In [ ]:

```
import sympy as sp
```

坐标及参数定义

相机内参矩阵：

$$\mathbf{K} = \begin{bmatrix} \alpha_x & 0 & x_0 \\ 0 & \alpha_y & y_0 \\ 0 & 0 & 1 \end{bmatrix}$$

定义归一化平面坐标$\mathbf{p}_n$、考虑畸变的归一化坐标$\mathbf{p}_n^d$和场景坐标（即相机坐标系下的坐标）$\mathbf{p}_s$，且有：

$$\mathbf{p}_n = \begin{bmatrix} x_n \\ y_n \\ 1 \end{bmatrix} = \frac{1}{z_s}\mathbf{p}_s = \frac{1}{z_s}\begin{bmatrix} x_s \\ y_s \\ z_s \end{bmatrix}$$

$$\mathbf{p}_n^d = \begin{bmatrix} x_n(1 + k_1 r_n^2 + k_2 r_n^4) + 2p_1 x_n y_n + p_2(r_n^2 + 2x_n^2) \\ y_n(1 + k_1 r_n^2 + k_2 r_n^4) + p_1(r_n^2 + 2y_n^2) + 2p_2 x_n y_n \\ 1 \end{bmatrix}$$

对畸变过程，记为：

$$\mathbf{p}_n^d = D(\mathbf{p}_n)$$

定义世界坐标$\mathbf{p}_w$，有：

$$\mathbf{p}_s = \mathbf{T}_{sw}\mathbf{p}_w$$

且：

$$\mathbf{T}_{cw} = \mathrm{Exp}(\xi) \tag{1}$$

$$\xi = \begin{bmatrix} \rho \\ \phi \end{bmatrix} \tag{2}$$

ρ代表平移、ϕ代表旋转。

定义像素坐标$\mathbf{p}_p$和考虑畸变的像素坐标$\mathbf{p}_p^d$，且有：

$$\mathbf{p}_p^d = \mathbf{K}\mathbf{p}_n^d$$

为了方便，若$\mathbf{p}$是齐次坐标，则求导的时候记$\mathbf{p}$为非齐次坐标，例如：

$$\mathbf{p}_n^d = \begin{bmatrix} x_n^p \\ y_n^p \end{bmatrix}, \mathbf{p}_p^d = \begin{bmatrix} x_p^p \\ y_p^p \end{bmatrix}$$

优化问题

优化问题

我们想通过非线性优化调整每幅图像与相机的相对位姿ξ、3D点的世界坐标$\mathbf{P}_w$。记$\mathbf{m} = \begin{bmatrix} u_{obs} \\ v_{obs} \end{bmatrix}$为实际测量到的像素坐标，$\mathbf{r}$为残差，有：

$$\mathbf{r} = \mathbf{p}_p^d - \mathbf{m} = \begin{bmatrix} x_p^d - u_{obs} \\ y_p^d - v_{obs} \end{bmatrix}$$

为了优化这些参数，我们需要推导残差对参数的雅克比矩阵$\frac{\partial \mathbf{r}}{\partial \xi}$、$\frac{\partial \mathbf{r}}{\partial \mathbf{P}_w}$。

In [ ]:
```
alpha_x, alpha_y, x0, y0 = sp.symbols('alpha_x, alpha_y, x_0, y_0')
k1, k2, p1, p2 = sp.symbols('k1, k2, p1, p2')
xn, yn, rn, pn = sp.symbols('x_n, y_n, r_n, p_n')
xdn, ydn, pdn = sp.symbols('x_n^d, y_n^d, p_n^d')
K = sp.Matrix([[alpha_x, 0, x0], [0, alpha_y, y0], [0, 0, 1]])
pn = sp.Matrix([[xn], [yn], [1]])
pdn = sp.Matrix([[xdn], [ydn], [1]])
display(K, pn, pdn)
```

$$\begin{bmatrix} \alpha_x & 0 & x_0 \\ 0 & \alpha_y & y_0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_n \\ y_n \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_n^d \\ y_n^d \\ 1 \end{bmatrix}$$

In [ ]:
```
xdn = xn * (1+k1*rn**2 + k2*rn**4) + 2*p1*xn*yn + p2*(rn**2 + 2*xn**2)
ydn = yn * (1+k1*rn**2 + k2*rn**4) + p1*(rn**2 + 2*yn**2) + 2*p2*xn*yn
pdn = sp.Matrix([[xdn], [ydn], [1]])
pdp = K * pdn

pdp.row_del(2)
param_vector = sp.Matrix([[alpha_x], [alpha_y], [x0], [y0], [k1], [k2], [p1], [p2]])
```

Out[ ]:

$$\begin{bmatrix} 2p_1x_ny_n + p_2\left(r_n^2 + 2x_n^2\right) + x_n\left(k_1r_n^2 + k_2r_n^4 + 1\right) & 0 & 1 & 0 & \alpha_x r_n^2 x_n & \alpha_x r_n^4 x_n & 2\alpha_x x_n y_n & \alpha_x\left(r_n^2 + 2x_n^2\right) \\ 0 & p_1\left(r_n^2 + 2y_n^2\right) + 2p_2x_ny_n + y_n\left(k_1r_n^2 + k_2r_n^4 + 1\right) & 0 & 1 & \alpha_y r_n^2 y_n & \alpha_y r_n^4 y_n & \alpha_y\left(r_n^2 + 2y_n^2\right) & 2\alpha_y x_n y_n \end{bmatrix}$$

下面推导$\frac{\partial \mathbf{r}}{\partial \xi}$ 由于：

$$\frac{\partial \mathbf{r}}{\partial \xi} = \frac{\partial \mathbf{p}_p^d}{\partial \mathbf{p}_n} \frac{\partial \mathbf{p}_n}{\partial \mathbf{p}_s} \frac{\partial \mathbf{p}_s}{\partial \xi}$$

故只需分别推导$\frac{\partial \mathbf{p}_p^d}{\partial \mathbf{p}_n}$、 $\frac{\partial \mathbf{p}_n}{\partial \mathbf{p}_s}$、 $\frac{\partial \mathbf{p}_s}{\partial \xi}$

In [ ]:
```
pn.row_del(2)
```

In [ ]:
```
pdp.jacobian(pn)
```

Out[ ]:

$$\begin{bmatrix} \alpha_x \left(k_1 r_n^2 + k_2 r_n^4 + 2p_1 y_n + 4p_2 x_n + 1\right) & 2\alpha_x p_1 x_n \\ 2\alpha_y p_2 y_n & \alpha_y \left(k_1 r_n^2 + k_2 r_n^4 + 4p_1 y_n + 2p_2 x_n + 1\right) \end{bmatrix}$$

故

$$\frac{\partial \mathbf{p}_p^d}{\partial \mathbf{p}_n} = \begin{bmatrix} \alpha_x \left(k_1 r_n^2 + k_2 r_n^4 + 2p_1 y_n + 4p_2 x_n + 1\right) & 2\alpha_x p_1 x_n \\ 2\alpha_y p_2 y_n & \alpha_y \left(k_1 r_n^2 + k_2 r_n^4 + 4p_1 y_n + 2p_2 x_n + 1\right) \end{bmatrix}$$

In [ ]:
```
xs, ys, zs, ps = sp.symbols('x_s, y_s, z_s, p_s')
ps = sp.Matrix([[xs], [ys], [zs]])
xn = xs / zs
yn = ys / zs
pn = sp.Matrix([[xn], [yn]])
pn.jacobian(ps)
```

Out[ ]:

$$\begin{bmatrix} \frac{1}{z_s} & 0 & -\frac{x_s}{z_s^2} \\ 0 & \frac{1}{z_s} & -\frac{y_s}{z_s^2} \end{bmatrix}$$

故

$$\frac{\partial \mathbf{p}_n}{\partial \mathbf{p}_s} = \begin{bmatrix} \frac{1}{z_s} & 0 & -\frac{x_s}{z_s^2} \\ 0 & \frac{1}{z_s} & -\frac{y_s}{z_s^2} \end{bmatrix}$$

对$\frac{\partial \mathbf{p}_s}{\partial \xi}$的推导不能利用sympy完成，需要手动推导:

$$\begin{aligned} \frac{\partial \mathbf{p}_s}{\partial \delta\xi} &= \frac{\partial(\mathbf{T}_{sw}\mathbf{p}_w)}{\partial \delta\xi} && (3) \\ &= \lim_{\delta\xi \to 0} \frac{\exp(\delta\xi^\wedge)\exp(\xi^\wedge)\mathbf{p}_w - \exp(\xi^\wedge)\mathbf{p}_w}{\delta\xi} && (4) \\ &= \begin{bmatrix} \mathbf{I}_{3\times3} & -(\mathbf{R}_{sw}\mathbf{p}_w + \mathbf{t}_{sw})^\wedge_{3\times3} \\ \mathbf{0}^T_{1\times3} & \mathbf{0}^T_{1\times3} \end{bmatrix} && (5) \\ &= \begin{bmatrix} \mathbf{I}_{3\times3} & -\lfloor \mathbf{p}_s \rfloor^\wedge_{3\times3} \\ \mathbf{0}^T_{1\times3} & \mathbf{0}^T_{1\times3} \end{bmatrix} && (6) \end{aligned}$$

注意：在这里，$\mathbf{p}_s$和$\mathbf{p}_w$为齐次坐标，因为我们用到了齐次变换矩阵。若$\mathbf{p}_s$为非齐次坐标形式，则

$$\frac{\partial \mathbf{p}_s}{\partial \delta\xi} = \begin{bmatrix} \mathbf{I}_{3\times3} & -\lfloor \mathbf{p}_s \rfloor_{3\times3} \end{bmatrix}$$

故

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial \xi} &= \frac{\partial \mathbf{p}_p^d}{\partial \mathbf{p}_n} \frac{\partial \mathbf{p}_n}{\partial \mathbf{p}_s} \frac{\partial \mathbf{p}_s}{\partial \xi} && (7) \\ &= \begin{bmatrix} \alpha_x \left(k_1 r_n^2 + k_2 r_n^4 + 2p_1 y_n + 4p_2 x_n + 1\right) & 2\alpha_x p_1 x_n \\ 2\alpha_y p_2 y_n & \alpha_y \left(k_1 r_n^2 + k_2 r_n^4 + 4p_1 y_n + 2p_2 x_n + 1\right) \end{bmatrix} && (8) \\ &\quad \cdot \begin{bmatrix} \frac{1}{z_s} & 0 & -\frac{x_s}{z_s^2} \\ 0 & \frac{1}{z_s} & -\frac{y_s}{z_s^2} \end{bmatrix} && (9) \\ &\quad \cdot \begin{bmatrix} \mathbf{I}_{3\times3} & -\lfloor \mathbf{p}_s \rfloor_{3\times3} \end{bmatrix} && (10) \end{aligned}$$

下面推导 $\frac{\partial \mathbf{r}}{\partial \mathbf{p}_w}$: 由于 $\frac{\partial \mathbf{p}_s}{\partial \mathbf{P}_w} = \mathbf{R}_{sw} = Exp(\phi)$，则

$$
\begin{aligned}
\frac{\partial \mathbf{r}}{\partial \xi} &= \frac{\partial \mathbf{p}_p^d}{\partial \mathbf{p}_n} \frac{\partial \mathbf{p}_n}{\partial \mathbf{p}_s} \frac{\partial \mathbf{p}_s}{\partial \mathbf{p}_w} && (11) \\
&= \begin{bmatrix} \alpha_x \left(k_1 r_n^2 + k_2 r_n^4 + 2p_1 y_n + 4p_2 x_n + 1\right) & 2\alpha_x p_1 x_n \\ 2\alpha_y p_2 y_n & \alpha_y \left(k_1 r_n^2 + k_2 r_n^4 + 4p_1 y_n + 2p_2 x_n + 1\right) \end{bmatrix} && (12) \\
&\quad \cdot \begin{bmatrix} \frac{1}{z_s} & 0 & -\frac{x_s}{z_s^2} \\ 0 & \frac{1}{z_s} & -\frac{y_s}{z_s^2} \end{bmatrix} && (13) \\
&\quad \cdot \mathbf{R}_{sw} && (14)
\end{aligned}
$$