The Hopf algebra of Symmetric functions and some generalizations

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What's Hopf algebra ?

Let $\mathbb{K}$ be a field.

An algebra is a triple (A, M, u), where

A: a vector space over $\mathbb{K}$

M: $A \otimes A \longrightarrow A$ (product)

u: $\mathbb{K} \longrightarrow A$ (unit)

$$
\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "I \otimes M"] \arrow[d, "M \otimes I"'] & A \otimes A \arrow[d, "M"] \\
A \otimes A \arrow[r, "M"] & A
\end{tikzcd}
$$

associativity of M:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

$$
\begin{tikzcd}
 & A \otimes A \arrow[dd, "M"] & \\
\mathbb{K} \otimes A \arrow[ur, "u \otimes I"] \arrow[dr] & & A \otimes \mathbb{K} \arrow[ul, "I \otimes u"'] \arrow[dl] \\
 & A &
\end{tikzcd}
$$

unitary property:

$$u(1_{\mathbb{K}}) \cdot a = a \cdot u(1_{\mathbb{K}}) = a$$

A **Coalgebra** is a triple (C, Δ, ε), where

C : a vector space over $\mathbb{K}$

Δ : $C \longrightarrow C \otimes C$ (coproduct)

ε : $C \longrightarrow \mathbb{K}$ (counit)

$$
\begin{tikzcd}
C \otimes C \otimes C & & C \otimes C \arrow[ll, "I \otimes \Delta"'] \\
C \otimes C \arrow[u, "\Delta \otimes I"] & & C \arrow[ll, "\Delta"'] \arrow[u, "\Delta"']
\end{tikzcd}
$$

(coassociativity)

$$
\begin{tikzcd}
 & C \otimes C \arrow[dl, "\varepsilon \otimes I"'] \arrow[dr, "I \otimes \varepsilon"] & \\
\mathbb{K} \otimes C & & C \otimes \mathbb{K} \\
 & C \arrow[ul] \arrow[ur] \arrow[uu, "\Delta"'] &
\end{tikzcd}
$$

(counitary property)

For $c \in C$, if $\Delta(c) = \sum_i c_{1i} \otimes c_{2i}$, we use the

Sweedler notation: $\Delta(c) = \sum_c c_{(1)} \otimes c_{(2)}$

by suppressing the index i.

C is **cocommutative** if for any $c \in C$,

$$\sum_c c_{(1)} \otimes c_{(2)} = \sum_c c_{(2)} \otimes c_{(1)}$$

Using the Sweedler notation, we have

$$\sum_c \Delta(c_{(1)}) \otimes c_{(2)} = \sum_c c_{(1)} \otimes \Delta(c_{(2)}) \quad \text{(Coassociativity)}$$

and

$$\sum_c \varepsilon(c_{(1)}) c_{(2)} = \sum_c c_{(1)} \varepsilon(c_{(2)}) = c \quad \text{(counitary property)}$$

Bialgebra

Let (B, M, u) be an algebra and (B, Δ, ε) be a coalgebra.

If both Δ and ε are algebra morphism, then B is called a **bialgebra**.

In other words, B is a bialgebra if for any $h, g \in B$,

$$\Delta(g \cdot h) = \sum_{g,h} g_{(1)} h_{(1)} \otimes g_{(2)} h_{(2)}$$

and

$$\varepsilon(g \cdot h) = \varepsilon(g) \cdot \varepsilon(h)$$

where $\Delta(g) = \sum_{g} g_{(1)} \otimes g_{(2)}$ and $\Delta(h) = \sum_{h} h_{(1)} \otimes h_{(2)}$.

Hopf algebra

Suppose that $(B, M, u, \Delta, \varepsilon)$ is a bialgebra. For any $f, g \in Hom(B, B)$, define

$$f * g = M \circ (f \otimes g) \circ \Delta \in Hom(B, B)$$

Then $Hom(B, B)$ is an algebra with the product $*$ and the unit $u \circ \varepsilon$.

Definition. Suppose that $(H, M, u, \Delta, \varepsilon)$ is a bialgebra. An element

$$S \in Hom(H, H)$$

is called an **antipode** for H if it is inverse under the convolution $*$ to the identity map $I: h \to h$. A bialgebra with an antipode is called a **Hopf algebra**.

A graded vector space $V = \bigoplus_{n \geq 0} V_n$ is finite-type if for any n, $\dim V_n < +\infty$. V is connected if $V_0 \cong \mathbb{K}$.

graded algebra: $A = \bigoplus_{n \geq 0} A_n$ and $M(A_i \otimes A_j) \subseteq A_{i+j}$

graded coalgebra: $C = \bigoplus_{n \geq 0} C_n$ and $\Delta(C_m) \subseteq \bigoplus_{j=0}^{m} C_j \otimes C_{m-j}$

Suppose that $H = \bigoplus_{n \geq 0} H_n$ is a graded Hopf algebra with product $\bullet$ and coproduct Δ. Then its graded dual

$$H^* = \bigoplus_{n \geq 0} H_n^*$$

is also a Hopf algebra with product $\bullet$ and coproduct Δ given by:

$$\langle \alpha^* \bullet \beta^*, h \rangle = \langle \alpha^* \otimes \beta^*, \Delta(h) \rangle = \sum_h \langle \alpha^*, h_{(1)} \rangle \langle \beta^*, h_{(2)} \rangle$$

$$\langle \Delta(\alpha^*), h \otimes g \rangle = \langle \alpha^*, h \bullet g \rangle$$

where $\alpha^*, \beta^* \in H^*$, $h, g \in H$, and $\Delta(h) = \sum_h h_{(1)} \otimes h_{(2)}$.

$$\alpha^* \cdot \beta^* = \sum_\gamma c_\gamma \gamma^* \qquad \text{if } \Delta(\gamma) = c_\gamma \, \alpha \otimes \beta + \cdots\cdots$$

$$\Delta(\gamma^*) = \sum c_{\alpha^*\beta^*} \, \alpha^* \otimes \beta^* \qquad \text{if } \alpha \cdot \beta = c_{\alpha^*\beta^*} \gamma + \cdots\cdots$$

The Hopf algebra $\Lambda = \mathrm{Sym}$ of symmetric functions

$\Lambda(x)$ is the $\mathfrak{S}_\infty$-invariant subalgebra of $\mathbb{Q}[[x_1, x_2, \cdots\cdots]] \triangleq \mathbb{Q}[[x]]$, and

the coproduct on $\Lambda(x)$ is given by the "doubling variables trick:"

$$\Delta: \quad \Lambda(x) \longrightarrow \Lambda(x,y) \hookrightarrow \Lambda(x) \otimes \Lambda(y) \cong \Lambda(x) \otimes \Lambda(x)$$

Where $(x, y) = (x_1, x_2, \cdots\cdots, \quad y_1, y_2, \cdots\cdots)$

proposition: We have:
$$\Delta(m_\lambda) = \sum_{\substack{\mu,\nu \\ \mu \cup \nu = \lambda}} m_\mu \otimes m_\nu$$

$$\Lambda = \mathbb{Q}[e_1, e_2, \cdots\cdots], \qquad \Delta(e_k) = \sum_{i+j=k} e_i \otimes e_j$$

$$\Lambda = \mathbb{Q}[h_1, h_2, \cdots\cdots], \qquad \Delta(h_k) = \sum_{i+j=k} h_i \otimes h_j$$

$$\Lambda = \mathbb{Q}[p_1, p_2, \cdots\cdots], \qquad \Delta(p_k) = 1 \otimes p_k + p_k \otimes 1$$

$$s_\lambda = \sum_T x^T$$

$$s_\mu \cdot s_\nu = \sum_\lambda c_{\mu\nu}^\lambda s_\lambda$$

$$s_\lambda(x,y) = \sum_{\mu \subseteq \lambda} s_\mu(x)\, s_{\lambda/\mu}(y)$$

$$= \sum_{\mu \subseteq \lambda} s_\mu(x) \cdot \left[\sum_\nu c_{\mu\nu}^\lambda s_\nu(y) \right]$$

$$= \sum_{\mu,\nu} c_{\mu\nu}^\lambda s_\mu(x)\, s_\nu(y)$$

Where $c_{\mu\nu}^\lambda$ is the Littlewood-Richardson coefficient.

So Sym is a self-dual Hopf algebra.

The Hopf algebra QSym of quasi-symmetric functions

$f(x_1, x_2, \cdots) \in \mathbb{Q}[[x_1, x_2, \cdots]]$ is quasi-symmetric if

$$[x_{i_1}^{n_1} x_{i_2}^{n_2} \cdots x_{i_k}^{n_k}] f = [x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}] f$$

for any $i_1 < i_2 < \cdots < i_k$ and $n_1, n_2, \cdots, n_k \in \mathbb{P}$.

Given a composition $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k)$, let

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$$

and let $\quad QSym^n = \left\{ \sum_{\alpha \vDash n} c_\alpha M_\alpha : \quad c_\alpha \in \mathbb{Q} \right\}$

Then we have: $\quad QSym = \bigoplus_{n \geq 0} QSym^n$.

Given two compositions $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_k)\vDash m$ and $\beta=(\beta_1,\beta_2,\cdots\beta_\ell)\vDash n$, a composition $\gamma\vDash m+n$ is a shuffle of α and β, denoted by $\gamma\in\alpha\sqcup\!\sqcup\,\beta$, if there exists $1\le i_1<i_2<\cdots<i_k\le k+\ell$, such that

$$\alpha=(\gamma_{i_1},\gamma_{i_2},\cdots,\gamma_{i_k}) \text{ and } \beta=(\gamma_{j_1},\gamma_{j_2},\cdots\gamma_{j_\ell}),$$

where $\{j_1,j_2,\cdots,j_\ell\}_< = [k+\ell]\setminus\{i_1,\cdots,i_k\}$.

A composition $\gamma\vDash m+n$ is a quasi-shuffle of α and β, denoted by $\gamma\in\alpha\overset{q}{\sqcup\!\sqcup}\beta$, if γ is obtained by first shuffling components α_i and β_j and then replacing any number of pairs of consecutive components α_i and β_j by $\alpha_i+\beta_j$.

Proposition. For $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_k)$, $\beta=(\beta_1,\beta_2,\cdots,\beta_\ell)$, we have

$$M_\alpha\cdot M_\beta=\sum_{\gamma\in\alpha\,\overset{\otimes}{\sqcup\!\sqcup}\,\beta}M_\gamma$$

Let $(x,y)=(x_1<x_2<\cdots<y_1<y_2<\cdots)$, then we can define:

$$\Delta:\ QSym(x)\longrightarrow QSym(x,y)\hookrightarrow QSym(x)\otimes QSym(y)\cong QSym(x)\otimes QSym(x)$$

Proposition. For a composition $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_k)$, we have:

$$\Delta(M_\alpha)=\sum_{i=0}^{\ell}M_{(\alpha_1,\alpha_2,\cdots,\alpha_i)}\otimes M_{(\alpha_{i+1},\cdots,\alpha_k)}$$

The Hopf algebra NSym of non-commutative symmetric functions

$$NSym = \mathbb{Q}\langle H_1, H_2, \cdots \rangle$$

is the free non-commutative algebra on generators $\{H_1, H_2, \cdots\cdots\}$ with coproduct determined by:

$$\Delta(H_n) = \sum_{i=0}^{n} H_i \otimes H_{n-i}$$

For a composition $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k) \vdash n$, let $H_\alpha = H_{\alpha_1} H_{\alpha_2} \cdots H_{\alpha_k}$ and let

$$NSym^n = \left\{ \sum_{\alpha \vdash n} c_\alpha H_\alpha : \quad c_\alpha \in \mathbb{Q} \right\}$$

Then

$$NSym = \bigoplus_{n \geq 0} NSym^n .$$

Proposition. $NSym \cong (QSym)^*$

With the dual pairing $\langle H_\alpha, M_\beta \rangle = \delta_{\alpha,\beta}$.

Note that:

$$Sym = \mathbb{Q}[h_1, h_2, \cdots\cdots]$$

and

$$\Delta(h_n) = \sum_{i=0}^{n} h_i \otimes h_{n-i}$$

So the map π:

$$\begin{aligned} NSym &\longrightarrow Sym \\ H_n &\longrightarrow h_n \end{aligned}$$

is a Hopf surjection, which is adjoint to the inclusion $Sym \hookrightarrow QSym$

- The Hopf algebra NCSym of Symmetric functions in noncommuting variables

Let $\mathbb{Q}\langle\langle x\rangle\rangle = \mathbb{Q}\langle\langle x_1, x_2, \cdots\cdots\rangle\rangle$ be the associative algebra of formal power series in noncommuting variables $x_1, x_2, \cdots\cdots$. For any $m \in \mathbb{P}$, the symmetric group $\mathfrak{S}_n$ acts on $\mathbb{Q}\langle\langle x\rangle\rangle$ by

$$\sigma \cdot f(x_1, x_2, \cdots\cdots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \cdots\cdots). \quad (1)$$

Define the algebra of symmetric functions in noncommuting variables, NCSym, to be the subalgebra consisting of all $f \in \mathbb{Q}\langle\langle x\rangle\rangle$ which are bounded degree and invariant under the action of $\mathfrak{S}_m$ for all $m \in \mathbb{P}$.

$$f = x_1x_2x_1x_2 + x_2x_1x_2x_1 + x_1x_3x_1x_3 + x_3x_1x_3x_1 + x_2x_3x_2x_3 + x_3x_2x_3x_2 + \cdots$$

$$= \sum_{i \neq j} x_i x_j x_i x_j$$

$$= M_{\{\{1,3\},\{2,4\}\}}$$

Given a set partition $\pi \vdash [n]$, define the monomial symmetric function m_π in noncommuting variables by

$$m_\pi = \sum_{(i_1, i_2, \cdots, i_n)} x_{i_1} x_{i_2} \cdots x_{i_n}$$

where the sum is over all n-tuples $(i_1, i_2, \cdots, i_n)$ with $i_j = i_k$ if and only if j and k are in the same block in π.

Let Π_n denote the set of partitions of $[n]$. Set $\Pi_0 = \{\emptyset\}$. Then:

$$NCSym = \bigoplus_{n \geq 0} \mathbb{Q}\{M_\pi \mid \pi \in \Pi_n\}.$$

let $x = \{x_1, x_2, \cdots\}$ and $y = \{y_1, y_2, \cdots\}$ be two sets of noncommuting variables, with the relations $x_i y_j = y_j x_i$.

The coproduct of NCSym is defined by

$$\Delta: \quad NCSym(x) \longrightarrow NCSym(x, y) \hookrightarrow NSym(x) \otimes NSym(y) \cong NSym(x) \otimes NSym(x)$$

Proposition. For any $\pi \in \Pi_m$ and $\sigma \in \Pi_n$, we have

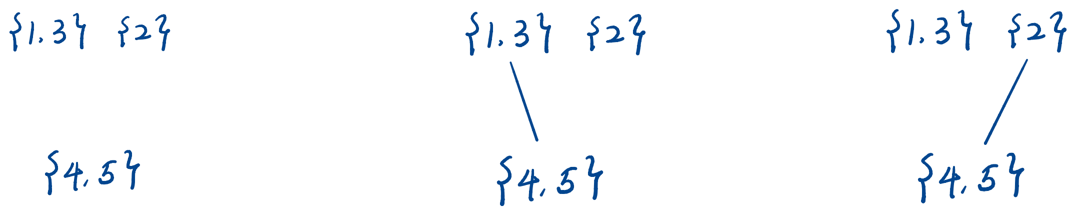
(1) $$M_\pi \cdot M_\sigma = \sum_{\theta \in R(\pi,\sigma)} M_{\theta(\pi,\sigma)}$$

where $R(\pi,\sigma)$ denotes the set of all matchings between π and $\sigma+m$, and for a matching θ, $\theta(\pi,\sigma)$ denotes the set partition of $[m+n]$ obtained from π and $\sigma+m$ by combining B_i and C_j+m if $\{B_i, C_j+m\}$ is an edge in θ.

(2) $$\Delta(M_\pi) = \sum_{\substack{(\pi_1,\pi_2) \\ \pi_1 \uplus \pi_2 = \pi}} M_{st(\pi_1)} \otimes M_{st(\pi_2)}$$

where $st(\sigma)$ denotes the standardization of σ which is defined to be a set partition obtained from σ by substituting the smallest element by 1, the second smallest element by 2 and so on.

Example. Let $\pi = \{\{1,3\},\{2\}\} \in \Pi_3$, $\sigma = \{\{1,2\}\} \in \Pi_2$. Then $\theta(\pi,\sigma)$ consists of the following three matchings:



So

$$M_\pi \cdot M_\sigma = M_{\{\{1,3\},\{2\},\{4,5\}\}} + M_{\{\{1,3,4,5\},\{2\}\}} + M_{\{\{1,3\},\{2,4,5\}\}}.$$

$$\Delta(M_\pi) = 1 \otimes M_{\{\{1,3\},\{2\}\}} + M_{\{\{1,2\}\}} \otimes M_{\{\{1\}\}} + M_{\{\{1\}\}} \otimes M_{\{\{1,2\}\}} + M_{\{\{1,3\},\{2\}\}} \otimes 1$$

Other well known Hopf algebras in Combinatorics

(1) the Incidence Hopf algebra of ranked posets (Ehrenborg)

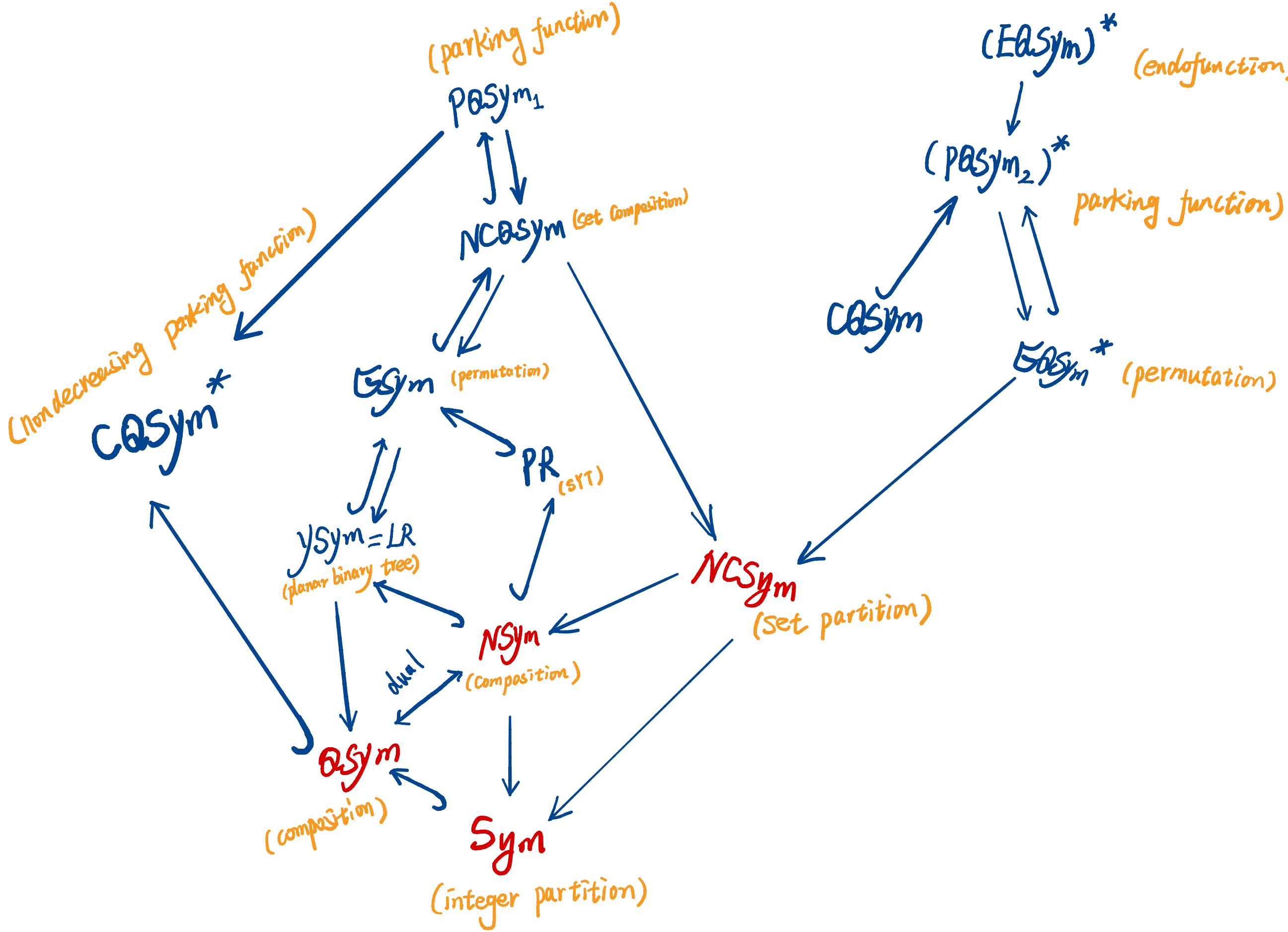
(2) the Hopf algebra of graphs (William R. Schmitt)

(3) the Malvenuto-Reutenauer Hopf algebra of permutations

(4) the Loday-Ronco Hopf algebra of planar binary tree

(5) the Grossman-Larson Hopf algebra of trees

⋮



(parking function)
PQSym1
NCQSym (set composition)
(nondecreasing parking function)
CQSym*
𝔖Sym (permutation)
PR (SYT)
YSym=LR
(planar binary tree)
dual
NSym
(composition)
QSym
(composition)
Sym
(integer partition)
NCSym
(set partition)
(EQSym)*
(endofunction)
(PQSym2)*
parking function)
CQSym
𝔖QSym*
(permutation)

For $\pi = \{A_1, A_2, \cdots A_i\} \in \Pi_m$, $\sigma = \{B_1, B_2, \cdots, B_j\} \in \Pi_n$, define

$$\pi \circ \sigma = \{A_1 \cup (B_1 + m), A_2 \cup (B_2 + m), \cdots\} \in \Pi_{m+n}$$ (split product)

$$\pi | \sigma = \{A_1, A_2, \cdots, A_i, B_1 + m, B_2 + m, \cdots, B_j + m\}$$ (slash product)

A partition π is unsplitable (resp. atomic) if it can't be written as

$\pi = \sigma \circ \tau$ (resp. $\pi = \sigma | \tau$) for non-empty set partitions σ and τ.

Let $\mathcal{U}_n$ (resp. $\mathcal{A}_n$) denote the set of unsplitable (resp. atomic) partitions of size n. Set $\mathcal{U} = \bigcup_{n \geq 0} \mathcal{U}_n$, $\mathcal{A} = \bigcup_{n \geq 0} \mathcal{A}_n$.

Theorem:

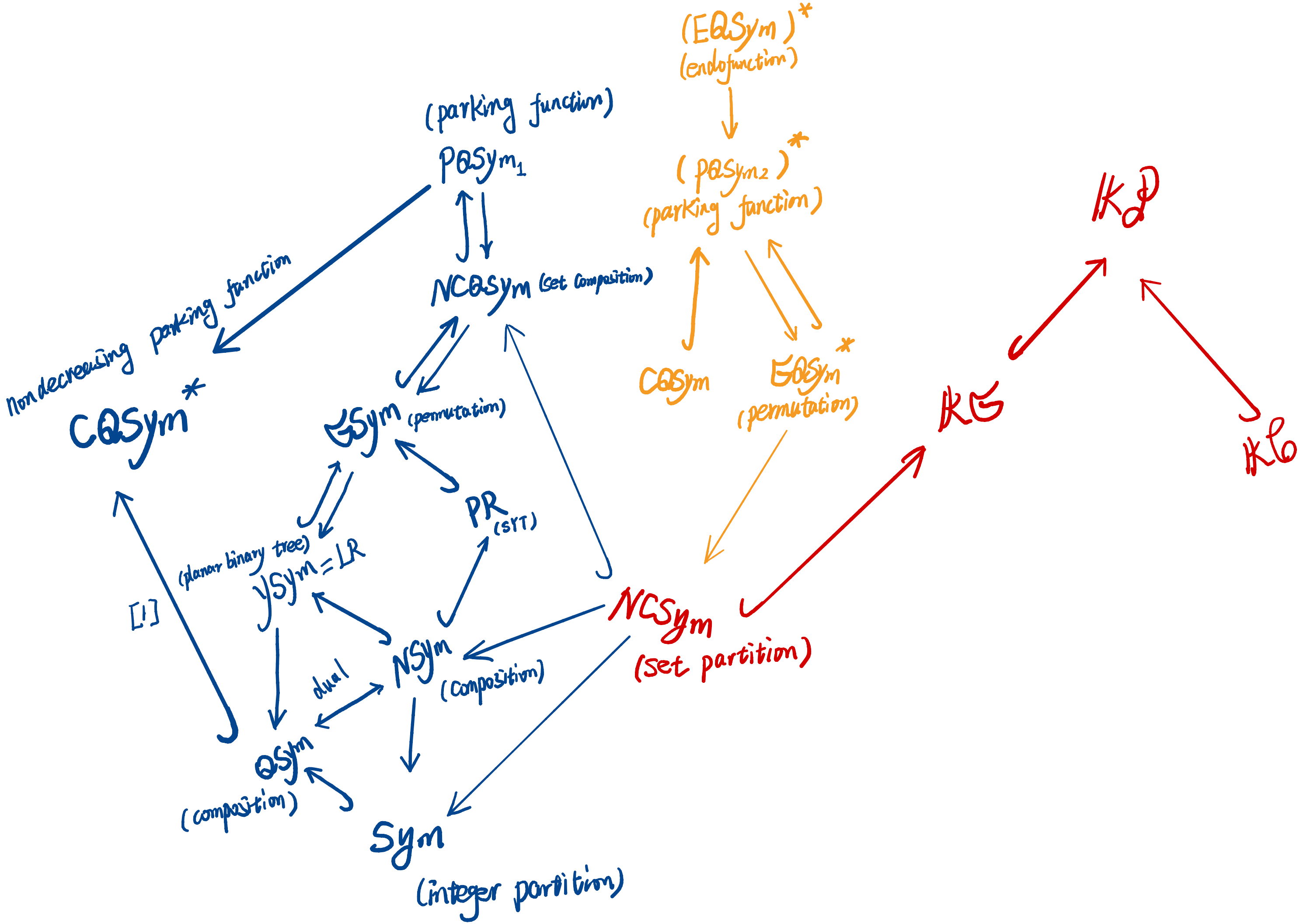
(1) (Wolf 1936) The algebra NCSym is freely generated by $\{m_\pi \mid \pi \in \mathcal{U}\}$.

(2) (N. Bergeron etc. 2005) The algebra NCSym is freely generated by $\{p_\pi \mid \pi \in \mathcal{A}\}$,

where $p_\pi = \sum_{\sigma \geq \pi} m_\sigma$.

Theorem (Chen-Li-Wang) For any $\pi \in \mathcal{A}_n \setminus \mathcal{U}_n$, suppose that $\pi = \sigma \circ (\delta \mid \tau)$ where σ is unsplitable and τ is atomic. Let $\hat{\pi} = (\sigma \circ \delta) \mid \tau$. Then

$$\varphi: \pi \mapsto \hat{\pi} \text{ is a bijection from } \mathcal{A}_n \setminus \mathcal{U}_n \longrightarrow \mathcal{U}_n \setminus \mathcal{A}_n.$$


(EQSym)*
(endofunction)
(parking function)
PQSym$_1$
(PQSym$_2$)*
(parking function)
$\mathbb{K}\mathcal{P}$
nondecreasing parking function
NCQSym (set composition)
CQSym*
CQSym
$\mathfrak{S}$QSym*
(permutation)
$\mathbb{K}\mathfrak{S}$
$\mathfrak{S}$Sym (permutation)
$\mathbb{K}\mathcal{C}$
PR
(SYT)
(planar binary tree)
YSym = LR
[1]
NCSym
NSym
(set partition)
dual
(composition)
QSym
(composition)
Sym
(integer partition)

Let P_n the set of parking functions of length n and let

$$\mathbb{K}\mathcal{P} = \bigoplus_{n \geq 0} \mathbb{K}\{M_a \mid a \in P_n\}$$

$$a = 56357622315 \in P_{11} \longleftrightarrow F_a = (15, 223, 3576, 56)$$

$$M_a \cdot M_b = \sum_{\theta \in R(a,b)} M_{\theta(a,b)}$$

$$\Delta(M_a) = \sum_{\substack{a', a'' \\ F_{a'} \uplus F_{a''} = F_a}} M_{park(a')} \otimes M_{park(a'')}$$

Theorem: (Li, 2015) $\mathbb{K}\mathcal{P}$ is a graded, connected and cocommutative Hopf algebra with unit M_ε and counit given by

$$\varepsilon(M_a) = \begin{cases} 1, & \text{if } a = \phi; \\ 0, & \text{otherwise}. \end{cases}$$

Theorem: (Li, 2015)

(1) The algebra $\mathbb{K}\mathcal{P}$ is freely generated by $\{M_a \mid a \text{ is unsplitable}\}$;

(2) The algebra $\mathbb{K}\mathcal{P}$ is freely generated by $\{Q_a \mid a \text{ is atomic}\}$,

Where $Q_a = \sum_{b \geq a} M_b$.

Theorem: (Li, 2015) For each $n \geq 0$, let

$$\mathcal{N}_n = \{ a \in \mathcal{P}_n \mid \text{each word in } F_a \text{ is nondecreasing} \}$$

$$\mathcal{D}_n = \{ a \in \mathcal{P}_n \mid F_a = (w_1, w_2, \cdots, w_k),\ w_i \cap w_j = \phi \}$$

$$\mathfrak{S}_n = \{ a \in \mathcal{P}_n \mid a \text{ is a permutation of } [n] \}$$

$$\mathcal{C}_n = \{ a \in \mathcal{P}_n \mid a = a_1 a_2 \cdots a_n \text{ with } a_1 \geq \cdots \geq a_n \}$$

Then: (Li, 2015)

(1) The subspaces $\mathbb{K}\mathcal{N} = \bigoplus_{n \geq 0} \mathbb{K}\{ M_a \mid a \in \mathcal{N}_n \}$, $\mathbb{K}\mathcal{D} = \bigoplus_{n \geq 0} \mathbb{K}\{ M_a \mid a \in \mathcal{D}_n \}$

$\mathbb{K}\mathfrak{S} = \bigoplus_{n \geq 0} \mathbb{K}\{ M_a \mid a \in \mathfrak{S}_n \}$, $\mathbb{K}\mathcal{C} = \bigoplus_{n \geq 0} \mathbb{K}\{ Q_a \mid a \in \mathcal{C}_n \}$

are free Hopf subalgebras of $\mathbb{K}\mathcal{P}$.

(2) The subspace $\mathbb{K}\Pi = \bigoplus_{n \geq 0} \mathbb{K}\{ M_a \mid a \in \mathfrak{S}_n \cap \mathcal{N}_n \}$ is a free Hopf subalgebra isomorphic to NCSym.

Theorem: (Li, 2020) For any $a \in \mathcal{A}E_n \setminus \mathcal{U}E_n$, suppose that $a = f_1 \circ (f' | g)$, where f_1 is unsplitable and g is atomic. Let

$$\hat{a} = (f_1 \circ f') | g$$

Then the map $a \to \hat{a}$ is a bijection from $\mathcal{A}E_n \setminus \mathcal{U}E_n$ to $\mathcal{U}E_n \setminus \mathcal{A}E_n$.

References

1. E. Abe, Hopf algebras, Cambridge University Press, Cambridge (1980)

2. M. Aguiar and F. Sottile, Structure of the Malvenuto-Reutenauer Hopf algebra of permutations, Adv. Math. 191, 225–275, 2005

3. M. Aguiar and F. Sottile, Structure of the Loday-Ronco Hopf algebra of planar binary trees, J. Algebra, 295, 473–511, 2006.

4. N. Bergeron and M. Zabrocki, The Hopf algebras of symmetric functions and quasi-symmetric functions in non-commutative variables are free and co-free, J. Algebra Appl. 8 581–600, 2009

5. W.Y.C. Chen, T.X.S. Li and D.G.L. Wang, A bijection between atomic partions and unsplitable partitions, Electron. J. Combin. 18, 7 pp. 2001

6. R. Ehrenborg, On posets and Hopf algebras, Adv. Math. 119, 1-25, 1996

7. D. Grinberg and V. Reiner, Hopf algebras in Combinatorics, arXiv: 1409.8356v5

8. T.X.S. Li, The monomial basis and the Q-basis of the Hopf algebra of parking functions, J. Algebr. Combin. 42(2), 473-496, 2015

9. C. Malvenuto and C. Reutenauer, Duality between quasi-symmetric functions and the Solomon descent algebra, Journal of Algebra, 177, 967-982, 1995

10. J.-C. Novelli and J.-Y. Thibon, Hopf algebras and dendriform structures arising from parking functions, Fundamenta Mathematicae, 193, 189-241, 2007

11. B. Sagan and M. Rosas, Symmetric functions in noncommuting variables, Trans. Am. Math. Soc. 358 (1), 215-232, 2006

Thank you!