NONEXISTENCE OF SOLUTIONS OF CERTAIN PDES MATH-M-541 PARTIAL DIFFERENTIAL EQUATIONS II

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1. Introduction

In this presentation we discuss some nonexistence of solutions of certain partial differential equations. This topic complements well the topics discussed throughout the course, i.e. existence, uniqueness, and regularity theory. We follow the presentation in [1, Chapter 9.4].

2. Nonlinear Parabolic Equation

2.1. **Quadratic nonlinearity.** We consider the following initial-boundary value problem (ibvp) in a bounded domain Ω and time $0 \le t \le T$

$$\begin{cases} u_t - \Delta u = u^2 & \text{in } \Omega \times (0,T] \\ u = g & \text{on } \Omega \times \{t=0\} \\ u = 0 & \text{on } \partial\Omega \times [0,T]. \end{cases} \tag{2.1}$$

So g is the initial condition, and has 0 Dirichlet boundary condition. We know that the heat operator has a smoothing effect, while the quadratic nonlinearity has a "blow-up" effect. We can see this through the ODE $u' = u^2$, which has solution $1/(c-t)$ that goes to infinity in finite time. Thus it is reasonable to expect that for large enough initial data g, certain quantities related to the solution u will tend to infinity within time T. We formalize the result in the following theorem.

Theorem 2.2 (Blow-up for large data)**.** *Let $\lambda_1 > 0$ be the principle eigenvalue of $-\Delta$ in $H_0^1(\Omega)$, and $w_1 > 0$ be the corresponding eigenfunction with $\int_\Omega w_1 = 1$. Assume that $g \ge 0$ and*

$$\int_\Omega g w_1 dx > \lambda_1. \tag{2.3}$$

Then there cannot exist a smooth solution u to (2.1) *for all times $T > 0$.*

Remark 2.4. Recall properties of the principle eigenvalue and its eigenfunction, that λ_1 is simple, w_1 is smooth if $\partial\Omega$ is smooth, and the eigenfunctions $\{w_k\}_k$ form a basis of $L^2(\Omega)$. Thus the assumption (2.3) tells the first Fourier coefficient (up to $\|w_1\|_{L^2}$) of g.

Proof. Suppose u is a smooth solution to (2.1). Since g cannot be identically zero, u is not constant. By the strong maximal principle for parabolic equations ($u_t - \Delta u \ge 0$), the minimum 0 is not attained in Ω over $[0,T]$, i.e. $u > 0$. Define

$$\eta(t) := \int_\Omega u(x,t) w_1(x) dx.$$

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By the Cauchy-Schwarz inequality

$$\begin{aligned}\eta^2 &= \left(\int_\Omega (uw_1^{1/2})w_1^{1/2}dx\right)^2 \leq \int_\Omega u^2 w_1 dx \int_\Omega w_1 dx \\ &= \int_\Omega u^2 w_1 dx.\end{aligned}$$

Taking the time derivative of η we get

$$\begin{aligned}\frac{d\eta}{dt} &= \int_\Omega u_t w_1 dx = \int_\Omega (\Delta u + u^2) w_1 dx \\ &= \int_\Omega u\Delta w_1 + u^2 w_1 dx = \int_\Omega u(-\lambda_1)w_1 + u^2 w_1 dx \\ &= -\lambda_1 \eta + \int_\Omega u^2 w_1 dx \\ &\geq -\lambda_1 \eta + \eta^2.\end{aligned}$$

Let $\xi(t) := e^{\lambda_1 t}\eta(t)$ we have

$$\frac{d\xi}{dt} = e^{\lambda_1 t}\frac{d\eta}{dt} + e^{\lambda_1 t}\lambda_1\eta(t) \geq e^{\lambda_1 t}\eta^2 = e^{-\lambda_1 t}\xi^2.$$

So

$$\frac{d\xi/dt}{\xi^2} = \frac{d}{dt}\frac{-1}{\xi(t)} \geq e^{-\lambda_1 t},$$

and integrating from 0 to t

$$\frac{-1}{\xi(t)} \geq \frac{-1}{\xi(0)} + \frac{1-e^{-\lambda_1 t}}{\lambda_1},$$

rearranging we get

$$\xi(t) \geq \frac{\xi(0)\lambda_1}{\lambda_1 - \xi(0)(1-e^{-\lambda_1 t})}$$

if the denominator is not zero. By assumption (2.3) we have

$$\xi(0) = \eta(0) = \int_\Omega u(x,0)w_1 dx = \int_\Omega g w_1 dx > \lambda_1.$$

Thus, for

$$t^* = -\frac{1}{\lambda_1}\log\left(\frac{\eta(0)-\lambda_1}{\eta(0)}\right) < \infty,$$

we have

$$\lambda_1 - \xi(0)(1-e^{-\lambda_1 t^*}) = \lambda_1 - \eta(0)\left(1 - \frac{\eta(0)-\lambda_1}{\eta(0)}\right) = \lambda_1 - \lambda_1 = 0,$$

and therefore for some $0 < t_* \leq t^*$, as $t \nearrow t_*$

$$\eta(t) = e^{-\lambda_1 t}\xi(t) \to \infty.$$

Recall that the quantity $\eta(t) = \int_\Omega u(x,t)w_1(x)dx$ is the first Fourier coefficient under the basis $\{w_k\}_k$. In this case the coefficient goes to infinity, and we say u blows up at time t_*. $\square$

2.2. **For small data.** We now consider the following initial value problem (ivp) in $\mathbb{R}^n$

$$\begin{cases} u_t - \Delta u = u^p & \text{in } \mathbb{R}^n \times (0, T] \\ u = g \in C_c^\infty & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{2.5}$$

Since we are in an unbounded domain, integrability conditions are much stronger than that in a bounded domain. Thus we can expect the condition on the initial data g for the solution to blow up is much weaker. In fact, we will show that given appropriate p, then for any nontrivial initial data, the solution will blow up in finite time.

Theorem 2.6 (Blow-up for small data)**.** *Assume g is not identically zero. Let*

$$1 < p < \frac{n+2}{n}. \tag{2.7}$$

Then there cannot exist a nonnegative, integrable, and smooth solution u to (2.5) *for all times.*

Proof. Recall the fundamental solution of the heat equation at time $s > 0$

$$\Phi(x, s) = \frac{1}{(4\pi s)^{n/2}} e^{-|x|^2/4s},$$

with

$$\int_{\mathbb{R}^n} \Phi(x, s) = 1,$$

and that

$$\Delta\Phi = \frac{\partial \Phi}{\partial s} = -\frac{n}{2s}\Phi + \frac{|x|^2}{4s^2}\Phi \geq -\frac{n}{2s}\Phi.$$

Define

$$\eta(t) := \int_{\mathbb{R}^n} u(x, t)\Phi(x, s)dx.$$

By the Hölder inequality

$$\begin{aligned} \eta^p &= \left(\int_{\mathbb{R}^n} (u\Phi^{1/p})\Phi^{1/q} dx \right)^p \leq \int_{\mathbb{R}^n} u^p \Phi dx \left(\int_{\mathbb{R}^n} \Phi dx \right)^{p/q} \\ &= \int_{\mathbb{R}^n} u^p \Phi dx. \end{aligned}$$

Taking the time derivative and let $\lambda := n/2s$, we get

$$\begin{aligned} \frac{d\eta}{dt} &= \int_{\mathbb{R}^n} u_t \Phi dx = \int_{\mathbb{R}^n} (\Delta u + u^p)\Phi dx \\ &= \int_{\mathbb{R}^n} u\Delta\Phi + u^p \Phi dx \geq \int_{\mathbb{R}^n} u(-\lambda)\Phi + u^p\Phi dx \\ &\geq -\lambda\eta + \eta^p. \end{aligned}$$

Let $\xi(t) := e^{\lambda t}\eta(t)$ we have

$$\frac{d\xi}{dt} = e^{\lambda t}\frac{d\eta}{dt} + e^{\lambda t}\lambda\eta(t) \geq e^{\lambda t}\eta^p = e^{-\lambda(p-1)t}\xi^p.$$

So

$$\frac{d\xi/dt}{\xi^p} = \frac{d}{dt}\frac{-1}{(p-1)\xi^{p-1}(t)} \geq e^{-\lambda(p-1)t},$$

and integrating from 0 to t

$$\frac{-1}{\xi^{p-1}(t)} \geq \frac{-1}{\xi^{p-1}(0)} + \frac{1-e^{-\lambda(p-1)t}}{\lambda},$$

rearranging we get

$$\xi^{p-1}(t) \geq \frac{\xi^{p-1}(0)\lambda}{\lambda - \xi^{p-1}(0)(1-e^{-\lambda(p-1)t})}$$

if the denominator is not zero. Similar to the previous proof, $\eta(t) \to \infty$ in finite time if

$$\xi(0) = \eta(0) > \lambda^{1/(p-1)},$$

fully expanding out we get

$$\frac{1}{(4\pi)^{n/2}} \int_{\mathbb{R}^n} g e^{-|x|^2/4s} dx > s^{n/2} \left(\frac{n}{2s}\right)^{1/(p-1)} = cs^{\frac{n}{2}-\frac{1}{p-1}}$$

for some $c > 0$. So for any g not identically zero we can choose s large enough so that the above is valid if $\frac{n}{2} - \frac{1}{p-1} < 0$, or $1 < p < \frac{n+2}{n}$, which is precisely the assumption (2.7). Note that $0 < \Phi \leq 1$ and we assume $u \geq 0$, the blow-up of $\eta(t)$ implies

$$\int_{\mathbb{R}^n} |u| \geq \int_{\mathbb{R}^n} u\Phi = \eta \to \infty,$$

which means u is not integrable. □

Remark 2.8. There is another interpretation of of the quantity $\eta(t)$. Recall that the Fourier transform $\mathcal{F}$ defined on $L^2(\mathbb{R}^n)$ is an isometric isomorphism, and that the Gaussian $\psi = e^{-|x|^2/2}$ is an eigenfunction of $\mathcal{F}$

$$\mathcal{F}(\psi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} e^{-|x|^2/2} dx = (2\pi)^{n/2}\psi.$$

In fact, we can find a sequence of eigenfunctions ψ_n with complex eigenvalues λ_n, with the Gaussian being ψ_0. The eigenfunctions $\{\psi_n\}_n$, also known as Hermite functions, form an orthogonal basis of $L^2(\mathbb{R}^n)$. Thus $\eta(t)$ can be seen as a weighted (by s) Fourier coefficient of u under $\{\psi_n\}_n$.

References

[1] Evans, L. C. (2010) *Partial Differential Equations.* 2nd Edition, American Mathematical Society.