HOMEWORK 14

Due date:

Exercise 7.1, ($\delta = \sqrt{-5}$ in problem 7.1), 7.3, (in exercise 7.3, R is the integer ring of $\mathbb{Q}(\sqrt{-26})$), Page 410 of Artin's book,

Let F be an algebraic number field and $\mathcal{O}_F$ be its ring of integers. Let $\mathfrak{a} \subset \mathcal{O}_F$ be a nonzero ideal. Recall that $\mathcal{O}_F/\mathfrak{a}$ is finite. We have defined

$$\mathrm{Nm}(\mathfrak{a}) = |\mathcal{O}_F/\mathfrak{a}|,$$

which is a positive integer.

Problem 1. *Suppose* $[F : \mathbb{Q}] = n$. *For* $a \in \mathbb{Z}$, *show that* $\mathrm{Nm}(a\mathcal{O}_F) = a^n = \mathrm{Nm}_{F/\mathbb{Q}}(a)$.

This is proved in class. Repeat it here.

Problem 2. *Let* $\mathfrak{a}, \mathfrak{b}$ *are two coprime nonzero ideals. Show that* $\mathrm{Nm}(\mathfrak{a}\mathfrak{b}) = \mathrm{Nm}(\mathfrak{a})\mathrm{Nm}(\mathfrak{b})$.

This is a consequence of Chinese remainder theorem.

Problem 3. *Let* $\mathfrak{p}$ *be a nonzero prime ideal of* $\mathcal{O}_F$. *Show that* $\mathrm{Nm}(\mathfrak{p}) = p^f$ *for some prime integer* $p \in \mathbb{Z}$ *and some positive integer* f.

Problem 4. *Let* F *be an algebraic number field and* $\mathcal{O}_F$ *be its ring of integers. From Hurwitz lemma, there exists an integer* $M = M_F$ *such that for any* $\alpha, \beta \in \mathcal{O}_F$ *with* $\beta \neq 0$, *there exists an integer* t *with* $1 \leq t \leq M$ *and an element* $\omega \in \mathcal{O}_F$ *such that*

$$|\mathrm{Nm}_{F/\mathbb{Q}}(t\alpha - \omega\beta)| < |\mathrm{Nm}_{F/\mathbb{Q}}(\beta)|.$$

Re-examine the proof given in class and try to find an explicit form of M_F. *For the field* $F = \mathbb{Q}(\sqrt{-13})$, *find an explicit* $M = M_F$. *The constant* M *should be as small as possible.*

We know that $M_F > 1$ since $\mathcal{O}_F$ is not a PID when $F = \mathbb{Q}(\sqrt{-13})$. Is $M = 2$ enough? If so, prove it. If not, find one counter example and try the next one.

Problem 5. *Let* F *be an algebraic number field. Show that* $\mathcal{O}_F$ *is a PID iff for every* $\alpha \in F, \alpha \notin \mathcal{O}_F$, *there exists* $\beta, \gamma \in \mathcal{O}_F$ *such that*

$$0 < |\mathrm{Nm}_{F/\mathbb{Q}}(\alpha\beta - \gamma)| < 1.$$

Problem 6. *Let* K *be an algebraic number field. Show that there exists a finite extension* L/K *such that for every ideal* $\mathfrak{a} \subset \mathcal{O}_K$, *the ideal* $\mathfrak{a}\mathcal{O}_L$ *is principal in* $\mathcal{O}_L$.

Hint: use finiteness of class numbers. See this link for a solution.

Given a matrix $A = (a_{i,j}) \in \mathrm{Mat}_{n\times n}(\mathbb{C})$, recall that the Hilbert-Schimidt norm is defined to be

$$||A||_{HS} = \sqrt{\sum_{i,j} |a_{ij}|^2}.$$

Problem 7. *Show that*

$$||A + B||_{HS} \leq ||A||_{HS} + ||B||_{HS}$$

for all $A, B \in \mathrm{Mat}_{n\times n}(\mathbb{C})$.