

# SOLVING THE BLACK-SCHOLES-MERTON EQUATION

YUTAN ZHANG, ALEX HIMONAS

## 1. THE BLACK-SCHOLES-MERTON MODEL

In this presentation we formulate the Black-Scholes-Merton model for European call options, and find the solution to the BSM PDE. A European call option is a financial contract that gives the owner the right but not the obligation to buy a share of stock at a specified price  $K$  that can only be exercised at future date  $T$  specified in the contract. The predetermined price  $K$  is called the *strike price*, and the time  $T$  at which the option can be exercised can be referred to as *maturity date*. Since the owner has the right but not the obligation to exercise the option, the payoff of the option at time  $T$  is  $V(S_T, T) = [S_T - K]^+ \doteq \max\{S_T - K, 0\}$ , since when  $S_T < K$ , the owner would let the option expire. We wish to find the fair price for the European call option  $V(S_t, t)$  at times  $t \in [0, T]$ .

For the BSM model, we assume that the price of a stock  $S_t$  satisfies the following log-normal stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad t \geq 0, \quad (1.1)$$

where  $r$  is the risk-free interest rate (such as government bond return),  $\sigma$  the volatility of the stock price, and  $W_t$  a Wiener process (Brownian motion) starting at zero. We consider a European call option for this stock with strike price  $K$  and exercise time  $T$ .

The main idea of Black, Scholes, and Merton was to think of  $V(s, t)$  as a *deterministic* quantity that depends only on the (assumed) stock price at that time  $s = S_t$  and the time remaining until maturity date  $T - t$ . Indeed,  $V(S_t, t)$  is the stochastic process because of  $S_t$ , and  $V(s, t)$  is deterministic upon  $s$  and  $T - t$ .

By replicating portfolio and Itô's Lemma (see [2]), we can show that  $V(s, t)$  satisfies the Black-Scholes-Merton partial differential equation

$$\boxed{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} + rs \frac{\partial V}{\partial s} - rV = 0, \quad t \in [0, T], \quad s > 0.} \quad (1.2)$$

It is also subject to the terminal condition (for call option payoff at time  $T$ )

$$\boxed{V(s, T) \doteq \lim_{t \rightarrow T^-} V(s, t) = [s - K]^+.} \quad (1.3)$$

## 2. SOLVING THE BSM EQUATION

Here we present a different approach than in [2]. To solve system (1.2, 1.3), we first note that  $\tau = T - t$  is a more natural variable than  $t$  since the value of the option is determined backwards

from time  $T$  to time 0, and it makes (1.3) into an initial condition. Then (1.2) becomes

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} + rs \frac{\partial V}{\partial s} - rV, \quad \tau \in [0, T]. \quad (2.1)$$

We also note that the right hand side is Cauchy-Euler (equi-dimensional) in the variable  $s$ , so we make the change of variable  $x = \ln s$  and  $V(s, \tau) = G(x, \tau)$ . Thus

$$\frac{\partial V}{\partial s} = \frac{1}{s} \frac{\partial G}{\partial x} \text{ and } \frac{\partial^2 V}{\partial s^2} = \frac{1}{s^2} \left( \frac{\partial^2 G}{\partial x^2} - \frac{\partial G}{\partial x} \right).$$

Substituting  $V(x, \tau)$  with  $G(x, \tau)$  into (2.1) we get

$$\frac{\partial G}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 G}{\partial x^2} + \left( r - \frac{1}{2}\sigma^2 \right) \frac{\partial G}{\partial x} - rG. \quad (2.2)$$

We see that (2.2) is linear constant coefficient and parabolic. To get rid of  $-rG$ , we make the change of variable  $G(x, \tau) = e^{-r\tau}U(x, \tau)$ . Thus

$$\frac{\partial G}{\partial \tau} = \frac{\partial U}{\partial \tau} - rG \Rightarrow \frac{\partial U}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 U}{\partial x^2} + \left( r - \frac{1}{2}\sigma^2 \right) \frac{\partial U}{\partial x}. \quad (2.3)$$

To get rid of  $(r - \sigma^2/2)U_x$ , we make the change of variable  $x \rightarrow \xi = x + (r - \sigma^2/2)\tau$ , so  $u(\xi(x, \tau), \tau) = U(x, \tau)$ . So we have

$$\frac{\partial U}{\partial x} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} = \frac{\partial u}{\partial \xi}, \quad \frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2}, \quad \frac{\partial U}{\partial \tau} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial \tau} + \frac{\partial u}{\partial \tau} = \left( r - \frac{1}{2}\sigma^2 \right) \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \tau}.$$

Substituting back into (2.3), the terms with  $u_\xi$  cancel, and we get

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial \xi^2}.$$

To rescale the factor  $\sigma^2/2$ , we let  $\eta = \sigma^2\tau/2$ , so  $u_\tau = u_\eta\sigma^2/2$ , we get

$$\frac{\partial u}{\partial \eta} = \frac{\partial^2 u}{\partial \xi^2}. \quad (2.4)$$

Therefore, we have the following change of variables

$$\begin{cases} u(\xi, \eta) = e^{r\tau}V(s, t) \\ \eta(\tau) = \frac{\sigma^2}{2}\tau \\ \xi(s, \tau) = \ln s + \left( r - \frac{\sigma^2}{2} \right) \tau \\ \tau(t) = T - t \end{cases}$$

and (1.2) is transformed into the following standard heat equation form

$$\frac{\partial u}{\partial \eta} - \frac{\partial^2 u}{\partial \xi^2} = 0. \quad (2.5)$$

It is subjected to the new initial condition determined by (1.3)

$$u(\xi, 0) = \lim_{\eta \rightarrow 0^+} e^{r\tau}V(s, t) = [e^\xi - K]^+. \quad (2.6)$$

Condition (2.6) is obtained since  $\eta \rightarrow 0^+$  implies  $\tau \rightarrow 0^+$ , and  $\ln s = \xi - (r - \sigma^2/2)\tau$ , so we have  $s = e^\xi \exp(-(r - \sigma^2/2)\tau) \rightarrow e^\xi$ . We solve the system (2.5, 2.6) using the following Theorem.

**Theorem 2.1** (Solution of Heat equation [1]). *Consider the system of equations*

$$\partial_t u - \partial_x^2 u = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad (2.7)$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}. \quad (2.8)$$

If  $u_0(x) \in L^1(\mathbb{R})$ , then the system admits a unique solution  $u(x, t)$  given by

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{1}{4t}(x-y)^2} u_0(y) dy. \quad (2.9)$$

In fact, if  $\exp(-x^2)u_0(x) \in L^1(\mathbb{R})$ , (2.9) remains the solution to the system.

Applying Theorem 2.1, we have the solution for system (2.5, 2.6)

$$\begin{aligned} u(\xi, \eta) &= \frac{1}{\sqrt{4\pi\eta}} \int_{\mathbb{R}} e^{-\frac{1}{4\eta}(\xi-y)^2} [e^y - K]^+ dy \\ &= \frac{1}{\sqrt{4\pi\eta}} \int_{\ln K}^{\infty} (e^y - K) \exp\left[-\frac{(y-\xi)^2}{4\eta}\right] dy \\ &= \frac{1}{\sqrt{4\pi\eta}} \int_{\ln K}^{\infty} \exp\left[y - \frac{(y-\xi)^2}{4\eta}\right] dy - \frac{1}{\sqrt{4\pi\eta}} \int_{\ln K}^{\infty} K \exp\left[-\frac{(y-\xi)^2}{4\eta}\right] dy. \end{aligned}$$

Since the model is based on standard normal distribution, it is natural to express the solution in the form of  $N(x)$ , where

$$N(x) \doteq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy.$$

We examine the term inside the first exponential.

$$\begin{aligned} y - \frac{(y-\xi)^2}{4\eta} &= \frac{4\eta y - (y-\xi)^2}{4\eta} - (\xi + \eta) + (\xi + \eta) \\ &= \frac{4(y-\xi)\eta - (y-\xi)^2 - 4\eta^2}{4\eta} + (\xi + \eta) \\ &= -\frac{(y-\xi-2\eta)^2}{4\eta} + (\xi + \eta). \end{aligned}$$

Let  $p = \frac{y-\xi-2\eta}{\sqrt{2\eta}}$ , so  $dp = \frac{dy}{\sqrt{2\eta}}$ , the first term becomes

$$\begin{aligned} \frac{e^{\xi+\eta}}{\sqrt{4\pi\eta}} \int_{\ln K}^{\infty} \exp\left[-\frac{(y-\xi-2\eta)^2}{4\eta}\right] dy &= \frac{e^{\xi+\eta}}{\sqrt{2\pi}} \int_{\frac{\ln(K)-\xi-2\eta}{\sqrt{2\eta}}}^{\infty} e^{-\frac{p^2}{2}} dp \\ &= \frac{e^{\xi+\eta}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\xi+2\eta-\ln K}{\sqrt{2\eta}}} e^{-\frac{p^2}{2}} dp = e^{\xi+\eta} N\left(\frac{\xi+2\eta-\ln K}{\sqrt{2\eta}}\right). \end{aligned}$$

Similarly (more easily) for the second term, let  $q = \frac{y-\xi}{\sqrt{2\eta}}$ , so  $dq = \frac{dy}{\sqrt{2\eta}}$ , we have

$$\begin{aligned} -\frac{K}{\sqrt{4\pi\eta}} \int_{\ln K}^{\infty} \exp\left(-\frac{(y-\xi)^2}{4\eta}\right) dy &= -\frac{K}{\sqrt{2\pi}} \int_{\frac{\ln(K)-\xi}{\sqrt{2\eta}}}^{\infty} e^{-\frac{q^2}{2}} dq \\ &= -\frac{K}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\xi-\ln K}{\sqrt{2\eta}}} e^{-\frac{q^2}{2}} dq = -KN\left(\frac{\xi-\ln K}{\sqrt{2\eta}}\right). \end{aligned}$$

Therefore, we have

$$u(\xi, \eta) = e^{\xi+\eta} N\left(\frac{\xi+2\eta-\ln K}{\sqrt{2\eta}}\right) - KN\left(\frac{\xi-\ln K}{\sqrt{2\eta}}\right).$$

Changing back from  $u(\xi, \eta)$  to  $V(s, \tau)$  via the change of variables, we have

$$\begin{aligned} V(s, \tau) &= \exp\left[-r\tau + \ln(s) + \left(r - \frac{\sigma^2}{2}\right)\tau + \frac{\sigma^2}{2}\tau\right] N\left(\frac{\ln s + \left(r - \frac{\sigma^2}{2}\right)\tau + \sigma^2\tau - \ln K}{\sigma\sqrt{\tau}}\right) \\ &\quad - e^{-r\tau} KN\left(\frac{\ln s + \left(r - \frac{\sigma^2}{2}\right)\tau - \ln K}{\sigma\sqrt{\tau}}\right) \\ V(s, \tau) &= sN\left(\frac{\ln(s/K) + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right) - e^{-r\tau} KN\left(\frac{\ln(s/K) + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right). \end{aligned}$$

Therefore, we arrive at the solution to the system (1.2, 1.3) known as the BSM formula

$$\boxed{V(s, \tau) = sN(d_+) - e^{-r\tau} KN(d_-), \quad d_{\pm} = \frac{\ln(s/K) + \left(r \pm \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \quad \tau = T - t.} \quad (2.10)$$

We can see that (2.10) satisfies the terminal condition (1.3) as

$$\lim_{t \rightarrow T^-} d_{\pm} = \lim_{\tau \rightarrow 0^+} d_{\pm} = \begin{cases} +\infty, & s > K \\ 0, & s = K, \text{ and } \lim_{\tau \rightarrow 0^+} e^{-r\tau} = 1, \\ -\infty, & s < K \end{cases}$$

therefore

$$V(s, T) \doteq \lim_{t \rightarrow T^-} V(s, t) = \begin{cases} sN(\infty) - KN(\infty) = s - K, & s > K \\ sN(0) - KN(0) = (s - K)/2 = 0, & s = K \\ sN(-\infty) - KN(-\infty) = 0, & s < K \end{cases} = [s - K]^+.$$

## REFERENCES

- [1] A. Himonas. *Lecture Notes in Partial Differential Equations*. Lecture Notes, 2023, University of Notre Dame.
- [2] A. Himonas and T. Cosimano. *Mathematical Methods in Finance and Economics*. Lecture Notes (eBook), 2023, University of Notre Dame.