INTRODUCTION TO THE FEYNMAN-KAC FORMULA

YUTAN ZHANG

Abstract. This paper provides a brief introduction to the Feynman-Kac formula. We start with an intuitive connection between diffusion processes and solution to the heat equation and provide sufficient probability background. We then derive the Feynman-Kac formula in different cases and generalities. We finish by showing its applications in option pricing.

Contents

1. The heat and diffusion equation

The heat equation is a well-known parabolic partial differential equation (PDE) which has been studied since Joseph Fourier. It takes the form $\partial_t u - \Delta u = 0$, where Δu is the Laplacian. For the Cauchy problem (also known as initial value problem) to the heat equation on $\mathbb{R}^n \times \mathbb{R}^+$

$$\begin{cases} \partial_t u - \Delta u = 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ u(x,0) = f(x) \in H^s(\mathbb{R}^n), \end{cases} \tag{1.1}$$

its solution can be obtained via Fourier transform and is given by

$$u(x,t) = \int_{\mathbb{R}^n} K(x-y,t) f(y) dy, \tag{1.2}$$

where

$$K(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} \tag{1.3}$$

is called the fundamental solution (heat kernel) to the heat equation. Note that the heat kernel (1.3) is nothing but the probability density function of the normal (Gaussian) distribution. In fact, equation (1.1) is also known as the diffusion equation, where u is the probability density function of a particle diffusing in a medium. The equation describing two phenomena coincide precisely because heat is

Date: December 12, 2024 .

nothing but the diffusion of thermal energy through the exchange of microscopic particle collisions and other interactions, prime examples of Brownian motion.

In 1827, a botanist by the name of Robert Brown was observing grains of pollen suspended in a fluid and began to notice that they would move erratically. The pollen seemed to change path and move about in a random way. Similar phenomena have been seen throughout history. Nonetheless, the person who is credited with the true discovery is Robert Brown–hence the name, Brownian motion [6]. In 1900, Louis Bachelier modeled the stochastic process in his doctoral thesis when analyzing the price of options. Albert Einstein derived many PDEs and ODEs to model Brownian motion and came up with the following result, that the probability that a particle is in a given interval $[a, b]$ is given by

$$\mathbb{P}(a \leq B_t \leq b) = \frac{1}{\sqrt{2\pi t}} \int_a^b e^{-x^2/2t} dx.$$

We have already noted that this definition is similar to the normal distribution with $\sigma = 1$. The only difference is the time dependence of this function. One can see that as time increases, the particles spread out more and more, yet always remain in a normal distribution. Additionally, in this model all the particles are starting at $x = 0$.

Norbert Wiener made advances in the formalization of stochastic processes and stochastic calculus, and proved the existence of Brownian motion. This field was further advanced by Kiyoshi Itô, who created a method for handling stochastic differential equations. Richard Feynamn and Mark Kac established the link between parabolic PDEs (like the heat equation) and stochastic processes through a representation of solutions now known as the **Feynman-Kac formula**. This has profound applications in quantum mechanics and financial mathematics, which we will also briefly discuss.

2. Probability Preliminaries

In this section we state the mathematical foundations that will be needed for the derivation of the Feynman-Kac formula. To keep the presentation short and self-contained, we will not go through the axiomatic setup and prove the existence of Wiener process, one type of Brownian motion. We will give definitions and provide necessary preliminaries to understand important results of the Itô integral and Itô's formula. For background, rigorous proofs, and examples, see [2, 4, 5].

Definition 2.1. *A* ***Wiener process*** $\{W(t)\}_{t\geq 0}$ *is a family of random variables* $W(t) : \Omega \to \mathbb{R}$, *where* Ω *is a probability space with measure* $\mathbb{P}$ *satisfying the following properties:*

1. $W(0) = 0$.
2. $\forall \omega \in \Omega$, *the map* $t \mapsto W(\omega, t)$ *is continuous from* $[0, \infty)$ *into* $\mathbb{R}$ *with probability one.*
3. $\forall t > s \geq 0$, $W(t) - W(s)$ *is normally distributed with mean* 0 *and variance* $t - s$, *i.e.,*

$$\mathbb{P}(a \leq W(t) - W(s) \leq b) = \frac{1}{\sqrt{2\pi(t-s)}} \int_a^b e^{-x^2/2(t-s)} dx.$$

4. $\forall 0 = t_0 < t_1 < t_2 < \ldots < t_n$, *the increments* $W(t_{i+1}) - W(t_i)$ *are independent random variables.*

Remark 2.2. $W(t)$ is also denoted by W_t. The notation $W(\omega, t)$ and $W_t(\omega)$ stress that for each t, it is a random variable defined on a sample space Ω.

Remark 2.3. The sample paths are continuous but nowhere differentiable, as seen by

$$\mathbb{E}\left[\left(\frac{W(t+\Delta t) - W(t)}{\Delta t}\right)^2\right] = \frac{\Delta t}{(\Delta t)^2} = \frac{1}{\Delta t} \xrightarrow{\Delta t \downarrow 0} \infty.$$

We provide a brief construction of Wiener process as the limit of random walks. Let $\{X_j\}_j$ be independent identically distributed random variables with $\mathbb{P}(X_j = 1) = \mathbb{P}(X_j = -1) = 1/2$, so that $\mathbb{E}[X_j] = 0$ and $\text{var}[X_j] = 1$. Let Δx and Δt be positive. Define $S_0 = 0$ and

$$S_{k\Delta t} = \sum_{j=1}^{k} X_j \Delta x,$$

and linearly interpolate in time. So S_t is a random walk with properties $\mathbb{E}[S_{k\Delta t}] = 0$,

$$\text{var}[S_{k\Delta t}] = \frac{(\Delta x)^2}{\Delta t} k\Delta t,$$

and increments of Δt are independent random variables. We want the variance at time t to equal t, i.e. $(\Delta x)^2/\Delta t = 1$. For each $n \in \mathbb{Z}^+$, let $\Delta t = 1/n$ and $\Delta x = 1/\sqrt{n}$, and we define $\{S_t^n\}_t$ as above. Explicitly writing out the interpolation we have

$$\begin{aligned}
S_t^n &= (\lfloor nt \rfloor + 1 - nt) S^n_{\lfloor nt \rfloor/n} + (nt - \lfloor nt \rfloor) S^n_{(\lfloor nt \rfloor + 1)/n} \\
&= S^n_{\lfloor nt \rfloor/n} + (nt - \lfloor nt \rfloor) \left(S^n_{(\lfloor nt \rfloor + 1)/n} - S^n_{\lfloor nt \rfloor/n} \right) \\
&= \frac{\sqrt{\lfloor nt \rfloor}}{\sqrt{n}} \frac{1}{\sqrt{\lfloor nt \rfloor}} \sum_{j=1}^{\lfloor nt \rfloor} X_j + \frac{nt - \lfloor nt \rfloor}{\sqrt{n}} X_{\lfloor nt \rfloor + 1}.
\end{aligned}$$

Note that

$$\frac{\sqrt{\lfloor nt \rfloor}}{\sqrt{n}} \to \sqrt{t} \quad \text{and} \quad \frac{nt - \lfloor nt \rfloor}{\sqrt{n}} X_{\lfloor nt \rfloor + 1} \to 0,$$

and by the central limit theorem we have

$$\frac{1}{\sqrt{\lfloor nt \rfloor}} \sum_{j=1}^{\lfloor nt \rfloor} X_j \xrightarrow{d} Z \sim \mathcal{N}(0, 1).$$

Therefore we have

$$S_t^n \xrightarrow{d} \sqrt{t} Z = W_t$$

where W_t satisfies all properties in Definition 2.1.

Next we give the preliminaries to understand Itô's formula, the stochastic calculus counterpart of the chain rule.

Definition 2.4. *A **simple stochastic process** is a function of the form*

$$f(\omega, t) = \sum_{j=0}^{n-1} \alpha_j(\omega) \mathbf{1}_{[t_j, t_{j+1})}(t),$$

where $0 = t_0 < t_1 < \ldots < t_n = T$, $\alpha_j(\omega)$ random variables, and $\mathbf{1}_{[a,b)}(t) = 1$ if $a \leq t < b$ and else 0.

Definition 2.5. *The **Itô integral** for a **simple** stochastic process $f(\omega, t)$ is defined by*

$$\int_0^T f(\omega, t) dW_t = \sum_{j=0}^{n-1} \alpha_j(\omega)(W_{t_{j+1}}(\omega) - W_{t_j}(\omega)).$$

Remark 2.6. For simple processes this is very similar to how we define the Riemann-Stieltjes integral. But note that the following procedure for general processes is similar to how we construct the Lebesgue integral, i.e. taken as an appropriate limit of simple functions.

Definition 2.7. *The **Itô integral** for a **general** stochastic process $f(\omega, t)$ that is square integrable*

$$\mathbb{E}\left[\int_0^T f^2(\omega,t)dt\right] < \infty$$

is given by

$$\int_0^T f(\omega,t)dW_t(\omega) \overset{L^2(\Omega)}{=} \lim_{n\to\infty}\int_0^T f_n(\omega,t)dW_t(\omega),$$

where $f_n(\omega, t)$ is a sequence of simple stochastic processes converging to $f(\omega, t)$ in $L^2(\Omega)$.

Proposition 2.8. *For a square integrable stochastic process $f(\omega, t)$, we have*

$$\mathbb{E}\left[\int_0^T f(\omega,t)dW_t(\omega)\right] = 0.$$

Proof. This is true for simple stochastic processes by property 3 of Definition 2.1 of mean zero. Then pass to the L^2 limit. □

Example 2.9. Let $f(\omega, t) = W_t(\omega)$ be a Wiener process, then

$$\int_0^T W_t(\omega)dW_t(\omega) = \frac{1}{2}W_t^2(\omega) - \frac{1}{2}T.$$

Remark 2.10. Note the similarity between the above example and the following Riemann integral

$$\int_0^T tdt = \frac{1}{2}T^2.$$

Definition 2.11. *An **Itô process** is a stochastic process X_t on Ω of the form*

$$X_t = X_0 + \int_0^t u(\omega,s)ds + \int_0^t v(\omega,s)dW_s,$$

where $v(\omega, s)$ satisfies the square integrability condition

$$\mathbb{E}\left[\int_0^t v^2(\omega,s)ds\right] < \infty.$$

*The stochastic integral is also written as a **stochastic differential equation** of the form*

$$dX_t = udt + vdW_t.$$

Proposition 2.12 (Itô's formula)**.** *Let X_t be an Itô process*

$$dX_t = udt + vdW_t,$$

and $g(x,t) \in C^2(\mathbb{R}\times[0,T])$. Then $Y_t = g(X_t, t)$ is also an Itô process with

$$dY_t = \frac{\partial g}{\partial t}(X_t,t)dt + \frac{\partial g}{\partial x}(X_t,t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(X_t,t)(dX_t)^2$$

where $(dt)^2 = dtdW_t = dW_t dt = 0$ and $(dW_t)^2 = dt$.

Here we only give an intuitive and formal derivation of Itô's formula. For a rigorous proof, see [5, Theorem 4.1.2, pp.44-48]. We write out the total derivative of g and the Taylor expansions

$$\begin{aligned} dg &= \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial t}dt = \lim_{\Delta x\downarrow 0} g(x+\Delta x,t) - g(x,t) + \lim_{\Delta t\downarrow 0} g(x,t+\Delta t) - g(x,t) \\ &= \lim_{dx,dt}\frac{\partial g}{\partial x}dx + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dx)^2 + \frac{\partial g}{\partial t}dt + \frac{1}{2}\frac{\partial^2 g}{\partial t^2}(dt)^2 + R, \end{aligned}$$

where R is the Taylor remainder term. Substitute $x = X_t$ and $Y_t = g(X_t, t)$ so that $dx = dX_t$ we get

$$\begin{aligned} dY_t = \lim_{dx,dt} \frac{\partial g}{\partial x}(udt + vdW_t) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(u^2(dt)^2 + 2uvdtdW_t + v^2(dW_t)^2) \\ + \frac{\partial g}{\partial t}dt + \frac{1}{2}\frac{\partial^2 g}{\partial t^2}(dt)^2 + R. \end{aligned}$$

Now $(dt)^2$ and $dtdW_t$ goes to zero faster than dt, so we set them to be zero. $(dW_t)^2 = dt$ is obtained by property 3 of Definition 2.1. Assuming the remainder R also goes to zeroas we take the limit, we get

$$dY_t = \left(\frac{\partial g}{\partial t} + u\frac{\partial g}{\partial x} + \frac{1}{2}v^2\frac{\partial^2 g}{\partial x^2}\right)dt + v\frac{\partial g}{\partial x}dW_t$$

as desired.

3. The Feynman-Kac Formula

We are now ready to present and prove several versions of the Feynman-Kac formula.

3.1. For the heat equation with cooling.

Theorem 3.1 (Feynman-Kac formula). *Let $q(x)$ be a non-negative continuous function, and $f(x)$ be bounded continuous. Suppose $u(x,t)$ is a bounded function that solves the Cauchy problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} = -q(x)u, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+ \\ u(x,0) = f(x) \in C_b(\mathbb{R}). \end{cases} \tag{3.2}$$

Then

$$u(x,t) = \mathbb{E}\left[\exp\left\{-\int_0^t q(X_s)ds\right\}f(X_t)\right] \tag{3.3}$$

where $dX_t = dW_t$ and $X_0 = x$.

Proof. Fix $t > 0$. Consider the stochastic process

$$Y_s = g(X_s, s) = \exp\left\{-\int_0^s q(X_\tau)d\tau\right\}u(X_s, t-s).$$

Since u is a solution to (3.2), with the non-negativity of q and boundedness of u, we have $g \in C^2$. By Itô's formula, we have

$$\begin{aligned} dY_s &= \left(\frac{\partial g}{\partial s} + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}\right)ds + \frac{\partial g}{\partial x}dW_s \\ &= -q(X_s)u(X_s, t-s)\exp\left\{-\int_0^s q(X_\tau)d\tau\right\}ds + \frac{\partial u}{\partial s}(X_s, t-s)\exp\left\{-\int_0^s q(X_\tau)d\tau\right\}ds \\ &\quad + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(X_s, t-s)\exp\left\{-\int_0^s q(X_\tau)d\tau\right\}ds + \frac{\partial u}{\partial x}(X_s, t-s)\exp\left\{-\int_0^s q(X_\tau)d\tau\right\}dW_s. \end{aligned}$$

Since u satisfies (3.2), we have $u_s + u_{xx}/2 - qu = -u_t + u_{xx}/2 - qu = 0$, so the ds terms sum up to zero. Thus

$$dY_s = \frac{\partial u}{\partial x}(X_s, t-s)\exp\left\{-\int_0^s q(X_\tau)d\tau\right\}dW_s.$$

Integrate both sides from zero to t, we get

$$Y_t - Y_0 = \int_0^t \frac{\partial u}{\partial x}(X_s, t-s)\exp\left\{-\int_0^s q(X_\tau)d\tau\right\}dW_s.$$

Taking expectations, by Proposition 2.8,

$$\mathbb{E}\left[\int_0^t \frac{\partial u}{\partial x}(X_s, t-s)\exp\left\{-\int_0^s q(X_\tau)d\tau\right\}dW_s\right] = 0,$$

so we have

$$\mathbb{E}[Y_t] = \mathbb{E}[Y_0] = \mathbb{E}[u(X_0, t)] = u(x, t).$$

Note that

$$Y_t = \exp\left\{-\int_0^t q(X_\tau)d\tau\right\}u(X_t, 0),$$

therefore

$$u(x,t) = \mathbb{E}\left[\exp\left\{-\int_0^t q(X_s)ds\right\}f(X_t)\right].$$

□

3.2. **Slight generalization.**

Theorem 3.4 (Feynman-Kac formula). *Let $q(x)$ be a non-negative continuous function, and $f(x)$, $\mu(x)$, $\sigma(x)$ be bounded continuous. Suppose $u(x,t)$ is a bounded function that solves the Cauchy problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mu(x)\dfrac{\partial u}{\partial x} - \dfrac{1}{2}\sigma^2(x)\dfrac{\partial^2 u}{\partial x^2} + q(x)u = 0, \\ u(x,0) = f(x) \in C_b(\mathbb{R}). \end{cases} \tag{3.5}$$

Then

$$u(x,t) = \mathbb{E}\left[\exp\left\{-\int_0^t q(X_s)ds\right\}f(X_t)\right] \tag{3.6}$$

where $dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$ and $X_0 = x$.

Proof. Similar to before, consider

$$Y_s = g(X_s, s) = \exp\left\{-\int_0^s q(X_\tau)d\tau\right\}u(X_s, t-s).$$

By Itô's formula, we have

$$\begin{aligned} dY_s &= \left(\frac{\partial g}{\partial s} + \frac{1}{2}\sigma^2\frac{\partial^2 g}{\partial x^2}\right)ds + \frac{\partial g}{\partial x}dX_s \\ &= -q(X_s)u(X_s, t-s)\exp\left\{-\int_0^s q(X_\tau)d\tau\right\}ds - \frac{\partial u}{\partial t}(X_s, t-s)\exp\left\{-\int_0^s q(X_\tau)d\tau\right\}ds \\ &\quad + \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2}(X_s, t-s)\exp\left\{-\int_0^s q(X_\tau)d\tau\right\}ds + \mu\frac{\partial u}{\partial x}(X_s, t-s)\exp\left\{-\int_0^s q(X_\tau)d\tau\right\}ds \\ &\quad + \sigma(X_s)\frac{\partial u}{\partial x}(X_s, t-s)\exp\left\{-\int_0^s q(X_\tau)d\tau\right\}dW_s. \end{aligned}$$

The ds terms sum up to zero again because of (3.5), so we have

$$dY_s = \sigma(X_s)\frac{\partial u}{\partial x}(X_s, t-s)\exp\left\{-\int_0^s q(X_\tau)d\tau\right\}dW_s.$$

The rest follows exactly as the proof of Theorem 3.1. □

3.3. **Generalization in higher dimensions.**

Most of the theory and proof can be generalized to higher dimensions easily with some linear algebra. Here we will only show the main ideas and results. Readers interested in the rigorous setup are strongly encouraged to consult [5].

Proposition 3.7 (Generalized Itô's formula)**.** *Let X_t be be an n-dimensional Itô process of the form*

$$d\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1m} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nm} \end{pmatrix} \begin{pmatrix} dW_1 \\ \vdots \\ dW_m \end{pmatrix}.$$

Let $g(x,t) = (g_1(x,t), \dots, g_p(x,t))$ be a map from $C^2(\mathbb{R}^n \times \mathbb{R}^+])$ into $\mathbb{R}^p$. Then $Y_t = g(X_t, t)$ is also an Itô process, whose components Y_k, $1 \le k \le p$, is given by

$$dY_k = \frac{\partial g_k}{\partial t}(X,t)dt + \sum_i \frac{\partial g_k}{\partial x_i}(X,t)dX_i + \frac{1}{2}\sum_{i,j} \frac{\partial^2 g_k}{\partial x_i \partial x_j}(X,t)dX_i dX_j$$

where $(dt)^2 = dtdW_t = dW_t dt = 0$ and $dW_i dW_j = \delta_{ij}dt$, and δ_{ij} is the Kronecker delta.

Definition 3.8. *A (time-homogeneous)* ***Itô diffusion*** *is a stochastic process $X(\omega, t) : \Omega \times \mathbb{R}^+ \to \mathbb{R}^n$ satisfying a stochastic differential equation of the form*

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x,$$

where W_t is m-dimensional Wiener process, $\mu : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and together need to be Lipschitz continuous, i.e., $|\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| \le C|x - y|$. See [5, Theorem 5.2.1].

Theorem 3.9. *Let X_t be an Itô diffusion in $\mathbb{R}^n$ given by Definition 3.8. If $f \in C_0^2(\mathbb{R}^n)$, then there is a* ***generator*** *$\mathcal{A}$ (see* [5, Definition 7.3.1]*) given by*

$$\mathcal{A} = \sum_i \mu_i(x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j} (\sigma\sigma^T)_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j}.$$

Proof. See [5, Lemma 7.3.2, pp.118-119]. □

Theorem 3.10 (The Feynman-Kac formula)**.** *Let X_t be an Itô diffusion in $\mathbb{R}^n$ with generator $\mathcal{A}$. Let $f \in C_0^2(\mathbb{R}^n)$ and $q \in C(\mathbb{R}^n)$, with q being lower bounded. Consider*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathcal{A}u - qu, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ u(x,0) = f(x). \end{cases} \tag{3.11}$$

If $u(x,t) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R})$ is bounded on $K \times \mathbb{R}^n$ for every compact $K \subset \mathbb{R}$ and u solves (3.11)*, then*

$$u(x,t) = \mathbb{E}\left[\exp\left\{-\int_0^t q(X_s)ds\right\} f(X_t)\right]. \tag{3.12}$$

Remark 3.13. The above theorem is part (b) of Theorem 8.2.1 from [5, pp.137-139]. Part (a) states that given formula (3.12), there exists an associated PDE given by (3.11) such that (3.12) is a solution of, which is the opposite direction. In other words, it establishes a bilateral connection between the "average of" stochastic processes and parabolic PDEs.

4. Heuristic Interpretation and Applications

4.1. **Heuristics of the heat equation version.** We recall the formula (3.3)

$$u(x,t) = \mathbb{E}\left[\exp\left\{-\int_0^t q(X_s)ds\right\} f(X_t)\right]$$

where $dX_t = dW_t$ and $X_0 = x$. If there is no cooling term $q(x)$, the solution at position x is simply the expectation of a Brownian motion starting at x applied to the initial condition f. Roughly speaking, the heat (or particle distribution in a medium) at position x and time t is the average of possible values of the initial condition at the end positions of Brownian motions starting at x. In other words, if particles have an initial distribution f, and we let each particle diffuse freely, the average of all possible end positions of particles at time t is the distribution of particles at time t. If we include the cooling term, $u' = -qu$ has an exponential decay effect on the solution, hence taking the form in (3.3).

A further remark is as follows. If we consider the simple heat equation ivp

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+ \\ u(x,0) = f(x). \end{cases}$$

It is a well-posed equation, i.e. there exists a unique solution. We have two formulations of the solution from the classical PDE side and stochastic side, thus they have to be the same. Explicitly,

$$u(x,t) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} e^{-|x-y|^2/2t} f(y)dy = \mathbb{E}\left[f(X_t)\right], \quad dX_t = dW_t, X_0 = 0.$$

More generally, the expectation translates to the average of end positions of particles which follow the stochastic process X_t. We would naturally expect it to satisfy the PDE that reflects the dynamics of the X_t. This is precisely the result of the Feynman-Kac theorem.

4.2. **The Black-Scholes-Merton model.** This is a paradigmatic application of the Feynman-Kac formula. The basis of the Black-Scholes-Merton (BSM) model is that stock prices satisfy the **log-normal** model

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad t \geq 0, \tag{4.1}$$

where r is the risk-free interest rate, σ is the volatility of the stock, and W_t is a Wiener process (with respect to the *risk-neutral measure*) starting at zero. Consider a European call option for this stock with strike price K and exercise time T. Its pay-off at time T is $V(S_T, T) = [S_T - K]^+ \doteq \max\{S_T - K, 0\}$. Black and Scholes derived the BSM equation and formula for the price of the call option V using continuous hedging argument and CAPM model in their original paper [1]. We can also derive using replicating portfolio or as a limit of the binomial model [2].

Here we will use the Feynman-Kac formula to obtain the BSM equation from the valuation formula. The **risk-neutral measure** is a probability measure such that the expected return of the stock μ is the risk-free rate r. We can choose such a measure because the value of μ depends on the risk preference. The higher the level of risk aversion by investors, the higher μ will be for the stock. Thus, the simple assumption that all investors are risk neutral can be made, and the expected return on all investment assets is the risk-free rate r [3].

With the risk-neutral measure, the present value of any cash flow would just be its discounted expected value at the risk-free rate. In particular, the value of the call option is

$$V(S_t, t) = e^{-r(T-t)}\mathbb{E}\left[[S_T - K]^+\right]. \tag{4.2}$$

We can obtain the explicit formula for $V(S_t, t)$ by straightforward integration against the log-normal distribution, whose cdf is

$$F_X(x) = \Phi\left(\frac{\ln x - \mathbb{E}[X]}{\mathrm{var}[X]}\right)$$

where Φ is the standard normal cdf. Setting $\tau = T - t$ so that τ is the time to maturity, we have

$$\begin{aligned} V(S_t, t) &= e^{-r\tau} \int_K^\infty S_T - K dF(S_T) \\ &= e^{-r\tau} \int_K^\infty S_T dF(S_T) - e^{-r\tau} K \int_K^\infty dF(S_T). \end{aligned}$$

For the first integral, note that at time t the value of S_T follows log-normal distribution with mean $\ln S_t + (r - \sigma^2/2)\tau$ and variance $\sigma^2\tau$. Therefore

$$\begin{aligned}\int_K^\infty S_T dF(S_T) &= \exp\left(\ln S_t + (r - \frac{\sigma^2}{2})\tau + \frac{\sigma^2\tau}{2}\right) \cdot \Phi\left(\frac{-\ln K + \ln S_t + (r - \frac{\sigma^2}{2})\tau + \sigma^2\tau}{\sigma\sqrt{\tau}}\right)\\ &= e^{r\tau} S_t \Phi(d_+),\end{aligned}$$

where

$$d_+ = \frac{\ln(S_t/K) + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}.$$

The second integral is easier

$$\begin{aligned}\int_K^\infty dF(S_T) &= 1 - \Phi\left(\frac{\ln K - \ln S_t - (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\right)\\ &= 1 - \Phi(-d_-)\\ &= \Phi(d_-),\end{aligned}$$

where

$$d_- = \frac{\ln(S_t/K) + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}.$$

Combining the terms, we get the **Black-Scholes-Merton pricing formula**

$$V(S_t, t) = S_t\Phi(d_+) - e^{-r\tau} K\Phi(d_-). \tag{4.3}$$

Apply the Feynman-Kac theorem (specifically Theorem 3.4) to (4.1) and (4.2), we see that $V(s,t)$, where $s = S_t$ is the stock price at time t, must satisfy the ivp

$$\begin{cases}\dfrac{\partial V}{\partial \tau} - \dfrac{1}{2}\sigma^2 s^2 \dfrac{\partial^2 V}{\partial s^2} - rs\dfrac{\partial V}{\partial s} + rV = 0, \quad \tau \in (0, T]\\ V(s, \tau = 0) = [s - K]^+.\end{cases}$$

Substituting τ back to t we get the **Black-Scholes-Merton partial differential equation**

$$\begin{cases}\dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 s^2 \dfrac{\partial^2 V}{\partial s^2} + rs\dfrac{\partial V}{\partial s} - rV = 0, \quad t \in [0, T)\\ V(s, t = T) = [s - K]^+.\end{cases} \tag{4.4}$$

References

[1] Black, F. and Scholes, M. (1973) "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy* 81, no.3: 637-54. `https://doi.org/10.1086/260062`.

[2] Himonas, A. and Cosimano T. (2023) *Mathematical Methods in Finance and Economics*. University of Notre Dame.

[3] Hull, J. C. (2022) *Options, Futures and Other Derivatives*. 11th Edition, Pearson, New York.

[4] Nicolaescu, L. (2021) *A Graduate Course in Probability*. University of Notre Dame.

[5] Øksendal, B. (2003) *Stochastic Differential Equations: An Introduction with Applications*. 6th Edition, Springer.

[6] Tabor, D. (1991) *Gases, Liquids and Solids: And Other States of Matter*. 3rd Edition, Cambridge University Press.

Yutan Zhang, Department of Mathematics, Indiana University Bloomington, Bloomington, IN 47405
Email address: `zhangyut@iu.edu`