DERIVATION AND SOLUTION OF THE BLACK-SCHOLES-MERTON PDE

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ABSTRACT. The Black-Scholes-Merton partial differential equation is used to define the relationship between a European option's value, strike price, risk-free interest rate, volatility, and the price of the underlying asset. In this paper, we review the background and history behind the equation. Then, we derive the equation from foundations of Stochastic calculus and Brownian motion, for which a closed-form solution is found. We then show the utility of the model by reviewing multiple extensions and applications, including adaptation to American options and solving for price and implied volatility. Overall, the model is a powerful application of mathematical methods for information about financial derivatives, but it is hindered by its assumptions and the limitations of algorithms in solving nonlinear equations.

1. INTRODUCTION

The sophisticated behavior of markets has long attracted interest by mathematicians and scientists. Equations and models are developed in attempt to define or approximate the processes which govern this behavior. In this paper, we review the Black-Scholes-Merton model, which estimates the theoretical value of option contracts based on other investment information.

In this paper, we cover the BSM model in depth and show its usage. We begin with a historical review, followed by an overview of stochastic calculus. We use this to inform our derivation of the BSM model, which is then solved using a change-of-variables procedure. We then show applications of the model, including an extension to American options and sample calculations. We conclude by discussing the continued usage of the model.

1.1. **Background.** Option contracts, like other financial derivatives, have been around for hundreds of years. An option contract is simply the right to buy or sell an amount of an asset, usually stock, at a certain price at a later date. This asset is called the"underlying". An option to buy stocks is called a "call", while an option to sell stocks is called a "put". Options have an expiration date; for European-style options described by the BSM model, the options must be "exercised", or redeemed, on this date (and not before).

Options are sold by a "writer" to a "holder". The holder pays a fee, called a "premium", to the writer in exchange for the right either purchase or sell the underlying at a certain price, called the "strike price." This is often determined with reference to the current trading price of the underlying itself, called the "spot price."[1]

For example, suppose a person buys a call from a writer. The writer hopes that the spot price of the underlying won't be higher than the strike price plus the premium of the option on expiration date–that is, the call is "out-of-the-money", and expires worthless. If instead the strike price is higher, then the holder can exercise the call to buy the underlying at the strike price, a lower price than the spot price–the call is "in-the-money". The underlying is usually supplied by the writer. The situation is reversed for puts–a put is "out-of-the-money" when the strike price is higher than the spot, and "in-the-money"

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when the spot is higher than the strike. In the latter case, the holder sells the underlying at the strike price at expiry.

1.2. **History.** In 1900, Louis Bachelier defended his thesis in mathematics, analyzing the price of options based on probability theory. This was the first mathematical treatment of its kind in this field. Concurrently and in years following, Norbert Wiener made advances in stochastic calculus and proved the existence of Brownian motion. This field was further advanced by Kiyoshi Itō, who created a method for handling stochastic differential equations. As knowledge advanced, interest in mathematical economics and Bachelier's work slowly waxed. In 1967, Edward Thorpe and Sheen Kassouf published a book in which they derived an equation for the value of a financial derivative called a warrant. Thorpe and Kassouf's method was empirical, but Paul Samuelson and Robert Merton derived a very similar equation rigorously the next year. Fischer Black and Myron Scholes, at the time working at a consulting company with Wells Fargo as a client, adapted that work to derive a differential equation describing the value of options, as well as its solution. They also defined a number of parameters–called "The Greeks"–which allowed them to describe the sensitivity of an option's price to other variables.

Their model was made with under a number of assumptions, some of which were unrealistic. Merton, who heard about their method at a conference in 1970, thought this precluded the validity of their result, but derived a similar equation using stochastic calculus and Itō's method. The three then worked together and published a paper and an article in 1973 documenting their results.[2]

In 1997, Merton and Scholes won the Nobel Prize in Economics for their work. Unfortunately, Black was not given a Nobel Prize as he was dead at the time and thus ineligible.

In the 70s and 80s, the BSM model was used to calculate the value of options and generate returns for traders, who were able to buy undervalued or sell overpriced options. More recently, however, the formula has been inverted, so that the volatility of the underlying asset may be found, called the "implied volatility." Surprisingly, implied volatility has been found to be correlated with other parameters of the options (namely, the distance between its strike price and the underlying's current price), which contradicts one of the model's assumptions, that volatility describes a feature of the underlying and is independent of other options parameters. Under this assumption, we would expect to see that options on the same underlying with the same expiration would have the same volatility, a "flat" graph; instead it is parabolic, in a phenomenon known as the "volatility smile." Nonetheless, authors have argued for the continued use of the BSM model in practical applications.[3]

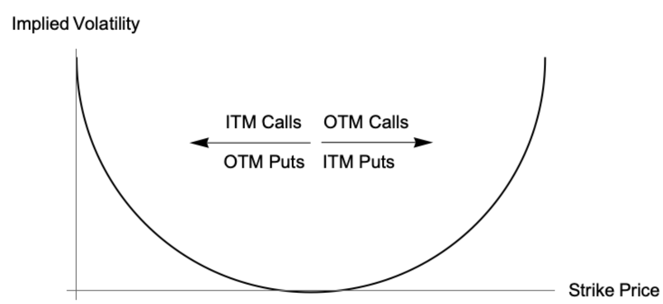

FIGURE 1.1. The "volatility smile"; implied volatility increases towards extrema in strike price.

1.3. **The Greeks.** Components of the BSM have been individually named and are often used by traders and academics to study an option's sensitivity to market conditions. Each of these components is assigned a Greek symbol, and they are colloquially known as "the Greeks". This section provides some information about each.

1.3.1. *Delta* (Δ). Delta represents the change of the option's price with respect to the underlying's price, $\Delta = \frac{\partial V}{\partial S}$. For traders, delta provides an approximate probability the option will expire in-the-money. It also provides an important metric for creating a delta-neutral position, in which the combined deltas of a portfolio is zero. This means, within a certain range of market movements, the investor's position does not change. A delta-netural portfolio may be created for hedging purposes.

1.3.2. *Theta* (Θ). Theta represents the change of the option's price with respect to time, $\Theta = \frac{\partial V}{\partial t}$. Theta is higher for options at-the-money and lower for options in- and out-of-the-money. It is used by traders to indicate theta-decay, or time-decay, the phenomenon that an option loses value as time goes on, all else being equal. Options lose value over time because there is less time for the underlying's price to move in a favorable direction, so the option is more likely to expire worthless. This is important to traders, as they must consider how the option will decline in value if they plan to buy or sell it for a profit. For example, even if an option's value increases due to market movements, theta-decay may cause its value to be too low to make a profit when sold.

1.3.3. *Gamma* (Γ). Gamma represents the change of the option's delta with respect to the underlying's price, $\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 V}{\partial S^2}$. Like theta, it is higher for options at-the-money and lower for options in- and out-of-the-money. It also generally increases nearer to expiry. Traders use this to measure the stability of the option's delta, which could be important for strategies relying on specific values of delta. Using both is sometimes referred to as delta-gamma-hedging.

1.3.4. *Vega* (ν)*)*. Vega[1] represents the change of the option's value with respect to the underlying's implied volatility, $\nu = \frac{\partial V}{\partial \sigma}$. It does not actually feature explicitly in the BSM. Traders can use vega to assess the exposure of their portfolios to market movements, as it indicates the direction an option's price is expected to move with increased volatility.

1.3.5. *Rho* (ρ). Rho represents the change of the option's value with respect to the risk-free rate of interest, $\rho = \frac{\partial V}{\partial r}$. Like vega, does not actually feature explicitly in the BSM. It is generally considered the least important of the Greeks.

1.3.6. *Minor Greeks.* Other measures, such as lambda, epsilon, color, convexity, etc. are also used in options trading and analysis. They are often second- or third-order partial derivatives of the option's price delta, or other parameter with respect to another. They are less widely known or used than the main Greeks, which directly concern the parameters of the BSM.[4]

2. Stochastic Calculus Basics

In this section we introduce the necessary mathematical foundations.

To understand the basic notions of stock behaviour we make an intuitive assumption. Namely, we indicate that the stock behaviour is random. But if it is random, how do we model it?

In 1827, a botanist by the name of Robert Brown was observing grains of pollen suspended in a fluid and began to notice that they would move erratically. The pollen seemed to change path and move about in a random way. Similar phenomena have been seen throughout history. The Roman philosopher Lucretius observed dust particles floating through the air and noted their random movement. Similarly, in 1785 Jan Ingenhousz observed coal dust on the surface of alcohol–again noting its random movement. Nonetheless, the person who is credited with the true discovery is Robert Brown–hence the name, Brownian motion.[5] A simulation of Brownian motion in one dimension is shown in Figure 2.1.

Of course, this quantity was of great interest to mathematicians. So, in consequence, Einstein derived many PDEs and ODEs to model this phenomena and came up with the following result:

[1] "Vega" is not actually the name of the Greek character ν, which is called "nu". The name "Vega" seems to come from traders associating Greek letters with Latin counterparts and, evidently, not knowing Greek.

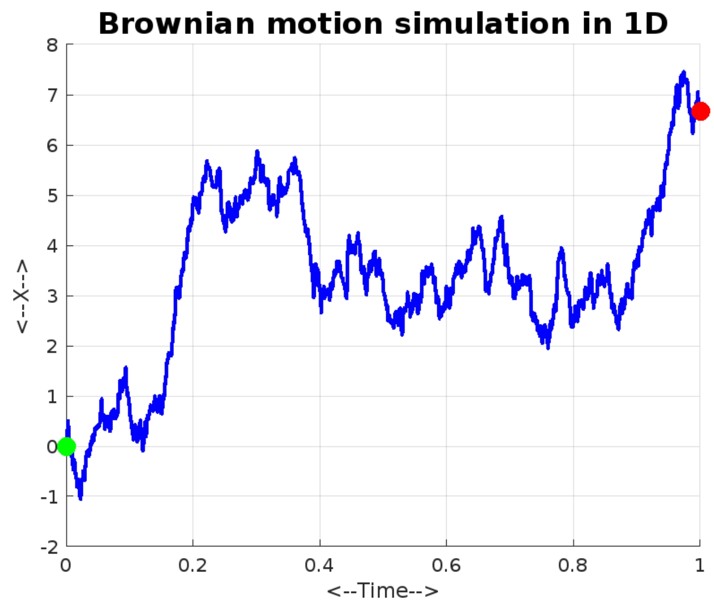

FIGURE 2.1. A simulation of Brownian motion (a generalized Wiener process).

The probability that a particle is in a given interval $[a, b]$ is given by

$$\mathbb{P}(a \le B_t \le b) = \frac{1}{\sqrt{2\pi t}} \int_a^b e^{-\frac{1}{2t}x^2} dx.$$

Interestingly, we note that this definition is strikingly similar to the normal distribution with $\sigma = 1$. Namely, the only difference is the time dependence of this function. A graph of the distribution of particles modeled by Brownian motion is shown in Figure 2.2. One can see that as time increases, the particles spread out more and more, yet always remain in a normal distribution. Additionally, (in this model) all the particles are starting at $x = 0$. Thus, the particle distribution is the Dirac delta function, meaning that the probability of finding the particle at $t = 0$ is given by

$$\begin{cases} \mathbb{P}(x) = 0 | x \neq 0 \\ \mathbb{P}(x) = 1 | x = 0 \end{cases}$$

We now define probability space, but we do not go in depth into measure theory.

Definition 2.1. *A* ***probability space*** *is a triple* $(\Omega, \mathcal{F}, P)$*, sometimes simply denoted by* Ω*, where* Ω *is the sample space,* $\mathcal{F}$ *is the* σ*-algebra on* Ω*, and* P *is the probability measure. A* ***σ-algebra*** *(or* σ*-field) on a set* X *is a nonempty collection of subsets of* X *closed under complement and countable unions. A* ***probability measure*** *is a measure* P *on* Ω *where* $P(\Omega) = 1$.

Remark 2.1. P is a real-valued function on the power set of Ω satisfying:

1. $P(\varnothing) = 0$ and $P(\Omega) = 1$,
2. $P(X) \le P(Y)$ if $X \subset Y$,
3. If $\{X_i\}$ are pairwise disjoint where $i \in I$ is countable, then

$$P\left(\bigcup_{i \in I} X_i\right) = \sum_{i \in I} P(X_i).$$

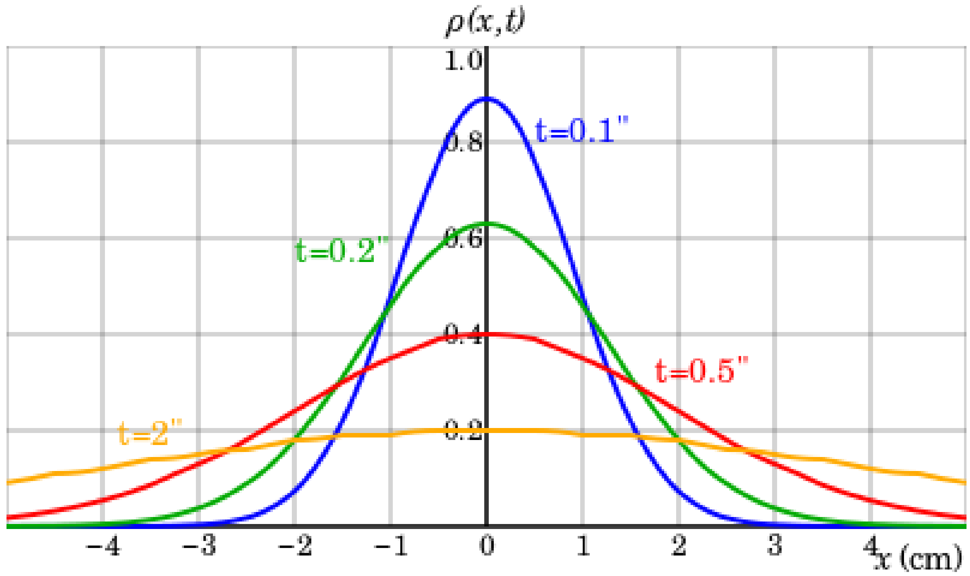

FIGURE 2.2. Diffusion of particles modeled by Brownian motion (reconstruction of a figure from [6]).

In essence, a probability space is simply a space containing all possible outcomes and their probabilities. Although a simple definition, it is important for future use. We now rigorously define a type of Brownian motion.

Definition 2.2. *A* ***Wiener process*** $\{W(t)\}_{t\geq 0}$ *is a family of random variables* $W(t) : \Omega \to \mathbb{R}$*, where* Ω *is a probability space satisfying the following properties:*

1. $W(0) = 0$.
2. $\forall \omega \in \Omega$*, the map* $t \mapsto W(\omega, t)$ *is continuous from* $[0, \infty)$ *into* $\mathbb{R}$ *with probability one.*
3. $\forall t > s \geq 0$*,* $W(t) - W(s)$ *is normally distributed with mean* 0 *and variance* $t - s$*, i.e.,*

$$\mathbb{P}(a \leq W(t) - W(s) \leq b) = \frac{1}{\sqrt{2\pi(t-s)}} \int_a^b e^{-\frac{1}{2(t-s)}x^2} dx.$$

4. $\forall 0 = t_0 < t_1 < t_2 < \ldots < t_n$*, the increments* $W(t_{i+1}) - W(t_i)$ *are independent random variables.*

Remark 2.2. $W(t)$ is also denoted by W_t. The notation $W(\omega, t)$ and $W_t(\omega)$ stress that for each t, it is a random variable defined on a sample space Ω.

Remark 2.3. The sample paths are continuous but nowhere differentiable.

Next we introduce the notion of integrals for stochastic processes. Here we only state the following definition, properties and results of the Itô's integral. For proofs and examples, reference [7].

Definition 2.3. *A* ***simple stochastic process*** *is a function of the form*

$$f(\omega, t) = \sum_{j=0}^{n-1} \alpha_j(\omega) \mathbf{1}_{[t_j, t_{j+1})}(t), \tag{2.1}$$

where $0 = t_0 < t_1 < \ldots < t_n = T$*,* $\alpha_j(\omega)$ *random variables, and* $\mathbf{1}_{[a,b)}(t) = 1$ *if* $a \leq t < b$ *and else* 0*.*

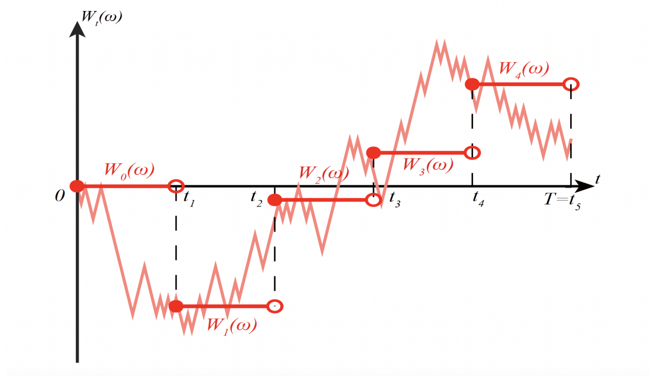

FIGURE 2.3. Example of a simple stochastic process from a general process.

Definition 2.4. *The* ***Itô integral*** *for a* ***simple*** *stochastic process $f(t)$ like (2.1) is given by*

$$\int_0^T f(t)dW_t = \sum_{j=0}^{n-1} \alpha_j(\omega)(W_{t_{j+1}}(\omega) - W_{t_j}(\omega)). \tag{2.2}$$

Remark 2.4 (Financial Interpretation)*.* If we think of W_t as the price of one share of an asset at time t, the times $t_0, t_1, \ldots, t_{n-1}$ as the trading dates, and $f(t_0), f(t_1), \ldots, f(t_{n-1})$ as the number of shares in the portfolio at each trading date and held until the next trading date, then the Itô integral gives the **gain** from trading at all dates.

Definition 2.5. *The* ***Itô integral*** *for a* ***general*** *stochastic process $f(\omega, t)$ that is square integrable*

$$\mathbb{E}\left[\int_0^T f^2(\omega, t)dt\right] < \infty$$

is given by

$$\int_0^T f(\omega, t)dW_t(\omega) \overset{L^2(\Omega)}{=} \lim_{n\to\infty} \int_0^T f_n(\omega, t)dW_t(\omega), \tag{2.3}$$

where $f_n(\omega, t)$ is a sequence of simple stochastic processes converging to $f(\omega, t)$ in L^2.

Example 2.1. Let $f(\omega, t) = W_t(\omega)$ be a Wiener process, then

$$\int_0^T W_t(\omega)dW_t(\omega) = \frac{1}{2}W_t^2(\omega) - \frac{1}{2}T. \tag{2.4}$$

Remark 2.5*.* Note the similarity between (2.4) and the following Riemann integral

$$\int_0^T tdt = \frac{1}{2}T^2.$$

Definition 2.6. *A* ***stochastic integral*** *is a stochastic process X_t on Ω of the form*

$$X_t = X_0 + \int_0^t u(\omega, s)ds + \int_0^t v(\omega, s)dW_s, \tag{2.5}$$

where $v(\omega, s)$ satisfies the square integrability condition

$$\mathbb{E}\left[\int_0^t v^2(\omega, s)ds\right] < \infty.$$

The stochastic integral is also written as a ***stochastic differential equation*** *of the form*

$$dX_t = udt + vdW_t. \tag{2.6}$$

Proposition 2.1 (Itô's formula)**.** *Let X_t be a stochastic integral*

$$dX_t = udt + vdW_t,$$

and $g(x,t) \in C^2(\mathbb{R} \times [0,T])$. Then $Y_t = g(X_t, t)$ is a stochastic integral with

$$dY_t = \frac{\partial g}{\partial t}(X_t,t)dt + \frac{\partial g}{\partial x}(X_t,t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(X_t,t)(dX_t)^2. \tag{2.7}$$

Remark 2.6. We use the following formal rules for computing $(dX_t)^2$:

$$(dt)^2 = dtdW_t = dW_tdt = 0 \text{ and } (dW_t)^2 = dt,$$

so we have

$$\begin{aligned}(dX_t)^2 &= (udt + vdW_t)(udt + vdW_t)\\ &= u^2(dt)^2 + uvdtdW_t + vudW_tdt + v^2(dW_t)^2\\ &= v^2dt.\end{aligned}$$

3. Stock and Option Basics

At first glance, a generalized Wiener process (or Brownian motion) seems to be adaptable for a model of stock prices. Like stocks prices, these processes take on only positive values and exhibit noise. However, Wiener processes assume a constant expected drift rate, whereas a stock's expected return is constant with a variable drift rate. This is because the stock's expected return required by an investor is independent of the stock's price. For example, if an investor desires a 20% return on a stock, we assume that goal does not change whether the stock price is \$10 or \$100. Still, Wiener processes are adaptable for stock price modeling.

We now formally derive the **log-normal stochastic differential equation** that models stock prices. Assume that on average, the price of a share of stock S_t grows with continuously compounded interest at a constant risk-free rate a, then it satisfies the following differential equations:

$$\frac{dS_t}{dt} = aS_t, \text{ or } \frac{dS_t}{S_t} = adt.$$

This is the **log-linear** model for stock prices. Next, we assume that there is a up and down "normal" random motion with expected value of zero and standard deviation σ called volatility. To model volatility, we add to the right-hand side of the equation the term σdW_t, which is often referred to as "white noise". Indeed, σdW_t captures the normal random motion from Definition 2.2 property 3, that dW_t is normally distributed with mean 0 and variance dt. Thus, we obtain the equation:

$$\frac{dS_t}{S_t} = adt + \sigma dW_t,$$

which is also written in the form

$$dS_t = aS_tdt + \sigma S_tdW_t, \quad t \geq 0, \tag{3.1}$$

and this is the **log-normal** model for stock prices.

To solve equation (3.1), we apply Itô's formula with $g(x,t) = \ln x$, $x > 0$, and $X_t = S_t$. Then

$$Y_t = \ln S_t, \quad \frac{\partial g}{\partial t} = 0, \quad \frac{\partial g}{\partial x} = \frac{1}{x}, \quad \frac{\partial^2 g}{\partial x^2} = -\frac{1}{x^2}.$$

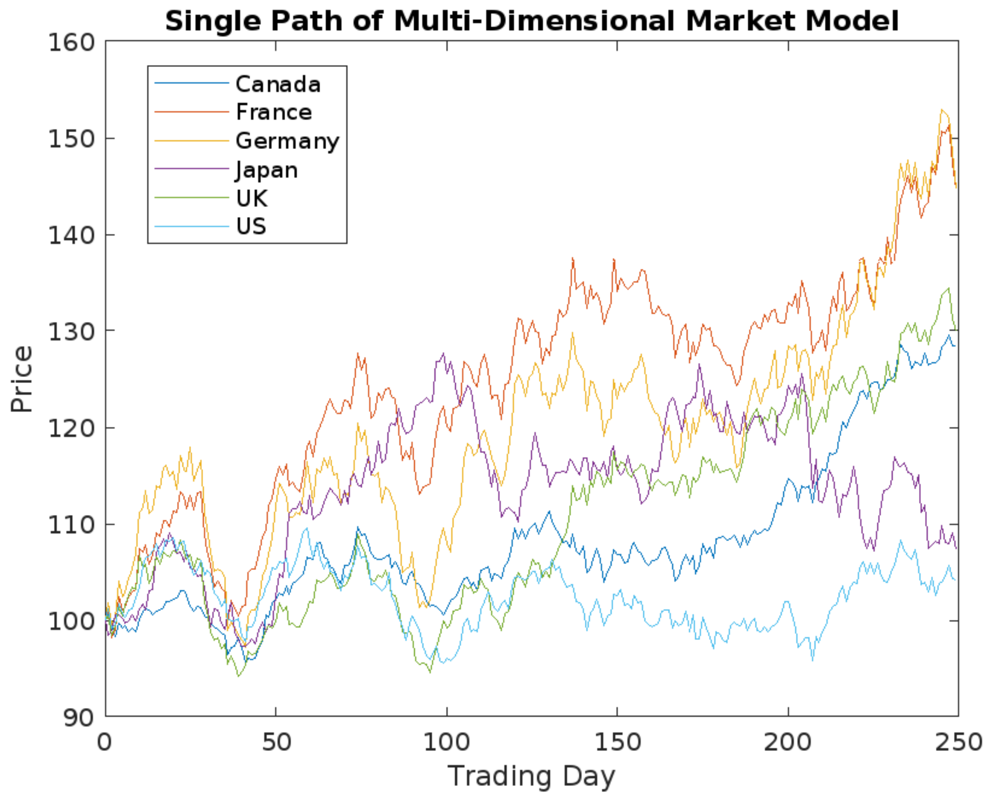

FIGURE 3.1. Simulation of prices of several equity markets, based on indices. Graphs show clear Brownian motion.

By Itô's formula, we get

$$\begin{aligned} d(\ln S_t) &= \frac{dS_t}{S_t} - \frac{1}{2}\frac{1}{S_t^2}(dS_t)^2 \\ &= \frac{dS_t}{S_t} - \frac{1}{2}\frac{1}{S_t^2}(aS_t dt + \sigma S_t dW_t)^2 && \text{(equation (3.1))} \\ &= \frac{dS_t}{S_t} - \frac{1}{2}\frac{1}{S_t^2}\sigma^2 S_t^2 dt && \text{(Remark (2.6))} \\ &= \left(a - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t. && (dS_t/S_t) \end{aligned}$$

From Definition 2.6, this is the differential form of the stochastic integral

$$\ln S_t = \ln S_0 + \int_0^t \left(a - \frac{1}{2}\sigma^2\right) d\tau + \int_0^t \sigma dW_\tau,$$

solving this yields

$$S_t = S_0 \exp\left[(a - \frac{1}{2}\sigma^2)t + \sigma W_t\right]. \tag{3.2}$$

This is the solution to the **log-normal** model for the stock.

Some limitations of this model are the assumption of a constant volatility rate and the lack of discontinuous jumps. As discussed earlier, the volatility rate of actual stocks varies, possibly stochastically. Stock prices also jump drastically in response to news, events, or market conditions, which would need to be modeled as discontinuities.[8] Since the generalized Wiener process is continuous, this is not feasible–but perhaps the attempt to model the complexities of real-world conditions is unreasonable anyway.

4. Main Theorem

We now state the main result of this paper.

Theorem 1. *Suppose that the stock price S_t satisfies the log-normal stochastic differential equation*

$$dS_t = aS_t dt + \sigma S_t dW_t, \quad t \geq 0, \tag{4.1}$$

where a is the stock's mean rate of growth, σ is the volatility of the stock, and W_t is a Wiener process starting at zero. Consider a European call option for this stock with strike price K and exercise time T. Therefore, its pay-off at time T is

$$V(S_T, T) = [S_T - K]^+ \doteq \max\{S_T - K, 0\}. \tag{4.2}$$

*Denote by $V(s,t)$ the value of the call option at time t if the stock price is $S_t = s$, and r the risk-free interest rate. Then, $V(s,t)$ satisfies the **Black-Scholes-Merton partial differential equation***

$$\boxed{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} + rs\frac{\partial V}{\partial s} - rV = 0, \quad t \in [0,T], \quad s \geq 0.} \tag{4.3}$$

*Under the **terminal condition***

$$V(s,T) \doteq \lim_{t\to T^-} V(S,T) = [s-K]^+, \tag{4.4}$$

*the **boundary condition***

$$V(0,t) \doteq \lim_{s\to 0^+} V(s,t) = 0, \forall t \in [0,T], \tag{4.5}$$

and the growth condition

$$\lim_{t\to\infty} [V(s,t) - (s - Ke^{-r(T-t)})] = 0, \tag{4.6}$$

the solution to the PDE (4.3) with conditions (4.4), (4.5), and (4.6) is given by

$$V(s,t) = s\Phi(d_+(s,T-t)) - Ke^{-r(T-t)}\Phi(d_-(s,T-t)), \quad 0 \leq t < T, \quad s > 0, \tag{4.7}$$

where

$$d_\pm(s,\tau) \doteq \frac{1}{\sigma\sqrt{\tau}}\left[\ln\frac{s}{K} + (r \pm \frac{1}{2}\sigma^2)\tau\right]$$

and Φ the standard normal distribution

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy.$$

5. Derivation of the Black-Scholes-Merton PDE

In this section we derive the Black-Scholes-Merton partial differential equation based on the assumptions of Theorem 1. Our goal is to determine the pay-off function $V(s,t)$. For this we consider the idea of replicating portfolio.

5.1. **Replicating Portfolio.** Assume that at time $t = 0$ we begin with a portfolio of value X_0. At any time $t \geq 0$, we invest in the following two financial instruments:

- A money market with a constant interest rate r,
- A stock whose time evolution S_t is modeled by equation (4.1).

Denote by X_t the value of our portfolio at time t, we construct a portfolio such that its value is equal to the value of the call option at that time, i.e.,

$$X_t = V(S_t, t). \tag{5.1}$$

Assume that at each time t, our portfolio consists of Δ_t shares of stock, and therefore, $X_t - \Delta_t S_t$ in the money market. Then the differential dX_t is given by

$$dX_t = \Delta_t dS_t + r(X_t - \Delta_t S_t)dt. \tag{5.2}$$

Remark 5.1. Compare equation (5.2) with the following discrete time formula for a better understanding

$$X_{n+1} = \Delta_n S_{n+1} + (1 + r)(X_n - \Delta_n S_n).$$

Considering the discounted (present) values of X_t and $V(S_t, t)$, equality (5.1) becomes

$$e^{-rt} X_t = e^{-rt} V(S_t, t), \tag{5.3}$$

taking the differentials, we have

$$d(e^{-rt} X_t) = d(e^{-rt} V(S_t, t)). \tag{5.4}$$

5.2. **Computing the differential** $d(e^{-rt} X_t)$**.** We use Itô's formula with $g(x, t) = e^{-rt} x$. Then

$$\begin{aligned}
d(e^{-rt} X_t) &= \frac{\partial g}{\partial t}(X_t, t)dt + \frac{\partial g}{\partial x}(X_t, t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(X_t, t)(dX_t)^2 \\
&= -re^{-rt} X_t dt + e^{-rt} dX_t \\
&= -re^{-rt} X_t dt + e^{-rt}(\Delta_t dS_t + r(X_t - \Delta_t S_t)dt) && \text{(equation (5.2))} \\
&= \Delta_t e^{-rt}(dS_t - rS_t dt) \\
&= \Delta_t e^{-rt}(aS_t dt + \sigma S_t dW_t - rS_t dt). && \text{(assumption (4.1))}
\end{aligned}$$

Therefore,

$$d(e^{-rt} X_t) = (a - r)e^{-rt} \Delta_t S_t dt + \sigma e^{-rt} \Delta_t S_t dW_t. \tag{5.5}$$

5.3. **Computing the differential** $d(e^{-rt} V(S_t, t))$**.** We use Itô's formula with $g(x, t) = e^{-rt} V(x, t)$.

$$\begin{aligned}
&\frac{\partial g}{\partial t}(x, t) = -re^{-rt} V(x, t) + e^{-rt}\frac{\partial V}{\partial t}(x, t), \\
&\frac{\partial g}{\partial x}(x, t) = e^{-rt}\frac{\partial V}{\partial x}(x, t), \quad \frac{\partial^2 g}{\partial x^2}(x, t) = e^{-rt}\frac{\partial^2 V}{\partial x^2}(x, t).
\end{aligned}$$

Then,

$$\begin{aligned}
d(e^{-rt} V(S_t, t)) =& d(g(S_t, t)) = \frac{\partial g}{\partial t}(S_t, t)dt + \frac{\partial g}{\partial x}(S_t, t)dS_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(S_t, t)(dS_t)^2 \\
&= \left(-re^{-rt} V(S_t, t) + e^{-rt}\frac{\partial V}{\partial t}(S_t, t)\right) dt \\
&\quad + e^{-rt}\frac{\partial V}{\partial x}(S_t, t)dS_t + \frac{1}{2}e^{-rt}\frac{\partial^2 V}{\partial x^2}(S_t, t)(dS_t)^2.
\end{aligned}$$

By assumption (4.1) and Remark (2.6), we have $(dS_t)^2 = \sigma^2 S_t^2 dt$, so this gives

$$\begin{aligned} d(e^{-rt}V(S_t,t)) =& e^{-rt}\left(-rV(S_t,t) + \frac{\partial V}{\partial t}(S_t,t) + aS_t\frac{\partial V}{\partial x}(S_t,t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial x^2}(S_t,t)\right) dt \\ &+ e^{-rt}\sigma S_t \frac{\partial V}{\partial x}(S_t,t)dW_t. \end{aligned} \tag{5.6}$$

5.4. **Equating the differentials.** From relations (5.5) and (5.6), and the equality (5.4), we have

$$(a-r)e^{-rt}\Delta_t S_t = e^{-rt}\left(-rV(S_t,t) + \frac{\partial V}{\partial t}(S_t,t) + aS_t\frac{\partial V}{\partial x}(S_t,t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial x^2}(S_t,t)\right),$$

$$\sigma e^{-rt}\Delta_t S_t = e^{-rt}\sigma S_t \frac{\partial V}{\partial x}(S_t,t).$$

The second relation gives the **Delta-hedging formula**

$$\Delta_t = \frac{\partial V}{\partial x}(S_t,t). \tag{5.7}$$

Substituting into the first relation gives the **Black-Scholes-Merton partial differential equation**

$$\frac{\partial V}{\partial t}(S_t,t) + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2}(S_t,t) + rs\frac{\partial V}{\partial s}(S_t,t) - rV(S_t,t) = 0, \tag{5.8}$$

considering $V = V(s,t)$, we obtain the BSM of the form in equation (4.3).

6. Solving the Black-Scholes-Merton PDE

From the derivation above, the BSM boundary value problem is given by

$$\begin{cases} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} + rs\frac{\partial V}{\partial s} - rV = 0, \quad t \in [0,T], \quad s \geq 0, \\ V(s,T) \doteq \lim_{t\to T^-} V(S,T) = [s-K]^+, \\ V(0,t) \doteq \lim_{s\to 0^+} V(s,t) = 0, \qquad \forall t \in [0,T], \\ \lim_{t\to\infty}[V(s,t) - (s - Ke^{-r(T-t)})] = 0. \end{cases} \tag{6.1}$$

Problems like these may often be solved cleanly by transforming them into other PDEs for which the solution is well-known. In this case, we will transform the equation into the heat equation with an initial condition, whose general form is

$$\begin{cases} u_t - u_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0, \\ u(x,0) = u_0(x), \end{cases} \tag{6.2}$$

and whose solution is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{1}{4t}|x-y|^2} u_0(y)dy. \tag{6.3}$$

To achieve this, we apply the **change of variables**:

$$\boxed{V(s,t) = e^{-r\tau}G(x,y),} \tag{6.4}$$

where $x(s,t)$, $y(s,t)$ are functions of s, t, and $\tau = T - t$.

We begin by taking the partial derivatives necessary to transform the BSM PDE. Using the chain and product rules, we obtain:

$$\frac{\partial V}{\partial t} = re^{-r\tau}G + e^{-r\tau}[G_x x_t + G_y y_t]$$
$$\frac{\partial V}{\partial s} = e^{-r\tau}[G_x x_s + G_y y_s]$$
$$\frac{\partial^2 V}{\partial s^2} = e^{-r\tau}[G_{xx}x_s^2 + G_{xy}x_s y_s + G_x x_{ss} + G_{yx}x_s y_s + G_{yy}y_s^2 + G_y y_{ss}]$$

Substituting these equations into (4.3), we have

$$re^{-r\tau}G + e^{-r\tau}[G_x x_t + G_y y_t] + \frac{1}{2}\sigma^2 s^2 e^{-r\tau}[G_{xx}x_s^2 + G_{xy}x_s y_s + G_x x_{ss} + G_{yx}x_s y_s + G_{yy}y_s^2 + G_y y_{ss}] + rse^{-r\tau}[G_x x_s + G_y y_s] - re^{-r\tau}G = 0, \quad \tau \in [0,T], \quad s \geq 0$$

Dividing out the e^{-rt} term and combining like terms:

$$\frac{\sigma^2 s^2}{2}x_s^2 G_{xx} + \frac{\sigma^2 s^2}{2}x_s y_s G_{xy} + \frac{\sigma^2 s^2}{2}y_s^2 G_{yy} + [\frac{\sigma^2 s^2}{2}x_{ss} + rsx_s + x_t]G_x + [\frac{\sigma^2 s^2}{2}y_{ss} + rsy_s + y_t]G_y = 0.$$

To obtain the form of the heat equation $G_y - G_{xx} = 0$, their coefficients must be equal. Additionally, we cannot have G_{yy}, G_x, or G_{xy} terms with nonzero coefficients. Therefore, the following conditions must be met:

1. $\frac{\sigma^2 s^2}{2}x_s^2 + \frac{\sigma^2 s^2}{2}y_{ss} + rsy_s + y_t = 0$
2. $\frac{\sigma^2 s^2}{2}x_s y_s = 0$
3. $\frac{\sigma^2 s^2}{2}y_s^2 = 0$
4. $\frac{\sigma^2 s^2}{2}x_{ss} + rsx_s + x_t = 0$

The first condition is derived from the coefficients of the persisting heat equation, while the rest are derived from setting the other coefficients to zero. From condition (3), since $\sigma, s \geq 0$, for any case other than the trivial one, y must be independent of s; i.e. $y = y(t)$. Under this stipulation, conditions (2) and (3) are annihilated, and the remaining reduce to

1. $\frac{\sigma^2 s^2}{2}x_s^2 + y_t = 0$
2. $\frac{\sigma^2 s^2}{2}x_{ss} + rsx_s + x_t = 0$

The new condition (1) implies that $\frac{\sigma^2 s^2}{2}x_s^2$ is not a function of s, which also implies that sx_s is not a function of s. So, x must have an $\ln s$ term. But x is also a function of t. Therefore we may make another change of variables and obtain

$$x(s,t) = \ln(s) + \gamma\tau \implies x_s = \frac{1}{s}$$
$$\implies x_t = -\gamma \qquad (\tau = T - t),$$

where γ is some constant. Substituting this into the above conditions, we obtain

$$\frac{\sigma^2}{2} + y_t = 0 \implies y(t) = \frac{\sigma^2 \tau}{2}$$
$$\frac{-\sigma^2}{2} + r - \gamma = 0 \implies \gamma = r - \frac{\sigma^2}{2}.$$

With these definitions, we now have that

$$\frac{\sigma^2 s^2}{2}x_s^2 = -\frac{\sigma^2 s^2}{2}y_{ss} - rsy_s - y_t$$

These coefficients may then be divided out of the PDE; since the rest are equal to zero, we obtain the heat equation

$$G_y - G_{xx} = 0,$$

where $x = x(s,t)$ and $y = y(t)$ and subject to the boundary condition

$$G|_{\tau=0} = G(x(s,T), y(T)) = [e^x - K]^+.$$

Now we apply Equation (6.3) to obtain the solution with the changed variables

$$G(x,y) = \frac{1}{\sqrt{4\pi y}} \int_{-\infty}^{\infty} f(u) e^{\frac{-(u-x)^2}{4y}} du \tag{6.5}$$

To define f, we use the above boundary condition. We also note that the value of the option must be positive, so that $G|_{\tau=0} = 0$ for $x < \ln K$. Hence we obtain

$$G(x,y) = \frac{1}{\sqrt{4\pi y}} \int_{\ln(K)}^{\infty} (e^u - K) e^{\frac{-(u-x)^2}{4y}} du. \tag{6.6}$$

The next step is to undo the change of variables, i.e. transform $G(x,y) \to V(s,t)$.

To do this, we apply the definition of $V(s,t)$ from (6.4) and the variable transformations of x and y. This results in

$$V(s,t) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{\ln K}^{\infty} e^u e^{\frac{-(u-\ln s-(r-\frac{\sigma^2}{2})\tau)^2}{2\sigma^2\tau}} du - \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{\ln K}^{\infty} K e^{\frac{-(u-\ln s-(r-\frac{\sigma^2}{2})\tau)^2}{2\sigma^2\tau}} du$$

To make sense of this equation, we may try to transform the integrals into ones that are easier to solve. Since we recall from the first section that the BSM equation is derived from stochastic processes (whose distributions are strikingly similar to the normal distribution), it makes sense to transform these integrals into ones that are of the same form as the Gaussian distribution, namely,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$$

The first step in this process is to transform the exponent of the first equation into an exponent of the form y^2. To do this, let us multiply out terms, and then add and subtract $ln(s) + r\tau$

$$\begin{aligned}
u + \frac{-(u-\ln s-(r-\frac{\sigma^2}{2})\tau)^2}{2\sigma^2\tau} &= \frac{2\sigma^2\tau u - [u^2 - 2[\ln s + (r-\frac{\sigma^2}{2})\tau]u + [\ln s + (r-\frac{\sigma^2}{2}\tau]^2]}{2\sigma^2\tau} \\
&= \frac{-u^2 + 2[\ln s + (r-\frac{\sigma^2}{2})\tau]u - [\ln s + (r-\frac{\sigma^2}{2}\tau]^2}{2\sigma^2\tau} \\
&= \frac{-u^2 + 2[\ln s + (r-\frac{\sigma^2}{2})\tau]u - \ln(s)^2 - 2(r-\frac{\sigma^2}{2})\tau\ln(s)}{2\sigma^2\tau} - \\
&\quad \frac{(r^2 - 2r\frac{\sigma^2}{2}r + \frac{\sigma^4}{4})\tau^2 + 2(\ln s + r\tau)\sigma^2\tau - 2(\ln s + r\tau)\sigma^2\tau}{2\sigma^2\tau} \\
&= \frac{-u^2 + 2[\ln s + (r-\frac{\sigma^2}{2})\tau]u - \ln(s)^2 + 2(r-\frac{\sigma^2}{2})\tau\ln(s)}{2\sigma^2\tau} - \\
&\quad \frac{(r^2 - 2r\frac{\sigma^2}{2}r + \frac{\sigma^4}{4})\tau^2 - 2(\ln s + r\tau)\sigma^2\tau}{2\sigma^2\tau} \\
&= \frac{-(u-\ln s-(r+\frac{\sigma^2}{2})\tau)^2}{2\sigma^2\tau} + \ln s + r\tau
\end{aligned}$$

So, now we have found a way to make the integral look like a normal distribution integral. Then, we simply let $p = \frac{(u - \ln s - (r + \frac{\sigma^2}{2})\tau)}{\sqrt{\sigma^2\tau}}$ which, by a change of vars, transforms the first integral into

$$\frac{s}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\ln(s/K)+(r+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}} e^{\frac{-p^2}{2}} dp$$

Now, we make a similar change of variables to the second integral to create a different form of the normal distribution. Letting $w = \frac{(u - \ln s - (r - \frac{\sigma^2}{2})\tau)}{\sqrt{\sigma^2\tau}}$ we see that the second integral becomes

$$\frac{Ke^{-r\tau}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\ln(s/K)+(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}} e^{\frac{-w^2}{2}} dp$$

We recall the definition of the normal distribution and denote it as $\Phi(x)$. Then, the first integral that we have transformed is now equal to

$$s\Phi(\frac{\ln(s/K) + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})$$

and the second integral is equal to

$$Ke^{-r\tau}\Phi(\frac{\ln(s/K) + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})$$

So, combining these two expressions, we get an explicit definition for the solution of the BSM PDE in terms of the normal distribution

$$\boxed{V(s,t) = s\Phi\left(\frac{\ln(s/K) + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\right) - Ke^{-r\tau}\Phi\left(\frac{\ln(s/K) + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\right).}$$

But, what makes this different from the heat equation? In essence, the BSM PDE and the Heat equation are the same *equation*, but they have different *boundary conditions*, which is the reason that all of this tricky calculus and change of variables have to be used to solve it. While the heat equation has the conditions along the x-axis and the t-axis, the BSM PDE instead has them along the t-axis and the line $V_T(s) = [S - K]^+$. This is also shown in Figure 6.1.

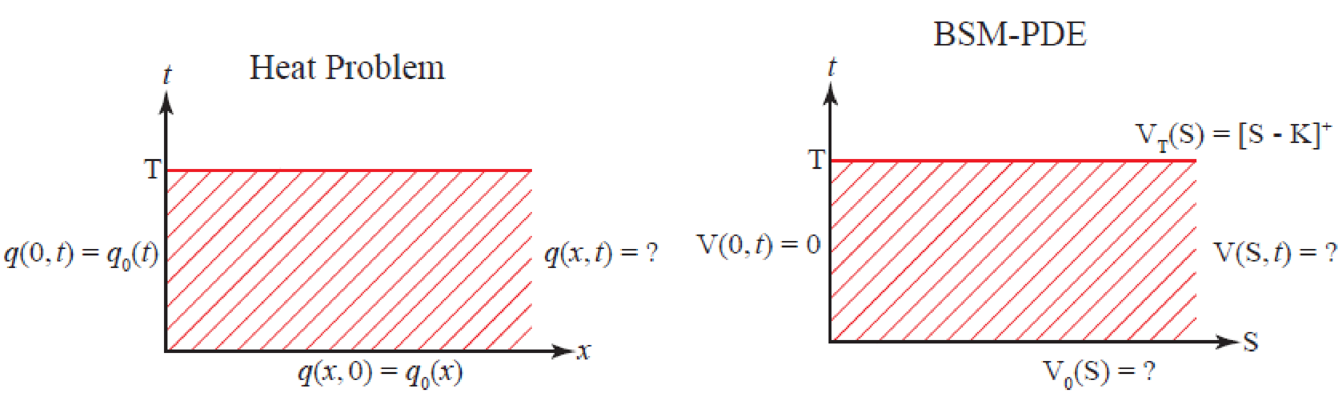

FIGURE 6.1. Differences Between the Heat Problem and the BSM Problem

7. Extension to American Options

American options differ from European options in that they may be exercised any time before the expiration date. This often makes them more valuable, as there is a greater window of opportunity to exercise them. American options are traded on indexes and are more common today than European options, which are usually traded over-the-counter. In this section, we present an approximation of American options using the BSM.

Using a similar derivation, we obtain the *inequality*

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \leq 0, \quad t \in [0,T], \quad S \geq 0,$$

subject to the same boundary conditions, as well as an additional *free boundary condition* $V(S,t) \geq V(S,T)$. This signifies the option price at time t for a given price S is greater than or equal to the option payoff at expiry T for the same given price. This is consistent with our expectations; before expiry, the option has more opportunities to move in-the-money.[9]

In general, this inequality doesn't have a closed-form solution, but approximations exist. The price of an American call option may be approximated as a sum of a European call option and an early exercise premium $\epsilon(S,t)$:

$$C_A(S,t) = C_E(S,t) + \epsilon(S,t), \tag{7.1}$$

where $C_A(S,t)$ is the price of the American call and $C_E(S,t)$ is the price of the option if it were a European call. From similar methods as above, the PDE for the early exercise premium is

$$\frac{\partial \epsilon}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \epsilon}{\partial S^2} + rb\frac{\partial \epsilon}{\partial S} - r\epsilon = 0,$$

where b is the cost of carrying the underlying (typically $b < r$), and other terms are defined as above. After some simplifications and applying the change of variables $\epsilon(S,t) = f(t)g(S,f)$, where $f(t) = 1 - \epsilon^{-r\tau}$, this equation can be written as

$$S^2 g_{SS} + \frac{2b}{\sigma^2} S g_S - \frac{2r}{\sigma^2(1-e^{-r\tau})} g - \frac{2b}{\sigma^2}(1-f)g_f = 0.$$

To get a closed-form solution, Barone-Adesi and Whaley approximate the last term of the left-hand side to be zero.[10] This is reasonable for American options with short times to expiration, since as $\tau \to 0, g_f \to 0$, so the term disappears. Then we have

$$S^2 g_{SS} + \frac{2b}{\sigma^2} S g_S - \frac{2r}{\sigma^2(1-e^{-r\tau})} g = 0,$$

which is an second-order ordinary differential equation. Its general solution is

$$\begin{aligned} g(S) &= a_1 S^{q_+} + a_2 S^{q_-}, \quad \text{where} \\ q_\pm &= \frac{-(N-1) \pm \sqrt{(N-1)^2 + 4M/f}}{2}, \end{aligned}$$

where $M = \frac{2r}{\sigma^2}$ and $N = \frac{2b}{\sigma^2}$. Since $M/f > 0, q_+ > 0$ and $q_- < 0$. Undoing the change of variables and substituting this back into Equation (7.1), we find that

$$C_A(S,t) = C_E(S,t) + f(t)[a_1 S^{q_+} + a_2 S^{q_-}].$$

We can now find the value of a_1 and a_2 by imposing some constraints. We first notice that, if $a_2 \neq 0$, as $S \to 0, a_2 S^{q_-} \to \infty$ since $q_- < 0$. But this would mean that $C_A \to \infty$ as $S \to 0$, which is clearly not true, as a call will expire worthless if its underlying loses all its value. Therefore we impose $a_2 = 0$, and the above equation reduces to

$$C_A(S,t) = C_E(S,t) + f(t)a_1 S^{q_+}.$$

To constrain a_1, we introduce the critical price S^*. This is the price imposed by the boundary of $S-K$ on $f(t)a_1S^{q_+}$, where $S-K$ is the proceeds earned if the call is exercised early when $S > K$. Beyond this point, the payoff of the call is only determined by this difference. The value of S^* can be found using the equation

$$S^* - K = C_E(S^*,t) + f(t)a_1 S^{*q_+}.$$

Due to the complexity of this equation, S^* must be determined iteratively, for which some algorithms have been given.[9, 10, 11] When known, an approximation of the solution of Equation (7.1) is found to be

$$\begin{cases} C_A(S,t) & = C_E(S,t) + A_1(\frac{S}{S^*})^{q_+}, \quad \text{when } S < S^* \\ C_A(S,t) & = S-K, \quad \text{when } S \geq S^*, \end{cases}$$

$$\text{where } A_1 = \frac{S^*}{q_+}\{1 - e^{(b-r)\tau}N[\ln\frac{S^*}{K} + (b+0.5\sigma^2)\tau]/\sigma\sqrt{\tau}\}.$$

An approximation for an American put can be derived in a similar way. For the put, since the payoff will become large as $S \to 0$, we have the flipped situation

$$P_A(S,t) = P_E(S,t) + f(t)a_2S^{q_-},$$

where P_A is the value of the American put and P_E is the value of the European put. We may find the critical price for this situation iteratively, S^{**}, to obtain

$$\begin{cases} P_A(S,t) & = P_E(S,t) + A_2(\frac{S}{S^{**}})^{q_-}, \quad \text{when } S > S^{**} \\ P_A(S,t) & = S-K, \quad \text{when } S \leq S^{**}, \end{cases}$$

$$\text{where } A_2 = -\frac{S^{**}}{q_-}\{1 - e^{(b-r)\tau}N[-\ln\frac{S^{**}}{K} + (b+0.5\sigma^2)\tau]/\sigma\sqrt{\tau}\}.$$

The above two systems give quadratic approximations for the value of American options. Other authors expand their methods to obtain new models.[11, 12]

8. Price & Implied Volatility Calculation

8.1. **Price Calculation.** We will now use the BSM to calculate the price of an option.

Suppose we want to buy a call on adidas AG (XE:ADS) with an expiry of 1 year. The call has the following parameters:[2]

- the spot price S is €124.26, the closing stock price on 12/2/22;
- the expiration date is 12/21/23, so $T = 1$, or 1 year;
- the strike price X is €76;
- the risk-free interest rate 5%;
- the volatility σ is 0.03.

[2]Values were derived from multiple sources. The stock price was taken from Barron's. The strike price and expiration date were chosen based on options that could be traded on Eurex. The risk-free interest rate was derived from the implied interest rate for US zero-coupon bonds issued on 12/02/22 and with expiration for 52 weeks and the forward exchange rate of the Euro and dollar, in accordance with the method given by PwC.[13] The volatility was chosen based on the underlying's slow volatility for the date 12/02/22.

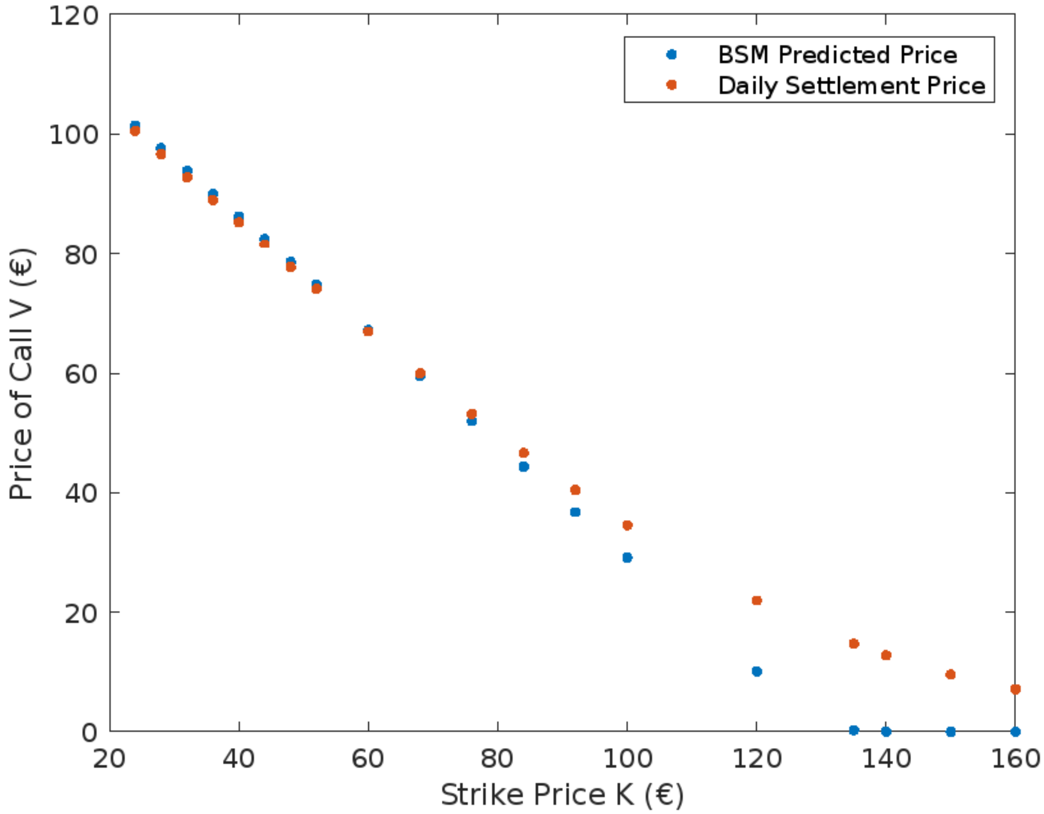

FIGURE 8.1. Comparison of BSM predicted and actual prices of the adidas AG call option.

Using the solution formula given by Equation (4.7), we receive a price $V =$ €51.97. This represents a percent error of 2.3% from the actual daily settlement price listed on Eurex, which was €53.19.

For this strike price, the BSM yields a value close to the daily settlement value. However, at other strike prices, the two values quickly diverge. This is shown in Figure 8.1.

These inaccuracies may be due to the estimates of the risk-free interest rate and the volatility. Additionally, in this calculation, the volatility rate was assumed to be constant for all strike prices, when in reality it differs (see the above section on the volatility smile). Instead, a more instructive approach may be the calculation of implied volatility, which can be done since the market price of the option is known.

8.2. **Implied Volatility Calculation.** Often we know the market price of an option V_{mkt} for given values of $K, S, r,$ and T. However, σ is often an unknown value, as there is no direct way to measure its value. The solution equation of the BSM may be inverted to find a value of σ, the implied volatility:

$$V_{mkt} = sN(d_+(s, T-t)) - Ke^{-r(T-t)}d_-(s, T-t)).$$

While this equation is nonlinear, we know that it is increasing with respect to σ, since $\frac{\partial V}{\partial \sigma} = S\sqrt{T}\frac{dN}{dd_1} > 0$. This matches real expectations; if a stock has a higher volatility, it is more likely to make large moves before the expiration date of an option, which means it is more likely to be widely different from the option's strike price. Hence, a higher σ signifies a higher probability for the option to be exercisable at expiry, so the option is more valuable. Because the equation is increasing with respect to σ, we know there is a unique solution for σ to satisfy the equation.

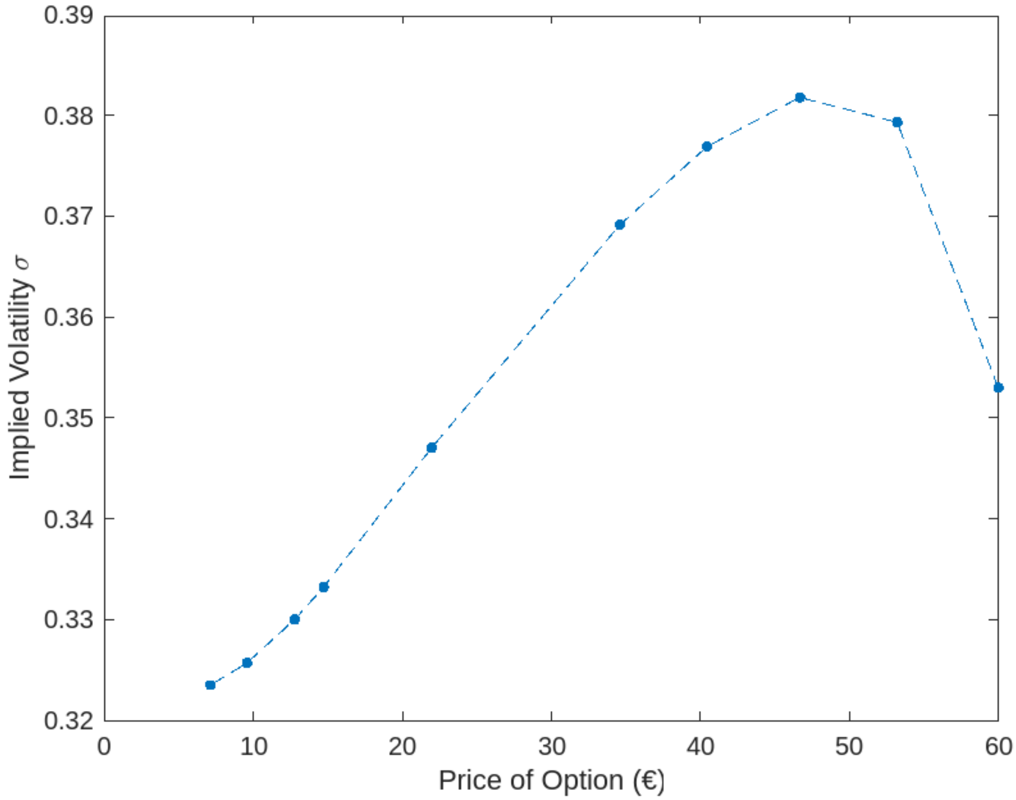

Figure 8.2. Graph of implied volatility versus option value using the fzero algorithm. Some values for which a numerical solution could not be found are omitted.

Although the equation is not easily invertible, a value of σ may be solved numerically. The MATLAB function fzero.m uses an algorithm that combines the bisection, secant, and inverse quadratic interpolation methods to find a numerical solution; this was used to find values for σ of the adidas AG stock, using the daily settlement value as the price of the option. The results are plotted in Figure 8.2.[3] The graph is not increasing because the strike price also changes for each option price. It confirms the phenomenon expected, that the volatility changes for different option prices, even though we would expect that the volatility would be the same, since it is tied to the underlying and not the option in its definition.

Although the implied volatility varies for different prices, it often stays within a given range–here, from 0.3-0.4. Knowing the implied volatility of a stock quantifies the uncertainty of the underlying stock, which can be used to gauge market sentiment and expected trends. Calculations like these are often made by traders and firms wishing to obtain data on the market and construct portfolios with greater chances of profit.

9. Summary & Implications

We have reviewed the history, derivation, solution, and application of the Black-Scholes-Merton partial differential equation. The equation arose after increased interest in expressing financial processes mathematically, and its derivation heavily relies on the Brownian motion of stocks and the validity of Stochastic calculus in analyzing such processes. Brownian motion and Wiener processes characterize

[3]Some prices were omitted because the algorithm didn't find values of σ for these prices. This does not mean that a unique solution doesn't exist, but just that the algorithm couldn't find values for it.

random motion. The equation can be solved using a change of variables to pare it down to a heat equation with a boundary condition, for which the solution is well known. Reversing the change of variables and analyzing the solution based on Gaussian distributions yields a closed-form solution that relates the value of an option to its strike price, risk-free interest rate, expiry date, and the volatility and spot price of the underlying asset. The equation and solution can also be extended to approximate the value of American options, which can be exercised earlier than the expiry date. We also show how the BSM can calculate the price of options directly, and how it can be used to find the implied volatility of an underlying.

The BSM model is powerful, and still receives wide use in simulations of market conditions. It has also been highly influential, as terms and properties of the model are used by traders characterize and communicate about assets (such as "The Greeks"). However, it is limited by its assumptions that no dividends are paid by the stock and the underlying has a constant volatility across prices and time. Its solution is also continuous, which cannot accommodate large movements in prices from external events in short times that would appear as nearly discontinuous jumps. Calculations of implied volatility must be done numerically, and are not always achievable by many algorithms. A more practical concern is that European options, which the BSM equation models, receive limited interest today. Normally European options are traded over-the-counter, while American options are traded on standardized indexes.

To address these and other limitations, authors cited here and elsewhere have either modified the BSM or proposed using it in new and creative ways. In fact, implied volatility calculations first arose after the model's utility for predicting prices had become obsolete. The fundamental genius underlying the model aids its ability to be extended to other areas. Doubtless other companies have developed models which are based on the BSM in their own analyses, although those models are propriety and not open to the public. The BSM has had a lasting academic impact in the study and discussion of options, and a lasting cultural impact among traders and the ways they think about options. For these reasons, we surmise that the BSM will receive continued interest and study for years to come.

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