行列式的计算方法总结

张彪

天津师范大学

zhang@tjnu.edu.cn

1. 利用定义计算

$$\begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 2 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & n-1 & \cdots & 0 & 0 \\ n & 0 & \cdots & 0 & 0 \end{vmatrix} \qquad \begin{vmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & n-1 \\ n & 0 & 0 & \cdots & 0 \end{vmatrix}$$

2. 利用行列式性质把行列式化为上、下三角形行列式.

$$\begin{vmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{vmatrix} \qquad \begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix}$$

有时也可化成斜上、斜下三角形行列式

$$\begin{vmatrix} b & \cdots & b & b & a \\ b & \cdots & b & a & b \\ b & \cdots & a & b & b \\ \vdots & & \vdots & \vdots & \vdots \\ a & \cdots & b & b & b \end{vmatrix} = \begin{vmatrix} b & \cdots & b & b & a+(n-1)b \\ b & \cdots & b & a & a+(n-1)b \\ b & \cdots & a & b & a+(n-1)b \\ \vdots & & \vdots & \vdots & \vdots \\ a & \cdots & b & b & a+(n-1)b \end{vmatrix}$$

$$= (a+(n-1)b) \begin{vmatrix} b & \cdots & b & b & 1 \\ b & \cdots & b & a & 1 \\ b & \cdots & a & b & 1 \\ \vdots & & \vdots & \vdots & \vdots \\ a & \cdots & b & b & 1 \end{vmatrix} = (a+(n-1)b) \begin{vmatrix} b & \cdots & b & b & 1 \\ 0 & \cdots & 0 & a-b & 0 \\ 0 & \cdots & a-b & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ a-b & \cdots & 0 & 0 & 0 \end{vmatrix}$$

$$=(-1)^{\frac{n(n-1)}{2}}(a+(n-1)b)(a-b)^{n-1}$$

3. 行列式按一行 (一列) 展开, 或按多行 (多列) 展开 (Laplace 定理)

设 $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 4 \\ 1 & 1 & 2 & 3 \\ 2 & 3 & 4 & 4 \end{pmatrix}$, **计算** $A_{41} + A_{42} + A_{43} + A_{44}$ **和** $A_{14} + 3A_{24} + 2A_{34} + 4A_{44}$.

3. 行列式按一行（一列）展开，或按多行（多列）展开（Laplace 定理）

设 $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 4 \\ 1 & 1 & 2 & 3 \\ 2 & 3 & 4 & 4 \end{pmatrix}$，计算 $A_{41} + A_{42} + A_{43} + A_{44}$ 和 $A_{14} + 3A_{24} + 2A_{34} + 4A_{44}$.

解

$$A_{41} + A_{42} + A_{43} + A_{44} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 4 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0, \qquad A_{14} + 3A_{24} + 2A_{34} + 4A_{44} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 3 \\ 1 & 1 & 2 & 2 \\ 2 & 3 & 4 & 4 \end{vmatrix} = 0.$$

$$D_{2n} = \begin{vmatrix} a_n & & & & & b_n \\ & \ddots & & & \iddots & \\ & & a_1 & b_1 & & \\ & & c_1 & d_1 & & \\ & \iddots & & & \ddots & \\ c_n & & & & & d_n \end{vmatrix} = \prod_{i=1}^{n} (a_i d_i - b_i c_i) \, .$$

4. 箭头形行列式或者可以化为箭头形的行列式

$$\begin{vmatrix} a_0 & 1 & 1 & \cdots & 1 \\ 1 & a_1 & 0 & \cdots & 0 \\ 1 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \cdots & a_n \end{vmatrix} = \begin{vmatrix} a_0 - \sum_{i=1}^{n} \frac{1}{a_i} & 1 & 1 & \cdots & 1 \\ 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{vmatrix}$$

$$= \left(a_0 - \sum_{i=1}^{n} \frac{1}{a_i} \right) a_1 a_2 \cdots a_n$$

$$
\begin{vmatrix}
x_1 - m & x_2 & x_3 & \cdots & x_n \\
x_1 & x_2 - m & x_3 & \cdots & x_n \\
x_1 & x_2 & x_3 - m & \cdots & x_n \\
\vdots & \vdots & \vdots & & \vdots \\
x_1 & x_2 & x_3 & \cdots & x_n - m
\end{vmatrix}
$$

$$
= \begin{vmatrix}
x_1 - m & x_2 & x_3 & \cdots & x_n \\
m & -m & 0 & \cdots & 0 \\
m & 0 & -m & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
m & 0 & 0 & \cdots & 0 - m
\end{vmatrix}
$$

$$
= \begin{vmatrix}
\sum_{i=1}^n x_i - m & x_2 & x_3 & \cdots & x_n \\
0 & -m & 0 & \cdots & 0 \\
0 & 0 & -m & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \cdots & 0 - m
\end{vmatrix}
= \left(\sum_{i=1}^{n} x_i - m \right) (-m)^{n-1}
$$

$$
\begin{vmatrix}
1 & 1 & \cdots & 1 & a_0 \\
0 & 0 & \cdots & a_1 & 1 \\
\vdots & \vdots & & \vdots & \vdots \\
0 & a_{n-1} & \cdots & 0 & 1 \\
a_n & 0 & \cdots & 0 & 1
\end{vmatrix}
$$

$$
= \begin{vmatrix}
1 & 1 & \cdots & 1 & a_0 - \sum_{i=1}^{n} \frac{1}{a_i} \\
0 & 0 & \cdots & a_1 & 0 \\
\vdots & \vdots & & \vdots & \vdots \\
0 & a_{n-1} & \cdots & 0 & 0 \\
a_n & 0 & \cdots & 0 & 0
\end{vmatrix}
$$

$$
=(-1)^{\frac{n(n-1)}{2}} a_1 a_2 \cdots a_n \left(a_0 - \sum_{i=1}^{n} \frac{1}{a_i} \right)
$$

$$
\begin{vmatrix}
b & \cdots & b & b & a \\
b & \cdots & b & a & b \\
b & \cdots & a & b & b \\
\vdots & & \vdots & \vdots & \vdots \\
a & \cdots & b & b & b
\end{vmatrix}
=
\begin{vmatrix}
b & b & \cdots & b & a \\
0 & 0 & \cdots & a-b & b-a \\
\vdots & \vdots & & \vdots & \vdots \\
0 & a-b & \cdots & 0 & b-a \\
a-b & 0 & \cdots & 0 & b-a
\end{vmatrix}
$$

$$
=
\begin{vmatrix}
b & b & \cdots & b & a+(n-1)b \\
0 & 0 & \cdots & a-b & 0 \\
\vdots & \vdots & & \vdots & \vdots \\
0 & a-b & \cdots & 0 & 0 \\
a-b & 0 & \cdots & 0 & 0
\end{vmatrix}
$$

$$
=(-1)^{\frac{n(n-1)}{2}}(a+(n-1)b)(a-b)^{n-1}
$$

5. 逐行逐列相加减

行列式特点是每相邻两行 (列) 之间有许多元素相同. 用逐行 (列) 相减可以化出零.

$$\begin{vmatrix} 1 & 2 & 2 & \cdots & 2 \\ 2 & 2 & 2 & \cdots & 2 \\ 2 & 2 & 3 & \cdots & 2 \\ \vdots & \vdots & \vdots & & \vdots \\ 2 & 2 & 2 & \cdots & n \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 \\ 2 & 2 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & n-2 \end{vmatrix} = (-2)(n-2)!$$

$$\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 2 & 3 & \cdots & n-1 & n \\ 3 & 3 & 3 & \cdots & n-1 & n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-1 & n-1 & n-1 & \cdots & n-1 & n \\ n & n & n & \cdots & n & n \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & -1 & 0 \\ n & n & n & \cdots & n & n \end{vmatrix} = (-1)^{n-1} n.$$

$$
\begin{vmatrix}
1 & 2 & 3 & \cdots & n-1 & n \\
2 & 3 & 4 & \cdots & n & 1 \\
3 & 4 & 5 & \cdots & 1 & 2 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
n-1 & n & 1 & \cdots & n-3 & n-2 \\
n & 1 & 2 & \cdots & n-2 & n-1
\end{vmatrix}
=
\begin{vmatrix}
1 & 2 & 3 & \cdots & n-1 & n \\
1 & 1 & 1 & \cdots & 1 & 1-n \\
1 & 1 & 1 & \cdots & 1-n & 1 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
1 & 1 & 1-n & \cdots & 1 & 1 \\
1 & 1-n & 1 & \cdots & 1 & 1
\end{vmatrix}
$$

$$
=
\begin{vmatrix}
\frac{n(n+1)}{2} & 2 & 3 & \cdots & n-1 & n \\
0 & 1 & 1 & \cdots & 1 & 1-n \\
0 & 1 & 1 & \cdots & 1-n & 1 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 1 & 1-n & \cdots & 1 & 1 \\
0 & 1-n & 1 & \cdots & 1 & 1
\end{vmatrix}
= \frac{n(n+1)}{2}
\begin{vmatrix}
1 & 1 & \cdots & 1 & 1-n \\
1 & 1 & \cdots & 1-n & 1 \\
\vdots & \vdots & & \vdots & \vdots \\
1 & 1-n & \cdots & 1 & 1 \\
1-n & 1 & \cdots & 1 & 1
\end{vmatrix}
$$

$$
= \cdots\cdots
$$

$$
\begin{vmatrix}
a+x_1 & a & a & \cdots & a \\
a & a+x_2 & a & \cdots & a \\
a & a & a+x_3 & \cdots & a \\
\vdots & \vdots & \vdots & & \vdots \\
a & a & a & \cdots & a+x_n
\end{vmatrix}
$$

$$
= \begin{vmatrix}
1 & a & a & a & \cdots & a \\
0 & a+x_1 & a & a & \cdots & a \\
0 & a & a+x_2 & a & \cdots & a \\
0 & a & a & a+x_3 & \cdots & a \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & a & a & a & \cdots & a+x_n
\end{vmatrix}
$$

7. 利用数学归纳法证明行列式

$$\begin{vmatrix} \alpha+\beta & \alpha\beta & 0 & \cdots & 0 & 0 \\ 1 & \alpha+\beta & \alpha\beta & \cdots & 0 & 0 \\ 0 & 1 & \alpha+\beta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha+\beta & \alpha\beta \\ 0 & 0 & 0 & \cdots & 1 & \alpha+\beta \end{vmatrix} = \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}$$

证明: 利用第二数学归纳法: 若 $n=1$, 则 $D_1=\alpha+\beta=\frac{\alpha^2-\beta^2}{\alpha-\beta}$, 成立.

假设结论对所有小于 n 的都成立.

将 D_n 按第一行 (或第一列) 展开得 $D_n=(\alpha+\beta)D_{n-1}-\alpha\beta D_{n-2}$,

利用归纳假设可得

$D_n=(\alpha+\beta)D_{n-1}-\alpha\beta D_{n-2}=(\alpha+\beta)\frac{\alpha^n-\beta^n}{\alpha-\beta}-\alpha\beta\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}=\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}$.

8. 利用递推公式

解: 按第一行展开得 $D_n = (\alpha+\beta)D_{n-1} - \alpha\beta D_{n-2}$, 将此式化为:

$$D_n - \alpha D_{n-1} = \beta\left(D_{n-1} - \alpha D_{n-2}\right) \quad (1)$$
$$D_n - \beta D_{n-1} = \alpha\left(D_{n-1} - \beta D_{n-2}\right) \quad (2)$$

利用公式 (1) 得

$$D_n - \alpha D_{n-1} = \beta\left(D_{n-1} - \alpha D_{n-2}\right) = \cdots = \beta^{n-2}\left(D_2 - \alpha D_1\right) = \beta^n,$$

利用公式 (2) 得

$$D_n - \beta D_{n-1} = \alpha\left(D_{n-1} - \beta D_{n-2}\right) = \cdots = \alpha^{n-2}\left(D_2 - \beta D_1\right) = \alpha^n,$$

解得 $D_n = \begin{cases} \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}, & \alpha \neq \beta \\ (n+1)\alpha^n, & \alpha = \beta \end{cases}$

$$D_n = \begin{vmatrix} 7 & 2 & & & \\ 5 & 7 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 7 & 2 \\ & & & 5 & 7 \end{vmatrix} = 7D_{n-1} - 10D_{n-2}$$

9. 拆项法：将行列式的其中一行 (列) 拆成两个数的和

要点：分解成两个容易求的行列式的和.

$$D_n=\begin{vmatrix} a & b & b & \cdots & b & b \\ c & a & b & \cdots & b & b \\ c & c & a & \cdots & b & b \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ c & c & c & \cdots & a & b \\ c & c & c & \cdots & c & a \end{vmatrix}=\begin{vmatrix} c+a-c & b & b & \cdots & b & b \\ c & a & b & \cdots & b & b \\ c & c & a & \cdots & b & b \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ c & c & c & \cdots & a & b \\ c & c & c & \cdots & c & a \end{vmatrix}$$

$$=\begin{vmatrix} c & b & b & \cdots & b & b \\ c & a & b & \cdots & b & b \\ c & c & a & \cdots & b & b \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ c & c & c & \cdots & a & b \\ c & c & c & \cdots & c & a \end{vmatrix}+\begin{vmatrix} a-c & b & b & \cdots & b & b \\ 0 & a & b & \cdots & b & b \\ 0 & c & a & \cdots & b & b \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & c & c & \cdots & a & b \\ 0 & c & c & \cdots & c & a \end{vmatrix}$$