Derivation of the Heston Model Option Pricing Formula

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Options

An option is a contract that gives the holder the right, but not the obligation, to buy or sell an underlying asset at a specified price and time. Options are commonly used to manage risk or speculate on movement of securities.

There are two main types of options: calls and puts.

- A call gives the holder the right to buy the underlying asset at a specified price
- A put gives the holder the right to sell the underlying at a set price.

Black, Scholes, and Merton

Fischer Black

Myron Scholes

Robert Merton

Creators of the BSM model

Heston

In the year 1993, the Heston model for option pricing was introduced in a paper published by Steve Heston, a finance professor hailing from the University of Maryland.

It addresses some of the unrealistic assumptions of BSM - that the volatility of an underlying asset is constant - by modeling the volatility of the underlying as a stochastic process.

Steven Heston

Brownian motion

We give the important results for Itô integrals and Itô's formula. Financial interpretation of Itô integral – if W_t is the asset price, $f(t_0), f(t_1), \ldots, f(t_{n-1})$ as the number of shares in the portfolio held until the next trading date, then the Itô integral gives the gain from trading at all dates.

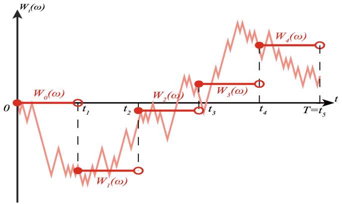

Figure 1: Example of a simple stochastic process from a general process.

Itô's Lemma

Theorem (Generalized Itô's formula)

Let X_t be be an n-dimensional Itô process of the form

$$d\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} dt + \begin{pmatrix} v_{11} & \cdots & v_{1m} \\ \vdots & \ddots & \vdots \\ v_{n1} & \cdots & v_{nm} \end{pmatrix} \begin{pmatrix} dW_1 \\ \vdots \\ dW_m \end{pmatrix}.$$

Let $g(t,x) = (g_1(t,x), \ldots, g_p(t,x))$ be a map from $C^2([0,\infty) \times \mathbb{R}^n])$ into $\mathbb{R}^p$. Then $Y_t = g(t, X_t)$ is also an Itô process, whose components Y_k, $1 \le k \le p$ is given by

$$dY_k = \frac{\partial g_k}{\partial t}(t,X)dt + \sum_i \frac{\partial g_k}{\partial x_i}(t,X)dX_i + \frac{1}{2}\sum_{i,j} \frac{\partial^2 g_k}{\partial x_i \partial x_j}(t,X)dX_i dX_j.$$

Basic Fourier Analysis

Theorem

For $f \in L^1(\mathbb{R}^n)$, the Fourier transform $\mathcal{F}$ is the operator

$$\mathcal{F} : f \to \widehat{f} \text{ such that } \widehat{f}(\xi) \doteq \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(x)dx.$$

The inverse operator of $\mathcal{F}$, the inverse Fourier transform is given by

$$\mathcal{F}^{-1} : f \to \tilde{f} \text{ such that } \tilde{f}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} f(\xi)d\xi.$$

If $\widehat{f}, f \in L^1$, then $\mathcal{F}^{-1}(\mathcal{F}f) = f$, and if $\tilde{f}, f \in L^1$, then $\mathcal{F}(\mathcal{F}^{-1}f) = f$.

Basic PDE

Theorem

Let L be a linear partial differential operator. The fundamental solution E is the solution to the $LE = \delta(x)$, where everything is taken in the sense of distributions in $\mathcal{D}'(\mathbb{R}^n)$.

If E is the fundamental solution of $LE = \delta$, then for $f \in L_0^1(\mathbb{R}^n)$, $u = E * f$, taken in the sense of convolution of distributions, is the solution to $Lu = f$.

The fundamental solution obtained from $LE = \delta$ can also be used to solve the homogeneous Cauchy problem $Lu = 0$ subjected to $u|_{t=0} = f$, where the solution is given by $u = E * f$.

Log-normal stock

Assume that the price of a share of stock S_t satisfies the **log-normal** model:

$$dS_t = aS_t dt + \sigma S_t dW_t, \quad t \geq 0.$$

where a is the stock's mean rate of growth, σ is the volatility of the stock, and W_t is a Wiener process starting at zero. Consider a European call option for this stock with strike price K and exercise time T. Therefore, its pay-off at time T is

$$F(S_T, T) = [S_T - K]^+ \doteq \max\{S_T - K, 0\}.$$

The BSM equation

By replicating portfolio that consists of Δ_t shares of stock and $X_t - \Delta_t S_t$ in the money market, we have $dX_t = \Delta_t dS_t + r(X_t - \Delta_t S_t)dt$. Considering the present values of X_t and $V(S_t, t)$ and taking differentials on both sides, we have

$$d(e^{-rt}X_t) = d(e^{-rt}F(S_t, t)).$$

Apply the Itô's formula to both sides and equating, we get the **Delta-hedging formula**

$$\Delta_t = \frac{\partial F}{\partial x}(S_t, t).$$

Substituting back we get the **Black-Scholes-Merton PDE**

$$\frac{\partial F}{\partial t}(S_t, t) + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2}(S_t, t) + rs\frac{\partial F}{\partial s}(S_t, t) - rF(S_t, t) = 0.$$

The BSM formula

Apply the following change of variables

$$u(\xi,\eta) = e^{r\tau}F(s,t), \qquad \eta(\tau) = \frac{\sigma^2}{2}\tau,$$

$$\xi(s,\tau) = \ln s + \left(r - \frac{\sigma^2}{2}\right)\tau, \qquad \tau(t) = T - t,$$

the system is transformed into the following standard heat equation Cauchy problem

$$\frac{\partial u}{\partial \eta} - \frac{\partial^2 u}{\partial \xi^2} = 0, \quad \xi \in \mathbb{R}, \eta \in (0, \sigma^2 T/2]$$

$$u(\xi,0) = \lim_{\eta\to 0^+} e^{r\tau}F(s,t) = [e^{\xi} - K]^+.$$

The BSM formula (cont.)

Now we use the solution formula for the Cauchy problem of the heat equation to find $u(\xi, \eta)$. Substituting the original variables back and rearranging, we get the famous **Black-Scholes-Merton pricing formula**

$$F(s, \tau) = sN(d_+) - e^{-r\tau} KN(d_-), \tag{1}$$

$$d_\pm = \frac{\ln(s/K) + \left(r \pm \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \ \tau = T - t, \tag{2}$$

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy.$$

1. History
2. Math Preliminaries
3. The BSM Model
4. **The Heston Model**
5. Data Testing
6. References

Assumptions

The dynamics of the **Heston model** is given by

$$\begin{cases} dS_t = \mu S_t dt + \sqrt{v_t} S_t dZ_1 \\ dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dZ_2 \end{cases} \tag{3}$$

where v_t denotes the variance of the stock at time t, S_t the stock price at t, and Z_1 and Z_2 are Wiener processes of asset price and volatility respectively. We also denote by ρ the correlation between the two Brownian motions Z_1, Z_2, more precisely

$$dZ_1 dZ_2 = \rho dt, \;\; \rho \in (-1, 1).$$

Replicating portfolio

Let $F(S_t, v_t, t)$ denote the price of an option. Since there are two state variables S and v, we consider the replicating portfolio consists of three independent assets:

- The option of interest F^1 on an underlying stock,
- A different option F^2 on the same stock,
- That underlying stock S_t satisfying (3).

Denote by X_t the value of our portfolio at time t, we have

$$X_t = F_t^1 - aF_t^2 - \Delta_t S_t \quad \forall t$$

It should be emphasized that unlike the BSM case where the portfolio replicates the option of interest, we now construct a **risk-free portfolio** by investing in the options.

Replicating portfolio (cont.)

Taking differentials of X we get

$$dX = dF^1 - adF^2 - \Delta dS.$$

Now we apply the generalized Itô's formula to get dF^1 and dF^2. Since we are replicating a risk-free portfolio and with no arbitrage, X must satisfy

$$dX = rXdt,$$

where r is the risk-free interest rate. To achieve this, the dS and dv terms must be zero, so we get

$$a = \frac{\partial F^1/\partial v}{\partial F^2/\partial v} \text{ and } \Delta = \frac{\partial F^1}{\partial S} - a\frac{\partial F^2}{\partial S},$$

analogous to the Delta-hedging formula of BSM.

Heston pricing equation

We define the Heston generator

$$\mathcal{A}_H := rS\frac{\partial}{\partial S} + \kappa(\theta - v)\frac{\partial}{\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2}{\partial v^2} + \frac{1}{2}vS^2\frac{\partial^2}{\partial S^2} + \rho\sigma vS\frac{\partial^2}{\partial S\partial v}. \quad (4)$$

Simplifying using the operator $\mathcal{A}_H$, we get the following equality

$$\frac{\frac{\partial F^1}{\partial t} + \mathcal{A}_H F^1 - rF^1}{\partial F^1/\partial v} = \frac{\frac{\partial F^2}{\partial t} + \mathcal{A}_H F^2 - rF^2}{\partial F^2/\partial v}.$$

Setting $\Lambda(S, v, t)$, we produce the **Heston model pricing equation**

$$\boxed{\frac{\partial F}{\partial v}\Lambda(S, v, t) = \frac{\partial F}{\partial t} + \mathcal{A}_H F - rF.} \quad (5)$$

Choosing volatility risk premium

$$\frac{\partial F}{\partial t} + rS\frac{\partial F}{\partial S} + \frac{1}{2}vS^2\frac{\partial^2 F}{\partial S^2} - rF + \left[\kappa(\theta - v) - \Lambda(S,v,t)\right]\frac{\partial F}{\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 F}{\partial v^2} + \rho\sigma vS\frac{\partial^2 F}{\partial S\partial v} = 0. \quad (6)$$

Heston assumes that the volatility risk premium is proportional to v

$$\Lambda(S,v,t) = \lambda v.$$

Replacing Λ into (6) we get the following partial differential equation

$$\frac{\partial F}{\partial t} + rS\frac{\partial F}{\partial S} + \frac{1}{2}vS^2\frac{\partial^2 F}{\partial S^2} - rF + \left[\kappa(\theta - v) - \lambda v\right]\frac{\partial F}{\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 F}{\partial v^2} + \rho\sigma vS\frac{\partial^2 F}{\partial S\partial v} = 0. \quad (7)$$

Taking log-moneyness

We simplify (7) by taking the following change of variables

$$\tau = T - t, \;\; x = \ln S.$$

The second transform is obtained because (7) is Cauchy-Euler in the variable S, so setting $x = \ln S$ makes the S-dependent coefficients become constants in the equation by the chain rule. So we get

$$\begin{aligned} &-\frac{\partial F}{\partial \tau} + \frac{1}{2} v \frac{\partial^2 F}{\partial x^2} + \left(r - \frac{1}{2} v\right) \frac{\partial F}{\partial x} - rF \\ &\qquad + \rho \sigma v \frac{\partial^2 F}{\partial x \partial v} + \frac{1}{2} \sigma^2 v \frac{\partial^2 F}{\partial v^2} + \left[\kappa(\theta - v) - \lambda v\right] \frac{\partial F}{\partial v} = 0. \qquad (8) \end{aligned}$$

Taking Fourier transform

Taking Fourier transform with respect to x we have

$$-\frac{\partial \widehat{F}}{\partial \tau} + (i\xi r - r)\,\widehat{F} - \frac{1}{2}v(\xi^2 + i\xi)\widehat{F} + \left[\kappa(\theta - v) - \lambda v + i\xi\rho\sigma v\right]\frac{\partial \widehat{F}}{\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 \widehat{F}}{\partial v^2} = 0. \quad (9)$$

Let

$$\widehat{F}(\xi, v, \tau) = \exp\left[(i\xi r - r)\tau\right]\widehat{H}(\xi, v, \tau),$$

we get

$$\frac{\partial \widehat{H}}{\partial \tau} = \frac{1}{2}\sigma^2 v\frac{\partial^2 \widehat{H}}{\partial v^2} + \left[\kappa(\theta - v) - \lambda v + i\xi\rho\sigma v\right]\frac{\partial \widehat{H}}{\partial v} - \frac{1}{2}v(\xi^2 + i\xi)\widehat{H}. \quad (10)$$

European call options

We focus on a European call option with strike price K and exercise time T. Then, $F(S, v, t)$ is subject to the following boundary condition

$$F(S, v, T) = [S_T - K]^+, \tag{11}$$

applying the same change of variables we get

$$F(x, v, 0) = [e^x - K]^+. \tag{12}$$

We also take its Fourier transform

$$\begin{aligned}
\widehat{F}(\xi, v, 0) &= \int_{\ln K}^{\infty} e^{-ix\xi}(e^x - K)dx \\
&= \left(\frac{e^{(-i\xi+1)x}}{-i\xi + 1} - K\frac{e^{-ix\xi}}{-i\xi} \right) \Bigg|_{x=\ln K}^{\infty} \\
&= -\frac{K^{1-i\xi}}{\xi^2 + i\xi}, \quad \mathrm{Im}\xi < -1.
\end{aligned}$$

Boundary conditions

Financial claim	Payoff function	Payoff transform	Regularity
Call option	$[S_T - K]^+$	$-\dfrac{K^{1-i\xi}}{\xi^2 + i\xi}$	$\mathrm{Im}\xi < -1$
Put option	$[K - S_T]^+$	$-\dfrac{K^{1-i\xi}}{\xi^2 + i\xi}$	$\mathrm{Im}\xi > 0$
Covered call	$\min[S_T, K]$	$\dfrac{K^{1-i\xi}}{\xi^2 + i\xi}$	$-1 < \mathrm{Im}\xi < 0$
Delta function	$\delta_0\left(\ln \dfrac{S_T}{K}\right)$	$K^{-i\xi}$	$\mathbb{C}$
Money market	1	$2\pi\delta(\xi)$	$\mathbb{C}$

Table 1: Generalized Fourier transforms for selected financial claims.

Solution approach

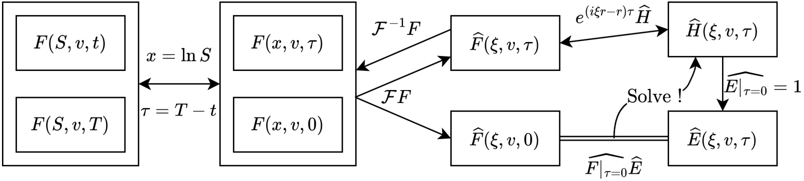

Figure 2: The solution map of Heston model using Fourier transform.

Fundamental transform

Recall that if E is the fundamental solution to $LE = 0, E|_{t=0} = \delta$, then $\widehat{u} = \widehat{E * f} = \widehat{E}\widehat{f}$ is the solution in Fourier transform. $\widehat{E}$ where $\widehat{E|_{t=0}} = 1$ is sometimes referred to as the *fundamental transform*. We assume that the fundamental transform is regular in ξ within a strip $\alpha < \mathrm{Im}\xi < \beta$. We call this strip the *fundamental strip of regularity*.

We wish to find fundamental transform to (10). Here we assume $\lambda = 0$. Let $\eta = \sigma^2\tau/2$, (10) becomes

$$\frac{\partial \widehat{E}}{\partial \eta} = v\frac{\partial^2 \widehat{E}}{\partial v^2} + \tilde{\kappa}(\tilde{\theta} - v)\frac{\partial \widehat{E}}{\partial v} - \tilde{c}(\xi) v\widehat{E}, \tag{13}$$

where

$$\tilde{\kappa} = \frac{2(\kappa - i\xi\rho\sigma)}{\sigma^2}, \quad \tilde{\theta} = \frac{\kappa\theta}{\kappa - i\xi\rho\sigma}, \quad \tilde{c}(\xi) = \frac{\xi^2 + i\xi}{\sigma^2}.$$

Fundamental transform (cont.)

(13) is parabolic in v, so we guess the solution

$$\widehat{E}(\xi, v, \eta) = \exp\left[C(\eta) + D(\eta)v\right], \tag{14}$$

with boundary condition $\widehat{E}(\xi, v, 0) = 1$, so $C(0) = D(0) = 0$. Take the derivatives of $\widehat{E}(\xi, v, \eta)$, substitute into (13) we get

$$\frac{\partial C}{\partial \eta} + \frac{\partial D}{\partial \eta} v = vD^2 + \tilde{\kappa}(\tilde{\theta} - v)D - \tilde{c}v.$$

Now equate terms with v and without v, we get the set of equations:

$$\begin{cases} \dfrac{\partial D}{\partial \eta} = D^2 - \tilde{\kappa}D - \tilde{c} \\ \dfrac{\partial C}{\partial \eta} = \tilde{\kappa}\tilde{\theta}D \end{cases} \tag{15}$$

Fundamental transform (cont.)

Note that the first equation of (15) is exactly in the form of a Riccati equation, for which the solution D is known. Then $C(\eta)$ can be found via integrating D from (16). We impose the boundary conditions on C and D and get

$$D(\eta) = \frac{\tilde{\kappa} + \tilde{d}}{2} \left(\frac{1 - e^{\tilde{d}\eta}}{1 - g e^{\tilde{d}\eta}} \right), \tag{16}$$

$$C(\eta) = \tilde{\kappa}\tilde{\theta} \left[\frac{\tilde{\kappa} + \tilde{d}}{2}\eta - \ln \left(\frac{1 - g e^{\tilde{d}\eta}}{1 - g} \right) \right], \tag{17}$$

where

$$g = \frac{\tilde{\kappa} + \tilde{d}}{\tilde{\kappa} - \tilde{d}}, \quad \tilde{d} = \sqrt{\tilde{\kappa} + 4\tilde{c}}.$$

Wrapping everything together

We now have found the fundamental transform $\widehat{E}(\xi, v, \tau)$

$$\begin{cases} \widehat{E} = \exp\left[C(\eta) + D(\eta)v\right] \\ C(\eta) = \tilde{\kappa}\tilde{\theta}\left[\dfrac{\tilde{\kappa}+\tilde{d}}{2}\eta - \ln\left(\dfrac{1-ge^{\tilde{d}\eta}}{1-g}\right)\right] \\ D(\eta) = \dfrac{\tilde{\kappa}+\tilde{d}}{2}\left(\dfrac{1-e^{\tilde{d}\eta}}{1-ge^{\tilde{d}\eta}}\right) \\ g = \dfrac{\tilde{\kappa}+\tilde{d}}{\tilde{\kappa}-\tilde{d}} \qquad \tilde{d} = \sqrt{\tilde{\kappa}+4\tilde{c}} \\ \tilde{\kappa} = \dfrac{2(\kappa - i\xi\rho\sigma)}{\sigma^2} \qquad \tilde{\theta} = \dfrac{\kappa\theta}{\kappa - i\xi\rho\sigma} \\ \eta = \dfrac{\sigma^2\tau}{2} \qquad \tau = T - t \end{cases}$$

Wrapping everything together (cont.)

For European call options,

$$\widehat{F}(\xi, v, 0) = -\frac{K^{1-i\xi}}{\xi^2 + i\xi}, \quad \mathrm{Im}\xi < -1.$$

Thus, we have found the **pricing formula for European call options under the Heston model**

$$\boxed{F(X, v, \tau, K) = -\frac{Ke^{-r\tau}}{2\pi} \int_{ik-\infty}^{ik+\infty} e^{i\xi X} \frac{\widehat{E}(\xi, v, \tau)}{\xi^2 + i\xi} d\xi} \tag{18}$$

where

$$X = \ln \frac{S}{K} + r\tau, \;\; \tau = T - t, \;\; \alpha < \mathrm{Im}k < -1.$$

Final remark

Figure 3 illustrates the strip of integration for both equations in the complex plane. We must note that there are more delicate considerations regarding the fundamental strip of regularity as well as potential singularities of the terms in (18) for which the integration needs to be taken care of. However, this is beyond the scope of this paper.

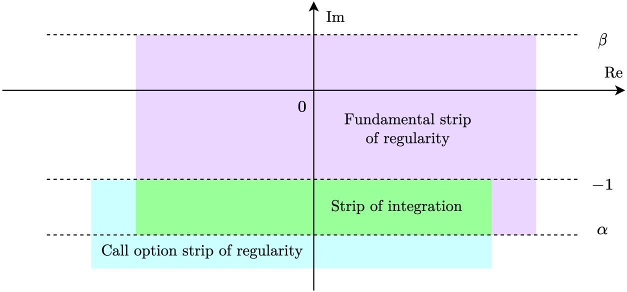

Figure 3: Illustration of the strip of integration of (18).

Estimation with loss function (MSE, RMSE)

Mean Error Sum of Squares (MSE):

$$\text{MSE} = \frac{1}{N} \sum_{t,k} w_{tk} (C_{tk} - C_{tk}^{\Theta})^2$$

Issue: underestimation for short maturity, out-of-the-money options.
Relative Mean Error Sum of Squares (RMSE):

$$\text{RMSE} = \frac{1}{N} \sum_{t,k} w_{tk} \frac{(C_{tk} - C_{tk}^{\Theta})^2}{C_{tk}}$$

Issue: overestimation for low market-price options.

Estimation with loss function (IVMSE)

Implied Volatility Mean Error Sum of Squares (IVMSE):

$$\text{IVMSE} = \frac{1}{N}\sum_{t,k} w_{tk}(IV_{tk} - IV_{tk}^{\Theta})^2,$$

Issue: Numerically intensive, requires a lot of computational power.
Improvement with ν - the sensitivity of the option price to change in volatility:

$$\frac{1}{N}\sum_{t,k} w_{tk}\frac{(C_{tk} - C_{tk}^{\Theta})^2}{\nu^2}.$$

Issue: comes at a cost of loss in precision.

MSE predictions

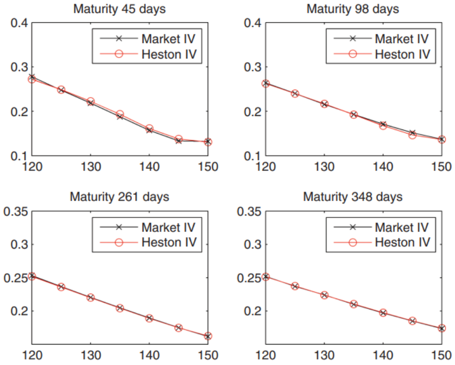

Figure 4: S&P500 Market and Heston implied volatilities estimated using MSE [12].

Maximum Likelihood Estimator (MLE)

The likelihood function: a function of the parameters that measures how well the observed data fit in the probability distribution according to the unknown parameters.
The MLE estimator of unknown parameter σ is defined as

$$\widehat{\sigma}_{MLE} = \arg\max_{\sigma\in\mathbb{R}} L(y_1, ..., y_n \mid \sigma).$$

Find r, θ, κ, ρ, σ that optimize

$$\ell(r,\kappa,\theta,\sigma,\rho) = \sum_{t=1}^{n} \ln\left(f\left(Q_{t+1}, v_{t+1}\right) \mid r,\kappa,\theta,\sigma,\rho\right).$$

where Q_{t+1} is asset return $(S_{t+1})/(S_t)$ and v_{t+1} is asset price variance.

MLE predictions

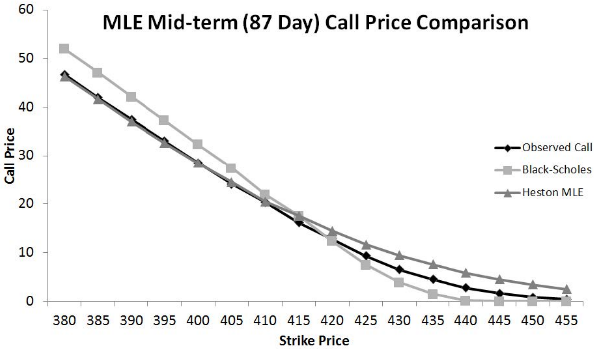

Figure 5: BSM, Heston, and observed predicted price of an option [3].

Summary

- The Heston model has been widely used in the financial industry for a variety of applications, including option pricing, risk management, and portfolio optimization.
- The Heston model improves upon the BSM model by incorporating stochastic volatility, which is a more realistic assumption for a number of financial markets.
- The Heston model does a better job of capturing the impact of changes in volatility on option prices compared to the BSM Model.

Thanks!

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