SOLVING THE BLACK-SCHOLES-MERTON EQUATION
MATH 80350, TOPICS IN ANALYSIS OF PDE

YUTAN ZHANG

1. The Black-Scholes-Merton Model

In this presentation we formulate the Black-Scholes-Merton model for European call options, and find the solution to the BSM PDE. A European call option is a financial contract that gives the owner the right but not the obligation to buy a share of stock at a specified price K that can only be exercised at future date T specified in the contract. The predetermined price K is called the *stike price*, and the time T at which the option can be exercised can be referred to as *maturity date*. Since the owner has the right but not the obligation to exercise the option, the payoff of the option at time T is $V(S_T, T) = [S_T - K]^+ \doteq \max\{S_T - K, 0\}$, since when $S_T < K$, the owner would let the option expire. We wish to find the fair price for the European call option $V(S_t, t)$ at times $t \in [0, T]$.

For the BSM model, we assume that the price of a stock S_t satisfies the following log-normal stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad t \geq 0, \tag{1.1}$$

where r is the risk-free interest rate (such as government bond return), σ the volatility of the stock price, and W_t a Wiener process (Brownian motion) starting at zero. We consider a European call option for this stock with strike price K and exercise time T.

The main idea of Black, Scholes, and Merton was to think of $V(s, t)$ as a *deterministic* quantity that depends only on the (assumed) stock price at that time $s = S_t$ and the time remaining until maturity date $T - t$. Indeed, $V(S_t, t)$ is the stochastic process because of S_t, and $V(s, t)$ is deterministic upon s and $T - t$.

By replicating portfolio and Itô's Lemma (see [2]), we can show that $V(s, t)$ satisfies the Black-Scholes-Merton partial differential equation

$$\boxed{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} + rs\frac{\partial V}{\partial s} - rV = 0, \quad t \in [0, T], \quad s > 0.} \tag{1.2}$$

It is also subject to the terminal condition (for call option payoff at time T)

$$\boxed{V(s, T) \doteq \lim_{t \to T^-} V(s, t) = [s - K]^+.} \tag{1.3}$$

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2. Solving the BSM Equation

Here we present a different approach than in [2]. To solve system (1.2, 1.3), we first note that $\tau = T - t$ is a more natural variable than t since the value of the option is determined backwards from time T to time 0, and it makes (1.3) into an initial condition. Then (1.2) becomes

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} + rs\frac{\partial V}{\partial s} - rV, \quad \tau \in [0, T]. \tag{2.1}$$

We also note that the right hand side is Cauchy-Euler (equi-dimensional) in the variable s, so we make the change of variable $x = \ln s$ and $V(s, \tau) = G(x, \tau)$. Thus

$$\frac{\partial V}{\partial s} = \frac{1}{s}\frac{\partial G}{\partial x} \text{ and } \frac{\partial^2 V}{\partial s^2} = \frac{1}{s^2}\left(\frac{\partial^2 G}{\partial x^2} - \frac{\partial G}{\partial x}\right).$$

Substituting $V(x, \tau)$ with $G(x, \tau)$ into (2.1) we get

$$\frac{\partial G}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 G}{\partial x^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial G}{\partial x} - rG. \tag{2.2}$$

We see that (2.2) is linear constant coefficient and parabolic. To get rid of $-rG$, we make the change of variable $G(x, \tau) = e^{-r\tau}U(x, \tau)$. Thus

$$\frac{\partial G}{\partial \tau} = \frac{\partial U}{\partial \tau} - rG \Rightarrow \frac{\partial U}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 U}{\partial x^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial U}{\partial x}. \tag{2.3}$$

To get rid of $(r - \sigma^2/2)U_x$, we make the change of variable $x \to \xi = x + (r - \sigma^2/2)\tau$, so $u(\xi(x, \tau), \tau) = U(x, \tau)$. So we have

$$\frac{\partial U}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{\partial u}{\partial \xi}, \ \frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2}, \ \frac{\partial U}{\partial \tau} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial \tau} + \frac{\partial u}{\partial \tau} = \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \tau}.$$

Substituting back into (2.3), the terms with u_ξ cancel, and we get

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial \xi^2}.$$

To rescale the factor $\sigma^2/2$, we let $\eta = \sigma^2\tau/2$, so $u_\tau = u_\eta \sigma^2 \tau/2$, we get

$$\frac{\partial u}{\partial \eta} = \frac{\partial^2 u}{\partial \xi^2}. \tag{2.4}$$

Therefore, we have the following change of variables

$$\boxed{\begin{cases} u(\xi, \eta) = e^{r\tau}V(s, t) \\ \eta(\tau) = \dfrac{\sigma^2}{2}\tau \\ \xi(s, \tau) = \ln s + \left(r - \dfrac{\sigma^2}{2}\right)\tau \\ \tau(t) = T - t \end{cases}}$$

and (1.2) is transformed into the following standard heat equation form

$$\boxed{\frac{\partial u}{\partial \eta} - \frac{\partial^2 u}{\partial \xi^2} = 0.} \tag{2.5}$$

It is subjected to the new initial condition determined by (1.3)

$$\boxed{u(\xi, 0) = \lim_{\eta \to 0^+} e^{r\tau} V(s, t) = [e^{\xi} - K]^+.} \tag{2.6}$$

Condition (2.6) is obtained since $\eta \to 0^+$ implies $\tau \to 0^+$, and $\ln s = \xi - (r - \sigma^2/2)\tau$, so we have $s = e^{\xi} \exp(-(r - \sigma^2/2)\tau) \to e^{\xi}$. We solve the system (2.5, 2.6) using the following Theorem.

Theorem 2.1 (Solution of Heat equation [1]). *Consider the system of equations*

$$\partial_t u - \partial_x^2 u = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{2.7}$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}. \tag{2.8}$$

If $u_0(x) \in L^1(\mathbb{R})$, then the system admits a unique solution $u(x, t)$ given by

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{1}{4t}(x-y)^2} u_0(y) dy. \tag{2.9}$$

In fact, if $\exp(-x^2) u_0(x) \in L^1(\mathbb{R})$, (2.9) remains the solution to the system.

Applying Theorem 2.1, we have the solution for system (2.5, 2.6)

$$\begin{aligned} u(\xi, \eta) &= \frac{1}{\sqrt{4\pi\eta}} \int_{\mathbb{R}} e^{-\frac{1}{4\eta}(\xi - y)^2} [e^y - K]^+ dy \\ &= \frac{1}{\sqrt{4\pi\eta}} \int_{\ln K}^{\infty} (e^y - K) \exp\left[-\frac{(y-\xi)^2}{4\eta}\right] dy \\ &= \frac{1}{\sqrt{4\pi\eta}} \int_{\ln K}^{\infty} \exp\left[y - \frac{(y-\xi)^2}{4\eta}\right] dy - \frac{1}{\sqrt{4\pi\eta}} \int_{\ln K}^{\infty} K \exp\left[-\frac{(y-\xi)^2}{4\eta}\right] dy. \end{aligned}$$

Since the model is based on standard normal distribution, it is natural to express the solution in the form of $N(x)$, where

$$N(x) \doteq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy.$$

We examine the term inside the first exponential.

$$\begin{aligned} y - \frac{(y-\xi)^2}{4\eta} &= \frac{4\eta y - (y-\xi)^2}{4\eta} - (\xi + \eta) + (\xi + \eta) \\ &= \frac{4(y-\xi)\eta - (y-\xi)^2 - 4\eta^2}{4\eta} + (\xi + \eta) \\ &= -\frac{(y - \xi - 2\eta)^2}{4\eta} + (\xi + \eta). \end{aligned}$$

Let $p = \frac{y - \xi - 2\eta}{\sqrt{2\eta}}$, so $dp = \frac{dy}{\sqrt{2\eta}}$, the first term becomes

$$\begin{aligned} \frac{e^{\xi+\eta}}{\sqrt{4\pi\eta}} \int_{\ln K}^{\infty} \exp -\frac{(y - \xi - 2\eta)^2}{4\eta} dy &= \frac{e^{\xi+\eta}}{\sqrt{2\pi}} \int_{\frac{\ln(K) - \xi - 2\eta}{\sqrt{2\eta}}}^{\infty} e^{-\frac{p^2}{2}} dp \\ = \frac{e^{\xi+\eta}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\xi + 2\eta - \ln K}{\sqrt{2\eta}}} e^{-\frac{p^2}{2}} dp &= e^{\xi+\eta} N\left(\frac{\xi + 2\eta - \ln K}{\sqrt{2\eta}}\right). \end{aligned}$$

Similarly (more easily) for the second term, let $q = \frac{y-\xi}{\sqrt{2\eta}}$, so $dq = \frac{dy}{\sqrt{2\eta}}$, we have

$$-\frac{K}{\sqrt{4\pi\eta}} \int_{\ln K}^{\infty} \exp - \frac{(y-\xi)^2}{4\eta} dy = -\frac{K}{\sqrt{2\pi}} \int_{\frac{\ln(K)-\xi}{\sqrt{2\eta}}}^{\infty} e^{-\frac{q^2}{2}} dq$$

$$= -\frac{K}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\xi - \ln K}{\sqrt{2\eta}}} e^{-\frac{q^2}{2}} dq = -KN\left(\frac{\xi - \ln K}{\sqrt{2\eta}}\right).$$

Therefore, we have

$$u(\xi,\eta) = e^{\xi+\eta} N\left(\frac{\xi + 2\eta - \ln K}{\sqrt{2\eta}}\right) - KN\left(\frac{\xi - \ln K}{\sqrt{2\eta}}\right).$$

Changing back from $u(\xi,\eta)$ to $V(s,\tau)$ via the change of variables, we have

$$V(s,\tau) = \exp\left[-r\tau + \ln(s) + \left(r - \frac{\sigma^2}{2}\right)\tau + \frac{\sigma^2}{2}\tau\right] N\left(\frac{\ln s + \left(r - \frac{\sigma^2}{2}\right)\tau + \sigma^2\tau - \ln K}{\sigma\sqrt{\tau}}\right)$$
$$- e^{-r\tau} KN\left(\frac{\ln s + \left(r - \frac{\sigma^2}{2}\right)\tau - \ln K}{\sigma\sqrt{\tau}}\right)$$

$$V(s,\tau) = sN\left(\frac{\ln(s/K) + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right) - e^{-r\tau} KN\left(\frac{\ln(s/K) + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right).$$

Therefore, we arrive at the solution to the system (1.2, 1.3) known as the BSM formula

$$\boxed{V(s,\tau) = sN(d_+) - e^{-r\tau} KN(d_-),\ d_\pm = \frac{\ln(s/K) + \left(r \pm \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}},\ \tau = T - t.} \tag{2.10}$$

We can see that (2.10) satisfies the terminal condition (1.3) as

$$\lim_{t\to T^-} d_\pm = \lim_{\tau \to 0^+} d_\pm = \begin{cases} +\infty, & s > K \\ 0, & s = K \\ -\infty, & s < K \end{cases}, \text{ and } \lim_{\tau\to 0^+} e^{-r\tau} = 1,$$

therefore

$$V(s,T) \doteq \lim_{t \to T^-} V(s,t) = \left.\begin{cases} sN(\infty) - KN(\infty) = s - K, & s > K \\ sN(0) - KN(0) = (s-K)/2 = 0, & s = K \\ sN(-\infty) - KN(-\infty) = 0, & s < K \end{cases}\right\} = [s-K]^+.$$

References

[1] A. Himonas. *Lacture Notes in Partial Differential Equations.* Lecture Notes, 2023, University of Notre Dame.

[2] A. Himonas and T. Cosimano. *Mathematical Methods in Finance and Economics.* Lecture Notes (eBook), 2023, University of Notre Dame.