Descent generating polynomials for $(n − 3)$- and $(n − 4)$-stack-sortable (pattern-avoiding) permutations

Philip B. Zhang [a],[*], Sergey Kitaev [b]

[a] *School of Mathematical Science & Institute of Mathematics and Interdisciplinary Sciences, Tianjin Normal University, Tianjin 300387, PR China*
[b] *Department of Mathematics and Statistics, University of Strathclyde, 26 Richmond Street, Glasgow G1 1XH, United Kingdom*

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ABSTRACT

In this paper, we find distribution of descents over $(n − 3)$- and $(n − 4)$-stack-sortable permutations in terms of Eulerian polynomials. Our results generalize the enumeration results by Claesson, Dukes, and Steingrímsson on $(n − 3)$- and $(n − 4)$-stack-sortable permutations. Moreover, we find distribution of descents on $(n − 2)$-, $(n − 3)$- and $(n − 4)$-stack-sortable permutations that avoid any given pattern of length 3, which extends known results in the literature on distribution of descents over pattern-avoiding 1- and 2-stack-sortable permutations. Our distribution results also give enumeration of $(n − 2)$-, $(n − 3)$- and $(n − 4)$-stack-sortable permutations avoiding any pattern of length 3. One of our conjectures links our work to stack-sorting with restricted stacks, and the other conjecture states that 213-avoiding permutations sortable with t stacks are equinumerous with 321-avoiding permutations sortable with t stacks for any t.

1. Introduction

Let n be a positive integer. A permutation of length n is a rearrangement of the set $[n] := \{1, 2, \ldots, n\}$ and ε denotes the empty permutation. For a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$ with $\pi_i = n$, the *stack-sorting operation* $\mathcal{S}$ is recursively defined as follows, where $\mathcal{S}(\varepsilon) = \varepsilon$:

$$\mathcal{S}(\pi) = \mathcal{S}(\pi_1 \cdots \pi_{i-1}) \mathcal{S}(\pi_{i+1} \cdots \pi_n) n.$$

A permutation π is *t-stack-sortable* if the permutation $\mathcal{S}^t(\pi)$, obtained by applying the stack-sorting operation t times, is the identity permutation. The study of $(t-)$stack-sortable permutations has attracted great attention in mathematics and theoretical computer science (see, e.g. [1–10]).

Clearly, all permutations are $(n−1)$-stack-sortable. Knuth [11] discovered that the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ count the number of permutations of length n whose resulting permutations are 1-stack-sortable permutations of length n. It turns out that 1-stack-sortable permutations are precisely 231-avoiding permutations, where a permutation $\pi_1\pi_2 \cdots \pi_n$ avoids a pattern $p = p_1 p_2 \cdots p_k$ (which is also a permutation) if there is no subsequence $\pi_{i_1} \pi_{i_2} \cdots \pi_{i_k}$ such that $\pi_{i_j} < \pi_{i_m}$ if and only if $p_j < p_m$ [12]. For the cases of $t = 2$ and $t = n − 2$, enumeration results for t-stack-sortable permutations were given by West [13], and for $t = n − 3$ and $t = n − 4$ by Claesson, Dukes, and Steingrímsson [7]. For an overview of properties of t-stack-sortable permutations, see [2,11,13] and [12, Sec 2.1]. More recent results related to stack-sortings can be found here [3,5,8,9,14].

In this paper, we consider the descent generating polynomials for (pattern-avoiding) t-stack-sortable permutations for certain values of t. Given a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$, the *descent statistic* on π is defined as the number of $i \in [n-1]$ such that $\pi_i > \pi_{i+1}$. Denote by $\mathfrak{S}_n$ the set of permutations of $[n]$. The *Eulerian polynomial $A_n(x)$* is defined as the generating function of the descent statistic over $\mathfrak{S}_n$, namely,

$$A_n(x) := \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{des}(\pi)}.$$

It is known that the Eulerian polynomials can be computed inductively by

$$A_0(x) = 1, \quad A_n(x) = \sum_{k=0}^{n-1} \binom{n}{k} A_k(x)(x-1)^{n-1-k}, \quad n \geq 1.$$

Also, for $n \geq 1$, the Eulerian polynomials can be computed as

$$A_n(x) = \sum_{k=1}^{n} k! S(n,k)(x-1)^{n-k}$$

where $S(n,k)$ is the *Stirling numbers of the second kind*.

Denote by $\mathcal{W}_{n,t}$ the set of t-stack-sortable permutations of $[n]$. Let

$$W_{n,t}(x) := \sum_{\pi \in \mathcal{W}_{n,t}} x^{\mathrm{des}(\pi)}$$

be the generating function for the descent statistic over t-stack-sortable permutations.

It is known that $W_{n,1}(x)$ are the Narayana polynomials

$$N_n(x) := \sum_{k=0}^{n-1} \frac{1}{n} \binom{n}{k} \binom{n}{k+1} x^k$$

and $W_{n,n-1}(x)$, being the set of all permutations of length n, are the Eulerian polynomials $A_n(x)$. Jacquard and Schaeffer [15] gave the following formula in the case of $t = 2$

$$W_{n,2}(x) = \sum_{k=0}^{n} \frac{(n+k)!(2n-k-1)!}{(k+1)!(n-k)!(2k+1)!(2n-2k-1)!} x^k,$$

which also counts certain *planar maps* and so-called $\beta(0,1)$-trees encoding them [12]. For the case of $t = n - 2$, Brändén [16, Sec. 5] proved that

$$W_{n,n-2}(x) = A_n(x) - x A_{n-2}(x). \tag{1}$$

Let $W_{n,t}^p(x)$ denote the generating function for descents over permutations in $\mathcal{W}_{n,t}$ that avoid a pattern p. It is known [3] that there are

$$\frac{1}{k+1} \binom{n-1}{k} \binom{n}{k} \tag{2}$$

p-avoiding permutations of length n with k descents if $p \in \{132, 213, 231, 312\}$, which in particular gives the distribution of descents over 1-stack-sortable permutations (which are precisely 231-avoiding permutations). On the other hand, the distribution of descents over 123-avoiding permutations is given by the following formula [1,3], where t and x correspond to the length and the number of descents,

$$1 + \sum_{n=1}^{\infty} t^n \sum_{\pi \in \mathcal{W}_{n,1}} x^{\mathrm{des}(\pi)} = \frac{-1 + 2tx + 2t^2 x - 2tx^2 - 4t^2 x^2 + 2t^2 x^3 + \sqrt{1 - 4tx - 4t^2 x + 4t^2 x^2}}{2tx^2(tx - 1 - t)}.$$

The distribution for the number of 321-avoiding permutations of length n with k descents [17, A091156] is

$$\frac{1}{n+1} \binom{n+1}{k} \sum_{j=0}^{n-2k} \binom{k+j-1}{k-1} \binom{n+1-k}{n-2k-j}. \tag{3}$$

The number of descents plus the number of ascents in a permutation of length n is $n-1$, and reading all 321-avoiding permutations in the reverse order gives all 123-avoiding permutations. Hence, we have the following distribution of descents over 123-avoiding permutations of length n,

$$\frac{1}{n+1} \binom{n+1}{k+2} \sum_{j=0}^{2k-n+2} \binom{n-k+j-2}{j} \binom{k+2}{n-k+j}. \tag{4}$$

A particular result of Bukata at el. [3] states that the number of 213-avoiding 1-stack-sortable (i.e. 213-avoiding and 231-avoiding) permutations of length n with k descents is given by $\binom{n-1}{k}$. Egge and Mansour [18] studied 132-avoiding 2-stack-sortable permutations.

In this paper, we give formulas for $W_{n,n-3}(x)$ and $W_{n,n-4}(x)$ in terms of Eulerian polynomials, which generalize the enumeration results of Claesson, Dukes, and Steingrímsson [7]. Moreover, we derive explicit formulas for $W_{n,n-2}^p(x)$, $W_{n,n-3}^p(x)$ and $W_{n,n-4}^p(x)$, where p is any permutation of length 3, which extend the studies in [1,3,18]. Setting $x = 1$ in these formulas gives us enumeration of p-avoiding $(n-2)$-, $(n-3)$- and $(n-4)$-stack-sortable permutations for $p \in \mathfrak{S}_3$ (see Table 1).

The paper is organized as follows. In Section 2 we list the main enumerative results in the paper and introduce the stack-sorting complexity. In Sections 3 and 4 we prove Theorems 1 and 2, respectively. In Section 5 we provide proofs of Theorems 3–8. Finally, in Section 6 we give some concluding remarks and state two conjectures.

2. The main results in the paper

The main results in this paper are stated in the following theorems.

Theorem 1. *For $n \geq 4$, we have that*

$$W_{n,n-3}(x) = A_n(x) - \frac{5}{2}xA_{n-2}(x) - \left(n - \frac{3}{2}\right)x(x+1)A_{n-3}(x). \tag{5}$$

Theorem 2. *For $n \geq 4$, we have that*

$$W_{n,n-4}(x) = \begin{cases} 1 & \text{if } n = 4 \\ x^4 + 10x^3 + 20x^2 + 10x + 1 & \text{if } n = 5 \\ \frac{1}{6}\Big(6A_n(x) - (25x + 1)A_{n-2}(x) & \text{if } n \geq 6 \\ \quad + \Big((-15n + 24)x^2 + (-15n + 30)x + 6\Big)A_{n-3}(x) \\ \quad + \Big(\left(-3n^2 + 12n + 1\right)x^3 + \left(-12n^2 + 48n - 25\right)x^2 \\ \quad + \left(-3n^2 + 12n - 13\right)x - 11 \Big) A_{n-4}(x) \\ \quad - 6\left(x^4 - 1\right)A_{n-5}(x)\Big). \end{cases}$$

Theorem 3. *We have that*

$$W_{n,t}^{231}(x) = \sum_{k=0}^{n-1} \frac{1}{k+1}\binom{n-1}{k}\binom{n}{k}x^k \quad \text{for } 1 \leq t \leq n - 1.$$

Theorem 4. *For $n \geq 4$, we have that*

$$W_{n,n-2}^{123}(x) = W_{n,n-1}^{123}(x) - x^{n-2};$$
$$W_{n,n-3}^{123}(x) = W_{n,n-1}^{123}(x) - (n-2)x^{n-3} - (n+1)x^{n-2};$$
$$W_{n,n-4}^{123}(x) = W_{n,n-1}^{123}(x) - \begin{cases} (2x + 11x^2 + x^3) & \text{if } n = 4 \\ (15x^2 + 16x^3) & \text{if } n = 5 \\ \left((n^2 - n - 3)x^{n-3} + \frac{n^2+n+2}{2}x^{n-2}\right) & \text{if } n \geq 6 \end{cases}$$

where $W_{n,n-1}^{123}(x) = \sum_{k\geq 0} \frac{1}{n+1}\binom{n+1}{k+2}\sum_{j=0}^{2k-n+2}\binom{n-k+j-2}{j-1}\binom{k+2}{n-k+j}x^k.$

Theorem 5. *For $n \geq 4$, we have that*

$$W_{n,n-2}^{321}(x) = W_{n,n-1}^{321}(x) - x;$$
$$W_{n,n-3}^{321}(x) = W_{n,n-1}^{321}(x) - (n+1)x - (n-3)x^2;$$
$$W_{n,n-4}^{321}(x) = W_{n,n-1}^{321}(x) - \frac{n^2+n+2}{2}x - \left(n^2 - n - 10\right)x^2 - \binom{n-4}{2}x^3;$$

where $W_{n,n-1}^{321}(x) = \sum_{k\geq 0}\frac{1}{n+1}\binom{n+1}{k}\sum_{j=0}^{n-2k}\binom{k+j-1}{k-1}\binom{n+1-k}{n-2k-j}x^k.$

Recall that the Narayana polynomials are defined as

$$N_n(x) = \sum_{k=0}^{n-1} \frac{1}{k+1} \binom{n-1}{k} \binom{n}{k} x^k$$

for $n \geq 1$ and $N_0(x) = 1$. It is known that

$$W_{n,n-1}^{213}(x) = W_{n,n-1}^{132}(x) = W_{n,n-1}^{312}(x) = N_n(x).$$

Theorem 6. *For $n \geq 4$, we have that*

$$W_{n,n-2}^{213}(x) = N_n(x) - x;$$
$$W_{n,n-3}^{213}(x) = N_n(x) - 3x - (2n-5)x^2;$$
$$W_{n,n-4}^{213}(x) = N_n(x) - (6x + (8n-26)x^2 + (2n-7)(n-3)x^3).$$

Theorem 7. *For $n \geq 4$, we have that*

$$W_{n,n-2}^{132}(x) = N_n(x) - xN_{n-2}(x);$$
$$W_{n,n-3}^{132}(x) = N_n(x) - xN_{n-2}(x) - 2(x+x^2)N_{n-3}(x);$$
$$W_{n,n-4}^{132}(x) = W_{n,n-3}^{132}(x) - \begin{cases} (3x + 3x^2 + x^3) & \text{if } n = 4 \\ (3x + 7x^2 + 3x^3) & \text{if } n = 5 \\ (3x + 8x^2 + 3x^3)N_{n-4}(x) & \text{if } n \geq 6 \end{cases}$$

Theorem 8. *For $n \geq 4$, we have that*

$$W_{n,n-2}^{312}(x) = N_n(x) - xN_{n-2}(x);$$
$$W_{n,n-3}^{312}(x) = N_n(x) - 2xN_{n-2}(x) - x(x+1)N_{n-3}(x);$$
$$W_{n,n-4}^{312}(x) = W_{n,n-3}^{312}(x) - \begin{cases} (3x + 3x^2 + x^3) & \text{if } n = 4 \\ \left(\begin{array}{l} xN_{n-2}(x) + x^2N_{n-3}(x) \\ +(2x + 2x^2 + x^3)N_{n-4}(x) \end{array}\right) & \text{if } n \geq 5 \end{cases}$$

Remark 9. Setting $x = 1$ in Theorems 3–8, we obtain respective enumeration of p-avoiding $(n-2)$-, $(n-3)$- and $(n-4)$-stack-sortable permutations for $p \in \mathfrak{S}_3$ that is summarized in Table 1.

Following [7], let the *(stack-sorting) complexity* of π be the smallest integer t such that π is t-stack-sortable and denote by $\mathcal{E}_{n,t}$ (resp., $\mathcal{E}_{n,t}^p$) the set of permutations of length n (resp., avoiding a pattern p) with complexity t. Denote by

$$E_{n,t}^p(x) = \sum_{\pi \in \mathcal{E}_{n,t}^p} x^{\mathrm{des}(\pi)},$$

the generating function for descents over $\mathcal{E}_{n,t}^p$, where p can be omitted. Then, we have that, for $2 \leq t \leq n-1$,

$$W_{n,t-1}^p(x) = W_{n,t}^p(x) - E_{n,t}^p(x) \tag{6}$$

where again p can be omitted (if there is no pattern to avoid). Hence, together with (1), in order to compute $W_{n,n-3}(x)$, it suffices to find $E_{n,n-2}(x)$. Similarly, once (5) is proved, the formula for $W_{n,n-4}(x)$ will be obtained from the formula for $E_{n,n-3}(x)$ given in Theorem 13. The same approach is applied to compute $W_{n,n-2}^p(x)$, $W_{n,n-3}^p(x)$, $W_{n,n-4}^p(x)$ for $p \in \mathfrak{S}_3$ in Section 5.

3. Proof of Theorem 1

In this section, we shall prove Theorem 1, which gives a formula for $W_{n,n-3}(x)$. By (6), it suffices to compute $E_{n,n-2}(x)$.

Let us first review the structure of $\mathcal{E}_{n,n-2}$. For convenience of representation of certain sets of permutations over the alphabet $\{1, 2, \ldots\}$, we use the computer science notation of *glob patterns*. An asterisk $(*)$ stands for any rearrangement of a subset of $[n]$ (including the empty permutation), and a question mark $(?)$ stands for any single element in $[n]$. For a word w over the alphabet $\{*, ?\} \cup \{1, 2, \ldots\}$, $\langle w \rangle$ denotes the set of permutations obtained from w by all possible replacements of $*$'s and $?$'s. Also, let

$$\langle w_1, \ldots, w_k \rangle := \langle w_1 \rangle \cup \cdots \cup \langle w_k \rangle.$$

Table 1

Enumeration of p-avoiding t-stack-sortable permutations of length $n \geq 4$ for $t \in \{n - 4, n - 3, n - 2\}$ where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number. For any p of length 3, $(n-1)$-stack-sortable permutations are counted by C_n. Note that we have the same formulas for $p = 213$ and $p = 321$. In Section 6 we state Conjecture 15 generalizing this observation along with Conjecture 16 linking our permutations to the 321-machine considered in [5].

p	$(n - 2)$-stack-sortable	$(n - 3)$-stack-sortable
123	13, 41, 131, 428, 1429, 4861, 16795, …	7, 33, 121, 416, 1415, 4845, 16777, …
	$C_n - 1$; [17, A001453]	$C_n - 2n + 1$
132	12, 37, 118, 387, 1298, 4433, 15366, …	8, 29, 98, 331, 1130, 3905, 13650, …
	$C_n - C_{n-2}$; [17, A280891]	$C_n - C_{n-2} - 4C_{n-3}$
213	13, 41, 131, 428, 1429, 4861, 16795, …	8, 34, 122, 417, 1416, 4846, 16778, …
	$C_n - 1$; [17, A001453]	$C_n - 2(n - 1)$
231	14, 42, 132, 429, 1430, 4862, 16796, …	14, 42, 132, 429, 1430, 4862, 16796, …
	C_n; [17, A000108]	C_n; [17, A000108]
312	12, 37, 118, 387, 1298, 4433, 15366, …	8, 28, 94, 317, 1082, 3740, 13078, …
	$C_n - C_{n-2}$; [17, A280891]	$C_n - 2(C_{n-2} + C_{n-3})$
321	13, 41, 131, 428, 1429, 4861, 16795, …	8, 34, 122, 417, 1416, 4846, 16778, …
	$C_n - 1$; [17, A001453]	$C_n - 2(n - 1)$

p	$(n - 4)$-stack-sortable
123	0, 11, 83, 361, 1340, 4747, 16653, …
	$C_n - \frac{3n^2 - n - 4}{2}$, $n \geq 6$
132	1, 16, 70, 261, 934, 3317, 11802, …
	$C_n - 14C_{n-4}$, $n \geq 6$
213	1, 16, 89, 365, 1341, 4744, 16645, …
	$C_n - (2n^2 - 5n + 1)$
231	1, 42, 132, 429, 1430, 4862, 16796, …
	C_n; [17, A000108]
312	1, 16, 65, 236, 838, 2969, 10559, …
	$C_n - (C_{n-2} + C_{n-3} + 5C_{n-4})$, $n \geq 5$
321	1, 16, 89, 365, 1341, 4744, 16645, …
	$C_n - (2n^2 - 5n + 1)$

West [13] characterized $\mathcal{E}_{n,n-2}$ for all $n \geq 4$ as follows,

$$\mathcal{E}_{n,n-2} = \mathfrak{S}_n \cap \langle \ *n2, \ *(n - 1)1n, \ *n1?, \ *n?1, \ *n*(n - 2)*(n - 1)1 \ \rangle. \tag{7}$$

For convenience, we let $A_n\langle w \rangle$ be the generating function for distribution of descents over permutations of $[n]$ in $\langle w \rangle$. Easy observations lead to the following equations, where the factor of x (resp., x^2) records the extra descent (resp., two descents),

$$A_n\langle *n2 \rangle = xA_{n-2}(x),$$
$$A_n\langle *(n - 1)1n \rangle = xA_{n-3}(x),$$
$$A_n\langle *n1? \rangle = (n - 2)xA_{n-3}(x),$$
$$A_n\langle *n?1 \rangle = (n - 2)x^2A_{n-3}(x).$$

We proceed by computing the remaining case of $A_n\langle *n*(n - 2)*(n - 1)1 \rangle$.

Lemma 10. *For any $n \geq 2$, we have that*

$$A_n\langle *n*(n - 1)* \rangle = \frac{1}{2}\big(A_n(x) + (x - 1)A_{n-1}(x)\big).$$

Proof. We note that the set of all permutations of $[n]$ is the disjoint union of the following four sets

$$\langle *n?*(n - 1)* \rangle \bigcup \langle *n(n - 1)* \rangle \bigcup \langle *(n - 1)?*n* \rangle \bigcup \langle *(n - 1)n* \rangle.$$

Since

$$A_n\langle *n(n - 1)* \rangle = xA_{n-1}(x), \quad A_n\langle *(n - 1)n* \rangle = A_{n-1}(x),$$
$$A_n\langle *n?*(n - 1)* \rangle = A_n\langle *(n - 1)?*n* \rangle,$$

it follows that

$$2A_n\langle * n ? * (n-1)* \rangle + xA_{n-1}(x) + A_{n-1}(x) = A_n(x),$$

and thus

$$A_n\langle * n ? * (n-1)* \rangle = \frac{1}{2}\big(A_n(x) - (x+1)A_{n-1}(x)\big).$$

From

$$\langle * n * (n-1)* \rangle = \langle * n ? * (n-1)* \rangle \bigcup \langle * n(n-1)* \rangle$$

we have that

$$A_n\langle * n * (n-1)* \rangle = A_n\langle * n ? * (n-1)* \rangle + A_n\langle * n(n-1)* \rangle$$
$$= \frac{1}{2}\left(A_n(x) + (x-1)A_{n-1}(x)\right).$$

This completes the proof. □

Now we are able to obtain an expression for $E_{n,n-2}(x)$.

Theorem 11. *For all $n \geq 4$,*

$$E_{n,n-2}(x) = \frac{1}{2}x\big(3A_{n-2}(x) + (2n-3)(x+1)A_{n-3}(x)\big). \tag{8}$$

Proof. Note that for any permutation in $\mathfrak{S}_n \cap \langle * n*(n-2)*(n-1)1 \rangle$, the leftmost $n-2$ elements have the same pattern as any permutation in $\mathfrak{S}_{n-2} \cap \langle *(n-2)*(n-3)* \rangle$. It then follows from Lemma 10 that

$$A_n\langle * n*(n-2)*(n-1)1 \rangle = \frac{1}{2}x\left(A_{n-2}(x) + (x-1)A_{n-3}(x)\right)$$

where the factor of x corresponds to the descent $(n-1)1$. Summing up the polynomials corresponding to the five words in (7), we obtain the desired result. This completes the proof. □

Since $W_{n,n-3}(x) = W_{n,n-2}(x) - E_{n,n-2}(x)$, formula (5) follows immediately from (1) and (8), which completes the proof of Theorem 1.

4. Proof of Theorem 2

In this section, we shall prove Theorem 2, which gives a formula for $W_{n,n-4}(x)$. By (6), it suffices to compute $E_{n,n-3}(x)$.

Our proof is based on the classification of permutations in $\mathcal{E}_{n,n-3}$, listed in Tables 2 and 3, that were given by Claesson, Dukes, and Steingrímsson [7]. We shall compute the descent generating polynomial $A_n\langle w \rangle$ for each type of w. To do this, we divide all the cases into *four* classes.

Class 1: All cases in Table 2 except case 1(d)

The descent generating functions for the cases in Table 2 except 1(d) can be obtained directly. For example, consider type 4(a), namely

$$*n??1, \text{ but not } *n(n-2)(n-1)1.$$

Since the letters ?? can be chosen arbitrarily from $\{2, 3, \ldots, n-1\}$, and ordering them in decreasing order results in an extra descent, we obtain that

$$A_n\langle *n??1 \rangle = \left(\binom{n-2}{2}x^3 + \binom{n-2}{2}x^2\right)A_{n-4}(x),$$

and hence the descent generating polynomial for case 4(a) is

$$\left(\binom{n-2}{2}x^3 + \left(\binom{n-2}{2} - 1\right)x^2\right)A_{n-4}(x).$$

Class 2: 1(d), 5(a), 5(b), 5(e), 5(f), and 5(g)

The descent generating functions for cases 1(d), 5(a), 5(b), 5(e), 5(f), and 5(g) can be derived from Lemma 10. For instance, in the case of 5(a), the type $*nA(n-2)B(n-1)2$, where $A \cup B \neq \emptyset$, can be translated into the form

$$*n*(n-2)*(n-1)2, \quad \text{but not} \quad *n(n-2)(n-1)2,$$

and hence we obtain the descent generating polynomial as desired.

Class 3: 5(c)

To deal with case 5(c), we need the following lemma.

Table 2

Permutations in $\mathcal{E}_{n,n-3}$ for $n \geq 4$.

Case	Type of w	Descent gen. polynomial $A_n\langle w \rangle$
$\pi_n = n$		
1(a)	$*(n-1)2n$	$xA_{n-3}(x)$
1(b)	$*(n-1)?1n$	$(n-3)x^2A_{n-4}(x)$
1(c)	$*(n-1)1?n$	$(n-3)xA_{n-4}(x)$
1(d)	$*(n-1)*(n-3)*(n-2)1n$	$\frac{1}{2}x\left(A_{n-3}(x) + (x-1)A_{n-4}(x)\right)$
1(e)	$*(n-2)1(n-1)n$	$xA_{n-4}(x)$
$\pi_{n-1} = n$		
2(a)	$*n3$	$xA_{n-2}(x)$
2(b)	$*(n-1)1n?$, where $? = 4, 5, \ldots, n-2$	$(n-5)x^2A_{n-4}(x)$
2(c)	$*(n-2)1n(n-1)$	$x^2A_{n-4}(x)$
$\pi_{n-2} = n$		
3(a)	$*n2?$, where $? \neq 1$	$(n-3)xA_{n-3}(x)$
3(b)	$*n?2$, where $? \neq 1$	$(n-3)x^2A_{n-3}(x)$
$\pi_{n-3} = n$		
4(a)	$*n??1$, but not $*n(n-2)(n-1)1$	$\left(\binom{n-2}{2}x^3 + (\binom{n-2}{2} - 1)x^2\right)A_{n-4}(x)$
4(b)	$*n?1?$	$(n-2)(n-3)x^2A_{n-4}(x)$
4(c)	$*n1??$	$\binom{n-2}{2}(x + x^2)A_{n-4}(x)$
4(d)	$*n(n-2)(n-1)2$	$x^2A_{n-4}(x)$

Table 3

Permutations in $\mathcal{E}_{n,n-3}$ for $n \geq 4$.

Case	Type of w	Descent generating polynomial $A_n\langle w \rangle$
$\pi_{n-i} = n$ and $i > 3$		
5(a)	$*nA(n-2)B(n-1)2$, where $A \cup B \neq \emptyset$	$\frac{1}{2}x\left(A_{n-2}(x) + (x-1)A_{n-3}(x)\right) - x^2A_{n-4}(x)$
5(b)	$*n*(n-3)*(n-1)(n-2)1$	$\frac{1}{2}x^2\left(A_{n-3}(x) + (x-1)A_{n-4}(x)\right)$
5(c)	$*n*(n-1)*(n-3)*(n-2)1$	$\frac{1}{6}x\left(A_{n-2}(x) + 3(x-1)A_{n-3}(x) + 2(x-1)^2A_{n-4}(x)\right)$
5(d)	$*(n-1)*n*\begin{Bmatrix} (n-3)*(n-4) \\ (n-4)*(n-3) \end{Bmatrix}*(n-2)1$	$\frac{1}{12}\left(A_{n-2}(x) + 6(x^2-1)A_{n-3}(x) -(12x^3 + x^2 - 2x - 11)A_{n-4}(x) +6(x^4-1)A_{n-5}(x)\right)$
5(e)	$*n*(n-2)*(n-1)\begin{Bmatrix} 1? \\ ?1 \end{Bmatrix}$	$(n-4)(\frac{x}{2} + \frac{x^2}{2})\left(A_{n-3}(x) + (x-1)A_{n-4}(x)\right)$
5(f)	$*n*(n-3)*(n-1)1(n-2)$	$\frac{1}{2}x\left(A_{n-3}(x) + (x-1)A_{n-4}(x)\right)$
5(g)	$*n*(n-3)*(n-2)1(n-1)$	$\frac{1}{2}x\left(A_{n-3}(x) + (x-1)A_{n-4}(x)\right)$
5(h)	$*(n-2)*n*\begin{Bmatrix} (n-3)*(n-4) \\ (n-4)*(n-3) \end{Bmatrix}*(n-1)1$	$\frac{1}{12}\left(A_{n-2}(x) + 6(x^2-1)A_{n-3}(x) -(12x^3 + x^2 - 2x - 11)A_{n-4}(x) +6(x^4-1)A_{n-5}(x)\right)$

Lemma 12. *For all $n \geq 3$,*

$$A_n\langle *n*(n-1)*(n-2)* \rangle = \frac{1}{6}\left(A_n(x) + 3(x-1)A_{n-1}(x) + 2(x-1)^2A_{n-2}(x)\right).$$

Proof. We note that the set of all permutations of $[n]$ is the disjoint union of the following three sets, where abc forms a permutation of $\{n-2, n-1, n\}$:

(i) $\langle *a\,?*b\,?*c* \rangle$;

(ii) $\langle *a\,b\,?*c* \rangle$ or $\langle *a\,?*b\,c* \rangle$;.

(iii) $\langle *a\,b\,c* \rangle$.

Then, we obtain the following formulas for these three types

$$A(\mathfrak{S}_n \cap \langle *a\,b\,?*c* \rangle) = x^{\chi(a>b)} \times \frac{1}{2}\left(A_{n-1}(x) + (x-1)A_{n-2}(x)\right),$$

$$A(\mathfrak{S}_n \cap \langle *a\,?*b\,c* \rangle) = x^{\chi(b>c)} \times \frac{1}{2}\left(A_{n-1}(x) + (x-1)A_{n-2}(x)\right),$$

$$A(\mathfrak{S}_n \cap \langle *a\,b\,c* \rangle) = x^{\chi(a>b)+\chi(b>c)} \times \frac{x}{2}\left(A_{n-1}(x) + (x-1)A_{n-2}(x)\right),$$

where $\chi(\cdot)$ is the function whose value is 1 if the statement is true and 0 otherwise. Hence, it follows that

$$6 \times A(\mathfrak{S}_n \cap \langle *a\,?*b\,?*c* \rangle) + 6(x + 1) \times \frac{1}{2}\left(A_{n-1}(x) + (x - 1)A_{n-2}(x)\right)$$
$$+(x^2 + 4x + 1)A_{n-2}(x) = A_n(x).$$

Therefore, we obtain that

$$A_n\langle *a\,?*b\,?*c* \rangle = \frac{1}{6}\left(A_n(x) - 3(x - 1)A_{n-1}(x) + 2(x^2 + x + 1)A_{n-2}(x)\right).$$

In order to obtain the desired formula, we note that the set of permutations in $\mathfrak{S}_n \cap \langle *n*(n-1)*(n-2)* \rangle$ is the union of the following four cases

$$\langle *n\,?*(n-1)\,?*(n-2)* \rangle \bigcup \langle *n(n-1)\,?*(n-2)* \rangle$$
$$\bigcup \langle *n\,?*(n-1)(n-2)* \rangle \bigcup \langle *n(n-1)(n-2)* \rangle.$$

Hence, it follows that

$$A_n\langle *n*(n-1)*(n-2)* \rangle = \frac{1}{6}\left(A_n(x) - 3(x - 1)A_{n-1}(x) + 2(x^2 + x + 1)A_{n-2}(x)\right)$$
$$+ 2 \times \frac{x}{2}\left(A_{n-1}(x) + (x - 1)A_{n-2}(x)\right) + x^2 A_{n-2}(x)$$
$$= \frac{1}{6}\left(A_n(x) + 3(x - 1)A_{n-1}(x) + 2(x - 1)^2 A_{n-2}(x)\right).$$

This completes the proof. □

Class 4: 5(d) and 5(h)

For the remaining cases of 5(d) and 5(h), we can obtain the desired formulas by following similar lines of the proof of Lemma 12.

Now we complete the proof of the formulas of the descent generating polynomials of all types listed in Tables 2 and 3. By summing up all these polynomials, we obtain the following expression for $E_{n,n-3}(x)$.

Theorem 13. *For all $n \geq 6$, the descent generating polynomial of permutations in $\mathcal{E}_{n,n-3}$ is*

$$E_{n,n-3}(x) = \frac{1}{6}\Big((10x + 1)A_{n-2}(x) + \big((9n - 15)x^2 + (9n - 21)x - 6\big)A_{n-3}(x)$$
$$+ \big((3n^2 - 12n - 1)x^3 + (12n^2 - 48n + 25)x^2$$
$$+ (3n^2 - 12n + 13)x + 11\big)A_{n-4}(x) + 6(x^4 - 1)A_{n-5}(x)\Big).$$

Since $W_{n,n-4}(x) = W_{n,n-3}(x) - E_{n,n-3}(x)$, Theorem 2 follows immediately from (5) and Theorem 13.

5. Proofs of Theorems 3–8

We note that Theorem 3 is an immediate corollary to formula (2) and the observation that 1-stack-sortable permutations are precisely 231-avoiding permutations.

For the other theorems, we can follow the same steps as in the proofs of Theorems 1 and 2, and apply suitable known formulas for $W_{n,n-1}^p$ in question. Namely, we will iterate (6) for $t = n - 1, n - 2, n - 3$ for each pattern p in question. Additionally, we need to make sure that the forbidden pattern does not occur in them. Below, we provide brief (omitting justifications for easier parts) proofs of Theorems 4–8, where ↓ and ↑ stand for decreasing and increasing permutations, respectively.

5.1. Proof of Theorem 4

Considering (7) we can see that for $n \geq 4$,

$$\mathcal{E}_{n,n-2}^{123} = \mathfrak{S}_n \cap \langle\; \downarrow n2, \; \downarrow n1?, \; \downarrow n?1, \; n(n-2) \downarrow (n-1)1 \;\rangle \tag{9}$$

where, for example, $*(n-1)1n$ disappears in the unrestricted set $\mathcal{E}_{n,n-2}$ because $n \geq 4$ and any element in the $*$ along with $(n-1)$ and n would form the forbidden pattern 123. Also, because of $(n-1)$, the leftmost two $*$'s in $*n*(n-2)*(n-1)1$ in $\mathcal{E}_{n,n-2}$ must be empty, while the remaining $*$ must be a decreasing permutation. And so on.

It is easy to see that $A_n\langle \downarrow n2 \rangle = x^{n-2}$, $A_n\langle \downarrow n1? \rangle = (n-2)x^{n-3}$, $A_n\langle \downarrow n?1 \rangle = (n-2)x^{n-2}$ and $A_n\langle n(n-2) \downarrow (n-1)1 \rangle = x^{n-2}$, and hence from (9) we have

$$E_{n,n-2}^{123}(x) = (n - 2)x^{n-3} + nx^{n-2}.$$

Table 4
Permutations in $\mathcal{E}^{123}_{n,n-3}$ for $n \geq 6$.

Case	Type of w	Descent generating polynomial $A_n\langle w \rangle$
2(a)	$\downarrow n3$	x^{n-2}
3(b)	$\downarrow n{?}2$, where ${?} \neq 1$	$(n-3)x^{n-2}$
4(a)	$\downarrow n{??}1$, but not 4231	$\frac{(n-2)(n-3)}{2}x^{n-2} + (n-3)x^{n-3}$ (cases: $2n{??}1$, $\downarrow n{?}21$, $\downarrow n2{?}1$)
4(b)	$\downarrow n{?}1{?}$	$\frac{n(n-3)}{2}x^{n-3}$ (consider the cases $2 \in \downarrow$ and $2 \notin \downarrow$)
4(c)	$\downarrow n1ba$, where $b > a$	$\frac{(n-2)(n-3)}{2}x^{n-2}$
5(a)	$n(n-2) \downarrow (n-1)2$, $\downarrow \neq \emptyset$	x^{n-2}
5(b)	$n(n-3) \downarrow (n-1)(n-2)1$	x^{n-2}
5(c)	$n(n-1)(n-3) \downarrow (n-2)1$	x^{n-2}
5(d)	$(n-1)n(n-3)(n-4) \downarrow (n-2)1$	x^{n-3}
5(e)	$n(n-2) \downarrow (n-1) \begin{Bmatrix} 1? \\ ?1 \end{Bmatrix}$	$(n-4)(x^{n-3} + x^{n-2})$
5(f)	$n(n-3) \downarrow (n-1)1(n-2)$	x^{n-3}
5(h)	$(n-2)n(n-3)(n-4) \downarrow (n-1)1$	x^{n-3}

Now, since $\mathcal{E}_{n,n-1} = \mathfrak{S}_n \cap \langle {*}n1 \rangle$ [13], $\mathcal{E}^{123}_{n,n-1} = \mathfrak{S}_n \cap \langle \downarrow n1 \rangle$ and $E^{123}_{n,n-1}(x) = x^{n-2}$. Since $\mathcal{W}_{n,n-1} = \mathfrak{S}_n$, formula (4) can be used to find $W^{123}_{n,n-1}(x)$, and then iteration of (6) for $t = n-1, n-2$ gives the desired formulas for $W^{123}_{n,n-2}(x)$ and $W^{123}_{n,n-3}(x)$.

To complete the proof of Theorem 4, we need to compute $E^{123}_{n,n-3}(x)$ and apply (6) for $t = n-3$. We analyse Tables 2 and 3 presenting $\mathcal{E}_{n,n-3}$. We see that to avoid the pattern 123, the following cases are impossible when $n \geq 6$: 1(a), 1(b), 1(c), 1(d), 1(e), 2(b) (there is no place for 3), 2(c), 3(a), 4(d) (there is no place for 1 even for $n = 4$), 5(g). The remaining cases are listed in Table 4, along with several brief comments, and they give the desired formula for $W^{123}_{n,n-4}(x)$.

5.2. Proof of Theorem 5

Considering (7) we can see that for $n \geq 4$,

$$\mathcal{E}^{321}_{n,n-2} = \mathfrak{S}_n \cap \langle\, 1 \uparrow n2,\ An2,\ \uparrow (n-1)1n,\ \uparrow n1{?}\, \rangle \tag{10}$$

where, the element 1 in A is not in the leftmost position, while all other elements in A are increasing (because of the element 2). Indeed, for example, in the unrestricted set $\mathcal{E}_{n,n-2}$ ${*}(n-1)1n$ becomes $\uparrow (n-1)1n$ because of the element 1, and ${*}n{?}1$ disappears because no matter what ${?}$ is, $n{?}1$ is an occurrence of the pattern 321.

It is easy to see that $A_n\langle 1 \uparrow n2 \rangle = x$, $A_n\langle An2 \rangle = (n-3)x^2$, $A_n\langle \uparrow (n-1)1n \rangle = x$ and $A_n\langle \uparrow n1{?} \rangle = (n-2)x$, and hence from (10) we have

$$E^{321}_{n,n-2}(x) = nx + (n-3)x^2.$$

Now, since $\mathcal{E}_{n,n-1} = \mathfrak{S}_n \cap \langle {*}n1 \rangle$ [13], $\mathcal{E}^{321}_{n,n-1} = \mathfrak{S}_n \cap \langle \uparrow n1 \rangle$ and $E^{321}_{n,n-1}(x) = x$. Since $\mathcal{W}_{n,n-1} = \mathfrak{S}_n$, formula (3) can be used to find $W^{321}_{n,n-1}(x)$, and then iteration of (6) for $t = n-1, n-2$ gives the desired formulas for $W^{321}_{n,n-2}(x)$ and $W^{321}_{n,n-3}(x)$.

To complete the proof of Theorem 5, we need to compute $E^{321}_{n,n-3}(x)$ and apply (6) for $t = n-3$. We analyse Tables 2 and 3 presenting $\mathcal{E}_{n,n-3}$. We see that to avoid the pattern 321, none of the cases in Table 3 are possible, and the following cases are impossible too: 1(b), 1(d), 3(b), 4(a), 4(b) and 4(d). The remaining cases are listed in Table 5 and they give the desired formula for $W^{321}_{n,n-4}(x)$. Here are our comments for three (more involved) cases in Table 5.

1(a) Because of the element 2, all elements but 1 in ${*}$ must increase. The case of $1 \uparrow (n-1)2n$ gives the term of x, and placing 1 differently in ${*}$ gives the term of $(n-4)x^2$.

2(a) Because of the element 3, all elements greater than 3 in ${*}$ must be increasing. Note that if ${*}$ begins with 2, then we can place 1 in $(n-3)$ places that corresponds to the term of $(n-3)x^2$. Otherwise, to avoid the pattern 321, 1 must precede 2. We distinguish four possible cases here: $12 \uparrow n3$ giving x; ${*}12{*}n3$, where to the left of 12 there is at least one element that gives $(n-4)x^2$; $1{*}2{*}n3$, where there is at least one element between 1 and 2, that gives $(n-4)x^2$, and the remaining case of 1 preceding 2, 1 being not the leftmost element, and 1 and 2 staying not together, which gives the term of $\binom{n-4}{2}x^3$.

3(a) Because of the element 2, all elements in ${*}$ that are greater than 3 must be increasing and ${?} \neq 1$, because otherwise $n21$ is an occurrence of the pattern 321. The case of $1 \uparrow n2{?}$ gives the term of $(n-3)x$ and the remaining cases of placing 1 differently and choosing ${?}$ give the term of $(n-3)(n-4)x$.

Table 5
Permutations in $\mathcal{E}_{n,n-3}^{321}$ for $n \geq 4$.

Case	Type of w	Descent gen. pol. $A_n\langle w \rangle$
1(a)	$*(n-1)2n$	$x + (n-4)x^2$
1(c)	$\uparrow (n-1)1?n$	$(n-3)x$
1(e)	$\uparrow (n-2)1(n-1)n$	x
2(a)	$*n3$	$x + (3n-11)x^2 + \binom{n-4}{2}x^3$
2(b)	$\uparrow (n-1)1n?$, where $? = 4, 5, \ldots, n-2$	$(n-5)x^2$
2(c)	$\uparrow (n-2)1n(n-1)$	x^2
3(a)	$*n2?$, where $? \neq 1$	$(n-3)x + (n-3)(n-4)x^2$
4(c)	$\uparrow n1??$, where $??$ is increasing	$\binom{n-2}{2}x$

Table 6
Permutations in $\mathcal{E}_{n,n-3}^{213}$ for $n \geq 5$.

Case	Type of w	Descent gen. polynomial $A_n\langle w \rangle$
2(a)	$\uparrow n3$	x
3(a)	$\uparrow n23$	x
3(b)	$1 \uparrow n?2$	$(n-3)x^2$ $(n-3$ choices for $?)$
4(a)	$\uparrow n??1$, but not $\uparrow n(n-2)(n-1)1$	$(n-4)x^2 + \binom{n-2}{2}x^3$
4(b)	$\uparrow n?12$	$(n-3)x^2$ $(n-3$ choices for $?)$
4(c)	$\uparrow n123$ and $\uparrow n132$	$x + x^2$
4(d)	$\uparrow n(n-2)(n-1)2$	x^2
5(a)	$AnB(n-2)(n-1)2$, $AB = \uparrow$, $B \neq \emptyset$	$(n-4)x^2$
5(b)	$AnB(n-3)(n-1)(n-2)1$, $AB = \uparrow$	$(n-4)x^3$
5(c)	$AnB(n-1)C(n-3)(n-2)1$, $ABC = \uparrow$	$\binom{n-3}{2}x^3$
5(d)	$A(n-1)nB(n-4)(n-3)(n-2)1$ $AB = \uparrow$	$(n-5)x^2$
5(e)	$AnB(n-2)(n-1)\begin{Bmatrix} 12 \\ ?1 \end{Bmatrix}$, $AB = \uparrow$	$(n-4)\left(x^2 + (n-4)x^3\right)$

5.3. Proof of Theorem 6

Considering (7) we can see that for $n \geq 4$,

$$\mathcal{E}_{n,n-2}^{213} = \mathfrak{S}_n \cap \langle \uparrow n2, \uparrow n12, \uparrow n?1, AnB(n-2)(n-1)1 \rangle \tag{11}$$

where to avoid the pattern 213, $AB = \uparrow$. Note that the pattern $*(n-1)1n$ in $\mathcal{E}_{n,n-2}$ disappears because it contains the occurrence $(n-1)1n$ of the pattern 213.

It is easy to see that $A_n\langle \uparrow n2 \rangle = x$, $A_n\langle \uparrow n12 \rangle = x$, $A_n\langle \uparrow n?1 \rangle = (n-2)x^2$ (choosing $?$ in $(n-2)$ ways) and $A_n\langle \uparrow n \uparrow (n-2)(n-1)1 \rangle = (n-3)x^2$ (choosing placement of n in $(n-3)$ ways), and hence from (11) we have

$$E_{n,n-2}^{213}(x) = 2x + (2n-5)x^2.$$

Now, since $\mathcal{E}_{n,n-1} = \mathfrak{S}_n \cap \langle *n1 \rangle$ [13], $\mathcal{E}_{n,n-1}^{213} = \mathfrak{S}_n \cap \langle \uparrow n1 \rangle$ and $E_{n,n-1}^{213}(x) = x$. Since $\mathcal{W}_{n,n-1} = \mathfrak{S}_n$, formula (2) can be used to find $W_{n,n-1}^{213}(x)$, and then iteration of (6) for $t = n-1, n-2$ gives the desired formulas for $W_{n,n-2}^{213}(x)$ and $W_{n,n-3}^{213}(x)$.

To complete the proof of Theorem 6, we need to compute $E_{n,n-3}^{213}(x)$ and apply (6) for $t = n-3$. We analyse Tables 2 and 3 presenting $\mathcal{E}_{n,n-3}$ using the observation that to the left of a large element, smaller elements must be in increasing order to avoid the pattern 213. We see that to avoid the pattern 213, the following cases are not possible: 1(a)–1(e), 2(b), 2(c), 5(f)–5(h) (as well as the first subcase of 5(d)). The remaining cases are listed in Table 6 and they give the desired formula for $W_{n,n-4}^{213}(x)$. Here are our comments for some cases in Table 6.

4(a) Suppose $a < b$. Then two possibilities are $\uparrow nab1$ and $\uparrow nba1$. Note that in the latter case any choice of a, b gives a 213-avoiding permutation (justifying the term of $\binom{n-2}{2}x^3$), while in the former case to avoid 213, we must have $b = a + 1$. Since in the former case choosing $a = n-2$ is not an option by the 4(a) description, a can be chosen from $\{2, 3, \ldots, n-3\}$, justifying the term of $(n-4)x^2$.

5(a) $(n-4)$ is the number of ways to insert n in AB of length $n-4$ so that B is non-empty.

5(c) $\binom{n-3}{2}$ is the number of ways to insert n and $n-1$ in ABC of length $n-6$. Any insertion will result in 3 descents in the resulting permutation.

5(d) $(n-5)$ is the number of ways to insert $(n-1)n$ in AB of length $n-5$. Any insertion will result in 2 descents in the outcome permutation.

Table 7
Permutations in $\mathcal{E}_{n,n-3}^{132}$ for $n \geq 6$.

Case	Type of w	Descent gen. polynomial $A_n\langle w \rangle$
1(b)	$*(n-1)21n$	$x^2 N_{n-4}(x)$
1(c)	$*(n-1)12n$	$x N_{n-4}(x)$
1(d)	$(n-1)*(n-3)*(n-2)1n$	$x^2 N_{n-4}(x)$
1(e)	$*(n-2)1(n-1)n$	$x N_{n-4}(x)$
4(a)	$*n231$ and $*n321$	$(x^2 + x^3) N_{n-4}(x)$
4(b)	$*n213$ and $*n312$	$2x^2 N_{n-4}(x)$
4(c)	$*n123$	$x N_{n-4}(x)$
5(c)	$n(n-1)*(n-3)*(n-2)1$	$x^3 N_{n-4}(x)$
5(d)	$(n-1)n*\left\{\begin{array}{c}(n-3)*(n-4)\\(n-4)*(n-3)\end{array}\right\}*(n-2)1$	$x^2 N_{n-4}(x)$
5(e)	$n*(n-2)*(n-1)\left\{\begin{array}{c}12\\21\end{array}\right\}$	$(x^2 + x^3) N_{n-4}(x)$
5(g)	$n*(n-3)*(n-2)1(n-1)$	$x^2 N_{n-4}(x)$

5(e) In $AnB(n-2)(n-1)12$ we have 2 descents and n can be placed in $(n-4)$ ways in the permutation AB of length $n-5$. On the other hand, independently to the choices of placing n in $AnB(n-2)(n-1)?1$ with 3 descents, we can choose ? in $(n-4)$ ways (any choice will result in a 231-avoiding permutation).

5.4. Proof of *Theorem 7*

Considering (7) we can see that for $n \geq 4$,

$$\mathcal{E}_{n,n-2}^{132} = \mathfrak{S}_n \cap \langle\ *(n-1)1n,\ *n12,\ *n21,\ n*(n-2)*(n-1)1\ \rangle \tag{12}$$

where $*$ represents any 132-avoiding permutation in the first three patterns, and $*(n-2)*$ represents any non-empty permutation on elements $\{2, 3, \ldots, n-2\}$. Indeed, for example, in the unrestricted set $\mathcal{E}_{n,n-2}$, the pattern $*n2$ is not taken to the set $\mathcal{E}_{n,n-2}^{132}$ as any permutation corresponding to it contains the pattern 132 (formed by the elements in $\{1, 2, n\}$). Also, the leftmost $*$ in $*n*(n-2)*(n-1)1$ must be empty as otherwise any element in it along with n and $n-1$ will form an occurrence of the pattern 132. Moreover, $? = 2$ in $*n1?$ or otherwise the element in place of $?$ along with 2 and n will form an occurrence of the pattern 132.

It is easy to see that $A_n\langle *(n-1)1n \rangle = x N_{n-3}(x)$, $A_n\langle *n12 \rangle = x N_{n-3}(x)$, $A_n\langle *n21 \rangle = x^2 N_{n-3}(x)$ and, as $n \geq 4$, $A_n\langle n*(n-2)*(n-1)1 \rangle = x^2 N_{n-3}(x)$. Hence, it follows from (12) that

$$E_{n,n-2}^{132}(x) = 2(x + x^2) N_{n-3}(x).$$

Now, since $\mathcal{E}_{n,n-1} = \mathfrak{S}_n \cap \langle *n1 \rangle$ [13], $\mathcal{E}_{n,n-1}^{132} = \langle *n1 \rangle$, where $*$ is any 132-avoiding permutation on $\{2, 3, \ldots, n-1\}$, and $E_{n,n-1}^{132}(x) = x N_{n-2}(x)$. Since $\mathcal{W}_{n,n-1} = \mathfrak{S}_n$, formula (2) can be used to find $W_{n,n-1}^{132}(x)$, and then iteration of (6) for $t = n-1, n-2$ gives the desired formulas for $W_{n,n-2}^{132}(x)$ and $W_{n,n-3}^{132}(x)$.

To complete the proof of Theorem 7, we need to compute $E_{n,n-3}^{132}(x)$ and apply (6) for $t = n-3$. We analyse Tables 2 and 3 presenting $\mathcal{E}_{n,n-3}$. We see that to avoid the pattern 132, none of the following cases are possible: 1(a), 2(a), 2(b), 2(c), 3(a), 3(b), 4(d), 5(a), 5(b), 5(f), 5(h); in particular, in 4(d) the elements in $\{1, 2, n\}$ form the pattern 132. The remaining cases are listed in Table 7 and they give the desired formula for $W_{n,n-4}^{132}(x)$. We comment three cases in Table 7.

1(b) x^2 records the descents $(n-1)2$ and 21, and $*$ is any permutation on $\{3, 4, \ldots, n-2\}$.
1(d) No matter if $*$ is empty or not in $(n-1)*(n-3)$, $(n-1)$ will be involved in a descent, and $(n-2)1$ is the other descent recorded by x^2. Also, $*(n-3)*$ is any non-empty 132-avoiding permutation on $\{2, 3, \ldots, n-3\}$.
5(d) Note that $*(n-3)*(n-4)*$ or $*(n-4)*(n-3)*$ can be any non-empty 132-avoiding permutation on $\{2, 3, \ldots, n-3\}$. This gives the term of $x^2 N_{n-4}(x)$.

5.5. Proof of *Theorem 8*

Considering (7) we can see that for $n \geq 4$,

$$\mathcal{E}_{n,n-2}^{312} = \mathfrak{S}_n \cap \langle\ 1*n2,\ *(n-1)1n,\ *n?1\ \rangle \tag{13}$$

where $*$ can be any 312-avoiding permutation over the respective set. Note that the pattern $*n1?$ in $\mathcal{E}_{n,n-2}$ disappears because it contains the occurrence $n1?$ of the pattern 312.

It is easy to see that $A_n\langle 1*n2 \rangle = A_n\langle *(n-1)1n \rangle = x N_{n-3}(x)$. Computing $A_n\langle *n?1 \rangle$ requires the following lemma.

Lemma 14. *The descent generating polynomial for* 312*-avoiding permutations of length n that end with*

- *an ascent is* $N_{n-1}(x)$;
- *a descent is* $N_n(x) - N_{n-1}(x)$.

Proof. To avoid the pattern 312, any permutation ending with an ascent must end with the largest element n. But then the element n does not contribute any descent and is independent from the rest of the permutation hence proving the first claim. The second claim is straightforward as any permutation either ends with an ascent or with a descent. □

Now, $A_n\langle *n?1\rangle = xA_{n-1}\langle *(n-1)?\rangle$ where $*?$ can be any 312-avoiding permutation of length $(n-2)$. If $*?$ ends with an ascent, an extra descent will be created after inserting $(n-1)$, so by Lemma 14 descents are counted by $xN_{n-3}(x)$. If $*?$ ends with a descent, inserting $(n-1)$ does not create an extra descent, so by Lemma 14 descents are counted by $N_{n-2}(x) - N_{n-3}(x)$. Hence

$$A_n\langle *n?1\rangle = xN_{n-2}(x) + x(x-1)N_{n-3}(x),\tag{14}$$

and from (13) we have

$$E^{312}_{n,n-2}(x) = xN_{n-2}(x) + x(x+1)N_{n-3}(x).$$

Now, since $\mathcal{E}_{n,n-1} = \mathfrak{S}_n \cap \langle *n1\rangle$ [13], $\mathcal{E}^{312}_{n,n-1} = \mathfrak{S}_n \cap \langle *n1\rangle$ and $E^{312}_{n,n-1}(x) = xN_{n-2}(x)$. Since $\mathcal{W}_{n,n-1} = \mathfrak{S}_n$, formula (2) can be used to find $W^{312}_{n,n-1}(x)$, and then iteration of (6) for $t = n-1, n-2$ gives the desired formulas for $W^{312}_{n,n-2}(x)$ and $W^{312}_{n,n-3}(x)$.

To complete the proof of Theorem 8, we need to compute $E^{312}_{n,n-3}(x)$ and apply (6) for $t = n-3$. We analyse Tables 2 and 3 presenting $\mathcal{E}_{n,n-3}$ using the observation that to the right of a large element, smaller elements must be in decreasing order to avoid the pattern 312. We see that to avoid the pattern 312, the following cases are not possible: 1(c), 1(d), 2(b), 3(a), 4(b)–4(d), 5(a-h), and also the restriction "but not ..." in 4(a) can be removed because $n(n-2)(n-1)$ forms the pattern 312. The remaining cases are listed in Table 8 ($*$ there denotes any 312-avoiding permutation over the respective set) and they give the desired formula for $W^{312}_{n,n-4}(x)$. We comment on three cases in Table 8.

1(b) $A_n\langle *(n-1)?1n\rangle = A_{n-1}\langle *(n-1)?\rangle$ and is given by (14).
3(b) $A_n\langle 1*n?2\rangle = A_{n-1}\langle *(n-1)?1\rangle$ and is given by (14).
4(a) $A_n\langle *nba1\rangle = xA_{n-1}\langle *(n-1)ba\rangle$. To compute $A_{n-1}\langle *(n-1)ba\rangle$, note that the elements $1, 2 \ldots, a-1$ in $*$ must be in the leftmost $a-1$ positions (or else there will be an occurrence of the pattern 312). Hence, the permutation formed by the smallest $a-1$ elements is independent from the rest of a 312-avoiding permutation π of length $n-1$, and we have

$$A_{n-1}\langle *(n-1)ba\rangle = \sum_{a=1}^{n-3} W^{312}_{a-1,a-2}(x)A_{n-a}\langle *(n-a)?1\rangle$$

$$= \sum_{a=1}^{n-3} N_{a-1}(x)\,(x(x-1)N_{n-a-3}(x) + xN_{n-a-2}(x))$$

where $*(n-a)?1$ is the pattern formed by the $n-a$ largest elements in π and formula (14) can be applied. By using the following recurrence relation of Narayana polynomials (see [19, Theorem 2.2])

$$N_n(x) = N_{n-1}(x) + x\sum_{i=0}^{n-2} N_i(x),\, N_{n-1-i}(x)\text{ for }n \geq 1,$$

we get that

$$A_{n-1}\langle *(n-1)ba\rangle = (N_{n-3}(x) + (x-1)N_{n-4}(x)) + (N_{n-2}(x) - N_{n-3}(x))$$

$$= N_{n-2}(x) + (x-2)N_{n-3}(x) + (x-1)^2N_{n-4}(x).$$

6. Concluding remarks

Table 1 confirms the following conjecture in the cases of $t \in \{n-1, n-2, n-3, n-4\}$, while the case of $t = 1$ is equivalent to avoiding two patterns (additionally the pattern 231 in each case) and is known, and is easy to see to be true [12].

Conjecture 15. *The number of* 213*-avoiding t-stack-sortable permutations of length n is the same as that of* 321*-avoiding t-stack-sortable permutations of length n for any $t \geq 0$ and $n \geq 0$.*

Table 8
Permutations in $\mathcal{E}_{n,n-3}^{312}$ for $n \geq 4$.

Case	Type of w	Descent gen. polynomial $A_n\langle w \rangle$
1(a)	$1*(n-1)2n$	$xN_{n-4}(x)$
1(b)	$*(n-1)?1n$	$xN_{n-3}(x) + x(x-1)N_{n-4}(x)$
1(e)	$*(n-2)1(n-1)n$	$xN_{n-4}(x)$
2(a)	$12*n3$ and $21*n3$	$x(x+1)N_{n-4}(x)$
2(c)	$*(n-2)1n(n-1)$	$x^2N_{n-4}(x)$
3(b)	$1*n?2$	$xN_{n-3}(x) + x(x-1)N_{n-4}(x)$
4(a)	$*nba1, \ b > a$	$xN_{n-2}(x) + x(x-2)N_{n-3}(x) + x(x-1)^2N_{n-4}(x)$

Table 9
The number of t-stack-sortable p-avoiding permutations of length $n \geq 1$ for $p \in \{213, 321\}$.

$t = 1$	1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...
$t = 2$	1, 2, 5, 13, 34, 89, 233, 610, 1597, 4181, 10946, ...
$t = 3$	1, 2, 5, 14, 41, 122, 365, 1094, 3281, 9842, 29525, ...
$t = 4$	1, 2, 5, 14, 42, 131, 417, 1341, 4334, 14041, 45542, ...
$t = 5$	1, 2, 5, 14, 42, 132, 428, 1416, 4744, 16016, 54320, ...

Conjecture 15 has also been confirmed computationally for $t \in \{2, 3, 4, 5\}$ for small values of n, and the respective numbers, along with those corresponding to $t = 1$, are given in Table 9. Interestingly, Table 9 matches the table in Section 4 in [5], which leads to the following conjecture related to so-called 321-machine that we state in terms of pattern avoidance (again, based on Section 4 in [5]).

Conjecture 16. *For any $t \geq 1$, t-stack-sortable p-avoiding permutations of length $n \geq 0$, for $p \in \{213, 321\}$, are in one-to-one correspondence with permutations of length n avoiding the patterns 132 and $12\cdots(t+2)$ simultaneously.*

Research directions for t-stack-sortable permutations include the study of unimodality, log-concavity and real-rootedness of the polynomials $W_{n,t}(x)$ for any $1 \leq t \leq n-1$. We will not define these notions here, instead referring the interested readers to [2,20–23]. In particular, Bóna [2] proved the symmetry and unimodality of $W_{n,t}(x)$ by giving combinatorial bijections and injections. Based on numerical evidence, Bóna further conjectured log-concavity and real-rootedness of $W_{n,t}(x)$. Brändén [20] strengthened Bóna's result by showing its γ-positivity via a combinatorial group action. Since no exact formulas are known for $W_{n,t}(x)$ in general, the remaining log-concavity and real-rootedness problems seem to be very hard. The real-rootedness of Narayana polynomials and Eulerian polynomials are well-known [24] and the real-rootedness of $W_{n,n-2}(x)$ was proved by Brändén [16] and Zhang [10].

In this paper, we obtained expressions for $W_{n,t}(x)$ in the cases of $t = n-3$ and $t = n-4$ in terms of Eulerian polynomials. There are several known approaches to study the unimodality, γ-positivity, log-concavity and real-rootedness for Eulerian polynomials, see [20,21] for example. In particular, Brändén [20] proved combinatorially γ-positivity for $W_{n,t}(x)$ for any t that also implies unimodality of $W_{n,t}(x)$. However, we still cannot prove the log-concavity and real-rootedness for $W_{n,t}(x)$. The formulas derived in this paper could potentially be useful to answer the questions, for example, if more analytic properties of Eulerian polynomials and/or the stack-sorting operation are discovered.

Finally, p-avoiding t-stack-sortable permutations can be studied for longer patterns p, or indeed for other types of patterns p (*consecutive*, *vincular*, *bivincular*, etc. [12], not to be defined here). Such studies are interesting in their own right, but also may bring interesting connections to other combinatorial objects.

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Data availability

No data was used for the research described in the article.

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