Graph Optimization
Lab session 2

Exercize 1. Consider a multicommodity flow problem defined over a directed graph $G = (N, A)$. Each arc is associated with a capacity u_{ij} and a cost per unit of flow c_{ij}. The set of demands K represents different commodities to be routed on the graph, each described by three parameters, a source node $s_k \in N$, a destination node $t_k \in N$, and an amount of flow d_k to be sent from the source node to the destination node. The problem consists in deciding the routing of each demand with the aim of minimizing the overall cost. Consider the path-based formulation of the problem:

$$\min \sum_{p \in P} c_p x_p \tag{1}$$

$$\sum_{p \in P_k} x_p = d_k, \qquad \forall k \in K \tag{2}$$

$$\sum_{p \in P_{ij}} x_p \leq u_{ij}, \qquad \forall (i,j) \in A \tag{3}$$

$$x_p \geq 0, \tag{4}$$

where P is the set of paths on the graph, P_{ij} is the set of paths including arc (i, j), and c_p is the path cost $c_p = \sum_{(i,j) \in p} c_{ij}$. The problem can be solved through column generation. The dual constraint associated with each primal x_p variable is :

$$\sigma_k + \sum_{(i,j) \in p} \mu_{ij} \leq c_p,$$

where σ_k is the dual variable associated with (2) and μ_{ij} is the dual variable associated with (3). The pricing problem for each demand k can be formulated as:

$$\begin{aligned} &\min \sum_{(i,j) \in A} g_{ij} z_{ij} \\ &\sum_{(j,i) \in A} z_{ji} - \sum_{(i,j) \in A} z_{ij} = b_i, \forall i \in N \\ &z_{ij} \in \{0, 1\} \quad \forall (i,j) \in A, \end{aligned}$$

where $g_{ij} = c_{ij} - \mu_{ij}$ and

$$b_i = \begin{cases} -1, i = s_k \\ 1, i = t_k \\ 0, \text{otherwise} \end{cases}$$

1. Write an AMPL script that solves the multicommodity problem using column generation.
2. Solve the following instance (described in the `multicommodity.dat`).

   The graph has 6 nodes and 12 arcs. Arcs set and arc costs are described un the following table:

arc	cost	arc	cost
(1,2)	10	(5,2)	4
(1,4)	10	(3,5)	6
(1,5)	10	(3,6)	4
(1,3)	6	(4,5)	4
(2,3)	4	(4,6)	6
(2,4)	4	(5,6)	10

The arc capacity is equal to 4 for all $(i, j) \in A$.

The commodities are given in the following table, where s_k denotes the source node of commodity k, t_k the destination node, and d_k the demand:

k	s_k	t_k	d_k
1	1	4	4
2	1	5	5
3	1	6	3
4	5	6	3

Start from the restricted master problem defined by the following paths:

source	destination	path
1	4	(1,4)
1	4	(1,2),(2,4)
1	5	(1,5)
1	5	(1,2),(2,3),(3,5)
1	6	(1,2),(2,4),(4,6)
5	6	(5,6)
5	6	(5,2)(2,4)(4,6)