * G)(\xi)$
Time shift	$f(t - t_0)$	$e^{-2\pi i \xi t_0} F(\xi)$
Freq shift	$e^{2\pi i \xi_0 t} f(t)$	$F(\xi - \xi_0)$
Differentiation	$\frac{df(t)}{dt}$	$2\pi i \xi F(\xi)$
Mult. by t	$t f(t)$	$\frac{i}{2\pi} \frac{dF(\xi)}{d\xi}$
Time scaling	$f(at)$	$\frac{1}{\lvert a \rvert} F(\xi / a)$

+ Parseval's Identity $\int_{-\infty}^{\infty} \lvert f(t) \rvert^2 dt = \int_{-\infty}^{\infty} \lvert F(\xi) \rvert^2 d\xi$.

f energy = F energy.

Discrete FT in 1D

$$\begin{cases} F(k) \equiv \sum_{n=0}^{N-1} f(n)\, e^{-\frac{2\pi i}{N} kn} \\ f(n) \equiv \frac{1}{N} \sum_{k=0}^{N-1} F(k)\, e^{\frac{2\pi i}{N} kn} \end{cases}$$

n analogous to t.

k analogous to ξ

Fourier Transform in 2D.

The idea: To decompose the image function $f(x,y)$ to a linear combination of harmonic functions (weights given by $F(u,v)$)
(sin & cos, or more generally orthogonal functions).

u,v are spatial frequencies:

$$
\begin{cases}
F(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, e^{-2\pi i(xu+yv)}\, dx\, dy, \\
f(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v)\, e^{2\pi i(xu+yv)}\, du\, dv.
\end{cases}
$$

(linear systems are like gates that control how much power at each frequency is passed through)

Because of Euler' formula, ~~to~~ $e^{iz} = \cos z + i \sin z$

$$e^{-2\pi i(xu+yv)} = \cos(-2\pi i\, xu) + i \sin(-2\pi i\, xu)$$

real part — imaginary part.

$F(u,v)$ gives the weights of harmonic components in the linear combination.

Discrete FT in 2D

$$
\begin{cases}
F(u,v) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) \exp\left[-2\pi i \left(\frac{mu}{M} + \frac{nv}{N}\right)\right], \\
u = 0, 1, \ldots, M-1,\ v = 0, 1, \ldots, N-1. \\
f(m,n) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) \exp\left[2\pi i \left(\frac{mu}{M} + \frac{nv}{N}\right)\right]. \\
m = 0, 1, \ldots, M-1,\ n = 0, 1, \ldots, N-1.
\end{cases}
$$