Group Actions, Pólya's Theorem, and Applications in Tempered Music

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Abstract

This report summarizes the work of Franck Jedrzejewski, *Mathematical Theory of Music* [1]. It covers Chapter 2.1 – 2.3 of the book. We introduce the notion of a group action, the Burnside Lemmas, cycle index, and Pólya's Enumeration Theorem. We use these in the classification of pitch-class sets. In particular, we cover the cyclic classification proposed by Edmond Costère (1954), and the dihedral classification proposed by Hanson (1960), Zalewski and Forte (1972).

1 Group Actions

We have studied the permutation group acting on a finite set of integers, and we have proved Cayley's Theorem that every group is isomorphic to a group of permutations. Instead of restricting the set the groups acts on, we now reformulate this same idea by group actions.

Definition 1. *A* group action *of a multiplicative group G with identity* 1 *on a set X is a mapping $\cdot : G \times X \to X$ given by $(g, x) \mapsto g \cdot x$ such that:*
(1) $\forall x \in X$, $1 \cdot x = x$,
(2) $\forall g, h \in G$ *and* $x \in X$, $(gh) \cdot x = g \cdot (h \cdot x)$.

Example 1. The symmetric group S_n *acting on* the set $A = \{1, 2, \ldots, n\}$ is a group action. The identity action is the identity map $\epsilon : A \to A$.

Example 2 (Cayley's Theorem)**.** Let G be a group, and $g \in G$. Define a function $\cdot : G \times G \to G$ by $g \cdot x = gx$ for all $x \in G$.[1] We can show by this group action that every group is isomorphic to a group of permutations.

Remark. The group action induces an equivalence relation $x \sim y$ if $\exists g \in G$ such that $y = g \cdot x$.

Definition 2. *The* orbits *of G on X are the equivalence classes $G(x)$ of the above equivalence relation given by $G(x) = \{g \cdot x \mid g \in G\}$. The* set of all orbits *is denoted by $G\backslash\backslash X = \{G(x) \mid x \in X\}$. We say the action is* transitive *(or G acts* transitively*) if there is only one orbit.*

Remark. If there is only one orbit, that means $G(x) = X$ for all $x \in X$. This implies that for $x_1, x_2, y \in X$, $\exists g_1, g_2 \in G$ such that $x_2 = g_1 \cdot x_1$ and $y = g_2 \cdot x_2$. This implies $y = g_2 \cdot (g_1 \cdot x_1) = (g_2 g_1) \cdot x_1$, thus the action is transitive.

Definition 3. *The* stabilizer *of $x \in X$ is the set $G_x = \{g \in G \mid g \cdot x = x\}$. The set of all* fixed points *of $g \in G$ is the set $X_g = \{x \in X \mid g \cdot x = x\}$.*

Remark. G_x is a subgroup of G (denoted by $G_x \leq G$).

[1] In language of Gallian [2], define function $T_g : G \to G$ by $T_g(x) = gx$.

Proof. Refer to Gallian p.121 Exercise 43 [2]. $\square$

Example 3. Consider a group action on itself by *conjugation* $G \times G \to G$ by $(g, x) \mapsto gxg^{-1}$. The orbit $G(x) = \{gxg^{-1} \mid g \in G\}$ is the *conjugacy class* of x. The stabilizer of x is the set of commutative elements in G,

$$G_x = \{g \in G \mid gxg^{-1} = x\} = \{g \in G \mid gx = xg\} = C_G(x)$$

in other words, the centralizer of x.

Definition 4. *Let G be a group and H be a nonempty subset of G. The* left coset *of H in G containing g is the set $gH = \{gh \mid h \in H\}$. We use $|gH|$ to denote the number of elements in the set gH.*

Example 4. Consider a subgroup $H \leq G$ that acts by multiplication from the right on G. The orbit $H(g)$ is the left coset gH.

Definition 5. *Let $H \leq G$. The* index *of H in G is the number of distinct left cosets of H in G. We denote by $|G : H|$.*

Remark. The index of $H = \{1\}$ in G is the order of G, $|G|= |G : \{1\}|$.

Theorem 1 (Lagrange)**.** *Let G be a finite group and H be a subgroup of G. Then $|G : H|= |G|/|H|$.*

Proof. See Gallian pp.153–154 Theorem 7.1 and Corollary 1 [2]. $\square$

Theorem 2. *Let group G act on set X. The order of the orbit of $x \in X$ is equal to the index of the stabilizer of x in G, $|G(x)| = |G : G_x|$.*

Proof. Note that by Theorem 1, this is a reformulation of the Orbit-Stabilizer Theorem, $|G|= |G(x)||G_x|$. See Gallian pp.159–160 Theorem 7.4 [2]. $\square$

Theorem 3. *A group action of a finite group G on a set X induces a group homomorphism $\phi : G \to S_X$ given by $\phi(g) = \varphi_g$, where $\varphi_g(x) = g \cdot x$, $x \in X$. φ_g is called a* permutation representation *of G on X.*

Proof. $\forall g_1, g_2 \in G$, $\varphi_{g_1 g_2}(x) = (g_1 g_2) \cdot x = g_1(g_2 \cdot x) = g_1(\varphi_{g_2}(x)) = \varphi_{g_1} \circ \varphi_{g_2}(x)$. So $\phi(g_1 g_2) = \varphi_{g_1 g_2} = \varphi_{g_1} \circ \varphi_{g_2} = \phi(g_1) \circ \phi(g_2)$, ϕ is a group homomorphism. $\square$

Remark. Theorem 3 tells us that every group action on a set has the same group structure as the permutation group (or its subgroup) on that set. We shall use this homomorphism in the following sections when considering group actions, and write the actions in cycle notation as in permutation groups.

In the chord classification problem, we want to count the number of non-equivalent elements of X under specific group action G (or equivalence classes, Definition 2), thus we want the number of G-orbits.

Lemma 4 (Burnside)**.** *Let G be a finite multiplicative group acting on a finite set X. The number of G-orbits ($|G\backslash\backslash X|$) is the number of average fixed points,*

$$|G\backslash\backslash X|= \frac{1}{|G|} \sum_{g \in G} |X_g|$$

Proof.

$$\begin{aligned}
|G\backslash\backslash X| &= \sum_{x\in X}\frac{1}{|G(x)|} = \frac{1}{|G|}\sum_{x\in X}\frac{|G|}{|G(x)|} = \frac{1}{|G|}\sum_{x\in X}|G_x| && \text{(Definition 2; Theorem 2)}\\
&= \frac{1}{|G|}\sum_{x\in X}\sum_{g\in G_x}1 = \frac{1}{|G|}\sum_{\substack{(x,g)\\ \in X\times G_x}}1 = \frac{1}{|G|}\sum_{\substack{g\cdot x=x\\ (x,g)\in X\times G}}1 = \frac{1}{|G|}\sum_{\substack{(x,g)\\ \in X_g\times G}}1 = \frac{1}{|G|}\sum_{g\in G}\sum_{x\in X_g}1 && \text{(Definition 3)}\\
&= \frac{1}{|G|}\sum_{g\in G}|X_g| && \text{(Definition of order)}
\end{aligned}$$

□

In order to introduce Pólya's Enumeration Theorem, we need a generalized version of this lemma.

Definition 6. *Let R be a commutative ring such that $\mathbb{Q}$ is a subring of R, and G be finite group acting on a finite set X. A* weight function *is a function $w : X \to R$ that is constant on each G-orbit,*

$$\forall x \in X, \quad w(g\cdot x) = w(x) \quad \forall g \in G$$

The weight of an orbit *$w(G(x))$ is the weight of any element of the orbit, $w(G(x)) = w(x)$.*

Remark. Note that for any element x_1, x_2 in the same orbit, $w(x_1)$ always equals $w(x_2)$ by Definitions 2 and 6. We also note the significance of the weight function that it is preserved in each orbit, making it a good candidate for counting.

Lemma 5 (Generalize Burnside)**.** *The sum of weights of G-orbits is the average number of weighted fixed points,*

$$\sum_{u\in G\backslash\backslash X} w(u) = \frac{1}{|G|}\sum_{g\in G}\sum_{x\in X_g} w(x)$$

Proof.

$$\begin{aligned}
\sum_{u\in G\backslash\backslash X} w(u) &= \sum_{u\in G\backslash\backslash X}\sum_{x\in u}\frac{w(u)}{|u|} && \text{(Definition 2)}\\
&= \sum_{u\in G\backslash\backslash X}\sum_{x\in u}\frac{w(u)}{|G(x)|} && (x\in u, u = G(x))\\
&= \frac{1}{|G|}\sum_{x\in X}\frac{|G|}{|G(x)|}w(x) && \text{(Orbits form a partition)}\\
&= \frac{1}{|G|}\sum_{x\in X}|G_x|w(x) && \text{(Theorem 2)}\\
&= \frac{1}{|G|}\sum_{x\in X}\sum_{g\in G_x} w(x) && \text{(Definition of order)}\\
&= \frac{1}{|G|}\sum_{g\in G}\sum_{x\in X_g} w(x) && \text{(Definition 3)}
\end{aligned}$$

□

Remark. Note that we recover the original version of Burnside Lemma by setting $w(x) = 1$ for all $x \in X$.

2 Pólya's Enumeration

In this section, X, Y are finite sets, R is a commutative ring that contains $\mathbb{Q}$ as a subring, and G is a group acting on X, $\cdot : G \to X$. In order to introduce Pólya's Enumeration Theorem, we need the following definitions of configurations and cycle index.

Definition 7. *The set of* configurations Y^X *is the set of functions* $f : X \to Y$.

Remark. The group action $\cdot : G \to X$ induces a group action of G on Y^X,

$$* : G \times Y^X \to Y^X \text{ given by } (g, f) \mapsto f \circ \varphi_g^{-1}$$

where φ_g is the permutation representation of G on X (Theorem 3), and $\circ$ is function composition.

Lemma 6. *The weight function* $h : Y \to R$ *induces a weight function on* Y^X,

$$\omega : Y^X \to R \text{ given by } \omega(f) = \prod_{x \in X} h(f(x))$$

Proof. To see that ω is a weight function, $\forall g \in G$,

$$\omega(g * f) = \prod_{x \in X} h(g * f(x)) = \prod_{x \in X} h(f(\varphi_g^{-1}(x))) = \prod_{x \in X} h(f(x)) = \omega(f)$$

□

Definition 8. *Let G be a group acting on set X. The* cycle index *of the action is a polynomial* $P_{(G,X)} \in \mathbb{Q}[t_1, \ldots, t_{|X|}]$ *defined by*

$$P_{(G,X)}(t_1, \ldots, t_{|X|}) = \frac{1}{|G|} \sum_{g \in G} \prod_{k=1}^{|X|} t_k^{j_k(\bar{g})}$$

where permutation $\bar{g}$ is in the form of a product of disjoint (independent) cycles, and $j_k(\bar{g})$ is the number of cycles of length k of $\bar{g}$.

Remark. Note that cycle index is an extension of the generalized Burnside's Lemma when we consider the action as a permutation in its cycle form.

Example 5. Let $G = S_3$ be the group of permutation of the set $\{1, 2, 3\}$. S_3 has 6 elements, $s_1 = (1)(2)(3)$, $s_2 = (1)(23)$, $s_3 = (12)(3)$, $s_4 = (13)(2)$, $s_5 = (123)$, $s_6 = (132)$. For s_1, the number of cycle of length k is $j_1(s_1) = 3$, $j_2(s_1) = 0$, $j_3(s_1) = 0$. Similarly, $j_1(s_2) = 1$, $j_2(s_2) = 1$, $j_3(s_2) = 0$, $\ldots$, $j_1(s_6) = 0$, $j_2(s_6) = 0$, $j_3(s_6) = 1$. Thus the cycle index is the polynomial

$$P_{(S_3,X)}(t_1, t_2, t_3) = 16(t_1^3 + 3t_1t_2 + 2t_3).$$

Theorem 7 (**Pólya**). *The sum of weights of G-orbits on Y^X is given by*

$$\sum_{u \in G \backslash\backslash Y^X} \omega(u) = \frac{1}{|G|} \sum_{g \in G} \prod_{k=1}^{|X|} \left(\sum_{y \in Y} h(y)^k \right)^{j_k(\bar{g})}$$

where $h : Y \to R$ is a weight function as in Lemma 6, $\bar{g}$ and $j_k(\bar{g})$ are given as Definition 8.

Proof. By the generalized Burnside Lemma 5, it suffices to show that

$$\sum_{f\in Y_g^X} \omega(f) = \prod_{k=1}^{|X|} \left(\sum_{y\in Y} h(y)^k\right)^{j_k(\bar{g})}.$$

By Lemma 6,

$$\sum_{f\in Y_g^X} \omega(f) = \sum_{f\in Y_g^X} \prod_{x\in X} h(f(x))$$

But f is constant in Y_g^X by Definition 3, therefore, it is constant on the cycles of the permutation representation of G by Theorem 3. So we get

$$\begin{aligned}
\sum_{f\in Y_g^X} \omega(f) &= \sum_{f\in Y_g^X} \prod_{u\in <g>\backslash\backslash X} h(f(u))^{|u|} && (<g>=\{g^k\},\ |u| \text{ the order of orbit})\\
&= \prod_{u\in <g>\backslash\backslash X} \sum_{y\in Y} h(y)^{|u|} && (\text{independent operations, } f(u)=y, \forall f)\\
&= \prod_{k=1}^{|X|} \left(\sum_{y\in Y} h(y)^k\right)^{j_k(\bar{g})} && (\text{always } j_k(\bar{g}) \text{ cycles of length } k,\ |u| = \mathrm{lcm}(\text{cycle lengths}))
\end{aligned}$$

□

Remark. Since conjugate elements have the same number of k-cycles in their decomposition, it is sufficient to compute the cycle index over the conjugacy classes

$$P_{(G,X)}(t_1,\ldots,t_{|X|}) = \frac{1}{|G|}\sum_{c\in C} |c| \prod_{k=1}^{|X|} t_k^{j_k(\bar{g}_c)}$$

where C is the set of all conjugacy classes.

If for the weights of the elements of Y we take power of an independent variable z, the power series in the Pólya's formula are called the *configuration counting series* (left) and the *figures counting series* (right). The coefficient of order k of the left hand side is the number of classes in Y^X of weight z^k, and the coefficients of the right hand side are the numbers of elements of Y of weight z^k.

Example 6. Continuing example 5. Let $Y=\{0,1\}$, $X=\{1,2,3\}$, and $G=S_3$. The weight function is defined on Y by $h(0)=1$, $h(1)=z$. Substituting t_k by $1+z^k$ (details in next section), we get the sum of weights of G-orbits by Pólya's Theorem (7) is the following series

$$P_{(S_3,X)}(z) = 1+z+z^2+z^3$$

We generalize the above example, we have the following result for S_n.

Proposition 8. *The cycle index of symmetric group S_n of set $X=\{1,2,\ldots,n\}$ is given by*

$$P_{(S_n,X)}(t_1,\ldots,t_n) = \sum_j \prod_k \frac{1}{j_k!}\left(\frac{t_k}{k}\right)^{j_k}$$

where the sum is taken over all $j=(j_1,\ldots,j_n)$ verifying that $\sum_{k=1}^n kj_k = n$.

3 Classification in Tempered Music

In order to introduce the classification of chords, we begin by some definitions in tempered music.

Definition 9. *In classical temperament, the twelve pitch classes* $(C, C\sharp, D, \ldots, B\flat, B)$ *are identified with* $\mathbb{Z}_{12}$ *by* $C = 0$, $C\sharp = 1$, $D = 2$, $\ldots$, $B\flat = 10$, $B = 11$. *A* pitch class set (pcset) *is a subset of* $\mathbb{Z}_{12}$. *A pitch class set of cardinality* k *is called a pcset of length* k *or a* k-chord.

Remark. This definition can be generalized to any temperament of n pitch classes.

In the chord counting problem of n-tone music, the set X is identified with $\mathbb{Z}_n$, and the group G is the cyclic $\mathcal{C}_n$, dihedral $\mathcal{D}_n$, symmetric $\mathcal{S}_n$, or affine groups $\mathcal{A}_n$. In this section we will only consider the cyclic group $\mathcal{C}_n$ and dihedral group $\mathcal{D}_n$. A pitch class set $A \subset X$ corresponds to a characteristic function F taking the pitches in the set to 1 and the remaining notes to 0,

$$F : X = \mathbb{Z}_n \to Y = \{0, 1\}, F(x) = \begin{cases} 1 & \textit{if } x \in A \\ 0 & \textit{if } x \notin A \end{cases}$$

Note there is a bijection between the pcsets and functions F. The set of all pcsets is therefore Y^X (Definition 7), we denote it by $F(\mathbb{Z}_n)$.

Remark. The group G acts on $F(X)$ and induces an equivalence relation on k-chords.

Proof. By Remark of Definition 7, G acts on Y^X, i.e., $F(X)$. Two k-chords are equivalent if their characteristic functions are equivalent, $f_1 \sim f_2 \in F(X)$ if $\exists g \in G, f_1 = g \circ f_2$. This is clearly an equivalence relation.

□

Remark. The weight function on Y is given by $h(y) = \begin{cases} z & \textit{if } y = 1 \\ 1 & \textit{if } y = 0 \end{cases}$. By Lemma 6, it induces a weight function on $F(\mathbb{Z}_n)$ given by $\omega(f) = z^k + 1$ if f corresponds to a k-chord.

Definition 10. *Let* G *be a group acting on* $F(\mathbb{Z}_n)$. *The elements of the quotient set* $F(\mathbb{Z}_n)/G$ *defines the* pcsets classes *or simply* sets classes *relatively to the action of the group* G.

For the dihedral group $\mathcal{D}_n$, *the elements of* $F(\mathbb{Z}_n)/\mathcal{D}_n$ *are called the set classes under the action of the dihedral group or simply the* d-classes.

For the cyclic group $\mathcal{C}_n$, *the elements of* $F(\mathbb{Z}_n)/\mathcal{C}_n$ *are called the set classes under the action of the cyclic group or simply the* c-classes *or* musical assemblies.

Remark. By factoring out the group action G, we are effectively treating the elements of $F(X)$ in the same equivalence class induced by G as the same element in $F(X)/G$.

We now define some of the most common actions on pcsets in tempered music.

Definition 11. *A* transposition *is a mapping* $T_a(x) : \mathbb{Z}_n \to \mathbb{Z}_n$ *via* $T_a(x) = x + a \mod n$.

Remark. Two pcsets are equivalent under $\mathcal{C}_n$ if they are reducible to each other by transposition.

Definition 12. *The* inversion *is the mapping* $I : \mathbb{Z}_n \to \mathbb{Z}_n$ *via* $I(x) = -x \mod n$.

Definition 13. *The* inversion of order a *is defined by the composition* $I_a := T_a \circ I$, $I_a : \mathbb{Z}_n \to \mathbb{Z}_n$ *via* $I_a(x) = -x + a \mod n$.

Remark. Two pcsets are equivalent under $\mathcal{D}_n$ if they are reducible to the same form by T_a or I_a.

Example 7. Consider the pcset $A = \{F, B, D\sharp, G\sharp\} = \{3, 5, 8, 11\}$ (known as the "Tristan chord"). If we apply T_{-3} to A, we get $B = \{0, 2, 5, 8\}$ (half-diminished 7th chord), then A and B are equivalent under $\mathcal{C}_n$ and $\mathcal{D}_n$. If we apply I_{-1} to A, we get $C = \{0, 3, 6, 8\}$ (dominant 7th chord), then A and C are equivalent under $\mathcal{D}_n$ only. Note however, that B and C are not equivalent.

Note that if two pcsets are equivalent under $\mathcal{D}_n$, then they are equivalent under $\mathcal{C}_n$.

Recall that the Burnside's Lemma gives the number orbits or equivalence classes of a set under a group action. Now if we substitute the t_k's in the cycle index by the weight function $\omega(f) = z^k + 1$ and apply Pólya's Theorem 7, we get the sum of the weights of the equivalence classes (k-chord classes) under G. The number of k-chords classes is therefore the coefficient of z^k in the cycle index.

Example 8. Consider the group $\mathcal{C}_3$ acting on $\mathbb{Z}_3$. $|\mathcal{C}_3| = 3$, and we can write out the elements in cycle notation (012), (210), (0)(1)(2). By Definition 8, we can easily compute the cycle index to be

$$P_{(\mathcal{C}_3,\mathbb{Z}_3)}(t_1, t_2, t_3) = \frac{1}{3}(t_1^3 + 2t_3).$$

We generalize the above example to get the following proposition.

Proposition 9. *The cycle index of the cyclic group $\mathcal{C}_n$ is the polynomial*

$$P_{(\mathcal{C}_n,\mathbb{Z}_n)}(t_1, \dots, t_n) = \frac{1}{n} \sum_{d|n} \varphi\left(\frac{n}{d}\right) t_{n/d}^d$$

where φ is the Euler totient function.

Recall that the *Euler totient function* φ for an integer m is the number of positive integers not greater than and relatively prime to m. The first few values are: $\varphi(1) = 1$, $\varphi(2) = 1$, $\varphi(3) = 2$, $\varphi(4) = 2$, $\varphi(5) = 4$, $\varphi(6) = 2$, $\varphi(7) = 6$, $\varphi(8) = 4$, $\varphi(9) = 6$, $\varphi(10) = 4$, $\varphi(11) = 10$, $\varphi(12) = 4$, etc.

Example 9. For $n = 12$, the cycle index is

$$P_{(\mathcal{C}_{12},\mathbb{Z}_{12})}(t_1, \dots, t_{12}) = \frac{1}{12}(t_1^{12} + t_2^6 + 2t_3^4 + 2t_4^3 + 2t_6^2 + 4t_{12})$$

Substituting t_k by $1 + z^k$, we get the following counting series

$$P_{(\mathcal{C}_{12},\mathbb{Z}_{12})}(z) = 1 + z + 6z^2 + 19z^3 + 43z^4 + 66z^5 + 80z^6 + 66z^7 + 43z^8 + 19z^9 + 6z^{10} + z^{11} + z^{12}$$

Each coefficient of z^k is the number of k-chord classes under the $\mathcal{C}_n$ group action. There are, for example, 6 intervals and 19 trichords under $\mathcal{C}_{12}$.

Example 10. Consider the group $\mathcal{D}_3$ acting on $\mathbb{Z}_3$. $|\mathcal{D}_3| = 6$, and we can write out the elements in cycle notation (012), (210), (0)(1)(2), (0)(12), (1)(20), (2)(01). Note that the first 3 elements are just from $\mathcal{C}_3$. We can calculate the cycle index to be

$$P_{(\mathcal{D}_3,\mathbb{Z}_3)}(t_1, t_2, t_3) = \frac{1}{6}(t_1^3 + 2t_3 + 3t_1t_2) = \frac{1}{2}P_{(\mathcal{C}_3,\mathbb{Z}_3)} + \frac{1}{2}t_1t_2$$

Now consider the group $\mathcal{D}_4$ acting on $\mathbb{Z}_4$. $|\mathcal{D}_4| = 8$, we write out he elements in cycle notation (0123), (13)(20), (1032), (0)(1)(2)(3), (01)(23), (12)(30), (1)(3)(20), (0)(2)(13). Note again that the first 4 elements are just from $\mathcal{C}_4$. We calculate the cycle index to be

$$P_{(\mathcal{D}_4,\mathbb{Z}_4)}(t_1, t_2, t_3, t_4) = \frac{1}{8}(t_1^4 + 3t_2^2 + 2t_4 + 2t_1^2t_2) = \frac{1}{2}P_{(\mathcal{C}_4,\mathbb{Z}_4)} + \frac{1}{4}(t_1^2t_2 + t_2^2).$$

We generalize the above example to get the following proposition.

Proposition 10. *The cycle index of the dihedral group $\mathcal{D}_n$ is given by*

$$P_{(\mathcal{D}_n,\mathbb{Z}_n)}(t_1,\dots,t_n) = \begin{cases} \frac{1}{2}P_{(\mathcal{C}_n,\mathbb{Z}_n)} + \frac{1}{2}t_1t_2^{(n-1)/2} & \textit{if } n \textit{ is odd} \\ \frac{1}{2}P_{(\mathcal{C}_n,\mathbb{Z}_n)} + \frac{1}{4}(t_1^2t_2^{(n-2)/2} + t_2^{n/2}) & \textit{if } n \textit{ is even} \end{cases}$$

Example 11. For $n = 12$, the cycle index is

$$P_{(\mathcal{D}_{12},\mathbb{Z}_{12})}(t_1,\dots,t_{12}) = \frac{1}{24}(t_1^{12} + t_2^6 + 2t_3^4 + 2t_4^3 + 2t_6^2 + 4t_{12}) + \frac{1}{4}t_1^2t_2^5 + \frac{1}{4}t_2^6$$

Substituting t_k by $1 + z^k$, we get the following counting series

$$P_{(\mathcal{D}_{12},\mathbb{Z}_{12})}(z) = 1 + z + 6z^2 + 12z^3 + 29z^4 + 38z^5 + 50z^6 + 38z^7 + 29z^8 + 12z^9 + 6z^{10} + z^{11} + z^{12}$$

Each coefficient of z^k is the number of k-chord classes under the $\mathcal{D}_n$ group action. There are, for example, 12 trichords under $\mathcal{D}_{12}$.

Summarizing the number of assemblies from each class, we get the following table.

k	k-chords	$\mathcal{C}_n$ (Costère)	$\mathcal{D}_n$ (Forte)
1	Unison	1	1
2	Intervals	6	6
3	Trichords	19	12
4	Tetrachords	43	29
5	Pentachords	66	38
6	Hexachords	80	50
7	Heptachords	66	38
8	Octachords	43	29
9	Enneachords	19	12
10	Decachords	6	6
11	Endecachords	1	1
12	Dodecachords	1	1
	Total	351	223

Table 1: Classification of pitch-class sets under $\mathcal{C}_n$ and $\mathcal{D}_n$ [1]

Thus far, we have seen the classification of chords under the groups $\mathcal{C}_n$ and $\mathcal{D}_n$, the most common actions on musical notes, through Pólya's Enumeration Theorem. There are more complicated actions such as the symmetric group $\mathcal{S}_n$, and the affine groups $\mathcal{A}_n$. The results of these classifications can be used in analysis of pieces which traditional tonal functions are more or less abandoned. Similar methods can be used in the analysis musical motifs, where there are more parameters. Interested readers should consult Jedrzejewski [1].

References

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