Introduction to the Feynman-Kac Formula

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Heat and diffusion

The **heat equation**, also known as the **diffusion equation**

$$\frac{\partial u}{\partial t} - \Delta u = 0,$$

where Δu is the Laplacian, was developed by Joseph Fourier to model heat in a given region. It also models particles diffusing through a medium, where u represents the probability density function associated with the position of a single particle.

Joseph Fourier

Solution to the heat equation ivp

The solution to the Cauchy problem (initial value problem) of the heat equation in $\mathbb{R}^n \times \mathbb{R}^+$

$$\begin{cases} \partial_t u - \Delta u = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ u(x,0) = f(x) \in H^s(\mathbb{R}^n) \end{cases}$$

is given by

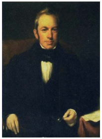
$$u(x,t) = \int_{\mathbb{R}^n} K(x-y,t) f(y) dy,$$

where

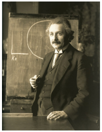
$$K(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}$$

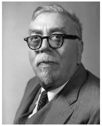
is called the **fundamental solution** (heat kernel) to the heat equation.

Brownian motion and probability


Robert Brown


Louis Bachelier


Albert Einstein


Norbert Wiener


Kiyoshi Itô


Richard Feynman


Mark Kac

Wiener process

Definition

A **Wiener process** $\{W(t)\}_{t\geq 0}$ is a family of random variables $W(t) : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$, satisfying the following properties:

1. $W(0) = 0$.
2. $\forall \omega \in \Omega$, the map $t \mapsto W(\omega, t)$ is continuous from $[0, \infty)$ into $\mathbb{R}$ with probability one.
3. $\forall t > s \geq 0$, $W(t) - W(s)$ is normally distributed with mean 0 and variance $t - s$, i.e.,
$$\mathbb{P}(a \leq W(t) - W(s) \leq b) = \frac{1}{\sqrt{2\pi(t-s)}} \int_a^b e^{-x^2/2(t-s)} dx.$$
4. $\forall 0 = t_0 < t_1 < t_2 < \ldots < t_n$, the increments $W(t_{i+1}) - W(t_i)$ are independent random variables.

Construction via random walk

Define $S_0 = 0$ and $S_{k\Delta t} = \sum_{j=1}^{k} X_j \Delta x$ and linearly interpolate in time. So

S_t is a random walk with properties $\mathbb{E}[S_{k\Delta t}] = 0$, $\mathrm{var}[S_{k\Delta t}] = \frac{(\Delta x)^2}{\Delta t} t$.
For each $n \in \mathbb{Z}^+$, let $\Delta t = 1/n$ and $\Delta x = 1/\sqrt{n}$, and we define $\{S_t^n\}_t$ as above. Explicitly writing out the interpolation

$$
\begin{aligned}
S_t^n &= (\lfloor nt \rfloor + 1 - nt) S^n_{\lfloor nt \rfloor / n} + (nt - \lfloor nt \rfloor) S^n_{(\lfloor nt \rfloor + 1)/n} \\
&= \frac{\sqrt{\lfloor nt \rfloor}}{\sqrt{n}} \frac{1}{\sqrt{\lfloor nt \rfloor}} \sum_{j=1}^{\lfloor nt \rfloor} X_j + \frac{nt - \lfloor nt \rfloor}{\sqrt{n}} X_{\lfloor nt \rfloor + 1} \\
&\xrightarrow{d} \sqrt{t} Z = W_t
\end{aligned}
$$

by the central limit theorem.

Itô integral

Definition

The **Itô integral** for a **simple** stochastic process $f_n(\omega, t)$ is defined by

$$\int_0^T \sum_{j=0}^{n-1} \alpha_j(\omega)\mathbf{1}_{[t_j,t_{j+1})}(t)dW_t = \sum_{j=0}^{n-1} \alpha_j(\omega)(W_{t_{j+1}}(\omega) - W_{t_j}(\omega)).$$

For a **general** stochastic process $f(\omega, t)$ that is square integrable

$$\int_0^T f(\omega, t)dW_t(\omega) \overset{L^2(\Omega)}{=} \lim_{n\to\infty} \int_0^T f_n(\omega, t)dW_t(\omega),$$

where $f_n(\omega, t)$ is a sequence of simple stochastic processes converging to $f(\omega, t)$ in $L^2(\Omega)$.

Itô process

Definition

An **Itô process** is a stochastic process X_t on Ω of the form

$$X_t = X_0 + \int_0^t u(\omega, s)ds + \int_0^t v(\omega, s)dW_s,$$

where $v(\omega, s)$ satisfies the square integrability condition

$$\mathbb{E}\left[\int_0^t v^2(\omega, s)ds\right] < \infty.$$

The stochastic integral is also written as a **stochastic differential equation** of the form

$$dX_t = udt + vdW_t.$$

Itô's lemma

Proposition

Let X_t be an Itô process

$$dX_t = udt + vdW_t,$$

and $g(x,t) \in C^2(\mathbb{R} \times [0,T])$. Then $Y_t = g(X_t, t)$ is also an Itô process with

$$dY_t = \frac{\partial g}{\partial t}(X_t,t)dt + \frac{\partial g}{\partial x}(X_t,t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(X_t,t)(dX_t)^2$$

where $(dt)^2 = dtdW_t = dW_tdt = 0$ and $(dW_t)^2 = dt$.

Heat equation version

Theorem (Feynman-Kac formula)

Let $q(x)$ be a non-negative continuous function, and $f(x)$ be bounded continuous. Suppose $u(x,t)$ is a bounded function that solves the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} = -q(x)u, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+ \\ u(x,0) = f(x) \in C_b(\mathbb{R}). \end{cases} \tag{1}$$

Then

$$u(x,t) = \mathbb{E}\left[\exp\left\{-\int_0^t q(X_s,s)ds\right\} f(X_t)\right] \tag{2}$$

where $dX_t = dW_t$ and $X_0 = x$.

Heuristics

Roughly speaking, if particles have an initial distribution f, and we let each particle diffuse freely, the average of all possible end positions of particles at time t is the distribution of particles at time t. The cooling term $u' = -qu$ adds an exponential decay effect.
If we consider the following simple heat equation ivp

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+ \\ u(x,0) = f(x). \end{cases}$$

It is a well-posed equation, i.e. there exists a unique solution. We have two formulations of the solution, thus they have to be the same.

$$\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} e^{-|x-y|^2/2t} f(y)dy = \mathbb{E}\left[f(X_t)\right], \quad dX_t = dW_t, X_0 = x.$$

Sketch of proof 1

Fix $t > 0$. Consider the stochastic process

$$Y_s = g(X_s, s) = \exp\left\{-\int_0^s q(X_\tau)d\tau\right\}u(X_s, t-s).$$

Apply Itô's formula

$$\begin{aligned}
dY_s &= \left(\frac{\partial g}{\partial s} + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}\right) ds + \frac{\partial g}{\partial x} dW_s \\
&= -\, qu \exp\left\{-\int_0^s q d\tau\right\}ds + (-1)\frac{\partial u}{\partial t}\exp\left\{-\int_0^s q d\tau\right\}ds \\
&\quad + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\exp\left\{-\int_0^s q d\tau\right\}ds + \frac{\partial u}{\partial x}\exp\left\{-\int_0^s q d\tau\right\}dW_s.
\end{aligned}$$

Sketch of proof 2

Use the fact that u solves (1) and integrate to get

$$Y_t - Y_0 = \int_0^t \frac{\partial u}{\partial x}(X_s, t-s) \exp\left\{-\int_0^s q(X_\tau)d\tau\right\} dW_s.$$

Taking expectations, and by property of the Itô integral, we have

$$\mathbb{E}[Y_0] = \mathbb{E}[Y_t],$$

or

$$u(x,t) = \mathbb{E}\left[\exp\left\{-\int_0^t q(X_s)ds\right\} f(X_t)\right].$$

For Itô diffusion processes in one-dimension

Theorem (Feynman-Kac formula)

Let $q(x)$ be a non-negative continuous function, and $f(x)$, $\mu(x)$, $\sigma(x)$ be bounded continuous. Suppose $u(x,t)$ is a bounded function that solves the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mu(x)\dfrac{\partial u}{\partial x} - \dfrac{1}{2}\sigma^2(x)\dfrac{\partial^2 u}{\partial x^2} + q(x)u = 0, \\ u(x,0) = f(x) \in C_b(\mathbb{R}). \end{cases} \tag{3}$$

Then

$$u(x,t) = \mathbb{E}\left[\exp\left\{-\int_0^t q(X_s)ds\right\} f(X_t)\right] \tag{4}$$

where $dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$ and $X_0 = x$.

The Black-Scholes-Merton model

The basis of the Black-Scholes-Merton (BSM) model is that stock prices satisfy the **log-normal** model

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad t \geq 0, \tag{5}$$

where r is the risk-free interest rate, σ is the volatility of the stock, and W_t is a Wiener process (under the *risk-neutral measure*). Consider a European call option for this stock with strike price K and exercise time T. Its pay-off at time T is $V(S_T, T) = [S_T - K]^+ \doteq \max\{S_T - K, 0\}$. The value of the call option is simply the discounted expected value of V_T

$$V(S_t, t) = e^{-r(T-t)}\mathbb{E}\left[[S_T - K]^+\right]. \tag{6}$$

The Black-Scholes-Merton PDE and formula

Since $dS_t = rS_t dt + \sigma S_t dW_t$, by the Feynman-Kac formula, V must satisfy the **Black-Scholes-Merton partial differential equation**

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 s^2 \dfrac{\partial^2 V}{\partial s^2} + rs\dfrac{\partial V}{\partial s} - rV = 0, \quad t \in [0,T) \\ V(s,T) = [s-K]^+. \end{cases} \tag{7}$$

We can apply a series of changes of variables to transform into the heat equation, and explicitly obtain the **Black-Scholes-Merton formula**

$$V(S,t) = S_t\Phi(d_+) - e^{-r\tau}K\Phi(d_-),\ d_\pm = \frac{\ln\frac{S_t}{K} + \left(r \pm \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}},\ \tau = T-t. \tag{8}$$

We can also obtain the BSM formula via direct evaluation of (6).

References

[1] Black, F. and Scholes, M. (1973) “The Pricing of Options and Corporate Liabilities.” *Journal of Political Economy* 81, no.3: 637-54. https://doi.org/10.1086/260062.

[2] Himonas, A. and Cosimano T. (2023) *Mathematical Methods in Finance and Economics*. University of Notre Dame.

[3] Hull, J. C. (2022) *Options, Futures and Other Derivatives*. 11th Edition, Pearson, New York.

[4] Nicolaescu, L. (2021) *A Graduate Course in Probability*. University of Notre Dame.

[5] Øksendal, B. (2003) *Stochastic Differential Equations: An Introduction with Applications*. 6th Edition, Springer.

[6] Tabor, D. (1991) *Gases, Liquids and Solids: And Other States of Matter*. 3rd Edition, Cambridge University Press.