DERIVATION OF THE HESTON MODEL OPTION PRICING FORMULA

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Abstract. This paper provides an introductory study of option pricing using the Heston model. This paper begins with an introduction that provides a background on the history of option pricing and the mathematical preliminaries necessary for understanding the Heston model, including stochastic calculus, Fourier analysis, complex analysis, and partial differential equations.

We begin with a rapid review of the Black-Scholes-Merton formula, discussing its assumptions and limitations, and introduce the stochastic volatility model. We then present the derivation of the Heston model pricing equation and the solution formula for European call options using Fourier transform and the fundamental solution. We also discuss the limitations of the Heston model and compares it to the constant volatility stock model, specifically in terms of option pricing formula. Finally, we discuss the practical applications of the Heston model in the industry and its implications. The limitations of the model are also highlighted. The study provides valuable insights into stock pricing using the Heston model and is of interest to academics and professionals in finance and related fields.

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1. Introduction

The sophisticated behavior of markets has long attracted interest by mathematicians and scientists. Equations and models are developed in attempt to define or approximate the processes which govern this behavior. In this paper, we cover the Heston model in depth and show its usage. We begin with a historical review, followed by an overview of stochastic calculus and Fourier Analysis. We use these to inform our derivation of the BSM equation, which is then solved using Fourier transformation. We then show applications of the model, compare it to the original BSM model under log-normal stock pricing. We conclude by discussing the continued usage of the model.

1.1. **Background.** Option contracts, like other financial derivatives, have been around for hundreds of years. An option contract is simply the right to buy or sell an amount of an asset, usually stock, at a certain price at a later date. This asset is called the "underlying". An option to buy stocks is called a "call", while an option to sell stocks is called a "put". Options have an expiration date; for European-style options, the options must be "exercised", or redeemed, on this date (and not before).

Options are sold by a "writer" to a "holder". The holder pays a fee, called a "premium", to the writer in exchange for the right either purchase or sell the underlying at a certain price, called the "strike price." This is often determined with reference to the current trading price of the underlying itself, called the "spot price" [2].

For example, suppose a person buys a call from a writer. The writer hopes that the spot price of the underlying won't be higher than (approximately) the strike price plus the premium of the option on expiration date–that is, the call is "out-of-the-money", and expires worthless. If instead the strike price is higher, then the holder can exercise the call to buy the underlying at the strike price, a lower price than the spot price–the call is "in-the-money". The underlying is usually supplied by the writer. The situation is reversed for puts–a put is "out-of-the-money" when the strike price is higher than the spot, and "in-the-money" when the spot is higher than the strike. In the latter case, the holder sells the underlying at the strike price at expiry.

1.2. **History.** In 1900, Louis Bachelier defended his thesis in mathematics, analyzing the price of options based on probability theory. This was the first mathematical treatment of its kind in this field. Concurrently and in years following, Norbert Wiener made advances in stochastic calculus and proved the existence of Brownian motion. This field was further advanced by Kiyoshi Itō, who created a method for handling stochastic differential equations. As knowledge advanced, interest in mathematical economics and Bachelier's work slowly waxed. In 1967, Edward Thorpe and Sheen Kassouf published a book in which they derived an equation for the value of a financial derivative called a warrant. Thorpe and Kassouf's method was empirical, but Paul Samuelson and Robert Merton derived a very similar equation rigorously the next year. Fischer Black and Myron Scholes, at the time working at a consulting company with Wells Fargo as a client, adapted that work to derive a differential equation describing the value of options, as well as its solution [1]. They also defined a number of parameters–called "The Greeks"–which allowed them to describe the sensitivity of an option's price to other variables.

In the year 1993, the Heston model for option pricing was introduced in a paper published by Steve Heston, a finance professor hailing from the University of Maryland [4]. The model itself is rooted in the Black-Scholes model; however, it addresses some of the unrealistic assumptions of this model. The Black-Scholes model assumes that the volatility of an underlying asset is constant, which is not the case. Empirical evidence suggests the market-implied volatility of an asset varies over time, and for options on the underlying, vary with strike price. The Heston model improves upon this assumption by modeling the volatility of the underlying as a stochastic process. It models this volatility as a square-root process, which shall be elaborated upon later in this paper.

The Heston model has become popular in recent years as it addresses key criticisms associated with the Black-Scholes model. It is able to price a wider array of options, including exotic options with varying expiry conditions and options on assets for which returns cannot be modeled with a log-normal distribution. The Heston model is used by many financial institutions, traders, and hedge funds to both price options as well as manage risk.

2. Mathematical Preliminaries

In this section we state the mathematical foundations that will be needed for the derivation of the option pricing formula under the BSM model and the Heston model.

2.1. **Stochastic calculus.** To understand the basic notions of stock behaviour we make an intuitive assumption. Namely, we indicate that the stock behaviour is random. But if it is random, how do we model it? In 1827, a botanist by the name of Robert Brown was observing grains of pollen suspended in a fluid and began to notice that they would move erratically. The pollen seemed to change path and move about in a random way. Similar phenomena have been seen throughout history. The Roman philosopher Lucretius observed dust particles floating through the air and noted their random movement. Similarly, in 1785 Jan Ingenhousz observed coal dust on the surface of alcohol–again noting its random movement. Nonetheless, the person who is credited with the true discovery is Robert Brown–hence the name, Brownian motion [15].

One type of Brownian motion that is axiomatically defined is called a Wiener process, usually denoted by W_t. To keep the presentation short and self-contained, we will not go through the axiomatic setup for stochastic processes X_t and stochastic integrals. We will give the important results for Itô integrals and Itô's formula. For background, proofs and examples, see [8, 11].

There is one financial interpretation of Itô integral worth noting. Let W_t be a Wiener process, like Figure 2.1. If we think of W_t as the price of one share of an asset at time t, the times $t_0, t_1, \ldots, t_{n-1}$ as the trading dates, and $f(t_0), f(t_1), \ldots, f(t_{n-1})$ as the number of shares in the portfolio at each trading date and held until the next trading date, then the Itô integral gives the gain from trading at all dates.

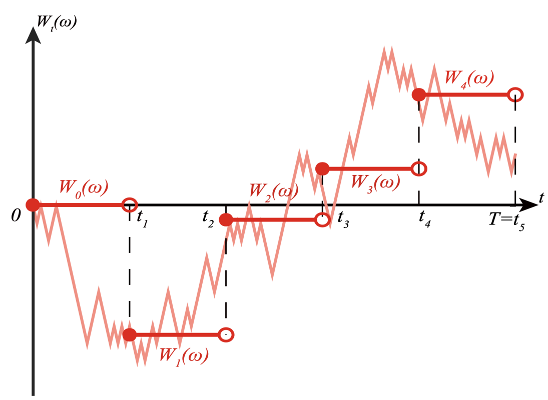

Figure 2.1. Example of a simple stochastic process from a general process [8].

Definition 2.1. *A stochastic integral is a stochastic process X_t on Ω of the form*

$$X_t = X_0 + \int_0^t u(\omega, s)ds + \int_0^t v(\omega, s)dW_s,$$

where $v(\omega, s)$ satisfies the square integrability condition

$$\mathbb{E}\left[\int_0^t v^2(\omega, s)ds\right] < \infty.$$

The stochastic integral is also written as a stochastic differential equation of the form

$$dX_t = udt + vdW_t.$$

Proposition 2.1 (Itô's formula)**.** *Let X_t be a stochastic integral*

$$dX_t = udt + vdW_t,$$

and $g(x,t) \in C^2(\mathbb{R} \times [0,T])$. Then $Y_t = g(X_t, t)$ is a stochastic integral with

$$dY_t = \frac{\partial g}{\partial t}(X_t, t)dt + \frac{\partial g}{\partial x}(X_t, t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(X_t, t)(dX_t)^2.$$

Proposition 2.2 (Generalized Itô's formula)**.** *Let X_t be be an n-dimensional Itô process of the form*

$$d\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} dt + \begin{pmatrix} v_{11} & \cdots & v_{1m} \\ \vdots & \ddots & \vdots \\ v_{n1} & \cdots & v_{nm} \end{pmatrix} \begin{pmatrix} dW_1 \\ \vdots \\ dW_m \end{pmatrix}.$$

Let $g(t,x) = (g_1(t,x), \ldots, g_p(t,x))$ be a map from $C^2([0,\infty) \times \mathbb{R}^n])$ into $\mathbb{R}^p$. Then $Y_t = g(t, X_t)$ is also an Itô process, whose components Y_k, $1 \leq k \leq p$ is given by

$$dY_k = \frac{\partial g_k}{\partial t}(t, X)dt + \sum_i \frac{\partial g_k}{\partial x_i}(t, X)dX_i + \frac{1}{2}\sum_{i,j}\frac{\partial^2 g_k}{\partial x_i \partial x_j}(t, X)dX_i dX_j.$$

Remark 2.1. We use the following formal rules for computing $(dX_t)^2$:

$$(dt)^2 = dtdW_t = dW_t dt = 0 \text{ and } dW_i dW_j = \delta_{ij}dt,$$

where δ_{ij} is the indicator function and is 1 when $i = j$ and 0 otherwise.

Now we state the key theorems which allows for finding the pricing equation to the Heston model. This is undoubtedly the most technical part of this paper. Any subsequent parts that invoke any part of the theorem or stochastic differential equations will not be elaborated. Readers interested in the rigorous setup are strongly encouraged to consult [11].

Definition 2.2. *A (time-homogeneous) Itô diffusion is a stochastic process $X(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}^n$ satisfying a stochastic differential equation of the form*

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad t \geq s, X_s = x,$$

where W_t is m-dimensional Brownian motion, $b : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n \times m}$, and together need to be Lipschitz continuous, i.e., $|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq C|x - y|$. See [11, Theorem 5.2.1].

Theorem 2.3. *Let X_t be an Itô diffusion in $\mathbb{R}^n$ given by Definition 2.2. If $f \in C_0^2(\mathbb{R}^n)$, then there is a generator $\mathcal{A}$ (see* [11, Definition 7.3.1, Lemma 7.3.2]*) given by*

$$\mathcal{A} = \sum_i b_i(x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j}.$$

2.2. **PDE fundamentals.** Partial differential equations (PDE) and their solutions are a major field in mathematical analysis. One common method of solving PDEs is using Fourier transform. Thus, we state the definition and prove relevant properties of the Fourier theory, with certain results from distribution theory, that we will used in solving the Heston model.

Definition 2.3. *For $f \in L^1(\mathbb{R}^n)$, the Fourier transform*[1] $\mathcal{F}$ *is the operator*

$$\mathcal{F} : f \to \widehat{f} \text{ such that } \widehat{f}(\xi) \doteq \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(x) dx. \tag{2.1}$$

Definition 2.4. *The inverse operator of $\mathcal{F}$, the inverse Fourier transform is given by*

$$\mathcal{F}^{-1} : f \to \tilde{f} \text{ such that } \tilde{f}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} f(\xi) d\xi. \tag{2.2}$$

Lemma 2.4. *The Fourier transform of the one-dimensional Gaussian is given by* $(\lambda > 0)$

$$\widehat{e^{-\frac{\lambda}{2}x^2}}(\xi) = \left(\frac{2\pi}{\lambda}\right)^{1/2} e^{-\frac{1}{2\lambda}\xi^2}.$$

Proof. The proof itself is not so important, but it involves integration along lines in the complex plane, which in later sections will be utilized to recover option pricing formula from its Fourier transform. So it is good to get familiar with the complex plane and see similar techniques rigorously done here.

Directly apply the definition we have

$$\begin{aligned}
\widehat{e^{-\frac{\lambda}{2}x^2}}(\xi) &= \int_{\mathbb{R}} e^{-ix\xi} e^{-\frac{\lambda}{2}x^2} dx \\
&= \int_{\mathbb{R}} \exp\left[-\frac{\lambda}{2}\left(x^2 + \frac{2ix\xi}{\lambda}\right)\right] dx \\
&= \int_{\mathbb{R}} \exp\left[-\frac{\lambda}{2}\left(x^2 + \frac{2\xi}{\lambda} ix + (\frac{\xi}{\lambda} i)^2\right) - \frac{\xi^2}{2\lambda}\right] dx \\
&= e^{-\frac{1}{2\lambda}\xi^2} \int_{\mathbb{R}} \exp\left[-\frac{\lambda}{2}\left(x + \frac{\xi}{\lambda} i\right)^2\right] dx.
\end{aligned}$$

For such integrals, it is easier to evaluate using techniques from complex analysis. Consider $f(z) = \exp\left(-\frac{\lambda}{2}z^2\right)$, the last desired integral is equivalent to integrating $f(z)$ along $z = x + \frac{\xi}{\lambda}i$ with $\mathrm{Re}(z) = x$ from $-\infty$ to ∞. We consider the following closed rectangular path

$$C^R = C_1^R + C_2^R + C_3^R + C_4^R,$$

where $R \in \mathbb{R}^+$, $C_1^R : (-R + \frac{\xi}{\lambda}i \to R + \frac{\xi}{\lambda}i)$, $C_2^R : (R + \frac{\xi}{\lambda}i \to R)$, $C_3^R : (R \to -R)$, $C_4^R : (-R \to -R + \frac{\xi}{\lambda}i)$ are straight paths. Figure 2.2 illustrates the contour C^R.

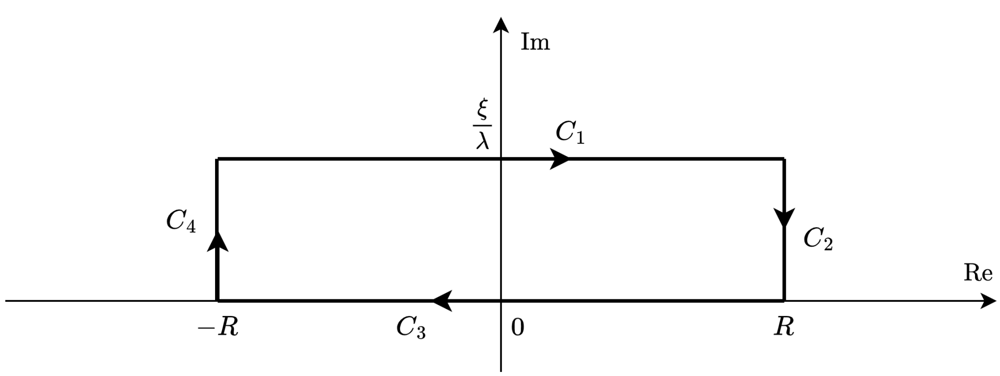

FIGURE 2.2. The rectangular contour of integration.

[1] There are different normalization conventions regarding the Fourier transform, including unitary and non-unitary (difference of 2π in exponential or constant). This paper follows the convention used in [7].

Since $f(z)$ is entire, by Cauchy's Theorem, we have for any $R \in \mathbb{R}^+$

$$\int_{C^R} f(z)dz = \left(\int_{C_1^R} + \int_{C_2^R} + \int_{C_3^R} + \int_{C_4^R} \right) f(z)dz = 0.$$

So by deformation of contours, we can take the limit $R \to \infty$, and $\int_{C^\infty} f(z)dz = 0$. We note that $\int_{C_1^R} f(z)dz$ is the desired integral with real part restricted to $[-R, R]$ and $\int_{C_3^R} f(z)dz$ is just $\int_R^{-R} f(x)dx$. For $\int_{C_2^R} f dz$ and $\int_{C_4^R} f dz$, we note the following

$$\begin{aligned}
\int_{C_2^R} f(z)dz &= \int_{\xi/\lambda}^{0} e^{-\frac{\lambda}{2}(R+is)^2} ds = \int_{\xi/\lambda}^{0} e^{\frac{\lambda s^2}{2}} e^{-\frac{\lambda R^2}{2}} e^{-i\lambda R s} ds \\
&\leq \sup_{s \in C_2^R} \left| \frac{\xi}{\lambda} \right| \left| e^{\frac{\lambda s^2}{2}} \right| \left| e^{-\frac{\lambda R^2}{2}} \right| \left| e^{-i\lambda R s} \right| \\
&\lesssim \left| e^{-\frac{\lambda R^2}{2}} \right| \xrightarrow{R \to \infty} 0.
\end{aligned}$$

Similarly we have $\int_{C_4^R} f(z)dz \to 0$ as $R \to 0$. So from Cauchy's Theorem we get

$$\int_{C_1^R} f(z)dz + \int_R^{-R} f(x)dx = 0 \xrightarrow[R \to \infty]{f|_{C_1^R}} \int_{\mathbb{R}} \exp\left[-\frac{\lambda}{2} \left(x + \frac{\xi}{\lambda} i \right)^2 \right] dx = \int_{\mathbb{R}} e^{-\frac{\lambda}{2} z^2} dx.$$

Apply the change of variable $y = (\lambda/2)^{1/2} x$ we have

$$\int_{\mathbb{R}} e^{-\frac{\lambda}{2} z^2} dx = \left(\frac{2}{\lambda} \right)^{1/2} \int_{\mathbb{R}} e^{-y^2} dy = \left(\frac{2\pi}{\lambda} \right)^{1/2}.$$

Substituting back this value completes the proof. $\square$

Theorem 2.5. *If $\widehat{f}, f \in L^1$, then $\mathcal{F}^{-1}(\mathcal{F} f) = f$, and similarly if $\tilde{f}, f \in L^1$, then $\mathcal{F}(\mathcal{F}^{-1} f) = f$.*

Proof. We will only prove $f = \mathcal{F}^{-1}(\widehat{f})$ for the one-dimensional case when f is a priori known to be continuous. The other direction and the general cases are similar. For a comprehensive treatment with less restrictions, see [7] and [13, Chapter 4].

Since $\lim_{\varepsilon \to 0} e^{-\varepsilon \xi^2} = 1$, by the Dominated Convergence Theorem, we have

$$\begin{aligned}
\frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \widehat{f}(\xi) d\xi &= \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} e^{-\varepsilon \xi^2} \widehat{f}(\xi) d\xi && \text{(integrand dominated by } \|\widehat{f}\|) \\
&= \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ix\xi} e^{-\varepsilon \xi^2} e^{-iy\xi} f(y) dy d\xi && \text{(Fubini)} \\
&= \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{\mathbb{R}} f(y) \left(\int_{\mathbb{R}} e^{-i(y-x)\xi} e^{-\varepsilon \xi^2} d\xi \right) dy.
\end{aligned}$$

By Lemma 2.4, let $\lambda = 2\varepsilon$, we have

$$\int_{\mathbb{R}} e^{-i(y-x)\xi} e^{-\varepsilon \xi^2} d\xi = \left(\frac{\pi}{\varepsilon} \right)^{1/2} \exp\left[-\frac{1}{4\varepsilon} (y-x)^2 \right].$$

So we have

$$\begin{aligned}
\frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \widehat{f}(\xi) d\xi &= \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{\mathbb{R}} f(y) \left(\frac{\pi}{\varepsilon} \right)^{1/2} \exp\left[-\frac{1}{4\varepsilon} (y-x)^2 \right] dy \\
&= \lim_{\varepsilon \to 0} \frac{1}{\sqrt{2\pi\varepsilon}} \int_{\mathbb{R}} f(y) \exp\left[-\frac{1}{4\varepsilon} (y-x)^2 \right] dy.
\end{aligned}$$

Now make the change of variable $w = (y-x)/(2\sqrt{\varepsilon})$, so $dy = 2\sqrt{\varepsilon}dw$, we have

$$\begin{aligned}
\frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\widehat{f}(\xi)d\xi &= \lim_{\varepsilon\to 0}\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-w^2} f(x+2\sqrt{\varepsilon}w)dw \\
&= \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-w^2}\lim_{\varepsilon\to 0} f(x+2\sqrt{\varepsilon}w)dw && \text{(integrand dominated by } \|f\|) \\
&= \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-w^2} f(x)dw && (f \text{ continuous}) \\
&= f(x)\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-w^2}dw \\
&= f(x). && \text{(Gaussian integral)}
\end{aligned}$$

So the proof is now complete. □

Lemma 2.6. *If $f(x), x\in\mathbb{R}$ vanishes at $x=\pm\infty$, and $\partial f/\partial x$ exists and is Fourier transformable, then*

$$\widehat{\partial_x f}(\xi) = i\xi\widehat{f}(\xi).$$

Proof. We prove by direct calculation using integration by parts.

$$\begin{aligned}
\widehat{\partial_x f}(\xi) &= \int_{\mathbb{R}} e^{-ix\xi}\partial_x f(x)dx \\
&= e^{-ix\xi}f(x)\Big|_{-\infty}^{\infty} - \int_{\mathbb{R}}\partial_x\left(e^{-ix\xi}\right)f(x)dx \\
&= -(-i\xi)\int_{\mathbb{R}} e^{-ix\xi}f(x)dx \\
&= i\xi\widehat{f}(\xi).
\end{aligned}$$

□

Lemma 2.7. *The Fourier transform of the Delta distribution is 1, i.e., $\widehat{\delta}_0 = 1$.*

Proof. For any test function $\phi\in\mathcal{D}(\mathbb{R}^n) := C_0^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned}
\left\langle\widehat{\delta},\phi\right\rangle := \left\langle\delta,\widehat{\phi}\right\rangle &:= \widehat{\phi}(0) \\
&= \int_{\mathbb{R}^n} e^{-i\langle x,0\rangle}\phi(x)dx \\
&= \langle 1,\phi\rangle.
\end{aligned}$$

□

Definition 2.5. *If $f,g\in L^1(\mathbb{R}^n)$, the convolution $f*g$ is defined by*

$$f*g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

Corollary 2.8.
1. *If $\widehat{f}=0$ and $f\in L^1$, then $f=0$.*
2. *If $f\in L^2(\mathbb{R}^n)$ the $\widehat{f}\in L^2(\mathbb{R}^n)$ and the Parseval–Plancherel identity holds $\|\widehat{f}\|_{L^2} = (2\pi)^{n/2}\|f\|_{L^2}$, this is the "isometry" on $L^2(\mathbb{R}^n)$.*
3. *$\widehat{f*g} = \widehat{f}\widehat{g}$.*

Proof. 1 is trivial. 2 relies on the density of the Schwartz space $\mathcal{S}$ in L^2 and the completeness of L^2, which we refer to [7, Chapter 2]. We now prove 3.

$$\begin{aligned}
(\widehat{f*g})(\xi) &= \int_{\mathbb{R}} e^{-ix\xi}\left(\int_{\mathbb{R}} f(y)g(x-y)dy\right)dx \\
&= \int_{\mathbb{R}} e^{-iy\xi}e^{-i(x-y)\xi}\int_{\mathbb{R}} f(y)g(x-y)dydx \\
&= \int_{\mathbb{R}} e^{-iy\xi}\left(\int_{\mathbb{R}} e^{-i(x-y)\xi}g(x-y)dx\right)f(y)dy && \text{(Fubini)} \\
&= \int_{\mathbb{R}} e^{-iy\xi}\widehat{g}(\xi)f(y)dy \\
&= \widehat{g}(\xi)\int_{\mathbb{R}} e^{-iy\xi}f(y)dy = \widehat{f}(\xi)\widehat{g}(\xi).
\end{aligned}$$

□

Now we introduce the idea of fundamental solution to PDEs, which is important and very useful for elliptic and parabolic PDEs. As with the case of Itô's formula and Feynman-Kac Theorem, we only present the main results. Readers interested in the rigorous setup of distribution theory and functional analysis bases and their proofs may refer [7, Chapter 2] and [16].

Definition 2.6. *Let L be a linear partial differential operator. The fundamental solution E is the solution to the $LE = \delta(x)$, where everything is taken in the sense of distributions in $\mathcal{D}'(\mathbb{R}^n)$.*

Proposition 2.9. *If E is the fundamental solution of $LE = \delta$, then for $f \in L^1_0(\mathbb{R}^n)$, $u = E * f$, taken in the sense of convolution of distributions, is the solution to $Lu = f$.*

Corollary 2.10. *The fundamental solution obtained from $LE = \delta$ can also be used to solve the homogeneous Cauchy problem $Lu = 0$ subjected to $u|_{t=0} = f$, where the solution is given by $u = E * f$.*

Remark 2.2. When we consider the Cauchy problem $Lu = 0$ with $u|_{t=0} = f$, suppose E is the fundamental solution to $LE = 0, E|_{t=0} = \delta$, then by Corollaries 2.8 and 2.10, $\widehat{u} = \widehat{E}\widehat{f}$ is the solution in Fourier transform. $\widehat{E}$ where $\widehat{E|_{t=0}} = 1$ is sometimes referred to as the *fundamental transform*.

3. A Rapid Review of the BSM Formula

3.1. **Log-normal stock.** At first glance, a Brownian motion seems to be adaptable for a model of stock prices. Like stocks prices, these processes take on only positive values and exhibit noise. However, Wiener processes assume a constant expected drift rate, whereas a stock's expected return is constant with a variable drift rate. This is because the stock's expected return required by an investor is independent of the stock's price. For example, if an investor desires a 20% return on a stock, we assume that goal does not change whether the stock price is \$10 or \$100. Still, Wiener processes are adaptable for stock price modeling.

We now state the log-normal stochastic differential equation that models stock prices, which is the basis for BSM model. Assume that on average, the price of a share of stock S_t grows with continuously compounded interest at a constant rate a. Next, we assume that there is a up and down "normal" random motion with expected value of zero and standard deviation σ called volatility. Thus, we obtain

$$\frac{dS_t}{S_t} = adt + \sigma dW_t, \text{ or } dS_t = aS_tdt + \sigma S_tdW_t, \quad t \geq 0, \tag{3.1}$$

and this is the **log-normal** model for stock prices, and is the basis for the BSM model. Figure 3.1 illustrates a simulation of a log-normal stock price.

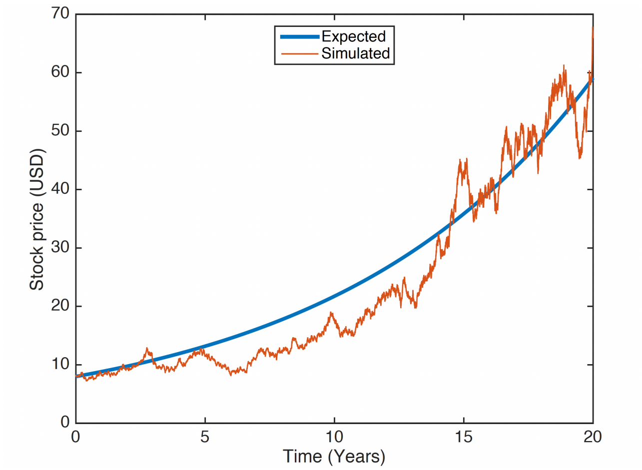

FIGURE 3.1. Simulation of a stock price under the log-normal model [8].

3.2. **The BSM equation and formula.** Suppose that the stock price S_t satisfies (3.1), where a is the stock's mean rate of growth, σ is the volatility of the stock, and W_t is a Wiener process starting at zero. Consider a European call option for this stock with strike price K and exercise time T. Therefore, its pay-off at time T is $F(S_T, T) = [S_T - K]^+ \doteq \max\{S_T - K, 0\}$. Denote by $F(s,t)$ the value of the call option at time t if the stock price is $S_t = s$, and r the risk-free interest rate. We consider the idea of replicating portfolio to derive the BSM equation like in [8]. Assume that at time $t = 0$ we begin with a portfolio of value X_0. At any time $t \geq 0$, we invest in the following two financial instruments:

- A money market with a constant interest rate r,
- A stock whose time evolution S_t is modeled by (3.1).

Denote by X_t the value of our portfolio at time t, we construct a portfolio such that its value is equal to the value of the call option at that time, i.e.,

$$X_t = F(S_t, t).$$

Assume that at each time t, our portfolio consists of Δ_t shares of stock, and therefore, $X_t - \Delta_t S_t$ in the money market. Then the differential dX_t is given by

$$dX_t = \Delta_t dS_t + r(X_t - \Delta_t S_t)dt.$$

Considering the present values of X_t and $V(S_t, t)$ and taking differentials on both sides, we have

$$d(e^{-rt}X_t) = d(e^{-rt}F(S_t, t)).$$

Apply the Itô's formula to both sides and equating, we get the **Delta-hedging formula**

$$\Delta_t = \frac{\partial F}{\partial x}(S_t, t). \tag{3.2}$$

Substituting back we get the **Black-Scholes-Merton partial differential equation**

$$\frac{\partial F}{\partial t}(S_t, t) + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2}(S_t, t) + rs\frac{\partial F}{\partial s}(S_t, t) - rF(S_t, t) = 0,$$

considering $F = F(s,t)$, we obtain the following BSM equation

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2} + rs\frac{\partial F}{\partial s} - rF = 0, \quad t \in [0, T), \quad s \geq 0. \tag{3.3}$$

It is subjected to the terminal condition considering the pay-off at time T

$$F(s,T) \doteq \lim_{t\to T^-} F(S,T) = [s-K]^+. \tag{3.4}$$

Apply the following change of variables

$$u(\xi,\eta) = e^{r\tau}F(s,t),\ \ \eta(\tau) = \frac{\sigma^2}{2}\tau,\ \ \xi(s,\tau) = \ln s + \left(r - \frac{\sigma^2}{2}\right)\tau,\ \ \tau(t) = T-t,$$

the system (3.3, 3.4) is transformed into the following standard heat equation Cauchy problem

$$\frac{\partial u}{\partial \eta} - \frac{\partial^2 u}{\partial \xi^2} = 0, \quad \xi \in \mathbb{R}, \eta \in (0, \sigma^2 T/2]$$
$$u(\xi,0) = \lim_{\eta\to 0^+} e^{r\tau}F(s,t) = [e^\xi - K]^+.$$

Now we use the solution formula for the Cauchy problem of the heat equation (see [6]) to find

$$u(\xi,\eta) = \frac{1}{\sqrt{4\pi\eta}}\int_{\ln K}^{\infty}(e^y - K)\exp\left[-\frac{(y-\xi)^2}{4\eta}\right]dy.$$

Substituting the original variables back and rearranging, we get the solution to (3.3, 3.4), the famous **Black-Scholes-Merton pricing formula**

$$F(s,\tau) = sN(d_+) - e^{-r\tau}KN(d_-),\ d_\pm = \frac{\ln(s/K) + \left(r \pm \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}},\ \tau = T-t, \tag{3.5}$$

where $N(x)$ the standard normal distribution

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}dy.$$

3.3. **The Greeks.** Components of the BSM have been individually named and are often used by traders and academics to study an option's sensitivity to market conditions. Each of these components is assigned a Greek symbol, and they are colloquially known as "the Greeks". This section provides some information about each.

3.3.1. *Delta* (Δ). Delta represents the change of the option's price with respect to the underlying's price, $\Delta = \frac{\partial F}{\partial S}$. For traders, delta provides an approximate probability the option will expire in-the-money. It also provides an important metric for creating a delta-neutral position, in which the combined deltas of a portfolio is zero. This means, within a certain range of market movements, the investor's position does not change. A delta-neutral portfolio may be created for hedging purposes.

3.3.2. *Theta* (Θ). Theta represents the change of the option's price with respect to time, $\Theta = \frac{\partial F}{\partial t}$. Theta is higher for options at-the-money and lower for options in- and out-of-the-money. It is used by traders to indicate theta-decay, or time-decay, the phenomenon that an option loses value as time goes on, all else being equal. Options lose value over time because there is less time for the underlying's price to move in a favorable direction, so the option is more likely to expire worthless. This is important to traders, as they must consider how the option will decline in value if they plan to buy or sell it for a profit. For example, even if an option's value increases due to market movements, theta-decay may cause its value to be too low to make a profit when sold.

3.3.3. *Gamma* (Γ). Gamma represents the change of the option's delta with respect to the underlying's price, $\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 F}{\partial S^2}$. Like theta, it is higher for options at-the-money and lower for options in- and out-of-the-money. It also generally increases nearer to expiry. Traders use this to measure the stability of the option's delta, which could be important for strategies relying on specific values of delta. Using both is sometimes referred to as delta-gamma-hedging.

3.3.4. *Vega* (ν)*).* Vega[2] represents the change of the option's value with respect to the underlying's implied volatility, $\nu = \frac{\partial F}{\partial \sigma}$. It does not actually feature explicitly in the BSM. Traders can use vega to assess the exposure of their portfolios to market movements, as it indicates the direction an option's price is expected to move with increased volatility.

3.3.5. *Rho* (ρ). Rho represents the change of the option's value with respect to the risk-free rate of interest, $\rho = \frac{\partial F}{\partial r}$. Like vega, rho does not actually feature explicitly in the BSM. It is generally considered the least important of the Greeks.

4. Assumptions on the Heston Model

Some limitations of the BSM model are the assumption of a constant volatility rate and the lack of discontinuous jumps. The volatility of actual stocks vary, possibly stochastically. Stock prices also jump drastically in response to news, events, or market conditions, which would need to be modeled as discontinuities [9]. Since the generalized Wiener process is continuous, this is not feasible–but perhaps the attempt to model the complexities of real-world conditions is unreasonable anyway.

The Heston Model is considered a more accurate representation of the market compared to simpler models such as the BSM model. In the Heston model, the volatility of the stock is not constant but varies over time in a random fashion, following a specific stochastic process. This variation in volatility is an important factor in modeling financial markets as it reflects the uncertainty and risk associated with the asset's price movements. In [12], the following assumptions are made for the Heston model:

1. The asset price follows a stochastic process known as a geometric Brownian motion, which is the same process used in the Black-Scholes model.
2. The volatility of asset price is also a stochastic process, which follows a mean-reverting square-foot diffusion process. It means that the volatility fluctuates randomly around a long-term average value and tends to return to this value over time.
3. The asset price and volatility can be correlated.
4. The Efficient Market Hypothesis holds, so there are no arbitrage opportunities in the market.

Thus, the dynamics of the **Heston model** is given by

$$\begin{cases} dS_t = \mu S_t dt + \sqrt{v_t} S_t dZ_1 \\ dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dZ_2 \end{cases} \tag{4.1}$$

where v_t denotes the variance of the stock at time t, S_t the stock price at t, μ the unconditional mean return of the stock, $\theta > 0$ the unconditional mean volatility towards which v_t tends, $\kappa > 0$ the constant coefficient of volatility mean reversion, $\sigma > 0$ the constant volatility of volatility v_t, and Z_1 and Z_2 are Wiener processes of asset price and volatility respectively. We also denote by ρ the correlation between the two Brownian motions Z_1, Z_2, more precisely

$$dZ_1 dZ_2 = \rho dt, \ \ \rho \in (-1, 1). \tag{4.2}$$

We note the following properties of the volatility process v_t. When $\sigma^2 > 2\kappa\theta$, 0 is attainable boundary and is strongly reflecting, so the volatility is never negative, and ∞ is an unattainable boundary, meaning that volatility cannot grow arbitrarily large.

[2] "Vega" is not actually the name of the Greek character ν, which is called "nu". The name "vega" seems to come from traders associating Greek letters with Latin counterparts and, evidently, not knowing Greek.

For convenience later on, we write Z_1, Z_2 in terms of two uncorrelated Brownian motions

$$dZ_1 = dW_1$$
$$dZ_2 = \rho dW_1 + \sqrt{1-\rho^2}dW_2$$

with $dW_1 dW_2 = 0$, such that the system (4.1) is transformed into the matrix form

$$\begin{pmatrix} dS \\ dv \end{pmatrix} = \begin{pmatrix} \mu S \\ \kappa(\theta - v) \end{pmatrix} dt + \begin{pmatrix} \sqrt{v}S & 0 \\ \sigma\rho\sqrt{v} & \sigma\sqrt{1-\rho^2}\sqrt{v} \end{pmatrix} \begin{pmatrix} dW_1 \\ dW_2 \end{pmatrix}, \tag{4.3}$$

which is an Itô diffusion that we can solve later.

5. The Heston Model Pricing Equation

Let $F(S_t, v_t, t)$ denote the price of an option. We apply the usual replicating portfolio method to derive the pricing equation as in the BSM model. Since there are two state variables S and v, we consider the replicating portfolio consists of three independent assets:

- The option of interest F^1 on an underlying stock,
- A different option F^2 on the same stock,
- That underlying stock S_t satisfying (4.1).

Denote by X_t the value of our portfolio at time t, we have

$$X_t = F_t^1 - aF_t^2 - \Delta_t S_t \quad \forall t$$

with weights a and Δ both functions of t. The subscript t denoting the value at time t will be omitted from now on for simplicity. It should be emphasized that unlike the BSM case where the portfolio replicates the option of interest, we now construct a **risk-free portfolio** by investing in the options. Taking differentials of X we get

$$dX = dF^1 - adF^2 - \Delta dS. \tag{5.1}$$

Now we apply the generalized Itô's formula (Proposition 2.2 and Remark 2.1) to get for $i = 1, 2$

$$\begin{aligned} dF^i &= \frac{\partial F^i}{\partial t}dt + \frac{\partial F^i}{\partial v}dv + \frac{\partial F^i}{\partial S}dS + \frac{1}{2}\left(\frac{\partial^2 F^i}{\partial S^2}(dS)^2 + \frac{\partial^2 F^i}{\partial v^2}(dv)^2 + 2\frac{\partial^2 F^i}{\partial S \partial v}dSdv\right) \\ &= \frac{\partial F^i}{\partial t}dt + \frac{\partial F^i}{\partial v}dv + \frac{\partial F^i}{\partial S}dS + \frac{1}{2}vS^2\frac{\partial^2 F^i}{\partial S^2}dt + \frac{1}{2}\sigma^2 v\frac{\partial^2 F^i}{\partial v^2}dt + \rho\sigma vS\frac{\partial^2 F^i}{\partial S\partial v}dt. \end{aligned}$$

Substituting back to (5.1) we get

$$\begin{aligned} dX = &\left[\frac{\partial F^1}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 F^1}{\partial S^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2 F^1}{\partial v^2} + \rho\sigma vS\frac{\partial^2 F^1}{\partial S\partial v}\right]dt \\ &- a\left[\frac{\partial F^2}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 F^2}{\partial S^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2 F^2}{\partial v^2} + \rho\sigma vS\frac{\partial^2 F^2}{\partial S\partial v}\right]dt \\ &+ \left[\frac{\partial F^1}{\partial S} - a\frac{\partial F^2}{\partial S} - \Delta\right]dS + \left[\frac{\partial F^1}{\partial v} - a\frac{\partial F^2}{\partial v}\right]dv. \end{aligned}$$

Since we are replicating a risk-free portfolio and assuming no arbitrage, X must satisfy

$$dX = rXdt, \tag{5.2}$$

where r is the risk-free interest rate. To achieve this, the dS and dv terms must be zero, so we get

$$a = \frac{\partial F^1/\partial v}{\partial F^2/\partial v} \text{ and } \Delta = \frac{\partial F^1}{\partial S} - a\frac{\partial F^2}{\partial S},$$

analogous to the Delta-hedging formula (3.2). Substituting a and Δ back to (5.2) we get

$$\begin{aligned}\left[\frac{\partial F^1}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 F^1}{\partial S^2}+\frac{1}{2}\sigma^2 v\frac{\partial^2 F^1}{\partial v^2}+\rho\sigma vS\frac{\partial^2 F^1}{\partial S\partial v}\right]&\\ -\frac{\partial F^1/\partial v}{\partial F^2/\partial v}\left[\frac{\partial F^2}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 F^2}{\partial S^2}+\frac{1}{2}\sigma^2 v\frac{\partial^2 F^2}{\partial v^2}+\rho\sigma vS\frac{\partial^2 F^2}{\partial S\partial v}\right]&\\ =r\left[F^1-\frac{\partial F^1/\partial v}{\partial F^2/\partial v}\cdot F^2-\frac{\partial F^1}{\partial S}\cdot S\ +\frac{\partial F^1/\partial v}{\partial F^2/\partial v}\cdot\frac{\partial F^2}{S}\cdot S\right]&,\end{aligned}$$

rearranging terms with coefficient $\frac{\partial F^1/\partial v}{\partial F^2/\partial v}$, we get

$$\begin{aligned}\frac{\partial F^1}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 F^1}{\partial S^2}+\frac{1}{2}\sigma^2 v\frac{\partial^2 F^1}{\partial v^2}+\rho\sigma vS\frac{\partial^2 F^1}{\partial S\partial v}-rF^1+rS\frac{\partial F^1}{\partial S}&\\ =\frac{\partial F^1/\partial v}{\partial F^2/\partial v}\left[\frac{\partial F^2}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 F^2}{\partial S^2}+\frac{1}{2}\sigma^2 v\frac{\partial^2 F^2}{\partial v^2}+\rho\sigma vS\frac{\partial^2 F^2}{\partial S\partial v}-rF^2+rS\frac{\partial F^2}{\partial S}\right]&,\end{aligned}$$

dividing both sides by $\partial F^1/\partial v$, we get

$$\begin{aligned}\frac{\frac{\partial F^1}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 F^1}{\partial S^2}+\frac{1}{2}\sigma^2 v\frac{\partial^2 F^1}{\partial v^2}+\rho\sigma vS\frac{\partial^2 F^1}{\partial S\partial v}-rF^1+rS\frac{\partial F^1}{\partial S}}{\partial F^1/\partial v}&\\ =\frac{\frac{\partial F^2}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 F^2}{\partial S^2}+\frac{1}{2}\sigma^2 v\frac{\partial^2 F^2}{\partial v^2}+\rho\sigma vS\frac{\partial^2 F^2}{\partial S\partial v}-rF^2+rS\frac{\partial F^2}{\partial S}}{\partial F^2/\partial v}&,\end{aligned}$$

adding $\kappa(\theta-v)$ to both sides we get

$$\begin{aligned}\frac{\frac{\partial F^1}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 F^1}{\partial S^2}+\frac{1}{2}\sigma^2 v\frac{\partial^2 F^1}{\partial v^2}+\rho\sigma vS\frac{\partial^2 F^1}{\partial S\partial v}-rF^1+rS\frac{\partial F^1}{\partial S}+\kappa(\theta-v)\frac{\partial F^1}{\partial v}}{\partial F^1/\partial v}&\\ =\frac{\frac{\partial F^2}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 F^2}{\partial S^2}+\frac{1}{2}\sigma^2 v\frac{\partial^2 F^2}{\partial v^2}+\rho\sigma vS\frac{\partial^2 F^2}{\partial S\partial v}-rF^2+rS\frac{\partial F^2}{\partial S}+\kappa(\theta-v)\frac{\partial F^2}{\partial v}}{\partial F^2/\partial v}&.\end{aligned}$$

Next, to simplify presentation, we recall that the Heston model (4.1) is an Itô diffusion in the form

$$\begin{pmatrix} dS\\ dv\end{pmatrix}=\begin{pmatrix}\mu S\\ \kappa(\theta-v)\end{pmatrix}dt+\begin{pmatrix}\sqrt{v}S & 0\\ \sigma\rho\sqrt{v} & \sigma\sqrt{1-\rho^2}\sqrt{v}\end{pmatrix}\begin{pmatrix}dW_1\\ dW_2\end{pmatrix},$$

by Theorem 2.3 we have

$$\begin{pmatrix}\sqrt{v}S & 0\\ \sigma\rho\sqrt{v} & \sigma\sqrt{1-\rho^2}\sqrt{v}\end{pmatrix}\begin{pmatrix}\sqrt{v}S & 0\\ \sigma\rho\sqrt{v} & \sigma\sqrt{1-\rho^2}\sqrt{v}\end{pmatrix}^T=\begin{pmatrix}vS^2 & \rho\sigma vS\\ \rho\sigma vS & \sigma^2 v\end{pmatrix},$$

thus we can compute its generator

$$\mathcal{A}=\mu S\frac{\partial}{\partial S}+\kappa(\theta-v)\frac{\partial}{\partial v}+\frac{1}{2}\left(vS^2\frac{\partial^2}{\partial S^2}+\sigma^2 v\frac{\partial^2}{\partial v^2}+2\rho\sigma vS\frac{\partial^2}{\partial S\partial v}\right).$$

Note the similarity between $\mathcal{A}$ and the numerators of the long equality. We define the new operator

$$\mathcal{A}_H:=rS\frac{\partial}{\partial S}+\kappa(\theta-v)\frac{\partial}{\partial v}+\frac{1}{2}\sigma^2 v\frac{\partial^2}{\partial v^2}+\frac{1}{2}vS^2\frac{\partial^2}{\partial S^2}+\rho\sigma vS\frac{\partial^2}{\partial S\partial v}. \tag{5.3}$$

Simplifying using the operator $\mathcal{A}_H$, we get the following equality

$$\frac{\frac{\partial F^1}{\partial t}+\mathcal{A}_H F^1-rF^1}{\partial F^1/\partial v}=\frac{\frac{\partial F^2}{\partial t}+\mathcal{A}_H F^2-rF^2}{\partial F^2/\partial v}.$$

We note that the left side is an expression only of F^1, and the right side only of F^2. Since we assume F^1 and F^2 are independent, we conclude that both sides are equal to a function independent of F, we denote by $\Lambda(S,v,t)$. Equating $\Lambda(S,v,t)$ to either side produces the **Heston model pricing equation**

$$\frac{\partial F}{\partial v}\Lambda(S,v,t) = \frac{\partial F}{\partial t} + \mathcal{A}_H F - rF. \tag{5.4}$$

Expanding out $\mathcal{A}_H$ and rearranging, we get the following partial differential equation

$$\frac{\partial F}{\partial t} + rS\frac{\partial F}{\partial S} + \big[\kappa(\theta - v) - \Lambda(S,v,t)\big]\frac{\partial F}{\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 F}{\partial v^2} + \frac{1}{2}vS^2\frac{\partial^2 F}{\partial S^2} + \rho\sigma vS\frac{\partial^2 F}{\partial S\partial v} - rF = 0. \tag{5.5}$$

This is equation (6) in [4]. Note that when v is constant, all terms with $\frac{\partial}{\partial v}$ disappear, and we get back the BSM option pricing PDE

$$\frac{\partial F}{\partial t} + rS\frac{\partial F}{\partial S} + \frac{1}{2}vS^2\frac{\partial^2 F}{\partial S^2} - rF = 0,$$

where $v = \sigma_{BS}^2$ is the constant variance, thus we can view the log-normal stock model and the BSM equation as a special case of the Heston model, that is, when the volatility is constant.

There is a special meaning to the function $\Lambda(S,v,t)$, that it represents the volatility risk premium. volatility risk premium is the difference between expected future volatility of some asset and the current implied volatility. For strike prices that are out of the money, volatility may need to increase beyond the current implied volatility in order for the strike to be realized. This is a profitable opportunity for volatility sellers due to risk aversions from investors. Most investors are willing to sacrifice some of their return for more stability.

On the right hand side, recall that the expression $-rF$ is the funding cost of market maker issuing an option F, or, in other works, the cost of borrowing money for the option; $\frac{\partial F}{\partial t}$ represents the time decay of the option (Theta from the BSM Greeks, see 3.3.2); and $\mathcal{A}_H F$ reflects the changes or sensitivities of the option due to the dynamics of the underlying and the volatility process.

6. Solving the Heston Model for European Call Options

To solve (5.5), we must specify the term $\Lambda(S,v,t)$ to obtain the full PDE, and the boundary condition it is subjected to, which is determined by the type of option in consideration.

6.1. The pricing PDE.

We start with Λ. Heston assumes that the volatility risk premium is proportional to v [4]. As discussed above, with higher implied volatility, there should be higher volatility risk premiums due to increased risk aversion in investor's behaviors. Therefore, we need to impose an assumption on how volatility impacts volatility risk premium. We assume that for simplicity of the model, Heston proposed to assume that the relationship is positively proportional (by a constant).

This makes economical sense because the market price of risk refers to the additional compensation or return that investors require for taking on additional risk. Unlike interest rates, volatility is not directly trade-able. Instead, people hold options or variance swaps as volatility sensitive products to hedge the volatility variability. The amount of products held are linearly proportional to the risk. Thus we have the linear relationship

$$\Lambda(S,v,t) = \lambda v.$$

Replacing Λ into (5.5) we get the following partial differential equation

$$\frac{\partial F}{\partial t} + rS\frac{\partial F}{\partial S} + \big[\kappa(\theta - v) - \lambda v\big]\frac{\partial F}{\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 F}{\partial v^2} + \frac{1}{2}vS^2\frac{\partial^2 F}{\partial S^2} + \rho\sigma vS\frac{\partial^2 F}{\partial S\partial v} - rF = 0. \tag{6.1}$$

We simplify (6.1) by taking the following change of variables similar to the BSM case

$$\tau = T - t, \;\; x = \ln S.$$

We first note that $\tau = T - t$ is a more natural variable than t since the value of the option is determined backwards from time T to time t. The second transform is obtained because (6.1) is Cauchy-Euler (equi-dimensional) in the variable S, so setting $x = \ln S$ makes the S-dependent coefficients become constants in the equation by the chain rule. So we get

$$-\frac{\partial F}{\partial \tau} + \frac{1}{2} v \frac{\partial^2 F}{\partial x^2} + \left(r - \frac{1}{2} v\right) \frac{\partial F}{\partial x} + \rho \sigma v \frac{\partial^2 F}{\partial x \partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 F}{\partial v^2} + \big[\kappa(\theta - v) - \lambda v\big] \frac{\partial F}{\partial v} - rF = 0. \tag{6.2}$$

We note that this cannot be further simplified and solved like BSM. We follow the method of [10] and [12, Chapter 4]. It is precisely taking $x = \ln S$ that allows for this method, because we transformed the original domain from $S \in \mathbb{R}_{\geq 0}$ to $x \in \mathbb{R}$, which allows us to take the Fourier transform. By Lemma 2.6, taking Fourier transform with respect to x and rearranging, we have

$$-\frac{\partial \widehat{F}}{\partial \tau} + (i\xi r - r)\,\widehat{F} - \frac{1}{2} v(\xi^2 + i\xi)\widehat{F} + \big[\kappa(\theta - v) - \lambda v + i\xi\rho\sigma v\big] \frac{\partial \widehat{F}}{\partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 \widehat{F}}{\partial v^2} = 0. \tag{6.3}$$

This is significantly simpler than (6.2) because $\widehat{F}$ has only derivatives with respect to τ and v, and not x as in (6.1) or (6.2). We apply a change of dependent variable similar to the BSM case and write

$$\widehat{F}(\xi, v, \tau) = \exp\big[(i\xi r - r)\tau\big] \widehat{H}(\xi, v, \tau),$$

substituting $\widehat{H}$ back to (6.3) we get

$$\frac{\partial \widehat{H}}{\partial \tau} = \frac{1}{2}\sigma^2 v \frac{\partial^2 \widehat{H}}{\partial v^2} + \big[\kappa(\theta - v) - \lambda v + i\xi\rho\sigma v\big] \frac{\partial \widehat{H}}{\partial v} - \frac{1}{2} v(\xi^2 + i\xi)\widehat{H}. \tag{6.4}$$

6.2. **Boundary condition.** We focus our attention on the case of a European call option with strike price K and exercise time T. Then, $F(S, v, t)$ is subject to the following boundary condition

$$F(S, v, T) = [S_T - K]^+, \tag{6.5}$$

applying the same change of variables we get

$$F(x, v, 0) = [e^x - K]^+. \tag{6.6}$$

We also take its Fourier transform

$$\begin{aligned} \widehat{F}(\xi, v, 0) &= \int_{\mathbb{R}} e^{-ix\xi} [e^x - K]^+ dx \\ &= \int_{\ln K}^{\infty} e^{-ix\xi}(e^x - K) dx \\ &= \left(\frac{e^{(-i\xi+1)x}}{-i\xi + 1} - K \frac{e^{-ix\xi}}{-i\xi} \right) \Bigg|_{x=\ln K}^{\infty} \end{aligned}$$

Note that $\exp ix\xi$ defines a unit circle in the complex plane. For convergence of both $\exp -ix\xi$ and $\exp{(-i\xi + 1)x}$ as $x \to \infty$, we must bound the modules of both terms, i.e.,

$$\begin{aligned} \left| e^{(-i\xi+1)x} \right| &= \exp \mathrm{Re}(-i\xi + 1)x \\ &= \exp x(1 - \mathrm{Re}(i\xi)) \\ &= \exp x(1 + \mathrm{Im}\xi) < \infty \text{ as } x \to \infty, \end{aligned}$$

so we get $\mathrm{Im}\xi < -1$. The other term gives $\mathrm{Im}\xi < 0$. Thus we have the transformed initial condition

$$\widehat{F}(\xi, v, 0) = -\frac{K^{1-i\xi}}{\xi^2 + i\xi}, \quad \mathrm{Im}\xi < -1. \tag{6.7}$$

Like in [10], we also provide in Table 6.1 the generalized Fourier transforms of selected financial claims.

Financial claim	Payoff function	Payoff transform	Strip of regularity
Call option	$[S_T - K]^+$	$-\dfrac{K^{1-i\xi}}{\xi^2 + i\xi}$	$\mathrm{Im}\xi < -1$
Put option	$[K - S_T]^+$	$-\dfrac{K^{1-i\xi}}{\xi^2 + i\xi}$	$\mathrm{Im}\xi > 0$
Covered call	$\min\left[S_T, K\right]$	$\dfrac{K^{1-i\xi}}{\xi^2 + i\xi}$	$-1 < \mathrm{Im}\xi < 0$
Delta function	$\delta_0\left(\ln \dfrac{S_T}{K}\right)$	$K^{-i\xi}$	$\mathbb{C}$
Money market	1	$2\pi\delta(\xi)$	$\mathbb{C}$

Table 6.1. Generalized Fourier transforms for selected financial claims.

There is a remark when recovering the initial condition from the Fourier transform. When we introduced the Fourier transform pair $\mathcal{F}$ and $\mathcal{F}^{-1}$, they both integrate along $\mathbb{R}^n$. This cannot be applied to $\widehat{F}$ because it is not well-defined on $\mathbb{R}$. To recover F, we must use the generalized (complex) Fourier transform (the construction uses a similar contour integration seen in the proof of Lemma 2.4, see [14, Chapter 4] for reference). We integrate along the strip of regularity, which for European call options is $\mathrm{Im}\xi < -1$ under the convention in this paper. So we have, for some $k \in \mathbb{C}$ where $\mathrm{Im}k < -1$

$$F(x, v, \tau) = \frac{1}{2\pi} \int_{ik-\infty}^{ik+\infty} e^{ix\xi} \widehat{F}(\xi, v, \tau) d\xi.$$

6.3. **The fundamental transform.** We are now to solve system (6.4, 6.7). We use the notion of fundamental transform like in [10]. From Remark 2.2, once we know the fundamental transform $\widehat{E}$, all we need to do is multiply by the payoff transform $\widehat{F|_{\tau=0}}$ to get $\widehat{H}$. We then multiply by $\exp\left[(i\xi r - r)\tau\right]$ to get $\widehat{F}$. Lastly, we take the inverse Fourier transform along the strip of regularity and change the variables back and get $F(S, v, t)$, the desired value of the option. Figure 6.1 illustrates the overall approach we take to find the solution F.

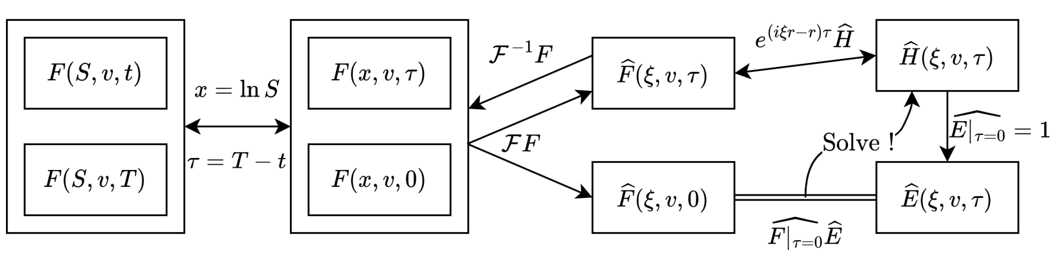

Figure 6.1. The solution map of Heston model using Fourier transform.

To make this procedure work, we must find a strip where a fundamental solution exists. We define

Definition 6.1. *We say the Cauchy problem (6.4) is* regular[3] *if there exists a fundamental transform to (6.4) which is regular as a function of ξ within a strip $\alpha < \mathrm{Im}\xi < \beta$, where α and β are real numbers. We call this strip the* fundamental strip of regularity*.*

The call option payoff transform exists for $\mathrm{Im}\xi < -1$, therefore, the call option solution exists only under the assumption that the fundamental strip of regularity has $\alpha < -1$. That is, we assume the fundamental strip of regularity and the strip of regularity for payoff transforms intersect. If they don't, then this solution method does not work, but there is always alternative formula that exists, see [10]. In typical cases, $\alpha < -1$ and $\beta > 0$, and we shall continue with this approach.

We assume from now on $\lambda = 0$ for the fundamental transform $\widehat{E}$. Then the PDE for $\widehat{E}$ becomes:

$$\frac{\partial \widehat{E}}{\partial \tau} = \frac{1}{2}\sigma^2 v \frac{\partial^2 \widehat{E}}{\partial v^2} + \big[\kappa(\theta - v) + i\xi\rho\sigma v\big]\frac{\partial \widehat{E}}{\partial v} - \frac{1}{2}v(\xi^2 + i\xi)\widehat{E}. \tag{6.8}$$

This PDE can be converted into a form similar to Riccati equation[4] for which the solution is known. Define $\eta = \sigma^2\tau/2$, (6.8) becomes

$$\begin{aligned}\frac{\partial \widehat{E}}{\partial \eta} &= v\frac{\partial^2 \widehat{E}}{\partial v^2} + \frac{2}{\sigma^2}\big[\kappa(\theta - v) + i\xi\rho\sigma v\big]\frac{\partial \widehat{E}}{\partial v} - \frac{\xi^2 + i\xi}{\sigma^2}v\widehat{E} \\ &= v\frac{\partial^2 \widehat{E}}{\partial v^2} + \tilde{\kappa}(\tilde{\theta} - v)\frac{\partial \widehat{E}}{\partial v} - \tilde{c}(\xi)v\widehat{E},\end{aligned} \tag{6.9}$$

where

$$\tilde{\kappa} = \frac{2(\kappa - i\xi\rho\sigma)}{\sigma^2}, \;\; \tilde{\theta} = \frac{\kappa\theta}{\kappa - i\xi\rho\sigma}, \;\; \tilde{c}(\xi) = \frac{\xi^2 + i\xi}{\sigma^2}.$$

The second line in (6.9) is a parabolic equation in v, and has a solution of the form[5]

$$\widehat{E}(\xi, v, \eta) = \exp\big[C(\eta) + D(\eta)v\big], \tag{6.10}$$

with boundary condition $\widehat{E}(\xi, v, 0) = 1$, so $C(0) = D(0) = 0$. Take the derivatives of $\widehat{E}(\xi, v, \eta)$, substitute into (6.9), and cancel $\widehat{E}$ on both sides, we get

$$\frac{\partial C}{\partial \eta} + \frac{\partial D}{\partial \eta}v = vD^2 + \tilde{\kappa}(\tilde{\theta} - v)D - \tilde{c}v.$$

Now equate terms with v and without v, we get the set of equations:

$$\begin{cases} \dfrac{\partial D}{\partial \eta} = D^2 - \tilde{\kappa}D - \tilde{c} \\ \dfrac{\partial C}{\partial \eta} = \tilde{\kappa}\tilde{\theta}D \end{cases} \tag{6.11}$$

Note that the first equation of (6.11) is exactly in the form of a Riccati equation. Therefore, we can set up the second order ordinary differential equation (ODE) [5]

$$w'' + \tilde{\kappa}w' - \tilde{c}w = 0$$

and define

$$\alpha = \frac{-\tilde{\kappa} + \tilde{d}}{2}, \beta = \frac{-\tilde{\kappa} - \tilde{d}}{2}, \text{ where } \tilde{d} = \sqrt{\tilde{\kappa} + 4\tilde{c}}.$$

[3] A function $f(z)$ is *regular* if it is analytic and single-valued in a domain.

[4] We will not cover Riccati equations or their solution methods here. Interested readers are encouraged to consult [5, Chapter 4] for a comprehensive treatment of Riccati's equations.

[5] This is a reasonable guess because differentiating with respect to η produces a v in front of $\partial_v^i \widehat{E}$ by the chain rule.

Therefore, the solution to this Riccati equation is

$$D(\eta) = -\frac{K\alpha e^{\alpha\eta} + \beta e^{\beta\eta}}{Ke^{\alpha\eta} + e^{\beta\eta}}.$$

Considering the initial condition $D(0) = 0$, we obtain

$$K = \frac{\tilde{\kappa} + \tilde{d}}{-\tilde{\kappa} + \tilde{d}},$$

which gives

$$D(\eta) = \frac{\tilde{\kappa} + \tilde{d}}{2} \left(\frac{1 - e^{\tilde{d}\eta}}{1 - ge^{\tilde{d}\eta}} \right), \tag{6.12}$$

where

$$g = -K = \frac{\tilde{\kappa} + \tilde{d}}{\tilde{\kappa} - \tilde{d}}.$$

Now looking at the second equation of (6.11), $C(\eta)$ can be found via integrating D from (6.12).

$$\begin{aligned}
\int_0^\eta D(y)dy &= \frac{\tilde{\kappa} + \tilde{d}}{2} \int_0^\eta \left(\frac{1 - e^{\tilde{d}y}}{1 - ge^{\tilde{d}y}} \right) dy + K_1 \\
&= \frac{\tilde{\kappa} + \tilde{d}}{2d} \left[\tilde{d}\eta + \frac{1-g}{g} \ln \left(\frac{1 - ge^{\tilde{d}\eta}}{1-g} \right) \right] + K_1 \\
&= \left[\frac{\tilde{\kappa} + \tilde{d}}{2}\eta - \ln \left(\frac{1 - ge^{\tilde{d}\eta}}{1-g} \right) \right] + K_1,
\end{aligned}$$

where K_1 is an integrating constant. Substituting in initial condition $C(0) = 0$, we get $K_1 = 0$. Hence,

$$C(\eta) = \tilde{\kappa}\tilde{\theta} \left[\frac{\tilde{\kappa} + \tilde{d}}{2}\eta - \ln \left(\frac{1 - ge^{\tilde{d}\eta}}{1-g} \right) \right]. \tag{6.13}$$

6.4. **Wrapping everything together.** We follow Figure 6.1. Thus far, we have found the fundamental transform $\widehat{E}(\xi, v, \tau)$, where it is given by

$$\begin{cases}
\widehat{E} = \exp\left[C(\eta) + D(\eta)v\right] \\
C(\eta) = \tilde{\kappa}\tilde{\theta} \left[\dfrac{\tilde{\kappa} + \tilde{d}}{2}\eta - \ln \left(\dfrac{1 - ge^{\tilde{d}\eta}}{1-g} \right) \right] \\
D(\eta) = \dfrac{\tilde{\kappa} + \tilde{d}}{2} \left(\dfrac{1 - e^{\tilde{d}\eta}}{1 - ge^{\tilde{d}\eta}} \right) \\
g = \dfrac{\tilde{\kappa} + \tilde{d}}{\tilde{\kappa} - \tilde{d}} \qquad \tilde{d} = \sqrt{\tilde{\kappa} + 4\tilde{c}} \\
\tilde{\kappa} = \dfrac{2(\kappa - i\xi\rho\sigma)}{\sigma^2} \qquad \tilde{\theta} = \dfrac{\kappa\theta}{\kappa - i\xi\rho\sigma} \\
\eta = \dfrac{\sigma^2\tau}{2} \qquad \tau = T - t
\end{cases}$$

Now, recall that $\widehat{H}$ can be obtained from simply multiplying $\widehat{E}$ by the payoff transform since $\widehat{E}$ is the fundamental transform. So we have for European call options

$$\widehat{H}(\xi, v, \tau) = -\left(\frac{K^{1-i\xi}}{\xi^2 + i\xi} \right) \widehat{E}(\xi, v, \tau).$$

Recall also that

$$\widehat{F}(\xi, v, \tau) = \exp\left[(i\xi r - r)\tau\right]\widehat{H}(\xi, v, \tau).$$

Taking the generalized inverse Fourier transform along the strip of regularity of both $\widehat{E}$ and the payoff transform, we finally have for some $k \in \mathbb{C}$ where $\alpha < \mathrm{Im}k < -1$, the **pricing formula for European call options under the Heston model**

$$F(x, v, \tau, K) = \frac{1}{2\pi}\int_{ik-\infty}^{ik+\infty} e^{ix\xi}\exp\left[(i\xi r - r)\tau\right]\left(-\frac{K^{1-i\xi}}{\xi^2 + i\xi}\right)\widehat{E}(\xi, v, \tau)d\xi, \text{ where } x = \ln S. \tag{6.14}$$

If we let $X = \ln(S/K) + r\tau$, then F can be written in the form

$$F(X, v, \tau, K) = -\frac{Ke^{-r\tau}}{2\pi}\int_{ik-\infty}^{ik+\infty} e^{i\xi X}\frac{\widehat{E}(\xi, v, \tau)}{\xi^2 + i\xi}d\xi, \quad \alpha < \mathrm{Im}k < -1. \tag{6.15}$$

Figure 6.2 illustrates the strip of integration for both equations in the complex plane.

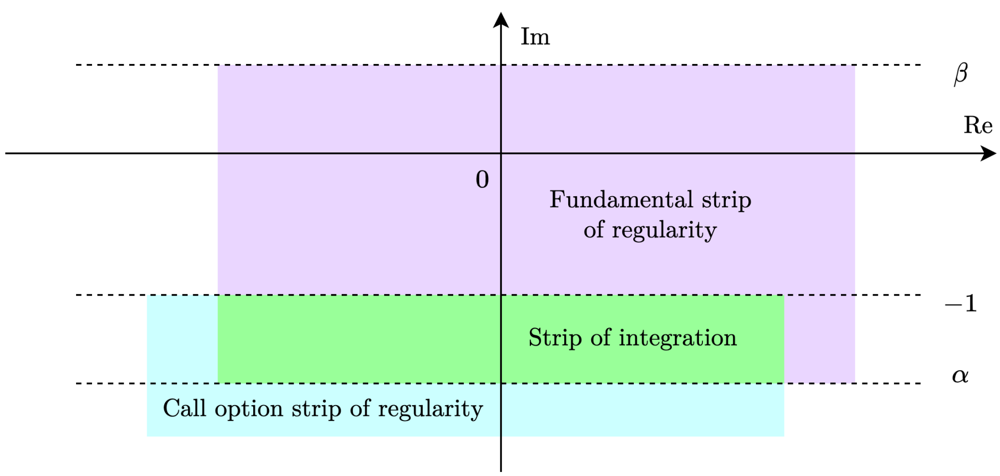

FIGURE 6.2. Illustration of the strip of integration of (6.14) and (6.15).

We must note that there are more delicate considerations regarding the fundamental strip of regularity as well as potential singularities of the terms in (6.14) and (6.15) as we are working in the complex domain, for which the integration needs to be taken care of. However, this is beyond the scope of this paper. Interested readers can consult [10] for detailed analysis.

7. Data Testing of the Heston Model

While we have the Heston model, it still requires a significant amount of work to apply it. In order to use the Heston model, we need to know the parameters required in the model. In theory, we are given parameters such as the mean volatility and volatility of volatility. However, in real-life situations, these parameters are not given. Therefore, we need a mechanism to produce estimated parameters. In this section, we discuss two separate parameter estimation – the method using loss function and maximum likelihood estimation.

7.1. **Estimation using Loss Function.** One of the most common approach to produce estimation of the parameters in Heston model is through **Loss Function**. Loss Function captures how much the predicted value deviates from the actual value, and to optimize an estimation, we want to minimize the loss function. There are many different kinds of loss functions, here we will introduce three:

1. Mean Error Sum of Squares (MSE)
2. Relative Mean Error Sum of Squares (RMSE)
3. Implied Volatility Mean Error Sum of Squares (IVMSE)

In order to estimate the parameters, we need to know the market price of options maturing at τ_t with Strike price K_k, denoted as $C(\tau_t, K_k) = C_{tk}$ and a corresponding model price $C(\tau_t, K_k; \Theta) = C_{tk}^{\Theta}$. Therefore, the loss function method is only useful when we have a reliable set of data on quoted option prices in the market.

When using **MSE** to obtain optimal estimation, we are minimizing the function

$$\frac{1}{N}\sum_{t,k} w_{tk}(C_{tk} - C_{tk}^{\Theta})^2,$$

where N denotes the number of quotes used, and w_{tk} denotes the weight assigned to each quote.

MSE minimizes the squared difference between the quoted and model prices. However, deep out-of-the-money options and short-maturity options contribute very little to the function, so the optimization will be more well-fitting for the long-maturity, in-the-money options (call options with strike price less than stock price and put options with strike price above stock price). Therefore, we have an underestimation for certain options.

RMSE slightly improves on the underestimation of out-of-the-money, short-maturity options from MSE. When using RMSE to obtain optimal estimation, we are minimizing the function

$$\frac{1}{N}\sum_{t,k} w_{tk}\frac{(C_{tk} - C_{tk}^{\Theta})^2}{C_{tk}}$$

with N, w_{tk} the same as that of MSE. Since we are dividing C_{tk}, options with low market value will contribute a lot more to the function and thus we have an overestimation for these options.

Therefore, for both MSE and RMSE we assign weights to the individual data points to mitigate the over(under)-estimation problems. However, the assignments of the weights can often be subjective and we face the potential of manipulating data sets and over-fitting.

Aside from using the option prices, we could also use implied volatility to produce loss functions, such as **IVMSE**. We will minimize the loss function

$$\frac{1}{N}\sum_{t,k} w_{tk}(IV_{tk} - IV_{tk}^{\Theta})^2,$$

where IV_{tk} denotes the quoted implied volatility of the option and IV_{tk}^{Θ} denotes the model implied volatility. Using implied volatility improves on the over(under)-estimation issues of MSE and RMSE because it no longer depends on the option prices. However, it introduced a new issue of calculation intensity, since for each quote, we will need to compute the implied volatility. From BSM, we know that the bisection method is helpful in getting the implied volatility, but for large data sets, bisection method would require too much computational power. Therefore, one of the plausible ways is to use the ν greek in the BSM model. In BSM, ν denotes the sensitivity of the option price to change in volatility, so we could rewrite the IVMSE loss function into:

$$\frac{1}{N}\sum_{t,k} w_{tk}\frac{(C_{tk} - C_{tk}^{\Theta})^2}{\nu^2}.$$

While using ν is better than using bisection method to get implied volatility in terms of computational power, it comes at a cost of losing precision in the prediction power of IVMSE .

Therefore, we have discussed how three commonly used loss functions: MSE, RMSE, IVMSE predict parameters in the Heston model and their respective weaknesses. Figure 7.1 shows the prediction of implied volatilities from MSE method compared to the observed market implied volatilities, and we could see that the prediction is more accurate for long-maturity options, which aligns with our previous discussion on MSE weaknesses.

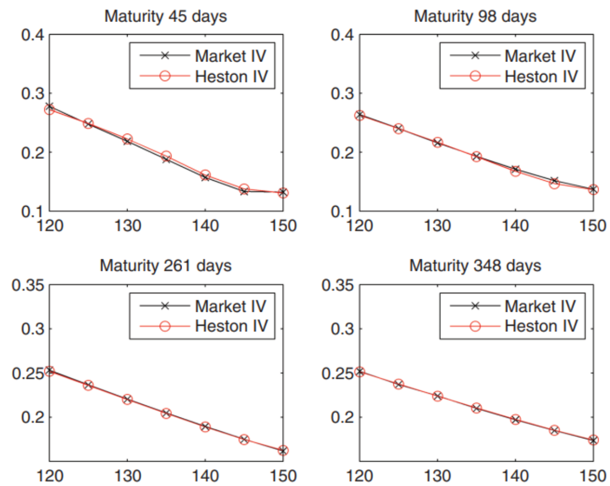

FIGURE 7.1. S&P500 Market and Heston implied volatilities estimated using MSE [12].

7.2. **Data testing using Maximum Likelihood Estimator.** Maximum Likelihood Estimator, or MLE, refers to an estimator that selects values for parameters that maximizes the likelihood function and then optimize the parameter estimates. For example, in the discrete case, let $\{y_i\}_{i=1}^n$ be the sample observations of random variable $\{Y_i\}_{i=1}^n$ whose distribution depends on a parameter σ. Suppose $\{Y_i\}$ is a set of discrete random variables, the likelihood function (of the data) denoted $L(y_1, ..., y_n \mid \sigma)$ is defined to be the joint probability mass function evaluated at y_i. In short, the likelihood function is a function of the parameters that measures how well the observed data fit in the probability distribution according to the unknown parameters. The prior distribution is analyzed through historical data.

With likelihood function defined, the MLE of unknown parameter σ is defined as

$$\widehat{\sigma}_{MLE} = \arg\max_{\sigma\in\mathbb{R}} L(y_1, ..., y_n \mid \sigma).$$

where $\arg\max$ is the maximizer of the function. Note that the log function is strictly increasing, therefore, maximizing $L(y_1, ..., y_n \mid \sigma)$ with respect to σ is equivalent to maximizing $\ln(L(y_1, ..., y_n \mid \sigma))$. Therefore,

$$\widehat{\sigma}_{MLE} = \arg\max_{\sigma\in\mathbb{R}} \ln(L(y_1, ..., y_n \mid \sigma)).$$

In the case of using Heston model to find option pricing, we could consider it to be a problem of finding the MLE of the parameters r, θ, κ, ρ, σ, based on the prior distribution of our parameters. Solving for MLE, we take partial derivatives of the likelihood function with respect to each parameters and set it to 0, call the resulting vector of the parameters $\widehat{\alpha}_{MLE}$. The likelihood function of the data is defined as the product of the probability density functions of each observation.

$$\ell(r, \kappa, \theta, \sigma, \rho) = \sum_{t=1}^{n} \ln(f(Q_{t+1}, v_{t+1}) \mid r, \kappa, \theta, \sigma, \rho).$$

where Q_{t+1} is asset return $(S_{t+1})/(S_t)$ and v_{t+1} is asset price variance.

In Dunn's paper, the authors utilized real data from market on 36 different options. They used the MLE method to estimate for Heston model parameters, and calculated the respective predictions on the option prices. Comparing the predictions from Heston model with the predictions from BSM and the observed data from market, the results showed that statistically, the Heston model produced more accurate results [3]. In Figure 7.2 we present a graph which displays the observed price of an option, the price of the option as predicted by BSM, and the price of an option as predicted by the Heston model using MLE. As shown below, the Heston model seems to perform better than the BSM model.

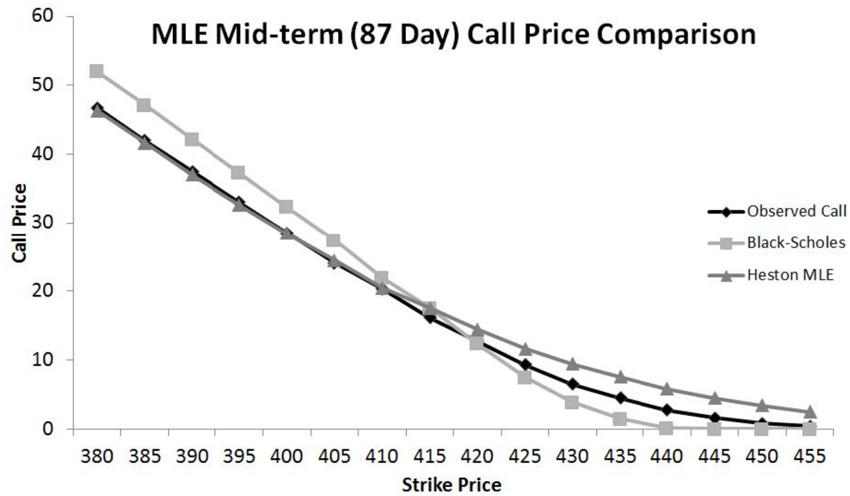

Figure 7.2. BSM, Heston, and observed predicted price of an option [3].

Through visual inspection, we see promising results regarding the application of Heston model to option price estimation. The estimates of the Heston Model are closer than the estimates of the Black-Scholes model to the actual call option prices in our data sample.

8. Summary

Although there's no evidences of Heston model being used directly in companies' systems due to privacy concerns, there are traces of the model being actively used in the industry by examples of its appearance in financial platforms. Heston model is provided in leading financial software. Bloomberg is a leading financial software company that provide a financial terminal to over 300,000 users worldwide. It's Derivative Library (DLIB) is a comprehensive platform which is part of the terminal for pricing and analyzing derivatives, structured products and dynamic strategies, and Heston model is provided among other quantitative models.

The Heston model has been widely used in the financial industry for a variety of applications, including option pricing, risk management, and portfolio optimization. It has also been used in academic research to study the behavior of financial markets and to develop new financial models.

The Heston model improves upon the Black-Scholes model by incorporating stochastic volatility, which is a more realistic assumption for a number of financial markets. This allows the Heston model to capture both the volatility smile and volatility term structure. The Heston model does a better job of capturing the impact of changes in volatility on option prices compared to the Black-Scholes Model. All in all, the Heston model can be seen as an improved version of the Black-Scholes model as it addresses a number of concerns with the latter.

One relevant assumption that still is up for discussion is the usage of a continuous time model for the stock price. In reality, stock movement is not a continuous random variable, rather it is discrete. However, we do not see this assumption as a major problem in the reality of financial markets.

Overall, the Heston model is a valuable tool for financial analysts and researchers who are interested in understanding and predicting the behavior of financial markets and the prices of financial instruments such as options. In the future, it would be beneficial to compare the results of the Heston model to those of other stochastic volatility models

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