History, Derivation, and Solution to the Black-Scholes-Merton PDE

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I. INTRODUCTION

A. Options

- An **option contract** is simply the right to buy or sell an amount of an asset, usually stock, at a certain price at a later date

- Options have an expiration date; for European-style options described by the BSM model, the options must be "exercised", or redeemed, on this date (and not before).

- Options are sold by a "writer" to a "holder". The holder pays a fee, called a "premium", to the writer in exchange for the right either purchase or sell the underlying at a certain price, called the "strike price." This is often determined with reference to the current trading price of the underlying itself, called the "spot price."

- The writer hopes that the spot price of the underlying won't be higher than the strike price plus the premium of the option on expiration date–that is, the call is "out-of-the-money", and expires worthless. If instead the strike price is higher, then the holder can exercise the call to buy the underlying at the strike price, a lower price than the spot price–the call is "in-the-money".

B. History

- **1900** Louis Bachelier analyzed the price of options based on probability theory in his thesis.
- **1923** Norbert Wiener made advances in stochastic calculus and proved the existence of Brownian motion.
- **1944** Kiyoshi Itô created a method for handling stochastic differential equations and created ways to integrate over stochastic processes
- **1967** Edward Thorpe and Sheen Kassouf published a book in which they derived an equation for the value of a financial derivative called a warrant.
- **1945** Paul Samuelson and Robert Merton derived a very similar equation rigorously and mathematically.
- **1968** Fischer Black and Myron Scholes, at the time working at a consulting company with Wells Fargo as a client, derived a differential equation describing the value of options, as well as its solution and additional measures called "The Greeks".
- **1973** Merton (a mathematician) found Black and Scholes assumptions to be too broad and worked with them to derive a similar equation using stochastic calculus and Itō's method which they published.
- **1997** Merton and Scholes recieved the nobel prize in Economics for their work

I. INTRODUCTION

B. History

(a) Fischer Black

(b) Myron Scholes

(c) Robert Merton

Figure 1: Creators of the BSM PDE

Historical use of the BSM Equation

- In the 70s and 80s, the BSM model was used to calculate the value of options and generate returns for traders, who were able to buy undervalued or sell overpriced options.
- Recently, the formula has been inverted, so that the volatility of the underlying asset may be found, called the "implied volatility."
- This contradicts one of the model's assumptions, that volatility describes a feature of the underlying and is independent of other options parameters.
- While this assumption would guarantee a flat volatility (i.e. the same for all strike prices), however, instead it is parabolic, in a phenomenon known as the "volatility smile."

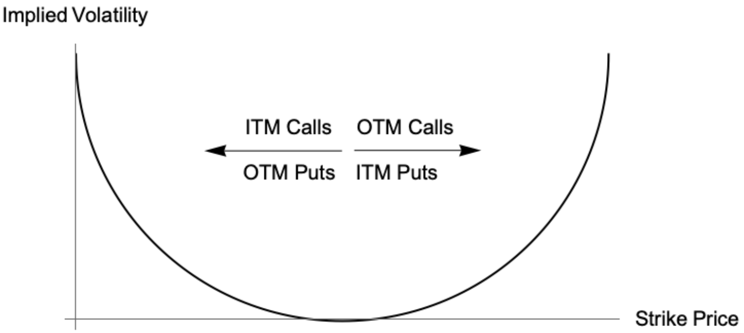

Figure 2: The "volatility smile"; implied volatility increases towards extrema in strike price.

I. INTRODUCTION
D. The Greeks

- **Delta** (Δ) Delta represents the change of the option's price with respect to the underlying's price, $\Delta = \frac{\partial V}{\partial S}$. For traders, delta provides an approximate probability the option will expire in-the-money.

- **Theta** (Θ) Theta represents the change of the option's price with respect to time, $\Theta = \frac{\partial V}{\partial t}$. It is used by traders to indicate theta-decay, or time-decay, the phenomenon that an option loses value as time goes on, all else being equal.

- **Gamma** (Γ) Gamma represents the change of the option's delta with respect to the underlying's price, $\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 V}{\partial S^2}$. Traders use this to measure the stability of the option's delta, which could be important for strategies relying on specific values of delta.

- **Vega** (ν)**)** Vega represents the change of the option's value with respect to the underlying's implied volatility, $\nu = \frac{\partial V}{\partial \sigma}$. Traders can use vega to assess the exposure of their portfolios to market movements, as it indicates the direction an option's price is expected to move with increased volatility.

- **Rho** (ρ) Rho represents the change of the option's value with respect to the risk-free rate of interest, $\rho = \frac{\partial V}{\partial r}$. It is generally considered the least important of the Greeks.

II. BASIC STOCHASTIC CALCULUS

A. Brownian Motion

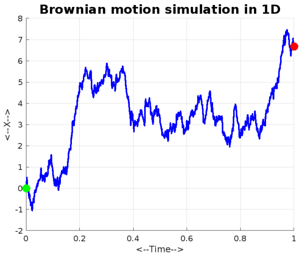

Figure 3: 1D Brownian Motion

To derive the BSM equation it is first important to figure out how stocks behave. Since they behave randomly, they are modeled by **Brownian Motion**

- **~50 BCE** Roman philosopher Lucretius observed dust particles floating through the air and noted their random movement.
- **1785** Jan Ingenhousz observed coal dust on the surface of alcohol in a similar motion.
- **1827** A botanist by the name of Robert Brown (who is credited with the phenomenon) was observing grains of pollen suspended in a fluid and began to notice that they would move erratically.

II. BASIC STOCHASTIC CALCULUS

A. Brownian Motion

As this was of great interest to Mathematicians, Albert Einstein started working on these problems and came up with the following result:
The probability that a particle is in a given interval $[a, b]$ is given by

$$\mathbb{P}(a \leq B_t \leq b) = \frac{1}{\sqrt{2\pi t}} \int_a^b e^{-\frac{1}{2t}x^2}\, dx.$$

Interestingly, we note that this definition is strikingly similar to the normal distribution with $\sigma = 1$. Namely, the only difference is the time dependence of this function.

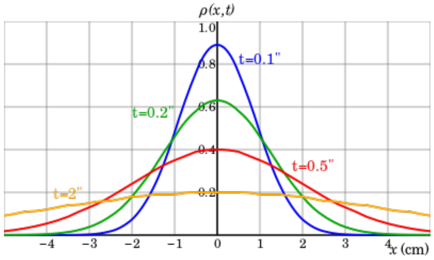

Figure 4: Diffusion of particles modeled by Brownian motion.

II. BASIC STOCHASTIC CALCULUS

B. Probability Space and Wiener Process

Definition 1. *A* ***probability space*** *is a triple $(\Omega, \mathcal{F}, P)$, sometimes simply denoted by Ω, where Ω is the sample space, $\mathcal{F}$ is the σ-algebra on Ω, and P is the probability measure. A* ***σ-algebra*** *(or σ-field) on a set X is a nonempty collection of subsets of X closed under complement and countable unions. A* ***probability measure*** *is a measure P on Ω where $P(\Omega) = 1$.*

Definition 2. *A* ***Wiener process*** *$\{W(t)\}_{t \geq 0}$ is a family of random variables $W(t) : \Omega \to \mathbb{R}$, where Ω is a probability space satisfying the following properties:*

1. *$W(0) = 0$.*

2. *$\forall \omega \in \Omega$, the map $t \mapsto W(\omega, t)$ is continuous from $[0, \infty)$ into $\mathbb{R}$ with probability one.*

3. *$\forall t > s \geq 0$, $W(t) - W(s)$ is normally distributed with mean 0 and variance $t - s$, i.e.,*

$$\mathbb{P}(a \leq W(t) - W(s) \leq b) = \frac{1}{\sqrt{2\pi(t-s)}} \int_a^b e^{-\frac{1}{2(t-s)}x^2} dx.$$

4. *$\forall 0 = t_0 < t_1 < t_2 < \ldots < t_n$, the increments $W(t_{i+1}) - W(t_i)$ are independent random variables.*

Definition 3. *A **simple stochastic process** is a function of the form*

$$f(\omega, t) = \sum_{j=0}^{n-1} \alpha_j(\omega) \mathbf{1}_{[t_j, t_{j+1})}(t),$$

where $0 = t_0 < t_1 < \ldots < t_n = T$*,* $\alpha_j(\omega)$ *random variables, and* $\mathbf{1}_{[a,b)}(t) = 1$ *if* $a \leq t < b$ *and else* 0*.*

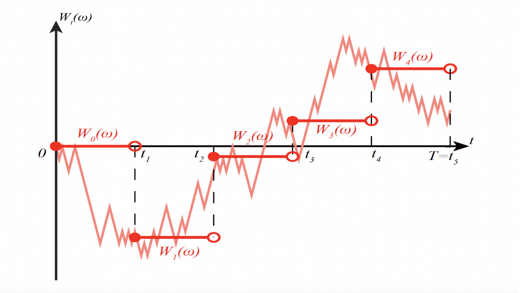

Figure 5: Example of a simple stochastic process from a general process.

Definition 4. *The **Itô integral** for a simple stochastic process* $f(t)$ *is given by*

$$\int_0^T f(t) dW_t = \sum_{j=0}^{n-1} \alpha_j(\omega)(W_{t_{j+1}}(\omega) - W_{t_j}(\omega)).$$

Definition 5. *The* ***Itô integral*** *for a general stochastic process $f(\omega, t)$ that is square integrable*

$$\mathbb{E}\left[\int_0^T f^2(\omega, t)dt\right] < \infty$$

is given by

$$\int_0^T f(\omega, t)dW_t(\omega) \overset{L^2(\Omega)}{=} \lim_{n\to\infty} \int_0^T f_n(\omega, t)dW_t(\omega),$$

where $f_n(\omega, t)$ is a sequence of simple stochastic processes converging to $f(\omega, t)$ in L^2.

Definition 6. *A* ***stochastic integral*** *is a stochastic process X_t on Ω of the form*

$$X_t = X_0 + \int_0^t u(\omega, s)ds + \int_0^t v(\omega, s)dW_s,$$

where $v(\omega, s)$ satisfies the square integrability condition

$$\mathbb{E}\left[\int_0^t v^2(\omega, s)ds\right] < \infty.$$

The stochastic integral is also written as a ***stochastic differential equation*** *of the form*

$$dX_t = udt + vdW_t.$$

Proposition 1 (Itô's formula)**.** *Let X_t be a stochastic integral*

$$dX_t = udt + vdW_t,$$

and $g(x,t) \in C^2(\mathbb{R} \times [0,T])$. Then $Y_t = g(X_t, t)$ is a stochastic integral with

$$dY_t = \frac{\partial g}{\partial t}(X_t, t)dt + \frac{\partial g}{\partial x}(X_t, t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(X_t, t)(dX_t)^2.$$

Remark 1. We use the following formal rules for computing $(dX_t)^2$:

$$(dt)^2 = dtdW_t = dW_t dt = 0 \text{ and } (dW_t)^2 = dt,$$

so we have

$$\begin{aligned}
(dX_t)^2 &= (udt + vdW_t)(udt + vdW_t) \\
&= u^2(dt)^2 + uvdtdW_t + vudW_t dt + v^2(dW_t)^2 \\
&= v^2 dt.
\end{aligned}$$

III. STOCK PRICING MODEL

A. Log-normal model

Assume that on average, the price of a share of stock S_t grows with continuously compounded interest at a constant risk-free rate a, then it satisfies the following differential equations:

$$\frac{dS_t}{dt} = aS_t, \text{ or } \frac{dS_t}{S_t} = adt.$$

This is the **log-linear** model for stock prices. To model volatility, we add to the right-hand side of the equation the term σdW_t, which is often referred to as "white noise". Thus, we obtain the equation:

$$\frac{dS_t}{S_t} = adt + \sigma dW_t, \text{ or}$$
$$dS_t = aS_t dt + \sigma S_t dW_t, \quad t \geq 0,$$

and this is the **log-normal** model for stock prices.

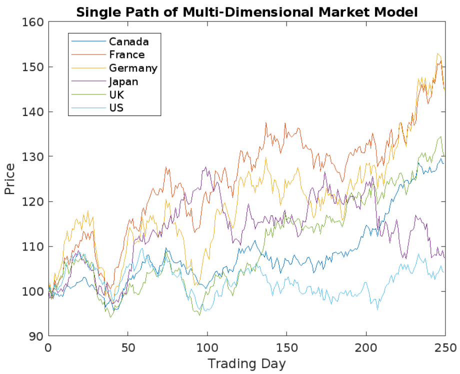

Figure 6: Simulation of prices of several equity markets.

To solve the equation, we apply Itô's formula with $g(x,t) = \ln x$, $x > 0$, and $X_t = S_t$. Then

$$\begin{aligned} d(\ln S_t) &= \frac{dS_t}{S_t} - \frac{1}{2}\frac{1}{S_t^2}(dS_t)^2 \\ &= \frac{dS_t}{S_t} - \frac{1}{2}\frac{1}{S_t^2}(aS_t dt + \sigma S_t dW_t)^2 \\ &= \frac{dS_t}{S_t} - \frac{1}{2}\frac{1}{S_t^2}\sigma^2 S_t^2 dt \\ &= \left(a - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t. \end{aligned}$$

From Definition 6, this is the differential form of the stochastic integral

$$\ln S_t = \ln S_0 + \int_0^t \left(a - \frac{1}{2}\sigma^2\right) d\tau + \int_0^t \sigma dW_\tau,$$

solving this yeilds

$$S_t = S_0 \exp\left[(a - \frac{1}{2}\sigma^2)t + \sigma W_t\right].$$

This is the solution to the **log-normal** model for the stock.

IV. THEOREM

A. BSM equation

Theorem 1. *Suppose that the stock price S_t satisfies the log-normal stochastic differential equation*

$$dS_t = aS_t dt + \sigma S_t dW_t, \quad t \geq 0, \tag{1}$$

where a is the stock's mean rate of growth, σ is the volatility of the stock, and W_t is a Wiener process starting at zero. Consider a European call option for this stock with strike price K and exercise time T. Therefore, its pay-off at time T is

$$V(S_T, T) = [S_T - K]^+ \doteq \max\{S_T - K, 0\}. \tag{2}$$

Denote by $V(s,t)$ the value of the call option at time t if the stock price is $S_t = s$, and r the risk-free interest rate. Then, $V(s,t)$ satisfies the ***Black-Scholes-Merton partial differential equation***

$$\boxed{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} + rs\frac{\partial V}{\partial s} - rV = 0, \quad t \in [0, T], \quad s \geq 0.} \tag{3}$$

IV. THEOREM

B. BSM solution

*Under the **terminal and boundary conditions***

$$V(s,T) \doteq \lim_{t\to T^-} V(S,T) = [s-K]^+, \quad V(0,t) \doteq \lim_{s\to 0^+} V(s,t) = 0, \forall t \in [0,T], \tag{4}$$

and the growth condition

$$\lim_{t\to\infty} [V(s,t) - (s - Ke^{-r(T-t)})] = 0, \tag{5}$$

the solution to the PDE (3) with conditions (4) and (5) is given by

$$\boxed{V(s,t) = s\Phi(d_+(s,T-t)) - Ke^{-r(T-t)}\Phi(d_-(s,T-t)), \quad 0 \le t < T, \quad s > 0,} \tag{6}$$

where

$$\boxed{d_\pm(s,\tau) \doteq \frac{1}{\sigma\sqrt{\tau}}\left[\ln\frac{s}{K} + (r \pm \frac{1}{2}\sigma^2)\tau\right], \ \textit{and } \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy.}$$

V. DERIVATION OF BSM

A. Replicating portfolio

Our goal is to determine the pay-off function $V(s,t)$. For this we consider the idea of replicating portfolio. Assume that at time $t=0$ we begin with a portfolio of value X_0. At any time $t \geq 0$, we invest in the following two financial instruments:

- A money market with a constant interest rate r,
- A stock whose time evolution S_t is modeled by equation (1).

Denote by X_t the value of our portfolio at time t, we construct a portfolio such that its value is equal to the value of the call option at that time, i.e., $X_t = V(S_t,t)$, or discounted $e^{-rt}X_t = e^{-rt}V(S_t,t)$.
Assume that at each time t, our portfolio consists of Δ_t shares of stock, and therefore, $X_t - \Delta_t S_t$ in the money market. Then the differential dX_t is given by

$$dX_t = \Delta_t dS_t + r(X_t - \Delta_t S_t)dt.$$

Remark 2. Compare equation with the following discrete time formula for a better understanding

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n).$$

B. Computing $d(e^{-rt}X_t)$

We use Itô's formula with $g(x,t) = e^{-rt}x$. Then

$$\begin{aligned} d(e^{-rt}X_t) &= \frac{\partial g}{\partial t}(X_t,t)dt + \frac{\partial g}{\partial x}(X_t,t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(X_t,t)(dX_t)^2 \\ &= -re^{-rt}X_t dt + e^{-rt}dX_t \\ &= -re^{-rt}X_t dt + e^{-rt}(\Delta_t dS_t + r(X_t - \Delta_t S_t)dt) \\ &= \Delta_t e^{-rt}(dS_t - rS_t dt) \\ &= \Delta_t e^{-rt}(aS_t dt + \sigma S_t dW_t - rS_t dt). && \text{(assumption (1))} \end{aligned}$$

Therefore,

$$d(e^{-rt}X_t) = (a-r)e^{-rt}\Delta_t S_t dt + \sigma e^{-rt}\Delta_t S_t dW_t.$$

We use Itô's formula with $g(x,t) = e^{-rt}V(x,t)$.

$$\frac{\partial g}{\partial t}(x,t) = -re^{-rt}V(x,t) + e^{-rt}\frac{\partial V}{\partial t}(x,t),$$
$$\frac{\partial g}{\partial x}(x,t) = e^{-rt}\frac{\partial V}{\partial x}(x,t), \quad \frac{\partial^2 g}{\partial x^2}(x,t) = e^{-rt}\frac{\partial^2 V}{\partial x^2}(x,t).$$

Then,

$$\begin{aligned} d(e^{-rt}V(S_t,t)) =& d(g(S_t,t)) = \frac{\partial g}{\partial t}(S_t,t)dt + \frac{\partial g}{\partial x}(S_t,t)dS_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(S_t,t)(dS_t)^2 \\ =& \left(-re^{-rt}V(S_t,t) + e^{-rt}\frac{\partial V}{\partial t}(S_t,t)\right)dt + e^{-rt}\frac{\partial V}{\partial x}(S_t,t)dS_t + \frac{1}{2}e^{-rt}\frac{\partial^2 V}{\partial x^2}(S_t,t)(dS_t)^2. \end{aligned}$$

By assumption (1) and Remark (1), we have $(dS_t)^2 = \sigma^2 S_t^2 dt$, so this gives

$$d(e^{-rt}V(S_t,t)) = e^{-rt}\left(-rV(S_t,t) + \frac{\partial V}{\partial t}(S_t,t) + aS_t\frac{\partial V}{\partial x}(S_t,t) + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial x^2}(S_t,t)\right)dt + e^{-rt}\sigma S_t\frac{\partial V}{\partial x}(S_t,t)dW_t.$$

D. Equating the differentials

From $e^{-rt}X_t = e^{-rt}V(S_t,t)$, we have $d(e^{-rt}X_t) = d(e^{-rt}V(S_t,t))$. So

$$(a-r)e^{-rt}\Delta_t S_t = e^{-rt}\left(-rV(S_t,t) + \frac{\partial V}{\partial t}(S_t,t) + aS_t\frac{\partial V}{\partial x}(S_t,t) + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial x^2}(S_t,t)\right),$$

$$\sigma e^{-rt}\Delta_t S_t = e^{-rt}\sigma S_t\frac{\partial V}{\partial x}(S_t,t).$$

The second relation gives the **Delta-hedging formula**

$$\Delta_t = \frac{\partial V}{\partial x}(S_t,t). \tag{7}$$

Substituting into the first relation gives the **Black-Scholes-Merton partial differential equation**

$$\frac{\partial V}{\partial t}(S_t,t) + \frac{1}{2}\sigma^2 s^2\frac{\partial^2 V}{\partial s^2}(S_t,t) + rs\frac{\partial V}{\partial s}(S_t,t) - rV(S_t,t) = 0, \tag{8}$$

considering $V = V(s,t)$, we obtain the BSM of the form in equation (3).

VI. SOLVING THE BSM

A. Change of Variables

From the derivation above, the BSM boundary value problem is given by

$$\begin{cases} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} + rs\frac{\partial V}{\partial s} - rV = 0, \quad t \in [0,T], \quad s \geq 0, \\ V(s,T) \doteq \lim_{t\to T^-} V(S,T) = [s-K]^+, \\ V(0,t) \doteq \lim_{s\to 0^+} V(s,t) = 0, \qquad \forall t \in [0,T], \\ \lim_{t\to\infty}[V(s,t) - (s - Ke^{-r(T-t)})] = 0. \end{cases} \tag{9}$$

We aim to create a heat equation, i.e. one of the form:

$$\begin{cases} u_t - u_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0, \\ u(x,0) = u_0(x), \end{cases} \tag{10}$$

as we know their solution. So, we first apply the change of variables,

$$\boxed{V(s,t) = e^{-r\tau}G(x,y),} \tag{11}$$

where $x(s,t),\ y(s,t)$ are functions of s,t, and $\tau = T - t$.

B. Expansion of Change of Variables

Now, let us find the terms present in the BSM PDE,

$$\frac{\partial V}{\partial t} = re^{-r\tau}G + e^{-r\tau}[G_x x_t + G_y y_t]$$

$$\frac{\partial V}{\partial s} = e^{-r\tau}[G_x x_s + G_y y_s]$$

$$\frac{\partial^2 V}{\partial s^2} = e^{-r\tau}[G_{xx}x_s^2 + G_{xy}x_s y_s + G_x x_{ss} + G_{yx}x_s y_s + G_{yy}y_s^2 + G_y y_{ss}]$$

Which, when plugged in, yields,

$$re^{-r\tau}G + e^{-r\tau}[G_x x_t + G_y y_t] + \frac{1}{2}\sigma^2 s^2 e^{-r\tau}[G_{xx}x_s^2 + G_{xy}x_s y_s + G_x x_{ss} + G_{yx}x_s y_s + G_{yy}y_s^2 + G_y y_{ss}] + rse^{-r\tau}[G_x x_s + G_y y_s] - re^{-r\tau}G = 0, \quad \tau \in [0,T], \quad s \geq 0$$

Which simplifies to

$$\frac{\sigma^2 s^2}{2}x_s^2 G_{xx} + \frac{\sigma^2 s^2}{2}x_s y_s G_{xy} + \frac{\sigma^2 s^2}{2}y_s^2 G_{yy} + [\frac{\sigma^2 s^2}{2}x_{ss} + rsx_s + x_t]G_x + [\frac{\sigma^2 s^2}{2}y_{ss} + rsy_s + y_t]G_y = 0.$$

C. Transformation to Heat Equation

To obtain the form of the heat equation $G_y - G_{xx} = 0$, their coefficients must be equal. Additionally, we cannot have G_{yy}, G_x, or G_{xy} terms with nonzero coefficients. Therefore, the following conditions must be met:

1. $\frac{\sigma^2 s^2}{2} x_s^2 + \frac{\sigma^2 s^2}{2} y_{ss} + rsy_s + y_t = 0$
2. $\frac{\sigma^2 s^2}{2} x_s y_s = 0$
3. $\frac{\sigma^2 s^2}{2} y_s^2 = 0$
4. $\frac{\sigma^2 s^2}{2} x_{ss} + rsx_s + x_t = 0$

However, since y is independent of s this results in the simplification to:

1. $\frac{\sigma^2 s^2}{2} x_s^2 + y_t = 0$
2. $\frac{\sigma^2 s^2}{2} x_{ss} + rsx_s + x_t = 0$

The, we take another change of vars. to simplify the equation further:

$$
\begin{aligned}
x(s,t) = \ln(s) + \gamma\tau \implies x_s &= \frac{1}{s} \\
\implies x_t &= -\gamma \qquad (\tau = T - t),
\end{aligned}
$$

where γ is some constant. Substituting this into the above conditions, we obtain

$$
\begin{aligned}
\frac{\sigma^2}{2} + y_t = 0 &\implies y(t) = \frac{\sigma^2\tau}{2} \\
\frac{-\sigma^2}{2} + r - \gamma = 0 &\implies \gamma = r - \frac{\sigma^2}{2}.
\end{aligned}
$$

Now,

$$\frac{\sigma^2 s^2}{2} x_s^2 = -\frac{\sigma^2 s^2}{2} y_{ss} - rsy_s - y_t$$

Which is simply the heat equation!

$$G_y - G_{xx} = 0,$$

D. Solving the Heat Equation

First, we note that the BC is:

$$G|_{\tau=0} = G(x(s,T), y(T)) = [e^x - K]^+.$$

Then, applying the solution to the heat equation we get,

$$G(x,y) = \frac{1}{\sqrt{4\pi y}} \int_{-\infty}^{\infty} f(u) e^{\frac{-(u-x)^2}{4y}} du \tag{12}$$

which, when the boundary condition is plugged into f, yields,

$$G(x,y) = \frac{1}{\sqrt{4\pi y}} \int_{\ln(K)}^{\infty} (e^u - K) e^{\frac{-(u-x)^2}{4y}} du. \tag{13}$$

This is because the value of the option must be positive, so that $G|_{\tau=0} = 0$ for $x < \ln K$. Although we have now solved a form of the BSM equation, we still do not have an explicit solution.
Now, the next step is to undo the change of variables, i.e. transform $G(x,y) \to V(s,t)$.

E. Reverse Change of Vars.

Plugging in the definition of $V(s,t)$ and the variable transformations of x and y we get,

$$V(s,t) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{\ln K}^{\infty} e^u e^{\frac{-(u-\ln s-(r-\frac{\sigma^2}{2})\tau)^2}{2\sigma^2\tau}} du - \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{\ln K}^{\infty} K e^{\frac{-(u-\ln s-(r-\frac{\sigma^2}{2})\tau)^2}{2\sigma^2\tau}} du$$

We recall the Normal Distribution,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{y^2}{2}} dy$$

So, let us try to transform the above equation into a more manageable form in terms of the Normal Distribution. We note,

$$\begin{aligned} u + \frac{-(u - \ln s - (r - \frac{\sigma^2}{2})\tau)^2}{2\sigma^2\tau} &= \frac{-u^2 + 2[\ln s + (r - \frac{\sigma^2}{2})\tau]u - \ln(s)^2 - 2(r - \frac{\sigma^2}{2})\tau\ln(s)}{2\sigma^2\tau} - \\ &\quad \frac{(r^2 - 2r\frac{\sigma^2}{2}r + \frac{\sigma^4}{4})\tau^2 + 2(\ln s + r\tau)\sigma^2\tau - 2(\ln s + r\tau)\sigma^2\tau}{2\sigma^2\tau} \\ &= \frac{-(u - \ln s - (r + \frac{\sigma^2}{2})\tau)^2}{2\sigma^2\tau} + \ln s + r\tau \end{aligned}$$

So, now we have found a way to make the integral look like a normal distribution integral. Then, we simply let $p = \frac{(u - \ln s - (r + \frac{\sigma^2}{2})\tau)}{\sqrt{\sigma^2 \tau}}$ which, by a change of vars, transforms the first integral into

$$\frac{s}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\ln(s/K)+(r+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}} e^{\frac{-p^2}{2}}\, dp$$

Now, we make a similar change of variables to the second integral to create a different form of the normal distribution. Letting $w = \frac{(u - \ln s - (r - \frac{\sigma^2}{2})\tau)}{\sqrt{\sigma^2 \tau}}$ we see that the second integral becomes

$$\frac{Ke^{-r\tau}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\ln(s/K)+(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}} e^{\frac{-w^2}{2}}\, dp$$

Combining these two expressions, we get,

$$\boxed{V(s,t) = s\Phi\left(\frac{\ln(s/K) + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\right) - Ke^{-r\tau}\Phi\left(\frac{\ln(s/K) + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\right).}$$

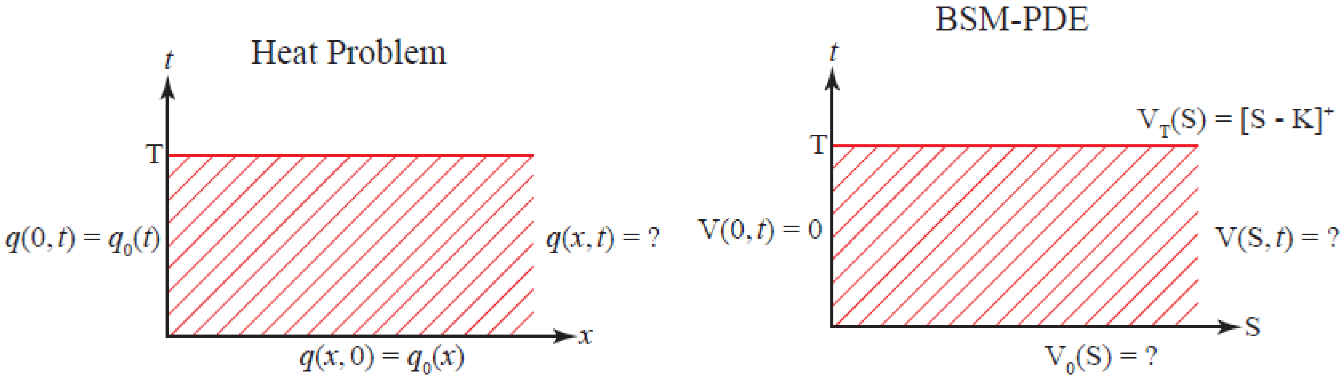

Figure 7: Differences Between the Heat Problem and the BSM Problem

VII. EXTENSION TO AMERICAN OPTIONS
A. Introduction and Derivation

American options differ from European options in that they may be exercised any time before the expiration date. This often makes them more valuable, as there is a greater window of opportunity to exercise them. Using a similar derivation, we obtain the *inequality*

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \leq 0, \quad t \in [0,T], \quad S \geq 0,$$

subject to the same boundary conditions, as well as an additional *free boundary condition* $V(S,t) \geq V(S,T)$. The price of an American call option may be approximated as a sum of a European call option and an early exercise premium $\epsilon(S,t)$:

$$C_A(S,t) = C_E(S,t) + \epsilon(S,t), \tag{14}$$

From similar methods as above, the PDE for the early exercise premium is

$$\frac{\partial \epsilon}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \epsilon}{\partial S^2} + rb\frac{\partial \epsilon}{\partial S} - r\epsilon = 0,$$

where b is the cost of carrying the underlying (typically $b < r$), and other terms are defined as above.

After some simplifications and applying the change of variables $\epsilon(S,t) = f(t)g(S,f)$, where $f(t) = 1 - \epsilon^{-r\tau}$, this equation can be written as

$$S^2 g_{SS} + \frac{2b}{\sigma^2} S g_S - \frac{2r}{\sigma^2(1-e^{-r\tau})} g - \frac{2b}{\sigma^2}(1-f) g_f = 0.$$

Making the assumption that the last term of the left-hand side is zero, we obtain a second-order ordinary differential equation. Its general solution is

$$g(S) = a_1 S^{q_+} + a_2 S^{q_-}, \quad \text{where}$$
$$q_\pm = \frac{-(N-1) \pm \sqrt{(N-1)^2 + 4M/f}}{2},$$

where $M = \frac{2r}{\sigma^2}$ and $N = \frac{2b}{\sigma^2}$. Since $M/f > 0, q_+ > 0$ and $q_- < 0$. Undoing the change of variables and substituting this back into Equation (14), we find that

$$C_A(S,t) = C_E(S,t) + f(t)[a_1 S^{q_+} + a_2 S^{q_-}].$$

To constrain a_1, we introduce the critical price S^*. This is the price imposed by the boundary of $S - K$ on $f(t)a_1S^{q+}$, where $S - K$ is the proceeds earned if the call is exercised early when $S > K$. Beyond this point, the payoff of the call is only determined by this difference. The value of S^* can be found using the equation

$$S^* - K = C_E(S^*, t) + f(t)a_1S^{*q+}.$$

Then, we find the American put by an iterative process and the assumption that the payoff will become large as $S \to 0$. We also note S^{**} is simply the American equivalent to $S*$. Then, the BSM solution is:

$$\begin{cases} P_A(S,t) & = P_E(S,t) + A_2(\frac{S}{S^{**}})^{q-}, \quad \text{when } S > S^{**} \\ P_A(S,t) & = S - K, \quad \text{when } S \leq S^{**}, \end{cases}$$

$$\text{where } A_2 = -\frac{S^{**}}{q_-}\{1 - e^{(b-r)\tau}N[-\ln\frac{S^{**}}{K} + (b + 0.5\sigma^2)\tau]/\sigma\sqrt{\tau}\}.$$

The above two systems give quadratic approximations for the value of American options.

VIII. EXTENSION TO AMERICAN OPTIONS

A. Price Calculation

One application (its original application) of the BSM is to determine the price of a certain call option. Here, we treat the following example with the BSM analytically.
Given: Suppose we want to buy a call on adidas AG (XE:ADS) with an expiry of 1 year. The call has the following parameters:

- the spot price S is €124.26, the closing stock price on 12/2/22;
- the expiration date is 12/21/23, so $T = 1$, or 1 year;
- the strike price X is €76;
- the risk-free interest rate 5%;
- the volatility σ is 0.03.

Then, we get the following result for the actual option versus the BSM prediction:

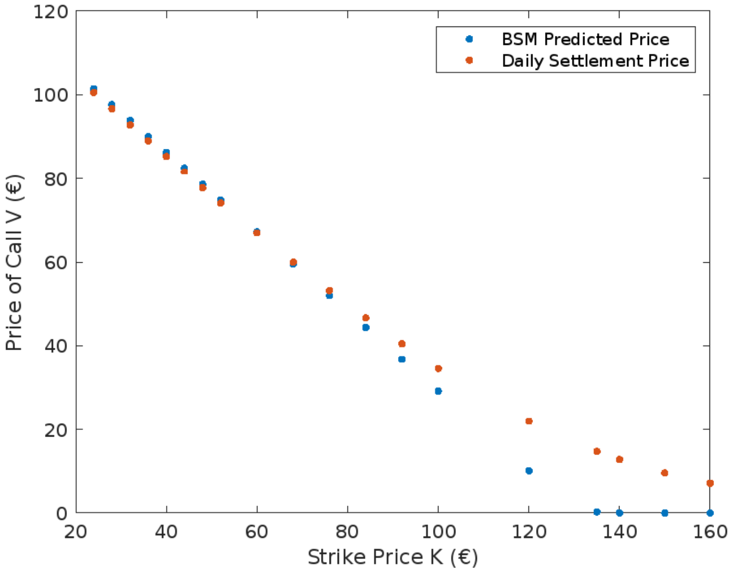

Figure 8: Comparison of BSM predicted and actual prices of the adidas AG call option.

B. Implied Volatility Calculation

We may also invert the BSM equation (albeit with some difficulty) to create an equation that models the volatility of a stock based on its price.
This is the inverted equation in order to find a value of σ, the implied volatility:

$$V_{mkt} = sN(d_+(s, T-t)) - Ke^{-r(T-t)}d_-(s, T-t)).$$

While this equation is nonlinear, we know that it is increasing with respect to σ, since $\frac{\partial V}{\partial \sigma} = S\sqrt{T}\frac{dN}{dd_1} > 0$.

Because the equation is increasing with respect to σ, we know there is a unique solution for σ to satisfy the equation.

So, in the same case as above, we used MATLABs solver to find the value σ across a given price range.

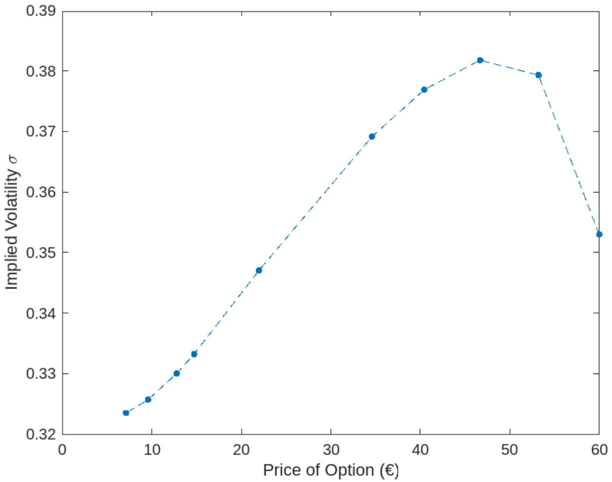

Figure 9: Graph of implied volatility versus option value using the fzero algorithm. Some values for which a numerical solution could not be found are omitted.

IX. REFERENCES

[1] James Chen, *European Options.* (31 January 2022) Investopedia.

[2] Szpiro, G. G. *Pricing the Future: finance, physics, and the 300-year journey to the black-scholes equation: A story of genius and Discovery.* (2011). Basic Books.

[3] Weatherall, J. (2017). *The Peculiar Logic of the Black-Scholes Model.* SSRN Electronic Journal . doi:10.2139/ssrn.2976242.

[4] Adam Hayes, *What Are Greeks in Finance and How Are They Used?* (10 March 2022) Investopedia.

[5] Tabor, D. *Gases, Liquids and Solids: And Other States of Matter.* (1991). 3rd Edition, Cambridge: Cambridge University Press. p.120.

[6] Lavenda, Bernard H. *Nonequilibrium Statistical Thermodynamics.* (1985) John Wiley & Sons, p.20 ISBN: 978-0-471-90670-4.

[7] Himonas, A. and Cosimano T., *Mathematical Methods in Finance and Economics.* Lecture Notes, University of Notre Dame.

[8] Hull, J.C. *Options, Futures and Other Derivatives.* (2012) 8th Edition, Prentice-Hall, Upper Saddle River.

[9] Fatone, L. , Mariani, F. , Recchioni, M. and Zirilli, F., *The Barone-Adesi Whaley Formula to Price American Options Revisited.* (2015) Applied Mathematics, 6, 382-402. doi: 10.4236/am.2015.62036.

[10] Barone-Adesi, G. and Whaley, R.E., *Efficient Analytic Approximation of American Option Values.* (1987) The Journal of Finance, 42, 301-320. http://dx.doi.org/10.1111/j.1540-6261.1987.tb02569.x

IX. REFERENCES CONT.

[11] Petter Bjerksund, Gunnar Stensland, *Closed-form approximation of American options.* (1993) Scandinavian Journal of Management, Volume 9, Supplement 1, Pages S87-S99, ISSN 0956-5221, https://doi.org/10.1016/0956-5221(93)90009-H.

[12] Ju, Nengjiu, and Zhong Rui, *An Approximate Formula for Pricing American Options.* (Winter 1999) Journal of Derivatives.

[13] PwC. *Risk Free Interest Rates.* (31 Jul 2022) US Stock-Based Compensation Guide, Section 9.5.