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PREFACE

Nilpotent Lie algebras have played an important role over the last years : either in the domain at Algebra when one considers its role in the classification problems of Lie algebras, or in the domain of geometry since one knows the place of nilmanifolds in the illustration, the description and representation of specific situations.

The first fondamental results in the study of nilpotent Lie algebras are obvisouly, due to Umlauf. In his thesis (Leipzig, 1991), he presented the first non trivial classifications. The systematic study of real and complex nilpotent Lie algebras was independently begun by Dixmier and Morozov. Complete classifications in dimension less than or equal to six were given and the problems regarding superior dimensions brought to light, such as problems related to the existence from seven up, of an infinity of non isomorphic complex nilpotent Lie algebras. One can also find these losts (for complex and real algebras) in the books about differential geometry by Vranceanu. A more formal approach within the frame of algebraic geometry was developed by Michèle Vergne. The variety of Lie algebraic laws is an affine algebraic variety and the nilpotent laws constitute a Zariski's closed subset. In this view the role of irreducible components appears naturally as well the determination or estimate of their numbers.

Theoritical physicists, interested in the links between diverse mechanics have developed the idea of contractions of Lie algebras (Segal, Inonu, Wigner). That idea was in fact very convenient in the determination of components. Others tried to combine it with the formal notion about deformations which was introduced by Gerstenhaber.

Truly the link between these two notions was only described a few years ago. One can read it in this book. The linear approach to deformations is situated in the frame of spaces of cohomology : a deformation is a formal series which formally satisfies Jacobi's identities. The linear part of it is thus a cocycle for values in the adjoint module (see Chevalley cohomology). Hence the problem of the existence in a given cocycle, of a deformation having that cocycle as its linear part. In this frame rigid algebras appear also naturally. They are the ones which are invariant by deformations. Nijenhuis and Richardson have shown that the laws which have a trivial second group of Chevalley's cohomology are rigid. However they are not all of the same kind : Nijenhuis and Richardson give an example of a rigid law having a non-trivial cohomology (in fact, as we will see it here, the class of rigid laws with a non trivial cohomology is far from being small).

It is important to know about rigid laws : the orbit (i.e the set of all isomorphic laws) of a rigid is an open subset in the variety of Lie algebraic laws and its closure is an irreducible component of Zariski. An estimate of the number of isomorphic classes of rigid laws gives thus an estimate of the number of irreducible components. This approach has been taken by Michèle Vergne in the frame of nilpotent laws. She has developed a cohomology adapted to nilpotent deformations, described components for large dimensions and introduced incidentally the notion of filiform algebras. Unfortunately we do not know yet of rigid nilpotent laws, and the problem of the existence of rigid nilpotent Lie algebras in the subvariety of nilpotent laws is also being considered.

The aim of this book is to put together all the results which were obtained either in the cohomological study of nilpotent algebras or in the problems of classification and the problems of deformations. The notion of derivations of nilpotent Lie algebras is also eludied. It elips naturally into the cohomological domain (derivations are linted to the first group of cohomology). Their interest is not only algebraic : Riemanian spaces with a left invariant homogeneous metric often have an isometric group the algebra of which can be identified with one of orthogonal derivations. The study of infinitesimal frame : it has the advantage of simplyfying the approach and thus eliminate its formal aspect. It also allows to solve the geometrical problems : what are the conditions for a cocycle (or for a formal tangent vector) to be integrable, mean the first term of a deformation (or so that this formal vector be a vector of the cone of tangents). One can in the infinitesimal scape describe the duality contraction - deformation perfectly. We focus also on two classes of nilpotent Lie algebras, which are more or less models of nilpotent Lie algebras : those are filiform algebras (the least commutative ones) and the characteristically nilpotent Lie algebras (all the derivations are nilpotent).

This book was made possible thanks for the collaboration of Professor Yu. Khakimdjanov, who worked at the UHA as a visiting professor during the years 1991-94. We hope that this fruitful exchange will continue .... .

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Colmar - Mulhouse
23 october 1994

CHAPTER 1

LIE ALGEBRAS. GENERALITIES

The aim of this chapter is to recall some fundamental notions concerning finite-dimensional Lie algebras. We also present the principal classes of these algebras and mainly the more interesting classes of nilpotent Lie algebras. First we study Lie algebras on an arbitrary field, although the larger part of the book is devoted to complex nilpotent Lie algebras.

I. LIE ALGEBRAS. GENERALITIES

I.1. Notions of Lie algebras

Definition 1. *A Lie algebra* $\mathfrak{g}$ *over a field* K *is a vector space on* K *with a bilinear mapping* $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *denoted* $(x,y) \to [x,y]$ *and called the bracket of* $\mathfrak{g}$ *and satisfying*:

(1) $[x,x] = 0 \ , \forall\, x \in \mathfrak{g}$,

(2) $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0 \ , \ \forall \ x, y, z \in \mathfrak{g}$.

Remarks

1. The identity (2) is called the *Jacobi identity.*

2. The relation (1) implies the anticommutativity of the multiplication of $\mathfrak{g}$:

$$[x, y] = -[y, x] \quad \forall\ x, y \in \mathfrak{g}.$$

In fact, we have

$$0 = [x+y\ , x+y] = [x,y] + [y,x]\ .$$

Conversely, if the characteristic of the field K is different from 2, anticommutativity of the bracket implies $[x,x] = 0$.

I.2. Examples

1. Every vector space $\mathfrak{a}$ with the bracket $[x,y] = 0$, for all x and y in $\mathfrak{a}$, is a Lie algebra, called *an Abelian Lie algebra.*

2. Let V be the real vector space $\mathbb{R}^3$. The bracket defined by the cross-product of the vectors of $\mathbb{R}^3$ defines on $\mathbb{R}^3$ a non-Abelian Lie algebra.

3. Let $\mathcal{M}_n(K)$ be the space of n-square matrices on K. The multiplication

$$[A,B] = AB - BA$$

satisfies conditions 1 and 2. Then, $\mathcal{M}_n(K)$ with this bracket is a Lie algebra, and denoted as gl(n,K).

4. Let V be a vector space on K. The vector space End V of the endomorphisms of V with the bracket $[f,g] = f \circ g - g \circ f$ is a Lie algebra, denoted as gl(V,K) (or gl(V) if the field is clearly specified). This Lie algebra is called the hint of *general linear algebra..*

5. Let $\mathcal{A}$ be an associative algebra on K whose multiplication is $\circ$. We can provide $\mathcal{A}$ for the structure of a Lie algebra by putting

$$[x,y] = x \circ y - y \circ x \quad , \quad \forall\, x, y \in \mathcal{A}.$$

Note that examples 3 and 4 are of this type.

6. Let V be a (2k+1)-dimensional vector space and $(e_1 ,..., e_{2k+1})$ a basis of V. The brackets defined by

$$[e_{2i}, e_{2i+1}] = e_1 \qquad \text{for} \qquad i = 1, \ldots, k.$$

(other brackets except those obtained by anticommutativity are 0) endows V with the structure of a Lie algebra.

This Lie algebra plays an important role in the theory of nilpotent Lie algebras. It is called the Heisenberg algebra and noted H_k.

I.3. Lie subalgebras

Definition 2. *Let* $\mathfrak{g}$ *be a Lie algebra on* K. *A subspace* $\mathfrak{g}_1$ *of* $\mathfrak{g}$ *is called a Lie subalgebra if* $[x, y] \in \mathfrak{g}_1$ *whenever* $x, y \in \mathfrak{g}_1$. *We note that a Lie subalgebra of a Lie algebra is also a Lie algebra for the induced multiplication.*

Examples

1. Every subspace of an Abelian Lie algebra is an Abelian subalgebra.
2. The subspace of gl(n,K) constituted of all upper triangular matrices (a_{ij}) with $a_{ij} = 0$ if $i < j$ is a subalgebra of gl(n,K).
3. The subspace of gl(n,K) constituted of all diagonal matrices is also a subalgebra.

I.4. Classical Lie algebras

1. *The special linear Lie algebra* sl(n,K) , K = $\mathbb{R}$ or $\mathbb{C}$.

The set of matrices $A = (a_{ij})$ of gl(n,K) with trace zero, that is

$$\operatorname{tr} A = \sum_{i=1}^{n} a_{ii} = 0 ,$$

is a Lie subalgebra of gl(n,K).

2. *The special orthogonal Lie algebra* so(n,K) , K = $\mathbb{R}$ or $\mathbb{C}$.

Let so(n,K) be the space

$$so(n,K) = \left\{A = (a_{ij}) \in gl(n,K) \quad : \ {}^tA = -A\right\}$$

where ${}^tA$ is the transposed matrix of A.

3 *The symplectic Lie algebra* $sp(2n,K)$, $K = \mathbb{R}$ or $\mathbb{C}$.

Let $sp(2n,K) = \left\{A = (a_{ij}) \in gl(2n,K) \ : {}^tAJ = -JA\right\}$

where $$J = \begin{pmatrix} 0 & Id_n \\ -Id_n & 0 \end{pmatrix}.$$

Every matrix A of $sp(2n,K)$ can be decomposed as

$$A = \begin{pmatrix} M & N \\ P & Q \end{pmatrix},$$

where N and P are n-square symmetric matrices and the matrices M and Q satisfy ${}^tM = -Q$. Then, $sp(2n, K)$ is a Lie subalgebra of $gl(n,K)$ called a *symplectic algebra*..

Remark. It is easy to enlarge these examples by considering, not only the set of matrices, but endomorphisms of a given finite-dimensional vector space. Therefore, we can define the following Lie algebras

(1) $sl(V) = \{f \in \text{End } V : tr(f) = 0\}$, where $tr(f)$ is the trace of f, that is the sum of the diagonal entries of a matrix of f (this is independent of the choice of basis of f).

(2) $so(V) = \{f \in \text{End } V \,/ < f(v),w > = -< v,f(w) >,\ \forall\, v, w \in V \text{ and} \det f = 1\}$ where $<,>$ is a nondegenerate symetric bilinear form and det f is the determinant of f.

(3) $sp(V) = \{f \in \text{End } V \,/ <\Omega(f(v),w) = -\Omega(v,f(w))>,\ \forall\, v, w \in V\}$, where Ω is a nondegenerate bilinear antisymmetric form on V with $\dim V = 2n$.

I.5. Ideals

Definition 3. *A Lie subalgebra* $\mathfrak{I}$ *of* $\mathfrak{g}$ *is called an ideal of* $\mathfrak{g}$ *if* $[\mathfrak{I},\mathfrak{g}] \subset \mathfrak{I}$ *that is* $[x,y] \in \mathfrak{I}$ *for all* $x \in \mathfrak{I}$ *and* $y \in \mathfrak{g}$.

Of course, there are some Lie subalgebras which are not ideals.

Examples

1. Let $\mathfrak{g}$ be a Lie algebra. Then $\{0\}$ is an ideal.

2. Let $\mathfrak{g}$ be a Lie algebra. The *center*

$$Z(\mathfrak{g}) = \{x \in \mathfrak{g} : [z, y] = 0 \quad \forall\, y \in \mathfrak{g}\}$$

is an ideal of $\mathfrak{g}$.

3. Let $\mathfrak{h}$ be a subalgebra of a Lie algebra $\mathfrak{g}$. The *normalizer* of $\mathfrak{h}$ in $\mathfrak{g}$

$$N_{\mathfrak{g}}(\mathfrak{h}) = \left\{x \in \mathfrak{g} : \left[x, \mathfrak{h}\right] \subset \mathfrak{h}\right\}$$

is a subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. Then $\mathfrak{h}$ is an ideal of $N_{\mathfrak{g}}(\mathfrak{h})$.

4. Let $\mathfrak{g} = gl(n, \mathbb{C})$ and $\mathfrak{h} = sl(n, \mathbb{C})$. Then $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. In fact, every matrix A can be written as $A = aI + A_1$, where A_1 is in $sl(n, \mathbb{C})$ and I is the identity matrix. Let B be in $sl(n, \mathbb{C})$. We have

$$[A, B] = [aI + A_1, B] = [A_1, B] \in sl(n, \mathbb{C})\,.$$

Remark. If $\mathfrak{I}$ and $\mathfrak{J}$ are ideals of a Lie algebra $\mathfrak{g}$, then $\mathfrak{I} + \mathfrak{J} = \{x + y\,, x \in \mathfrak{I} \text{ and } y \in \mathfrak{J}\}$, $[\mathfrak{I}, \mathfrak{J}] = \{\Sigma\,[x_1, y_2]\,; x_1 \in \mathfrak{I}\,, y_2 \in \mathfrak{J}\}$ and $\mathfrak{I} \cap \mathfrak{J}$ are ideals of $\mathfrak{g}$.

I.6. Quotient Lie algebra

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{I}$ an ideal of $\mathfrak{g}$. The quotient vector space $\mathfrak{g}/\mathfrak{I}$ is a Lie algebra for the multiplication defined by $[x + \mathfrak{I}\,,\ y + \mathfrak{I}] = [x, y] + \mathfrak{I}$. This is unambiguous since if $x + \mathfrak{I} = x' + \mathfrak{I}\,, y + \mathfrak{I} = y' + \mathfrak{I}$, then $[x' + \mathfrak{I}\,, y' + \mathfrak{I}] = [x\,, y] + \mathfrak{I}$.

I.7. Homomorphisms

Let $\mathfrak{g}$ and $\mathfrak{g}_1$ be two Lie algebras on K. A linear mapping

$$\varphi : \mathfrak{g} \to \mathfrak{g}_1$$

is called a *homomorphism* if $\varphi[x,y] = [\varphi(x), \varphi(y)]$ for all x and y in $\mathfrak{g}$. We note that the first bracket is in $\mathfrak{g}$ and the second is in $\mathfrak{g}_1$.

As usual, φ is an isomorphism if it is a bijectif homomorphism and φ is an automorphism if φ is an isomorphism and $\mathfrak{g} = \mathfrak{g}_1$.

Examples

1. Let V be a n-dimensional K vector space. Then if we choose a basis of V, the mapping

$$\begin{aligned} \varphi : gl(V) &\to gl(n, K) \\ f &\to \varphi(f), \end{aligned}$$

where $\varphi(f)$ is the matrix of f corresponding to the given basis, is an isomorphism between these two Lie algebras.
For this, we find an isomorphism between sl(V) and sl(n, K), between so(V) and so(n, K), and also between sp(V) and sp(2p, K) (n = 2p).

2. Let $\mathfrak{I}$ be an ideal of the Lie algebra $\mathfrak{g}$. The canonical map

$$\begin{aligned} \pi : \mathfrak{g} &\to \mathfrak{g}/\mathfrak{I} \\ x &\to x + \mathfrak{I} \end{aligned}$$

is a homomorphism of $\mathfrak{g}$ onto $\mathfrak{g}/\mathfrak{I}$. Its Kernel

$$\text{Ker } \pi = \{x \in \mathfrak{g} : \pi(x) = 0\}$$

is equal to $\mathfrak{I}$.

Proposition 1. *Let* $\varphi : \mathfrak{g} \to \mathfrak{g}_1$ *be a homomorphism of Lie algebras. Then* Ker φ *is an ideal of* $\mathfrak{g}$ *and for every ideal* $\mathfrak{I}$ *of* $\mathfrak{g}$, $\mathfrak{I} \subset$ Ker φ, *there exists an unique homomorphism* $\psi : \mathfrak{g}/\mathfrak{I} \to \mathfrak{g}_1$ *such that the following diagram*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, dashed, "\varphi"] \arrow[dr, phantom, "\pi"] & \mathfrak{g}_1 \\ & \mathfrak{g}/\mathfrak{I} \arrow[u, "\psi"'] \end{tikzcd}$$

commutes, where π *is the canonical homomorphism.*

We note that, if $\mathfrak{I}$ = Ker φ, then $\mathfrak{g}/$Ker φ is isomorphic to Im φ, the image of φ. The proof is standard and very easy.

I.8. Direct sum of Lie algebras

Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie algebras over K. We can build a new Lie algebra by

$$\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2 = \{(x_1, x_2) : x_1 \in \mathfrak{g}_1 \ , x_2 \in \mathfrak{g}_2\}$$

with the multiplication

$$[(x_1, x_2) , (y_1, y_2)] = ([x_1, x_2] ,[y_1, y_2]) .$$

We can remark that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are ideals of $\mathfrak{g}$. We shall say that $\mathfrak{g}$ is the direct sum of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ and we shall denote $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$.

It is easy to generalize this construction in order to define the direct sum

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \dots \oplus \mathfrak{g}_p$$

of the Lie algebras $\mathfrak{g}_1 , \dots, \mathfrak{g}_p$ over K.

Now suppose that $\mathfrak{I}$ and $\mathfrak{J}$ are ideals of a given Lie algebra $\mathfrak{g}$, such that $\mathfrak{g} = \mathfrak{I} + \mathfrak{J}$ (direct sum of vector subspaces). Then, $\mathfrak{g}$ is isomorphic to the direct sum $\mathfrak{I} \oplus \mathfrak{J}$.

II. DERIVATIONS OF LIE ALGEBRAS

II.1. Definition and examples

Definition 4. *Let $\mathfrak{g}$ be a Lie algebra over* K. *A linear endomorphism* f *of $\mathfrak{g}$ is called a derivation of $\mathfrak{g}$ if it satisfies*

$$[f(x), y] + [x, f(y)] - f[x, y] = 0, \quad \forall\, x, y \in \mathfrak{g} .$$

It is easy to see that the set Der $\mathfrak{g}$ of all derivations of $\mathfrak{g}$ is a vector subspace of End $\mathfrak{g}$.

Proposition 2. Der $\mathfrak{g}$ *is a Lie algebra over* K *for the bracket* $[f, g] = f \circ g - g \circ f$.

Proof.

$$
\begin{aligned}
(f \circ g - g \circ f)\,[x, y] &= f \circ g\,[x, y] - g \circ f\,[x,y] \\
&= f[[g(x),y] + [x,g(y)]] - g[[f(x),y] + [x,f(y)]] \\
&= [f \circ g(x),y] + [g(x),f(y)] + [f(x),g(y)] + [x,f \circ g(y)] \\
&\quad - [g \circ f(x),y] - [f(x),f(y)] - [g(x),f(y)] - [x,g \circ f(y)] \\
&= [(f \circ g - g \circ f)\,(x), y] + [x,(f \circ g - g \circ f)\,(y)]
\end{aligned}
$$

Then, Der $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(\mathfrak{g})$.

Examples

1. If $\mathfrak{g}$ is an Abelian Lie algebra, then every endomorphism of $\mathfrak{g}$ is a derivation.

2. Let $\mathfrak{g}$ be a Lie algebra and f an endomorphism of $\mathfrak{g}$ satisfying $f[x, y] = 0$, $\forall\, x, y \in \mathfrak{g}$, and $f(\mathfrak{g}) \subset Z(\mathfrak{g})$. Then f is a derivation of $\mathfrak{g}$.

II.2. Inner derivations

Let $\mathfrak{g}$ be a Lie algebra over K and $x \in \mathfrak{g}$. The endomorphism of $\mathfrak{g}$ defined by $y \to [x, y]$ is denoted ad x. The Jacobi identity implies that ad x is a derivation of $\mathfrak{g}$ for all x in $\mathfrak{g}$, called *ithe nner derivation.* All others derivations of $\mathfrak{g}$ are called *outer.*

Proposition 3. *The linear map* $x \to \mathrm{ad}\ x$ *is a homomorphism of* $\mathfrak{g}$ *into the Lie algebra* Der $\mathfrak{g}$.

In fact

$$
\begin{aligned}
\mathrm{ad}\,[x, y]\,(y) &= [[x,y], z] = [x, [y,z]] - [y, [x,z]] \\
&= (\mathrm{ad}\, x \circ \mathrm{ad}\, y - \mathrm{ad}\, y \circ \mathrm{ad}\, x)\,(y) \\
&= [\mathrm{ad}\, x, \mathrm{ad}\, y]\,(z)
\end{aligned}
$$

Proposition 4. *The set of inner derivations of* $\mathfrak{g}$ *is an ideal of* Der $\mathfrak{g}$.

Proof. Let f be in Der $\mathfrak{g}$ and x in $\mathfrak{g}$. We have, for all y in $\mathfrak{g}$,

$$\begin{aligned}[f, \mathrm{ad}\, x](y) &= f[x, y] - \mathrm{ad}\, x(f(y)) \\ &= [f(x), y] + [x, f(y)] - [x, f(y)] \\ &= [f(x), y] \\ &= \mathrm{ad}(f(x))(y)\end{aligned}$$

Then $[f, \mathrm{ad}\, x] = \mathrm{ad}(f(x))$ for all x in $\mathfrak{g}$.

II.3. Characteristic ideals

比理想更严格.

An ideal $\mathfrak{I}$ of a Lie algebra $\mathfrak{g}$ is called characteristic if $f(\mathfrak{I})$ is included in $\mathfrak{I}$ for all derivations f of $\mathfrak{g}$. We note that every ideal is invariant for any inner derivation of $\mathfrak{g}$. But, generally, it is not invariant for any outer derivation. We can see easily that every characteristic ideal $\mathfrak{J}$ of a characteristic ideal $\mathfrak{I}$ of $\mathfrak{g}$ is also a characteristic ideal of $\mathfrak{g}$. Likewise, if $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are characteristic ideals of $\mathfrak{g}$, $[\mathfrak{I}_1, \mathfrak{I}_2]$ is also a characteristic ideal of $\mathfrak{g}$.

首先注意到一个事实:
a如果是b的理想,b是c的理想.那么a不一定是c的理想.
反例:李代数是哈森伯格代数[x1,x2]=[x3,x4]=x5,
a=span{x2,x4} b={x2,x4,x5} c={x1....x5} 那么a不是c理想.

因为假设a是b的特征理想,
b是c的特征理想.
那么c的刀子限制在b上是一个b的刀子.所以c的刀子限制在b上是保a的.
所以c的刀子是保a的,因为限制a比限制b更小,所以更对了.

II.4. Derivations of a direct sum of Lie algebras

We want to compute the derivations of a finite direct sum $\mathfrak{g} = \oplus\, \mathfrak{g}_i$ of Lie algebras $\mathfrak{g}_i$ over K. We can consider every derivation f_i of $\mathfrak{g}_i$ as a derivation of $\mathfrak{g}$ by :

$$f_i(x) = f_i(x_1, \ldots, x_p) = f_i(x_i) \quad \text{for all} \quad x = (x_1, \ldots, x_p) \ \text{in} \ \mathfrak{g}.$$

就是作用在其他分量上是0.

Lemma. *For every* i, j *such that* $1 \leq i, j \leq p$, *any endomorphism* f *of* $\mathfrak{g}$ *satisfying* $f(\mathfrak{g}_k) = 0$ *if* $k \neq i$, $f(\mathfrak{g}_i) \subset Z(\mathfrak{g}_j)$ *and* $f[(\mathfrak{g}_i), (\mathfrak{g}_i)] = 0$ *is a derivation of* $\mathfrak{g}$.

The proof is obvious.

The set of all derivations of $\mathfrak{g}$ described in this lemma is denoted $\mathfrak{D}(\mathfrak{g}_i, \mathfrak{g}_j)$.

Theorem 1. *If* $\mathfrak{g}$ *is a direct sum* $\mathfrak{g} = \bigoplus_{i=1}^{p} \mathfrak{g}_i$ *of Lie algebras over* K, *then*

$$\mathrm{Der}\, \mathfrak{g} = \bigoplus_{i=1}^{p} \mathrm{Der}\, \mathfrak{g}_i \oplus \left(\bigoplus_{i \neq j} \mathfrak{D}(\mathfrak{g}_i, \mathfrak{g}_j) \right)$$

Proof. From the lemma, we have

$$\mathrm{Der}\,\mathfrak{g} \supset \oplus_{i=1}^{p} \mathrm{Der}\,\mathfrak{g}_i \oplus \left(\oplus_{i\neq j}\mathcal{D}(\mathfrak{g}_i, \mathfrak{g}_j)\right)$$

Now we prove the converse inclusion :

一个映射不是映射到自己的空间就是映射到其他的空间

Let f be in Der $\mathfrak{g}$. Then $f = \sum_{i=1}^{p} f_i + \sum_{i\neq 1}^{p} f_{ij}$ where $f_i \in \mathrm{End}\,\mathfrak{g}_i$, $f_{ij} \in \mathrm{Hom}(\mathfrak{g}_i, \mathfrak{g}_j)$ are defined by $f_i = \pi_i \circ f \circ \pi_i$, $f_{ij} = \pi_j \circ f \circ \pi_i$ where π_i is the canonical projection of $\mathfrak{g}$ on $\mathfrak{g}_i$. These endomorphisms f_i, f_{ij} are derivations of $\mathfrak{g}$. We shall prove that $f_{ij} \in \mathcal{D}(\mathfrak{g}_i, \mathfrak{g}_j)$. Let $x = [y, z]$ in $[\mathfrak{g}_i, \mathfrak{g}_i]$. Then

$$f_{ij}(x) = f_{ij}[y, z] = [f_{ij}(y), z] + [y, f_{ij}(z)] = 0 \quad \text{if} \quad i \neq j \quad \text{and} \quad f_{ij}[\mathfrak{g}_i, \mathfrak{g}_i] = 0 .$$

Suppose now that $f_{ij}(\mathfrak{g}_i)$ is not contained in $Z(\mathfrak{g}_j)$. There is $x \neq 0$ in $\mathfrak{g}_i$ such that $y = f_{ij}(x)$ $y \notin Z(\mathfrak{g}_j)$. We choose z in $\mathfrak{g}_j$ satisfying $[z, y] \neq 0$. Then

$$[f_{ij}, \mathrm{ad}\, z](x) = [z, y] \neq 0$$

and the nontrivial derivation $[f_{ij}, \mathrm{ad}\, z]$ is in $\mathcal{D}(\mathfrak{g}_i, \mathfrak{g}_j)$. However, this derivation is inner (Proposition 4). This is impossible if $i \neq j$.

II.5. Semidirect sum of Lie algebras

Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie algebras over K. A Lie algebra $\mathfrak{g}$ over K is called a *semidirect sum of* $\mathfrak{g}_1$ by $\mathfrak{g}_2$ if $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ (direct sum of vector spaces) where $\mathfrak{g}_1$ is an ideal and $\mathfrak{g}_2$ a subalgebra of $\mathfrak{g}$.

Notation. We shall also denote the semidirect sum by

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

and in this decomposition <u>first</u> the ideal is always layed.
So, we have

$$[\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_1 \ , \ [\mathfrak{g}_1, \mathfrak{g}_2] \subset \mathfrak{g}_1 \ , \ [\mathfrak{g}_2, \mathfrak{g}_2] \subset \mathfrak{g}_2 \ .$$

The direct sum corresponds to the particular case where $\mathfrak{g}_2$ is also an ideal. We use the same notation which we will summarize if it is necessary.

Let $\mathfrak{g}$ be a semi direct sum of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. For every $x \in \mathfrak{g}_2$, ad $x(\mathfrak{g}_1) \subset \mathfrak{g}_1$.

Then the restriction of ad x to $\mathfrak{g}_1$ is a derivation of $\mathfrak{g}_1$. This defines a homomorphism

$$\theta : \mathfrak{g}_2 \to \mathrm{Der}\, \mathfrak{g}_1$$
$$x \to \mathrm{ad}\, x .$$

Conversely, the datum of the Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_2$ and of a homomorphism

$$\theta : \mathfrak{g}_2 \to \mathrm{Der}\, \mathfrak{g}_1$$

of Lie algebras permits the defining of a Lie algebra $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ (direct sum of vector space) with the bracket :

$$[(x_1, x_2), (y_1, y_2)] = \big([x_1, y_1] + \theta(x_2)(y_1) - \theta(y_2)(x_1), [x_2, y_2]\big) .$$

In this manner, $\mathfrak{g}$ appears as the semidirect sum $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. The trivial case $\theta = 0$ corresponds to the direct sum of $\mathfrak{g}_1$ and $\mathfrak{g}_2$.

III. NILPOTENT AND SOLVABLE LIE ALGEBRAS

III.1. Derived sequences, central sequences

Let $\mathfrak{g}$ be a Lie algebra over K (K = $\mathbb{R}$ or $\mathbb{C}$). We put $\mathcal{D}^0\mathfrak{g} = \mathfrak{g}$; $\mathcal{D}^1\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$ and, more generally, $\mathcal{D}^{k+1}\mathfrak{g} = [\mathcal{D}^k\mathfrak{g}, \mathcal{D}^k\mathfrak{g}]$, for every $k \geq 0$. All these subspaces are ideals of $\mathfrak{g}$ and we have the following decreasing sequence, called the *derived sequence* of $\mathfrak{g}$:

$$\mathfrak{g} = \mathcal{D}^0\mathfrak{g} \supset \mathcal{D}^1\mathfrak{g} \supset \dots \supset \mathcal{D}^k\mathfrak{g} \supset \dots$$

The *descending central sequence of* $\mathfrak{g}$:

$$\mathfrak{g} = \mathcal{C}^0\mathfrak{g} \supset \mathcal{C}^1\mathfrak{g} \supset \dots \supset \mathcal{C}^k\mathfrak{g} \supset \dots$$

is defined with the following ideals :

$$\left\{ \begin{array}{ll} \mathcal{C}^0\mathfrak{g} = \mathfrak{g} \ , & \\ \mathcal{C}^{k+1}\mathfrak{g} = [\mathcal{C}^k\mathfrak{g}, \mathfrak{g}] & \forall\, k \geq 0 \ . \end{array} \right\}$$

Note that this sequence decreases more slowly that the derived sequence and the Lie algebra $\mathcal{C}^{i+1}\mathfrak{g}/\mathcal{C}^{i+2}\mathfrak{g}$ is an ideal of $\mathcal{C}^i\mathfrak{g}/\mathcal{C}^{i+2}\mathfrak{g}$ contained in its center.

The *ascending central sequence* is given by

$$\mathcal{C}_0(\mathfrak{g}) = 0\,,$$
$$\mathcal{C}_k(\mathfrak{g}) = \left\{x \in \mathfrak{g} \ :\ [x,\mathfrak{g}] \subset \mathcal{C}_{k-1}(\mathfrak{g})\right\} \quad \forall\ k \geq 1\,.$$

We have

$$\{0\} = \mathcal{C}_0(\mathfrak{g}) \subset \mathcal{C}_1(\mathfrak{g}) \subset \ldots \subset \mathcal{C}_k(\mathfrak{g}) \subset \ldots$$

We note that $\mathcal{C}_1(\mathfrak{g})$ is the center of $\mathfrak{g}$ and $\mathcal{C}_{k+1}(\mathfrak{g})$ is the reciprocal image for the canonical embedding of $\mathfrak{g}$ on $\mathfrak{g}/\mathcal{C}_k(\mathfrak{g})$ of the center of $\mathfrak{g}/\mathcal{C}_k(\mathfrak{g})$.

III.2. Definition of solvable Lie algebras

Definition 5. *A Lie algebra* $\mathfrak{g}$ *is called solvable if there is an integer less* k *such that* $\mathfrak{D}^k\mathfrak{g} = \{0\}$.

Examples.

1. Every Abelian Lie algebra is solvable.

2. The Lie algebra of the affine group of a straight line is solvable because it is generated by two independent vectors x and y satisfying $[x, y] = x$.

3. The subalgebra of $gl(n,\mathbb{C})$ composed of triangular matrices is solvable.

4. Let V be a vector space of dimension n and $D = \left(0 = V_0 \subset V_1 \subset \ \ldots \ \subset V_n = V\right)$ a flag of V that is a sequence of fitted-together vector subspaces. Let $\mathfrak{b}(D) = \left\{ f \in \operatorname{End} V \ :\ f(V_i) \subset V_i \ \ \forall\, i \right\}$. Then $\mathfrak{b}(D)$ is a solvable Lie algebra for the bracket

$$[f, g] = f \circ g - g \circ f\ .$$

The matrices of the elements of $\mathfrak{b}(D)$ relative to an adapted basis of the flag are triangular. This algebra is similar to that described in example 3.

Proposition 5. *The Lie algebra* $\mathfrak{g}$ *is solvable if and only if there is a descending sequence of ideals of* $\mathfrak{g}$

$$\mathfrak{g} = I_0 \supset I_1 \supset \ldots \supset I_k = \{0\}$$

such that $[I_j, I_j] \subset I_{j+1}$ *for* $0 \le j \le k-1$.

Proof. For the necessary condition, we take $I_j = \mathcal{D}^j\mathfrak{g}$.

Sufficient condition : we have $\mathcal{D}^1\mathfrak{g} \subset I_1$ and, more generally, $\mathcal{D}^j\mathfrak{g} \subset I_j$.

As $I_k = \{0\}$, then $\mathcal{D}^k\mathfrak{g}$ is equal to 0.

Proposition 6. *Let* $\mathfrak{g}$ *be a Lie algebra.*

(a) *If* $\mathfrak{g}$ *is solvable, every subalgebra and every quotient algebra are solvable.*

(b) *If* $\mathfrak{I}$ *is a solvable ideal of* $\mathfrak{g}$ *such that* $\mathfrak{g}/\mathfrak{I}$ *is solvable, then* $\mathfrak{g}$ *is also solvable.*

(c) *If* $\mathfrak{I}$ *and* $\mathfrak{J}$ *are solvable ideals of* $\mathfrak{g}$, *then* $\mathfrak{I} + \mathfrak{J}$ *is also solvable.*

Proof.

(a) If L is a subalgebra of $\mathfrak{g}$, then $\mathcal{D}^i L$ is contained in $\mathcal{D}^i\mathfrak{g}$.

(b) Let $\pi : \mathfrak{g} \to \mathfrak{g} / \mathfrak{I}$ be the canonical homomorphism. As it exists, an integer k such as $\mathcal{D}^k(\mathfrak{g}/\mathfrak{I}) = 0$, then $\pi(\mathcal{D}^k\mathfrak{g}) = 0$. Thus $\mathcal{D}^k\mathfrak{g} \subset \mathfrak{I}$ and as $\mathfrak{I}$ is solvable, $\mathcal{D}^k\mathfrak{g}$ is also solvable and therefore $\mathfrak{g}$ is solvable.

(c) We can deduce this result from the property (b) using the fact that the ideal $\mathfrak{I}+\mathfrak{J} / \mathfrak{J}$ is isomorphic to $\mathfrak{I} / \mathfrak{I}\cap\mathfrak{J}$.

Property (c) implies that there exists a unique maximal solvable ideal of $\mathfrak{g}$. This ideal is called *the radical of* $\mathfrak{g}$.

III.3. Definition of nilpotent Lie algebras

Definition 6. *A Lie algebra* $\mathfrak{g}$ *is called nilpotent if there is an integer* k *such that* $\mathcal{C}^k\mathfrak{g} = \{0\}$. *The smallest integer* k *such that* $\mathcal{C}^k\mathfrak{g} = \{0\}$ *is called the nilindex (or the nilpotency class) of* $\mathfrak{g}$.

Examples.

1. Every Abelian algebra is nilpotent with a nilindex equal to 1.

2. The Heisenberg algebra H_k defined in the basis $(e_1, e_2, ..., e_{2k+1})$ by

$$[e_{2i-1}, e_{2i}] = e_{2k+1} \quad i = 1, ..., k$$

(the undefined brackets being nul) has a nilindex equal to 2.

3. The n-dimensional algebra defined in a basis $(e_1, ..., e_n)$ by the brackets

$$[e_1, e_i] = e_{i+1} \quad 2 \le i \le n-1$$

is nilpotent with a nilindex equal to $n - 1$.

4. The subalgebra of $gl(n,K)$ made up by the upper triangular nilpotent matrices (that is to say, the matrices $A = (a_{ij})$ such that $a_{ij} = 0$ for $i \le j$) is nilpotent.

5. Let V be a vectorial space and $D = (V_0 \subset V_1 \subset ... \subset V_n = V)$ is a flag of length n. The Lie algebra

$$\mathfrak{n}(D) = \{f \in \mathrm{End}(V) \;:\; f(V_i) \subset V_{i-1}\}$$

is nilpotent. We note that the endomorphisms belonging to $\mathfrak{n}(D)$ are written in an adapted basis to the flag D in a triangular form.

Proposition 7. *Let $\mathfrak{g}$ be a Lie algebra.*

(a) If $\mathfrak{g}$ is nilpotent, then every subalgebra, every quotient algebra is nilpotent.

(b) Let $Z(\mathfrak{g})$ be the center of $\mathfrak{g}$. Then if $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, $\mathfrak{g}$ is also nilpotent.

(c) If $\mathfrak{g}$ is nilpotent, its center $Z(\mathfrak{g})$ is notnrivial.

(d) If $\mathfrak{I}$ and $\mathfrak{J}$ are nilpotent ideals of $\mathfrak{g}$, then $\mathfrak{I} \cap \mathfrak{J}$ and $\mathfrak{I} + \mathfrak{J}$ also are nilpotent.

Proof.

(a) This is the same as the proof in the previous section.

(b) Suppose that $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent. There is an integer k such that $\mathcal{C}^k(\mathfrak{g}/Z(\mathfrak{g})) = 0$ so $\mathcal{C}^k\mathfrak{g}$ is contained in $Z(\mathfrak{g})$ and $\mathcal{C}^{k+1}\mathfrak{g} = \{0\}$.

(c) If g is nilpotent with a nilindex equal to k, then $\mathcal{C}^{k+1}\mathfrak{g} \neq \{0\}$ and $Z(\mathfrak{g}) \supset \mathcal{C}^{k-1}\mathfrak{g}$.

(d) Suppose that the nilindex of I is k and this one of J is 1. From the Jacobi identities, $\mathcal{C}^p(I+J)$ is included in $\mathcal{C}^{[p/2]+1}(J)$ where $[p/2]$ denotes the integer part of $p/2$. As $k-1$ is less than $\dim \mathfrak{g}$ for $p = 2\dim \mathfrak{g}$, then $\mathcal{C}^{p+1}(I+J) = \{0\}$.

Proposition 8. *Every nilpotent Lie algebra is solvable.*

In fact, $\mathcal{C}^i\mathfrak{g} \subset \mathcal{D}^i\mathfrak{g}$.

Proposition 9. *A Lie algebra* $\mathfrak{g}$ *is nilpotent if and only if there is a descending sequence of ideals*

$$\mathfrak{g} = I_0 \supset I_1 \supset \dots \supset I_k = \{0\}$$

such as $[\mathfrak{g}, I_j] \subset I_{j+1} \quad 0 \leq j \leq k-1$.

The proof is similar to the one shown in the solvable case.

Proposition 10. *A Lie algebra* $\mathfrak{g}$ *is nilpotent with a nilindex* k *if and only if for every* $x_0, x_1, \dots, x_k \in \mathfrak{g}$, *we have*

$$[x_0, [x_1, [x_2, \dots [x_{k-1}, x_k]]] \dots \dots] = 0 .$$

This follows directly from the definition.

III.4. Engel's Theorem

Let $\mathfrak{g}$ be a Lie algebra and let $x \in \mathfrak{g}$. Recall that ad x is the endomorphism of $\mathfrak{g}$ defined by $\mathrm{ad}\, x\,(y) = [x, y]$.

Theorem 2. *A Lie algebra* $\mathfrak{g}$ *is nilpotent if and only if* ad x *is nilpotent for every* x *in* $\mathfrak{g}$.

The proof is based on the following lemma (Engel's lemma) :

Lemma 1. *Let* V *be a nontrivial vector space on* K *and* $\mathfrak{g}$ *a subalgebra of* gl(V). *Suppose that every* x *in* $\mathfrak{g}$ *is a nilpotent endomorphism of* V. *Then there is* $v \neq 0, v \in V$, *such that* $x(v) = 0$, $\forall x \in \mathfrak{g}$.

Proof of the lemma. We use an induction on the dimension n of $\mathfrak{g}$. The lemma is true for $n = 0$. Suppose that it is true for a dimension less than n. We begin by showing that $\mathfrak{g}$ contains a codimension 1 ideal. Let $\mathfrak{h}$ be a subalgebra of dimension m of $\mathfrak{g}$. Let $x \in \mathfrak{h}$; it is a nilpotent endomorphism of V. This implies that the endomorphism ad x of $\mathfrak{h}$ is also nilpotent. Indeed, the terms of the decomposition of $(\mathrm{ad}\, x)^p(y)$ have the form $x^i y x^k$ with $i+k = p$. As $x^p = 0$ for some power p, ad x is also nilpotent. Now let $\sigma(x) : \mathfrak{g}/_{\mathfrak{h}} \to \mathfrak{g}/_{\mathfrak{h}}$ the endomorphism defined from ad x by projection on the quotient. As ad x is nilpotent, $\sigma(x)$ is also nilpotent. Then, by hypothesis, there is a $Y \in \mathfrak{g} / \mathfrak{h}$ such that $\sigma(x)(Y) = 0$ for all x in $\mathfrak{h}$. Let y in $\mathfrak{g}$ be a representation of Y. Then [x , y] is in $\mathfrak{h}$ for all $x \in \mathfrak{h}$ and $\mathfrak{h}$ is an ideal of a m+1-dimensional subalgebra of $\mathfrak{g}$. If we iterate this process, we find a codimension 1 ideal of $\mathfrak{g}$. Let $\mathfrak{h}$ be this ideal. We choose $a \in \mathfrak{g} - \mathfrak{h}$. The set $U = \{v \in V : x(V) = 0 \ \forall x \in \mathfrak{h}\}$ is not nul. As $x(a(v)) = [x , a](v) = 0 , \forall v \in U$. The space U is a-invariant. By the induction hypothesis, there is $u \in U$, $u \neq 0$ such as $a(u) = 0$. As $x(u) = 0 \ \forall x \in U$ and $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{C} a$, the lemma is proved.

Proof of the theorem. The necessary condition follows from Pproposition 6. For proving the sufficient condition, we use an induction. Suppose that the theorem is true for the Lie algebras of dimensions less than n. Let $\mathfrak{g}$ be an n-dimensional Lie algebra such that ad x is nilpotent for all x. From Engel's lemma, it exists that $y \in \mathfrak{g}$, $y \neq 0$ such that $[x,y] = 0 \ \forall x \in \mathfrak{g}$. This prove that the center $Z(\mathfrak{g})$ of $\mathfrak{g}$ is nontrivial. Let $\mathfrak{g}_1$ the quotient algebra $\mathfrak{g} / Z(\mathfrak{g})$, by hypothesis $\mathfrak{g}_1$ is nilpotent and from Proposition 7, $\mathfrak{g}$ is nilpotent.

Corollary 1. *Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{I}$ an ideal of $\mathfrak{g}$. If the quotient Lie algebra $\mathfrak{g} / \mathfrak{I}$ is nilpotent and if f or all x in $\mathfrak{g}$, the restriction of* ad x *to $\mathfrak{I}$ is nilpotent, then the Lie algebra $\mathfrak{g}$ is also nilpotent.*

The proof is contained in the proof of Engel's theorem.

Corollary 2. *Let $\mathfrak{g}$ be a Lie subalgebra of* gl(n,K) *whose elements are nilpotent. Then $\mathfrak{g}$ is nilpotent.*

III.5. Lie's theorem

The field K is algebraic closed and of characteristic 0.

Let $\mathfrak{g}$ be a Lie algebra on K and V a vector space on K. A *linear representation* of $\mathfrak{g}$ on V is a homomorphism of the algebras

$$\Phi : \mathfrak{g} \to gl(V),$$

i.e. is a linear map satisfying

$$\Phi[x, y]_{\mathfrak{g}} = [\Phi(x), \Phi(y)]_{gl(V)} = \Phi(x) \circ \Phi(y) - \Phi(y) \circ \Phi(x)$$

with x and y in $\mathfrak{g}$ ($[\ ,\ ]_{\mathfrak{g}}$ is the bracket in $\mathfrak{g}$). Then the vector space V is a $\mathfrak{g}$-module by putting

$$x.v = \Phi(x)(v) \qquad x \in \mathfrak{g} \ \text{ and } \ v \in V .$$

Naturally, we have an equivalence between the notions of the linear representations of $\mathfrak{g}$ and the $\mathfrak{g}$-modules.

Theorem 3 (Lie's theorem). *Let $\mathfrak{g}$ be a solvable Lie algebra on K and Φ a linear representation of $\mathfrak{g}$ in a vector space V. Then there exists a flag $D = (V = V_0 \supset V_1 \supset \ldots \supset V_n = \{0\})$ of V such that $\mathfrak{b}(D) \supset \Phi(g)$ where $\mathfrak{b}(D)$ is the Lie algebra of endomorphisms of V related with the flag D.*

This algebra $\mathfrak{b}(D)$ is described in example 4 (1.2). We can also enunciate the theorem as :

Theorem 4. *With the hypothesis of Lie's theorem, if $V \neq \{0\}$, there is a vector $v \neq 0$ in V such that*

$$\Phi(x)(v) = \lambda(x)v, \qquad \forall x \in \mathfrak{g} .$$

where λ is a linear form on $\mathfrak{g}$.

The equivalence between these theorems can be proved using an induction on the dimension of V.

Proof of Theorem 3.
First, we prove the following lemma :

Lemma. *Let $\mathfrak{g}$ be a Lie algebra on K and $\mathfrak{I}$ an ideal of $\mathfrak{g}$. Let V be a $\mathfrak{g}$-module and $v \in V$, $v \neq 0$ such that, for every $a \in \mathfrak{I}$, one has $a.v = \lambda(a).v$ where λ is a given linear form on $\mathfrak{I}$. Then $\lambda([x,a]) = 0$ $\forall\, x \in \mathfrak{g}$ and $a \in \mathfrak{I}$.*

Proof of lemma. One chooses a vector $x \in \mathfrak{g}$. Let V_i be the subspace of V generated by the vector $x.v$, $x.(x.v) = x^2.v$, ..., $x(x...(x.v)) = x^{i-1}.v$. These subspaces define the ascending sequence

$$\{0\} = V_0 \subset V_1 \subset ... \subset V_{i-1} \subset V_i .$$

Let p be the smallest integer satisfying $V_p = V_{p+1}$. The sequence (V_i) is stationary from V_p. One has $\dim V_p = p$. We shall show that $ax^i.v \equiv \lambda(a)x^i \pmod{V_i}$ for any $i \leq 0$ and a in $\mathfrak{I}$. (Recall that $x^i.v = x.x...x.v$ and $x^0.v = v$.) This relation is true for $i = 0$. Suppose that the relation is true for every i_0, $i_0 \geq i - 1 \geq 0$. Then

$$ax^i.v = a.x.x^{i-1}.v = x.a.x^{i-1}.v - [x,a]x^{i-1}.v = x.\lambda(a)x^{i-1}.v + x.v_{i-1} - \lambda[x,a]x^{i-1}.v + w_{i-1}$$

with v_{i-1} and w_{i-1} in V_{i-1}.
As $x.V_{i-1} \subset V_i$ and $\mathfrak{I}.V_i \subset V_i$, we have $x.v_{i-1} \in V_i$ and $\lambda[x,a]\, x^{i-1}.v \in V_i$.
Then

$$\begin{aligned} a\, x^i.v &\equiv x.\lambda(a)\, x^{i-1}.v \pmod{V_i} \\ &\equiv \lambda(a)\, x^i.v \pmod{V_i} . \end{aligned}$$

This proves that $ax^i.v \equiv \lambda(a)x^i.v \pmod{V_i}$. So, the matrix of $\Phi(a)$ related with a basis of V adapted to the flag of V, where $\Phi(a)(v) = a.v$, is triangular. The elements of the diagonal are equal to $\lambda(a)$. Then $\mathrm{Tr}\,(\Phi(a)) = n\lambda(a)$, and as $[a,x] \in I$, $\forall\, x \in \mathfrak{g}$, we have $\mathrm{Tr}\,\Phi[a,x] = n\lambda[a,x]$. But $\mathrm{Tr}\,\Phi[a,n] = \mathrm{Tr}(\Phi(a) \circ \Phi(x) - \Phi(x) \circ \Phi(a)) = 0$, and the linear form λ satisfies $\lambda[a,x] = 0$. This proves the lemma.

Returning to the proof of Theorem 3, one uses an induction on the dimension of $\mathfrak{g}$. If $\dim \mathfrak{g} = 0$, the theorem is true. Suppose $\dim \mathfrak{g} \neq 0$. As $\mathfrak{g}$ is solvable, we have

$$\mathcal{D}\mathfrak{g} = \mathcal{D}^1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g} \text{ and } \mathcal{D}\mathfrak{g} \neq \mathfrak{g} .$$

Let $\mathfrak{I}$ be a codimension 1 subspace of $\mathfrak{g}$ and $\mathfrak{I}$ containing $\mathcal{D}\mathfrak{g}$. It is an ideal of $\mathfrak{g}$

because $\mathfrak{I} \supset \mathcal{D}\mathfrak{g} \supset [\mathfrak{l}, \mathfrak{g}]$. From the hypothesis, there is a nontrivial vector v in V and a linear form λ on $\mathfrak{I}$ such that

$$\Phi(a)(v) = \lambda(a)v \ , \ \forall a \in \mathfrak{I} .$$

We have $W = \{ w \in V \text{ such that } \Phi(a)(v) = \lambda(a)(av) \quad \forall a \in \mathfrak{I} \}$. It is a nontrivial subspace of V. As before in the previous lemma, we have

$$\lambda[x,a] = 0 \quad \forall x \in \mathfrak{g} \text{ and } a \in \mathfrak{I} .$$

But

$$\begin{aligned} \Phi(a) \circ \Phi(x)(w) &= \Phi[a,x](w) + \Phi(x) \circ \Phi(a)(w) \\ &= \lambda[a,x]\, w + \Phi(x)\big(\lambda(a)w\big) \\ &= \lambda(a)\, \Phi(x)(w) \ , \end{aligned}$$

then $\Phi(x)(w) \in W$ and W is invariant in respect to the action of $\mathfrak{g}$. Let $x \in \mathfrak{g}$, $x \notin \mathfrak{I}$. As $\Phi(x)(W)$ is contained in W, there is an eigenvector $v_0 \neq 0$, $v_0 \notin W$. This vector is also an eigenvector for every linear operator $\Phi(y)$, $y \in Kx + I = \mathfrak{g}$. This proves the lemma.

Consequences

1. We can deduce from Lie's theorem the existence of a basis such that the matrices of the endomorphisms with respect to this basis are triangular.
2. We consider the adjoint representation of $\mathfrak{g}$. Then as before in the previous case, there is a basis of $\mathfrak{g}$ such that every adjoint operator ad x admits a matricial triangular representation with respect to this basis.

Corollary 3. *The Lie algebra $\mathfrak{g}$ is solvable if and only if the derived algebra $\mathcal{D}(\mathfrak{g})$ is nilpotent.*

Proof. We suppose that $\mathfrak{g}$ is solvable. From Lie's theorem applied to the adjoint representation, we deduce that there exists a sequence

$$\mathfrak{g} \supset \mathfrak{I}_1 \supset \mathfrak{I}_2 \supset \dots \supset \mathfrak{I}_n = \{0\} .$$

where $\mathfrak{I}_i$ are ideals of $\mathfrak{g}$. If $x \in \mathcal{D}^1\mathfrak{g}$, then ad $x(\mathfrak{I}_i)$ is contained in $\mathfrak{I}_{i+1}$. The endomorphism ad x is nilpotent. Its restriction to $\mathcal{D}^1\mathfrak{g}$ is also nilpotent. From Engel's theorem, $\mathcal{D}^1\mathfrak{g}$ is nilpotent.

III.6. Cartan Criterion for a solvable Lie algebra

Theorem 5. *Let* $\mathfrak{g}$ *be a subalgebra of* $\mathfrak{gl}(V)$, *where* V *is a finite-dimensional space over* K. *Then* $\mathfrak{g}$ *is solvable if and only if*

$$\mathrm{Tr}(xy) = 0 \quad \forall x \in \mathfrak{g} \ \textit{and} \ \forall y \in \mathcal{D}\mathfrak{g} .$$

Proof. Suppose that $\mathrm{Tr}(xy) = 0$ for all y in $\mathfrak{g}$ and x in $\mathcal{D}\mathfrak{g}$. For proving the solvability of $\mathfrak{g}$, it is sufficient to prove that $\mathcal{D}\mathfrak{g}$ is nilpotent. From Engel's lemma, we can prove that x is nilpotent in $\mathcal{D}\mathfrak{g}$. This is a consequence of the following lemma :

Lemma. *Let* $\mathfrak{g}$ *be a subalgebra of* $\mathfrak{gl}(V)$, *and* $M = \{x \in \mathfrak{gl}(V) : [x, \mathfrak{g}] \subset [\mathfrak{g}, \mathfrak{g}]\}$. *If* $x \in M$ *satisfies* $\mathrm{Tr}(xy) = 0$ *for all* y *in* M, *then* x *is nilpotent.*

Proof of the lemma. As x is an endormorphism of V, it can be written as $x = x_s + x_n$, where x_s is the semi- simple part and x_n is the nilpotent part with $[x_s, x_n] = 0$. Suppose that $(v_1, ..., v_n)$ is a basis of eigenvectors of x_s and let $\lambda_1, ..., \lambda_n$ be the corresponding eigenvalues. Let E be the Q-vector space spanned by $\lambda_1, ..., \lambda_n$. We have to prove that any linear function $f : E \to Q$ is zero. Let f be in E^*. We consider $y \in \mathfrak{gl}(V)$ such that $y(v_i) = f(\lambda_i)\, v_i$ and we note by $\{\varphi_{ij}\}$ the basis of $\mathfrak{gl}(V)$ defined from $\varphi_{ij}(v_i) = v_j$, $\varphi_{ij}(v_k) = 0, k \neq i$. Then $\mathrm{ad}\, x_s(\varphi_{ij}) = (\varphi_i - \varphi_j)\, \varphi_{ij}$ and $\mathrm{ad}\, y(\varphi_{ij}) = (f(\varphi_i) - f(\varphi_j))\, \varphi_{ij}$. From the Lagrange interpolation, we can find a polynomial $P(X)$ in $K[X]$ without a constant term such that $P(\varphi_i - \varphi_j) = f(\varphi_i) - f(\varphi_j)$ for all i, j, and $\mathrm{ad}\, y = P(\mathrm{ad}\, x_s)$. As $\mathrm{ad}\, x_s$ is the semisimple part of $\mathrm{ad}\, x$, from the Jordan decomposition of endomorphisms, we can write $\mathrm{ad}\, x_s$ as a polynomial in $\mathrm{ad}\, x$ without a constant term. Therefore, $\mathrm{ad}\, y$ is also a polynomial in $\mathrm{ad}\, x$ without a constant term. From the hypothesis, $\mathrm{ad}\, x(\mathfrak{g})$ is in $[\mathfrak{g}, \mathfrak{g}]$ and this implies that $\mathrm{ad}\, y(\mathfrak{g})$ is in $[\mathfrak{g}, \mathfrak{g}]$, thus y is in M and $\mathrm{Tr}(x.y) = 0$. So we get $\Sigma\, \lambda_i f(\lambda_i) = 0$ and $f(\Sigma\, \lambda_i f(\lambda_i)) = \Sigma\, f(\lambda_i)^2 = 0$. The number $f(\lambda_i)$ is rational, this gives $f(\lambda_i) = 0$ and $f = 0$.

We can return to the proof of the Cartan criterion.

As $\mathfrak{g} \subset M$, the lemma implies that every x in $[\mathfrak{g}, \mathfrak{g}]$ with $\mathrm{Tr}(x.y) = 0$ $\forall y \in M$ is nilpotent. But, $Tr([u, v]\ z) = Tr(u[v, z]) = Tr([v, z]u)$ for every u, v, z in $gl(V)$. In particular, if $z \in M$, $u, v \in \mathfrak{g}$, we have $[v, z] \in [\mathfrak{g}, \mathfrak{g}]$ and $Tr([v, z]\ u) = 0$. This gives $Tr([u, v]\ z) = 0$ and $Tr(x\,y) = 0 \quad \forall\, y \in M$. From the lemma, x is nilpotent this ends the proof after we have seen that the converse is a direct consequence of Lie's theorem.

Corollary. *Let* $\mathfrak{g}$ *be a Lie algebra satisfying* $Tr(\text{ad } x \text{ ad } y) = 0$ *for all* $x \in [\mathfrak{g}, \mathfrak{g}]$ *and* $y \in \mathfrak{g}$. *Then* $\mathfrak{g}$ *is solvable.*

IV. SEMISIMPLE LIE ALGEBRAS

The principal topics of this book concern the study of nilpotent Lie algebras. So we shall give only a short survey of the theory of semisimple Lie algebras (the reader interested in this subject can read the classical books, for example [Se1]). The aim of this survey is to present the necessary tools in order to study some important classes of nilpotent subalgebras of semisimple complex Lie algebras.

Throughout this section and in those following (except where otherwise specified), the field K is the complex field $\mathbb{C}$.

IV.1. Semisimplicity and radical

Definition 7. *A non-Abelian Lie algebra* $\mathfrak{g}$ *is called simple if its only ideals are* $\mathfrak{g}$ *and* $\{0\}$.

Examples

1. $sl(2, \mathbb{C})$ is simple.
2. We will see later that every classical Lie algebra $sl(n, \mathbb{C})$, $so(n, \mathbb{C})$, $sp(2n, \mathbb{C})$ (see I.4) are simple.

Recall that the radical of a Lie algebra $\mathfrak{g}$ is the unique maximal solvable ideal of $\mathfrak{g}$. It is denoted by $\mathrm{rad}\,\mathfrak{g}$.

Definition 8. *The Lie algebra $\mathfrak{g}$ is called semisimple if* $\mathrm{rad}\,\mathfrak{g} = \{0\}$.

Proposition 11. *Let $\mathfrak{g}$ be a Lie algebra. Then $\mathfrak{g}/\mathrm{rad}\,\mathfrak{g}$ is semisimple.*

Corollary. *If $\mathfrak{g}$ is a semisimple Lie algebra, then it doesn't have nonzero Abelian ideals.*

IV.2. Killing form

Let $\mathfrak{g}$ be a Lie algebra. A bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is said to be invariant if we have :

$$B([x, y], z) + B(x, [y, z]) = 0 \text{ for all } x, y, z \in \mathfrak{g}.$$

For example, the bilinear form $B(x,y) = \mathrm{Tr}(\mathrm{ad}\, x \circ \mathrm{ad}\, y)$ is invariant and symmetric. This form is called the *Killing form.*

Proposition 12. *Let $\mathfrak{I}$ be an ideal of the Lie algebra $\mathfrak{g}$. Then the orthogonal space $\mathfrak{I}^{\perp}$ of $\mathfrak{I}$ with respect to he Killing form B is an ideal of $\mathfrak{g}$.*

This results directly from the invariance of B.

Theorem 6 (Cartan Killing Criterion). *A Lie algebra $\mathfrak{g}$ is semisimple if and only if its Killing form B is nondegenerate.*

Proof. Let $\mathfrak{I}$ be the Kernel of B. It is an ideal of $\mathfrak{g}$. Suppose that $\mathfrak{g}$ is semisimple. If $x \in \mathfrak{I}$, then $B(x, y) = \mathrm{Tr}(\mathrm{ad}\, x \circ \mathrm{ad}\, y) = 0$ for all y in $\mathfrak{g}$, hence, in particular, if y is in $\mathfrak{D}\,\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. By Cartan's criterion, the subalgebra $\mathrm{ad}\,\mathfrak{I}$ of $\mathfrak{gl}(\mathfrak{g})$ is solvable. Since $\mathrm{ad}\,\mathfrak{I} = \mathfrak{I}/Z(\mathfrak{g})$, then $\mathfrak{I}$ is also solvable. As $\mathfrak{g}$ is semisimple, $\mathfrak{I} = \{0\}$.
Conversely. Suppose B is nondegenerate; we take an Abelian ideal $\mathfrak{a}$ of $\mathfrak{g}$. We

consider $f = \mathrm{ad}\, x \circ \mathrm{ad}\, y$ for $x \in \mathfrak{a}$ and $y \in \mathfrak{g}$. Then $\mathfrak{a} \supset f(\mathfrak{g})$ and $f(\mathfrak{a}) = \{0\}$, this gives $f^2 = 0$ and $\mathrm{Tr}(f) = 0$. This proves that $\mathfrak{a} \subset \mathfrak{l} = \mathrm{Ker}\, B = \{0\}$.

Corollary 1. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{l}$ an ideal of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{l}^{\perp}$ (direct sum of Lie algebras) where $\mathfrak{l}^{\perp}$ is the orthogonal of $\mathfrak{l}$ with respect to the Killing form of $\mathfrak{g}$.*

Corollary 2. *A semisimple Lie algebra is isomorphic to a direct sum of simple Lie algebras.*

Corollary 3. *If $\mathfrak{g}$ is semisimple, then $\mathfrak{g} = \mathfrak{D}\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.*

Proposition 13. *Every derivation of a semisimple Lie algebra is inner.*

Proof. Let B be the Killing form of a semisimple Lie algebra $\mathfrak{g}$. It is nondegenerated. Let f be a derivation of $\mathfrak{g}$ and Φ the linear form on $\mathfrak{g}$ defined by $\Phi(x) = \mathrm{tr}(\mathrm{ad}\, x_0\, f)$. There is a vector u in $\mathfrak{g}$ such that $B(u, x) = \Phi(x)$ for all x in $\mathfrak{g}$. We put $g = f - \mathrm{ad}\, u$. It is a derivation of g and we have

$$
\begin{aligned}
\mathrm{tr}(\mathrm{ad}\, x \circ g) &= \mathrm{tr}(\mathrm{ad}\, x \circ f) - \mathrm{tr}(\mathrm{ad}\, x \circ \mathrm{ad}\, u) \\
&= \Phi(x) - B(x, u) = 0 \quad \text{for all } x \text{ in } \mathfrak{g}
\end{aligned}
$$

Then
$$
\begin{aligned}
B(g(x), y) &= \mathrm{tr}(\mathrm{ad}\, g(x) \circ \mathrm{ad}\, y) \\
&= \mathrm{tr}([g, \mathrm{ad}\, x] \circ \mathrm{ad}\, y) \\
&= \mathrm{tr}(g \circ \mathrm{ad}\, x \circ \mathrm{ad}\, y - \mathrm{ad}\, x \circ g \circ \mathrm{ad}\, y) \\
&= \mathrm{tr}(g \circ \mathrm{ad}\, x \circ \mathrm{ad}\, y - g \circ \mathrm{ad}\, y \circ \mathrm{ad}\, x) \\
&= \mathrm{tr}(g \circ [\mathrm{ad}\, x, \mathrm{ad}\, y]) \\
&= \mathrm{tr}(g \circ \mathrm{ad}[x, y]) = 0,
\end{aligned}
$$

Hence, $g(x) = 0$ for all x in $\mathfrak{g}$ and $f = \mathrm{ad}\, u$. This proves the proposition.

IV.3. Complete reducibility of representations

First, we recall the notions of representations of Lie algebra $\mathfrak{g}$ and the corresponding notion of $\mathfrak{g}$ -modules.

Let $\mathfrak{g}$ be a Lie algebra. A *representation* of $\mathfrak{g}$ in a complex vector space V is a homomorphism of Lie algebras $\Phi : \mathfrak{g} \to gl(V)$. A vector space V endowed with an operation $\mathfrak{g} \times V \to V$ (denoted x . v) is called a $\mathfrak{g}$*-module* if we have

(i) $(ax + by)v = a(xv) + b(yv)$, $a, b \in \mathbb{C}$, $x, y \in \mathfrak{g}$, $v \in V$,

(ii) $x(av + bw) = a(xv) + b(xw)$, $a, b \in \mathbb{C}$, $x, y \in \mathfrak{g}$, $v \in V$,

(iii) $[x, y]v = xyv - yxv$, $x, y \in \mathfrak{g}$, $v \in V$.

The notions of the representations of $\mathfrak{g}$ in V and $\mathfrak{g}$-modules are equivalent: if Φ is a representation of $\mathfrak{g}$ in V, then V is a $\mathfrak{g}$-module for the operator $x.v = \Phi(x)(v)$; conversely, if V is a $\mathfrak{g}$-module, then the map $\Phi : \mathfrak{g} \to gl(V)$ given by $\Phi(x)(v) = xv$ is a representation of $\mathfrak{g}$ in V.

Definition 9. *A* $\mathfrak{g}$*-module* V *is called simple if* $V \neq \{0\}$ *and* V *has no submodules other than* $\{0\}$ *and* V *which is called semisimple if it is the direct sum of simple submodules.*

In the language of representations, these notions correspond to *irreducible* and *completely reducible* representations.

Remark. $\mathfrak{g}$ may be considered as a $\mathfrak{g}$-module for the operation $xy = [x, y]$. Note that $\mathfrak{g}$ may be a semisimple $\mathfrak{g}$-module without being a semisimple Lie algebra; for example in the case $\mathfrak{g} = \mathbb{C}$.

Theorem 7 (H. Weyl). *Let* $\mathfrak{g}$ *be a semisimple Lie algebra, and* V *a finite-dimensional* $\mathfrak{g}$*-module. Then* V *is semisimple.*

Proof. See, for example, [Se 1] or [Hu].

IV.4. Reductive Lie algebras

We can define the nilradical of the Lie algebra $\mathfrak{g}$ as the largest nilpotent ideal of $\mathfrak{g}$. We can also define the notion of the nilpotent radical of $\mathfrak{g}$ as the intersection of the kernels of the finite dimension irreducible representations of $\mathfrak{g}$. This ideal is contained in the nilradical and we can prove that it coincides with $\mathfrak{D}\mathfrak{g} \cap \operatorname{rad} \mathfrak{g}$.

Definition 10. *A Lie algebra is called reductive if its adjoint representation is completely reducible.*

Proposition 14. *Let $\mathfrak{g}$ be a Lie algebra. The following conditions are equivalent:*

(a) $\mathfrak{g}$ *is reductive,*

(b) $\mathfrak{D}\mathfrak{g}$ *is semisimple,*

(c) $\mathfrak{g} = \mathfrak{D}\mathfrak{g} \oplus Z(\mathfrak{g})$,

(d) $\operatorname{rad} \mathfrak{g} = Z(\mathfrak{g})$,

(e) *the nilpotent radical of* $\mathfrak{g}$ *is* $\{0\}$.

IV.5 Levi's Theorem

Theorem 8 (Levi). *Every Lie algebra* $\mathfrak{g}$ *is the semi-direct sum of its radical* $\operatorname{rad} \mathfrak{g}$ *and a semisimple subalgebra* s :

$$\mathfrak{g} = \operatorname{rad} \mathfrak{g} \oplus s .$$

The proof of this theorem follows Bourbaki [Bou 1].

Proof.

1st case. : $[\mathfrak{g}, \operatorname{rad} \mathfrak{g}] = \{0\}$. Then $\operatorname{rad} \mathfrak{g}$ is the center of $\mathfrak{g}$ and the adjoint representation of $\mathfrak{g}$ is semisimple (we can identify this representation with the representation of $\mathfrak{g}/Z(\mathfrak{g}) = \mathfrak{g}/\operatorname{rad} \mathfrak{g}$ which is semisimple). Then $\mathfrak{D}\mathfrak{g}$ is also semisimple and $\mathfrak{g} = \operatorname{rad} \mathfrak{g} \oplus \mathfrak{D}\mathfrak{g}$.

2nd case : $[\mathfrak{g}, \operatorname{rad} \mathfrak{g}] \neq \{0\}$ and the only ideals of $\mathfrak{g}$ contained in $\operatorname{rad} \mathfrak{g}$ are $\{0\}$ and $\operatorname{rad} \mathfrak{g}$. Then $[\mathfrak{g}, \operatorname{rad} \mathfrak{g}] = \operatorname{rad} \mathfrak{g}$, $[\operatorname{rad} \mathfrak{g}, \operatorname{rad} \mathfrak{g}] = \{0\}$ and the center of $\mathfrak{g}$ is $\{0\}$.
Let M be defined by $M = \left\{f \in \operatorname{End} \mathfrak{g} : f(\mathfrak{g}) \subset \operatorname{rad} \mathfrak{g} \,;\, f(x) = \lambda(f)x \ \ \lambda(f) \neq 0 \ \ \forall x \in \operatorname{rad} \mathfrak{g}\right\}$

and $N = \{f \in \mathrm{End}\,\mathfrak{g} : f(\mathfrak{g}) \subset \mathrm{rad}\,\mathfrak{g} \text{ and } f(\mathrm{rad}\,\mathfrak{g}) = 0\}$.

Therefore, N is of codimension 1 in M. Let σ be the representation of $\mathfrak{g}$ in End $\mathfrak{g}$ given by

$$\sigma(x)\,f = [ad\,x, f] \quad \forall\, x \in \mathfrak{g},\ \forall\, f \in \mathrm{End}\,\mathfrak{g} \ .$$

We have $\sigma(x)\,M \subset N$, for all x in $\mathfrak{g}$. Moreover, if $x \in \mathrm{rad}\,\mathfrak{g}$, $y \in \mathfrak{g}$ and $f \in M$ then

$$(\sigma(x)\,f)\,(y) = -\,\lambda(f)\,[x, y] \quad \text{(because rad } \mathfrak{g} \text{ is Abelian)}$$

$$= -\,\mathrm{ad}(\lambda(f)\,x)\,(y)\ .$$

As the center of $\mathfrak{g}$ is $\{0\}$, the mapping $x \to \mathrm{ad}\,x$ defines a bijection φ of rad $\mathfrak{g}$ onto a subspace P of End $\mathfrak{g}$, stable under $\sigma(\mathfrak{g})$. As rad $\mathfrak{g}$ is Abelian, $P \subset N$ and $\sigma(x)M \subset P$ for all x in rad $\mathfrak{g}$. The representation of $\mathfrak{g}$ on $V = M/P$ derived from σ is zero on rad $\mathfrak{g}$. It defines a representation σ_1 of the semisimple algebra $\mathfrak{g}/\mathrm{rad}\,\mathfrak{g}$ on V. For all $y \in \mathfrak{g}/\mathrm{rad}\,\mathfrak{g}$ $\sigma'(y)\,(V) \subset N/P$. As $\dim V - \dim N/P = 1$, there exists $f_0 \in M$ with $\lambda(f_0) = -1$ and $\sigma(x)\,f_0 \in P$ for all x in $\mathfrak{g}$. The mapping $x \to \varphi^{-1}(\sigma(x)\,f_0)$ is linear

$\sigma(x)\,f_0 \in P$ for all x in $\mathfrak{g}$. The mapping $x \to \varphi^{-1}(\sigma(x)\,f_0)$ is a linear mapping of $\mathfrak{g}$ into

mapping of $\mathfrak{g}$ into rad $\mathfrak{g}$. Then its restriction on rad $\mathfrak{g}$ is the identity mapping. Hence the Kernel s of this mapping satisfies $\mathfrak{g} = \mathrm{rad}\,\mathfrak{g} + s$ (vectorial direct sum). As $s = \{x \in \mathfrak{g} : \sigma(x)\,f_0 = 0\}$, s is a subalgebra of $\mathfrak{g}$.

General case. We process by induction on the dimension n of rad $\mathfrak{g}$.

If $n = 0$, the theorem is clear. Suppose the theorem is true for all Lie algebras whose radicals have a dimension less than dim rad $\mathfrak{g}$. Following the first case, it suffices to consider the case where $[\mathfrak{g}, \mathrm{rad}\,\mathfrak{g}] \neq \{0\}$. But $[\mathfrak{g}, \mathrm{rad}\,\mathfrak{g}]$ corresponds to the nilradical of $\mathfrak{g}$, it is nilpotent, and its center c is nontrivial. We choose a minimal nonzero ideal $\mathfrak{m}$ of $\mathfrak{g}$ contained in c. The case $\mathfrak{m} = \mathrm{rad}\,\mathfrak{g}$ corresponds to the 2nd case. We suppose $\mathfrak{m} \neq \mathrm{rad}\,\mathfrak{g}$ and consider $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{m} = \mathfrak{g}$ the canonical mapping. Then $\mathrm{rad}\,\mathfrak{g}_1 = \mathrm{rad}\,\mathfrak{g}/\mathfrak{m}$. By the induction hypothesis, $\mathfrak{g}_1 = \mathrm{rad}\,\mathfrak{g}_1 \oplus s_1$ where s_1 is semisimple.

Let $\mathfrak{h}$ be $\mathfrak{h} = \pi^{-1}(s_1)$. It is a subalgebra of $\mathfrak{g}$ containing $\mathfrak{m}$, such that $\mathfrak{h}/\mathfrak{m} = s_1$ is semisimple. Its radical is $\mathfrak{m}$ and by the induction hypothesis, $\mathfrak{h} = \mathfrak{m} \oplus s$, where s is semisimple. This implies $\mathfrak{g} = \mathrm{rad}\,\mathfrak{g} \oplus \mathfrak{h} = \mathrm{rad}\,\mathfrak{g} + \mathfrak{m} + s = \mathrm{rad}\,\mathfrak{g} \oplus s$. This gives the

Levi decomposition.

We call the semisimple Lie subalgebra given in the Levi's theorem a *Levi subalgebra of* $\mathfrak{g}$.

Remarks. 1. If s_1 and s_2 are two semisimple subalgebras of $\mathfrak{g}$ such that

$$\operatorname{rad}\mathfrak{g} \oplus s_1 = \operatorname{rad}\mathfrak{g} \oplus s_2 = \mathfrak{g},$$

then there exists an automorphism σ of $\mathfrak{g}$ such that $\sigma(s_1) = s_2$. Moreover, we can choose σ of the form $\sigma = \exp(\operatorname{ad} x)$, where x in the nilradical of $\mathfrak{g}$. This result is due to Malcev.

2. From Levi's theorem, the general description of Lie algebras is deduced from the description of semisimple Lie algebras, solvable Lie algebras, and the representation of semisimple Lie algebras as derivations of the radical. The class of semisimple Lie algebras is well known (see later). But the class of solvable and nilpotent Lie algebras is very difficult to describe. Moreover, solvable Lie algebras exist that are never radical of an unsolvable Lie algebra with a nontrivial representation of the Levi subalgebra.

Example. Let $\mathfrak{g}$ be the 7-dimension nilpotent Lie algebra defined by

$$\begin{aligned}
&[e_1, e_i] = e_{i+1}, \qquad i = 2,\dots,6,\\
&[e_2, e_3] = e_7,\\
&[e_3, e_4] = -e_7,\\
&[e_2, e_4] = e_6,\\
&[e_2, e_3] = e_6 + e_7.
\end{aligned}$$

We shall see that every derivation of $\mathfrak{g}$ is nilpotent. Then Der $\mathfrak{g}$ does not contain a semisimple subalgebra.

V. ON THE CLASSIFICATION OF COMPLEX SEMISIMPLE LIE ALGEBRAS

By classification, in this book we mean the classification up to an isomorphism of Lie algebras. In this section we resume the classical work of Killing and Cartan about the classification of complex semisimple Lie algebras. Recall that the ground field is $\mathbb{C}$ and that the Lie algebras considered are finite-dimensional.

V.1. Regular elements. Cartan subalgebras

Let $\mathfrak{g}$ be a semisimple Lie algebra and $x \in \mathfrak{g}$, $x \neq 0$. We let $\mathfrak{g}^\lambda{}_x$ denote the nilspace of $\operatorname{ad} x - \lambda \operatorname{Id}$; that is

$$\mathfrak{g}^\lambda{}_x = \{y \in \mathfrak{g}\,;\, (\operatorname{ad} x - \lambda \operatorname{Id})^p(y) = 0 \quad \text{for some } p\}\ ,$$

where $\lambda \in \mathbb{C}$.

Definition 11. *The element* $x \in \mathfrak{g}$ *is called regular if the dimension of the space* $\mathfrak{g}^0{}_x$ *is minimal*:

$$\dim \mathfrak{g}^0{}_x = \min_{y \in \mathfrak{g}} \left(\dim \mathfrak{g}^0{}_y\right).$$

We note that $\left[\mathfrak{g}^\lambda{}_x, \mathfrak{g}^\mu{}_x\right] \subset \mathfrak{g}_x^{\lambda+\mu}$ and it follows that $\mathfrak{g}^0{}_x$ is a subalgebra of $\mathfrak{g}$.

Proposition 15. *If* x *is regular, then* $\mathfrak{g}^0{}_x$ *is Abelian and is equal to its normalizer in* $\mathfrak{g}$.

Recall that the normalizer of $\mathfrak{g}^0{}_x$ in $\mathfrak{g}$ is the set of the elements y of $\mathfrak{g}$ such that $[y\,, \mathfrak{g}^0{}_x] \subset \mathfrak{g}^0{}_x$.

Definition 12. *A Lie subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}$ *is called a Cartan subalgebra of* $\mathfrak{g}$ *if there is a regular element* x *in* $\mathfrak{g}$ *such that* $\mathfrak{h} = \mathfrak{g}^0{}_x$.

We can note that there always exists a nontrivial Cartan subalgebras for every semisimple Lie algebra. Moreover, all the Cartan subalgebras of a semisimple Lie algebra $\mathfrak{g}$ are isomorphic. They are also conjugated in the following sense : let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be two Cartan subalgebras ; there is an automorphism σ of $\mathfrak{g}$ of the form exp(ad y) with $y \in \mathfrak{g}$ such that $\mathfrak{h}_2 = \sigma(\mathfrak{h}_1)$.

Consequences

1. All the Cartan subalgebras of $\mathfrak{g}$ have the same dimension r. This dimension is called *the rank of* $\mathfrak{g}$.

2. A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a maximal Abelian subalgebra of $\mathfrak{g}$. Every element of $\mathfrak{h}$ is semisimple, i.e. the endomorphisms ad x for all x in $\mathfrak{h}$ are semisimple. As these endomorphisms commute, it can be diagonalize together.

3. A maximal Abelian subalgebra of $\mathfrak{g}$ is not usually a Cartan subalgebra of $\mathfrak{g}$. However, a maximal Abelian subalgebra of $\mathfrak{g}$ whose elements are semisimple, is a Cartan subalgebra.

V.2. Root systems

Throughout this section, $\mathfrak{g}$ denotes a complex semisimple Lie algebra and $\mathfrak{h}$ a fixed Cartan subalgebra of $\mathfrak{g}$. A *root* α of $\mathfrak{g}$ (relative to $\mathfrak{h}$) is a *nonzero* element $\alpha \in \mathfrak{h}^*$ (the dual of $\mathfrak{h}$) such that the space $\mathfrak{g}^\alpha$ defined by

$$\mathfrak{g}^\alpha = \{x \in \mathfrak{g} : [x, h] = \alpha(h)\, x \quad \forall\, h \in \mathfrak{h}\}$$

is not $\{0\}$.

If α is a root, we call $\mathfrak{g}^\alpha$ a *eigensubspace of* $\mathfrak{g}$, and an element of $\mathfrak{g}^\alpha$ is said to have a *weight* α.

We have :

1. $\mathfrak{g}^0 = \mathfrak{h}$.

2. $\left[\mathfrak{g}^\alpha, \mathfrak{g}^\beta\right] \subset \mathfrak{g}^{\alpha+\beta}$ if $\alpha + \beta$ is a root,

   $\left[\mathfrak{g}^\alpha, \mathfrak{g}^\beta\right] = \{0\}$ if not.

3. Let Δ be the set of the roots of $\mathfrak{g}$. Then

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}^{\alpha} \quad \text{(vectorial direct sum).}$$

4. $\operatorname{Dim} \mathfrak{g}^{\alpha} = 1$ for all α in Δ.

5. If B is the Killing form of $\mathfrak{g}$, then $B(\mathfrak{g}^{\alpha}, \mathfrak{g}^{\beta}) = 0$ when $\alpha + \beta \neq 0$.
6. The Killing form restricted to $\mathfrak{h}$ is nondegenerated. This last property implies that for every α in Δ, there is a vector H_{α} in $\mathfrak{h}$ such that $B(H, H_{\alpha}) = \alpha(H)$ for all H in $\mathfrak{h}$.

7. If $\alpha \in \Delta$, then $-\alpha \in \Delta$ and we have
$[\mathfrak{g}^{\alpha}, \mathfrak{g}^{-\alpha}] = \mathbb{C}H_{\alpha}$, $\alpha(H_{\alpha}) \neq 0$.

V.3. An order relation on the set of roots Δ

Let r be the rank of the semisimple Lie algebra $\mathfrak{g}$. It is possible to find a basis $(\alpha_1, ..., \alpha_r)$ of $\mathfrak{h}^*$ such that

1. $\alpha_i \in \Delta$, $i = 1, ..., r$,
2. Every root α can be decomposed as a sum
$\alpha = k_1 \alpha_1 + k_2 \alpha_2 + ... + k_r \alpha_r$,

and the coefficients k_i are all positive integers or all negative integers.

A root system $S = (\alpha_1, ..., \alpha_r)$ satisfying the previous conditions is called *a system of simple roots*.

Now, we fix a system of simple roots. The elements of this system are called *simple roots*. A root α is called positive (resp. negative) if the coefficients of α in decomposition in the simple roots are positive (resp. negative).

We denote Δ_+ the set of positive roots, and Δ_- the set of negative roots. Then:

$$\Delta = \Delta_+ \cup \Delta_- , \ \Delta_- = -\Delta_+ .$$

From a fixed simple roots system $S = (\alpha_1, ..., \alpha_r)$, we can provide $\mathfrak{h}^*$ with a partial relation by stating that $\gamma > \nu \Leftrightarrow \gamma - \nu$ has a decomposition as a sum of simple roots α_i with positive coefficients. Of course, Δ inherits this partial order relation.

V.4. Weyl Bases

We keep the notation of the preceding section. We construct a basis of $\mathfrak{g}$ adapted to the decomposition in eigensubspaces

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta^+} \mathfrak{g}^\alpha \right) \oplus \left(\bigoplus_{\alpha \in \Delta^-} \mathfrak{g}^\alpha \right)$$

and to the chosen system of simple roots $S = (\alpha_1, ..., \alpha_r)$.

For each i, we put $H_i = H_{\alpha_i}$ and we choose elements $X_i \in \mathfrak{g}^{\alpha_i}$, $Y_i \in \mathfrak{g}^{-\alpha_i}$ such that $[X_i, Y_i] = H_i$. Finally, we put $n(i, j) = \alpha_i(H_j)$. The matrix formed by the numbers $n(i,j)$ is called the *Cartan matrix* of the given system S. We note that $n(i, j)$ is an integer ≤ 0 if $i \neq j$ and $n(i, i) = 2$.

Theorem 9. *The vectors* (H_i, X_i, Y_i) $i = 1 ,..., r$ *satisfy the following conditions*

1. $\mathfrak{n}^+ = \sum_{\alpha \in \Delta^+} \mathfrak{g}^\alpha$ *is generated by the vectors* X_i ,

   $\mathfrak{n}^- = \sum_{\alpha \in \Delta^-} \mathfrak{g}^\alpha$ *is generated by the vectors* Y_i ,

   $\mathfrak{g}$ *is generated by* (X_i, Y_i, H_i) ,

2. $[H_i, H_j] = 0 \, ; [H_i, X_j] = n(i,j)\, X_j \, ; [H_i, Y_j] = -\, n(i,j)\, Y_j$,

   $[X_i, Y_i] = H_i \, ; [X_i, Y_j] = 0$ if $i \neq j$,

3. $(\mathrm{ad}\, X_i)^{-n(i,j)+1} (X_j) = 0$, $i \neq j$,

   $(\mathrm{ad}\, Y_i)^{-n(i,j)+1} (Y_j) = 0$, $i \neq j$.

The Weyl basis is a basis of $\mathfrak{g}$ defined by the vectors (X_i, Y_i, H_i) and the above relations. Its vectors are in $\mathfrak{h}$ or in the eigensubspace $\mathfrak{g}^{\alpha}$. If X_α is the corresponding vector in the Weyl basis, we have

$$[X_\alpha, X_\beta] = 0 \text{ if } \alpha + \beta \notin \Delta,$$

$$[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta} \text{ if } \alpha + \beta \in \Delta,$$

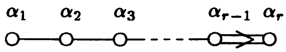
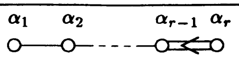
with $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$ and $(N_{\alpha,\beta})^2 = q(1-p)\alpha(H_\alpha)/2$ where p and q are defined as this : $\beta + n\alpha$ is a root for each n, $p \leq n \leq q$ and $\beta + k\alpha$ is not in Δ if $k \notin [p, q]$.

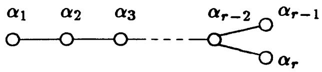
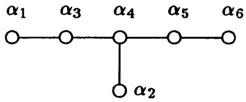
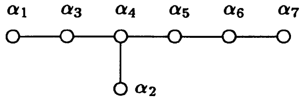
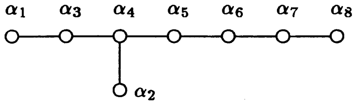
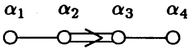
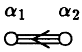
V.5. On the classification of complex simple Lie algebras.

Based on particular properties of the roots, the following properties are concentrated in the Dynkin diagram.

Let $S = (\alpha_1, ..., \alpha_r)$ be a simple roots system. These roots will be the vertices of a graph. Two vertices will be formed by $n(i, j) . n(j, i)$ edges. These integer numbers are equal to 0, 1, 2 or 3. We put the inequality sign > on the multiple edge to indicate which of the two adjacent roots is the longer; the absence of an inequality sign means that two adjacent roots have the same length. The length of a root α_i is given by $< \alpha_i, \alpha_i >$ where < , > is the restricted Killing form to $\mathfrak{h}^*$. Thus, the graph labelled is called the Dynkin diagram.

Theorem 10. *All the Dynkin diagrams of the simple complex Lie algebras are the following*:

Type of algebra	Dynkin diagram
A_r $(r \geq 1)$	
B_n $(n \geq 3)$	
C_n $(n \geq 2)$	

$D_n\ (n \geq 4)$	
E_6	
E_7	
E_8	
F_4	
G_2	

Remark. Indeed the Dynkin diagrams A_n ,..., E_8 correspond to roots systems and therefore to simple complex Lie algebras. We note that two isomorphic simple Lie algebras admit the same diagram and, conversely, two simple Lie algebras having the same Dynkin diagram are isomorphic. Hence, the classification of complex simple Lie algebras is deduced from theorem 9. It is sufficient to give the corresponding Lie algebra for each diagram :

- type A_r : $sl(r+1,\mathbb{C})$,
- type B_r : $so(2r+1,\mathbb{C})$, $r \geq 2$,
- type C_r : $sp(2r)$, $r \geq 3$,
- type D_r : $so(2r,\mathbb{C})$, $r \geq 4$.

These algebras are described in Section I.4. The diagrams E_6, E_7, E_8, F_4 and G_2 correspond to the *exceptional simple Lie algebras.* These algebra admit also matricial representations. For example, the simple Lie algebra G_2 may be presented as a derivation algebra of a Cayley algebra.

VI. THE NILRADICAL

VI.1. Definition

Theorem 11. *Let* $\mathfrak{g}$ *be a Lie algebra and* $\mathfrak{r} = \operatorname{rad}\mathfrak{g}$ *its radical. The following four sets are identical:*

(a) *the largest nilpotent ideal of* $\mathfrak{g}$,

(b) *the largest nilpotent ideal of* $\operatorname{rad}\mathfrak{g}$,

(c) *the set of* $x \in \operatorname{rad}\mathfrak{g}$ *such that* $\operatorname{ad}_{\mathfrak{g}} x$ *is nilpotent,*

(d) *the set of* $x \in \operatorname{rad}\mathfrak{g}$ *such that* $\operatorname{ad}_{\mathfrak{r}} x$ *is nilpotent.*

The ideal defined in this theorem is called the *nilradical of* $\mathfrak{g}$ and denoted $\operatorname{nil}\mathfrak{g}$.

Proposition 16. *Let* $\mathfrak{g}$ *be a Lie algebra and* f *a derivation of* $\mathfrak{g}$. *Then:*

$$f(\mathfrak{g}) \cap \operatorname{rad}\mathfrak{g} \subset \operatorname{nil}\mathfrak{g} .$$

This proposition is based on the following relation:

$$\operatorname{ad}(f(x)) = [f, \operatorname{ad} x] \in f(\operatorname{Der}\mathfrak{g}) \cap \operatorname{rad}(\operatorname{Der}\mathfrak{g}) \quad \text{for all } x \text{ in } \mathfrak{g}.$$

Corollary

1. *The nilpotent radical* $\mathcal{D}\mathfrak{g} \cap \operatorname{rad}\mathfrak{g}$ *of* $\mathfrak{g}$ *coincides with* $[\operatorname{rad}\mathfrak{g}, \mathfrak{g}]$, *hence it is included in* $\operatorname{nil}\mathfrak{g}$.
2. *For every* f *of* $\operatorname{Der}\mathfrak{g}$, $f(\operatorname{rad} g) \subset \operatorname{nil}\mathfrak{g}$.

VI.2. On the algebraic Lie algebras

Algebraic Lie algebras are the Lie algebras of algebraic Lie groups (see [Bo]). They constitute an interesting class of Lie algebras containing, in particular, semisimple Lie algebras and nilpotent Lie algebras. The algebra of all derivations of a Lie algebra is also algebraic. For these Lie algebras, we have a similar decomposition to Levi's decomposition.

Theorem 12. *Let $\mathfrak{g}$ be an algebraic Lie algebra. Then there is a reductive subalgebra $\mathfrak{a}$ such that* $\mathfrak{g} = \mathrm{nil}\,\mathfrak{g} \oplus \mathfrak{a}$ *(semidirect sum).*

VII. THE CLASSICAL INVARIANTS OF NILPOTENT LIE ALGEBRAS

The classification up to isomorphism of the Lie algebras is fundamental and a very difficult problem. It is probably one of the first problems that we encounte when understanding the structure of a set of Lie algebras. In the following sections, we shall give an algebraic structure to this set in order to present a general survey on the behaviour of Lie algebras. From the geometrical point of view, the classification of Lie algebras corresponds to a fibration of this set, the fiber being the isomorphic classes. We have seen that the classification of semisimple complex Lie algebras has been known ever since the works of Cartan and Killing. Modulo the study of derivations of solvable Lie algebras, the classification of the Lie algebras may be reduced by the Levi theorem to the classification of solvable Lie algebras and, in particular, of nilpotent Lie algebras. This shows that the study of nilpotent and solvable Lie algebras is very important. But it is very difficult to distinguish two non isomorphic solvable or nilpotent Lie algebras. Then it is interesting to know some invariants of these Lie algebras which are easy to present and to compute.

VII.1. The dimension of characteristic ideals and the nilindex

Let $\mathfrak{g}$ be a complex finite-dimensional nilpotent Lie algebra. Of course, dim $\mathfrak{g}$ is the simplest invariant. Then it is natural to classify the Lie algebras of a given dimension. Currently, we know the classifications of complex nilpotent Lie algebras of dimensions less than 8. These classifications are given in the next section. It is clear that the elaboration of these classifications is made using a finer invariant than the dimension of $\mathfrak{g}$.

(i) The nilindex of $\mathfrak{g}$

By definition, the nilindex of $\mathfrak{g}$ is the smallest positif integer s such that

$$\mathcal{C}^s \mathfrak{g} = \{0\}.$$

In particular, if s is the nilindex of $\mathfrak{g}$, we have $(\text{ad } X)^s = 0$ for all X in $\mathfrak{g}$.
We can also consider the minimum of the integer k such that $(\text{ad } X)^k = 0$ for all X in $\mathfrak{g}$.

Examples

1. If $s = 1$, then $\mathfrak{g}$ is Abelian. It is unique for a given dimension.

2. If $s = 2$, then we say that $\mathfrak{g}$ is two-step nilpotent. The Heisenberg algebra is of this type. We note that the classification of two-step nilpotent Lie algebras is unknown for dimensions greater than 9 (see Chapter 2).

3. If $s = \dim \mathfrak{g} - 1$, we say that $\mathfrak{g}$ is filiform. The study of these algebras can also be found in the Chapter 2.

(ii) The dimension of the characteristic ideals
Consider the derived sequence, the descending central sequence, and the ascending central sequence :

$$\mathfrak{g} = \mathcal{D}^0\mathfrak{g} \supset \mathcal{D}^1\mathfrak{g} \supset \ldots \supset \mathcal{D}^p\mathfrak{g} = \{0\},$$

$$\mathfrak{g} = \mathcal{C}^0\mathfrak{g} \supset \mathcal{C}^1\mathfrak{g} \supset \ldots \supset \mathcal{C}^s\mathfrak{g} = \{0\},$$

$$\{0\} = \mathcal{C}_0\mathfrak{g} \subset \mathcal{C}_1\mathfrak{g} \subset \ldots \subset \mathcal{C}_s\mathfrak{g} = \mathfrak{g} .$$

We note $d_i = \dim \mathcal{D}^i\mathfrak{g}$, $c^i = \dim \mathcal{C}^i\mathfrak{g}$ and $c_i = \dim \mathcal{C}_i\mathfrak{g}$.
Then the sequence of the dimensions $(n, d_1, \ldots, d_{p-1}, c^1, \ldots, c^{s-1}, c_1, \ldots, c_{s-1})$ is an invariant of $\mathfrak{g}$. The classification of complex nilpotent Lie algebras of dimension 6 (and less than 6) can be obtained using only this invariant.

VII.2. The characteristic sequence

Let X be in $\mathfrak{g}$. We denote c(X) the ordered sequence of a similitude invariant of the nilpotent operator ad X, which is the ordered sequence of the dimension of the Jordan blocks of this operator. In the set of these sequences, we can use the lexicographical order :

$$(c_1, c_2, \ldots, c_s) \geq (d_1, \ldots, d_p) \Leftrightarrow \exists\, i \text{ such that } c_j = d_j \text{ for } j < i \text{ and } c_i > d_i \ .$$

Let $c(\mathfrak{g})$ be defined by $c(\mathfrak{g}) = \underset{x \in \mathfrak{g}-\mathcal{D}\mathfrak{g}}{\mathrm{Max}} \{c(X)\}$.

This sequence is an invariant of $\mathfrak{g}$ called *the characteristic sequence.*

Examples

1. $c(\mathfrak{g}) = (n-1, 1)$

There is X in $\mathfrak{g}$ such that $(\mathrm{ad}\, X)^{n-2} \neq 0$, and the Jordan form of the matrix of ad X is

$$\begin{matrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ . & . & . & & \\ . & . & . & & \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{matrix}$$

The nilindex of $\mathfrak{g}$ is equal to n–1 and $\mathfrak{g}$ is filiform.

2. $c(\mathfrak{g}) = (1, 1, ..., 1)$

For every X in $\mathfrak{g} - \mathcal{D}\mathfrak{g}$, the Jordan matrix of ad X is zero. Then $\mathfrak{g}$ is Abelian.

3. $c(\mathfrak{g}) = (2, 1, ..., 1)$

This case corresponds to the Heisenberg algebra.

The classification of 7-dimensional complex nilpotent Lie algebras can be made using only this invariant (see the next Chapter).

VII.3. The rank of a nilpotent Lie algebra

In the Lie algebra Der $\mathfrak{g}$ of the derivations of the nilpotent Lie algebra $\mathfrak{g}$, we consider the maximal Abelian subalgebras constituted from semisimple elements. From Mostow's result [Mo], these algebras are conjugate under the action of the group of the inner automorphism of $\mathfrak{g}$. These Abelian algebras are sometimes called the maximal torus of derivations. So, the dimension of the maximal torus of Der $\mathfrak{g}$ is an invariant of $\mathfrak{g}$. It is called *the rank of* $\mathfrak{g}$.

We note that if the rank of $\mathfrak{g}$ is maximal, it is equal the dimension of $\mathfrak{g} - \mathcal{D}\mathfrak{g}$.

VII.4. Other invariants

Here are some of the invariants used in this book :

(i) the dimensions of cohomological spaces (for the trivial values in a cohomology, the cohomology with coefficients $\mathfrak{g}$-module, in the adjoint module, etc.),

(ii) the characteristic of the Lie algebra of derivations,

(iii) the dimension of the center,

(iv) Dixmier's invariant. It is the dimension of a maximal orbit for the coadjoint representation (see Chapter 7).

CHAPTER 2

SOME CLASSES OF NILPOTENT LIE ALGEBRAS

This chapter is devoted to some important classes of nilpotent Lie algebras. We also consider the problem of classification for the small dimensions.

I. FILIFORM LIE ALGEBRAS

I.1. Basic Notions. Examples

Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension n. Let $\mathcal{C}^k\mathfrak{g}$ be the central descending sequence of $\mathfrak{g}$.

Definition 1. *The Lie algebra* $\mathfrak{g}$ *is called filiform if*

$$\dim \mathcal{C}^k\mathfrak{g} = n - k - 1 \quad \textit{for} \quad k \geq 1 .$$

We can give another characterization of filiform Lie algebras.

Proposition 1. *The Lie algebra* $\mathfrak{g}$ *is filiform if and only if there is a nonzero vector* X *in* $\mathfrak{g} - \mathcal{D}\mathfrak{g}$ *such that the characteristic sequence* $c(X) = (n-1, 1)$.

Proof. Recall that $c(X)$ is the ordered sequence of the dimension of Jordan blocks of ad X. If $c(X) = (n-1, 1)$, there is a Jordan basis $(X = X_1, ..., X_n)$ of ad X satisfying

$$[X_1, X_i] = X_{i-1} \text{ for } i = 3, ..., n$$

and $\mathcal{C}^i\mathfrak{g}$ is generated by $\{X_2, X_3, ..., X_{n-i}\}$ for $2 \le i$. Then the Lie algebra $\mathfrak{g}$ is filiform. Conversely, we choose a basis $\{X_0, X_1, ..., X_{n-1}\}$ corresponding to the filtration

$$\mathfrak{g} = \mathcal{C}^0(\mathfrak{g}) \supset \mathcal{C}^1(\mathfrak{g}) \supset \mathcal{C}^2(\mathfrak{g}) \supset ... \supset \mathcal{C}^{n-1}(\mathfrak{g}) .$$

We can choose this basis so as to satisfy $c(X_0) = (n-1, 1)$.

Remark. The filiform Lie algebra have a maximal nilindex. These algebras are the "less" nilpotent.

Examples

1. Let L_n be the $(n+1)$-dimensional Lie algebra defined by $[X_0, X_i] = X_{i+1}$ for $i = 1, ..., n-1$ where $(X_0, ..., X_n)$ is a basis of L_n, the undefined brackets being zero. This algebra is filiform. This is, in a certain manner, the simplest filiform Lie algebra.

2. Let Q_n be the $(n+1)$-dimensional nilpotent Lie algebra defined in the basis $(X_0, ..., X_n)$ by

$$[X_0, X_i] = X_{i+1} \quad , \quad i = 1, ..., n-1 \ ; \ [X_i, X_{n-i}] = (-1)^i X_n \quad , \quad i = 1, ..., n-1 .$$

This algebra is also filiform.

3. Let $\mathcal{R}_n$ be defined in the basis $(X_0, ..., X_n)$ by

$$[X_0, X_i] = X_{i+1} \quad , \quad i = 1, ..., n-1 \quad ; \quad [X_1, X_i] = X_{i+2} \quad , \quad i = 2, ..., n-1 .$$

It is evidently filiform.

4. Let $\mathcal{W}_n$ be the Lie algebra whose brackets, in the basis $(X_0, ..., X_n)$, are :

$$[X_0, X_i] = X_{i+1} \quad , \quad i = 1, ..., n-1 \ ,$$

$$[X_i, X_j] = \frac{6\,(i-1)\,!\,(j-1)}{(i+j)\,!} X_{i+j+1} \quad , \quad 1 \le i < j \le n-1 \quad , \qquad i+j+1 \le n .$$

This algebra is filiform. It is isomorphic to the Lie algebra defined by :

$$[Y_i, Y_j] = (j-i)\, Y_{i+j} \qquad i+j \le n+1 \quad , \quad 1 \le i < j \le (n-1)$$

which is a finite quotient of the nilpotent part of the Witt algebra.

We shall see later that there are only two, up to isomorphism, (n+1)-dimensional Lie algebras, $n \geq 11$, whose brackets satisfy the relations :

$$[X_0, X_i] = X_{i+1}, \quad i = 1, \dots, n-1$$

$$[X_i, X_j] = a_{ij} X_{i+j+1} \quad \text{with} \quad 1 \leq i < j \leq n-1 \text{ and } i+j+1 \leq n .$$

These Lie algebras are $\mathcal{R}_n$ and $\mathcal{W}_n$.

I.2. Graded filiform Lie algebras

Let $\mathfrak{g}$ be a filiform Lie algebra. It is naturally filtered by the ideals $C^{i-1}\mathfrak{g}$ of the descending sequence (one put $C^i\mathfrak{g} = \mathfrak{g}$ for $i \leq 0$). Then we can associate to a filiform Lie algebra $\mathfrak{g}$ a graded Lie algebra, noted $\operatorname{gr} \mathfrak{g}$, which is also filiform. This algebra is defined by

$$\operatorname{gr} \mathfrak{g} = \Sigma C^{i-1}\mathfrak{g} / C^i\mathfrak{g} .$$

We denote $C^{i-1}\mathfrak{g} / C^i\mathfrak{g}$ by $\mathfrak{g}_i$. Then, we have $\operatorname{gr} \mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2 + \dots + \mathfrak{g}_n$ with $\dim \mathfrak{g}_1 = 2$, $\dim \mathfrak{g}_i = 1$ for $2 \leq i \leq n$ and $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for $i+j \leq n$.

Lemma. *There is a homogeneous basis* $(X_0, X_1, \dots, X_n)$ *of* $\operatorname{gr} \mathfrak{g}$ *such that*

$$X_0, X_1 \in \mathfrak{g}_1, \quad X_i \in \mathfrak{g}_i \quad i = 2, \dots, n,$$

$$[X_0, X_i] = X_{i+1}, \quad 1 \leq i \leq n-1 \ ; \ [X_0, X_n] = 0,$$

$$[X_1, X_2] = 0 \ ; \ [X_i, X_j] = 0 \text{ if } 1 \leq i < j \text{ and } i+j \neq n,$$

$$[X_i, X_{n-i}] = (-1)^i \alpha X_n \quad \textit{with} \quad \alpha = 0 \textit{ if } n \textit{ is even} .$$

Proof. Let Y_i be a nonzero vector in $\operatorname{gr} \mathfrak{g}$ for each i, $1 \leq i \leq n$. There is $Y \in \mathfrak{g}_1$, such that

$$[Y, Y_i] = f_i(Y) Y_{i+1} ,$$

the linear map $f_i : \mathfrak{g}_i \to \mathbb{C}$ being not identically null. There is $X_0 \in \mathfrak{g}_1$ such that $f_i(X_0) \neq 0$, $\forall$ i, $1 \leq i \leq n-1$. One can choose the vectors $X_i \in \mathfrak{g}_i$, with $X_i = \lambda_i Y_i$, $\lambda_i \neq 0$, so that they satisfy the conditions

$$[X_0, X_i] = X_{i+1}, \ 1 \leq i \leq n-1, \ [X_0, X_n] = 0.$$

As we have dim $\mathfrak{g}_1 = 2$, then $[X_0, X_2] = X_3$ and $[X_1, X_2] = aX_3$. We can affirm that there exists a unique $X_1 \in \mathfrak{g}_1$ satisfying $[X_1, X_2] = 0$. The other relations are determined by induction on the dimension of $\mathfrak{g}$. We consider the basis $(X_0, ..., X_n)$ of gr $\mathfrak{g}$ defined above. Then $(X_0, X_1, ..., X_{n-1})$ is a basis of the filiform graded algebra gr $\mathfrak{g}/\mathbb{C}X_n$ satisfying the relations of the lemma. Suppose that n is odd. We have $[X_i, X_j] = 0$, if $i+j < n-1$. One put $[X_i, X_{n-i}] = \alpha_i X_n$. Jacobi's identities imply $\alpha_i = -\alpha_{i+1}$ and then $\alpha_i = (-1)^i \alpha$. This gives the sought-after basis. Suppose n even. Than one has

$$[X_i, X_{n-i-1}] = (-1)^i \alpha X_{n-1} \qquad [X_i, X_{n-i}] = \alpha_i X_n$$

and the Jacobi identities imply

$$(-1)^i \alpha = \alpha_{i+1} + \alpha_i$$

and, thus, $\alpha_i = (-1)^{2+i} (\alpha_1 + (i-1)\alpha)$. As we have $\alpha_{n/2} = 0$, we deduce

$$\alpha_1 = \left(1 - \frac{n}{2}\right)\alpha \quad \text{and} \quad \alpha_i = (-1)^{i+1}\left(i - \frac{n}{2}\right)\alpha .$$

The Jacobi identities for the vectors $(X_1, X_{(n/2)-1}, X_{n-2})$ implies $\alpha = 0$ and $\alpha_i = 0$.

Corollary. *If n is odd, then there are only, up to isomorphism, two (n+1)-dimensional graded filiform Lie algebras. These algebras are* L_n *and* Q_n.

II. TWO-STEP NILPOTENT LIE ALGEBRAS

II.1. Definition and Examples

Definition 2. *A nilpotent Lie algebra* $\mathfrak{g}$ *is called two-step nilpotent if it satisfies* $C^2\mathfrak{g} = \{0\}$.

中心序列的第二项．也就是拿g跟g 做2个括积得到的结果．

Examples

1. The Abelian Lie algebra is two-step nilpotent.
2. The Heisenberg algebra H_p of dimension $2p+1$ is defined by the brackets :

$$[X_1, X_2] = [X_3, X_4] = \dots = [X_{2p-1}, X_{2p}] = X_{2p+1}$$

This Lie algebra is two-step nilpotent. It is, in a way, a model within these two-step nilpotent Lie algebras. It is characterized by the following property :

Proposition 2. *Every Lie algebra satisfying* $Z(\mathfrak{g}) = \mathcal{C}^1(\mathfrak{g})$ *and* $\dim Z(\mathfrak{g}) = 1$ *is isomorphic to the Heisenberg algebra.*

证明显然.

II.2. On the structure of the two-step nilpotent Lie algebras

生成元的个数一定比dimV大

Let V be a vector space complementary of $\mathcal{C}^1(\mathfrak{g})$ in $\mathfrak{g}$: $\mathfrak{g} = \mathcal{C}^1(\mathfrak{g}) \oplus V$. Note that $s = \dim V = \dim \mathfrak{g}/\mathcal{C}^1\mathfrak{g}$ is the minimum number of generators of $\mathfrak{g}$. As $\mathfrak{g}$ is two-step nilpotent, the derived algebra $\mathcal{C}^1(\mathfrak{g})$ is contained in the center $Z(\mathfrak{g})$.

Let U be the subspace $U = Z(\mathfrak{g}) \cap V$. It is an Abelian ideal of $\mathfrak{g}$. Consider a vector space complementary of U in V . We have

$$\mathfrak{g} = \mathcal{C}^1(\mathfrak{g}) \oplus U \oplus W, \qquad Z(\mathfrak{g}) = \mathcal{C}^1(\mathfrak{g}) \oplus U.$$

We can deduce that $\mathfrak{h} = \mathcal{C}^1(\mathfrak{g}) \oplus W$ is a two-step nilpotent subalgebra of $\mathfrak{g}$ satisfying $Z(\mathfrak{h}) = \mathcal{C}^1(\mathfrak{g})$ and $\mathcal{C}^1(\mathfrak{h}) = \mathcal{C}^1(\mathfrak{g})$.

就是加入一个括积平凡的部分U

Then every two-step nilpotent Lie algebra is a trivial extension of a two-step nilpotent algebra whose center is the derived algebra extended by an Abelian ideal. So one can suppose that the algebra $\mathfrak{g}$ satisfies the condition : $\mathcal{C}^1(\mathfrak{g}) = Z(\mathfrak{g})$.

Consider $(X_1, \dots, X_p, Y_1, \dots, Y_{n-p})$ as a basis of $\mathfrak{g}$ adapted to the decomposition :

$$\mathfrak{g} = \mathcal{C}^1(\mathfrak{g}) \oplus V = Z(\mathfrak{g}) \oplus V \qquad \text{with} \qquad \dim \mathcal{C}^1(\mathfrak{g}) = p .$$

Obviously, we have $[X_i, X_j] = [X_i, Y_k] = 0$. Then the Lie algebra $\mathfrak{g}$ is entirely defined by the only brackets :

$$[Y_i, Y_j] = \sum_{k=1}^{p} a_{ij}^k X_k \quad , \quad 1 \le i < j \le n-p .$$

$Y \in Z(g), X \in V$

Jacobi's identities are all satisfied and the constant a^k_{ij} are free. Intrinsically this means that the structure of $\mathfrak{g}$ is defined by a surjective linear map

$$\varphi : \Lambda^2 V \to Z(\mathfrak{g}) .$$

The dual point of view permits a concrete analysis. The dual map

$$\varphi^* : Z(\mathfrak{g})^* \to \Lambda^2(U)^*$$

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is injective. The data of this mapping (and then the data of the bracket of $\mathfrak{g}$) is the same as the data of a subspace of dimension p ($p = \dim Z(\mathfrak{g})$) in $\Lambda^2 V$. If $(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_{n-p})$ is the dual basis of a given basis $(X_1, \ldots, X_p, Y_1, \ldots, Y_{n-p})$, then the structural equations of $\mathfrak{g}$ (the dual Jacobi conditions) are :

$$d\alpha_k = \sum_{1 \le i < j \le n-p} a_{ij}^k \, \beta_i \wedge \beta_j \ .$$

The bilinear forms $(\theta_1 = d\alpha_1, \ldots, \theta_k = d\alpha_k)$ are two-forms on V. They generate the subspace $\varphi^*(Z(\mathfrak{g})^*)$. It appears clearly, in this writing, that the data of the two-forms θ_i is equivalent to the data of the constants (a^k_{ij}).

II.3. On the classification of the two-step nilpotent Lie algebras

At first sight, the problem of classifying two-step nilpotent Lie algebras appears to be easy to solve. Recall that this problem was first approached by Umlauf in 1891 in his thesis. Today, it is always an open problem. At the end of this chapter, we will give the rare results concerning this problem. If we introduce a new and strong hypothesis as the two-step nilpotency, we can hope for a simplification of this problem. Indeed, this is not the case. From the previous results, we can affirm :

Proposition 3. *The classification up to isomorphism of two-step nilpotent Lie algebras is equivalent to the classification of the orbits corresponding to the action of the linear group* $GL(p,\mathbb{C})$ *in the Grassmannian of the 2-spaces in the vector space* $\Lambda^2(\mathbb{C}^p)$.

表示在GL群作用同构意义下.

Now we consider the characteristic sequence of a two-step nilpotent Lie algebra $\mathfrak{g}$. As the nilindex of $\mathfrak{g}$ is 2 (if $\mathfrak{g}$ is not Abelian), then the characteristic sequence of $\mathfrak{g}$ has the following form :

$$s(\mathfrak{g}) = (2, 2, 2, \ldots, 2, 1, 1, \ldots, 1) \ .$$

Consider the class of 2-step nilpotent whose characteristic sequence is $(2, 2, \ldots, 2, 1)$. This class determines an open set in the algebraic set of two-step nilpotent Lie algebras. Suppose that $\dim \mathfrak{g} = 2p+1$. In this case, $\dim Z(\mathfrak{g}) = p$ and the brackets of $\mathfrak{g}$ are

$$[Y_i, Y_j] = \Sigma_{k=1,p} a^k_{ij} X_k, \ 1 \leq i < j \leq p+1$$

where $(X_1, ..., X_p, Y_1, ..., Y_{p+1})$ is a basis of $\mathfrak{g}$ and $(X_1, ..., X_p)$ a basis of $Z(\mathfrak{g})$.

We can verify that the parameters a^k_{ij} are free (all the Jacobi conditions are satisfied). Then this class of algebra is parametrized by the α^k_{ij} and is isomorphic to the vector space $Alt(\mathbb{C}^p)$ of the alternating bilinear forms on $\mathbb{C}^p$ with values on $\mathbb{C}^p$. So the classification of these two-step nilpotent Lie algebras is equivalent to the classification of the bilinear forms on $\mathbb{C}^p$ with values on $\mathbb{C}^p$ and contains the classification of all the Lie algebras of dimension p. We claim that this classification is impossible to establish.

III. CHARACTERISTICALLY NILPOTENT LIE ALGEBRAS

III.1. On Jacobson's theorem

Theorem 1. *Every Lie algebra on a field of characteristic zero having a nondegenerated derivation is nilpotent.*

Comments. A derivation f of a Lie algebra $\mathfrak{g}$ is defined by

$$df(X, Y) = [f(X), Y] + [X, f(Y)] - f [X, Y] = 0$$

(the operator d is related with the Chevalley cohomology). This derivation is nondegenerated if the linear map f is nondegenerated (the determinant is not 0). For example, if $\mathfrak{g}$ is semisimple, every derivation is inner, i.e. is it can be written as $f = ad\ X$ for some X in $\mathfrak{g}$. Such a derivation admits a nontrivial kernel, then every derivation in a semisimple Lie algebra is degenerated.

In his work, Jacobson presents the following problem : Is every nilpotent Lie algebra admitting a nondegenerated derivation ?

The answer to this problem is negative.

The first example of a nilpotent algebra whose derivations are degenerated was given by Dixmier and Lister in 1957 [D.L.]. It is the following 8-dimensional Lie algebra whose brackets are

$$[X_1, X_2] = X_5\,, \qquad [X_2, X_4] = X_6\,,$$
$$[X_1, X_3] = X_6\,, \qquad [X_2, X_6] = -X_7\,,$$
$$[X_1, X_4] = X_7\,, \qquad [X_3, X_4] = -X_5\,,$$
$$[X_1, X_5] = -X_8\,, \qquad [X_3, X_5] = -X_7\,,$$
$$[X_2, X_3] = X_8\,, \qquad [X_4, X_6] = -X_8\,.$$

This example was the starting point of a study of a new class of Lie algebras called characteristically nilpotent Lie algebras.

Definition 3. *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra and* Der $\mathfrak{g}$ *its derivations algebra. The Lie algebra* $\mathfrak{g}$ *is called characteristically nilpotent if every derivation* f *in* Der $\mathfrak{g}$ *is a nilpotent endomorphism.*

III.2. Characterization of characteristically nilpotent Lie algebras

Let $\mathfrak{g}^{[1]} = \mathrm{Der}\,\mathfrak{g}(\mathfrak{g}) = \{\, Y \in \mathfrak{g}$ such that $Y = f(X), f \in \mathrm{Der}\,\mathfrak{g}\,, X \in \mathfrak{g}\,\}$ and, more generally,

$$\mathfrak{g}^{[i]} = \mathrm{Der}\,\mathfrak{g}\,\left(\mathfrak{g}^{[i-1]}\right), \quad i > 1\ .$$

The Lie algebra $\mathfrak{g}$ is characteristically nilpotent if and only if there is $k \in \mathbb{N}$ such that

$$\mathfrak{g}^{[k]} = \{0\}.$$

This sequence $\mathfrak{g}^{[i]}$ generalizes the central descending sequence ; here we use the set of all derivations instead of the set of inner derivations.

Theorem 2 [LT]. *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra of a dimension more than* 2. *Then* $\mathfrak{g}$ *is characteristically nilpotent if and only if the Lie algebra* Der $\mathfrak{g}$ *is nilpotent.*

Note that the nilpotence of Der $\mathfrak{g}$ doesn't directly imply the nilpotence of its elements, but only the nilpotence of the adjoint operators. The proof of the theorem is based on the following lemma.

Lemma. *Let* L *be a nilpotent Lie algebra such that* $L = I_1 \oplus I_2$ *where* I_1 *and* I_2

are ideals, and $\mathcal{I}_1$ *is a central ideal. Then* Der L *is nonnilpotent.*

Proof. As $\mathcal{I}_1$ is central, one has $[\mathcal{I}_1, L] = \{0\}$. Let $x \neq 0$ be in $\mathcal{I}_1$, and U such that $\mathcal{I}_1 = U \oplus \mathbb{C}\{x\}$. As $\mathcal{I}_2$ is an ideal of L, it is nilpotent. Then its center is nontrivial. Let be $y \neq 0$ in $Z(\mathcal{I}_2)$. Let f_1 and f_2 be the derivations of L defined by

$$f_1(\mathcal{I}_2) = 0 \quad , \quad f_1(x) = y \quad , \quad f_1(U) = 0,$$

$$f_2(\mathcal{I}_2) = 0 \quad , \quad f_2(x) = x \quad , \quad f_2(U) = 0 .$$

Then we have $[f_1, f_2] = f_1$ and Der $\mathfrak{g}$ is not nilpotent.

Proof of the theorem. It is obvious that if $\mathfrak{g}$ is characteristically nilpotent, then Der $\mathfrak{g}$ is nilpotent. Now we prove the converse. Suppose Der $\mathfrak{g}$ nilpotent and $\dim \mathfrak{g} > 1$. Let α be in the dual space of Der $\mathfrak{g}$, and let V_α be the subspace

$$V_\alpha = \{ X \in \mathfrak{g} \text{ such that } \exists\, m \in N \text{ with } (f-\alpha(f)\,Id)^m(X) = 0 ,\ \forall f \in \text{Der}\,\mathfrak{g} \}.$$

If $V_\alpha \neq \{0\}$, we call α a *weight of* Der $\mathfrak{g}$. Note by Ω the set of weights of Der $\mathfrak{g}$. One has

$$\mathfrak{g} = \bigoplus_{\alpha\in\Omega} V_\alpha$$

$$\text{and } \quad [V_\alpha, V_\beta] \subset V_{\alpha+\beta} \quad \text{if} \quad \alpha+\beta \in \Omega \quad \text{or} \quad [V_\alpha, V_\beta] = \{0\} \quad \text{if not} .$$

As every ad X is a derivation of $\mathfrak{g}$, we deduce that each space V_α is an ideal of $\mathfrak{g}$. One deduces :

$$[V_\alpha, V_\beta] \subset V_\alpha \cap V_\beta \cap V_{\alpha+\beta} .$$

Then, $[V_\alpha, V_\beta] = \{0\}$ when $\alpha \neq 0$ or $\beta \neq 0$.

We consider the spaces

$$I_1 = \bigoplus_{\substack{\alpha\in\Omega \\ \alpha\neq 0}} V_\alpha \qquad \text{and} \qquad I_2 = V_0 .$$

Then $\mathfrak{g}$ is a direct vectorial sum of $\mathcal{I}_1$ and $\mathcal{I}_2$, which are ideals of $\mathfrak{g}$, $\mathcal{I}_1$ being central. The previous lemma implies $\mathcal{I}_1 = \{0\}$ or $\mathcal{I}_2 = \{0\}$.

If $\mathcal{I}_1 = \{0\}$, then $\mathfrak{g} = \mathcal{I}_2 = V_0$ and every derivation of $\mathfrak{g}$ is nilpotent.

if $\mathcal{I}_2 = \{0\}$, then $\mathfrak{g} = \mathcal{I}_1$. This implies that $\mathfrak{g}$ is Abelian and Der $\mathfrak{g} = gl(n,\mathbb{C})$.

As Der $\mathfrak{g}$ is nilpotent, we deduce that $n = 1$, and we find a contradiction.

III.3. Examples

(i) The smallest dimension where we meet an example of complex characteristically nilpotent Lie algebras is 7. This example has been given by Favre [Fa] :

$$
\begin{array}{ll}
[X_1,X_2] = X_3, & [X_1,X_6] = X_7, \\
[X_1,X_3] = X_4, & [X_2,X_3] = X_6, \\
[X_1,X_4] = X_5, & [X_2,X_4] = [X_2,X_5] = -[X_3,X_4] = X_7. \\
[X_1,X_5] = X_6, &
\end{array}
$$

Note that in dimension 7 there is one parameter family of nonisomorphic characteristically nilpotent Lie algebras. Contrary to Favre's example, these algebras are not filiform. This family is denoted by $\mathfrak{n}_7^{2,\alpha}$ in the list which will be presented in Section V.2 of this chapter.

(ii) In [Bou 1] we can read the following Lie algebra :

$$
\begin{array}{lll}
[X_1,X_2] = X_3, & [X_2,X_3] = X_5, & [X_3,X_4] = -X_7 + X_8, \\
[X_1,X_3] = X_4, & [X_2,X_4] = X_6, & [X_3,X_5] = -X_8. \\
[X_1,X_4] = X_5, & [X_2,X_5] = X_7, & \\
[X_1,X_5] = X_6, & [X_2,X_6] = 2X_8, & \\
[X_1,X_6] = X_8, & & \\
[X_1,X_7] = X_8, & &
\end{array}
$$

This algebra is characteristically nilpotent and not filiform.

III.4. Direct sum of characteristically nilpotent Lie algebras

Theorem 3. *Let* $\mathfrak{g} = \oplus_{i=1}^{p} \mathfrak{g}_i$ *be a Lie algebra where the* $\mathfrak{g}_i$ *are ideals of* $\mathfrak{g}$. *Then* $\mathfrak{g}$ *is characteristically nilpotent if and only if each ideal* $\mathfrak{g}_i$ *is characteristically nilpotent.*

Proof. We have seen, in a previous section, that

$$\mathrm{Der}\left(\bigoplus_{i=1}^{p} \mathfrak{g}_i\right) = \bigoplus_{i=1}^{p} (\mathrm{Der}\,\mathfrak{g}_i) \oplus \left(\bigoplus_{i \neq j} \mathfrak{D}(\mathfrak{g}_i,\mathfrak{g}_j)\right),$$

where

$$\mathfrak{D}(\mathfrak{g}_i, \mathfrak{g}_j) = \{ f \in \mathrm{End}\,\mathfrak{g} ,\ f(\mathfrak{g}_k) = 0 \text{ if } k \neq i,\ f(\mathfrak{g}_i) \subset Z(\mathfrak{g}_j) \text{ and } f[\mathfrak{g}_i, \mathfrak{g}_j] = 0 \}.$$

We suppose that each $\mathfrak{g}_i$ is characteristically nilpotent. In this case, we have $Z(\mathfrak{g}_i) \subset [\mathfrak{g}_i, \mathfrak{g}_i]$. Indeed, if it is not the case, there is a nonzero vector $X \in Z(\mathfrak{g})$ with $X \notin [\mathfrak{g}_i, \mathfrak{g}_i]$. If U is a complementary space of $\mathbb{C}X$ in $\mathfrak{g}_i$, we define the derivation f by $f(X) = X$ and $f(U) = 0$. This derivation is nonnilpotent. So we have $Z(\mathfrak{g}_i) \subset [\mathfrak{g}_i, \mathfrak{g}_i]$.
One deduces :

(i) if $f_i \in \mathrm{Der}\,(\mathfrak{g}_i)$ and $f_j \in \mathrm{Der}\,(\mathfrak{g}_j)$ with $i \neq j$, then $f_i \circ f_j = f_j \circ f_i$
(ii) $f_1 \circ f_2 = 0$ if $f_1 \in \mathfrak{D}(\mathfrak{g}_i, \mathfrak{g}_j)$ and $f_2 \in \mathfrak{D}(\mathfrak{g}_k, \mathfrak{g}_l)$.

As we have also $f(Z(\mathfrak{g})) \subset Z(\mathfrak{g})$ for all $f \in \mathrm{Der}\,\mathfrak{g}$, we deduce that every derivation is nilpotent.

Conversely, we suppose that $\mathfrak{g} = \bigoplus_{i=1}^{p} \mathfrak{g}_i$ is characteristically nilpotent.

Each derivation of $\mathfrak{g}_i$ can be extended naturally to a derivation of $\mathfrak{g}$. Then this derivation is nilpotent.

IV. STANDARD LIE ALGEBRAS

IV.1. Parabolic subalgebras of a semisimple algebra

Let $\mathfrak{g}$ be a semisimple complex Lie algebra.

Definition 4. *A subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ is called parabolic if it contains a Borel subalgebra of $\mathfrak{g}$.*

Recall what is a Borel subalgebra. First we fix the notation : $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, Δ the set of roots corresponding to $\mathfrak{h}$, S a basis of Δ (the simple roots), Δ_+ (resp. Δ_-) the set of positive (resp. negative) roots (recall that $\Delta = \Delta_+ \cup \Delta_-$) and

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \ / \ [X, H] = \alpha(H)X, \quad \forall H \in \mathfrak{h}\} .$$

The space $\mathfrak{g}_\alpha$ has a dimension one. We choose a nonnull vector X_α in $\mathfrak{g}_\alpha$. A *Borel subalgebra* $\mathfrak{b}$ of $\mathfrak{g}$ is a solvable maximal subalgebra of $\mathfrak{g}$. It is conjugate, up to an inner automorphism, with a subalgebra of the following type :

$$\mathfrak{b}' = \mathfrak{h} \oplus \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha .$$

The parabolic subalgebra are determined, up to an inner automorphism, by a subset S_1 of S. Let Δ_1 be the set of roots whose decomposition on S contains only elements of $S - S_1$. One denotes $\Delta_2 = \Delta - \Delta_1$, $\Delta_2{}^+ = \Delta_2 \cap \Delta_+$. Then the Lie algebra

$$\mathfrak{p} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta_2^+ \cup \Delta_1} \mathfrak{g}_\alpha$$

is parabolic and every parabolic subalgebra is conjugated with an algebra $\mathfrak{p}$.
We note that the nilradical of $\mathfrak{p}$ is

$$\mathfrak{n} = \sum_{\alpha \in \Delta_2^+} \mathfrak{g}_\alpha$$

and its reductive part is $\mathfrak{r} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta_1} \mathfrak{g}_\alpha$.

IV.2. Standard subalgebra

Definition 5. *A subalgebra of a semisimple Lie algebra is called standard if its normalizer is a parabolic subalgebra.*

These algebras were first studied by G.B. Gurevich [GU1] but in the case where the simple algebra $\mathfrak{g}$ is of type A_r. The motivation of this study is founded on the theory of complex homogeneous spaces : let M be a compact homogeneous space $M = G/H$ G being a complex Lie group and M a closed subgroup. If $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras corresponding to G and H, the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$ is parabolic. This result has been established by Tits. It permits us to translate the study of homogeneous complex spaces in the study of standard subalgebras. In order to simplify the writing, we shall call every standard subalgebra of a semisimple Lie algebra *standard algebra.*

We note that, for every standard algebra $\mathfrak{t}$ (not necessarly nilpotent) whose normalizer has the following form

$$\mathfrak{p} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta_1 \cup \Delta_2^+} \mathfrak{g}_\alpha .$$

(which is the canonical form of the parabolic algebras), one has :

$$\mathfrak{t} = \mathfrak{h}_1 \oplus \sum_{\alpha \in \Delta'} \mathfrak{g}_\alpha \quad \text{where} \quad \Delta' \subset \Delta \quad \text{and} \quad \mathfrak{h}_1 \subset \mathfrak{h} .$$

In the preceding chapter, we introduced that the partial order relation on the dual $\mathfrak{h}^*$ of the Cartan subalgebra $\mathfrak{h}$: $\omega_1 \geq \omega_2 \Leftrightarrow \omega_1 - \omega_2$ is a linear combination of simple roots with nonnegative coefficients.

Proposition 4. *Let* $\mathfrak{t}$ *be a standard algebra such that its normalizer can be expressed as* $\mathfrak{h} \oplus \sum_{\alpha \in \Delta_1 \cup \Delta_2^+} \mathfrak{g}_\alpha$. *Suppose that* α *and* β *are positive roots with* $\alpha \leq \beta$.
If the subspace $\mathfrak{g}_\alpha$ *is included in* $\mathfrak{t}$*, then* $\mathfrak{g}_\beta$ *is also in* $\mathfrak{t}$.

Proof. We consider in each subspace $\mathfrak{g}_\alpha$ a nonzero vector X_α. As $\gamma \geq \alpha$, one can write $\gamma = \alpha + \gamma_1 + ... + \gamma_k$ where the γ_i are positive roots. One can order again the terms of this sum $\gamma = \gamma_{i_1} + ... + \gamma_{i_s} + \alpha + \gamma_{i_{s+1}} + ... + \gamma_{i_k}$, such that every partially connected sum beginning with γ_{i_1} are positive roots. Each vector X_{γ_i} is in the normalizer of $\mathfrak{t}$. Then the image of the vector X_α by $\operatorname{ad} X_{\gamma_{i_k}} \circ \operatorname{ad} X_{\gamma_{i_{k-1}}} \circ ... \circ \operatorname{ad} X_{i_{\varphi+1}} \circ \operatorname{ad} X_{\gamma_{i_1} + ... + \gamma_{i_\varphi}}$ is in the space $\mathfrak{g}_\gamma$. This proves the proposition.

A description of standard algebras is presented in the general case in [KH 1]. Here we study only the nilpotent standard algebras.

IV.3. Nilpotent standard algebras

Let R be a subset of Δ_+ that consists of pairwise incomparable roots (for the partial order in $\mathfrak{h}^*$ given in the preceding section). One puts :

$$R_1 = \{\alpha \in \Delta_+ / \alpha \geq \beta \quad \text{for} \quad \beta \in R\}.$$

The subspace $\mathfrak{n} = \sum_{\alpha \in R_1} \mathfrak{g}_\alpha$ is a nilpotent subalgebra of $\mathfrak{g}$.

The normalizer of $\mathfrak{n}$ contains a Borel subalgebra. Then $\mathfrak{n}$ is a nilpotent standard subalgebra of $\mathfrak{g}$. We will say that $\mathfrak{n}$ *is the nilpotent standard subalgebra associated to* R. This process permits us to construct more easily such subalgebras. The following theorem shows that every standard algebra is of this type. So one obtains a complete description of these algebras.

Theorem 4. *Let* $\mathfrak{n}$ *be a nilpotent standard subalgebra whose normalizer has the form* $\mathfrak{h}_1 \oplus \sum_{\alpha \in \Delta_1 \cup \Delta_2^+} \mathfrak{g}_\alpha$ *. Then there is a subset* R *of* Δ_+ *whose elements are two by two incomparable and* $\mathfrak{n}$ *is the standard nilpotent subalgebra associated to* R.

Proof. As $\mathfrak{n}$ is an ideal of its normalizer, it is contained in the radical of this normalizer. But $\mathfrak{n}$ can be written as $\mathfrak{n} = \sum_{\alpha \in \Delta'} \mathfrak{g}_\alpha$ for some $\Delta' \subset \Delta$. One deduce $\Delta' \subset \Delta_2{}^+ \subset \Delta^+$. Let R be the subset of Δ' constituted of all minimal elements, always for the partial order law. These elements are pairwise incomparable. Then R defines $\mathfrak{n}$.

Corollary. *Every standard nilpotent algebra is conjugated to a nilpotent standard algebra associated to a set* R *of roots pairwise incomparable.*

IV.4. Structure of the normalizer of a nilpotent standard subalgebra

Definition 6. *A basic root* $\alpha \in S$ *is called extremal for* $\beta \in \Delta_+$ *if it satisfies* $\alpha = \beta$ *or* $\beta - \alpha \in \Delta$.

If $\beta \in \Delta_+$, then we note S^β the set of extremal roots for β.

Lemma 1. *Suppose that* $\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3$ *are roots such that* $\alpha_1 + \alpha_3$ *and* $\alpha_2 + \alpha_3$ *are nonzero. Then one of these combinations* $\alpha_1 + \alpha_3$ *or* $\alpha_2 + \alpha_3$ *is a root.*
The proof is a direct consequence of the Jacobi identity concerning $(X_{\alpha_1}, X_{\alpha_2}, X_{\alpha_3})$.

Lemma 2. *Let* α *and* β *be two positive roots such that* $\alpha + \beta \in \Delta$. *If* $\gamma \in S$ *is not extremal for* α, *and for* β, *then* γ *is not extremal for* $\alpha + \beta$.

Proof. We suppose γ to be extremal for $\alpha + \beta$. As $\gamma \in S$, then $\gamma \neq \alpha + \beta$ and $\alpha + \beta - \gamma \in \Delta$. From lemma 1, $\alpha - \gamma$ or $\beta - \gamma$ is a root and γ is extremal for α or for β. This is impossible.

Lemma 3. *Let* α *and* β *be two roots such that the decomposition of* α *as the sum of the basic roots contains all the roots of* S^β *(the set of extremal roots of* β*) with a multiplicity greater than that of* β. *Then* $\alpha \geq \beta$.

Proof. We put $\alpha = \beta + \gamma_1 + \ldots + \gamma_k - \xi_1 - \ldots - \xi_m$ with γ_i and ξ_j in S and $\gamma_i \neq \xi_j$ for every i and j. By hypothesis, the roots $\xi_1, \ldots, \xi_m$ are not in S^β. Suppose now that α is not greater (or equal to) than β (i.e. $m \geq 1$). We are going to show that $\nu = \beta + \gamma_1 + \ldots + \gamma_k$ is a root of Δ and that $\xi_1, \ldots, \xi_m$ are not in S^ν. For $k = 0$, the result is obvious. We suppose $k \geq 1$ and we make an induction on k. One has :

$$(*) \qquad (\beta,\beta) = (\beta,\alpha) + \sum_{i=1}^{k} (\beta,-\gamma_i) + \sum_{j=1}^{m} (\beta,\xi_j) > 0,$$

where $(\ ,\ )$ is the usual product on the set of the roots associated with the Killing-Cartan form.
As $(\beta,\alpha) \leq 0$ and $\sum_{j=1}^{m} (\beta,\xi_j) \leq 0$ (if not $\beta - \alpha$ and $\beta - \Sigma\,\xi_j$ are roots, this is opposite to $m \geq 1$ or to $\xi_1, \ldots, \xi_m \notin S^\beta$), from (*) there is an integer r, $1 \leq r \leq k$ such that one has $(\beta, -\gamma_r) > 0$. So $\beta + \gamma_r \in \Delta$. From Lemma 2, $\xi_1, \ldots, \xi_m \notin S^{\beta+\gamma_r}$. We apply the induction hypothesis for proving the previous assertion.
So, one has $(\nu,\nu) = (\nu,\alpha) + \sum_{i=1}^{m} (\nu,\xi_i) > 0$ and $\xi_1, \ldots, \xi_m \notin S^\nu$. One deduce $(\nu,\alpha) > 0$

and $\xi = \nu - \alpha = \xi_1 + ... + \xi_m$ is in the set Δ. We can suppose that each partial sum $\xi_1 + ... + \xi_r$ of ξ is in Δ. Let $\eta = \xi_1 + ... + \xi_{m-1}$. Then $\alpha = -\eta - \xi_m + \nu$. From Lemma 2, one has $\nu - \eta = \nu - \xi_1 - ... - \xi_{m-1}$ is a root. By a simple induction, we can affirm that $\nu - \xi_1 \in \Delta$. As $\xi \notin S^\nu$, this is impossible. We have proved the lemma.

Theorem 5. *Let $\mathfrak{n}$ be a standard nilpotent subalgebra associated to a set R of pairwise incomparable roots. Then the normalizer $\mathcal{N}(\mathfrak{n})$ of $\mathfrak{n}$ (it is parabolic because $\mathfrak{n}$ is standard) is defined by the subset*

$$S_1 = \bigcup_{\beta \in R} S^\beta \subset S.$$

Proof. Let $\mathfrak{r}$ be a parabolic subalgebra associated to the system S_1 and let $\mathfrak{n}$ be $\mathfrak{n} = \sum_{\alpha \in R_1} \mathfrak{g}_\alpha$ (R_1 is defined above). From Lemma 3, we can write $\mathfrak{r} \subset \mathcal{N}(\mathfrak{n})$.

Suppose that $\mathfrak{r} \neq \mathcal{N}(\mathfrak{n})$. As $\Delta = \Delta_1 \cup \Delta_2$, there is $\alpha \in \Delta_2^+$ such that $X_{-\alpha} \in \mathcal{N}(\mathfrak{n})$. This shows that the decomposition of α in simple roots contains one element ν of S_1 (by definition, ν is an extremal element for $\beta \in R$). But $\mathcal{N}(\mathfrak{n})$ is a parabolic subalgebra. Then $X_{-\nu} \in \mathcal{N}(\mathfrak{n})$. More so we have $\beta - \nu \in \Delta$ or $\beta = \nu$. If $\beta - \nu \in \Delta$, then

$[X_\beta, X_{-\nu}] = N.X_{\beta-\nu} \in \mathfrak{n}$ with $N \neq 0$. If $\beta = \nu$, then $[X_\beta, X_{-\nu}] = [X_\nu, X_{-\nu}] \in \mathfrak{n}$, but this vector is also in the Cartan subalgebra. These two cases lead to a contradiction.

Corollary. *Every system $R \subset \Delta^+$ of pairwise incomparable roots defines a nilpotent standard subalgebra*

$$\mathfrak{n} = \sum_{\alpha \in R_1} \mathfrak{g}_\alpha$$

whose normalizer is the parabolic subalgebra associated to the subsystem

$$S_1 = \bigcup_{\beta \in R} S^\beta.$$

Conversely, every nilpotent standard subalgebra is conjugated to a subalgebra $\mathfrak{n}$ associated to a subsystem $R \subset \Delta^+$ of pairwise incomparable roots.

IV.5. On the nilpotent algebras of maximal rank

Let $\mathfrak{g}$ be a semisimple complex Lie algebra and $\mathfrak{g}_+ = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ its nilpotent part. If $\mathfrak{n}$ is a standard nilpotent Lie algebra, then the quotient $\mathfrak{g}_+/\mathfrak{n}$ is also nilpotent.
This process permits us to construct a class of nilpotent Lie algebras. Every Lie algebra of this class, and every standard nilpotent Lie algebra is given by a subset $R \subset \Delta^+$ that consists of pairwise incomparable roots. If we consider the subset $R \subset \Delta^+$ whose elements are pairwise incomparable and containing (in the decomposition in simple roots) at least three distinct simple roots, then we obtain a subclass of nilpotent Lie algebras. These algebras are nothing but the nilpotent Lie algebras with maximal rank studied by G. Favre and L. Santharoubane [F.S.].

IV.6. Complete Standard Lie algebras

Definition 7. *A standard nilpotent algebra $\mathfrak{n}$ is called complete if it is the nilradical of its normalizer.*

It is easy to see that such an algebra is conjugated to a subalgebra associated to a subset of simple roots.

Theorem 6. *Let $\mathfrak{n}$ be a standard nilpotent Lie algebra associated to a system $R \subset \Delta^+$ of pairwise incomparable roots. The following propositions are equivalent:*
(i) $\mathfrak{n}$ *is an intersection of complete standard nilpotent subalgebras associated to some subsystems of* S
(ii) *every element* $\beta \in R$ *can be expressed as* $\alpha_1 + \dots + \alpha_m$ *with* $\alpha_1, \dots, \alpha_m \in S$ *and* $\alpha_i \neq \alpha_j$ *for* $i \neq j$.

Proof : (i) $\Rightarrow$ (ii).
We put

$$R = \{\beta_1, \dots, \beta_k\} \text{ and } \phi_R = \left\{ \bigcup_{i=1}^{k} \{\alpha_i\} / \alpha_i \in S^{\beta_i} \ i = 1, \dots, k \right\}.$$

Every element S' of ϕ_R is a subset of S and defines a standard nilpotent complete subalgebra $\mathfrak{n}_{S'}$. We easily show that $\mathfrak{n} \subset \mathfrak{n}_{S'}$ for all $S' \in \phi_R$. Then $\mathfrak{n}_1 = \cap \mathfrak{n}_{S'}$ for all $S' \in \phi_R$ satisfies $\mathfrak{n} \subset \mathfrak{n}_1$. Suppose that $\alpha \in \Delta^+$, $X_\alpha \in \mathfrak{n}_1$ and $X_\alpha \notin \mathfrak{n}$. The decomposition of α in simple roots does not contain roots of S^β when $\beta \in R$ (if not, we shall have $X_\alpha \in \mathfrak{n}$, from Lemma 3).

Consider $S' = \bigcup_{i=1}^{k}\{\alpha_i\}$ of ϕ_R, where $\alpha \neq \alpha_i$ for $i = 1,\dots,k$ (recall that $\alpha_i \in S^{\beta_i}$), then $X_\alpha \notin \mathfrak{n}_{S'}$; this is contrary to the choice of X_α.

(ii) $\Rightarrow$ (i).

Let be $\mathfrak{n} = \cap_{i=1,n}\mathfrak{n}_i$ where $\mathfrak{n}_i$ is a complete nilpotent stanadard subalgebra associated to the system S_i of simple roots. Consider a root $\beta \in R$. Suppose that its decomposition $\beta = k_1\alpha_1 + \dots + k_m\alpha_m$ contains a coefficient k_i strictly greater than 1. As the vector X_β is in $\mathfrak{n} = \cap_{i=1,n}\mathfrak{n}_i$, there is a root γ_i in S_i such that $\beta \geq \gamma_i$ for all i, $1 \leq i \leq n$. Then for the root $\beta = \alpha_1 + \dots + \alpha_m \in R^+$, we shall also have $\beta' \geq \gamma_i$. Thus, the vector X_β is in each $\mathfrak{n}_i$ for all i and, thus, is in $\mathfrak{n}$. This is impossible because $\beta' < \beta$.

Note. If the semisimple algebra $\mathfrak{g}$ is of type A_n, condition (ii) is necessarily satisfied. The decomposition of $\mathfrak{n}$ as an intersection of the complete subalgebra is always true. This result was proved by Gurevich [GU].

V. ON THE CLASSIFICATION OF NILPOTENT COMPLEX LIE ALGEBRAS

V.1. The classification in dimensions less than 6

The classification given in this section (and in the following sections) concerns only complex Lie algebras. By convention, the undefined brackets are null, except those which emanate from the anticommutativity.

Dimension 1

$\mathfrak{n}_1^1 = \mathfrak{a}_1$ = the Abelian algebra of dimension 1.

Dimension 2

$\mathfrak{n}_1^2 = \mathfrak{a}_1^2$ = the Abelian algebra of dimension 2.

Dimension 3

$\mathfrak{n}_1^3 = \mathfrak{a}_1^3$ = the Abelian algebra of dimension 3,
$\mathfrak{n}_2^3 = H_1$ = the Heisenberg algebra defined by $[X_1,X_2] = X_3$.

From dimension 4, we don't write the decomposable algebras, i.e. the Lie algebras which are direct sums of proper ideals.

Dimension 4

$\mathfrak{n}_1^4$: $[X_1,X_4] = X_3$; $[X_1,X_3] = X_2$. This is a filiform law.

Dimension 5

$\mathfrak{n}_1^5$: $[X_1,X_5] = X_4$; $[X_1,X_4] = X_3$; $[X_1,X_3] = X_2$,
$\mathfrak{n}_2^5$: $[X_1,X_5] = X_4$; $[X_1,X_4] = X_3$; $[X_1,X_3] = X_2$;
$[X_4,X_5] = X_2$.

These two laws are the 5 dimensional filiform laws.

$\mathfrak{n}_3^5$: $[X_1,X_5] = X_4$; $[X_1,X_3] = X_2$; $[X_2,X_3] = X_4$,
$\mathfrak{n}_4^5$: $[X_1,X_5] = X_4$; $[X_2,X_3] = X_4$,
$\mathfrak{n}_5^5$: $[X_1,X_5] = X_4$; $[X_1,X_3] = X_2$; $[X_3,X_5] = X_4$,
$\mathfrak{n}_6^5$: $[X_1,X_5] = X_4$; $[X_1,X_3] = X_2$,

Dimension 6

n_1^6 : $[X_1,X_6] = X_5$; $[X_1,X_5] = X_4$; $[X_1,X_4] = X_3$; $[X_1,X_3] = X_2$,

n_2^6 : $[X_1,X_6] = X_5$; $[X_1,X_5] = X_4$; $[X_1,X_4] = X_3$; $[X_1,X_3] = X_2$;
$[X_5,X_6] = X_2$,

n_3^6 : $[X_1,X_6] = X_5$; $[X_1,X_5] = X_4$; $[X_1,X_4] = X_3$; $[X_1,X_3] = X_2$;
$[X_3,X_6] = X_2$; $[X_4,X_5] = - X_2$,

n_4^6 : $[X_1,X_6] = X_5$; $[X_1,X_5] = X_4$; $[X_1,X_4] = X_3$; $[X_1,X_3] = X_2$;
$[X_5,X_6] = X_3$; $[X_4,X_6] = X_2$,

n_5^6 : $[X_1,X_6] = X_5$; $[X_1,X_5] = X_4$; $[X_1,X_4] = X_3$; $[X_1,X_3] = X_2$;
$[X_3,X_6] = X_2$; $[X_4,X_5] = - X_2$.

These laws are the filiform laws.

n_6^6 : $[X_1,X_6] = X_5$; $[X_1,X_5] = X_4$; $[X_1,X_4] = X_3$; $[X_2,X_6] = X_4$; $[X_2,X_5] = - X_3$

n_6^7 : $[X_1,X_6] = X_5$; $[X_1,X_5] = X_4$; $[X_1,X_4] = X_3$; $[X_2,X_6] = X_3$,

n_6^8 : $[X_1,X_6] = X_5$; $[X_1,X_5] = X_4$; $[X_1,X_4] = X_3$; $[X_2,X_6] = X_2$;
$[X_5,X_6] = X_3$,

n_6^9 : $[X_1,X_6] = X_5$; $[X_1,X_5] = X_4$; $[X_1,X_3] = X_2$; $[X_5,X_6] = X_3$;
$[X_4,X_6] = X_2$,

n_6^{10} : $[X_1,X_6] = X_5$; $[X_1,X_5] = X_3$; $[X_3,X_6] = X_2$; $[X_4,X_5] = X_2$,

n_6^{11} : $[X_1,X_6] = X_5$; $[X_1,X_5] = X_4$; $[X_1,X_3] = X_2$; $[X_2,X_3 = X_4$,

n_6^{12} : $[X_1,X_6] = X_5$; $[X_1,X_5] = X_4$; $[X_2X_3] = X_4$,

n_6^{13} : $[X_1,X_6] = X_5$; $[X_1,X_5] = X_4$; $[X_1,X_4] = X_3$; $[X_5,X_6] = X_2$,

n_6^{14} : $[X_1,X_6] = X_5$; $[X_5,X_6] = X_3$; $[X_4,X_6] = X_2$,

n_6^{15} : $[X_1,X_6] = X_5$; $[X_1,X_4] = X_3$; $[X_5,X_6] = X_3$; $[X_4,X_6] = X_2$,

n_6^{16} : $[X_1,X_6] = X_5$; $[X_1,X_4] = X_3$; $[X_5,X_6] = X_2$,

n_6^{17} : $[X_1,X_6] = X_5$; $[X_1,X_5] = X_4$; $[X_5,X_6] = X_2$; $[X_3,X_6] = X_4$,

n_6^{18} : $[X_1,X_6] = X_5$; $[X_1,X_5] = X_4$; $[X_3,X_6] = X_2$; $[X_5,X_6] = X_2$,

n_6^{19} : $[X_1,X_6] = X_5$; $[X_1,X_4] = X_3$; $[X_2,X_6] = X_3$,

n_6^{20} : $[X_1,X_6] = X_5$; $[X_1,X_4] = X_3$; $[X_4,X_6] = X_2$.

V.2. The classification in dimension 7

The previous tables show that, for dimensions less than 6, there is a finite number of isomorphism classes of complex nilpotent Lie algebras. But, for dimension 7 and more, we are going to find families with one or more parameters of nonisomorphic Lie algebras. This is the same as saying that the variety of isomorphism classes has nontrivial dimension. Finding one-parameter families of isomorphic classes of 7 dimensional nilpotent algebra is not the only difficulty. There is, in a variety of isomorphic classes, some isolated points. So every table of dimension 7 is very difficult to establsh. Now, we have 3 or 4 tables given by various authors. Unfortunately, it is not easy to compare these lists because of the techniques used and the invariants are very different (characteristic sequence, rank, extension). The links between the invariants used are not evident. We begin by giving all the one-parameter families.

Theorem 7. *There are six families with one parameter nilpotent Lie algebras of dimension 7, pairwise not isomorphic* :

$\mathfrak{n}_7^{1,\alpha}$: $[X_1,X_i] = X_{i-1}$ $\quad i = 3,\dots,7$
$[X_4,X_7] = \alpha X_2$ $\quad [X_5,X_7] = (1+\alpha)X_3$
$[X_5X_6] = X_2$ $\quad [X_6,X_7] = (1+\alpha)X_4$

$\mathfrak{n}_7^{2,\alpha}$: $[X_1,X_i] = X_{i-1}$ $\quad i = 4,\dots,7$
$[X_2,X_6] = X_3$ $\quad [X_2,X_7] = X_3 + X_4$
$[X_5X_7] = \alpha X_3$ $\quad [X_6,X_7] = \alpha X_4 + X_2$

$\mathfrak{n}_7^{3,\alpha}$: $[X_1,X_i] = X_{i-1}$ $\quad i = 3,4,5,7$
$[X_4,X_5] = X_6$ $\quad [X_4,X_7] = \alpha X_2$
$[X_5X_6] = X_2$ $\quad [X_5,X_7] = (1+\alpha)X_6$

$\mathfrak{n}_7^{4,\alpha}$: $[X_1,X_i] = X_{i-1}$ $\quad i = 3,4,5,7$
$[X_4,X_7] = \alpha X_2$ $\quad [X_5,X_6] = X_2$
$[X_5X_7] = (1+\alpha)X_3$

$$
\begin{array}{lll}
\mathfrak{n}_7^{5,\alpha} : [X_1,X_i] = X_{i-1} & i = 3,4,5,7 & \\
\phantom{\mathfrak{n}_7^{5,\alpha} :} [X_5,X_6] = [X_6,X_7] = X_2 & & [X_4,X_5] = \alpha X_2 \\
 & & [X_5 X_7] = X_3
\end{array}
$$

$$
\begin{array}{ll}
\mathfrak{n}_7^{6,\alpha} : [X_1,X_i] = X_{i-1} & i = 4,6,7 \\
\phantom{\mathfrak{n}_7^{6,\alpha} :} [X_4,X_7] = X_2 \ ; \ [X_2,X_4] = \alpha X_5 & \\
\phantom{\mathfrak{n}_7^{6,\alpha} :} [X_2 X_7] = [X_3 X_4] = X_5 &
\end{array}
$$

Remark.: Each Lie algebra of the family $\mathfrak{n}_7^{1,\alpha}$ is filiform, and the Lie algebras of the family $\mathfrak{n}_7^{2,\alpha}$ are characteristically nilpotent.

The classification of 7-dimensional nilpotent Lie algebras presented here is established using the characteristic sequence as the principal invariant. We don't give anything concerning the proof, the nonisomorphism between two algebras can be seen by using isomorphisms preserving the Jordan form of the characteristic vector, and isomorphisms transforming the characteristic vectors into characteristic vectors.

Lie algebras whose characteristic sequence is $c(\mathfrak{g}) = (6,1)$. Filiform Lie algebras.

$$
\begin{array}{ll}
\mathfrak{n}_7^{1,a} : [X_1,X_i] = X_{i-1} \ , \ 3 \le i \le 7 \ , & \\
\phantom{\mathfrak{n}_7^{1,a} :} [X_4,X_5] = aX_6 \ ; & [X_5,X_7] = (1+a)X_3 \quad a \neq 0 \ ; \\
\phantom{\mathfrak{n}_7^{1,a} :} [X_5 X_6] = X_2 \ ; & [X_6,X_7] = (1+a)X_4 \ ,
\end{array}
$$

$$
\begin{array}{ll}
\mathfrak{n}_7^{2} : [X_1,X_i] = X_{i-1} \ , \ 3 \le i \le 7 \ , & \\
\phantom{\mathfrak{n}_7^{2} :} [X_4,X_7] = X_2 \ ; & [X_5,X_7] = X_3 \ ; \\
\phantom{\mathfrak{n}_7^{2} :} [X_6,X_7] = X_4 \ , &
\end{array}
$$

$$
\begin{array}{ll}
\mathfrak{n}_7^{3} : [X_1,X_i] = X_{i-1} \ , \ 3 \le i \le 7 \ , & \\
\phantom{\mathfrak{n}_7^{3} :} [X_4,X_7] = X_2 \ ; & [X_5,X_7] = X_3 \ ; \\
\phantom{\mathfrak{n}_7^{3} :} [X_6,X_7] = X_2 + X_4 \ , &
\end{array}
$$

$\mathfrak{n}_7^4$: $[X_1,X_i] = X_{i-1}$, $3 \le i \le 7$,
$[X_4,X_7] = X_2$; $[X_5,X_6] = -X_2$;
$[X_5X_7] = X_2$; $[X_6,X_7] = X_3$,

$\mathfrak{n}_7^5$: $[X_1,X_i] = X_{i-1}$, $3 \le i \le 7$,
$[X_5,X_7] = X_2$; $[X_6,X_7] = X_2 + X_3$;

$\mathfrak{n}_7^6$: $[X_1,X_i] = X_{i-1}$, $3 \le i \le 7$,
$[X_5,X_7] = X_2$; $[X_6,X_7] = -X_3$,

$\mathfrak{n}_7^7$: $[X_1,X_i] = X_{i-1}$, $3 \le i \le 7$,
$[X_6,X_7] = X_2$,

$\mathfrak{n}_7^8$: $[X_1,X_i] = X_{i-1}$, $3 \le i \le 7$.

Lie algebra with characteristic sequence $c(\mathfrak{g}) = (5,1,1)$

$\mathfrak{n}_7^9$: $[X_1,X_i] = X_{i-1}$, $4 \le i \le 7$,
$[X_2,X_6] = \frac{1}{2} X_3$ $[X_2,X_7] = X_3 - \frac{1}{2} X_4$
$[X_5,X_6] = \frac{1}{2} X_3$ $[X_5,X_7] = X_3 - \frac{1}{2} X_4$
$[X_6,X_7] = \frac{1}{2} X_3 + \frac{1}{2} X_3$

$\mathfrak{n}_7^{10}$: $[X_1,X_i] = X_{i-1}$, $4 \le i \le 7$,
$[X_4,X_7] = X_2$; $[X_5,X_6] = -X_2$;
$[X_5X_7] = X_3$; $[X_6,X_7] = X_4$,

$\mathfrak{n}_7^{11}$: $[X_1,X_i] = X_{i-1}$, $4 \le i \le 7$,
$[X_4,X_7] = X_2$; $[X_5,X_6] = -X_2$;
$[X_6,X_7] = X_3$,

$\mathfrak{n}_7^{12}$: $[X_1,X_i] = X_{i-1}$, $4 \le i \le 7$,
$[X_4,X_7] = X_2$; $[X_5,X_6] = -X_2$;

$\mathfrak{n}_7^{13,a}$: $[X_1,X_i] = X_{i-1}$, $4 \le i \le 7$,
$[X_2,X_6] = X_3$; $[X_2,X_7] = X_3 + X_4$;
$[X_5,X_7] = aX_3$; $[X_6,X_7] = aX_4 + X_2$,

$\mathfrak{n}_7^{14}$: $[X_1,X_i] = X_{i-1}$, $4 \le i \le 7$,
$[X_2,X_6] = X_3$; $[X_2,X_7] = X_4$;
$[X_5,X_7] = X_3$; $[X_6,X_7] = X_2 + X_4$,

$\mathfrak{n}_7^{15}$: $[X_1,X_i] = X_{i-1}$ $4 \le i \le 7$,
$[X_2,X_6] = X_3$ $[X_2,X_7] = X_4$
$[X_6,X_7] = X_3$

$\mathfrak{n}_7^{16}$: $[X_1,X_i] = X_{i-1}$, $4 \le i \le 7$,
$[X_2,X_6] = \frac{1}{2} X_3$ $[X_2,X_7] = -\frac{1}{4} X_3 + \frac{1}{2} X_4$
$[X_5,X_7] = \frac{1}{4} X_3 - \frac{1}{2} X_4$ $[X_6,X_7] = -\frac{1}{2} X_2 - \frac{1}{2} X_3 + \frac{1}{4} X_4 - \frac{1}{2} X_5$
$[X_5,X_6] = -\frac{1}{2} X_3$

$\mathfrak{n}_7^{17}$: $[X_1,X_i] = X_{i-1}$, $4 \le i \le 7$,
$[X_2,X_6] = \frac{1}{2} X_3$ $[X_2,X_7] = -\frac{3}{4} X_3 + \frac{1}{2} X_4$
$[X_5,X_6] = -\frac{1}{2} X_3$ $[X_6,X_7] = \frac{1}{2} X_2 - \frac{1}{8} X_3 + \frac{1}{4} X_4 + \frac{1}{2} X_5$
$[X_5,X_7] = \frac{1}{4} X_3 + \frac{1}{2} X_4$

$\mathfrak{n}_7^{18}$: $[X_1,X_i] = X_{i-1}$, $4 \le i \le 7$,
$[X_2,X_7] = X_3$; $[X_5,X_6] = -X_3$;
$[X_5X_7] = X_3 - X_4$; $[X_6,X_7] = X_2 - 2X_3 + X_4 - X_5$,

$\mathfrak{n}_7^{19}$: $[X_1,X_i] = X_{i-1}$, $4 \le i \le 7$,
$[X_2,X_7] = X_3$; $[X_5,X_6] = X_3$;
$[X_5X_7] = X_4$; $[X_6,X_7] = X_2 + X_5$,

$\mathfrak{n}_7^{20}$: $[X_1,X_i] = X_{i-1}$, $4 \le i \le 7$,
$[X_2,X_7] = X_3$; $[X_6,X_7] = X_2$,

$\mathfrak{n}_7^{21}$: $[X_1,X_i] = X_{i-1}$, $4 \le i \le 7$,
$[X_2,X_7] = X_3$; $[X_5,X_6] = X_3$;
$[X_5X_7] = X_3 + X_4$; $[X_6,X_7] = X_2 + X_4 + X_5$,

$\mathfrak{n}_7^{22}$: $[X_1,X_i] = X_{i-1}$, $4 \le i \le 7$,
$[X_5,X_6] = X_3$; $[X_5,X_7] = X_3 + X_4$;
$[X_6X_7] = X_2 + X_4 + X_5$,

$\mathfrak{n}_7^{23}$: $[X_1,X_i] = X_{i-1}$, $4 \le i \le 7$,
$[X_2,X_6] = X_3$; $[X_2,X_7] = X_4$;
$[X_5,X_6] = X_3$; $[X_6,X_7] = X_2 + X_5$;
$[X_5X_7] = X_4$,

$\mathfrak{n}_7^{24}$: $[X_1,X_i] = X_{i-1}$, $4 \le i \le 7$,
$[X_5,X_6] = X_3$; $[X_5,X_7] = X_4$;
$[X_6,X_7] = X_2 + X_5$,

$\mathfrak{n}_7^{25}$: $[X_1,X_i] = X_{i-1}$, $4 \le i \le 7$
$[X_5X_7] = X_3$; $[X_6,X_7] = X_2 + X_4$,

$\mathfrak{n}_7^{26}$: $[X_1,X_i] = X_{i-1}$, $4 \le i \le 7$,
$[X_6,X_7] = X_2$,

$\mathfrak{n}_7^{27}$: $[X_1,X_i] = X_{i-1}$, $4 \le i \le 7$,
$[X_2,X_6] = X_3$; $[X_2,X_7] = X_4$;
$[X_5,X_6] = X_3$; $[X_6,X_7] = X_4 + X_5$;

$[X_5 X_7] = X_3 + X_4$,

n_7^{28} : $[X_1, X_i] = X_{i-1}$, $4 \leq i \leq 7$,

$[X_2, X_6] = X_3$; $[X_2, X_7] = X_3 + X_4$;

$[X_5, X_6] = X_3$; $[X_6, X_7] = X_5$;

$[X_5 X_7] = X_4$,

n_7^{29} : $[X_1, X_i] = X_{i-1}$, $4 \leq i \leq 7$,

$[X_2, X_7] = X_3$; $[X_5, X_6] = X_3$;

$[X_5 X_7] = X_4$; $[X_6, X_7] = X_5$,

n_7^{30} : $[X_1, X_i] = X_{i-1}$, $4 \leq i \leq 7$,

$[X_2, X_6] = X_3$; $[X_2, X_7] = X_4$;

$[X_5, X_7] = X_4$; $[X_6, X_7] = X_5$;

$[X_5, X_7] = X_3$,

n_7^{31} : $[X_1, X_i] = X_{i-1}$, $4 \leq i \leq 7$,

$[X_2, X_7] = X_3$; $[X_5, X_6] = X_3$;

$[X_5, X_7] = X_3 + X_4$; $[X_6, X_7] = X_4 + X_5$,

n_7^{32} : $[X_1, X_i] = X_{i-1}$, $4 \leq i \leq 7$,

$[X_2, X_7] = X_3$; $[X_5, X_6] = X_3$;

$[X_5, X_7] = X_4$; $[X_6, X_7] = X_5$,

n_7^{33} : $[X_1, X_i] = X_{i-1}$, $4 \leq i \leq 7$,

$[X_2, X_5] = X_3$; $[X_2, X_6] = X_4$;

$[X_2, X_7] = X_5$; $[X_6, X_7] = X_3$,

n_7^{34} : $[X_1, X_i] = X_{i-1}$, $4 \leq i \leq 7$,

$[X_2, X_5] = X_3$; $[X_2, X_6] = X_4$;

$[X_2, X_7] = X_3 + X_5$,

$\mathfrak{n}_7^{35}$: $[X_1,X_i] = X_{i-1}$, $4 \leq i \leq 7$,
$[X_2,X_5] = X_3$; $[X_2,X_6] = X_4$;
$[X_2,X_7] = X_5$,

$\mathfrak{n}_7^{36}$: $[X_1,X_i] = X_{i-1}$, $4 \leq i \leq 7$,
$[X_2,X_6] = X_3$; $[X_2,X_7] = X_4$;
$[X_5,X_7] = X_3$; $[X_6,X_7] = X_4$,

$\mathfrak{n}_7^{37}$: $[X_1,X_i] = X_{i-1}$, $4 \leq i \leq 7$,
$[X_2,X_6] = X_3$; $[X_2,X_7] = X_3+X_4$,

$\mathfrak{n}_7^{38}$: $[X_1,X_i] = X_{i-1}$, $4 \leq i \leq 7$,
$[X_2,X_6] = X_3$; $[X_2,X_7] = X_3+X_4$,

$\mathfrak{n}_7^{39}$: $[X_1,X_i] = X_{i-1}$, $4 \leq i \leq 7$,
$[X_2,X_7] = X_3$; $[X_5,X_7] = X_3$;
$[X_6,X_7] = X_4$,

$\mathfrak{n}_7^{40}$: $[X_1,X_i] = X_{i-1}$, $4 \leq i \leq 7$,
$[X_2,X_7] = X_3$; $[X_6,X_7] = X_3$,

$\mathfrak{n}_7^{41}$: $[X_1,X_i] = X_{i-1}$, $4 \leq i \leq 7$,
$[X_2,X_7] = X_3$,

$\mathfrak{n}_7^{42}$: $= \mathfrak{n}^1_6 \oplus a_1$ $\mathfrak{n}_7^{43}$: $= \mathfrak{n}^2_6 \oplus a_1$ $\mathfrak{n}_7^{44}$: $= \mathfrak{n}^3_6 \oplus a_1$

$\mathfrak{n}_7^{45}$: $= \mathfrak{n}^4_6 \oplus a_1$ $\mathfrak{n}_7^{46}$: $= \mathfrak{n}^5_6 \oplus a_1$

Lie algebras whose characteristic sequence is $c(\mathfrak{g}) = (4,1,1,1)$

$\mathfrak{n}_7^{47}$: $[X_1,X_i] = X_{i-1}$, $5 \le i \le 7$,
$[X_2,X_3] = X_4$; $[X_2,X_6] = X_4$;
$[X_2,X_7] = X_5$; $[X_6,X_7] = X_4$,

$\mathfrak{n}_7^{48}$: $[X_1,X_i] = X_{i-1}$, $5 \le i \le 7$,
$[X_2,X_3] = X_4$; $[X_2,X_6] = X_4$;
$[X_2,X_7] = X_5$,

$\mathfrak{n}_7^{49}$: $[X_1,X_i] = X_{i-1}$, $5 \le i \le 7$,
$[X_2,X_6] = X_4$; $[X_2,X_7] = X_5$;
$[X_3,X_7] = X_4$,

$\mathfrak{n}_7^{50}$: $[X_1,X_i] = X_{i-1}$, $5 \le i \le 7$,
$[X_2,X_7] = X_4$; $[X_6,X_7] = X_3$,

$\mathfrak{n}_7^{51}$: $[X_1,X_i] = X_{i-1}$, $5 \le i \le 7$,
$[X_2,X_3] = X_4$; $[X_6,X_7] = X_4$,

$\mathfrak{n}_7^{52}$: $[X_1,X_i] = X_{i-1}$, $5 \le i \le 7$,
$[X_2,X_3] = X_4$,

$\mathfrak{n}_7^{53}$: $= \mathfrak{n}^6_6 \oplus \mathfrak{a}_1$ $\mathfrak{n}_7^{54}$: $= \mathfrak{n}^8_6 \oplus \mathfrak{a}_1$ $\mathfrak{n}_7^{55}$: $= \mathfrak{n}^9_6 \oplus \mathfrak{a}_1$

$\mathfrak{n}_7^{56}$: $= \mathfrak{n}^{10}_6 \oplus \mathfrak{a}_1$ $\mathfrak{n}_7^{57}$: $= \mathfrak{n}^{11}_6 \oplus \mathfrak{a}_1$ $\mathfrak{n}_7^{58}$: $= \mathfrak{n}^1_5 \oplus \mathfrak{a}_2$

$\mathfrak{n}_7^{59}$: $= \mathfrak{n}^2_5 \oplus \mathfrak{a}_2$

Lie algebras whose characteristic sequence is $c(\mathfrak{g}) = (4,2,1)$

$\mathfrak{n}_7^{60}$: $[X_1,X_i] = X_{i-1}$, $i = 3, 4, 5, 7$,
$[X_3,X_5] = X_6$; $[X_4,X_5] = X_7$;
$[X_5,X_7] = X_6$,

$\mathfrak{n}_7^{61}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,5,7$,
$[X_3,X_5] = X_6$; $[X_4,X_5] = X_7$,

$\mathfrak{n}_7^{62}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,5,7$,
$[X_4,X_5] = X_6$; $[X_4,X_7] = aX_2$;
$[X_5,X_6] = X_2$; $[X_5,X_7] = (1+a)X_6$ $a \neq 0$,

$\mathfrak{n}_7^{63}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,5,7$,
$[X_4,X_5] = X_6$; $[X_4,X_7] = X_2$;
$[X_5,X_7] = X_3$,

$\mathfrak{n}_7^{64}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,5,7$,
$[X_4,X_5] = X_6$; $[X_5,X_7] = X_6$,

$\mathfrak{n}_7^{65}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,5,7$,
$[X_4,X_5] = X_6$; $[X_5,X_7] = X_2$,

$\mathfrak{n}_7^{66}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,5,7$,
$[X_4,X_5] = X_6$; $[X_5,X_7] = X_2 + X_6$,

$\mathfrak{n}_7^{67}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,5,7$,
$[X_4,X_5] = X_6$,

$\mathfrak{n}_7^{68}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,5,7$,
$[X_4,X_5] = X_2$; $[X_5,X_7] = X_6$;
$[X_6,X_7] = X_2$,

$\mathfrak{n}_7^{69}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,5,7$,
$[X_5,X_7] = X_6$; $[X_6,X_7] = X_2$,

$\mathfrak{n}_7^{70,a}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,5,7$,
$[X_4,X_5] = aX_2$, $[X_5,X_6] = X_2$,

$[X_5,X_7] = X_3$, $[X_6,X_7] = X_2$ $a \neq 0$;

$\mathfrak{n}_7^{71}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,5,7$,
$[X_4,X_5] = X_2$, $[X_6,X_7] = X_2$;

$\mathfrak{n}_7^{72}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,5,7$,
$[X_6,X_7] = X_2$;

$\mathfrak{n}_7^{73}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,5,7$,
$[X_4,X_7] = X_2$, $[X_5,X_6] = X_2$,
$[X_5,X_7] = 2X_3 + X_6$;

$\mathfrak{n}_7^{74}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,5,7$,
$[X_4,X_7] = X_2$, $[X_5,X_7] = X_3 + X_6$;

$\mathfrak{n}_7^{75}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,5,7$,
$[X_5,X_6] = X_2$, $[X_5,X_7] = X_3 + X_6$;

$\mathfrak{n}_7^{76}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,5,7$,
$[X_4,X_5] = X_2$, $[X_5,X_7] = X_6$;

$\mathfrak{n}_7^{77}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,5,7$,
$[X_5,X_7] = X_6$;

$\mathfrak{n}_7^{78,a}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,5,7$,
$[X_4,X_7] = aX_2$, $[X_5,X_6] = X_2$,
$[X_5,X_7] = (1+a)X_3$ $a \neq 0$;

$\mathfrak{n}_7^{79}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,5,7$,
$[X_4,X_5] = X_2$, $[X_4,X_7] = X_2$,
$[X_5,X_6] = X_2$, $[X_5,X_7] = 2X_3$;

$\mathfrak{n}_7^{80}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,5,7$,
$[X_4,X_7] = X_2$, $[X_5,X_7] = X_3$;

$\mathfrak{n}_7^{81}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,5,7$,
$[X_4,X_7] = X_2$, $[X_5,X_7] = X_2$;

$\mathfrak{n}_7^{82}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,5,7$,
$[X_4,X_5] = X_2$;

$\mathfrak{n}_7^{83}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,5,7$,
$[X_5,X_7] = X_2$;

$\mathfrak{n}_7^{84}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,5,7$;

Lie algebras whose characteristic is $c(\mathfrak{g}) = (3,3,1)$

$\mathfrak{n}_7^{85}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,6,7$,
$[X_4,X_7] = X_2$;

$\mathfrak{n}_7^{86}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,6,7$,
$[X_4,X_7] = X_2$, $[X_3,X_4] = X_5$;

$\mathfrak{n}_7^{87}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,6,7$;

$\mathfrak{n}_7^{88}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,6,7$,
$[X_3,X_4] = X_5$;

$\mathfrak{n}_7^{89}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,6,7$,
$[X_3,X_4] = X_5$, $[X_6,X_7] = X_2$;

$\mathfrak{n}_7^{90}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,6,7$,

$[X_3,X_7] = X_5$, $[X_4,X_7] = X_6$;

$\mathfrak{n}_7{}^{91}$: $[X_1,X_i] = X_{i-1}$, $i = 3, 4, 6, 7$,
$[X_3,X_4] = X_2 + X_5$, $[X_3,X_7] = X_5$,
$[X_4,X_7] = X_6$, $[X_6,X_7] = X_5$;

$\mathfrak{n}_7{}^{92}$: $[X_1,X_i] = X_{i-1}$, $i = 3, 4, 6, 7$,
$[X_3,X_4] = X_2 + X_5$, $[X_3,X_7] = X_5$,
$[X_4,X_7] = X_6$;

$\mathfrak{n}_7{}^{93}$: $[X_1,X_i] = X_{i-1}$, $i = 3, 4, 6, 7$,
$[X_3,X_4] = X_2$, $[X_6,X_7] = X_5$;

$\mathfrak{n}_7{}^{94}$: $[X_1,X_i] = X_{i-1}$, $i = 3, 4, 6, 7$,
$[X_3,X_4] = X_2$, $[X_6,X_7] = X_2 + X_5$;

$\mathfrak{n}_7{}^{95}$: $[X_1,X_i] = X_{i-1}$, $i = 3, 4, 6, 7$,
$[X_4,X_6] = X_2$, $[X_4,X_7] = X_3 + X_5$,
$[X_6,X_7] = X_5$;

$\mathfrak{n}_7{}^{96}$: $[X_1,X_i] = X_{i-1}$, $i = 3, 4, 6, 7$,
$[X_3,X_4] = X_2$, $[X_6,X_7] = X_2$;

$\mathfrak{n}_7{}^{97}$: $[X_1,X_i] = X_{i-1}$, $i = 3, 4, 6, 7$,
$[X_3,X_4] = X_2$, $[X_3,X_7] = X_5$,
$[X_4,X_6] = X_5$;

$\mathfrak{n}_7{}^{98}$: $[X_1,X_i] = X_{i-1}$, $i = 3, 4, 6, 7$,
$[X_3,X_4] = X_5$; $[X_3,X_7] = X_2$,
$[X_4,X_6] = X_2$;

$\mathfrak{n}_7{}^{99}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,6,7$,
$[X_3,X_4] = X_5$, $[X_4,X_7] = X_2$,
$[X_6,X_7] = X_5$;

$\mathfrak{n}_7{}^{100}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,6,7$,
$[X_3,X_7] = X_2$, $[X_4,X_6] = X_2$;

$\mathfrak{n}_7{}^{101}$: $[X_1,X_i] = X_{i-1}$, $i = 3,4,6,7$,
$[X_3,X_4] = X_2$;

Lie algebras whose characteristic sequence is $c(\mathfrak{g}) = (3,2,1,1)$

$\mathfrak{n}_7{}^{102}$: $[X_1,X_i] = X_{i-1}$, $i = 4,6,7$,
$[X_4,X_7] = X_2$;

$\mathfrak{n}_7{}^{103}$: $[X_1,X_i] = X_{i-1}$, $i = 4,6,7$,
$[X_4,X_7] = X_5$, $[X_6,X_7] = X_2$;

$\mathfrak{n}_7{}^{104}$: $[X_1,X_i] = X_{i-1}$, $i = 4,6,7$,
$[X_6,X_7] = X_2$;

$\mathfrak{n}_7{}^{105}$: $[X_1,X_i] = X_{i-1}$, $i = 4,6,7$,
$[X_4,X_7] = X_2$, $[X_2,X_7] = X_5$,
$[X_3,X_4] = X_5$;

$\mathfrak{n}_7{}^{106}$: $[X_1,X_i] = X_{i-1}$, $i = 4,6,7$,
$[X_3,X_4] = X_5$, $[X_4,X_7] = X_2$;

$\mathfrak{n}_7{}^{107}$: $[X_1,X_i] = X_{i-1}$, $i = 4,6,7$,
$[X_2,X_7] = X_3$, $[X_6,X_7] = X_5$;

$\mathfrak{n}_7^{108}$: $[X_1,X_i] = X_{i-1}$, $i = 4,6,7$,
$[X_2,X_4] = X_3$, $[X_2,X_7] = X_5$;

$\mathfrak{n}_7^{109}$: $[X_1,X_i] = X_{i-1}$, $i = 4,6,7$,
$[X_2,X_7] = X_3$ $[X_2,X_4] = X_5$;

$\mathfrak{n}_7^{110}$: $[X_1,X_i] = X_{i-1}$, $i = 4,6,7$,
$[X_2,X_4] = X_5$;

$\mathfrak{n}_7^{111}$: $[X_1,X_i] = X_{i-1}$, $i = 4,6,7$,
$[X_2,X_7] = X_5$;

$\mathfrak{n}_7^{112}$: $[X_1,X_i] = X_{i-1}$, $i = 4,6,7$,
$[X_2,X_4] = X_5$, $[X_6,X_7] = X_5$;

$\mathfrak{n}_7^{113}$: $[X_1,X_i] = X_{i-1}$, $i = 4,6,7$,
$[X_2,X_7] = X_5$, $[X_6,X_7] = X_5$;

$\mathfrak{n}_7^{114}$: $[X_1,X_i] = X_{i-1}$, $i = 4,6,7$,
$[X_2,X_7] = X_5$ $[X_3,X_4] = X_5$;

$\mathfrak{n}_7^{115}$: $[X_1,X_i] = X_{i-1}$, $i = 4,6,7$,
$[X_2,X_4] = X_6$, $[X_2,X_3] = X_5$;

$\mathfrak{n}_7^{116}$: $[X_1,X_i] = X_{i-1}$, $i = 4,6,7$,
$[X_2,X_7] = X_3$, $[X_6,X_7] = X_3$;

$\mathfrak{n}_7^{117}$: $[X_1,X_i] = X_{i-1}$, $i = 4,6,7$,
$[X_2,X_4] = X_3$;

$\mathfrak{n}_7^{118}$: $[X_1,X_i] = X_{i-1}$, $i = 4,6,7$,
$[X_2,X_4] = X_3$, $[X_3,X_4] = X_5$;

$\mathfrak{n}_7^{119}$: $[X_1,X_i] = X_{i-1}$, $i = 4,6,7$,
$[X_2,X_4] = X_3$, $[X_6,X_7] = X_3$;

$\mathfrak{n}_7^{120}$: $[X_1,X_i] = X_{i-1}$, $i = 4,6,7$,
$[X_2,X_3] = X_5$;

$\mathfrak{n}_7^{121}$: $[X_1,X_i] = X_{i-1}$, $i = 4,6,7$,
$[X_2,X_4] = X_3$; $[X_4,X_7] = X_5$;

$\mathfrak{n}_7^{122}$: $[X_1,X_i] = X_{i-1}$, $i = 4,6,7$,
$[X_2,X_4] = X_5$, $[X_3,X_7] = X_5$,
$[X_4,X_7] = X_5$;

$\mathfrak{n}_7^{123}$: $[X_1,X_i] = X_{i-1}$, $i = 4,6,7$,
$[X_2,X_7] = X_3$, $[X_4,X_7] = X_5$;

$\mathfrak{n}_7^{124}$: $[X_1,X_i] = X_{i-1}$, $i = 4,6,7$,
$[X_3,X_4] = X_5$, $[X_4,X_7] = X_2$;

$\mathfrak{n}_7^{125}$: $[X_1,X_i] = X_{i-1}$, $i = 4,6,7$,
$[X_2,X_3] = X_5$, $[X_2,X_4] = X_3 + X_6$,
$[X_4,X_7] = X_5$;

$\mathfrak{n}_7^{126}$: $[X_1,X_i] = X_{i-1}$, $i = 4,6,7$,
$[X_2,X_3] = X_5$, $[X_2,X_4] = X_3 + X_6$;

$\mathfrak{n}_7^{127}$: $[X_1,X_i] = X_{i-1}$, $i = 4,6,7$,
$[X_2,X_4] = aX_5$, $[X_2,X_7] = X_5$,
$[X_3,X_4] = X_5$, $[X_4,X_7] = X_2$ $a \neq 0$;

$\mathfrak{n}_7^{128}$: $[X_1,X_i] = X_{i-1}$, $i = 4,6,7$,
$[X_2,X_4] = X_5$, $[X_3,X_4] = X_5$,

$[X_4, X_7] = X_2$;

$\mathfrak{n}_7^{129} : = \mathfrak{n}_6^{13} \oplus \mathfrak{a}_1$ $\quad$ $\mathfrak{n}_7^{130} : = \mathfrak{n}_6^{14} \oplus \mathfrak{a}_1$ $\quad$ $\mathfrak{n}_7^{131} : = \mathfrak{n}_6^{19} \oplus \mathfrak{a}_1$

$\mathfrak{n}_7^{132} : = \mathfrak{n}_6^{20} \oplus \mathfrak{a}_1$ $\quad$ $\mathfrak{n}_7^{133} : = \mathfrak{n}_6^{17} \oplus \mathfrak{a}_1$ $\quad$ $\mathfrak{n}_7^{134} : = \mathfrak{n}_6^{18} \oplus \mathfrak{a}_1$

$\mathfrak{n}_7^{135} : = \mathfrak{n}^1_4 \oplus \mathfrak{n}^1_3$ $\quad$ $\mathfrak{n}_7^{136} : = \mathfrak{n}_6^{15} \oplus \mathfrak{a}_1$

Lie algebras whose characteristic sequence is c($\mathfrak{g}$) = (3,1,1,1,1)

$\mathfrak{n}_7^{137}$: $[X_1, X_i] = X_{i-1}$, $\quad i = 6, 7$,
$[X_2, X_3] = X_5$, $\quad [X_4, X_7] = X_5$;

$\mathfrak{n}_7^{138} : = \mathfrak{n}^3_5 \oplus \mathfrak{a}_2$ $\quad$ $\mathfrak{n}_7^{139} : = \mathfrak{n}_6^{22} \oplus \mathfrak{a}$ $\quad$ $\mathfrak{n}_7^{140} : = \mathfrak{n}_5^4 \oplus \mathfrak{a}_2$

$\mathfrak{n}_7^{141} : = \mathfrak{n}_5^5 \oplus \mathfrak{a}_2$

Lie algebras whose characteristic sequence is c($\mathfrak{g}$) = (2,2,2,1)

$\mathfrak{n}_7^{142}$: $[X_1, X_i] = X_{i-1}$, $\quad i = 3, 5, 7$,
$[X_3, X_5] = X_4$, $\quad [X_5, X_7] = X_2$;

$\mathfrak{n}_7^{143}$: $[X_1, X_i] = X_{i-1}$, $\quad i = 3, 5, 7$,
$[X_3, X_5] = X_6$, $\quad [X_5, X_7] = X_2$;

$\mathfrak{n}_7^{144}$: $[X_1, X_i] = X_{i-1}$, $\quad i = 3, 5, 7$,
$[X_5, X_7] = X_2$;

$\mathfrak{n}_7^{145}$: $[X_1, X_i] = X_{i-1}$, $\quad i = 3, 5, 7$;

Lie algebras whose characteristic sequence is c($\mathfrak{g}$) = (2,2,1,1,1)

$\mathfrak{n}_7^{146}$: $[X_1, X_i] = X_{i-1}$, $\quad i = 5, 7$,
$[X_2, X_3] = X_4$, $\quad [X_2, X_5] = X_6$;

$\mathfrak{n}_7{}^{147}$: $[X_1,X_i] = X_{i-1}$, $i = 5, 7$,
$[X_2,X_3] = X_4$;

$\mathfrak{n}_7^{148}$: $= \mathfrak{n}_6^{21} \oplus a$ $\mathfrak{n}_7^{149}$: $= \mathfrak{n}_6^{23} \oplus a_1$ $\mathfrak{n}_7^{150}$: $= \mathfrak{n}_6^{24} \oplus a_1$

$\mathfrak{n}_7^{151}$: $= \mathfrak{n}_5^{5} \oplus a_2$

Lie algebras whose characteristic sequence is c($\mathfrak{g}$) = (2,1,1,1,1,1)

$\mathfrak{n}_7{}^{152}$: $[X_1,X_7] = X_6$, $[X_2,X_3] = X_6$,
$[X_4,X_5] = X_6$;

$\mathfrak{n}_7^{153}$: $= \mathfrak{n}_5^{6} \oplus a_2$ $\mathfrak{n}_7^{154}$: $= \mathfrak{n}_3^{1} \oplus a_4$

Lie algebras whose characteristic sequence is c($\mathfrak{g}$) = (1,1,1,1,1,1,1)

$\mathfrak{n}_7{}^{155}$: a_7 .

V.3. Other classifications

In the next chapter, we shall present other classifications. In particular we shall describe all the filiform Lie algebras of rank nonzero, i.e. the uncharacteristically nilpotent Lie algebras (see Chapter 7).

CHAPTER 3

COHOMOLOGY OF LIE ALGEBRAS

I. BASIC NOTIONS

I.1. $\mathfrak{g}$-modules and representations

In the first chapter, we glanced at the notions of representations of Lie algebras and the corresponding notions of $\mathfrak{g}$-modules. Here we will study all those notions which can be utilized for the theories of cohomology. First, we recall all the definitions.

Let $\mathfrak{g}$ be a Lie algebra on a field K. An n-dimensional vector space V over the same field K is called a *$\mathfrak{g}$-module* if a bilinear map $\varphi : \mathfrak{g} \times V \to V$ satisfying the condition

$$\varphi([x, y], v) = \varphi(x,\varphi(y, v)) - \varphi(y, \varphi(x,v)), \quad \forall\, x, y \in \mathfrak{g} \ \text{ and } \ v \in V,$$

is given. We write gv instead of $\varphi(g,v)$, if no disarray is possible. In other words, a $\mathfrak{g}$-module V corresponds to a *representation of $\mathfrak{g}$ on* V, i.e. a homomorphism of Lie algebras $\Phi : \mathfrak{g} \to gl\,(V)$ defined by

$$\Phi(g)\,(v) = \varphi(g, v) = gv\,.$$

For any $\mathfrak{g}$-modules V and W, the vector spaces $V \oplus W$, $V \otimes W$ and Hom(V, W) can be considered as $\mathfrak{g}$-modules by putting :

$$g\,(v + w) = gv + gw\,,$$

$$g(v \otimes w) = gv \otimes w + v \otimes gw,$$
$$(gf)(v) = g(f(v)) - f(gv),$$

where $v \in V, w \in W, g \in \mathfrak{g}$ and $f \in \mathrm{Hom}(V, W)$.

The structure of a $\mathfrak{g}$-module on a tensor product induces one on the exterior products $\Lambda^p V$ (the quotient of the tensor product $\otimes^p V = V \otimes \ldots \otimes V$ by the subspace generated by all the $v \otimes v$). Note that $\Lambda V = \Sigma\, \Lambda^i V$ is a finite-dimensional (because V has a finite dimension) graded algebra over K with $\Lambda^0 V = K$, $\Lambda^1 V = V$. It is clear that $\Lambda^n V$ is one-dimensional ($n = \dim V$) and $\Lambda^k V = \{0\}$ for $k > n$. If W is a subspace of V, then $\Lambda^p W$ can be identified canonically with a subspace of $\Lambda^p V$.

Examples

(i) Let V be a vector space over K. It can be considered as a $\mathfrak{g}$-module for all Lie algebras $\mathfrak{g}$ by putting $gv = 0$ for all g in $\mathfrak{g}$ and v in V. This $\mathfrak{g}$-module is called trivial.

(ii) Let V be a $\mathfrak{g}$-module. An element v in V is called *$\mathfrak{g}$-invariant* if $gv = 0$ for all g in $\mathfrak{g}$. Let be $V^{\mathfrak{g}}$ the set of all $\mathfrak{g}$-invariant elements. Then $V^{\mathfrak{g}}$ is a $\mathfrak{g}$-submodule of V, which is trivial. This submodule is maximal within the trivial $\mathfrak{g}$-submodules of V.

(iii) Every Lie algebra $\mathfrak{g}$ can be considered as a $\mathfrak{g}$-module by putting $g_1 g_2 = [g_1, g_2]$ for all g_1 and g_2 in $\mathfrak{g}$. The corresponding representation of $\mathfrak{g}$ on $\mathfrak{g}$ is nothing but the *adjoint representation* $\Phi(g_1)(g_2) = \mathrm{ad}\, g_1 (g_2)$. This example plays an important role.

(iv) We consider $\mathfrak{g} = gl(n, K)$. The structure of a one-dimensional module on a vector space E_λ ($\lambda \in K$) is given by $fv = -\lambda\, \mathrm{Tr}(f)\, v$, $v \in E_\lambda$, where $\mathrm{Tr}(f)$ denotes the trace of the endomorphism f.

I.2. The space of cochains

Let $\mathfrak{g}$ be a Lie algebra and V a $\mathfrak{g}$-module. If $p \geq 1$, a *p-dimensional cochain of $\mathfrak{g}$ with values in* V, is a p-linear alternating mapping of $\mathfrak{g}^p$ into V. If $p = 0$, one defines a zero-dimensional cochain of $\mathfrak{g}$ as a constant function from $\mathfrak{g}$ to V. We denote $C^p(\mathfrak{g}, V)$

as the space of the p-cochain of $\mathfrak{g}$. Then, $C^p(\mathfrak{g}, V) = \mathrm{Hom}\ (\Lambda^p\mathfrak{g}, V)$ for $p \geq 1$, $C^0(\mathfrak{g}, V) = V$, and for $p < 0$, we put $C^p(\mathfrak{g}, V) = \{0\}$.
Let $C^*(\mathfrak{g}, V)$ be the direct sum of $C^p(\mathfrak{g}, V)$. The elements of $C^*(\mathfrak{g}, V)$ are called *cochains of* $\mathfrak{g}$ *with values in* V ; those of $C^p(\mathfrak{g}, V)$ are said to be of degree p.
We provide the space $C^p(\mathfrak{g}, V)$ of a $\mathfrak{g}$-module structure :

$$(x\Phi)\ (x_1, \ldots, x_p) = x\Phi\ (x_1, \ldots, x_p) - \sum_{1 \leq i \leq p} \Phi(x_1, \ldots, x_{i-1}, [x, x_i], \ldots, x_p)$$

for all $x, x_1, \ldots, x_p$ in $\mathfrak{g}$ and Φ in $C^p(\mathfrak{g}, V)$. This structure of a $\mathfrak{g}$-module can be expanded to the space $C^*(\mathfrak{g}, V)$. The corresponding representation will be denoted by θ.
For all y in $\mathfrak{g}$, we denote $i(y)$ as the endomorphism of $C^*(\mathfrak{g}, V)$ which maps each subspace $C^p(\mathfrak{g}, V)$ into $C^{p-1}(\mathfrak{g}, V)$ and given by

$$(i(y)\ \Phi)\ (x_1, \ldots, x_{p-1}) = \Phi(y, x_1, \ldots, x_{p-1}).$$

Lemma 1. *The representation* θ *of* $\mathfrak{g}$ *on* $C^*(\mathfrak{g}, V)$ *verifies* :

$$\theta(x) \circ i(y) - i(y) \circ \theta(x) = i[x, y] \text{ for all } x \text{ and } y \text{ in } \mathfrak{g}.$$

The proof is a direct consequence of the definition of θ.

I.3. The coboundary operator

Let $\mathfrak{g}$ be a Lie algebra and V a $\mathfrak{g}$-module. We define the endomorphism

$$d : C^*(\mathfrak{g}, V) \to C^*(\mathfrak{g}, V)$$

by putting

$$d\Phi(x) = x\Phi \quad \text{for} \quad \Phi \in C^0(\mathfrak{g}, V), \ \ x \in \mathfrak{g}$$

$$d\Phi\left(x_1, \ldots, x_{p+1}\right) = \sum_{1 \leq s \leq p+1} (-1)^{s+1} \left(x_s\Phi\right)\left(x_1, \ldots, \widehat{x}_s, \ldots, x_{p+s}\right) +$$

$$+ \sum_{1 \leq s < t \leq p+1} (-1)^{s+t} \Phi\left([x_s, x_t], x_1, \ldots, \widehat{x}_s, \ldots, \widehat{x}_t, \ldots, x_{p+1}\right)$$

for Φ in $C^p(\mathfrak{g}, V)$, $p \geq 1$, and $x_1, \ldots, x_{p+1}$ in $\mathfrak{g}$, where the symbol ^ over a letter means that it is omitted. This endomorphism d maps $C^p(\mathfrak{g}, V)$ into $C^{p+1}(\mathfrak{g}, V)$. Its restriction to $C^p(\mathfrak{g}, V)$ will be denoted d_p.

Theorem 1. *The endomorphism* d *verifies* $d^2 = 0$.

First we prove some identities.

Lemma 2. $\theta(x) = i(x) \circ d + d \circ i(x)$,

$$d \circ \theta(x) = \theta(x) \circ d \qquad \text{for all } x \text{ in } \mathfrak{g}.$$

Proof. The first identity is nothing but the definitions of the endomorphisms d and i. We prove the last by an induction on p. Let Φ be a 0-cochain. We have

$$\big(\theta(x)\, d\Phi\big)(y) = x\big(d\Phi\big)(y) - \big(d\Phi\big)([x,y]) = x\big(y\Phi\big) - [x\,,y]\,\Phi = y\big(x\Phi\big) - \big(d\theta(x)\,\Phi\big)(y)$$

We suppose now that the relation is true until the indice p−1, i.e. $d(\theta(x)\Phi) = \theta(x)\, d\Phi$ for all Φ in $C^{p-1}(\mathfrak{g}, V)$. One considers an element Φ of $C^p(\mathfrak{g}, V)$. We have

$$\begin{aligned} & i(y)\big(d\theta(x) - \theta(x)d\big)\Phi \\ &= \big(\theta(y) - di(y)\big)\theta(x).\Phi + \big(i([x,y] - \theta(x)\, i(y))\big)d\Phi \\ &= \theta(y)\theta(x)\Phi + d\big(i([x,y]) - \theta(x)i(y)\big) + \big(\theta([x,y]) - di([x,y])\Phi - \theta(x)\big(\theta(y) - di(y)\big)\big)\Phi \\ &= \big(\theta(x)d - d\theta(x)\big)\,\big(i(y)\Phi\big) \\ &= 0 \qquad \text{for all } y \text{ in } \mathfrak{g}. \end{aligned}$$

So $\big(d\theta(x) - \theta(x)d\big)\Phi = 0$ and the identity is verified.

Proof of the Theorem. From lemmas 1 and 2, we have :

$$i(x)dd\Phi = \big(\theta(x) - di(x)\big)d\Phi = d\theta(x)\Phi - d\big(\theta(x) - di(x)\big)\Phi = dd(i(x)\Phi).$$

One deduces : $dd\Phi = 0 \quad \forall\, \Phi \in C^0(\mathfrak{g},V)$ and, by induction,

$$dd\Phi = 0 \quad \forall\, \Phi \in C^p(\mathfrak{g},V).$$

So $dd = 0$.

I.4. The cohomology space

The restriction d_p of the coboundary operator d to the space of the p-cochaine $C^p(\mathfrak{g}, V)$ of the Lie algebra $\mathfrak{g}$ with values in V, has a kernel denoted $Z^p(\mathfrak{g}, V)$ whose elements are called *cocycles of degree* p *(or p-cocycles)* with values in V.

The elements of the image $B^p(\mathfrak{g}, V)$ of d_{p-1} are called *coboundaries of degree* p *(or* p-*coboundaries)* with values in V. From Theorem 1, we have $B^p(\mathfrak{g},V) \subset Z^p(\mathfrak{g}, V)$. The quotient space

$$H^p(\mathfrak{g}, V) = Z^p(\mathfrak{g}, V) / B^p(\mathfrak{g}, V)$$

is called the cohomology space of $\mathfrak{g}$ of degree p with values in V. If i = 0, we agree to take $B^0(\mathfrak{g}, V) = 0$, since there are no (–1)-cochains. In this case, we have $H^0(\mathfrak{g}, V) = Z^0(\mathfrak{g}, V)$. This space can be identified with the subspace $V^{\mathfrak{g}}$ of invariant elements of V.

The direct sum of the $H^p(\mathfrak{g}, V)$ for p in $\mathbb{N}$ is denoted by $H^*(\mathfrak{g}, V)$.

I.5. Exact sequence

Let α be a homomorphism of the $\mathfrak{g}$-module V into the $\mathfrak{g}$-module W. Using this, we can construct the homomorphism

$$\alpha_p : C^p(\mathfrak{g}, V) \to C^p(\mathfrak{g}, W)$$

defined by

$$\alpha_p(\Phi) = \alpha \circ \Phi$$

for each p. Then α can be extended to a K-linear mapping

$$\alpha^* : C^*(\mathfrak{g}, V) \to C^*(\mathfrak{g}, W) .$$

It is easy to see that :

$$\alpha_{p+1} \circ d_p = d_p \circ \alpha_p$$

and this implies that

$$\alpha^* \circ d = d \circ \alpha^* .$$

This last relation permits us to define a homomorphism of cohomology spaces

$$\tilde{\alpha} = H^*(\mathfrak{g}, V) \to H^*(\mathfrak{g}, W) .$$

It is called a *homomorphism associated to* α.

Consider an exact sequence of $\mathfrak{g}$-modules :

$$0 \to V_1 \xrightarrow{\alpha} V_2 \xrightarrow{\beta} V_3 \to 0 .$$

By exactness, we means that Im α = Ker β, Ker $\alpha = 0$ and Im $\beta = V_3$.

From this sequence, we can define another exact sequence of the cochain spaces :

$$0 \to C^*(\mathfrak{g}, V_1) \xrightarrow{\alpha^*} C^*(\mathfrak{g}, V_2) \xrightarrow{\beta^*} C^*(\mathfrak{g}, V_3) \to 0 \ .$$

In fact, let Φ be such that $\alpha^*(\Phi) = 0$, i.e. $\alpha \circ \Phi = 0$. Suppose that Φ is a p-cochain, then $\alpha(\Phi(v_1, ..., v_p)) = 0$ for all $v_1, ..., v_p$. As α is injective, $\Phi(v_1, ..., v_p) = 0$ for all $v_1, ..., v_p$ and $\Phi = 0$. The exactness of the terms of the sequence could be verified in this way. Passing to the cohomology spaces, we find the long sequence of homomorphisms

$$0 \to H^0(\mathfrak{g},V_1) \xrightarrow{\tilde{\alpha}_0} H^0(\mathfrak{g},V_2) \xrightarrow{\tilde{\beta}_0} H^0(\mathfrak{g},V_3) \xrightarrow{\delta_0} H^1(\mathfrak{g},V_1) \to$$

$$\xrightarrow{\tilde{\alpha}_1} H^1(\mathfrak{g},V_2) \xrightarrow{\tilde{\beta}_1} H^1(\mathfrak{g},V_3) \xrightarrow{\delta_1} H^2(\mathfrak{g},V_1) \to \dots\dots$$

We begin by explaining these notations. The homomorphism $\tilde{\alpha}_i$ is obtained from $\tilde{\alpha} : H^*(\mathfrak{g},V_1) \to H^*(\mathfrak{g},V_2)$ by the restriction of $H^i(\mathfrak{g},V_1)$. It is analogous for $\tilde{\beta}_i$. Now let $\tilde{\varphi}$ be in $H^p(\mathfrak{g},V_3)$ and φ in $Z^p(\mathfrak{g},V_3)$ be a representative of $\tilde{\varphi}$. As β is a surjective homomorphism, we can choose an element ψ in $C^p(\mathfrak{g},V_2)$ such that $\beta_p(\psi) = \varphi$. But $d\psi$ is in $Z^{p+1}(\mathfrak{g},V_2)$. As $d \circ \beta^* = \beta^* \circ d$, then $d\left(\beta_p\psi\right) = \beta_{p+1}(d\psi) = d\varphi = 0$ and $d\psi$ is in Ker $\beta_{p+1} = \mathrm{Im}\ \alpha_{p+1}$. So there is a cochain ρ in $C^{p+1}(\mathfrak{g},V_1)$ such that $\alpha_{p+1}(\rho) = d\psi$. But $d(\alpha_{p+1}(r)) = \alpha_{p+2}(d\rho) = 0$ then $d\rho = 0$. Hence, ρ is in $Z^{p+1}(\mathfrak{g},V_1)$ and the mapping δ_{p+1} is lefined by $\delta_p(\tilde{\varphi}) = \tilde{\rho}$, where $\tilde{\rho}$ is the class of ρ in $H^{p+1}(\mathfrak{g},V_1)$. This is well defined be cause the class $\tilde{\rho}$ of ρ doesn't depend of the choice of the representative φ in the class $\tilde{\varphi}$. In fact, we suppose that $\varphi + dv$ is another representative of $\tilde{\varphi}$. The cochain ψ_1 in in $C^p(\mathfrak{g},V_2)$, whose image by β_p is $\varphi + dv$, can be written as $\psi_1 = \psi + d\xi$. Then $d\psi_1 = d\psi$ and this defines the same cochain ρ in $C^{p+1}(\mathfrak{g},V_1)$.

Theorem 2. *Let*

$$0 \to V_1 \xrightarrow{\alpha} V_2 \xrightarrow{\beta} V_3 \to 0$$

be an exact sequence of $\mathfrak{g}$-modules. Then the long sequence of homomorphisms

$$0 \to H^0(\mathfrak{g},V_1) \xrightarrow{\widetilde{\alpha}_0} H^0(\mathfrak{g},V_2) \xrightarrow{\widetilde{\beta}_0} H^0(\mathfrak{g},V_3) \to$$

$$\xrightarrow{\delta_0} H^1(\mathfrak{g},V_1) \xrightarrow{\widetilde{\alpha}_1} H^1(\mathfrak{g},V_2) \xrightarrow{\widetilde{\beta}_1} H^1(\mathfrak{g},V_3) \xrightarrow{\delta_1} H^2(\mathfrak{g},V_1) \to \ldots$$

is exact.

Proof. It is sufficient to prove that $\widetilde{\alpha}_0$ is injective, Im δ_p = Ker $\widetilde{\alpha}_{p+1}$ and Im $\widetilde{\beta}_p$ = Ker δ_p. By definition,

$$H^0(\mathfrak{g},V_1) = Z^0(\mathfrak{g},V_1) = \{v \in V_1 : g.v_1 = 0 \ , \ \forall g \in \mathfrak{g}\} .$$

By hypothesis, α is injective. Then $\widetilde{\alpha}_0$ is also injective.

Let $\widetilde{\varphi}$ be in $H^p(\mathfrak{g},V_3)$. By the construction of δ_p, we have $\delta(\widetilde{\varphi}) = \widetilde{\rho}$ with $\widetilde{\rho}$ in $H^{p+1}(\mathfrak{g},V_1)$, and $\alpha_{p+1}(\rho) = d\psi$ with $\beta(\psi) = \varphi$, ψ in $C^p(\mathfrak{g},V_2)$. The class of cohomology of $\widetilde{\alpha}_{p+1}(\widetilde{\rho})$ is nonzero and Im δ_p is contained in Ker $\widetilde{\alpha}_{p+1}$.

Now we consider $\widetilde{\psi}$ in Ker $\widetilde{\alpha}_{p+1}$ and ψ as a representative of $\widetilde{\psi}$. We have $\alpha_{p+1}(\psi) = d\mu$ where μ is an element of $C^p(\mathfrak{g},V_2)$. Furthermore, one has :

$$d(\beta_p(\mu)) = \beta_{p+1}(d\mu) = \beta_{p+1}(\alpha_{p+1}(\psi)) = 0.$$

Then $\beta_p(\mu)$ is in $Z^p(\mathfrak{g},V_3)$. We denote by $\widetilde{v}$ the class of $\beta_p(\mu)$ in $H^p(\mathfrak{g},V_3)$. By the definition of δ_p, we have $\delta_p(\widetilde{v}) = \widetilde{\psi}$ then Im δ_p = Ker $\widetilde{\alpha}_{p+1}$.

We can prove the second relation Im $\widetilde{\beta}_p$ = Ker δ_p by using the same arguments, and the theorem is proved.

Corollary. *Let* W *be a sub-$\mathfrak{g}$-module of* V. *The canonical homomorphisms*

$$W \xrightarrow{i} V \xrightarrow{\pi} V/W$$

define the exact long sequence

$$0 \to H^0(\mathfrak{g},W) \xrightarrow{\widetilde{\alpha}_0} H^0(\mathfrak{g},V) \xrightarrow{\widetilde{\pi}_0} H^0(\mathfrak{g},V/W) \xrightarrow{\delta_0} H^1(\mathfrak{g},W) \xrightarrow{\widetilde{\alpha}_1} H^1(\mathfrak{g},V)$$

$$\xrightarrow{\widetilde{\pi}_1} H^1(\mathfrak{g},V/W) \xrightarrow{\delta_1} H^2(\mathfrak{g},W) \to \ldots \to .$$

II. SOME INTERPRETATIONS OF THE SPACE $H^i(\mathfrak{g},V)$ FOR $i = 0, 1, 2, 3$

II.1. The spaces $H^0(\mathfrak{g},V)$

We have seen that

$$H^0(\mathfrak{g},V) = \operatorname{Ker} d_0 = \{v \in V \,/\, g.v = 0 \ , \forall g \in \mathfrak{g}\} .$$

This is the space of invariant vectors for $\mathfrak{g}$.

Particular case. Suppose that $V = \mathfrak{g}$ is considered as $\mathfrak{g}$-module adjoint. Then, the space $H^0(\mathfrak{g},\mathfrak{g})$ is the center of $\mathfrak{g}$:

$$H^0(\mathfrak{g},\mathfrak{g}) = Z^0(\mathfrak{g},\mathfrak{g}) = Z(\mathfrak{g}) .$$

II.2. The spaces $H^1(\mathfrak{g},V)$

II.2.1. $V = K$

We consider K as a trivial $\mathfrak{g}$-module. The spaces $H^i(\mathfrak{g},K)$ are nothing but the cohomological spaces for the De Rham cohomology of the left invariant forms on the connected Lie group G associated to $\mathfrak{g}$.

For $i = 1$, we have $C^1(\mathfrak{g},K) = \mathfrak{g}^*$ and

$$H^1(\mathfrak{g},K) = \operatorname{Ker} d_1 = \{\alpha \in \mathfrak{g}^* : \alpha[x, y] = 0 \quad \forall\, x, y \in \mathfrak{g}\} .$$

In other words, $H^1(\mathfrak{g},K) = C^1(\mathfrak{g} / [\mathfrak{g},\mathfrak{g}] , K)$.

II.2.2. $V = \mathfrak{g}$ considered as the adjoint $\mathfrak{g}$-module

First interpretation. The space $Z^1(\mathfrak{g},\mathfrak{g})$ is the space of the derivations of $\mathfrak{g}$ and the space $B^1(\mathfrak{g},\mathfrak{g})$ is the space of the inner derivations of $\mathfrak{g}$. Then, the space $H^1(\mathfrak{g},\mathfrak{g})$ can be interpreted as the space of the "outer" derivations of the Lie algebra $\mathfrak{g}$. The particular case $H^1(\mathfrak{g},\mathfrak{g}) = \{0\}$ corresponds to the Lie algebra $\mathfrak{g}$ whose derivations are inner. This is the case for semisimple Lie algebras and for parabolic subalgebras of semisimple Lie algebras.

Second interpretation. *The space* $H^1(\mathfrak{g},\mathfrak{g})$ *can be interpreted as the set of the one-dimensional extensions classes of the Lie algebra* $\mathfrak{g}$.

A one-dimensional extension of $\mathfrak{g}$ is an exact sequence

$$0 \longrightarrow \mathfrak{g} \longrightarrow \tilde{\mathfrak{g}} \longrightarrow K \longrightarrow 0$$

of the homomorphism of Lie algebras, where K is viewed as an Abelian Lie algebra. Two such sequences,

$$0 \longrightarrow \mathfrak{g} \longrightarrow \tilde{\mathfrak{g}} \longrightarrow K \longrightarrow 0$$

and

$$0 \longrightarrow \mathfrak{g} \longrightarrow \tilde{\mathfrak{g}}_1 \longrightarrow K \longrightarrow 0,$$

are equivalent if there is a homomorphism $\varphi : \tilde{\mathfrak{g}} \longrightarrow \tilde{\mathfrak{g}}_1$ of Lie algebras such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{g} \arrow[r] \arrow[d, "id"] & \tilde{\mathfrak{g}} \arrow[r] \arrow[d, "\varphi"] & K \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & \mathfrak{g} \arrow[r] & \tilde{\mathfrak{g}}_1 \arrow[r] & K \arrow[r] & 0
\end{tikzcd}$$

is commutative.

Let α be a cocycle in $Z^1(\mathfrak{g},\mathfrak{g})$. To this cocycle corresponds the one-dimensional extension of $\mathfrak{g}$:

$$0 \longrightarrow \mathfrak{g} \xrightarrow{i} \tilde{\mathfrak{g}} = \mathfrak{g} \oplus K \xrightarrow{\pi_2} K \longrightarrow 0,$$

where $i : \mathfrak{g} \to \tilde{\mathfrak{g}} = \mathfrak{g} \oplus K$ is the natural injection $i(x) = (x, 0)$ and π_2 is the projection on K

$\pi_2(x, k) = k$. The Lie algebra law on $\mathfrak{g} \oplus K$ is given by:

$$[(x,a),(y,b)] = ([x,y] + b\alpha(x) - a\alpha(y), 0).$$

The Jacobi identity for this bracket is equivalent to α being a cocycle.

It is easy to see that with cohomologeous cocycles correspond equivalent one-dimensional extensions. Indeed, let $\alpha + d\theta$ be a cohomologeous cocycle to α, with θ in $C^0(\mathfrak{g},\mathfrak{g}) = \mathfrak{g}$. This cocycle defines a one-dimensional extension

$$0 \longrightarrow \mathfrak{g} \longrightarrow \tilde{\mathfrak{g}}_1 = \mathfrak{g} \oplus K \longrightarrow K$$

and the bracket of $\tilde{\mathfrak{g}}_1$ is given by the bracket

$$[(x,a),(y,b)] = \big([x,y] + b\big(\alpha+d\theta\big)(x) - a\big(\alpha+d\theta\big)(y), 0\big).$$

The mapping $\varphi : \tilde{\mathfrak{g}} \longrightarrow \tilde{\mathfrak{g}}_1$ defined by $\varphi(x,a) = \big(x+a\theta,a\big)$ is a homomorphism of Lie algebras which gives the equivalence between these two extensions.

II.3. The spaces $H^2(\mathfrak{g},V)$

II.3.1. The spaces $H^2(\mathfrak{g},K)$

We can look upon this space as a set of classes of the cohomology of the 2-left invariant forms on the connected Lie group G associated with $\mathfrak{g}$. Recall that if G is compact, then we obtain the De Rham cohomology of G. We can also interpret $H^2(\mathfrak{g},K)$ as the set of classes of one-dimensional central extensions of the Lie algebra $\mathfrak{g}$. A one-dimensional central extension of $\mathfrak{g}$ is an exact sequence

$$0 \longrightarrow K \longrightarrow \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0$$

of Lie algebras and their homomorphisms, in which the image of the homomorphism of K into $\tilde{\mathfrak{g}}$ is contained in the center of $\tilde{\mathfrak{g}}$.

Let α be in $Z^2(\mathfrak{g}, K)$. To α we can associate the extension

$$0 \longrightarrow K \xrightarrow{i} \tilde{\mathfrak{g}} = K \oplus \mathfrak{g} \xrightarrow{\pi_2} \mathfrak{g} \longrightarrow 0$$

where the bracket of $\tilde{\mathfrak{g}}$ is given by

$$[(a,x),(b,y)] = (\alpha[x,y],[x,y]) \ .$$

We easily verify that the Jacobi's identities for this bracket are equivalent to α be a cocycle, and two cohomologeous cocycles correspond to equivalent extensions. The trivial extension, i.e. the sequence which splits, corresponds to the zero of the space $H^2(\mathfrak{g}, K)$.

Remark. Let $(\omega_1,\dots, \omega_n)$ be a basis of $\mathfrak{g}^*$ and $d\omega_i = \Sigma\, C^i_{jk}\, \omega_j \wedge \omega_k$ the structural equations of $\mathfrak{g}$ (the definition of a Lie algebra by its constants of structure or by its structural equations will be developed in Chapter 5). If θ is in $Z^2(\mathfrak{g}, K)$, then θ is a 2-exterior form on $\mathfrak{g}$ satisfying $d\theta = 0$ (d is the coboundary operator for the cohomology $H^*(\mathfrak{g},K)$; it corresponds to the differential operator of differential (left or not) forms on the Lie group G associated to $\mathfrak{g}$).
We put $\theta = d\omega_{n+1}$. We define a Lie algebra $\mathfrak{g}_1$ of dimension dim $\mathfrak{g} + 1$, whose structural equations are $d\omega_i = \Sigma\, C^i_{jk}\, \omega_j \wedge \omega_k$, $i=1,\dots,n$, and $d\omega_{n+1} = 0$.
This extension corresponds to the preceding extension.

III.3.2. The spaces $H^2(\mathfrak{g},\mathfrak{g})$

The space $Z^2(\mathfrak{g},\mathfrak{g})$ can be interpreted as the space of the infinitesimal deformations of the Lie algebra $\mathfrak{g}$, and two cohomologeous cocycles define their equivalent infinitesimal deformations. This permits us to interpret the space $H^2(\mathfrak{g},\mathfrak{g})$ as the set of infinitesimal deformation classes of $\mathfrak{g}$. We will devote one chapter for the general study of deformations. However, in order to understand better this interpretation of $H^2(\mathfrak{g},\mathfrak{g})$, we give here some ways of approaching the topic.

Let $\mathfrak{g}$ be a Lie algebra whose bracket is denoted by μ . A *(formal) deformation of* $\mathfrak{g}$ is a one-parameter family of bilinear alternated mappings on the vector space underlying $\mathfrak{g}$ of the form :

$$\mu_t(x,y) = \mu(x,y) + t\varphi_1(x,y) + t^2\varphi_2(x,y) + \dots$$

such that (i) φ_i is a bilinear alternated mapping on $\mathfrak{g}$,

(ii) the Jacobi conditions for μ_t are fulfilled (formally).

This last condition means that all the coefficients of the formal serie

$$\mu_t(\mu_t(x,y),z) + \mu_t(\mu_t(y,z),x) + \mu_t(\mu_t(z,x),y)$$

are zero.

If we develop this formal identity, then the first relations are :

$$(a_1)\quad \mu(\mu(x,y),z) + \mu(\mu(y,z),x) + \mu(\mu(z,x),y) = 0$$

It is the Jacobi's identity for μ.

$$(a_2)\quad \mu(\varphi_1(x,y),z) + \mu(\varphi_1(y,z),x) + \mu(\varphi_1(z,x),y) + \varphi_1(\mu(x,y),z) +$$
$$+ \varphi_1(\mu(y,z),x) + \varphi_1(\mu(z,x),y) = d\varphi_1 = 0.$$

This relation is nothing but $d\varphi_1 = 0$ and φ_1 is in $Z^2(\mathfrak{g},\mathfrak{g})$.

So the first term φ_1 of the deformation μ_t of μ , called *the infinitesimal deformation associated to* μ_t is an element of $Z^2(\mathfrak{g},\mathfrak{g})$.

Two deformations μ_t and γ_t of μ are called equivalent if there is a family (f_t) of nondegenerate endomorphisms of the vector space $\mathfrak{g}$ of the form

$$f_t(x) = x + \sum_{i>1} t^i g(x),$$

where g_i is an endomorphism of the vector space $\mathfrak{g}$, such that

$$\gamma_t(x,y) = f_t^{-1}(\mu_t(f_t(x),f_t(y))),\quad \text{for all } x \text{ and } y \text{ in } \mathfrak{g}.$$

If we consider two equivalent deformations μ_t and γ_t of μ, then their infinitesimal deformations φ_1 and ψ_1 are cohomologeous cocycles, i.e. verify $\varphi_1 - \psi_1 \in B^2(\mathfrak{g},\mathfrak{g})$.

Another interpretation. The aim of the deformation theory is to give an interesting tool for a local and topological study of the variety of Lie algebras laws (see Chapter 5). But one big disadvantage a consequence of this approach, lives in the nature of the deformation : a deformation is not an element of this variety by a parametrized curve in this space throught out by the given point μ corresponding to a given Lie algebra $\mathfrak{g}$. We can process this directly by considering only "infinitesimal" deformations. For don't become dumbfounded by the infinitesimal deformation associated with a formal deformation, we call these little deformations *perturbations*. It is more gratifying to define this notion within the framework of Nonstandard Analysis. So we have the notion of a law infinitely close to a given law μ. Later, we will see that every perturbation γ of μ can be written as

$$\gamma(x,y) = \mu(x,y) + \varepsilon_1\varphi_1(x,y) + \varepsilon_1\varepsilon_2\varphi_2(x,y) + \ldots + \varepsilon_1\varepsilon_2 \ldots \varepsilon_k\varphi_k(x,y),$$

where k is less than $(n^3-n^2)/2$ ($n = \dim \mathfrak{g}$) and the bilinear alternated mappings φ_i are linearly independent.

As γ verifies the Jacobi identity (it is a Lie algebra law), this implies that φ_1 is in $Z^2(\mu,\mu)$. We again find the interpretation of $Z^2(\mu,\mu)$: it is the space of the first terms called the *differential of perturbation* γ.

II.4. The spaces $H^3(\mathfrak{g},\mathfrak{g})$

This space coincides with the space of the obstruction of the deformations of $\mathfrak{g}$. We consider an element φ of the space $H^2(\mathfrak{g},\mathfrak{g})$. Are the infinitesimal deformations of this class derived with respect to some deformations of $\mathfrak{g}$? It is not the general case and we shall indicate some necessary conditions related to $H^3(\mathfrak{g},\mathfrak{g})$ for this to take place.

Let μ_t be a deformation of the Lie algebra law μ of $\mathfrak{g}$ and φ_1 its infinitesimal deformation.

By writing $\mu_t = \mu + t\,\varphi_1 + t^2\,\varphi_2 + \ldots$, and using Jacobi's conditions for μ_t, we obtain

$$0 = \mu \circ \mu + t\mu \circ \varphi_1 + t^2\left(\frac{1}{2}\varphi_1 \circ \varphi_1 + \mu \circ \varphi_2\right) + \ldots + t^n\left(\sum_{i+j=n} \varphi_i \circ \varphi_j + \mu \circ \varphi_n\right),$$

where $\alpha \circ \beta$ is the 3-linear alternated mapping given by

$$(\alpha\circ\beta)(x,y,z) = \alpha(\beta(x,y), z) + \alpha(\beta(y,z), x) + \alpha(\beta(z,x), y) +$$
$$+ \beta(\alpha(x,y), z) + \beta(\alpha(y,z), x) + \beta(\alpha(z,x), y)$$

In particular, we have $\mu \circ \varphi = d\varphi$. The conditions related with φ_1 are

$$\left\{\begin{array}{l} d\varphi_1 = 0 \\ \frac{1}{2}\varphi_1 \circ \varphi_1 = -d\varphi_2 \\ \varphi_1 \circ \varphi_{2i} + \sum\limits_{k+l=1+2i} \varphi_k \circ \varphi_l = -d\varphi_{2i+1} \\ \varphi_1 \circ \varphi_{2i+1} + \sum\limits_{\substack{k+l=1+2i \\ k\neq l}} \varphi_k \circ \varphi_l + \frac{1}{2}\varphi_i \circ \varphi_i = -d\varphi_{2i+2} \end{array}\right\}$$

The first relation shows that φ_1 is in $Z^2(\mathfrak{g},\mathfrak{g})$. The second relation concerns the 3-cochains $\varphi_1 \circ \varphi_1$. We can see that if φ_1 is in $Z^2(\mathfrak{g},\mathfrak{g})$, then $\varphi_1 \circ \varphi_1$ is in $Z^3(\mathfrak{g},\mathfrak{g})$, and the cohomological class of $\varphi_1 \circ \varphi_1$ corresponds to the classical product on the classes of cohomology. We denote this by $[\varphi_1]^2$. So this second relation could be interpreted as a necessary condition for $\varphi_1 \in Z^2(\mathfrak{g},\mathfrak{g})$ to be a infinitesimal deformation of a deformation μ_t of μ is that $[\varphi_1]^2 = 0$.

As we can find examples showing that this condition is not sufficient, we can read the third relation as

$$\varphi_1 \circ \varphi_2 = -d\,\varphi_3 .$$

This shows that the 3-cochain $\varphi_1 \circ \varphi_2$ is a 3-coboundary (it is a 3-cocycle because φ_2 verifies $d\,\varphi_2 = \varphi_1 \circ \varphi_1$). Thus, the cohomological class of $\varphi_1 \circ \varphi_2$ vanishes in $H^3(\mathfrak{g},\mathfrak{g})$. From the construction of φ_2, this class doesn't depend of the choice of φ_2. It is the "Massey cube" of the class of φ_1. We can denote it by $[\varphi_1]^3$. Continuing the argument, we see that for the existence of the deformation it is necessary that all the Massey products of φ_1 vanish; all of these powers are in the space $H^3(\mathfrak{g},\mathfrak{g})$. (Note that the i^{th}-power product is defined when the k^{th}-power products are zero for $k < i$).

Conclusions. The conditions related to φ_1 show that all Massey powers are zero. It is clear that all these conditions are satisfied when $H^3(\mathfrak{g},\mathfrak{g}) = 0$. We can conclude that *if* $H^3(\mathfrak{g},\mathfrak{g}) = 0$, *every cocycle* φ_1 *in* $H^2(\mathfrak{g},\mathfrak{g})$ *is the infinitesimal deformation of a deformation (formal) of the law* μ *of* $\mathfrak{g}$.

An element φ in $\varphi 1$ is called integrable if there is a deformation μ_t of μ such that $\mu_t = \mu + t\,\varphi 1 + t^2\,\varphi 2 + \dots$

Remarks.

1. In fact, it is sufficient that a subspace of $H^3(\mathfrak{g},\mathfrak{g})$ vanishes for the integrability of any cocycle φ_1 in $Z^2(\mathfrak{g},\mathfrak{g})$. This subspace $H^3_I(\mathfrak{g},\mathfrak{g})$ has been studied by Rauch [Rau].

2. We can also write the necessary and sufficient conditions for the integrability of an

element φ_1 of $Z^2(\mathfrak{g},\mathfrak{g})$ in perturbation theory. Note that, in this case, we can entirely solve the conditions system on φ_1. As we can prove that the conditions of integrability for perturbation theory and deformation theory are the same, we find another approach of the space $H^3{}_f(\mathfrak{g},\mathfrak{g})$ (see Chapter 5).

III. COHOMOLOGY OF FILTERED AND GRADED LIE ALGEBRAS

III.1. Graded Lie algebras. Filtered Lie algebras

Definition 1. *A Lie algebra* $\mathfrak{g}$ *is called* $\mathbb{Z}$*-graded if it admits a decomposition*

$$\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$$

where the subspace $\mathfrak{g}_i$ *satisfies* $[\mathfrak{g}_i,\mathfrak{g}_j]\subset\mathfrak{g}_{i+j}$ *for all* i *and* j *in* $\mathbb{Z}$.

Note that $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$.

Definition 2. *A Lie algebra* $\mathfrak{g}$ *is called filtered if it satisfies :*

(i) $\mathfrak{g} = \bigcup_{i\in\mathbb{Z}} S_i$, *where* S_i *is a subspace of* $\mathfrak{g}$ *for all* i

(ii) $[S_i,S_j]\subset S_{i+j}$, *for all* i *and* j *in* $\mathbb{Z}$

(iii) $S_i\subset S_j$, *when* $i>j$ (*in this case ,the filtration is called ascending*).

A filtration $\mathfrak{g} = \bigcup_{i\in\mathbb{Z}} S_i$ of $\mathfrak{g}$ is called *finite* if there are n_1 and n_2 in $\mathbb{Z}$ such that

$$S_i = \mathfrak{g} \quad \text{if} \quad i\le n_1,$$
$$S_i = \{0\} \quad \text{if} \quad i\ge n_2.$$

Every $\mathbb{Z}$-graduation on $\mathfrak{g}$ defines a filtration by putting

$$S_k = F_k(\mathfrak{g}) = \bigoplus_{i\geq k} \mathfrak{g}_i .$$

This filtration is associated to the graduation $\mathfrak{g} = \oplus\, \mathfrak{g}_i$. Conversely, if $\mathfrak{g}$ is filtered, i.e. $\mathfrak{g} = \cup_{i\in\mathbb{Z}} S_i$, we construct a graduated Lie algebra, denoted as gr $\mathfrak{g}$, whose underlying subspace is isomorphic to $\mathfrak{g}$, by putting

$$\mathrm{gr}\ \mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \overline{S_i} \quad \text{with} \quad \overline{S_i} = \frac{S_i}{S_{i+1}} .$$

The structure of a Lie algebra on gr $\mathfrak{g}$ is given by

$$[\overline{x}, \overline{y}] = \overline{[x, y]} \quad \text{with} \quad \overline{x} \in \overline{S_i} \quad \text{and} \quad \overline{y} \in \overline{S_j} ,$$

x and y being representative of $\overline{x}$ and $\overline{y}$ in S_i and S_j.

Remark. The filtration associated to the graduation of gr $\mathfrak{g}$ is given by the sequence

$$\left\{ F_k(\mathrm{gr}\ \mathfrak{g}) = \bigoplus_{i\geq k} \overline{S_i} \right\} .$$

It is isomorphic to the filtration of the initial Lie algebra $\mathfrak{g}$ given by the sequence S_k.

III.2. Graded and filtered $\mathfrak{g}$-modules

Let $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\, \mathfrak{g}_i$ be a graduated Lie algebra and V a $\mathfrak{g}$-module.

We say that V is a graded $\mathfrak{g}$-module if it is a direct sum of subspaces V_i : $V = \oplus_{i\in\mathbb{Z}} V_i$ such that $\mathfrak{g}_i . V_j \subset V_{i+j}$ for all i, j in $\mathbb{Z}$.

Now we consider a filtered Lie algebra $\mathfrak{g} = \cup_{i\in\mathbb{Z}} S_i$, with the filtration descending.

A $\mathfrak{g}$-module V is called a filtered $\mathfrak{g}$-module if

(i) V is provided with a filtration

$$\ldots V_j \supset V_{j+1} \supset \ldots \quad j \in \mathbb{Z} \quad \text{with} \quad V = \bigcup_{i\in\mathbb{Z}} V_i$$

(the filtration is descending).

(ii) $S_i . V_j \subset V_{i+j}$.

This filtration is called *finite* if there are m_1 and m_2 in $\mathbb{Z}$ such that

$$V_i = V \quad \text{if} \quad i \leq m_1 ,$$
$$V_i = \{0\} \quad \text{if} \quad i \geq m_2 .$$

The link between filtration and graduation is established in the same way as for Lie algebras :

(1) Let $V = \oplus_{i\in\mathbb{Z}} V_i$ a graded $\mathfrak{g}$-module. We put $F_k(V) = \oplus_{i\geq k} V_i$.

Then $V = \cup_{k\in\mathbb{Z}} F_k(V)$ is a filtered $\mathfrak{g}$-module associated to the graded $\mathfrak{g}$-module V, $V = \oplus_{i\in\mathbb{Z}} V_i$, $\mathfrak{g}$ being considered as the filtered Lie algebra associated to the given graded Lie algebra.

(2) Let $V = \cup_{i\in\mathbb{Z}} V_i$ be a filtered $\mathfrak{g}$-module. We put

$$\operatorname{gr} V = \bigoplus_{j\in\mathbb{Z}} \overline{V_j} \quad \text{with} \quad \overline{V_j} = V_j / V_{j+1} .$$

We consider the graded Lie algebra $\operatorname{gr} \mathfrak{g} = \oplus_{i\in\mathbb{Z}} \overline{S_i}$. We provide $\operatorname{gr} V$ with a structure of a graded $\operatorname{gr} \mathfrak{g}$-module :

Let $\overline{\mathfrak{g}} \in \overline{S_i}$ and $\overline{v} \in V_j$; we put $\overline{\mathfrak{g}}.\overline{v} = \overline{\mathfrak{g}v}$ (this is correctly defined).

In fact, the filtration associated to this graded $\operatorname{gr} \mathfrak{g}$-module is isomorphic to the filtration of the $\mathfrak{g}$-module V.

III.3. Graduation and filtration of the cohomology spaces

Let $\mathfrak{g} = \oplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ be a graded Lie algebra and $V = \oplus_{i\in\mathbb{Z}} V_i$ a graded $\mathfrak{g}$-module.

We can graduate the space of the j-cochains by putting $C^j(\mathfrak{g},V) = \bigoplus_{k\in\mathbb{Z}} C^j_k(\mathfrak{g},V)$, where

$$C^j_k(\mathfrak{g},V) = \left\{ c \in C^j(\mathfrak{g},V) : c(a_{i_1},\dots,a_{i_j}) \in V_{i_1+\dots+i_j+k} , a_i \in \mathfrak{g}_i \right\} .$$

The coboundary operator d is in respect to the graduation, i.e. :

$$d\left(C^j_k(\mathfrak{g},V)\right) \subset C^{j+1}_k(\mathfrak{g},V) .$$

This permits us to graduate the cocycle and cobord spaces :

$$Z^j(\mathfrak{g},V) = \bigoplus_{k\in\mathbb{Z}} Z^j_k(\mathfrak{g},V)\,,$$

$$B^j(\mathfrak{g},V) = \bigoplus_{k\in\mathbb{Z}} B^j_k(\mathfrak{g},V)\,,$$

and then the cohomology spaces :

$$H^j(\mathfrak{g},V) = \bigoplus_{k\in\mathbb{Z}} H^j_k(\mathfrak{g},V), \quad \text{where} \quad H^j_k(\mathfrak{g},V) = Z^j_k(\mathfrak{g},V)\,/\,B^j_k(\mathfrak{g},V)\,.$$

Remark. The equalities are understood in an enlarged sense for avoid the use of isomorphisms.

Now we consider the filtrations.

Let $\mathfrak{g} = \bigcup_{i\in\mathbb{Z}} S_i$ be a filtered Lie algebra and $V = \bigcup_{i\in\mathbb{Z}} V_i$ a filtered $\mathfrak{g}$-module.

We filter the space of cochains $C^j(\mathfrak{g},V)$ by putting :

$$C^j(\mathfrak{g},V) = \bigcup_{k\in\mathbb{Z}} F_k\, C^j(\mathfrak{g},V)\,,$$

where

$$F_k\, C^j(\mathfrak{g},V) = \left\{c \in C^j(\mathfrak{g},V) : c(a_{i_1},\dots,a_{i_j}) \in V_{i_1+\dots+i_j+k}\,,\, a_i \in S_i\right\}\,.$$

As $d\big(F_k\, C^j(\mathfrak{g},V)\big) \subset F_k\, C^{j+1}(\mathfrak{g},V)$, we can provide the spaces of the cocycles of the coboundaries and of cohomology by the filtration :

$$Z^j(\mathfrak{g},V) = \bigcup_{k\in\mathbb{Z}} F_k\, Z^j(\mathfrak{g},V)\,,$$

$$B^j(\mathfrak{g},V) = \bigcup_{k\in\mathbb{Z}} F_k\, B^j(\mathfrak{g},V)\,,$$

$$H^j(\mathfrak{g},V) = \bigoplus_{k\in\mathbb{Z}} F_k\, H^j(\mathfrak{g},V) \quad \text{with} \quad F_k\, H^j(\mathfrak{g},V) = F_k Z^j\,/\,F_k B^j\,.$$

Note : The correspondence between graduation and filtration that we have considered for the $\mathfrak{g}$-modules and Lie algebras, passes on the spaces of cocycles, coboundaries, and cohomology.

IV. SPECTRAL SEQUENCES

IV.1. Definition

In this section we take a quick glance at the theory of spectral sequences.

Definition 3. *A spectral sequence is a sequence* $(E_r,d_r)_{r\in\mathbb{N}}$, *where* E_r *is a module (on a given ring) and* d_r *an endomorphism of* E_r*, such that*

(1) $E_r = \bigoplus\limits_{p,q\in\mathbb{Z}} E_r^{p,q}$,

(2) $d_r \circ d_r = 0$,

(3) $d_r(E_r^{p,q}) \subset E_r^{p+r,q-r+1}$,

(4) $E_{r+1}^{p,q} = \dfrac{\operatorname{Ker} d_r^{p,q}}{\operatorname{Im} d_r^{p-q,q+r-1}}$ *with* $d_r^{p,q}$ *is the restriction of* d_r *to* $E_r^{p,q}$.

This ends the theory.

Remark. We often add the condition $E_r^{p,q} = \{0\}$ if one of the two indices (or the two) is negative. Such a spectral sequence is denoted as being of the first quadrant.

IV.2. Some properties

Let m be an integer. We consider a spectral sequence $(E_r,d_r)_{r\in\mathbb{N}}$ such that $E_r^{p,q} = \{0\}$, if we have $p < m$ or $q < m$ (the case $m = 0$ corresponds to the spectral sequence of the first quadrant).

The condition (4) of the definition of a spectral means that E_{r+1} is the cohomology of the complex (E_r,d_r). Then $E_{r+1}^{p,q}$ is a subquotient of $E_r^{p,q}$. We show that $E_r^{p,q}$ is also a subquotient of $E_0^{p,q}$. For this we construct the sequence :

$$0 \subset B_0^{p,q} \subset \overline{B}_1^{p,q} \subset \ldots \subset \overline{B}_\infty^{p,q} \subset \overline{Z}_\infty^{p,q} \subset \ldots \subset \overline{Z}_1^{p,q} \subset Z_0^{p,q} \subset E_0^{p,q}$$

such that :

$$E_{r+1}^{p,q} \cong \frac{Z_r^{p,q}}{B_r^{p,q}} \cong \frac{\overline{Z}_r^{p,q}}{\overline{B}_r^{p,q}} .$$

We put $Z_r^{p,q} = \operatorname{Ker} d_r^{p,q}$,

$B_r^{p,q} = \operatorname{Im} d_r^{p-q,q+r-1}$,

$\varphi_r^{p,q} : Z_r^{p,q} \to \dfrac{Z_r^{p,q}}{B_r^{p,q}} = E_{r+1}^{p,q}$ the canonical projection,

$\overline{Z}_1^{p,q} = (\varphi_0^{p,q})^{-1} (Z_1^{p,q})$,

$\overline{B}_1^{p,q} = (\varphi_0^{p,q})^{-1} (B_1^{p,q})$.

Hence, we already have the following sequence :

$$0 \subset B_0^{p,q} \subset \overline{B}_1^{p,q} \subset \overline{Z}_1^{p,q} \subset Z_0^{p,q} \subset E_0^{p,q}$$

$$\text{with } E_2^{p,q} = \frac{Z_1^{p,q}}{B_1^{p,q}} \cong \frac{\overline{Z}_1^{p,q}}{\overline{B}_1^{p,q}} .$$

To continue the construction, one puts :

$$\overline{\varphi}_1^{p,q} : \overline{Z}_1^{p,q} \to E_2^{p,q} \cong \frac{\overline{Z}_1^{p,q}}{\overline{B}_1^{p,q}} \text{ the canonical projection ,}$$

after having identified the two last spaces

$$\overline{Z}_2^{p,q} = (\overline{\varphi}_1^{p,q})^{-1} (Z_2^{p,q}),$$

$$\overline{B}_2^{p,q} = (\overline{\varphi}_1^{p,q})^{-1} (B_2^{p,q}),$$

and iterating this process. To finish, we denote :

$$\overline{Z}_\infty = \bigcap_r \overline{Z}_r^{p,q},$$

$$\overline{B}_\infty = \bigcap_r \overline{B}_r^{p,q},$$

$$E_\infty^{p,q} = \frac{\overline{Z}_\infty^{p,q}}{\overline{B}_\infty^{p,q}} .$$

As $E_r^{p,q} = 0$, when $p < m$ or $q < m$, we have :

$$\overline{B}_{p-M}^{p,q} = \overline{B}_{p-M+1}^{p,q} = \dots = \overline{B}_\infty^{p,q},$$

$$\overline{Z}_{q-M+1}^{p,q} = \overline{Z}_{q-M+2}^{p,q} = \dots = \overline{Z}_\infty^{p,q},$$

then

$$E_\infty^{p,q} \cong \frac{\overline{Z}_r^{p,q}}{\overline{B}_r^{p,q}} \cong E_{r+1}^{p,q}$$

when $r \geq \max(q+1-M, p-M)$.

IV.3. Exact sequence associated to a filtered complex

We have previously defined the notion of a filtered $\mathfrak{g}$-module. Here we use the more general notion of a filtered module when the action of $\mathfrak{g}$ doesn't occur.

Definition 4. *A filtered module is a module* V *provided by a family of submodules* $\{F_kV\}_{k\in\mathbb{Z}}$ *such that* $F_sV \supset F_kV$ *if* $k > s$ *(descending filtration).*

In this work, we consider only the filtrations satisfying :

$$F_rV = V \quad \text{for} \quad r < m, \quad m \in \mathbb{Z}.$$

Definition 5. *A complex of modules*

$$A : 0 \to A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \xrightarrow{d^2} \dots$$

is called filtered if all the modules A^p *are filtered and if the coboundary operators respect these filtrations, that is* :

$$\left\{ \begin{array}{ll} A^p = A^p_M \supset A^p_{M+1} \supset \dots, & \textit{for all } p, \\ \textit{with } d^p(A^p_q) \subset A^{p+1}_q. & \end{array} \right\}$$

We can extract from such complex of modules the following complexes :

$$A_q : 0 \to A^0{}_q \xrightarrow{d^0} A^1{}_q \xrightarrow{d^1} A^2{}_q \xrightarrow{d^2} \dots, \qquad \forall\, q \geq m,$$

and A_{q+1} is a subcomplex of A_q and $A_m = A$.

We note $H^*(A)$ as the cohomology of the complex A, and $H^*(A_q)$ as the cohomology of the complex A_q :

$$H^p(A) = \frac{Z^p(A)}{B^p(A)} \quad ; \quad H^p(A_q) = \frac{Z^p(A_q)}{B^p(A_q)}.$$

To unburden the notations, we write :

$$Z^p(\text{resp. } B^p) \text{ the space } Z^p(A) \text{ (resp. } B^p(A)),$$

and

$$Z_q^p(\text{resp. } B_q^p) \text{ the space } Z^p(A_p) \text{ (resp. } B^p(A_q))$$

We put

$$H^p(A)_q = \frac{Z_q^p + B^p}{B^p} \ .$$

This is the canonical image of $H^p(A_q)$ in $H^p(A)$. We deduce a filtration of $H^p(A)$.

Now we can construct a spectral sequence associated to this filtration.
Let r be in $\mathbb{Z}$. For each p and q in $\mathbb{Z}$ satisfying $p \geq m$ and $p + q \geq 0$, we put :

$$Z_r^{p,q} = A_p^{p+q} \cap (d^{p+q})^{-1}\left(A_{q+r}^{p+q+1}\right),$$

$$B_r^{p,q} = A_p^{p+q} \cap d^{p+q-1}\left(A_{q-r}^{p+q-1}\right).$$

We note that $Z_r^{p,q}$ contains $B_{r-1}^{p,q}$ and $Z_{r-1}^{p+1,q-1}$.

Let be $E_r^{p,q} = Z_r^{p,q} \big/ B_{r-1}^{p,q} + Z_{r-1}^{p+1,q-1}$.

As d^{p+q} maps $Z_r^{p,q}$ in $Z_r^{p+r,q-r+1}$ and $B_{r-1}^{p,q} + Z_{r-1}^{p+r,q-1}$ in $B_{r-1}^{p+r,q-r+1} + Z_{r-1}^{p+r+1,q-r}$, then we reclaim, by passing through a quotient, a morphism

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1} \ .$$

In particular :

$$Z_{-1}^{p+1,q-1} = A_{p+1}^{p+q},$$

$$B_{-1}^{p,q} = A_{p+1}^{p+q},$$

$$Z_0^{p,q} = A_p^{p+1},$$

then $E_0^{p,q} = A_p^{p+q} \big/ A_{p+1}^{p+q}$.

The sequence is well constructed.

V. THE HOCHSCHILD-SERRE SPECTRAL SEQUENCE

Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ a subalgebra of g, and V a $\mathfrak{g}$-module. We put

$$F^pC^{p+q}(\mathfrak{g},V) = \left\{c \in C^{p+q}(\mathfrak{g},V) \,/\, c(a_1,\dots,a_{p+q}) = 0 \text{ for } a_1,\dots,a_{q+1} \in h\right\} .$$

We have $C^r(\mathfrak{g},V) = F^0C^r(\mathfrak{g},V) \supset \dots \supset F^rC^r(\mathfrak{g},V) \supset F^{r+1}C^r(\mathfrak{g},V) = 0$. Then

$$dF^pC^{p+q}(\mathfrak{g},V) \subset F^pC^{p+q+1}(\mathfrak{g},V) .$$

So we obtain a filtration of the complex $C^\bullet(\mathfrak{g},V)$. We denote by $\left(E_r^{p,q}, d_r^{p,q}\right)$ the spectral sequence corresponding to this filtration (the study of these spectral sequences has been developed in the previous section).

Then

$$E_0^{p,q} = \frac{F^pC^{p+q}(\mathfrak{g},V)}{F^{p+1}C^{p+q}(\mathfrak{g},V)}$$

iss isomorphic to

$$C^q\left(k, \operatorname{Hom}\left(\Lambda^p\left(\frac{g}{h}, V\right)\right)\right)$$

and we have

$$E_1^{p,q} = H^q\left(k\,;\operatorname{Hom}\left(\Lambda^p\left(\frac{g}{k}, V\right)\right)\right),$$

$$E_2^{p,0} = H^p(\mathfrak{g},k;V),$$

$$E_2^{p,q} = H^p\left(\frac{\mathfrak{g}}{k}, H^q(k,V)\right)$$

In this notation, $H^\bullet(\mathfrak{g},h;V)$ denotes the cohomology of $\mathfrak{g}$ modulo $\mathfrak{h}$ with values in V (or relative cohomology). The corresponding cochains verify :

$$c(a_1,\dots,a_q) = 0 \quad \text{if} \quad a_1 \in \mathfrak{h} \quad \text{and} \quad dc(a_1,\dots,a_{q+1}) = 0 \quad \text{if} \quad a_1 \in \mathfrak{h} .$$

It is the same as saying

$$C^q(\mathfrak{g},\mathfrak{h};V) = \operatorname{Hom}_k\left(\Lambda^q\left(\frac{\mathfrak{g}}{\mathfrak{h}}\right), V\right) .$$

The spectral sequence constructed in this way is called the Hochschild-Serre sequence [H-S].

Applications

1. The Hochschild-Serre exact sequence

Let $\mathfrak{l}$ be an ideal of $\mathfrak{g}$ and V a $\mathfrak{g}$-module. We suppose that

$$H^n(\mathfrak{l},V) = \{0\} \quad \text{for all } n \quad \text{satisfying} \quad 0 < n < m \quad \text{and} \quad m \geq 1 \ .$$

We consider the set $H^m(\mathfrak{l},V)^{\mathfrak{g}}$ of $\mathfrak{g}$-invariant elements of $H^m(\mathfrak{l},V)$. Each element of $H^m(\mathfrak{l},V)^{\mathfrak{g}}$ has a representative which is the restriction to $\mathfrak{l}$ of a cochain σ in $C^m(\mathfrak{g},V)$ such that $d\sigma$ belongs to $H^{m+1}\left(\frac{\mathfrak{g}}{\mathfrak{l}}, V^{\mathfrak{l}}\right)$ $\left(\text{where } V^{\mathfrak{l}} = \left\{v \in V : a.v = 0 \ \forall a \in \mathfrak{l}\right\}\right)$.

This element doesn't depend upon the choice of $H^m(\mathfrak{g},V)^{\mathfrak{g}}$. We denote t_{m+1} as the homomorphism

$$t_{m+1} : H^m(\mathfrak{l},V)^{\mathfrak{g}} \rightarrow H^{m+1}\left(\frac{\mathfrak{g}}{\mathfrak{l}}, V^{\mathfrak{l}}\right)$$

that we have constructed. Therefore, we have the exact sequence :

$$0 \rightarrow H^m\left(\frac{\mathfrak{g}}{\mathfrak{l}}, V^{\mathfrak{l}}\right) \xrightarrow[l_m]{} H^m(\mathfrak{g},V) \xrightarrow[r_m]{} H^m(\mathfrak{l},V)^{\mathfrak{g}} \xrightarrow[t_{m+1}]{} H^{m+1}\left(\frac{\mathfrak{g}}{\mathfrak{l}}, V^{\mathfrak{l}}\right) \xrightarrow[l_{m+1}]{} H^{m+1}(\mathfrak{g},V) \ .$$

In this sequence, the homomorphism l_m is defined by looking upon the cochains of $\frac{\mathfrak{g}}{\mathfrak{l}}$ in $V^{\mathfrak{l}}$ as cochains of $\mathfrak{g}$ in V.

As for the homomorphism r_m , this is a homomorphism of restriction.

We suppose now that $H^n(I,V) = 0$ for $2 \leq n \leq m$ and $m > 1$. In this case, we have the following exact sequence

$$H^m\left(\frac{\mathfrak{g}}{\mathfrak{l}}, V^{\mathfrak{l}}\right) \xrightarrow[l_m]{} H^m(\mathfrak{g},V) \xrightarrow[r'_m]{} H^{m-1}\left(\frac{\mathfrak{g}}{\mathfrak{l}}, H^{\mathfrak{l}}(\mathfrak{l},V)\right) \xrightarrow[d'_2]{} H^{m+1}\left(\frac{\mathfrak{g}}{\mathfrak{l}}, V^{\mathfrak{l}}\right) \xrightarrow[l_{m+1}]{} H^{m+1}(\mathfrak{g},V)$$

Here the homomorphism r'_m is the restriction to the first argument of the chosen cocycle representing the given cohomological class for the ideal I. The homomorphism d'_2 corresponds to $d_2 : E_2^{m-1,m} \rightarrow E_2^{m+1,0}$.

In particular, if

$$p = \dim\left(\frac{\mathfrak{g}}{I}\right) \quad \text{and} \quad q = \dim(I)\,,$$

then

$$H^{p+q}(\mathfrak{g},V) \cong H^p\left(\frac{\mathfrak{g}}{\mathfrak{l}}, H^q(\mathfrak{l},V)\right) .$$

2. Factorisation theorem

Suppose now that V is a finite-dimensional $\mathfrak{g}$-module such that $\mathfrak{g}/\mathfrak{l}$ is semisimple. Then

$$H^n(\mathfrak{g},V) \cong \sum_{i+j=n} H^i\left(\frac{\mathfrak{g}}{\mathfrak{l}}, K\right) \otimes H^j(\mathfrak{l},V)^{\mathfrak{g}} .$$

Example. Let $\mathfrak{g} = \mathbb{C}^{15} \oplus sl(2,\mathbb{C})$ be the semidirect sum corresponding to the irreducible representation of $sl(2,\mathbb{C})$ in $\mathbb{C}^{15}$. We put $V = \mathfrak{g}$ and $\mathfrak{l} = \mathbb{C}^{15}$ (considered as an Abelian Lie algebra). Then, we have $H^2(\mathfrak{g},\mathfrak{g}) \neq 0$.
This computation has been done by Richardson to describe a rigid Lie algebra (see Chapter 5) whose second cohomology space $H^2(\mathfrak{g},\mathfrak{g})$ is not trivial.

VI. COHOMOLOGICAL CALCULUS AND COMPUTERS

To end this chapter we shall discuss some logicials of formal calculus which have been developed. For example, Mapple and Mathematica, etc. These logicials permit us to use personal computers for the study of the cohomology of Lie algebras. Of course, the calculus related with the cohomology is linear, but the dimension of the underlying spaces and the rank of used matrices are very big, and this implies a quick limitation. With these logicials we can determine without toil, not only the dimensions of the cohomological spaces, but also the basis of these spaces and also for dimensions of large enough Lie algebras (this depends only of the possibility of the computer). Here we present a program using *Mathematica* which permits :
(1) the verification of the Jacobi identities;
(2) the calculus of the dimensions of $B^2(\mathfrak{g}, \mathfrak{g})$ and $Z^2(\mathfrak{g}, \mathfrak{g})$;
(3) the determination of a basis of $H^2(\mathfrak{g}, \mathfrak{g})$;
(4) the verification of the nilpotency.

VI.1 The MATHEMATICA Program

```
BeginPackage["ALN`"]

mu::usage      = "mu[_,_] is the ley of Lie algebra"
Adj::usage     = "Adj[Y] compute the adjoint of the vector Y.
  Adj[k] is the adjoint of the vector x[k]."
Jacobi::usage  = "Jacobi[ ] gives the Jacobi conditions."
NilPot::usage  = "NilPot[ ] gives the nilpotence conditions."
DimB2::usage   = "DimB2[ ] computes the dimension of B2(mu,mu)."
DimZ2::usage   = "DimZ2[ ] computes the dimension of Z2(mu,mu)."
DimH2::usage   = "DimH2[ ] computes the dimension of H2(mu,mu)."
BaseDer::usage = "BaseDer[ ] determines a base of Der(mu)."
BaseB2::usage  = "BaseB2[ ] determines a base of B2(mu,mu)."
BaseZ2::usage  = "BaseZ2[ ] determines a base of Z2(mu,mu)."
BaseH2::usage  = "BaseH2[ ] determines a base of H2(mu,mu)."

{`x,`n};

Begin["`Private`"]


mu[0, x_] := 0;
mu[x_, 0] := 0;
mu[x_, x_] := 0;
mu[x_, y_] := -mu[y, x] /; OrderedQ[{y,x}];
mu[x_+y_, z_] := mu[x, z] + mu[y, z];
mu[z_, x_+y_] := mu[z, x] + mu[z, y];
mu[a_ x_, y_] := a mu[x,y];
mu[x_, a_ y_] := a mu[x,y];


(* ADJOINTS *)

Coord[i_Integer, v_] := v /. Table[x[j]->If[j==i, 1, 0], {j,1,n}]
Coord[v_] := Table[Coord[i, v], {i, 1, n}]

Adj[i_Integer] := Adj[i] = Adj[x[i]]
Adj[v_] :=
Module[{adjv, f},
    adjv = Array[0, {n,n}];
    For[f = 1, f <= n, f++, adjv[[f]] = Coord[mu[v, x[f]]]];
    Transpose[adjv]
]
```

```
(* JACOBI CONDITIONS  *)

Jacobi[ ] :=
Module[{i, j, k, LRel},
    LRel = {};
    For[i = 1, i <= n-2, i++,
For[j = i+1, j <= n-1, j++,
    For[k = j+1, k <= n, k++,
LRel = Union[LRel, Jacobi[i, j, k]]
    ];
];
    ];
    LRel
]


Jacobi[i_, j_, k_] :=
Module[{LRel},
    LRel = Coord[mu[mu[x[i], x[j]], x[k]] +
 mu[mu[x[j], x[k]], x[i]] +
 mu[mu[x[k], x[i]], x[j]]
   ];
    Select[LRel, Function[!NumberQ[#]]]
]



(* NILPOTENCE CONDITIONS  *)

NilPot[ ] :=
Module[{i, LRel, cadj},
    LRel = {};
    For[i = 1, i <= n, i++, LRel = Union[LRel, NilPot[i]]];
    LRel
]


NilPot[i_Integer] :=
Module[{pol, lrel, laux, lambda},
    pol = Det[lambda IdentityMatrix[n] - Adj[i]];
    Select[Expand[CoefficientList[pol, lambda]],
   Function[!NumberQ[#]]]
]
```

```
(* RESOLUTION OF LINEAR SYSTEMS. ... *)

Desp[ec_]:=
Module[{ecx, lv, v, rs},
    ecx = Expand[ec];
    lv = Reverse[Flatten[Variables[ecx]]];
    If[Length[lv] > 0,
v = lv[[1]];
rs = Flatten[Solve[ecx == 0, v]][[1]];
Set[Evaluate[rs[[1]]], rs[[2]]]
    ]
]


(* DIMENSION OF SECOND COHOMOLOGICAL GROUP *)

DimB2[ ]:=
Module[{A, a, i, j, f, Df, Base},
    Base=Array[x, {n}];
    A=Array[a, {n,n}];
    f[v_]:=Coord[v] . A . Base;
    Df[i_, j_] := mu[f[x[i]], x[j]] +
  mu[x[i], f[x[j]]] -
  f[mu[x[i], x[j]]];

    For[i=1, i<=n, i++,
For[j=i+1, j<=n, j++,
    Map[Desp, Coord[Df[i,j]]]
]
    ];

    n^2-Length[Flatten[Variables[A]]]
]
```

```
DimZ2[ ]:=
Module[{A, TA, a, i, j, k, f, Df, Base},
    Base = Array[x, n];
    A = Array[a, {n,n,n}];
    TA = Transpose[A, {1,3,2}];
    f[u_, v_] := Coord[u] . TA . Coord[v] . Base;

    For[i = 1, i <= n - 1, i++,
For[j = i + 1, j <= n, j++,
    For[k = 1, k <= n, k++,
a[j,i,k] = -a[i,j,k]
    ]
]
    ];

    For[i = 1, i <= n, i++,
For[k = 1, k <= n, k++,
    a[i,i,k] = 0
]
    ];

    Df[i_Integer, j_Integer, k_Integer] :=
 mu[f[x[i], x[j]], x[k]] +
 mu[f[x[j], x[k]], x[i]] +
 mu[f[x[k], x[i]], x[j]] +
 f[mu[x[i], x[j]], x[k]] +
 f[mu[x[j], x[k]], x[i]] +
 f[mu[x[k], x[i]], x[j]];


    For[i = 1, i <= n - 2, i++,
For[j = i + 1, j <= n - 1, j++,
    For[k = j + 1, k <= n, k++,
Map[Desp, Coord[Df[i,j,k]]]
    ]
]
    ];

    Length[Flatten[Variables[A]]]
]


DimH2[ ] := DimZ2[ ] - DimB2[ ]
```

```
(* BASE OF H2 *)

LDif[genV_List, genL_List] :=
Module[{dim,  v, p, ini, i, j, base, genv, sol},
    d = Dimensions[First[genV]];
    base = RowReduce[Map[Flatten, genL]];
    genv = Map[Flatten, genV];
    dim = Length[genv[[1]]];

    p = {};
    For[i = 1, i <= Length[base], i++,
j = 1;
While[j <= dim && base[[i, j]] == 0, j++];
If[j > dim, base = Drop[base, {i}], AppendTo[p, j]];
    ];

    ini = Length[base] + 1;

    For[i = 1, i <= Length[genV], i++,
v = genv[[i]] - Sum[genv[[i, p[[j]]]] * base[[j]],
    {j, 1, Length[base]}
];
j = 1;
While[j <= dim && v[[j]] == 0, j++];
If[j <= dim,
    v = 1/v[[j]] * v;
    AppendTo[p, j];
    AppendTo[base, v];
];
    ];

    sol = base[[Range[ini, Length[base]]]];
    For[i = Length[d], i > 1, i--,
sol = Map[Function[Partition[#, d[[i]]]], sol]
    ];
    sol
]
```

```
BaseDer[ ] :=
Module[{A, a, i, j, f, Df, base},
    base = Array[x, {n}];
    A = Array[a, {n,n}];
    f[v_] := Coord[v] . A . base;
    Df[i_, j_] := mu[f[x[i]], x[j]] +
  mu[x[i], f[x[j]]] -
  f[mu[x[i], x[j]]];

    For[i = 1, i <= n, i++,
For[j = i+1, j <= n, j++,
    Map[Desp, Coord[Df[i,j]]]
]
    ];

    ParToBase[A]
]


ParToBase[lExPar_List]:=
Module[{lPar, dim, i, j},
    lPar = Flatten[Variables[lExPar]];
    dim = Length[lPar];
    VectorBase[i_] := lExPar /.
      Table[lPar[[j]] -> If[j == i, 1, 0],
    {j, 1, dim}
      ];
    Array[VectorBase, dim]
]



BaseB2[ ] :=
Module[{BBaseEnd},
    BaseEnd = Map[Function[Partition[#, n]],
  IdentityMatrix[n^2]
      ];
    Map[Delta, LDif[BaseEnd, BaseDer[ ]]]
]
```

```
Delta[A_List] :=
Module[{f, Df, base, Dfij},
    base = Array[x, {n}];
    f[v_] := Coord[v] . A . base;
    Df[i_, j_] := mu[f[x[i]], x[j]] +
  mu[x[i], f[x[j]]] -
  f[mu[x[i], x[j]]] ;

    Table[If[k == 1, Dfij = Df[i,j]];
  Coord[k, Dfij],
  {i,1,n}, {j,1,n}, {k,1,n}
    ]
]


BaseZ2[ ] :=
Module[{A, TA, a, i, j, k, f, Df, Base},
    Base = Array[x, n];
    A = Array[a, {n,n,n}];
    TA = Transpose[A, {1,3,2}];
    f[u_, v_] := Coord[u] . TA . Coord[v] . Base;

    For[i = 1, i <= n - 1, i++,
For[j = i + 1, j <= n, j++,
    For[k=1, k<=n, k++,
a[j,i,k]=-a[i,j,k]
    ]
]
    ];

    For[i = 1, i <= n, i++,
For[k = 1, k <= n, k++,
    a[i,i,k] = 0
]
    ];

    Df[i_Integer, j_Integer, k_Integer] :=
  mu[f[x[i], x[j]], x[k]] +
  mu[f[x[j], x[k]], x[i]] +
  mu[f[x[k], x[i]], x[j]] +
  f[mu[x[i], x[j]], x[k]] +
  f[mu[x[j], x[k]], x[i]] +
  f[mu[x[k], x[i]], x[j]];
```

```
    For[i = 1, i <= n - 2, i++,
For[j = i + 1, j <= n - 1, j++,
    For[k = j + 1, k <= n, k++,
Map[Desp, Coord[Df[i,j,k]]]
    ]
]
    ];

    ParToBase[A]
]



BaseH2[ ] := LDif[BaseZ2[ ], BaseB2[ ]]

End[ ]
EndPackage[ ]
```

VI.2. How to use this program

First, one introduces the constants of the structure by giving the dimensions of the Lie algebra and writing the nontrivial brackets :

$$\mathrm{mu}[x[i],x[j]] = C_{ij}^{k} x[k] .$$

For example, for the five-dimensional Heisenberg algebra, we write:

n = 5,

mu[x[2],x[3]] = x[1],

mu[x[4],x[5]] = x[1].

Now, the procedure Jacobi[ ] returns all the Jacobi equations which are not satisfied. In this example, Jacobi[ ] gives { }.

If we want compute the dimension of B^2 or Z^2 or H^2, it is sufficient to write dimB2[ ] or dimZ2[ ] . or dimH2[ ]

To explicitly obtain a basis for these spaces, write baseH2[ ], or baseZ2[ ] or baseB2[ ].

CHAPTER 4

COHOMOLOGY OF SOME NILPOTENT LIE ALGEBRAS

Today a few concrete results concerning cohomological calculations on nilpotent Lie algebras are revealed. In this chapter we present these calculations for some particular but nevertheless important classes of nilpotent Lie algebras : i.e. the filiform algebras and the nilradicals of parabolic algebras. We restrict ourselves here to the cohomology whose coefficients are in the adjoint module; some results concerning the cohomology with values in the field are given in the last chapter. As the Lie algebra of derivations related with the first cohomology group, we endeavour to describe this algebra.

I. DERIVATIONS OF SOME FILIFORM ALGEBRAS

Recall (see Chapter 2) that a (n+1)-dimension filiform algebra has a maximal index of nilpotency equal to n. Among all these algebras, only four play an important role. They are described in the basis $(e_0,e_1,...,e_n)$ by :

$$L_n : [e_0,e_i] = e_{i+1}, \qquad i = 1,...,n-1,$$

$$Q_n : [e_0,e_i] = e_{i+1}, \qquad i = 1,...,n-1 \text{ and } n=2k+1,$$
$$[e_i,e_{n-i}] = (-1)^i e_n, \qquad i = 1,...,n-1,$$

$$R_n : [e_0,e_i] = e_{i+1}\,, \qquad i = 1,\dots,n-1\,,$$
$$[e_1,e_i] = e_{i+2}\,, \qquad i = 2,\dots,n-2\,,$$

$$W_n : [e_0,e_i] = e_{2+1}\,, \qquad i = 1,\dots,n-1\,,$$
$$[e_i,e_j] = \frac{6(i-1)!(j-1)!(j-i)}{(i+j)!}\, e_{i+j+1} \quad \text{with} \begin{cases} 1 \le i,j \le n-2 \\ i+j+1 \le n\,. \end{cases}$$

I.1. The algebra of derivations of L_n

Proposition 1. *The linear mappings* ad e_0 , ad e_1 ,..., ad e_{n-1} ,h_2, h_3, ...,h_{n-1}, t_1, t_2 *and* t_3 *of the space* L_n, *with*

$h_k(e_i) = e_{i+k}$ *for any* $1 \le i \le n-k$,

$t_1(e_i) = e_i$ *for any* $1 \le i \le n$, $t_1(e_0) = 0$,

$t_2(e_0) = e_0$, $t_2(e_i) = (i-1)e_i$ *for any* $2 \le i \le n$, $t_3(e_0) = e_1$,

form a basis of Der L_n.

Proof. It is easy to verify that these mappings are the derivations of L_n and that they are linearly independent.

Let $d \in$ Der L_n . The Lie algebra L_n admits a natural graduation $L_n = \oplus_{i\in\mathbb{Z}}(L_n)_i$, where $(L_n)_i$ is the space generated by the vectors (e_i) for $2 \le i \le n$, while $(L_n)_1$ is generated by e_0 and e_1. The other spaces are reduced to $\{0\}$. This graduation is associated to the descending central sequence (see Chapter 2). The derivation d can be decomposed with respect to the graduation :

$$d = d_0 + d_1 + \dots + d_n \qquad \text{with} \qquad d_i \in \text{Der } L_n \;.$$

Consider $d_0 \in$ Der L_n. It is obvious that

$$d_0(e_k) = \left\{ \begin{array}{ll} \theta_0 e_0 + \theta'_0 e_1 & \text{if } k = 0 \\ \theta_1 e_0 + \theta'_1 e_0 & \text{if } k = 1 \\ \theta_k e_k & \text{if } 2 \le k \le n \end{array} \right\} ,$$

for some $\theta_0, \theta'_0, \theta_1, \theta'_1, \theta_2, \ldots, \theta_n \in K$. It follows from the equality

$$d_0[a, b] - [d_0(a), b] - [a, d_0(b)] = 0,$$

for $a = e_1$ and $b = e_k$ for $2 \le k < n$, that $\theta'_1 = 0$. This same equation, with $a = e_0$ and $b = e_k$ for $1 \le k < n$, yields $\theta_{k+1} = \theta_0 + \theta_k$. Considering the values (0, 1, 0), (1, 0, 0), and (0, 0, 1) for the triple $(\theta_0, \theta_1, \theta'_0)$, we arrive at t_1, t_2, and t_3, respectively. Thus, we have

$$d_0 = \theta_1 t_1 + \theta_2 t_2 + \theta'_0 t_3 .$$

Now consider $d_k \in \mathrm{Der}\, L_n$ with $k \ge 1$. It is obvious that

$$d_k(e_i) = \left\{ \begin{array}{ll} \tau_0 e_{k+1} & \text{for } i = 0 \\ \tau_i e_{k+i} & \text{for } 1 \le i \le n-k \end{array} \right\},$$

for some $\tau_0, \tau_1, \ldots, \tau_{n-k}$. Put

$$d'_k = d_k + \tau_0\, \mathrm{ad}\, e_k - \tau_1 h_k .$$

Then, $d'_k(e_0) = d'_k(e_1) = 0$. We will show that $d'_k = 0$. Indeed, suppose $d'_k \ne 0$. Then there exists an integer m such that $d'_k(e_m) \ne 0$ and $d'_k(e_i) = 0$ for $i < m$ (clearly $m \ge 2$). This implies that

$$d'_k[e_0, e_{m-1}] - [d'_k(e_0), e_{m-1}] - [e_0, d'_k(e_{m-1})] = d'_k(e_m) \ne 0 ,$$

which is impossible, proving the proposition.

Corollary 1. $\dim \mathrm{Der}\, L = 2n + 1$.

Corollary 2. $\dim H^1(L, L) = n + 1$.

In fact

$$H^1(L_n, L_n) = \frac{Z^1(L_n, L_n)}{B^1(L_n, L_n)}, \quad Z^1(L_n, L_n) = \mathrm{Der}(L_n) \text{ and } B^1(L_n, L_n) = \mathrm{ad}\, L_n ,$$

$\mathrm{Dim}\ B^1(L_n, L_n) = n$ because $\dim Z(L_n) = 1$.

Corollary 3. $\dim B^2(L_n, L_n) = n^2$.

In fact $B^2(L_n, L_n) = \{df \text{ for } f \in \mathrm{End}(L_n)\}$,

where $df(x,y) = -f[x,y] + [f(x),y] + [x,f(y)] \quad x, y \in L_n$.

I.2. The algebra of derivations of W_n

Proposition 2. *Let* h, t_1, t_2, t_3 *be the endomorphism of* W_n *defined by*

$$h(e_i) = (i+1)e_i \qquad 0 \le i \le n$$

$$t_1(e_i) = e_n \qquad t_1(e_i) = 0 \qquad i \ne 1$$

$$t_2(e_i) = e_{n-3+i} \quad i = 1,2,3 \qquad t_2(e_i) = 0 \qquad i \ge 4$$

$$t_3(e_i) = e_{n-2+i} \quad i = 1,2 \qquad t_3(e_i) = 0 \qquad i \ge 3.$$

Then, the endomorphism $\mathrm{ad}\, e_0$, $\mathrm{ad}\, e_1$, ..., $\mathrm{ad}\, e_{n-1}$, h, t_1, t_2 and t_3 *form a basis of* $\mathrm{Der}W_n$.

Proof. These endomorphisms are independents and belong to $\mathrm{Der}W_n$. Let d be a derivation of W_n. We can grade W_n by putting
We put $W_n = \oplus_{i\in\mathbb{Z}}(W_n)_i$, where $(W_n)_i$ is the space generated by e_{i-1} for $i \ge 1$ and $(W_n)_j = \{0\}$ for $j \ge n+2$ or $j \le 0$. This graduation is not associated to the filtration given by the filtration descending central sequence. The derivation d can be decomposed as $d = d_{-1} + d_0 + \dots + d_n$ with

$$d_i \in \mathrm{Der}(W_n)_i \qquad -1 \le i \le n,$$

$$d_i(W_n)_j \subset (W_n)_{i+j}.$$

Note that the ideals $C^i(W_n)$ are conserved by the derivations ; these ideals are characteristic. In the formal decomposition of d, $d = \oplus_{i\in\mathbb{Z}} d_i$, we have $d_k = 0$ for $k \le -2$.

By the same reasoning, we have $d_{-1}(e_i) = 0$ for $1 \le i \le n$. But as d_{-1} is a derivation, we don't have $d_{-1}(e_1) = \alpha e_0$ except the case $\alpha = 0$. This shows that $d_{-1} = 0$. So $d = d_0 + d_1 + \dots + d_n$. By adding, if necessary, a linear combination of the derivations $\mathrm{ad}\, e_0$, ..., $\mathrm{ad}\, e_{n-1}$, h and t_1, we can suppose that

$$d = d_0 + d_1 + \dots + d_{n-2}$$

and $\qquad d(e_0) = 0$.

We put $d_k(e_j) = \alpha_j^k e_{k+j}$ for $0 \le k \le n-2$ and $1 \le j \le n-k$.

The identity

$$d_k[e_i,e_j] = [d_k(e_i),e_j] + [e_i,d_k(e_j)]$$

gives

$$\alpha_1^k = \alpha_2^k = \dots = \alpha_{n-k}^k \quad \text{for} \quad i = 0$$

and

$$\alpha_1^k = 0 \quad \text{with } 0 \le k \le n-4 \quad \text{and} \quad k \ne 1 \quad \text{for} \quad i = 1 \quad \text{and} \quad j = 2 \, .$$

This proves the proposition.

Corollary 1. Dim $\mathrm{Der}(W_n) = n + 4$.

Corollary 2. Dim $H^1(W_n, W_n) = 4$.

Corollary 3. Dim $B^2(W_n, W_n) = n^2 + n - 3$.

I.3. The algebra of derivations of Q_n ($n = 2k+1$)

Proposition 3. *Let* $t_1, t_2, h_3, h_5, \dots, h_{2k-1}$ *be the elements of* $\mathrm{End}(Q_n)$ *defined by*

$$\begin{array}{lll} t_1(e_0) = -e_1 & & \\ t_1(e_i) = e_i & \text{if} & 1 \le i \le n-1 \\ t_1(e_n) = 2e_n & & \end{array}$$

$$\begin{array}{lll} t_2(e_0) = e_0 + e_1 & & \\ t_2(e_i) = (i-1)e_i & \text{if} & 1 \le i \le n-1 \\ t_2(e_n) = (n-2)e_n & & \end{array}$$

$$\begin{array}{lll} h_s(e_i) = e_{i+s} & \text{if} & 1 \le i \le n-s \\ h_s(e_i) = 0 & \text{if not.} & \end{array}$$

Then the endomorphisms ad e_i ($i = 0,\dots,n-1$), t_1, t_2 and $h_3, h_5, \dots, h_{2k-1}$ *form a basis of* $\mathrm{Der}(Q_n)$.

Proof. The independence and the existence of these endomorphisms in $\mathrm{Der}(Q_n)$ can be easily verified.

One grades Q_n by the filtration associated to the descending central sequence (as for L_n). If d is in $\mathrm{Der}(Q_n)$, it can be decomposed as

$$d = d_0 + \dots + d_{n-1}$$

with $d_i \in \mathrm{Der}(Q_n)$ and $d_i(Q_n)_j \subset (Q_n)_{i+j}$ for $0 \le i \le n-1$ and $i \le j \le n$.

The derivation d_0 verifies

$$d_0(e_0) = \alpha_0\, e_0 + \beta_0\, e_1$$
$$d_0(e_1) = \alpha_1\, e_0 + \beta_1\, e_1 \qquad \alpha_i,\ \beta_i \in \mathbb{C}$$
$$d_0(e_i) = \alpha_i\, e_i \qquad \text{if} \qquad i \geq 2\,.$$

As $d_0[e_1,e_i] = [d_0(e_1),e_i] + [e_1,d_0(e_i)]$, one obtains $d_1 = 0$.

This identity, calculated for the pair (e_0,e_i), gives $\alpha_{i+1} = \beta_1 + i\alpha_0$ (for $i = 1,\dots,n-2$) and $\alpha_n = \alpha_0 + \alpha_{n-1} - \beta_0$ (for $i = n-1$). For the pair (e_1,e_{n-1}), one obtains $\alpha_n = \beta_1 + \alpha_{n-1}$. Then, all the coefficients can be exprimed with only α_0 and β_1. The derivations corresponding to $(\alpha_0 = 1,\ \beta_1 = 0)$ and $(\alpha_0 = 0,\ \beta_1 = 1)$ are nothing but t_2 and t_1. Now, consider the derivations d_s with $n-1 \geq s \geq 1$.

One has :

$$d_s(e_0) = \alpha_0\, e_{s+1}$$
$$d_s(e_i) = \alpha_s\, e_{s+i} \qquad \text{if} \qquad 1 \leq i \leq n-k$$
$$d_s(e_i) = 0 \qquad \text{if not}\ .$$

Even if we add to d_s a multiple of ad e_s, we can suppose $\alpha_0 = 0$. The identity of the derivation applies to the pairs (e_0, e_i), and with $1 \leq i < n-s-1$ gives :

$$\alpha_1 = \alpha_2 = \dots = \alpha_{n-s}\ .$$

The same identity applies to the pairs (e_i,e_j) with $1 \leq i, j \leq n-s, i+j = n-s$, but for $s < n-1$ gives :

$$(-1)^i\alpha_1 = (-1)^j\alpha_1 \qquad \text{where} \qquad i+j = n-s\,.$$

If s is even, we have necessary $\alpha_1 = 0$ and $d_s = 0$. If s is odd, the identity is obviously satisfied because i and j have the same parity (recall that n is odd).

Then $\qquad d_s = 0 \qquad$ if s is even,

$\qquad\qquad d_s(e_i) = \alpha_1\, e_{s+i} \qquad$ if s is odd.

We can conclude that

$$d_{2s'+1} = h_{2s'+1} \quad \text{with} \quad 1 \leq s' \leq k-1 \quad \text{and recall that} \quad d_1 = \alpha_1 \,\text{ad}\, e_0\,.$$

The proposition is proved.

Corollary 1. Dim $\mathrm{Der}(Q_{2k+1}) = 3k+3$.

Corollary 2. Dim $H^1(Q_{2k+1}, Q_{2k+1}) = k+2$.

Corollary 3. Dim $B^2(Q_{2k+1}, Q_{2k+1}) = 4k^2 + 5k + 1$.

I.4. The algebra of derivations of R_n

Proposition 4. *Let* $t, h_2, h_3, ..., h_{n-1}$ *be the endomorphisms of* R_n *defined by*

$$t(e_i) = (i+1)e_i \ , \qquad 0 \le i \le n ,$$

$$h_k(e_i) = e_{i+k} \ , \quad 1 \le i \le n-k, \qquad h_k(e_0) = 0 ,$$

hen the endomorphisms ad e_i $(i = 0, 1, ..., n-1)$, t, $h_2, ..., h_{n-1}$ *constitute a basis of* $\mathrm{Der}(R_n)$.

Proof. Note that these endomorphisms are the derivations of R_n and form a free linear system. Let d be a derivation of R_n. We consider the graduation of R_n given by $R_n = \oplus_{i\in\mathbb{Z}}(R_n)_i$, where $(R_n)_i$ is the subspace of R_n generated by the vector e_{i-1}, $i=1..n+1$ and $(R_n)_j = \{0\}$ for $j \le 0$ and $j \ge n+2$. Again, we note that this graduation is not associated to the filtration defined by the central descending sequence. Each ideal $C^i(R_n)$ is stable for the derivations and is generated by the vectors $e_{i+1}, ..., e_n$ as soon as $i \ge 1$. Then, the derivation d can be written :

$$d = d_{-1} + d_0 + ... + d_n ,$$

with $d_i \in \mathrm{Der}(R_n)$, $d_i(R_n)_j \subset (R_n)_{i+j}$.

Consider $d_0 \in \mathrm{Der}(R_n)$. We have

$$d_0(e_i) = \alpha_i e_i \ , \qquad i = 0, 1, ..., n .$$

The identity

$$d_0[e_0,e_i] = [d_0(e_0),e_i] + [e_0,d(e_i)] ,$$

where $1 \le i \le n-1$ gives

$$\alpha_{i+1} = \alpha_0 + \alpha_i \ , \quad 1 \le i \le n-1 .$$

The same identity, applied to the pair (e_1,e_i) with $2 \le i \le n-2$, gives

$$\alpha_{i+2} = \alpha_1 + \alpha_i \ , \quad 2 \le i \le n-2 .$$

Then $\alpha_i = (i+1)\alpha_0$ and $d_0 = \alpha_0 t$. Now, we consider $d_k \in \mathrm{Der}(R_n)$, where $2 \le k \le n-1$. By adding, if it is necessary, a linear combination of the derivations ad e_1, ad e_2, ..., ad e_{n-2}, we can suppose that $d_k(e_0) = 0$.

As, we have :

$$d_k(e_j) = \alpha_j e_{i+j} \quad , \quad 1 \leq j \leq n-k \ ,$$

the identity of the derivation applied to the pair (e_0,e_j), gives

$$\alpha_1 = \alpha_2 = \dots = \alpha_{n-k} \ ,$$

then $d_k = \alpha_1 h_k$.

Note also that $d_n = \alpha \text{ ad } e_{n-1}$ for some α in $\mathbb{C}$. Now, it is sufficient to consider only the derivation d_1. By addition, if it is necessary, of a derivation of the type α ad e_0, we can suppose that $d_1(e_1) = 0$. This shows that d_1 is defined as follow :

$$d_1(e_i) = \alpha_i e_{i+1} \quad , \quad 0 \leq i \leq n-1 \quad , \quad \text{where} \quad \alpha_1 = 0 \ .$$

The identity of the derivation applied to the pair (e_0,e_i), where $i = 1, 2, \dots, n-2$, gives :

$$\alpha_2 = \alpha_1 = 0 \quad \text{and} \quad \alpha_j = (j-2)\alpha_0 \quad \text{if} \quad 3 \leq j \leq n-1 \ .$$

The same identity, applied to the pair (e_1,e_2), gives $\alpha_2 = \alpha_4$. Then, we deduce $\alpha_0 = 0$ and $d_1 = 0$. Q.E.D.

I.5. Derivations of the standard nilpotent algebra in $\mathfrak{g} = sl(r+1, \mathbb{C})$

Let $\mathfrak{n}$ be a standard subalgebra of $\mathfrak{g} = gl(r+1,\mathbb{C})$ defined by a subset R of Δ^+ composed of roots pairwise incomparable. We then have :

$$\mathfrak{n} = \sum_{\alpha \in \Delta'} \mathfrak{g}_\alpha \ , \text{ where } \Delta' = \left\{ \alpha \in \Delta^+ \mid \exists \gamma \in R \quad \text{with} \quad \alpha \geq \gamma \right\} \ .$$

Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ formed of all the diagonal matrices of $\mathfrak{g}$. The algebra of derivation Der $\mathfrak{n}$ of the algebra $\mathfrak{n}$ is naturally provided with a structure of $\mathfrak{h}$-module (see Chapter 3) :

$$(h.d)(x) = [h,d(x)] - d([h,x]) \ .$$

Then Der $\mathfrak{n}$ is the vectorial direct sum of the weighting subspaces V_λ, where the elements of V_λ are the vectors having λ as their weight. If d is a nontrivial element of V_λ, then $d(X_\alpha) = c.X_\beta$, where $\lambda = \beta - \alpha$. This permits us to consider only the weighting derivations for the description of Der $\mathfrak{n}$, i.e. the derivations belonging to V_λ for a weight λ of the type $\lambda = \beta - \alpha$, where α, β are in Δ.

Let $N_{\mathfrak{g}}\mathfrak{n}$ the normalizer of $\mathfrak{n}$ in $\mathfrak{g}$. We put

$$D_1 = \left\{d \in \operatorname{End}\mathfrak{n} \,/\, d = \operatorname{ad}x \quad \text{where} \quad x \in N_{\mathfrak{g}}\mathfrak{n}\right\},$$

$$D_2 = \left\{d \in \operatorname{End}\mathfrak{n} \,/\, d|_{[\mathfrak{n},\mathfrak{n}]} = 0 \ , \quad d(\mathfrak{n}) \subset Z(\mathfrak{g})\right\}.$$

It is clear that D_1 and D_2 are subsets of Der $\mathfrak{n}$.

Let $S = \{\alpha_1, \alpha_2, ..., \alpha_r\}$ be the ordered subset of the simple roots (see Chapter 1). One designates by α_{ij} the root $\alpha_i + ... + \alpha_{j-1}$ if $i < j$, or $-\alpha_j - \alpha_{j+1} - ... - \alpha_{i-1}$ if $i > j$.

Lemma 1. *Let $\mathfrak{n}$ be a standard subalgebra of $\mathfrak{g}$ defined by* R. *If* R *doesn't contain the roots of the form $\alpha_{1,k}$ or $\alpha_{s,r+1}$, then*

$$\operatorname{Der}\mathfrak{n} = D_1 + D_2 .$$

Proof. If $\mathfrak{n}$ is Abelian, the theorem is obvious. Suppose that $\mathfrak{n}$ is not Abelian. Let d be a derivation with weight $\lambda = \beta - \alpha$, where α and β are in Δ'. We distinguish three cases

(i) $\lambda = \beta - \alpha = 0$;

(ii) $\lambda = \beta - \alpha \in \Delta$;

(iii) $\lambda = \beta - \alpha \notin \Delta$.

Case (i). For all indices i such that $X_{\alpha_{i,r+1}}$ is in $\mathfrak{n}$, we choose an element h in $\mathfrak{h}$ such that ad h $(X_{\alpha_{i,r+1}}) = f\,(X_{\alpha_{i,r+1}})$. Then the derivation $d' = f - \operatorname{ad} h$ is rendered as zero for every vector $X_{\alpha_{i,r+1}}$ of $\mathfrak{n}$. We can suppose, by adding, of necessity, a derivation of D_2, that d' is zero on all the vectors X_α of $(\mathfrak{n}-[\mathfrak{n}-\mathfrak{n}]) \cap Z(\mathfrak{n})$. By using the fact that d' is a derivation, we can easily prove that $d'(X_\alpha) = 0$ for every X_α of $\mathfrak{n}$.

Case (ii). Let $\lambda = \beta - \alpha = \alpha_{k,s}$. By adding, if it is necessary, a linear combination of derivations of D_2 and ad e_j, we can suppose that $d(X_\tau) = 0$ if $\tau = \alpha_{s,m}$ where m is the smallest indice such that $X_{\alpha_{s,m}}$ is in $\mathfrak{n}$ (it is easy to see that such a root exists). Then we verify that $d(X_v) = 0$ for every X_v in $\mathfrak{n}$.

Case (iii). Let d be a derivation of $\mathfrak{n}$ such that $d(X_\alpha) = \operatorname{ad} X_\beta$, $\beta - \alpha \in \Delta$, $X_\alpha, X_\beta \in \mathfrak{n}$. Besides, suppose that d is not in D_2. Then

$$\lambda = \left(\lambda_\varphi + \alpha_{\varphi+1} + ... + \alpha_\tau - \alpha_p - ... - \alpha_u\right) \quad \text{with} \quad \varphi \le \tau \le p \le u$$

or

$$\lambda = (\alpha_\varphi+\alpha_{\varphi+1} +...+ \alpha_\tau - \alpha_p - ... - \alpha_u) \text{ with } \varphi<\tau<p\text{-}1 .$$

In each case, we can write λ as a difference of two roots $\lambda = \beta - \alpha$, and in a more distinguished manner :

$$\beta = (\alpha_\varphi+\alpha_{\varphi+1} +...+ \alpha_\tau) \text{ and } \alpha = (\alpha_l +...+ \alpha_n)$$

or

$$\beta = (\alpha_\varphi +...+ \alpha_{l-1}) \text{ and } \alpha = (\alpha_{l+1} +...+ \alpha_n)$$

in the first case, and

$$\beta = \pm(\alpha_\varphi +...+ \alpha_{l-1}) \text{ and } \alpha = \pm(\alpha_{t+1} +...+ \alpha_{l-1})$$

in the second case.

Except in the case where R admits a root of the form $\alpha_{1,k} = \alpha_1 +...+ \alpha_{k-1}$ (or of the form $\alpha_{\varphi,r+1} = \alpha_\varphi +...+ \alpha_r$) and if $\alpha = \alpha_1 +..+ \alpha_t$ (respectively, $\alpha = \alpha_t +...+ \alpha_2$), in other cases, we can always find a root vector X_γ in $\mathfrak{n}$ such that

$$d[X_\alpha,X_\gamma] \neq [d(X_\alpha),X_\gamma] + [X_\alpha,d(X_\gamma)].$$

This proves the lemma.

Let $\mathfrak{n}$ be a standard nilpotent subalgebra of $\mathfrak{g}$ defined by R, where R contains an element of the form $\alpha_{j,k}$. We put

$$R = \{\alpha_{\varphi_1,k_1},\alpha_{\varphi_2,k_2} , \dots , \alpha_{\varphi_m,k_m}\} ,$$

where $1 = \varphi_1 < \varphi_2 < ... < \varphi_m \leq r$, $k_1 < k_2 < \dots < k_m \leq r+1$, $\varphi_i \leq k_i$.

If t, s, p are integers satisfying

$$k_1 < t \leq s \leq k_2 \quad , \quad p > \varphi_m \quad , \quad X_{s,p} \in \mathfrak{n} ,$$

then the endomorphism of $\mathfrak{n}$ defined by

$$l(X_{\alpha_{1,t}}) = X_{s,p} \quad , \quad d(X_{\alpha_{1s}}) = X_{t,:}$$

is in Der $\mathfrak{n}$. We designate by D_3 the set of all these derivations.

Now suppose that R contains an element of the form $\alpha_{\varphi,r+1}$. We put

$$R = \{\alpha_{\varphi_1,k_1},\alpha_{\varphi_2,k_2} , \dots , \alpha_{\varphi_m,r+1}\}$$

with $\varphi_1 < \varphi_2 < ... < \varphi_m \leq r$, $k_1 < k_2 < \dots < k_m = r+1$, $\varphi_i \leq k_i$.

If t, s, p are integers satisfying

$$1 \leq k_1 \quad , \quad \varphi_{m-1} < t \leq p \leq \varphi_m \quad , \quad X_{s,t} \in \mathfrak{n} ,$$

then the endomorphism of $\mathfrak{n}$ defined by

$$d(X_{t,r+1}) = X_{l,p} \quad , \quad d(X_{p,r+1}) = X_{l,t}$$

is a derivation of $\mathfrak{n}$. We note by D_4 the set composed of all these derivations.

Theorem 1. *Let $\mathfrak{n}$ be a standard nilpotent subalgebra of $\mathfrak{g}$ defined by a system* R *of Δ^+ composed of roots pairwise incomparable. Then*

$$\mathrm{Der}\, \mathfrak{n} = D_1 + D_2 + D_3 + D_4 .$$

This theorem can be obtained directly by using the lemma and the definitions of D_3 and D_4.

II. COHOMOLOGY OF FILIFORM LIE ALGEBRAS

In this section, we shall describe the cohomology spaces with values in the adjoint module of the "model" filiform Lie algebra which is denoted L_n and which is defined, in the basis $(e_0,e_1,...,e_n)$, by the brackets $[e_0,e_i] = e_{i+1}$ $i = 1,...,n$. If $F_sZ^2(L_n,L_n)$ are the elements of the filtration obtained by the central descending and ascending sequences:

$$F_sZ^2\,(L_n,L_n) = \left\{\varphi \in Z^2\,(L_n,L_n) \ : \varphi((L_n)_i,(L_n)_j) \subset (L_n)_{i+j+s}\right\},$$

then the space $F_0Z^2(L_n,L_n)$ is identified as the "tangent spaces" at the point L_n to the set of nilpotent Lie algebras. This geometrical description will be developed in a later chapter.

We define 2-cochains $\varphi_{i,j}$ of a Lie algebra L_n in the adjoint representations by setting $\varphi_{i,j}\,(e_0,e_i) = e_j$ for $i, j = 1 ,..., n$. It is obvious that these cochains are cocycles. Therefore, for a description of the cocycle space $Z^2(L_n,L_n)$ it suffices to describe only the cocycles φ given by $\varphi(e_0,e_i) = 0$ for all $1 \leq i \leq n$. Since $L_n = \oplus\,(L_n)_i$ is a graded algebra, we can restrict ourselves to cocycles φ with $\varphi(L_n)_i\,(L_n)_j \subset (L_n)_k$

We remark that a 2-cochain φ satisfying $\varphi(e_0\,,e_i) = 0$ is a cocycle if and only if

$$\mathrm{ad}\, e_0(\varphi(e_i\,,e_j)) = \varphi(e_{i+1}\,,e_j) + \varphi(e_i\,,e_{j+1})$$

for all $1 \leq i, j \leq n$ (if $r > n$, then we set $e_r = 0$).

Proposition 5. *Let* (k,z) *be a pair of integers such that* $1 \leq k \leq n-1, 2k \leq s \leq n$; *then there is one and only one element* $\psi \in Z^2(L_n,L_n)$ *satisfying*

$$\psi(e_i,e_{i+1}) = \left\{ \begin{array}{ll} e_1 & \text{if } i = k, \\ 0 & \text{if not}, \end{array} \right\}$$

$$\psi(e_0,e_i) = 0, \quad 1 \leq i \leq n .$$

Proof. Let ψ be a cocycle satisfying the hypothesis of the proposition, and we have :

$$\psi(e_i,e_j) = \mathrm{ad}\, e_0(\psi(e_i,e_{j-1})) - \psi(e_{i+1},e_{j-1}) .$$

This relation shows that one of the following conditions is satisfied

$\psi(e_i,e_j) = 0$ if

(1) $k < i < j$;

(2) $i < j \leq k$;

(3) $k-i > j-k-1$.

Let $1 \leq i \leq k < j \leq n$. Then an induction with respect to $(j-k-1) - (k-i) = i+j-2k-1$ shows that

$$\psi(e_i,e_j) = (-1)^{k-i} C^{k-i}_{j-k-1} (\mathrm{ad}\, e_0)^{i+j-2k-1} e_1 .$$

This is deduced from the above relation and from the classical rule

$$C^{k-i}_{j-k-2} + C^{k-i-1}_{j-k-2} = C^{k-i}_{j-k-1} .$$

This shows the existence and the unicity of the cocycle ψ.

We now construct the cocycles $\psi_{k,s}$ for all $1 \leq k \leq n-1$ and $2k \leq s \leq n$ (we need $2k \leq s$ in order to ensure that $\varphi(e_i, e_{n+1}) = 0$ for all $1 \leq i \leq n$) by putting

$$\psi_{k,s}(e_i,e_{i+1}) = \left\{ \begin{array}{ll} e_s & \text{if } i = k \\ 0 & \text{if } i \neq k \end{array} \right\}$$

Proposition 6. *The cocycles* $\varphi_{i,j}$ *and* $\psi_{k,s}$ *with* $1 \leq i < j \leq n$ *and* $2k+1 \leq 1 \leq n$ *form a basis of* $F_0Z^2(L_n,L_n)$.

Proof. Let $\beta \in F_0Z^2(L\ ,L\ )$ be a cocycle compatible with the grading that satisfies

$\beta\,((L_n)_i,(L_n)_j) \subset (L_n)_{i+j}$. Subtracting, if necessary, the cocycles $\varphi_{i,j}$, we may assume that $\beta(e_0,e_i) = 0$ for all $1 \le i \le n$. If $\beta(e_i,e_{i+1}) = 0$ for all $1 \le i \le n-1$, then we have $\beta = 0$. Let k be the minimal positive integer such that $\beta(e_k,e_{k+1}) = \lambda e_s$ with $\lambda \neq 0, \lambda \in \mathbb{C}$ and $s \ge 2k+1$. Then the cocycle $\beta' = \beta - \lambda\psi_{k,s}$ satisfies the condition $\beta'(e_k,e_{k+1}) = 0$. If $\beta' \neq 0$, then we repeat the argument for β'. Continuing in this way, we eventually find that β is a linear combination of cocycles of the form $\psi_{k,s}$. The linear independence of the cocycles $\varphi_{i,j}$ and $\psi_{k,s}$ is obvious, and the proof is complete.

Corollary 1. *The dimension of the space* $F_0Z^2(L,L)$ *is equal to* $(3n^2-4n+1)/4$ *if* n *is odd, and to* $(3n^2-4n)/4$ *if* n *if even.*

Corollary 2. *The dimension of the space* $F_0H^2(L,L)$ *is equal to* $(n^2-2n-3)/4$ *if* n *is odd, and to* $(n^2-2n-4)/4$ *if* n *is even. A basis of* $F_0H^2(L,L)$ *is formed by the cohomology classes of the cocycles* $\psi_{k,s}$ *where* $1 \le k \le n, 4 \le s \le n$ *and* $s \ge 2k + 1$.

The proof of Corollary 2 can be deduced from the fact that the cocycle $\varphi_{i,j}$, $1 \le i < j \le n$, and the cocycle $\psi_{1,3}$ are elements of the space $F_0B^2(L_n,L_n)$. In fact, $\varphi_{i,j} = df$ and $\psi_{1,3} = dg$, where the endomorphisms f and g are defined by

$$f(e_m) = e_{j-1} \quad \text{if } m = i,$$
$$f(e_m) = 0 \quad \text{if not},$$

$$g(e_m) = e_0 \quad \text{if } m = i,$$
$$g(e_m) = 0 \quad \text{if not}.$$

The cohomology classes of the other elements of $F_0Z^2(L_n,L_n)$ are nonzero and form a free system.

III. COHOMOLOGY OF THE NILRADICALS OF PARABOLIC ALGEBRAS

This study gives rise to the works of Kostant which gave a description of the cohomology space $H^*(n,V)$, where $\mathfrak{n}$ is the nilradical of a parabolic subalgebra p of a complex semisimple Lie algebra $\mathfrak{g}$, and V is an irreducible $\mathfrak{g}$-module restricted to $\mathfrak{n}$. If

$\mathfrak{g}$ is simple, then for V, one can consider the Lie algebra $\mathfrak{g}$ with respect to the adjoint action, and we obtain a description of $H^{*}(\mathfrak{n},\mathfrak{g})$. However, for many applications (the study of the automorphisms of the Lie algebra $\mathfrak{n}$, the problem of classifying Lie algebras with a given radical, and algebraic Lie algebras with a given nilradical, the study of the deformations of the Lie algebra $\mathfrak{n}$, and the study of a variety of Lie structures of a vector space) it would be useful to have a description of the cohomology space $H^{*}(\mathfrak{n},\mathfrak{n})$. As Kostant observed by modifying his arguments somewhat, one could obtain a description of the space $H^1(\mathfrak{m},\mathfrak{m})$, where $\mathfrak{m}$ is the nilradical of a Borel subalgebra of a Lie algebra $\mathfrak{g}$. This would give a description of the derivation algebra Der $\mathfrak{m}$.

In the following two sections, we describe $H^1(\mathfrak{n},\mathfrak{n})$ and $H^2(\mathfrak{n},\mathfrak{n})$ for the nilradical $\mathfrak{n}$ of an arbitrary parabolic subalgebra $\mathfrak{p}$ of a complex simple Lie algebra $\mathfrak{g}$. The prototype is the Heisenberg algebra. To save on language, we will speak of the nilradical of parabolic algebras instead of the nilradical of parabolic subalgebras of simple complex algebras.

III.1. A decomposition of the groups $H^i(\mathfrak{n},\mathfrak{n})$ $i = 1,2$

III.1.1. The root filtration

Let us recall some classical notions. Let $\mathfrak{g}$ be a complex semisimple Lie algebra of finite dimension, $\mathfrak{h}$ a Cartan subalgebra of it, Δ a root system with respect to $\mathfrak{h}$, and S a basis of Δ (a system of simple roots). We have introduced a partial ordering in $\mathfrak{h}^*$ relative to which the positive elements are linear combinations of roots in S with positive coefficients. This ordering divides Δ into a set of positive roots Δ^+ and a set of negative roots Δ^-, so that $\Delta = \Delta^+ \cup \Delta^-$.
We denote by $\mathfrak{g}_\alpha$ the root space corresponding to the root α. We shall assume that for each $\alpha \in \Delta$, some nonzero element $e_\alpha \in \mathfrak{g}_\alpha$ is fixed.
Let S_1 be a subsystem of S, Δ_0 the set of roots that can be expressed in terms of S_1,

and $\Delta' = (\Delta - \Delta_0) \cap \Delta^+$. Then the subalgebra $\mathfrak{p} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta_0 \cup \Delta'} \mathfrak{g}_\alpha$ is parabolic, and

$$\mathfrak{s} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta_0} \mathfrak{g}_\alpha$$

is a reductive Levi subalgebra of it, and its nilradical has the form $\mathfrak{n} = \sum_{\alpha \in \Delta'} \mathfrak{g}_\alpha$.

Recall that any parabolic subalgebra is conjugate to the subalgebra $\mathfrak{p}$ for some subsystem S_1 of S.
Let $S = \{\alpha_1, \dots, \alpha_n\}$ and let $\beta \in \Delta$ .We consider a decomposition $k_1 \alpha_1 + \dots + k_n \alpha_n$ of the root β into simple roots. We call the integer $l(\beta) = k_1 + \dots + k_n$ the *length* of the root β. We consider the subset $S - S_1 = \{\alpha_{i_1}, \dots ,\alpha_{i_r}\}$. We call the integer $l_n(\beta) = k_{i_1} + \dots + k_{i_r}$ the $\mathfrak{n}$-length (or the S-S_1 length) of β. In the case $S = S - S_1$, the concepts of length and the $\mathfrak{n}$-length of roots obviously coincide. Since an arbitrary root decomposes into simple roots with either positive or negative coefficients, we have

$$l_n(\beta_1+\beta_2) = l_n(\beta_1) + l_n(\beta_2)$$

for any $\beta_1, \beta_2, \beta_1 + \beta_2 \in \Delta$.
We introduce the following notation :

$$\Delta_i = \{\beta \in \Delta / l_n(\beta) = i\} \;\; ; \;\; \mathfrak{g}_i = \sum_{\alpha \in \Delta_i} \mathfrak{g}_\alpha \, , \, i \neq 0 \;\; ; \;\; \mathfrak{g}_0 = h + \sum_{\alpha \in \Delta_0} \mathfrak{g}_\alpha = s \, .$$

The additivity of l_n shows that $\mathfrak{g} = \oplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ is a $\mathbb{Z}$-graded Lie algebra, where

$$\mathfrak{s} = \mathfrak{g}_0 \quad , \quad \mathfrak{n} = \oplus_{i \geq 0} \mathfrak{g}_i \quad , \quad \mathfrak{p} = \oplus_{i \geq 0} \mathfrak{g}_i \; .$$

Recall that the length of a negative root is negative, then the $\mathfrak{g}_i$ for $i < 0$ are not necessary zero. In fact, if $\mathfrak{g}_i \neq \{0\}$ for $i > 0$, then $\mathfrak{g}_{-i} \neq \{0\}$.
We call the resulting grading the *natural grading* of the Lie algebra $\mathfrak{g}$ with respect to $\mathfrak{n}$.
The subalgebra $\mathfrak{n} = \oplus_{i \in \mathbb{Z}} \mathfrak{n}_i$ is also $\mathbb{Z}$-graded if one sets

Setting

$$F_k(\mathfrak{g}) = \bigoplus_{i\geq k} \mathfrak{g}_i \quad , \quad F_k(\mathfrak{n}) = \bigoplus_{i\geq k} \mathfrak{n}_i \ ,$$

we obtain the filtrations $F(\mathfrak{g})$ and $F(\mathfrak{n})$ of the Lie algebras $\mathfrak{g}$ and $\mathfrak{n}$. We note that $F(\mathfrak{n})$ coincides with the natural filtration of the nilpotent Lie algebra $\mathfrak{n}$, i.e.,

$$F_1(\mathfrak{n}) = \mathfrak{n} \quad , \quad F_k(\mathfrak{n}) = [\mathfrak{n}, F_{k-1}(\mathfrak{n})] \ .$$

Remark. All subspaces $\mathfrak{g}_i$, $i \in \mathbb{Z}$, are s-modules with respect to the adjoint action (the subalgebra s coincides with $\mathfrak{g}_0$). Since s is reductive, these s-modules are semisimple. In addition, the $\mathfrak{n}$-modules $\mathfrak{n}$, $\mathfrak{g}$, and $\mathfrak{g}/\mathfrak{n}$ are semisimple s-modules. In what follows, we identify the s-module $\mathfrak{g}/\mathfrak{n}$ with the submodule $\oplus_{i\leq 0}\, \mathfrak{g}_i$ in $\mathfrak{g}$ complementary to $\mathfrak{n}$.

Now, the gradings of the Lie algebras $\mathfrak{n}$ and $\mathfrak{g}$ define $\mathbb{Z}$-gradings in the spaces of cochains, cocycles, and coboundaries :

$$C^j(\mathfrak{n},t) = \bigoplus_{i\in\mathbb{Z}} C^j_j(\mathfrak{n},t) \quad ;$$

$$Z^j(\mathfrak{n},t) = \bigoplus_{i\in\mathbb{Z}} Z_i(\mathfrak{n},t) \quad ; \quad B(\mathfrak{n},t) = \bigoplus_{i\in\mathbb{Z}} B^j_i(\mathfrak{n},t) \quad ;$$

where $t = \mathfrak{n}$, $\mathfrak{g}$, or $\mathfrak{g}/\mathfrak{n}$. This grading is compatible with the coboundary operator, and we also obtain a $\mathbb{Z}$-grading of the cohomology space :

$$H^j(\mathfrak{n},t) = \bigoplus_{i\in\mathbb{Z}} H^j_i(\mathfrak{n},t) \ .$$

Remark. In the space of cochains $C^j(\mathfrak{n},t,)$ where $t = \mathfrak{n}$, $\mathfrak{g}$ or $\mathfrak{g}/\mathfrak{n}$ a $\mathfrak{p}$-module structure can be introduced in the usual way :

$$(ac))\,(x_1, \dots, x_j) = ac(x_1, \dots, x_j) - \sum_{1\leq s\leq j} c(x_1, \dots, [a,x_s], \dots, x_j) \ ,$$

where $a \in \mathfrak{p}$ and $c \in C^j(\mathfrak{n},t)$. The spaces of cocycles $Z^j(\mathfrak{n},t)$ and coboundaries $B^j(\mathfrak{n},t)$ are $\mathfrak{p}$-submodules of the $\mathfrak{p}$-module $C^j(\mathfrak{n},t)$, and the space $H^j(\mathfrak{n},t)$ also acquires a $\mathfrak{p}$-module structure. All these $\mathfrak{p}$-modules can also be regarded as s-modules with respect to the same action, which are semisimple.

III.1.2. The decomposition theorem

Consider a short exact sequence

$$0 \to \mathfrak{n} \xrightarrow{\varphi} \mathfrak{g} \xrightarrow{\psi} \mathfrak{g}/\mathfrak{n} \to 0,$$

where φ is an imbedding and ψ is the natural homomorphism. This sequence generates a long exact sequence

$$\ldots \to H^{j-1}(\mathfrak{n},\mathfrak{g}/\mathfrak{n}) \xrightarrow{\delta^{\bullet}_{j-1}} H^j(\mathfrak{n},\mathfrak{n}) \xrightarrow{\varphi^{\bullet}_j} H^j(\mathfrak{n},\mathfrak{g}) \xrightarrow{\psi^{\bullet}_j} H^j(\mathfrak{n},\mathfrak{g}/\mathfrak{n}) \to \ldots .$$

We will note the space Im $\varphi^*_j \subset H^j(\mathfrak{n},\mathfrak{g})$ by $H^j_{\text{fund}}(\mathfrak{n},\mathfrak{n})$. It is called the **fundamental cohomology space**. We can consider the space $H^j_{\text{fund}}(\mathfrak{n},\mathfrak{n})$ as a sub-s-module of $H^j(\mathfrak{n},\mathfrak{n})$ by considering a supplementary of Im $d^{\bullet}_{j-1}(H^j(\mathfrak{n},\mathfrak{g}/\mathfrak{n}))$ in $H^j(\mathfrak{n},\mathfrak{n})$.

Theorem 2. *Let $\mathfrak{n}$ be the nilradical of a parabolic subalgebra $\mathfrak{p}$ of a complex simple Lie algebra $\mathfrak{g}$, defined by means of a subsystem $S_1 \subset S$ and different from those listed in Table 1. Then*

(a) $H^1(\mathfrak{n},\mathfrak{n}) \cong s \oplus H^1(\mathfrak{n},\mathfrak{g})$, *and*

(b) $H^2(\mathfrak{n},\mathfrak{n}) = H^j(\mathfrak{n},\mathfrak{g}.\mathfrak{n}) \oplus H^2_{\text{fund}}(\mathfrak{n},\mathfrak{n})$.

Proof. To prove (b), it is enough to show that

$$\delta^{\bullet}_1 : H^1(\mathfrak{n},\mathfrak{g}/\mathfrak{n}) \to H^2(\mathfrak{n},\mathfrak{n})$$

is injective, and to prove (a) it is enough to verify that

$$H^1(\mathfrak{n},\mathfrak{g}) - H^1_{\text{fund}}(\mathfrak{n},\mathfrak{n})$$

and that $\delta^{\bullet}_0$ is injective.

(a) According to a result of Kostant (Ko), as representatives of the primitive elements of the simple components of the semisimple s-module $H^1(\mathfrak{n},\mathfrak{g})$, one can take the following system of cocycles (which we shall henceforth identify with the corresponding elements in $H^1(\mathfrak{n},\mathfrak{g})$: $\{f_\alpha \mid \alpha \in S-S_1\}$, where

TABLE 1

$\mathfrak{g}$	$S \quad S_1$
A_1	α_1
$A_n, n \geq 2$	α_1
$A_n, n \geq 2$	α_n
$C_n, n \geq 2$	α_1

TABLE 2

$\mathfrak{g}$	$S \quad S_1$
$A_n, n \geq 2$	α_1
$C_n, n \geq 2$	α_1, α_2

$$f_\alpha(e_\gamma) = \left\{ \begin{array}{ll} e_{s_\alpha(\delta)} , & \text{if } \gamma = \alpha \\ 0 , & \text{if } \gamma \neq \alpha \end{array} \right\},$$

s_α is a reflection of the space s^* that carries Δ into itself and such that $s_\alpha(\alpha) = -\alpha$.

Definition 1. *A root α of S is called singular if $\delta-\alpha$ is a root where δ is maximal.*

If α is not a singular root, then α is not connected with a minimal root $(-\delta)$ in an extended Dynkin diagram ; in this case, $s_\alpha(\delta) = \delta$, and we have $f \in H^1_{fund}(\mathfrak{n},\mathfrak{n})$. If α is a singular root, then $s_\alpha(\delta) = \delta - 2\alpha$ for a simple Lie algebra $\mathfrak{g}$ of type C_n , and $s_\alpha(\delta) = \delta - \alpha$ for all remaining types of simple Lie algebras $\mathfrak{g}$. In all cases, except those listed in Table 1, $s_\alpha(\delta) \in \Delta'$, i.e., $e_{s_\alpha(\delta)} \in \mathfrak{n}$ and so $f \in H^1_{fund}(\mathfrak{n},\mathfrak{n})$.

The spaces of $\mathfrak{n}$-invariants elements in $\mathfrak{n}$ and in $\mathfrak{g}$ with respect to the adjoint action coincide (up to imbedding in $\mathfrak{g}$) and are equal to the center $z(\mathfrak{n})$. This means that $H^0(\mathfrak{n},\mathfrak{g}) = H^0(\mathfrak{n},\mathfrak{n})$ and the mapping δ^*_0 in the sequence is injective.

(b) We consider the given long exact sequence . We know that

$$\operatorname{Im} \varphi^*_1 = H^1_{fund}(\mathfrak{n},\mathfrak{n}) ,$$

and, from the proof of (a),

$$H^1_{fund}(\mathfrak{n},\mathfrak{n}) = H^1(\mathfrak{n},\mathfrak{g}).$$

This means that the mapping φ^*_1 is surjective and $\operatorname{Im} \psi^*_1 = 0$. Since the long sequence is exact, we have $\operatorname{Ker} \delta^*_1 = \operatorname{Im} \psi^*_1 = 0$, i.e., the mapping δ^*_1 is injective. This proves the theorem.

Remark. It is well known that $Z^1(\mathfrak{n},\mathfrak{n})$ coincides with the derivation algebra Der $\mathfrak{n}$. Theorem 1, in essence, gives a description of Der $\mathfrak{n}$ as an s-module.

Moreover, Theorem 1 reduces the descriptions of $H^2(\mathfrak{n},\mathfrak{n})$ to the description of $H^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$ and $H^2_{fund}(\mathfrak{n},\mathfrak{n})$ (except for the cases listed in Table 1). It turns out that for all $\mathfrak{n}$, except for those listed in Tables 1 and 2,

$$H^2_{fund}(\mathfrak{n},\mathfrak{n}) = H^2(\mathfrak{n},\mathfrak{g}) .$$

In addition, it turns out that $H^2{}_{\mathrm{fund}}(\mathfrak{n},\mathfrak{n})$ contains all the cocycles in $H^2(\mathfrak{n},\mathfrak{n})$ that preserve the filtration, i.e.,

$$\bigoplus_{i\geq 0} H^2_i(\mathfrak{n},\mathfrak{n}) \subset H^2_{\mathrm{fund}}(\mathfrak{n},\mathfrak{n}) \quad .$$

Now we examine the cases listed in Table 1 and describe the space $H^2(\mathfrak{n},\mathfrak{n})$. We denote simple roots by $\alpha_1 ,..., \alpha_n$ and number them according to Chapter 1.

(a) $\quad \mathfrak{g} = A_1 \quad$ and $\quad S \;\; S_1 = \{\alpha_1\}$.

The Lie algebra $\mathfrak{n}$ is one-dimensional, and so $H^2(\mathfrak{n},\mathfrak{n}) = 0$.

(b) $\mathfrak{g} = A_n$, $(n \geq 2)$ and $S \;\; S_1 = \{\alpha_1\}$ or $\{\alpha_n\}$.

The Lie algebra $\mathfrak{n}$ is Abelian and $H^2(\mathfrak{n},\mathfrak{n})$ consists of all skew-symmetric bilinear maps $f : \mathfrak{n} \wedge \mathfrak{n} \rightarrow \mathfrak{n}$.

(c) $\mathfrak{g} = C_n$, $(n \geq 2)$ and $S \;\; S_1 = \{\alpha_1\}$.

The Lie algebra $\mathfrak{n}$ is isomorphic to the Heisenberg algebra H_{n-1} and one can imbed it into A_n as the nilradical of the parabolic subalgebra defined by the subset S, $S_1 = \{\alpha_1,\alpha_n\}$ and use Theorem 1.

At the end of this chapter, we will carefully detail his example.

III.2. The calculation of the spaces $H^1(\mathfrak{n},\mathfrak{g}/\mathfrak{n})$

As shown in the previous section, the description of the space $H^2(\mathfrak{n},\mathfrak{n})$ reduces to the calculation of $H^1(\mathfrak{n},\mathfrak{g}/\mathfrak{n})$. In this section, we shall prove a number of lemmas that are necessary for the description of $H^1(\mathfrak{n},\mathfrak{g}/\mathfrak{n})$. In order to understand the idea of the calculations, we shall briefly describe their order.

(i) Some properties of cocycles

We saw that the spaces

$$C^k_j(\mathfrak{n},\mathfrak{t}) \quad , \quad Z^k_j(\mathfrak{n},\mathfrak{t}) \quad , \quad B^k_j(\mathfrak{n},\mathfrak{t}) \quad , \quad H^k_j(\mathfrak{n},\mathfrak{t})$$

are semisimple s-modules with respect to the usual action. As the $\mathfrak{n}$-module $\mathfrak{t}$, we can take $\mathfrak{n}$, $\mathfrak{g}$ or $\mathfrak{g}/\mathfrak{n}$. These spaces can also be regarded as $\mathfrak{h}$-modules with respect to the same action which are semisimple. We call elements of the components of the decomposition of these $\mathfrak{h}$-modules into simple submodules *weight* elements (cochains, cocycles, coboundaries, cohomologies). Let

$$g : e_{\alpha_1} \wedge \dots \wedge e_{\alpha_k} \to e_\beta \qquad (*)$$

be an element of $C^k_j(\mathfrak{n},\mathfrak{g})$. For any $h \in \mathfrak{h}$, we have

$$(h.g)\,(e_{\alpha_1},\dots,e_{\alpha_k}) = [h,e_\beta] - \sum_{i=1}^{k} \alpha_i\,(h) e_\beta = \left(\beta(h) - \sum_{i=1}^{k} \alpha_i\,(h)\right) e_\beta \, .$$

This means that g is a weight cochain with $\beta - \sum_1^n \alpha_i$. Since any cochain (resp. cocycle, coboundary, cohomology) in our case is a linear combination of cochains of the form (*), to describe the space of cochains (cocycles, coboundaries, cohomologies) it is enough to find all the weight cochains that correspond to a weight λ of the form $\beta - \sum_1^k \alpha_i$, where α_i and β are roots, $e_{\alpha_i} \in \mathfrak{n}$, and $e_\beta \in \mathfrak{t}$.

Lemma 1. *Let* f *be an element of* $C^1(\mathfrak{n}, \mathfrak{g})$ *such that* $f(\mathfrak{n})$ *does not lie in* $F_i\, \mathfrak{g}$ *for some* i *and let* $df(\mathfrak{n}) \subset F_i\, \mathfrak{g}$ *(d is the coboundary operator). Then there exists an element* $a \in \mathfrak{n}_1$ *such that* $f(a) \neq 0$.

Proof. Assume the contrary, let $f(\mathfrak{n}_1) = 0$ and let r be the smallest integer such that $f(\mathfrak{n}_r) \neq 0$. We choose an element $x \in \mathfrak{n}_r$ for which $f(x) \neq 0$. Since x can be expressed as a linear combination of root vectors and since f is linear, we can assume that $x = e_\beta$, where β is a root with $\mathfrak{n}$-length r. Since $r > 1$, we can decompose β into the sum $\beta_1 + \beta_2$, where $\beta_1 \in \Delta_k$, $\beta_2 \in \Delta_m$, $1 \leq k, m < r$, and $k + m = r$. We have

$$df\,(e_{\beta_1},e_{\beta_2}) = f[e_{\beta_1},e_{\beta_2}] - [f(e_{\beta_1}),e_{\beta_2}] - [e_{\beta_1},f(e_{\beta_2})]\, .$$

By the hypothesis of the lemma, $df\,(e_{\beta_1},e_{\beta_2}) \in F_i\,\mathfrak{g}$. Since $f[e_{\beta_1},e_{\beta_2}] = N_{\beta_1,\beta_2} \cdot f\,(e_\beta)$ is a non-zero element that does not lie in $F_i\mathfrak{g}$, this is possible only if $f(e_{\beta_1}) \neq 0$ or $f(e_{\beta_2}) \neq 0$. But this contradicts the fact that r is minimal. This proves the lemma.

Corollary 1. Let f *be a nonzero cocycle in* $Z^1(\mathfrak{n},\mathfrak{g}/\mathfrak{n})$. *Then there exists an element* $a \in \mathfrak{n}_1$, *such that* $f(a) \neq 0$, *where* a *can be chosen as a root element.*

Lemma 2. *Let* f *be a weight element in* $C^1_{-i}(\mathfrak{n},\mathfrak{g})$, *where* $i \geq 2$, *and let* df *be a nonzero element of* $Z^2(\mathfrak{n},\mathfrak{g})$ *that takes its values in* $\mathfrak{n}$. *Then there exists* $e_\alpha \in \mathfrak{n}_1$ *such that either* $f(e_\alpha) = ae_{-\alpha}$ *or* $f(e_\alpha) = ae_{-\beta}$, *where* $e_\beta \in \mathfrak{n}$, $\alpha + \beta \in \Delta$, *and* $\alpha \in \mathbb{C}$.

Proof. By Lemma 1, there exists an $e_\alpha \in \mathfrak{n}_1$ such that $f(e_\alpha) \neq 0$. Since f is a weight map in $C^1_{-i}(\mathfrak{n},\mathfrak{g})$, we have $f(e_\alpha) = ae_{-\beta}$ for some $\beta \in n_{i-1}$ and for some nonzero $a \in \mathbb{C}$. By the hypothesis of the lemma, $df(e_\alpha,e_\beta) \in \mathfrak{n}$. On the other hand, $df \in B^2_{-i}(\mathfrak{n},\mathfrak{g})$ and $df(e_\alpha, e_\beta) \in \mathfrak{n}_0 = s$, which is not contained in $\mathfrak{n}$. Therefore, $df(e_\alpha, e_\beta) = 0$, and so

$$f\left[e_\alpha,e_\beta\right] - \left[f(e_\alpha),e_\beta\right] - \left[e_\alpha,f(e_\beta)\right] = 0 .$$

If $\alpha + \beta \notin \Delta$, then $-a\left[e_{-\beta},e_\beta\right] - \left[e_\alpha,f(e_\beta)\right] = 0$. Since the first term on the left hand side of this equality is not equal to zero and lies in $\mathfrak{h}$, the second term must also lie in $\mathfrak{h}$. Consequently, $f(e_\beta) = be_{-\alpha}$ for some nonzero $b \in \mathbb{C}$. We have $ah_\beta = bh_\alpha$, where h_γ denotes the element $[e_\gamma, e_{-\gamma}]$ in $\mathfrak{h}$, $\gamma \in \Delta$. But this equality is possible only if $\alpha = \beta$ and $a = b$. This proves the lemma.

Corollary 2. *Let* f *be a weight cocycle in* $Z^1_{-i}(\mathfrak{n},\mathfrak{g}/\mathfrak{n})$ *where* $i \geq 2$. *Then there exists an* $e_\alpha \in \mathfrak{n}_1$ *such that either* $f(e_\alpha) = ae_{-\alpha}$, *or* $f(e_\alpha) = ae_{-\beta}$ *where* $e_\beta \in \mathfrak{n}$, $\alpha + \beta \in \Delta$ *and* $a \in \mathbb{C}$

Lemma 3. *Let* f *be a map in* $C^1_i(\mathfrak{n},\mathfrak{g})$, *where* $i \geq -1$. *Then* df *is a cocycle in* $Z^2(\mathfrak{n},\mathfrak{g})$ *that takes its values in* $\mathfrak{n}$.

Proof. It is clear that $f(F_2\mathfrak{n}) \subset F_1\mathfrak{n} = \mathfrak{n}$ and $f(\mathfrak{n}_1) \subset \mathfrak{n}_0 = s$. Therefore, $df(e_{\beta_1},e_{\beta_2}) \in \mathfrak{n}$ for any $e_{\beta_1}, e_{\beta_2} \in \mathfrak{n}$. This proves the lemma.

Corollary 3. $C^1_{-i}(n,\mathfrak{g}/\mathfrak{n}) = Z^1_{-1}(\mathfrak{n},\mathfrak{g}/\mathfrak{n})$.

This shows that the space $Z^1_j(\mathfrak{n},\mathfrak{g}/\mathfrak{n})$ is perfectly described.

Now we construct some cocycles in the space $Z^1_{-2}(\mathfrak{n},\mathfrak{g}/\mathfrak{n})$ which, as will be proved in the following subsections, generate this space.

Lemma 4. *Let* $e_\alpha \in \mathfrak{n}_1$ *be such that ,for any root* $\gamma \in \Delta_0$, *we have* $\alpha + \gamma \notin \Delta$, *and let* $f \in C^1(\mathfrak{n},\mathfrak{g}/\mathfrak{n})$ *be such that* $f(e_\alpha) = e_{-\alpha}$ *and* $f(e_\beta) = 0$ *for any* β, $\beta \neq \alpha$. *Then* $f \in Z^1(\mathfrak{n},\mathfrak{g}/\mathfrak{n})$.

Proof. It is enough to verify the containment $df(e_\nu,e_\tau) \in \mathfrak{n}$, in the case when one of the roots is equal to α (if $\nu = \tau = \alpha$, then obviously $df(e_\alpha, e_\alpha) = 0$). We have

$$df(e_\alpha,e_\tau) = [e_{-\alpha},e_\tau] \ .$$

The element $[e_{-\alpha},e_\tau]$ can fail to lie in $\mathfrak{n}$ only when $e_\tau \in \mathfrak{n}_1$ and $\tau - \alpha \in \Delta$. Let $\tau - \alpha = \gamma$. Since e_τ, $e_\alpha \in \mathfrak{n}_1$, we have $\gamma \in \Delta_0$ and $\alpha + \gamma = \tau \in \Delta$, which contradicts the hypothesis of the lemma. This proves the lemma.

Remark 6. For the hypothesis of the lemma to hold, it is necessary and sufficient that $\alpha \in S - S_1$ and that all the simple roots connected with α in the Dynkin diagram also lie in S-S_1. In the case when $\mathfrak{n}$ is the nilradical of a Borel subalgebra (i.e., in the case when $S = S_1 = S$), the hypothesis of the lemma holds for all elements $e_\alpha \in \mathfrak{n}_1$.

Lemma 5. *Let* $e_\alpha \in \mathfrak{n}_1$ *be such that the set* R_α *of roots* $\gamma \in \Delta_0$ *with* $\alpha + \gamma \in \Delta$ *is non-empty, where for any* $\gamma \in R_\alpha$ *we have* $2\alpha + \gamma \in \Delta$, *and let* $f \in C^1(\mathfrak{n},\mathfrak{g}/\mathfrak{n})$ *be such that*

$$f(e_\alpha) = e_{-\alpha}, \quad f(e_{2\alpha+\gamma}) = \frac{1}{N_{\alpha,\alpha+\gamma}} \cdot [e_{-\alpha},e_{\alpha+\gamma}]$$

with $f(e_\beta) = 0$ *for* $\beta \neq \alpha$, $2\alpha + \gamma$. *If* $2\alpha + \gamma = \alpha + (\alpha + \gamma)$ *is the unique decomposition i into the sum of two roots in* Δ_1 *for any* $\gamma \in R_\alpha$, *then* $f \in Z^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$.

Proof. It is enough to show that $df(e_\nu,e_\tau) \in n$ in the case when one of the roots ν or τ (say ν) is equal to α. We have

$$df(e_\alpha,e_\tau) = f[e_\alpha,e_\tau] - [e_{-\alpha},e_\tau] \ .$$

If $e_\tau \in \mathfrak{n}_i$, where $i \geq 2$, then obviously $df(e_\alpha,e_\tau) \in \mathfrak{n}$. But if $\tau \in \Delta_1$ and $\tau \neq \alpha + \gamma$, where $\gamma \in \Delta_0$, then $df(e_\alpha,e_\tau) = 0 \in \mathfrak{n}$. Let $\tau = \alpha + \gamma$. Then

$$df(e_\alpha,e_\tau) = N_{\alpha,\alpha+\gamma} f(e_{\alpha+\tau}) - [e_{-\alpha},e_\tau]$$

by the hypothesis of the lemma. This proves the lemma.

Remark. In the case when $\mathfrak{n}$ is the nilradical of a Borel subalgebra, there do not exist elements $e_\alpha \in \mathfrak{n}_1$ that satisfy the hypothesis of Lemma 5. Such elements also do not exist in the case $\mathfrak{g} = sl(n,\mathbb{C})$.

Lemma 6. Let $e_\alpha, e_\beta \in \mathfrak{n}_1$, $\alpha + \beta \in \Delta$, *be such that* $\beta + \gamma \notin \Delta$ *for any* $\gamma \in \Delta_0$, *and let* $f \in C^1(\mathfrak{n},\mathfrak{g}/\mathfrak{n})$ *be such that*

$$f(e_\alpha) = e_{-\beta} \quad , \quad f(e_{\alpha+\beta}) = \frac{1}{N_{\alpha,\beta}} \cdot [e_{-\beta}, e_\beta] \, ,$$

with $f(e_\sigma) = 0$ *or* $\sigma \neq \alpha,\ \alpha + \beta$. *Then* $f \in Z^1(\mathfrak{n},\mathfrak{g}/\mathfrak{n})$.

Proof. A direct check shows that $df(e_\alpha, e_\beta) = 0 \in \mathfrak{n}$. If neither of the roots τ or ν is equal to α or β, then obviously $df(e_\tau, e_\nu) = 0 \in \mathfrak{n}$. It remains to examine the case case when τ or ν (say τ) is equal to α (or β). We have

$$df(e_\alpha, e_\nu) = -[e_{-\beta}, e_\nu].$$

This expression can fail to lie in $\mathfrak{n}$ only in the case when $e_\nu \in n_1$ and $\nu - \beta = \gamma \in \Delta_0$. But this case is impossible since, by the hypothesis of the lemma, $\beta + \gamma \notin \Delta$ for any β where $\gamma \in \Delta_0$. This proves the lemma.

Lemma 7. *Let* $e_\alpha, e_\beta \in \mathfrak{n}_1$, $\alpha + \beta \in \Delta$, *be such that the set* R_β *of roots* $\gamma \in \Delta_0$ *with* $\beta + \gamma \in \Delta$ *is nonempty, where for any* $\gamma \in R_\beta$, *we have* $\alpha + \beta + \gamma \in \Delta$ *(except, perhaps, for the case* $\alpha = \beta + \gamma$*), and let*

$$f(e_\alpha) = e_{-\beta} \quad , \quad f(\varepsilon_{\alpha+\beta+\gamma}) = \frac{1}{N_{\alpha,\alpha+\gamma}} \cdot [e_{-\beta}, e_{\beta+\gamma}]$$

with $f(e_\sigma) = 0$ *for* $\sigma \neq \alpha, \alpha + \beta + \gamma$ *where* $\gamma \in R_\beta \cup \{0\}$. *If* $\alpha + \beta + \gamma = \alpha + (\beta+\gamma)$ *is the unique decomposition into the sum of two roots in* Δ_1 *for any* $\gamma \in R_\beta$, *then we have* $f \in Z^1(\mathfrak{n},\mathfrak{g}/\mathfrak{n})$.

Proof. It is clear that $df(e_\nu, e_\tau) = 0$ for all ν and τ, neither of which is equal to α or $\alpha + \beta + \gamma$, where $\gamma \in R_\beta$. Let $\nu = \alpha$ and $\tau \neq \alpha + \beta + \gamma$, where $\gamma \in R_\beta$. We have

$$d(e_\alpha, e_\tau) = f[e_\alpha, e_\tau] - [e_{-\beta}, e_\tau] \, .$$

If $e_\tau \in \mathfrak{n}_i$, where $i \geq 2$, then obviously $d(e_\alpha, e_\tau) \in \mathfrak{n}$. But if $e_\tau \in \mathfrak{n}_1$ and $\tau \neq \beta + \gamma$, where $\gamma \in \Delta_0$, then $f(e_\alpha, e_\tau) = 0$. Let $\tau = \beta + \gamma$ for some $\gamma \in \Delta_0$. In this case also

$$d(e_\alpha, e_{\beta+\gamma}) = N_{\alpha,\beta+\gamma} f(e_{\alpha+\beta+\gamma}) - [e_{-\beta}, e_{\beta+\gamma}] = 0$$

by the hypothesis of the lemma. In the remaining case, $\nu = \alpha + \beta + \gamma$, $\tau \neq \alpha$, we have

$$d(e_\nu, e_\tau) = c\,[e_\gamma, e_\tau] \in \mathfrak{n}$$

for some constant c. This proves the lemma.

Remark. In the case when $\mathfrak{n}$ is the nilradical of a Borel subalgebra, Lemma 6 is satisfied by any simple roots α, $\beta \in S$, connected by an edge in the Dynkin diagram, and there do not exist roots α and β that satisfy Lemma 7.

Lemma 8. *Let be $e_\alpha \in \mathfrak{n}_1$ be such that the set R_α of roots $\gamma \in \Delta_0$, where $\alpha + \gamma \in \Delta$, is nonempty but there exists $\gamma \in R_\alpha$ such that $2\alpha + \gamma \notin \Delta$ and let f be a weighted element in $C^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$ such that $f(e_\alpha) = a e_\alpha$, $a \neq 0$. Then $f \notin Z^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$.*

Proof. We choose a root $\gamma \in \Delta_0$ so that $\alpha + \gamma \in \Delta$ but $2\alpha + \gamma \notin \Delta$. We have

$$df(e_\alpha, e_{\alpha+\gamma}) = a[e_{-\alpha}, e_{\alpha+\gamma}] - [e_\alpha, f(e_{\alpha+\gamma})] .$$

If $f(e_{\alpha+\gamma}) \neq 0$, then, bearing in mind that f has weight -2α, we have $-\alpha + \gamma \in \Delta$ and $f(e_{\alpha+\gamma}) = b e_{-\alpha+\gamma}$ for some $b \neq 0$. But this contradicts Lemma 2, since $(\alpha + \gamma) + (\alpha - \gamma) = 2\alpha$ is not in Δ. Therefore, $f(e_{\alpha+\gamma}) = 0$ and

$$df(e_\alpha, e_{\alpha+\gamma}) = -aN_{-\alpha,\alpha+\gamma}\, e_\gamma \neq \alpha ,$$

where $e_\gamma \notin \mathfrak{n}$. This proves the lemma.

Lemma 9. *Let $e_\alpha \in \mathfrak{n}_1$ and $e_\beta \in \mathfrak{n}$, $\alpha + \beta \in \Delta$, be such that the set R_β of all the roots $\gamma \in \bigcup_{i\leq 0} \Delta_i$ with $\beta + \gamma \in \Delta'$ is nonempty, but there exists $\gamma \in R_\beta$ such that $\alpha+\beta+\gamma \notin \Delta$, $\alpha \neq \beta + \gamma$ and let f be a weight element in $Z^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$ such that $f(e_\alpha) = ae_{-\beta}$. Then $f \in B^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$.*

Proof. We consider an element $g = b\,\mathrm{ad}\, e_{-\alpha\angle-\beta}$ in $B^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$ and we choose b so that $g(e_\alpha) = f(e_\alpha)$. For the element $f_1 = f - g$ and for a root $\gamma \in R_\beta$ with $\beta + \gamma \in \Delta$ and also $\beta + \alpha + \gamma \notin \Delta$, $\alpha \neq \beta + \gamma$, we have

$$df_1(e_\alpha, e_{\beta+\gamma}) = -[e_\alpha, f_1(e_{\beta+\gamma})] .$$

If $f_1(e_{\beta+\gamma}) \neq 0$, then $f_1(e_{\beta+\gamma}) = ce_{-\alpha+\gamma}$ for some nonzero constant c, and we obtain

$$df_1(e_\alpha, e_{\beta+\gamma}) = -cN_{\alpha,-\alpha+\gamma}\, e_\gamma \notin \mathfrak{n},$$

whence $f \notin Z^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$, which contradicts the hypothesis. This proves the lemma ($df = df_1$) in the case under consideration.

Let $f_1(e_{\beta+\gamma}) = 0$. Since $\alpha + (\beta + \gamma) - \gamma = \alpha + \beta \in \Delta$ and $\alpha + (\beta + \gamma) \notin \Delta$, we have $\alpha - \gamma \in \Delta$ and

$$df'(e_{\alpha-\gamma}, e_{\beta+\gamma}) = N_{\alpha-\gamma,\beta+\gamma}\, f'(e_{\alpha+\beta}) - [f'(e_{\alpha-\gamma}), e_{\beta+\gamma}],$$

$$df'(e_\alpha, e_\beta) = N_{\alpha,\beta}\, f'(e_{\alpha+\beta}) - [e_\alpha, f'(e_\beta)].$$

If at least one of these expressions is nonzero, then $f' \notin Z^1(\mathfrak{n},\mathfrak{g}/\mathfrak{n})$, which is impossible. Therefore, both expressions are zero. If $f'(e_{\alpha+\beta}) \neq 0$, then

$$f_1(e_{\alpha-\gamma}) = be_{-\beta+\gamma} \neq 0, \quad f_1(e_\beta) = ce_{-\alpha} \neq 0,$$

and we have

$$[e_{-\beta-\gamma}, e_{\beta+\gamma}] = N[e_\alpha, e_{-\alpha}]$$

for some nonzero number N, i.e., $h_{\beta+\gamma} = -Nh_\alpha$ where $\alpha \neq \beta + \gamma$, which is impossible. Therefore $f'(e_{\alpha+\beta}) = 0$, whence $f'(e_\beta), f'(e_{\beta+\gamma}) = 0$.

Let us assume that $f'(e_\theta) = le_{-\nu}$ for some $e_\theta \in \mathfrak{n}_1$, where $l \neq 0$. Since $f'(e_\alpha) = f'(e_\beta) = 0$, we have $\theta \neq \alpha, \beta$. Since f is a weight element, we have $\theta + \nu = \alpha + \beta$ and $\alpha + \beta - \theta = \nu \in \Delta$. By Jacobi's identity, either $\alpha - \theta \in \Delta$ or $\beta - \theta \in \Delta$.

In the first case, $\theta = \alpha + \tau$ and $\nu = \beta - \tau$ for some $\tau \in \Delta_0$, and we have

$$df'(e_\theta, e_\nu) = -l[e_{\tau-\beta}, e_{\beta-\tau}] - [e_{\alpha+\tau}, f'(e_{\beta-\tau})].$$

This expression cannot be zero, since $h_{\beta-\tau} + ch_{\alpha+\tau} \neq 0$ in view of the fact that we have $\beta - \tau \neq \alpha + \tau$ (otherwise $\alpha + \beta = 2(\alpha + \tau) \in \Delta$, which is impossible). This means that $df'(e_\theta, e_\nu) \notin \mathfrak{n}$ and $f \notin Z^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$. We have a similar situation in the second case. Thus, if for some $e_\theta \in \mathfrak{n}_1$ we have $f'(e_\theta) \neq 0$, we obtain a contradiction. Let $f'(e_\theta) = 0$ for all $e_\theta \in \mathfrak{n}_1$. According to the corollary of Lemma 1, $f' = 0$. This is equivalent to the fact that $f - g \in B^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$. This proves the lemma.

Lemma 10. *Let* $e_\alpha \in \mathfrak{n}_1$ *and* $e_\beta \in \mathfrak{n}$, $\alpha + \beta \in \Delta$, *and suppose that there exists* $\gamma \in \left(\bigcup_{i \leq 0} \Delta_i\right) \cup \{0\}$ *such that* $\beta + \gamma, \alpha + \beta + \gamma \in \Delta$ *and* $\alpha + \beta + \gamma = \tau + \nu$ *for some* τ *and* ν *in* Δ' *different from* α *and* $\beta + \gamma$. *Then any cocycle* f *in* $Z^1(\mathfrak{n},\mathfrak{g}/\mathfrak{n})$ *such that* $f(e_\alpha) = e_{-\beta}$ *is cohomologous to zero.*

Proof. We first consider the case when $\alpha + \beta = \tau + \nu$, where $\tau, \nu \in \Delta'$, $\tau \neq \alpha, \beta$, and we consider an element $g = a \operatorname{ad} e_{-\alpha-\beta}$ in $B^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$; we choose the number a so that $f(e_\tau) = g(e_\tau)$ (it is clear that $\tau = \alpha + \beta - \nu$). For the cocycle $f' = f - g$, we have

$$f'(e_\tau, e_\nu) = N_{\tau,\nu} f'(e_{\alpha+\beta}) - [e_\tau, f'(e_\nu)] \ .$$

Since f' is a weight cocycle with weight $-\alpha - \beta$, we have

$$f'(e_\nu) = be_{-\tau} \quad , \quad f'(e_{\alpha+\beta}) = h$$

for some $b \in \mathbb{C}$ and $h \in \mathfrak{h}$. Thus, the element $f'(e_\tau, e_\nu)$, which lies in $\mathfrak{n}$, lies in $\mathfrak{h}$ as well and is therefore equal to zero. This is possible in only two cases :

(a) $f'(e_{\alpha+\beta}) = f'(e_\nu) = 0$;

(b) $f'(e_{\alpha+\beta}) = Nh_\tau$, $N \in \mathbb{C}$.

In the case (a), for any $\theta \in \Delta_1$ we have $f'(e_\theta) = 0$, and by Lemma 1 we have $f' = 0$, i.e., $f = g \in B^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$. For if we assume that $f'(e_\theta) \neq 0$, then $\mu = \alpha + \beta - \theta \in \Delta'$ and we have

$$f'(e_\theta, e_\mu) = N_1 h_\mu + N_2 h_\theta \in \mathfrak{n} \cap \mathfrak{h} = 0$$

for nonzero N_1 and N_2, which is impossible, since $\mu \neq \theta$.

In case (b), we have

$$f'(e_\alpha, e_\beta) = N_0 h_\tau - N' h_\beta - N'' h_\alpha$$

for some constants N_0, N' and N''. This expression lies in $\mathfrak{n} \cap \mathfrak{h}$ and is therefore equal to zero, i.e.

$$h_\tau = N_1 h_\beta + N_2 h_\alpha \tag{1}$$

for some nonzero numbers N_1 and N_2 (if, for example, $N_1 = 0$, then $h_\tau = N_2 h_\alpha$, where $\tau \neq \alpha$, which is impossible).

Similar arguments for ν instead of τ lead to the fact that either

$f \in B^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$, or

$$h_\nu = N_3 h_\beta + N_4 h_\alpha \tag{2}$$

for some nonzero numbers N_3 and N_4.

We can assume that the root vectors in the Lie algebra are chosen so that

$$h_\alpha = \sum_{i=1}^{n} k_i(\alpha) h_{\alpha_i} \ , \tag{3}$$

if

$$\alpha = \sum_{i=1}^{n} k_i(\alpha)\alpha_i \ , \tag{4}$$

is a decomposition of the positive root a into simple roots (the $k_i(\alpha)$ are nonnegative integers). It follows from (1)–(4) that

$$\tau = N_1\beta + N_2\,\alpha\,, \qquad \nu = N_3\beta + N_4\,\alpha \ ,$$

where, since $\tau + \nu = \alpha + \beta$ and $\tau \neq \alpha, \beta$, we have $N_3 = 1 - N_1$ and $N_4 = 1 - N_2$. In addition, the numbers N_1, N_2, N_3, and N_4 are different from zero and one, and we can assume they are rational.

Let us show that all the simple roots in the decomposition of $\alpha + \beta$ into simple roots occur with multiplicity ≥ 2. To do this, we consider several possible cases :

(1) $0 < N_1$, $N_2 < 1$. In this case, $0 < N_3, N_4 < 1$, and the roots τ and ν, each separately, contain all the simple roots in the decomposition of $\alpha + \beta$. This means that their sum $\tau + \nu = \alpha + \beta$ contains all these simple roots with multiplicity ≥ 2.

(2) $0 < N_1 < 1$ and $N_2 > 1$. All simple roots having multiplicity 1 in the decomposition of β are contained in α as well (otherwise these simple roots would occur in the decomposition of τ with nonintegral coefficients, which is impossible). In addition, since $0 < N_3 < 1$ and $N_4 < 0$, from the second equality in (5), all the simple roots in the decomposition of α are contained in β (since ν is a positive root, all the simple roots occur in the decomposition with multiplicity ≥ 0). The simple roots that occur in the decomposition of β but not in the decomposition of α, must have multiplicity ≥ 2 in β (otherwise, since $0 < N_1 < 1$, these roots would occur in the decomposition of τ with nonintegral coefficients). Thus, $\alpha + \beta$ contains all simple roots in its decomposition with multiplicity ≥ 2.

(3) $N_1 < 0$ and $0 < N_2 < 1$. In this case $N_3 = 1 - N_1 > 1$ and $0 < N_4 < 1$, and, by analogy with case (2), all the simple roots in the decomposition of $\alpha + \beta$ occur with multiplicity ≥ 2.

(4) $N_1 < 0$ and $N_2 > 1$. In this case from the first equality in (5), all roots in the decomposition of b occur in the decomposition of a. Since $N_3 > 0$ and $N_4 < 0$, by the

second equality in (13), all roots in the decomposition of α occur in the decomposition of β. Thus, α and β have the same simple roots in their decompositions into simple roots. Consequently, they all occur with multiplicity ≥ 2 in $\alpha + \beta$.

The cases $N_1, N_2 > 1$ and $N_1, N_2 < 0$ are impossible, since $\tau, \nu < \alpha + \beta$ (with respect to the partial order introduced in $\mathfrak{h}^*$) and since $\tau, \nu \in \Delta^+$.

We note that there exist three cases in all when a root from Δ^+ in the decomposition into simple roots contains all simple roots with multiplicity ≥ 2 :

(i) $\mathfrak{g} = E_8$ and $\alpha + \beta = \delta$ is a maximal root;

(ii) $\mathfrak{g} = F_4$ and $\alpha + \beta = \delta$ is a maximal root;

(iii) $\mathfrak{g} = G_2$ and $\alpha + \beta = \delta$ is a maximal root.

In all cases a singular root (see, for example, [KH2]), i.e., a simple root connected with a minimal root in an extended Dynkin diagram), occurs in the decomposition of δ with multiplicity 2, but all other positive roots occur with multiplicity ≤ 1. This means that in the decomposition of the roots, α, β, τ, and ν, a singular root occurs with multiplicity 1. This is possible only in the case when we have $N_1 + N_2 = 1$ in (5).
Since $N_1 \neq N_2$ (otherwise $\alpha + \beta = 2\tau \in \Delta$, which is impossible), it follows that

$$\begin{vmatrix} N_1 & N_2 \\ 1 - N_1 & 1 - N_2 \end{vmatrix} = N_1 - N_2 \neq 0$$

and we have

$$\beta = N'_1\, \tau + N'_2\, \nu \quad , \quad \alpha = (1 - N'_1)\, \tau + (1 - N'_2)\, \nu \ , \tag{6}$$

where $N'_1 = \dfrac{1 - N_1}{N_2 - N_1}$, $N'_2 = \dfrac{N_1}{N_1 - N_2}$, $N'_1 + N'_2 = 1$.

Let us consider a simple root in the decomposition of δ that occurs with multiplicity 3 (such roots exist in all three cases). This simple root α_0 can occur in the decomposition of α and β with multiplicities 0 and 3 or 1 and 2 (up to rearrangement). In the first case, τ contains (in the decomposition into simple roots) α_0 with multiplicity 1 or 2 (multiplicities 0 and 3 are impossible, since $N_2 \neq 0, 1$). Thus, up to replacing the equalities (5) by (6), we can assume that α and β contain α_0 with multiplicity 1 and 2, respectively. Then the number $N_1 + 2N_2$, which gives the multiplicity of the root α_0 in the decomposition of τ, can be equal to 0, 1, 2, or 3. Since $N_1 + N_2 = 1$, the only possible case is $N_2 = 2, N_1 = -1$ (or $N_2 = -1, N_1 = 2$). Assuming $\mathfrak{g} \neq G_2$, we consider a simple root β_0 in the decomposition of δ occurring with multiplicity 4. Denoting the multiplicities of the occurrence of β_0 in the decomposition of β and α by x and y, respectively, we have

$$-x + 2y = a_0 \ , \quad 2x - y = b_0 \ , \tag{7}$$

where the pair (a_0, b_0) can assume the values (4, 0), (3, 1), (1, 3), and (0, 4).
For all these values, system (7) does not admit integral solutions. We have concluded, therefore, that case (b) is impossible if $\mathfrak{g} \neq G_2$.
In the case $\mathfrak{g} = G_2$, the only situation in which (5) and (6) hold is

$$\alpha = \alpha_1 + \alpha_2 \ , \quad \beta = 2\,\alpha_1 + \alpha_2 \ , \quad \tau = \alpha_2 \ , \quad \nu = 3\,\alpha_1 + \alpha_2$$

(we use Bourbaki's notation [BOU3] for simple roots).
But in this case, for $\gamma = -\alpha_1$ we have $\beta + \gamma \in \Delta$ and $\alpha + \beta + \gamma \notin \Delta$, and by Lemma 9, f' is a nonzero element in $H^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$.
Thus, we have proved the lemma in the case when $\alpha + \beta = \tau + \nu$, where $\tau, \nu \in \Delta'$ and $\tau \neq \alpha, \beta$.

Now let $\alpha + \beta + \gamma = \tau + \nu$, where $\gamma \in \cup_{i\leq 0} \Delta_i$ and $\tau \neq \alpha$, $\beta + \gamma$ but the root $\alpha + \beta$ cannot be represented as a different sum of this type. We consider a cocycle g, $g = a \operatorname{ad} e_{-\alpha-\beta}$ in $Z^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$ cohomologous to zero and choose a such that $f(e_{\beta+\gamma}) = g(e_{\alpha+\gamma})$. For the difference $f' = f - g$ we have

$$f'(e_\tau, e_\nu) = N_1 f'(e_{\alpha+\beta+\gamma}) - [f'(e_\tau), e_\nu] - [e_\tau, f'(e_\nu)] = 0 \ .$$

If $f'(e_\tau) \neq 0$ (or $f'(e_\nu) \neq 0$), then $-\alpha - \beta + \tau = -\lambda \in \Delta$ and $f'(e_\tau) = c e_{-\alpha-\beta+\tau}$ for some nonzero $c \in \mathbb{C}$. This means that $\alpha + \beta = \tau + \lambda$ is a different representation of $\alpha + \beta$ as a sum of two roots in Δ, which is impossible by hypothesis. Therefore,

$$f'(e_\tau) = f'(e_\nu) = f'(e_{\alpha+\beta+\gamma}) = 0 \ ,$$

and we have

$$f'(e_\alpha, e_{\beta+\gamma}) = -[f'(e_\alpha), e_{\beta+\gamma}] = 0 \ ,$$

which is impossible only if $f'(e_\alpha) = 0$.

For any θ in Δ_1 different from α, we also have $f'(e_\theta) = 0$ (otherwise $\lambda = \alpha + \beta - \theta \in \Delta'$ and $\alpha + \beta = \theta + \lambda$, which is impossible by hypothesis). According to Lemma 1, the cocycle f' is equal to zero ; i.e., $f = f' + g$ lies in $B^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$. This proves the lemma.

Corollary. $H^1_i(\mathfrak{n}, \mathfrak{g}/\mathfrak{n}) = 0$ *if* $i < -2$.

Proof. Let f be a nonzero (weight) cocycle in $Z^1_i(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$ where $i < -2$. According to Lemma 1, there exist $\alpha \in \Delta_1$ and $\beta \in \cup_{i\geq 2}\Delta_i$, $\alpha + \beta \in \Delta$, such that $f(e_\alpha) = ae_{-\beta}$, $a \neq 0$. We consider the decomposition $\beta = \beta_1 + \beta_2$, where $\beta_1, \beta_2 \in \Delta'$. If one of the β_1 or β_2 (say β_1) is equal to α, then $\beta - \beta_2 = \alpha \in \Delta'$ although $\alpha + \beta - \beta_2 = 2\alpha \notin \Delta$. According to Lemma 9, f is cohomologous to zero, and in this case, the corollary is proved. But if neither β_1 nor β_2 is equal to α, then either $\alpha + \beta_1 \in \Delta$ or $\alpha + \beta_2 \in \Delta$ (since one has $\alpha + \beta = \alpha + \beta_1 + \beta_2 \in \Delta$). Setting $\tau = \alpha + \beta_1$ and $\nu = \beta_2$ in the first case and $\tau = \alpha + \beta_2$ and $\nu = \beta_1$ in the second case, we can apply Lemma 10. This proves the corollary.

The proof of the following lemma is similar to the proof of Lemma 10, and we omit it.

Lemma 11. *Let* $e_\alpha \in \mathfrak{n}_1$ *be such that the set* R_α *of roots* $\gamma \in \Delta_0$ *with* $\alpha + \gamma \in \Delta$ *is non-empty, where for any* $\gamma \in R_\alpha$, *we have* $2\alpha + \gamma \in \Delta$, *and we suppose that there exists* $\gamma \in R_\alpha \cup \{0\}$ *such that* $2\alpha + \gamma = \tau + \nu$ *for some* τ *and* ν *in* Δ_1 *different from* α *and* $\alpha + \gamma$. *If* $f \in Z^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$ *is a weight cocycle such that* $f(e_\alpha) = e_{-\alpha}$, *then* $f \in B^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$.

III.2.2. Description of the space $H^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$

For any root $\nu \in \Delta_1$, we set

$$R_\nu = \{\gamma \in \Delta_0 \,/\, \nu + \gamma \in \Delta\}$$

and we introduce the following notation :

$$\Omega_0 = \{(\alpha, h_\theta) \,/\, \alpha \in \Delta_1 \;,\; h_\theta = [e_\theta, e_{-\theta}] \;,\; \theta \in S\},$$

$$\Omega_1 = \{(\alpha, -\beta) \,/\, \alpha \in \Delta_1 \;,\; \beta \in \Delta_0\},$$

$$\Omega_2 = \left\{(\alpha,-\alpha) \,/\, \alpha \in \Delta_1 \ , \ R_\alpha = \varnothing\right\},$$

$$\Omega_3 = \left\{\begin{matrix}(\alpha,-\alpha) \,/\, \alpha \in \Delta_1 \ , \ R_\alpha^\alpha \neq \varnothing \ , \ 2\alpha+\gamma \in \Delta \ , \ 2\alpha+\gamma \neq \tau+\nu \\ \text{where } \tau, \nu \in \Delta_1 \text{ and } \tau \neq \alpha, \ \alpha+\gamma \quad \text{for all } \gamma \in R_\alpha\end{matrix}\right\},$$

$$\Omega_4 = \left\{(\alpha,-\beta) \,/\, \alpha, \beta \in \Delta_1 \ , \ \alpha+\beta \in \Delta \ , \ R_\beta = \varnothing\right\},$$

$$\Omega_5 = \left\{\begin{matrix}(\alpha,-\beta) \,/\, \alpha, \beta \in \Delta_1 \ , \ \alpha+\beta \in \Delta \ , \ R_\beta \neq \varnothing \ , \ \alpha+\beta+\gamma \in \Delta \\ \alpha+\beta+\gamma \neq \tau+\nu \text{ where } \tau,\nu \in \Delta_1 \text{ and } \tau \neq \alpha, \ \beta+\gamma \quad \text{for all } \gamma \in R_\beta\end{matrix}\right\}.$$

Remarks. 1. In the case $S = S_1 = S$ (i.e., in the case when $\mathfrak{n}$ is the nilradical of a Borel subalgebra), we have $\Omega_1 = \Omega_3 = \Omega_5 = \varnothing$. In addition, the pair $(\alpha,-\alpha)$ lies in Ω_2 if and only if $\alpha \in S$, and the pair $(\alpha, -\beta)$ lies in Ω_4 if and only if α and β lie in S and are connected by an edge in the Dynkin diagram.

2. For classical Lie algebras $\mathfrak{g}$ the conditions $2\alpha + \gamma \neq \tau + \nu$ and $\alpha + \beta + \gamma \neq \tau+\nu$ in the definitions of Ω_3 and Ω_5 are superfluous. They hold if the other conditions hold.

For each element $\omega_i \in \Omega_i$, where $0 \leq i \leq 5$, we consider the following cochains f_{ω_i} in $C^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$ defined as follows :

$$f_{\omega_0}(e_\tau) = \left\{\begin{matrix} h_\theta, & \text{if } \tau = \alpha \\ 0, & \text{if } \tau \neq \alpha \end{matrix}\right\}, \text{ where } \omega_0 = (\alpha,h_0) \in \Omega_0 ;$$

$$f_{\omega_1}(e_\tau) = \left\{\begin{matrix} e_{-\beta}, & \text{if } \tau = \alpha \\ 0, & \text{if } \tau \neq \alpha \end{matrix}\right\}, \text{ where } \omega_1 = (\alpha, -\beta) \in \Omega_1 ;$$

$$f_{\omega_2}(e_\tau) = \left\{\begin{matrix} e_{-\alpha}, & \text{if } \tau = \alpha \\ 0, & \text{if } \tau \neq \alpha \end{matrix}\right\}, \text{ where } \omega_2 = (\alpha, -\alpha) \in \Omega_2 ;$$

$$f_{\omega_3}(e_\tau) = \begin{cases} e_{-\alpha}, & \text{if } \tau = \alpha \\ \frac{1}{N_{\alpha,\alpha+\gamma}} \cdot [e_{-\alpha}, e_{\alpha+\gamma}], & \text{if } \tau = 2\alpha+\gamma \\ 0, \text{ if } \tau \neq \alpha, 2\alpha+\gamma \end{cases}, \text{ where } \omega_3 = (\alpha, -\alpha) \in \Omega_3, \gamma \in R_\alpha;$$

$$f_{\omega_4}(e_\tau) = \begin{cases} e_{-\beta}, & \text{if } \tau = \alpha \\ \frac{1}{N_{\alpha,\beta}} \cdot [e_{-\beta}, e_{\beta}], & \text{if } \tau = \alpha+\beta \\ 0, \text{ if } \tau \neq \alpha, \alpha+\beta \end{cases}, \text{ where } \omega_4 = (\alpha, -\beta) \in \Omega_4;$$

$$f_{\omega_5}(e_\tau) = \begin{cases} e_{-\beta}, & \text{if } \tau = \alpha \\ \frac{1}{N_{\alpha,\beta+\gamma}} \cdot [e_{-\beta}, e_{\beta+\gamma}], & \text{if } \tau = \alpha+\beta+\gamma \\ 0, \text{ if } \tau \neq \alpha, \alpha+\beta+\gamma \end{cases}, \text{ where } \omega_5 = (\alpha, -\beta) \in \Omega_5;$$

Let

$$E_i = \{ f_{\omega_i} \in C^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n}) / \ \omega_i \in \Omega_i \} \quad ,$$

where $1 \leq i \leq 6$, and let $E = \bigcup_{i=0}^{5} E_i$. It follows from Lemmas 3 - 7 that each element of E is contained in $Z^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$ and is not cohomologous to zero. These elements define nonzero elements in $H^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$. In what follows, for simplicity, we shall assume that $E \subset H^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$ and identify cocycles that differ by an element in $B^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$.

Theorem 3. *Let* $\mathfrak{n}$ *be the nilradical of a parabolic subalgebra* $\mathfrak{p}$ *of a complex simple Lie algebra* $\mathfrak{g}$ *defined by means of a subsystem* $S_1 \subset S$. *Then* $H^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n}) = \langle E \rangle$.

Proof. Let $\mathfrak{g}$ be a nonzero weight element of weight λ in $H^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$. According to Lemmas 9 and 10 (see also the corollary to the latter), $\mathfrak{g}$ lies in either $H^1_{-1}(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$ or $H^1_{-2}(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$. In the first case, obviously, $\mathfrak{g} \in \langle E_0, E_1 \rangle$ and the theorem is true.

Let $\mathfrak{g} \in H^1_{-2}(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$. We consider a root vector $e_\alpha \neq 0$ (the existence of such an element follows from Lemma 1). Then $\mathfrak{g}(e_\alpha) = ae_{-\beta}$ for some nonzero $a \in \mathbb{C}$ and for a root $\beta \in \Delta_1$ with $-\alpha - \beta = \lambda$. By Lemma 2, two cases are possible:
(a) $\beta = \alpha$;
(b) $\alpha + \beta \in \Delta$.

(a) According to Lemmas 8 and 11, this case can also be divided into two subcases:
(a_1) $R_\alpha = \varnothing$;
(a2) $R_\alpha \neq \varnothing$, but $2\alpha + \gamma \in \Delta$ and $2\alpha + \gamma \neq \tau + \nu$, where $\tau, \nu \in \Delta_1$ and $\tau, \nu \neq \alpha$, $\alpha + \gamma$ for all $\gamma \in R_\alpha$.
In subcase (a_1), we choose an element f in $\langle E_2 \rangle$ such that $f(e_\alpha) = ae_{-\alpha}$, and we consider the difference $g' = g - f$.
In subcase (a_2) we choose an analogous element f in $\langle E_3 \rangle$ and consider the difference $g' = g - f$. In both subcases, g' is an element of $H^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$ that vanishes at e_α and continues to vanish at those elements $e_\tau \in \mathfrak{n}_1$ at which g vanishes.

(b) According to Lemmas 9 and 10, this case also splits into two subcases, and in each of these subcases one can choose an element f in $\langle E_4 \rangle$ or $\langle E_5 \rangle$ such that the difference $g' = g - f$ vanishes at e_α and continues to vanish at all elements $e_\tau \in \mathfrak{n}_1$ for which $g(e_\tau) = 0$.
Thus, in all cases, the number of root vectors in $\mathfrak{n}_1$ at which g' vanishes is one less than the same number for g. If $g'(e_\tau) \neq 0$ for some $e_\tau \in \mathfrak{n}_1$, then, repeating the argument, we can find an f_1 in $E_2 \cup E_3 \cup E_4 \cup E_5$ for which $g'' = g' - f_1$ vanishes at e_τ. Continuing in this way, we find an element $g^{(k)} = g^{(k-1)} - f_{k-1}$ in $H^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$ that vanishes on all of $\mathfrak{n}_1$.
According to Lemma 1, this is possible only if $g^{(k)} = 0$. This means that g can be expressed as a linear combination of elements in E_2, E_3, E_4, and E_5. This proves the theorem.

III.2.3. A Basis of the space $H^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$

The elements of E, as shown in Theorem 2, are linear generators of the space $H^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$. In general, they can be linearly dependent in $H^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$, although as elements of $Z^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$ they are linearly independent (as is easy to see). Below, we shall prove several assertions that make it possible to find a basis of $H^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$ and sharpen Theorem 3.
We consider a partition $E = \cup E_\lambda$ of E into nonempty subsets E_λ consisting of mappings with the same weight λ.

Proposition 7. (a) *If* λ *is not a root, then* E_λ *is linearly independent in* $H^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$
(b) *If* λ *is a root in* Λ_{-1}, *then the system* E_λ *is linearly dependent in* $H^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$.
(c) *If* λ is *a root in* Δ_{-2}, *then* E_λ *is linearly independent in* $H^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$ *if and only if* $M_\lambda = \Omega_4 \cup \Omega_5$ *where* $M_\lambda = \{(\alpha - \beta) / \alpha,\beta \in \Delta_1 \ , \lambda = -\alpha - \beta\}$.

Proof. (a) The weight elements of $B^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$ obviously have the form $g = a \,\mathrm{ad}\, e_\alpha$, where $\alpha \in \cup_{i\le 0} \Delta_i$. Therefore, any nonzero linear combination of elements in E_λ cannot lie in $B^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$, since $\lambda \notin \Delta$.

(b) Let $g = \mathrm{ad}\, e_\lambda \in B^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$, where $\lambda \in \Delta_{-1}$. For any root α in Δ_1 for which $\alpha + \lambda = -\beta \in \Delta$, the pair $(\alpha,-\beta)$ lies in Ω_1. Therefore, one can find a linear combination f of elements in $E_0 \cup E_1$ such that the values of the maps f and g are equal at all root vectors $e_\alpha \in \mathfrak{n}_1$. We consider their difference $f' = f - g$, which lies in $Z^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$. The value of f' at any element in $\mathfrak{n}_1$, by the choice of f', is equal to zero. According to Lemma 1, this is possible only if $f'(\mathfrak{n}) \subset \mathfrak{n}$, i.e. if $f' = 0$. Thus a nonzero linear combination of elements in E_λ lies in $B^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$, i.e. is equal to zero in $H^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$.

(c) The proof of the sufficiency is completely analogous to the proof of (b). Let us prove the necessity. If $M_\lambda \neq \Omega_4 \cup \Omega_5$, then there exists a pair $(\alpha,-\beta)$ in M_λ, where $\lambda = -\alpha - \beta$, that does not lie in $\Omega_4 \cup \Omega_5$. For this pair, we have $g(e_\alpha) \neq N\, e_\beta \neq 0$, and any nonzero linear combination f of elements in E_1 is equal to zero at e_α. Since any weight element of weight λ in $B^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$ is a multiple of g, f cannot lie in $B^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$; that is, E_λ is linearly independent. This proves the lemma.

Corollary 1 (from the proof). *If* λ *is a root and* E_λ *is linearly dependent (i.e., either* $\lambda \in \Delta_{-1}$, *or* $\lambda \in \Delta_{-2}$ *and* $M_\lambda = \Omega_4 \cup \Omega_5$), *then :*
(a) If $\lambda \in \Delta_{-2}$, *any collection* E'_λ *that lies properly in* E_λ *is linearly independent.*
(b) If $\lambda \in \Delta_{-1}$ *any collection* $E'_\lambda \supset E_0 \cap E_\lambda$ *that lies properly in* E_λ *is linearly independent.*

Corollary 2. *If* $\mathfrak{n}$ *is the nilradical of a Borel subalgebra in a complex simple Lie algebra* $\mathfrak{g}$, *then* $\Omega_1 = \Omega_3 = \Omega_5 = \varnothing$, *and so* $E_1 = E_3 = E_5 = 0$. *In addition, any weight* λ *of a cocycle in* $H^1_{-2}(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$ *which is a root is equal to the sum of two simple roots. Consequently,* $M_\lambda = \Omega_4$.

III.3. Particular case : $\mathfrak{n}$ is the nilradical of a Borel subalgebra

Let $E_\lambda \neq \varnothing$. We consider the set

$$E'_\lambda = \left\{ \begin{array}{ll} E_\lambda \text{ if } \lambda \notin \Delta \text{ or } \lambda \in \Delta_{-2} \text{ , } M_\lambda \neq \Omega_4 \cup \Omega_5 & \\ E_\lambda - \{f\} \text{ , if } \lambda \in \Delta_{-1} \text{ or } \lambda \in \Delta_{-2} \text{ , } M_\lambda = \Omega_4 \cup \Omega_5 & , \end{array} \right\}$$

where f is an arbitrary chosen element of E_λ.
The next theorem follows from Proposition 7 and its corollary.

Theorem 4. *Let* $\mathfrak{n}$ *be the nilradical of a parabolic subalgebra of a complex simple Lie algebra* $\mathfrak{g}$ *defined by a subsystem* $S_1 \subset S$, *and let* Λ *be the set of weights (with respect to the adjoint action of* $\mathfrak{h}$) *of the cocycles in E. Then* $E' = \bigcup_{\lambda \in \Lambda} E'_\lambda$ *is a basis of the space* $H^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$.

Corollary 1. *If* $\mathfrak{g}$ *is a simple Lie algebra of rank* n *and* $\mathfrak{n}$ *is the nilradical of a Borel subalgebra, then* $\dim H^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n}) = n^2 + n - 1$.

Indeed, in this case $\Lambda = (-S) \cup \Delta_{-2} \cup (-2S)$ and

$$|E_\lambda| = \left\{ \begin{array}{l} n, \text{ if } \lambda \in -S, \\ 2, \text{ if } \lambda \in \Delta_{-2}, \\ 1, \text{ if } \lambda \in -2S, \end{array} \right\}, \quad |E'_\lambda| = \left\{ \begin{array}{l} n\text{-}1, \text{ if } \lambda \in -S, \\ 1, \text{ if } \lambda \in \Delta_{-2}, \\ 1, \text{ if } \lambda \in -2S, \end{array} \right\}$$

and we have $\dim H^1(\mathfrak{n},\mathfrak{g}/\mathfrak{n}) = n(n-1) + (n-1) + n = n^2 + n - 1$.

III.4. Particular case : $\mathfrak{n}$ is an Heisenberg algebra

Let H_k be an Heisenberg algebra with $k \geq 2$. It is isomorphic to the nilradical $\mathfrak{n}$ of a parabolic subalgebra of a simple algebra $\mathfrak{g}$ of type A_r $(r = k+1)$, defined by

$$S_1 = \{\alpha_1, \alpha_r\}.$$

It is easy to verify that, in this case, we have

$$\Omega_2 = \Omega_3 = \Omega_4 = \Omega_5 = \varnothing.$$

Then the space $H^1(\mathfrak{n},\mathfrak{g}/\mathfrak{n})$ is generated by the vectors of $E_0 \cup E_1$. As

$$\text{cardinal}\,(\Delta_1) = 2(r-1),$$

$$\text{cardinal}\,(\Delta_0) = r^2 - 3r + 2,$$

$$\text{cardinal}\,(S) = r,$$

we have

$$\text{cardinal}\,(E) = 2r^3 - 6r^2 + 8r - 4.$$

We deduce

$$\text{cardinal}\left(\bigcup_{\lambda\in\Lambda} E'_\lambda\right) = (2r^3 - 6r^2 + 8r - 4) - 2(r-1) = 2r^3 - 6r^2 + 6r - 2,$$

and

$$\dim H^1(\mathfrak{n},\mathfrak{g}/\mathfrak{n}) = 2r^3 - 6r^2 + 6r - 2.$$

III.5. Description of the space $H^2{}_{\text{fond}}(\mathfrak{n},\mathfrak{n})$

This description is based on the study of the s-module $H^j(\mathfrak{n},V)$, where V is a simple $\mathfrak{g}$-module whose structure has been given in Kostant's works (see [KO]).

III.5.1. **Description of the s-module $H^j(\mathfrak{n},V)$**

Let $\mathfrak{n}$ be the nilradical of a parabolic subalgebra of the simple complex algebra $\mathfrak{g}$, defied by $S_1 \subset S$. Consider a simple $\mathfrak{g}$-module V of dominating weight g (see Chapter 1) and the corresponding Weyl group of $\mathfrak{g}$ noted W.

We have seen that the spaces $C^j(n,V)$, $Z^j(n,V)$, $B^j(n,V)$ and $H^j(n,V)$ are provided to structures of s-modules, where s is the Levi subalgebra of $\mathfrak{g}$ given by

$$s = \mathfrak{h} + \sum_{\alpha \in \Delta_0} \mathfrak{g}_\alpha .$$

We put

$$U_\sigma = \sigma(\Delta^-) \cap \Delta^+, \text{ where } \sigma \in W ,$$

$$W_1 = \left\{ \sigma \in W \mid U_\sigma \subset \Delta' = \bigcup_{i \geq 1} \Delta_i \right\},$$

$$W^j{}_1 = W^j \cap W_1 ,$$

$$p = \frac{1}{2} \sum_{\gamma \in \Delta} \gamma .$$

We have

$$H^j(\mathfrak{n},V) = \sum_{\sigma \in W^j_1} W_\sigma ,$$

where W_σ is the simple s-module with dominating weight $\xi_\sigma = \sigma(p + \lambda) - p$ (λ is the weight of V).

Let $v_{\sigma(\lambda)}$ be a vector of V which is nonzero and which has a weight equal to $\sigma(\lambda)$ and let $(\alpha_1 ,..., \alpha_j)$ be the ordered elements of U_σ.

Then the cohomology classes of the cocycle f_σ defined by

$$f_\sigma(X_{\alpha_1} ,..., X_{\alpha_j}) = v_{\sigma(\lambda)}$$

is a primitive vector (i.e. a nonzero vector corresponding to the dominating weight) of the s-module W_σ.

Remarks. 1. For the description of these structures, Kostant shows that the dominating weight $\xi_\sigma = \sigma(p+\lambda) - p$ of the s-modules W_σ are all different from each other and distinct to the dominating weights of the simple components of the s-module $B^{\cdot}(n,V)$.

2. As $Z^{\cdot}(\mathfrak{n},V)$ is a semisimple s-module, there is a supplementary s-module to the s-module $B^{\cdot}(\mathfrak{n},V)$. This can be identified by $H^{\cdot}(\mathfrak{n},V)$.

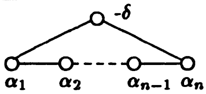
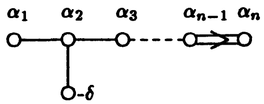
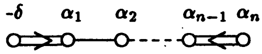
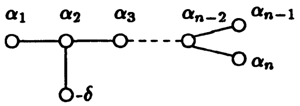
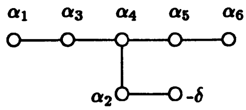
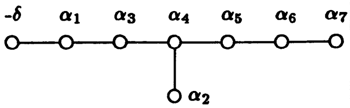
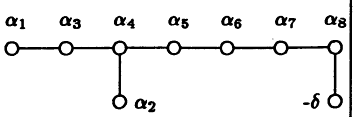
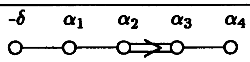
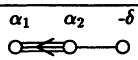
III.5.2. The space $H^2_{fond}(\mathfrak{n},\mathfrak{n})$

We recall that a root α is singular if $\delta - \alpha \in S$, where δ is a maximal root. The simple Lie algebras of type A_r (with $r > 2$) owns two singular roots denoted α_1 and α_r in Bourbaki's terminology. The other types have only one singular root. But in these cases, the multiplicity in the decomposition of δ of the singular root is equal to two.

Now if β is a positive root, then $\delta - \beta$ is a root if and only if the singular root (or one singular root in the A_r case) belongs to the decomposition of β, and its multiplicity is equal to 1 otherwise $\beta = \delta$.

Following is the table of maximal roots, of singular roots and the expanded Dynkin diagram for each type of simple complex Lie algebras. Also we precise a number $y(\mathfrak{g})$ which we will use later.

TABLE 1

Type of algebra	Dynkin extended diagram	Maximal root	Singular roots	$y(\mathfrak{g})$
A_n $(n \geq 2)$		$\alpha_1 + \alpha_2 + \cdots + \alpha_n$	α_1, α_n	5
B_n $(n \geq 3)$		$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n$	α_2	3
C_n $(n \geq 2)$		$2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n$	α_1	4
D_n $(n \geq 4)$		$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2}+$ $+\alpha_{n-1} + \alpha_n$	α_2	3
E_6		$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4+$ $+2\alpha_5 + \alpha_6$	α_2	2
E_7		$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4+$ $+3\alpha_5 + 2\alpha_6 + \alpha_7$	α_1	2
E_8		$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4+$ $+5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$	α_8	2
F_4		$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$	α_1	2
G_2		$3\alpha_1 + 2\alpha_2$	α_3	

We need some notions which were introduced before.

$$W^2 = \{\sigma \in W \,/\, \mathrm{card}\, U_\sigma = 2\}\,.$$

This set is composed of the elements σ of the Weyl group which are written as

$$\sigma = s_{\beta_1} s_{\beta_2} \quad \text{where} \quad \beta_1, \beta_2 \in S, \quad \beta_1 \neq \beta_2$$

(here s_β designates the associated reflexion to the root β).

As for the elements of $W^2{}_1 = W^2 \cap W_1$, they are of the form $\sigma = s_{\beta_1} \circ s_{\beta_2}$, where the the roots β_1 and β_2 satisfy one of the conditions

(a) $\beta_1 + \beta_2 \notin \Delta$, $\beta_1, \beta_2 \in S - S_1$,

(b) $\beta_1 + \beta_2 \in \Delta$, $\beta_1 \in S - S_1$.

In the first case, $U_\sigma = \{\beta_1, \beta_2\}$ and in the second case $U_\sigma = \{\beta_1, \beta_2 + k\beta_1\}$, where k is the largest integer such that $\beta_2 + k\beta_1 \in \Delta$.

Proposition 8. *The cardinal of $W^2{}_1$ is equal to*

$$\frac{r(r-1)}{2} + 2r - t_1 - t_2 + t_3$$

where t_1 is the number or roots of $S - S_1$ which are on the boundary of the Dynkin diagram, t_2 the number of pairs of roots of $S - S_1$ which are fastened in the Dynkin diagram and $t_3 = 0$ for the Lie algebras $\mathfrak{g}$ of type A_r, B_r, C_r, G_2, F_4 and $t_3 = 1$ for the other types.

The proof is obvious after we have noted that if β_1 and β_2 satisfy the conditions of case (a), the reflexions s_{β_1} and s_{β_2} commute.

Let $W^2{}_x$ be the subset of $W^2{}_1$ defined by

$$W^2_{\mathfrak{n}} = \left\{\sigma \in W^2{}_1 / \sigma(\delta) \in \Delta'\right\}\,.$$

In particular, if $\sigma \in W^2{}_{\mathfrak{n}}$, the vector $X_{\sigma(\delta)} \in \mathfrak{n}$, where δ is a maximal root.

Lemma 1. *If $\mathfrak{g}$ is of type C_r or A_r ($r \geq 2$) and if $S - S_1$ contains only one singular root or if $\mathfrak{g} = A_2$ and $S - S_1 = \{\alpha_1, \alpha_2\}$ then , $W^2{}_{\mathfrak{n}} = \varnothing$.*

Proof. We begin with the case $\mathfrak{g} = A_2$.

Then $W^2{}_1$ contain only two elements $s_{\alpha_1} \circ s_{\alpha_2}$ and $s_{\alpha_2} \circ s_{\alpha_1}$ which, when applied to the maximal root $\delta = \alpha_1 + \alpha_2$, give a negative root. So $W^2{}_n = \varnothing$.

In the other cases, $W^2{}_1 = \{s_{\beta_1} \circ s_{\beta_2}\}$, where $\beta_1 \in S_1$ is a singular root and β_2 is fastened to β_1 in the Dynkin diagram. This set is reduced to only one element (see Table 1). We have

$$s_{\beta_1} \circ s_{\beta_2}(\delta) = \left\{ \begin{array}{ll} \delta - \beta_1 \ \text{if} \ \mathfrak{g} = A_r & r > 2 \\ \delta - 2\beta_1 \ \text{if} \ \mathfrak{g} = C_r & r \geq 2 \end{array} \right\}.$$

As the roots $\delta - \beta_1$ (in the cases $\mathfrak{g} = A_2$) and $\delta - 2\beta_1$ (in the other cases) don't contain in their decomposition elements of $S - S_1$, then $s_{\beta_1} \circ s_{\beta_2}(\delta) \notin \Delta'$. Thus, $W^2{}_{\mathfrak{n}} = \varnothing$.

Lemma 2. *If* $\mathfrak{g} = A_r (r \geq 3)$ *and if* $S_1 = \{\alpha_1, \alpha_r\}$ *then*

$$W^2_{\mathfrak{n}} = \{s_{\alpha_1} \circ s_{\alpha_2} \ , \ s_{\alpha_r} \circ s_{\alpha_{r-1}}\} \ .$$

The proof is established by considering the tables of roots before the description of roots tied to the elements of $W^2{}_1$. We use same considerations for the following lemmas :

Lemma 3. *If* $\mathfrak{g} = C_2$ *and if* $S = \{\alpha_1, \alpha_2\}$, *then* $W^2_{\mathfrak{n}} = \{s_{\alpha_1} \circ s_{\alpha_2}\}$.

Lemma 4. *If* $\mathfrak{g} = C_r (r \geq 3)$ *and if* $S - S_1 = \{\alpha_1, \alpha_2\}$, *then*

$$W^2_{\mathfrak{n}} = \{s_{\alpha_1} \circ s_{\alpha_2} \ , \ s_{\alpha_2} \circ s_{\alpha_3}\} \ .$$

Lemma 5. *In all the other cases,* $W^2{}_{\mathfrak{n}} = W^2{}_1$.

We can deduce the following two theorems.

Theorem 4. *Let* $\mathfrak{n}$ *be the nilradical of a parabolic subalgebra of a simple complex Lie algebra which don't belong to Table 2. Then*

$$H^2{}_{\text{fond}}(\mathfrak{n},\mathfrak{n}) \cong H^2(\mathfrak{n},\mathfrak{g}) \ .$$

TABLE 2

$\mathfrak{g}$	$S\ S_1$	W_n^2	β_1	β_2	γ
A_2	α_1, α_2	$\varnothing$			
A_n	α_1 α_n	$\varnothing$ $\varnothing$			
A_n $n \geq 3$	$\alpha_1,\ \alpha_n$	$\varphi_{\alpha_1} \cdot \varphi_{\alpha_2}$ $\varphi_{\alpha_n} \cdot \varphi_{\alpha_{n-1}}$	α_1 α_n	$\alpha_1 + \alpha_2$ $\alpha_{n-1} + \alpha_n$	$\delta - \alpha_1$ $\delta - \alpha_n$
C_2	α_1, α_2	$\varphi_{\alpha_1} \cdot \varphi_{\alpha_2}$	α_1	$2\alpha_1 + \alpha_2$	α_2
C_3	α_1, α_2	$\varphi_{\alpha_1} \cdot \varphi_{\alpha_2}$ $\varphi_{\alpha_2} \cdot \varphi_{\alpha_3}$	α_1 α_2	$\alpha_1 + \alpha_2$ $2\alpha_2 + \alpha_3$	$2\alpha_2+\alpha_3$ $2\alpha_1+2\alpha_2+\alpha_3$
C_n $n \geq 4$	α_1, α_2	$\varphi_{\alpha_1} \cdot \varphi_{\alpha_2}$ $\varphi_{\alpha_2} \cdot \varphi_{\alpha_3}$	α_1 α_2	$\alpha_1 + \alpha_2$ $\alpha_2 + \alpha_3$	$2\alpha_2 + \alpha_3$ δ
C_n $n \geq 2$	α_1	$\varnothing$			

Theorem 6. *If* $\mathfrak{g}$ *is in the list of Table 2, then the* $\mathfrak{s}$*-module* $H^2_{fond}(\mathfrak{n},\mathfrak{n})$ *is the direct sum of* $\mathfrak{s}$*-submodules generated by the cocycles* v_i *of the following form :*

$$(e_{\beta_1}, e_{\beta_2}) \to e_j \ .$$

Now, one fixes for each complex simple Lie algebra the integer $y(\mathfrak{g})$ whose value is given in the Table I.

Theorem 7. *Suppose that* $\mathrm{card}(S_1) > y(\mathfrak{g})$. *Then*

$$H^2_0(\mathfrak{n},\mathfrak{n}) = 0 \quad , \quad F_i H^2(\mathfrak{n},\mathfrak{n}) \cong H^2_{fond}(\mathfrak{n},\mathfrak{n}) \ \ for \ \ i = 0.$$

Proof. Let $\sigma = s_{\beta_1} \circ s_{\beta_2}$ in $W^2_{\mathfrak{n}}$, β_1 and β_2 in S.
We have

$$U_\sigma = \{\beta_1, \beta_2 + k\beta_1\},$$

$\sigma(\delta) = \delta - k_2\,\beta_2 - k_1\,\beta_1$, where k_1 and k_2 are the greatest integers such that $\beta_2 + k\beta_1, \delta - k_2\,\beta_2, \delta - k_2\,\beta_1 - k_1\,\beta_1$ are roots. By using the roots tables, we can show that, if $\mathrm{card}(S_1) > y(\mathfrak{g})$, then

$$l_{S_1}(\sigma(\delta)) = l_{S_1}(\delta) - l_{S_1}(k_2\beta_2) - l_{S_1}(k_1\beta_1)$$

and

$$l_{S_1}(\sigma(\delta)) > l_{S_1}(\beta_1) + l_{S_1}(\beta_2 + k\beta_1)\,,$$

where $l_{S_1}(\alpha)$ appoints the $S - S_1$ length of α . This means that

$$l_{S_1}(\sigma(\delta)) > l_{S_1}(\beta_1) + l_{S_1}$$

where β_1 and β_2 are the roots such that $U_\sigma = \{\beta_1, \beta_2 + k\beta_1\}$.
We deduces that

$$W^2_0 = \{0\} \ \text{ and } \ W^2_{\mathfrak{n}} = \bigcup_{i \geq 0} W^2_i .$$

Now this theorem becomes a consequence of the two previous theorems.

III.6. The space $H^2(\mathfrak{n},\mathfrak{n})$: case of the nilradical of a Borel subalgebra

We suppose that $\mathfrak{n}$ is the nilradical of a Borel subalgebra ($S_1 = \varnothing$). In this case, the previous theorem can be formulated more precisely.

Theorem 8. (a) *If* $\mathfrak{g}$ *is a simple Lie algebra of rank* r *whose type is different to* A_i *for* $i = 1, 2, 3, 4, 5$, B_3, C_i, D_4 *and* G_2 *then* $H^2{}_0(\mathfrak{n},\mathfrak{n}) = 0$, $F_0H^2(\mathfrak{n},\mathfrak{n}) = F_1H^2(\mathfrak{n},\mathfrak{n}) = H^2{}_{fond}(\mathfrak{n},\mathfrak{n})$ and $\dim H^2{}_{fond}(\mathfrak{n},\mathfrak{n}) = (r^2 + r - 2)/2$

(b) *If* $\mathfrak{g}$ *is not of type pointed in* (a), *then the cocycles defined by* $\varphi(e_{\beta_1}, e_{\beta_2}) = e_\gamma$, *whose list is presented in Table 3, determine the basis of* $H^2{}_i(\mathfrak{n},\mathfrak{n})$ and $H^2{}_{fond}(\mathfrak{n},\mathfrak{n})$.

TABLE 3

$\mathfrak{g}$	v_i, u_i	W^2	β_1	β_2	γ	$H^2_{(i)}(n,n)$
A_4	v_1	$\varphi_{\alpha_1}\cdot\varphi_{\alpha_4}$	α_1	α_4	$\alpha_2+\alpha_3$	$W^2_{(0)}(n,n)$
	v_2	$\varphi_{\alpha_1}\cdot\varphi_{\alpha_2}$	α_1	$\alpha_1+\alpha_2$	$\alpha_2+\alpha_3+\alpha_4$	
	v_3	$\varphi_{\alpha_4}\cdot\varphi_{\alpha_3}$	α_4	$\alpha_3+\alpha_4$	$\alpha_1+\alpha_2+\alpha_3$	
	u_1	$\varphi_{\alpha_1}\cdot\varphi_{\alpha_3}$	α_1	α_3	$\alpha_2+\alpha_3+\alpha_4$	$H^2_{(1)}(n,n)$
	u_2	$\varphi_{\alpha_2}\cdot\varphi_{\alpha_4}$	α_2	α_4	$\alpha_1+\alpha_2+\alpha_3$	
	u_3	$\varphi_{\alpha_2}\cdot\varphi_{\alpha_3}$	α_2	$\alpha_2+\alpha_3$	δ	
	u_4	$\varphi_{\alpha_3}\cdot\varphi_{\alpha_2}$	α_3	$\alpha_2+\alpha_3$	δ	
A_5	v_1	$\varphi_{\alpha_2}\cdot\varphi_{\alpha_1}$	α_2	$\alpha_1+\alpha_2$	$\alpha_3+\alpha_4+\alpha_5$	$H^2_{(0)}(n,n)$
	v_2	$\varphi_{\alpha_4}\cdot\varphi_{\alpha_5}$	α_4	$\alpha_4+\alpha_5$	$\alpha_1+\alpha_2+\alpha_3$	
	u_1	$\varphi_{\alpha_1}\cdot\varphi_{\alpha_2}$	α_1	$\alpha_1+\alpha_2$	$\alpha_2\alpha_3+\alpha_4+\alpha_5$	$H^2_{(1)}(n,n)$
	u_2	$\varphi_{\alpha_1}\cdot\varphi_{\alpha_5}$	α_1	α_5	$\alpha_2+\alpha_3+\alpha_4$	
	u_3	$\varphi_{\alpha_5}\cdot\varphi_{\alpha_4}$	α_5	$\alpha_4+\alpha_5$	$\alpha_1\alpha_2+\alpha_3+\alpha_4$	
	u_4	$\varphi_{\alpha_1}\cdot\varphi_{\alpha_3}$	α_1	α_3	$\alpha_2\alpha_3+\alpha_4+\alpha_5$	$H^2_{(2)}(n,n)$
	u_5	$\varphi_{\alpha_1}\cdot\varphi_{\alpha_4}$	α_1	α_4	$\alpha_2\alpha_3+\alpha_4+\alpha_5$	
	u_6	$\varphi_{\alpha_2}\cdot\varphi_{\alpha_3}$	α_2	$\alpha_2+\alpha_3$	δ	
	u_7	$\varphi_{\alpha_2}\cdot\varphi_{\alpha_5}$	α_2	α_5	$\alpha_1\alpha_2+\alpha_3+\alpha_4$	
	u_8	$\varphi_{\alpha_3}\cdot\varphi_{\alpha_2}$	α_3	$\alpha_2+\alpha_3$	δ	
	u_9	$\varphi_{\alpha_3}\cdot\varphi_{\alpha_4}$	α_3	$\alpha_3+\alpha_4$	δ	
	u_{10}	$\varphi_{\alpha_3}\cdot\varphi_{\alpha_5}$	α_3	α_5	$\alpha_1\alpha_2+\alpha_3+\alpha_4$	
	u_{11}	$\varphi_{\alpha_4}\cdot\varphi_{\alpha_3}$	α_4	$\alpha_3+\alpha_4$	δ	
	u_{12}	$\varphi_{\alpha_2}\cdot\varphi_{\alpha_4}$	α_2	α_4	δ	$H^2_{(3)}(n,n)$
C_3	v_1	$\varphi_{\alpha_1}\cdot\varphi_{\alpha_2}$	α_1	$\alpha_1+\alpha_2$	$2\alpha_2+\alpha_3$	$H^2_{(0)}(n,n)$
	u_1	$\varphi_{\alpha_3}\cdot\varphi_{\alpha_1}$	α_3	α_1	$2\alpha_2+\alpha_3$	$H^2_{(1)}(n,n)$
	u_2	$\varphi_{\alpha_2}\cdot\varphi_{\alpha_3}$	α_2	$\alpha_3+2\alpha_2$	δ	
	u_3	$\varphi_{\alpha_3}\cdot\varphi_{\alpha_2}$	α_3	$\alpha_2+\alpha_3$	δ	$H^2_{(2)}(n,n)$

TABLE 3 (follow up)

$\mathfrak{g}$	v_i, u_i	W^2	β_1	β_2	γ	$H^2_{(i)}(n,n)$
	v_1	$\varphi_{\alpha_2}\cdot\varphi_{\alpha_1}$	α_2	$\alpha_1+\alpha_2$	$2\alpha_3+\alpha_4$	$H^2_{(0)}(n,n)$
	u_1	$\varphi_{\alpha_1}\cdot\varphi_{\alpha_2}$	α_1	$\alpha_1+\alpha_2$	$2\alpha_2+2\alpha_3+\alpha_4$	$H^2_{(2)}(n,n)$
	u_2	$\varphi_{\alpha_4}\cdot\varphi_{\alpha_3}$	α_1	α_3	$2\alpha_2+2\alpha_3+\alpha_4$	
	u_3	$\varphi_{\alpha_1}\cdot\varphi_{\alpha_4}$	α_1	α_4	$2\alpha_2+2\alpha_3+\alpha_4$	$H^2_{(3)}(n,n)$
C_4	u_4	$\varphi_{\alpha_3}\cdot\varphi_{\alpha_4}$	α_3	$\alpha_4+2\alpha_3$	δ	
	u_5	$\varphi_{\alpha_2}\cdot\varphi_{\alpha_3}$	α_2	$\alpha_2+\alpha_3$	δ	
	u_6	$\varphi_{\alpha_3}\cdot\varphi_{\alpha_2}$	α_3	$\alpha_2+\alpha_3$	δ	$H^2_{(4)}(n,n)$
	u_7	$\varphi_{\alpha_4}\cdot\varphi_{\alpha_3}$	α_4	$\alpha_3+\alpha_4$	δ	
	u_8	$\varphi_{\alpha_2}\cdot\varphi_{\alpha_4}$	α_2	α_4	δ	$H^2_{(5)}(n,n)$
	v_1	$\varphi_{\alpha_1}\cdot\varphi_{\alpha_2}$	α_1	$\alpha_1+\alpha_2$	$\alpha_2+2\alpha_3$	$H^2_{(0)}(n,n)$
	u_1	$\varphi_{\alpha_2}\cdot\varphi_{\alpha_1}$	α_2	$\alpha_1+\alpha_2$	$\alpha_1+\alpha_2+2\alpha_3$	$H^2_{(1)}(n,n)$
	u_2	$\varphi_{\alpha_2}\cdot\varphi_{\alpha_3}$	α_2	$\alpha_2+\alpha_3$	$\alpha_1+\alpha_2+2\alpha_3$	
	u_3	$\varphi_{\alpha_1}\cdot\varphi_{\alpha_3}$	α_1	α_3	δ	$H^2_{(3)}(n,n)$
	v_1	$\varphi_{\alpha_1}\cdot\varphi_{\alpha_2}$	α_1	$\alpha_1+\alpha_2$	$\alpha_2+2\alpha_3$	
	v_2	$\varphi_{\alpha_3}\cdot\varphi_{\alpha_2}$	α_3	$\alpha_2+\alpha_3$	$\alpha_1+\alpha_2+\alpha_4$	$H^2_{(0)}(n,n)$
B_3	v_3	$\varphi_{\alpha_4}\cdot\varphi_{\alpha_2}$	α_4	$\alpha_2+\alpha_4$	$\alpha_1+\alpha_2+\alpha_3$	
	u_1	$\varphi_{\alpha_2}\cdot\varphi_{\alpha_1}$	α_2	$\alpha_1+\alpha_2$	$\alpha_1+\alpha_2+\alpha_3+\alpha_4$	
	u_2	$\varphi_{\alpha_2}\cdot\varphi_{\alpha_3}$	α_2	$\alpha_2+\alpha_3$	$\alpha_1+\alpha_2+\alpha_3+\alpha_4$	$H^2_{(1)}(n,n)$
	u_3	$\varphi_{\alpha_2}\cdot\varphi_{\alpha_4}$	α_2	$\alpha_2+\alpha_4$	$\alpha_1+\alpha_2+\alpha_3+\alpha_4$	
	u_4	$\varphi_{\alpha_1}\cdot\varphi_{\alpha_3}$	α_1	α_3	δ	
	u_5	$\varphi_{\alpha_1}\cdot\varphi_{\alpha_4}$	α_1	α_4	δ	$H^2_{(3)}(n,n)$
	u_6	$\varphi_{\alpha_3}\cdot\varphi_{\alpha_4}$	α_3	α_4	δ	
G_2	u_1	$\varphi_{\alpha_1}\cdot\varphi_{\alpha_2}$	α_1	$\alpha_1+\alpha_2$	$\alpha_1+3\alpha_2$	$H^2_{(1)}(n,n)$

Proof. Point (a) emanates directly from Theorem 3 (III.5.2) and from Proposition (III.5.2).
Point (b) emanates from Theorems 1 and 2 (III.5.2).

Conclusion. We have decomposed the space $H^2(\mathfrak{n},\mathfrak{n})$ as :

$$H^2(\mathfrak{n},\mathfrak{n}) = H^2_{\text{fond}}(\mathfrak{n},\mathfrak{n}) \oplus H^1\left(\mathfrak{n},\mathfrak{g}/_{\mathfrak{n}}\right)$$

for every nilradical of a parabolic subalgebra (with some exceptions). But each spaces $H^2_{\text{fond}}(\mathfrak{n},\mathfrak{n})$ and $H^1(\mathfrak{n},\mathfrak{g}/_{\mathfrak{n}})$ is very precisely described in the previous sections. If we concentrate this study on the case when $\mathfrak{n}$ is the nilradical of a Borel subalgebra, and if $\mathfrak{g}$ is not of the types indicated in the previous theorem at the point (a), we reclaim the theorem of Leger–Luks.

Theorem 9.

$$\dim H^2(\mathfrak{n},\mathfrak{n}) = \frac{1}{2}\left(r^2 + r - 2\right) + r^2 + r - 1 = \frac{3}{2}\left(r^2 + r\right) - 2 \ .$$

III.7. The calculus of $H^2(\mathfrak{n},\mathfrak{n})$ when $\mathfrak{n}$ is the Heisenberg algebra

We continue the description of the cohomology of the Heisenberg algebra worked out in Section III.4. Let H_k be the $(2k + 1)$-dimensional Heisenberg algebra. It can be described as the nilradical of a parabolic subalgebra of the simple algebra A_{k+1} when we choose $S_1 = (\alpha_1,\alpha_{k+1})$. We have seen that $\dim H^1(\mathfrak{n},\mathfrak{g}/_{\mathfrak{n}}) = 2r^3 - 6r^2 + 6r - 2$. This permits us to compose the dimension of the space $H^2_{\text{fond}}(\mathfrak{n},\mathfrak{n})$ from the previous description,

$$H^2_{\text{fond}}(\mathfrak{n},\mathfrak{n}) = W_{\sigma_1} \oplus W_{\sigma_2}, \text{ where } \sigma_1 = s_{\alpha_1} \circ s_{\alpha_2} \ , \quad \sigma_2 = s_{\alpha_{k+1}} \circ s_{\alpha_k}$$

as soon as $k \geq 2$ (see Lemma 2, III.5.2 and Theorem 1, III.5.2).

Note that $\dim W_{\sigma_1} = \dim W_{\sigma_2}$.

The dimension of the simple s-module W_{σ_1} whose dominating weight λ is equal to $\lambda = (\delta - \alpha_1) - \alpha_1 - (\alpha_1 + \alpha_2) = \delta - 3\alpha_1 - \alpha_2$ can be comptuted from the Weyl formula

$$\dim V_{\alpha_1} = \prod_{\gamma \in \Delta_0^+} \left(1 + \frac{(\lambda,\gamma)}{(\rho,\gamma)}\right) = \frac{k(k^2-1)}{3}.$$

Then, we have

Theorem 10.

$$\dim H^2(H_k,H_k) = \frac{2k(k^{2?}-1)}{3} + 2(2k+1)^3 - 6(k+1)^2 + 6(k+1) - 2 \quad \text{for} \quad k \geq 3\,.$$

Remark. By a direct calculus, we can give the dimension of the space $H^2(H_1,H_1)$ (we can also find the dimension of these spaces by considering H_1 as the nilradical of a Borel subalgebra of A_2. As for the algebra H_2, the calculus is also made using the Weyl formula.

Theorem 11.

$$\boxed{\dim H^2(H_2,H_2) = 20}$$

$$\boxed{\dim H^2(H_1,H_1) = 5\ .}$$

Proof. In effect we have $\dim Z^2(H_1,H_1) = 0$, $\dim B^2(H_1,H_1) = 3$ and $\dim V_{\alpha_1} = 2$ if $k = 2$.

IV. DERIVATIONS OF SOME INFINITE-DIMENSIONAL TOPOGICALLY NILPOTENT LIE ALGEBRAS

IV.1. Infinite-dimensional standard algebras

We can describe the algebra of derivations of a subalgebra of the infinite-dimensional Lie algebra M composed of infinite complex matrices $A = (a_{ij})$ for $i,j \geq 1$ that contain only a finite number of nonzero elements. First, we expand the notion of a Borel subalgebra of Cartan subalgebras for M.

Let B be the subalgebra of M composed of upper triangular matrices, i.e. the matrices $A = (a_{ij})$ such that $a_{ij} = 0$ for $i > j$. This subalgebra is called a Borel subalgebra of M.

Definition 2. *A Lie subalgebra* L *of* M *is called standard if its normalizer*

$$\mathrm{Nor}(L) = \{X \in M : [X,L] \subset L\}$$

contains the Borel subalgebra B. *The subalgebra* L *is called a standard nilsubalgebra if each matrix of* L *is nilpotent.*

Note that a standard nilsubalgebra L of M is not generally nilpotent in the classical sense, i.e. the descending sequence is stationary. But every element of L is a nilpotent matrix. With an abuse of language, we shall speak of L as a *standard nilalgebra.*

IV.2. A topology in a standard nilalgebra

Let H be the subalgebra of M composed of diagonal matrices. It plays the role of a Cartan subalgebra.

Definition 3. *A root* α *of* M *with respect to* H *is a linear form of* H *satisfying*

$$\alpha(A) = a_{ii} - a_{jj} \qquad (i \neq j)\,.$$

We denote by Δ the set of these roots.

Definition 4. *A root* $\alpha \in \Delta$ *is simple if* $\alpha(A) = a_{ii} - a_{i+1,i+1}$. *We denote this simple root by* α_i.

The set of simple roots is naturally denoted by S. So $S = \{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_r, ...\}$. Each root $\alpha = \alpha_{ij}$, where $\alpha_{ij}(A) = a_{ii} - a_{jj}$, can be decomposed into the sum of simple roots :

$$\alpha_{ij} = \alpha_i + \alpha_{i+1} + ... + \alpha_{j-1} \quad \text{if } i < j \qquad \text{and}$$
$$\alpha_{ij} = -\alpha_j - \alpha_{j+1} ... - \alpha_{i-1} \quad \text{if } i > j.$$

The unit matrix (root vector) associated to a root α is denoted X_α and the unidimensional space generated by X_α is denoted M_α. We define a partial order on the set H^* of the linear form on H by putting $\lambda > 0$ if and only if λ can be expressed as a linear combination of simple roots with positive coefficients. It is clear that each root is equal to a linear combination of simple roots either with the coefficient 1 or with the coefficient -1, so that we obtain a patition of Δ into positive (Δ^+) and negative (Δ^-) roots: $\Delta = \Delta^+ \cup \Delta^-$.

Let R be the subset of Δ^+ that consists of the roots which are pairwise incomparable roots for the previous partial order relation. We put $L = \sum_{\beta \in \Delta'} M_\beta$ where Δ' is the set $\Delta' = \left\{\beta \in \Delta^+ / \exists \gamma \in R \quad \text{as} \quad \beta \geq \gamma\right\}$.

It is clear that L is a standard nilsubalgebra. Note that every standard nilsubalgebra of M can be obtained as this, from some set R of roots pairwise incomparable.

We consider in L, the natural filtration corresponding to the descending central sequence

$$C^0(L) = L \supset C^1(L) \supset C^2(L) \supset ......,$$

where $C^i(L) = \left[L, C^{i-1}(L)\right]$, $\quad i > 0$.

We provide L with a topology, considering the $C^i(L)$ as a basis of the neighborhood of 0. If R is a finite set, the Lie algebra L is nilpotent. Otherwise, L is not nilpotent but for every neighborhood U of 0, there is an integer n such that $C^n L \subset U$. In this case, we will say that L is *topologically nilpotent*.

IV.3. The weighting derivations of L

Let L be the standard nilsubalgebra determined by a subset $R \subset \Delta^+$ of pairwise incomparable roots.

Definition 5. *An operator* $f : L \to L$ *is called a weighting operator with weight* $\lambda \in H^*$ *if* $[h,f] = \lambda(h)f$ *with* $h \in H$ *and where* $[\ ,\ ]$ *designates the bracket in the Lie algebra of endomorphisms of the vector space* L.

For example, if f is a weighting operator satisfying $f(X_\alpha) = aX_\beta$, where $a \neq 0$, then its weight λ is equal to $\beta - \alpha$. Every operator of L can be formally decomposed into a sum (eventually infinite) of weighting operators.

Lemma 1. *Let* d *be a derivation of the algebra* L. *If* $d = \Sigma\, d_i$ *is a decomposition of* d *in the sum of the weighting operators, then each* d_i *is a derivation of* L.

Proof. For all X_α and X_β of L, we have

$$d[X_\alpha,X_\beta] = [d(X_\alpha),X_\beta] + [X_\alpha,d(X_\beta)].$$

The vectors $d[X_\alpha,X_\beta]$ and $[d(X_\alpha),X_\beta] + [X_\alpha,d(X_\beta)]$ can be decomposed in a finite number of root vectors. This means that only a finite number of weighting components of d are not equal to zero. But, two weighting operators, having distinct weights, transform a given root vector into two distinct root vectors (if the images are not zero). This proves the lemma.

This lemma permits us to restrict the study of derivations of L to the study of weighting derivations.

IV.4. Restriction to the finite-dimensional subalgebras

Let M(n) be the subalgebra of M composed of matrices $A = (a_{ij})$ satisfying $a_{ij} = 0$ for $i > n$ and $j > n$. This subalgebra can be identified as the Lie algebra $gl(n,\mathbb{C})$ of matrices of order n. Put $L(n) = L \cap M(n)$. It is a standard subalgebra of $gl(n,\mathbb{C})$ (before we have identified $M(n)$ to $gl(n,\mathbb{C})$) and it is defined by the system $R(n) = \{\alpha_{i,j} \in R / i,j \leq n\}$ (see Chapter 2). It is clear that the derived subalgebra of $L(n)$ is nothing but $\mathfrak{D}(L) \cap L(n)$, and we have

Lemma 1. *If* d *is a derivation of* L *such that* $d(L(n)) \subset L(n)$ *then the restriction of* d *to* L_n *denoted* by $d/L(N)$ *is a derivation of* $L(n)$.

Lemma 2. *Let* d *be a nonzero weighting derivation of* L*. Then there is an integer* N *such that* $d/L(n)$ *is a derivation of* $L(n)$ *for all* $n, n \geq N$.

Proof. From Lemma 1, it is sufficient to find an integer n such that $d(L(n)) \subset L(n)$.

Let X_α be a root vector such that $d(X_\alpha) = aX_\beta$ with $a \neq 0$. We consider the following three possible cases for weight $\gamma = \beta - \alpha$ of the derivation d:

First case. $\gamma = 0$

We choose n so that $X_\alpha \in L(n)$.

Second case. $\gamma \neq 0$ and $\gamma \in \Delta$.

(i) $\alpha = \alpha_{i,j}$ and $\beta = \alpha_{k,j}$, where $i < j$, $k < j$ and $i \neq k$.

We have $\gamma = \beta - \alpha = \alpha_{k,i}$.

Let be $n = \max(i,k)$. For every root α', β' such that $\beta' - \alpha' = \gamma$, we have $X_\alpha' \in L(n)$ and $X_\beta' \in L(n)$ and $d/_{L(n)} \in Der(L(n))$.

(ii) $\alpha = \alpha_{i,j}$, $\beta = \beta_{i,k}$, where $i < j$, $i < k$ and $j \neq k$.

Here we have $\gamma = \beta - \alpha = \alpha_{j,k}$. For n, we choose the maximum between j and k.

Third case. $\gamma \notin \Delta \cup \{0\}$.

The weight γ can be decomposed under the following forms :

$$\pm(\alpha_s + \alpha_{s+1} + \dots + \alpha_t - \alpha_l - \dots - \alpha_r) \text{ with } s \leq t < l \leq r,$$

or $$\pm(\alpha_s + \alpha_{s+1} + \dots + \alpha_t + \alpha_l + \dots + \alpha_r) \text{ with } s \leq t < l-1.$$

In each case, γ can be represented as a difference $\gamma = \beta - \alpha$ of two roots, and this at most as two distinct manners :

$\beta = \pm(\alpha_s + \alpha_{s+1} + ... + \alpha_t)$ and $\alpha = \pm(\alpha_l + ... + \alpha_r)$

or $\beta = \pm(\alpha_s + ... + \alpha_{l-1})$ and $\alpha = \pm(\alpha_{t+1} + ... + \alpha_l)$

in the first described cases and

$\beta = \pm(\alpha_s + ... + \alpha_r)$ and $\alpha = \pm(\alpha_{t+1} + ... + \alpha_{l-1})$

in the other case.

We choose N so as the root vectors X_α and X_β are in L(N).

Lemma 3. *Let* d *be a weighting derivation of* L *which is not of the form* ad x *for any* $x \in M$. *Then there is* N *such that* $d/L(n) \neq \text{ad } y$, $\forall\, y \in M(n)$ *for all* n, n > N.

Proof. Let γ be the weight of d. If γ is neither zero nor a root, it can be represented as a difference of two roots in no more than two different manners (see the proof of the preceding lemma). We choose N in such a way that the root vectors correspond to the roots defined by γ lying in L(N). We deduce the lemma from this.

Suppose that γ is a root and suppose that the lemma is not verified. Then for all N, there is n > N such that $d/L(n) = \text{ad } y$, $y \in M(n)$. As d is weighting, with weight γ, then $d/L(n) = c.\text{ad } X_\gamma$, $c \in \mathbb{C}$. By hypothesis, d is not equal to $c.\text{ad } X_\gamma$. Then there is $y \in L$ such that $d(y) \neq c.\text{ad } X_\gamma(y)$. But these two matrices lie in a space $L(n_0)$ for sufficiently large number n. Even if we choose $N = n_0$, a contradiction is unveiled.

The case where $\gamma = 0$ can be proved analogously. The lemma is proved.

Lemma 4. *Let* d *be a weighting derivation of* L *such that* $d/[L,L] \neq 0$. *Then there exists a positive integer* N *such that* d/L(n) *is a derivation of* L(n) *such that* $d/[L(n),L(n)] \neq 0$ *for all* n, n > N.

Proof. Let $d[x,y] \neq 0$, $x, y \in L$. We choose N as so to the matrices x and y lie in L(N). From Lemma 2, $d/_{L(N)}$ is a derivation of L(N).

Lemma 5. *Let* d *be a weighting derivation of* L *such that* $d(L) \not\subset Z(L)$, *where* Z(L) *is the center of* L. *Then there exists an integer* N *such that* $d(L(n)) \not\subset Z(L(n))$

for all $n, n \geq N$.

Proof. Let $x \in L$ such that $d(x) = y \notin Z(L)$. Let z be in L satisfying $[y,z] \neq 0$. We can find an integer N such that x, y and z lie in $L(N)$. By virtue of Lemma 4, we can regard $d/L(N)$ as a derivation. Moreover $d/L(N)(x) = y \notin Z(L(n))$. The lemma is proved.

IV.5. The structure of the algebra Der(L)

Let $\overline{M}$ be the Lie algebra of matrices $A = (a_{ij})$, $i,j \geq 1$, such that each line and each column contains only a finite number of nonzero elements. Then L is a subalgebra of $\overline{M}$. We note $N_{\overline{M}}(L)$ as the normalizer of L in $\overline{M}$. If $x \in N_{\overline{M}}(L)$, the operator $\operatorname{ad} x$ is an endomorphism of L which belongs to $\operatorname{Der}(L)$. Then the set $D_1 = \left\{\operatorname{ad} x \,/\, x \in N_{\overline{M}}(L)\right\}$ is a subset of $\operatorname{Der}(L)$. We put $D_2 = \left\{d \in \operatorname{End}(L) \,/\, d(L) \subset Z(L) \text{ and } d/_{[L,L]} = 0\right\}$.
Each element of D_2 is a derivation of L. Note that, if R is infinite, L is not nilpotent, $Z(L) = 0$ and $D_2 = \{0\}$.

Theorem 12. *Let* L *be a standard nilsubalgebra of* M *defined by a subsystem* $R \subset \Delta^+$ *of pairwise incomparable roots and let* R *not contain any roots of the form* $\alpha_{1,k}$. *Then* $\operatorname{Der} L = D_1 + D_2$.

Proof. Let d be a nontrivial derivation of L. Suppose that d doesn't belong to $D_1 + D_2$. From Lemmas 2 and 3, there is an integer N such that $d/L(n)$ is a nontrivial derivation of $L(n)$, for all $n, n > N$. The standard nilsubalgebra $L(n)$ of $gl(n,\mathbb{C})$ is defined by a finite system $R(n)$ of roots which doesn't contain any roots of the type $\alpha_{1,k}$. It follows from Lemmas 3, 4, and 5, that the derivation $d/L(n)$ doesn't belong to

$$D_1 \cap \operatorname{Der}(L(n)) + D_2 \cap \operatorname{Der}(L(n))$$

for sufficiently large n. However, in the finite dimension case, there are no such exist derivations (see Section I.5). This gives the theorem.

Suppose now that R is a finite set and contains roots of the form $\alpha_{1,k}$. We order the elements of R as $\alpha_{1,j_1}, \alpha_{i_2,j_2}, \ldots, \alpha_{i_s,j_s}$, where $1 < i_2 < \ldots < i_s$. For every pair of roots $\alpha = \alpha_{i,j}$ and $\beta = \beta_{k,m}$, where $j_1 \leq j \leq k \leq j_r$ and $i_s < m$, such that X_α and X_β lie in L, we define the operator of L, denoted $d_{\alpha,\beta}$, by

$$d_{\alpha\beta}(X_\alpha) = X_\beta \ , \ d_{\alpha\beta}(X_{\alpha_{1,k}}) = X_{\alpha_{j,m}} \text{ and } d_{\alpha\beta}(X_\gamma) = 0 \text{ where } \gamma \neq \alpha \text{ and } \gamma \neq \alpha_{1,k}.$$

Direct verification shows that $d_{\alpha\beta} \in$ Der L. We denote D_3 as the linear subspace spanned by these derivations.

Theorem 13. *Let* L *be a standard nilsubalgebra of* M *defined by a finite system* $R \subset \Delta^+$ *of pairwise incomparable roots. Suppose that* R *contains a root* $\alpha_{1,k}$. *Then*

$$\text{Der } L = D_1 + D_2 + D_3.$$

Proof. Suppose that there is $d \neq 0$, $d \in$ Der L such that $d \notin D_1 + D_2 + D_3$. We can assume that d is a weighting derivation. From Lemma 2, there is an integer N such that $d/L(n) \in \text{Der}(L(n))$ for all n, $n \geq N$. Then the standard nilsubalgebra L(n) of $gl(n,\mathbb{C})$ is determined by the system R(n) and d/L(n) is a derivation of Der L(n) which doesn't belong to $(D_1 + D_2 + D_3) \cap$ Der L(n). From Section I.5, this brings a contraction.

Q E D.

Finally, suppose that R is infinite. This implies the nonnilpotence of L, L is still topologically nilpotent, and its center Z(L) is reduced to {0}.

Lemma 1. *Let* d *be a weighting derivation such that* $d(X_\alpha) = aX_b$ *with* $\beta \neq \alpha$, $\beta - \alpha \notin \Delta$. *Then* $d = 0$.

Proof. We put $\alpha = \alpha_{i,j}$ and $\beta = \alpha_{k,m}$. By hypothesis, $j \neq m$ and $i \neq k$. We consider the root $\gamma = \alpha_{m,s}$ and we choose s to be large enough so that $X_\gamma \in$ L and $s > j$. This choice is possible because R contains an infinity of elements. As the set of inner derivations is an ideal of Der(L), then $[d,adX_\gamma] = d'$ is an inner derivation. We have

$$d'(X_\alpha) = (d \circ adX_\gamma - adX_\gamma \circ d)(X_\alpha) = aX_\tau$$

where $\tau = \gamma + \beta = \alpha_{k,s}$. The difference $\tau - \alpha$ is not a root because $k \neq i$ and $j \neq s$. If $a \neq 0$, we obtain a contradiction, d' being an inner derivation.

Lemma 2. *Let* d *be a weighting derivation such that* $d(X_\alpha) = aX_\beta$ *where* $\gamma = \beta - \alpha \in \Delta$ *and* $X_\gamma \in$ L. *Then* $d = c.\text{ad}\,X_\gamma$.

Proof. Let $\alpha = \alpha_{i,j}$, and $\beta = \alpha_{k,m}$. By assumption of the lemma, we have $m = j$ or $k = i$. We can suppose, without restriction, that $\alpha = \alpha_{i,j}$ and $\beta = \alpha_{k,j}$. So $\gamma = \alpha_{k,i}$. Let X_τ a root vector of L such that $\gamma + \tau \in \Delta$ but $\tau \neq \alpha$. We only need to show that $d(X_\tau) = c.adX_\gamma(X)$. The root τ can be either of the form $\alpha_{i,s}$, or $\alpha_{i,k}$. We will be satisfied to examine only the first case.We have

$$d(X_\tau) = b\, X_{\tau+\gamma},\quad c.adX_\gamma(X_\tau) = c\, X_{\tau+\gamma}.$$

We consider the derivation $d_1 = adX_\theta$, where $\theta = \alpha_{j,s}$ (if $s < j$, then $\theta = \alpha_{s,j}$) and $d' = [d,d_1]$. Then

$$d'(X_\alpha) = d{\circ}d_1(X_\alpha) - d_1{\circ}d(X_\alpha) = d(X_\tau) - d_1(X_\beta) = b\, X_{\tau+v} - c\, X_{\tau+v} = (b-c)X_{\tau+v}\ .$$

The weight of the derivation d' is equal to $\tau + \gamma - \alpha = \alpha_{k,s} - \alpha_{i,j}$, where $k \neq i$ and $s \neq j$ ($\alpha \neq \tau$ and $\beta \neq \alpha$) and is not a root. The weighting derivation d' has a weight which is neither a root nor a zero. From Lemma 1, $d' = 0$ and $c = 0$. This gives the lemma.

Theorem 14. Let L *be a standard subalgebra of* M *defined by an infinite system* R *of pairwise incomparable roots. Then* $\mathrm{Der}\, L = D_1$.

Proof. If R doesn't contain a root of the form $\alpha_{1,k}$, then Theorem 13 follows from Theorem 11 and the fact that $Z(L) = 0$.

Let $\alpha_{1,j_1} \in$ R and let d be a nonzero weighting derivation that doesn't lie in D_1. From Lemma 3 (IV.4), there is an integer N such that $d/L(n) \in \mathrm{Der}(L(n))$ and $d/L(n) \neq adx$ for all $x \in L(n)$ and for all n, $n \geq N$. From the studies of derivations of finite-dimensional standard algebras (§ I.5), the following two cases are possible :

(i) $d/_{[L(n),L(n)]} = 0$ and $d(L(n)) \subset Z(L(n))$,

(ii) $d/_{L(n)}(X_\alpha) = a.X_\beta$; $d/_{L(n)}(X_{\alpha'}) = a.X_{\beta'}$

$d/_{L(n)}(X_\gamma) = 0$ where $\alpha = \alpha_{1,j}$, $\beta = \alpha_{k,m}$,

$\alpha' = \alpha_{1,k}$, $\beta' = \alpha_{j,m}$, $\gamma \neq \alpha$ and $\gamma \neq \alpha'$,

$j_1 \leq j \leq k < j_2$ and $i_s < m$.

In this case, L(n) is supposed as being defined by the system

$$R(n) = \{\alpha_{1,j_1}, \alpha_{i_2,j_2}, \ldots, \alpha_{i_s,j_s}\},\ \text{with}\ 1 < i_2 < \ldots < i_s.$$

We choose an integer n sufficiently large for be sure that $[L(n),L(n)] \cap Z(L(n)) = \{0\}$ and $\alpha \neq \beta$ (it is possible that R is infinite). Then the weight of d is not equal to zero. From Lemma 1, if $\beta - \alpha \notin \Delta$, then $d = 0$, which is inconsistent.

If $\beta - \alpha \in \Delta$, from Lemma 2, then $d = c \operatorname{ad} X_\gamma$ and this is impossible because we have supposed $d \notin D_1$.

V. COHOMOLOGY OF THE INFINITE LIE ALGEBRA OF VECTOR FIELDS OF THE REAL STRAIGHTLINE

Consider the vector space V of infinite dimension provided with a countable basis $\{e_0,e_1,\dots,e_k,\dots\}$. It is possible to make a structure of a Lie algebra, noted L_0, from the bracket

$$[e_i,e_j] = (j-i)e_{i+j} \quad , \quad i, j \geq 0 .$$

We define the subalgebra L_k of L_0 as the subalgebra generated by the vectors $(e_k,e_{k+1},\dots,e_n,\dots)$. These algebra are provided with a natural graduation : the weight of vector e_j being the integer j. From this, we deduce a graduation of the cohomological spaces.

Remarks. (1) The algebras L_i are topologically nilpotent for the topology associated to the filtration generated by this graduation.

(2) A representation of these algebras is given by putting $e_i = x^{i+1} \dfrac{\partial}{\partial x}$.

For this, we consider these algebras as subalgebras of the algebra or vector fields on $\mathbb{R}$.

Proposition 9. *The spaces* $H^i(L_1,L_1)$ *and* $H^{i-1}(L_2,\mathbb{C})$ are *isomorphed as graded vectors spaces.*

We accept this proposition without proof ; for an example, one can read [FU].

As Goncharova has composed all the cohomological spaces of L_1 with values in $\mathbb{C}$ (see also [FU]), one deduces the following proposition.

Proposition 10. $\dim H^2(L_1,L_1) = 3$.

In fact this proposition has been proved directly by A. Fialovski [FI]. More precisely, he gave a precise description of the cocycles of a basis of $H^2(L_1,L_1)$.

Let f_{-2}, f_{-3} and f_{-4} be cocycles of weights -2, -3 and -4 defined by

$$f_{-2}(e_1,e_j) = 0, \quad \text{for all } j,$$
$$f_{-2}(e_2,e_3) = 4e_3,$$
$$f_{-2}(e_i,e_j) = \begin{cases} (-1)^i (j-i+2)\, e_{j+i-2}, & \text{as } i = 2,3 \text{ and } j \geq 4 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{-3}(e_1,e_j) = 0 \quad \forall j$$
$$f_{-3}(e_2,e_3) = 8e_2$$
$$f_{-3}(e_2,e_4) = 4e_3$$
$$f_{-3}(e_3,e_4) = -10e_4$$
$$f_{-3}(e_i,e_j) = \begin{cases} (-1)^i C_2^{i-2} (j-i+3)\, e_{j+i-3}, & \text{as } i = 2,3,4 \text{ and } j \geq 5 \\ 0, & \text{otherwise} \end{cases}$$

$$f_{-4}(e_1,e_j) = 0 \quad \forall j$$
$$f_{-4}(e_2,e_3) = -14e_1 \;;\; f_{-4}(e_2,e_4) = 0 \;;\; f_{-4}(e_2,e_5) = 8e_3$$
$$f_{-4}(e_3,e_4) = -24e_3 \;;\; f_{-4}(e_3,e_5) = -16e_4$$
$$f_{-4}(e_4,e_5) = 18e_5$$

$$f_{-4}(e_i,e_j) = \begin{cases} (-1)^i C_3^{i-2} (j-i+4)\, e_{j+i-4}, & \text{as } i = 2,3,4,5 \text{ and } j \geq 6 \\ 0, & \text{otherwise} \end{cases}$$

Then the classes of these cocycles form a basis of $H^2(L_1,L_1)$.

VI. ON THE COHOMOLOGY OF NILPOTENT LIE ALGEBRAS IN SMALL DIMENSION

By using the MATHEMATICA program given in Chapter 3, we can easely compute, for each nilpotent Lie algebras $\mathfrak{n}$ of dimension less than 7, the dimensions of the spaces $H^i(\mathfrak{n},\mathfrak{n})$ for $i = 0$, 1 and 2 and also the dimension of the spaces $H^i(\mathfrak{n},\mathbb{C})$ for $0 \leq i \leq \dim \mathfrak{n}$. Also this programm permits to obtain more informations as the basis of these spaces.

CHAPTER 5

THE ALGEBRAIC VARIETY OF THE LAWS OF LIE ALGEBRAS

I. THE VECTOR SPACE OF TENSORS $\mathcal{T}^n{}_{2,1}$

I.1. Definition

A tensor of type (2,1) on $\mathbb{C}^n$ is a bilinear mapping

$$T : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}^n$$

with values in $\mathbb{C}^n$.

We denote by $\mathcal{T}^n{}_{2,1}$ the set of alternated tensors. It is provided for a structure of a complex vector space of dimension $(n^3 - n^2)/2$

We fix a basis $\{e_1, ..., e_n\}$ of $\mathbb{C}^n$. The tensor T is defined by its constants of the structure T^k_{ij} given by

$$T(e_i,e_j) = \sum_{k=1}^{n} T^k_{ij}\, e_k .$$

The antisymmetry condition can be expressed by

$$T^k_{ij} = -T^k_{ji}$$

Identification of T with its structure constants determines, once the base $\{e_i\}$ is fixed, an isomorphism between $\mathcal{T}^n_{2,1}$ and $\mathbb{C}^{\frac{n^3-n^2}{2}}$.

I.2. Action of $GL(n,\mathbb{C})$. Classification of the tensors

Definition 1. *Two tensors* T and T' of $\mathcal{T}^n_{2,1}$ *are called isomorphic if it exists* $f \in GL(n,\mathbb{C})$ *such that*

$$T'(x,y) = f^{-1}(T(f(x),f(y))) \ , \ \forall x, y \in \mathbb{C}^n .$$

To simplify, we will note that $T' = f^{-1} \circ T(f, f)$. This formula corresponds to that for a change of basis. So the two tensors T and T' are isomorphic when the two tensors have the same constants of structure in comparison with the two bases of $\mathbb{C}^n$.

We will denote by O(T) (or G(T) if it is necessary to define the group G) the orbit of T under the action of $GL(n,\mathbb{C})$, i.e. is the set of isomorphic tensors of T.

To classify up an isomorphism, the tensors of $\mathbb{C}^n$ consists in determininge the whole orbits. Currently, only a few results are known about the classification. This can be easily explained. If we note $T_1, ..., T_n$ as the components of the tensor T, every T_i is a bilinear alternated form. The description of the tensors of $\mathcal{T}^n_{2,1}$ consists of simultaneously classifying *n* bilinear alternated forms. Except for the case where T_i commutes, we know almost nothing.about this classification.

Proposition 1. *If* $T = (T^k_{ij})$ *is a tensor of* $\mathcal{T}^n_{2,1}$ *and if* $f = (a_{ij})$ *is an isomorphism of* $\mathbb{C}^n$, *then the constants of structure of* $T' = f^{-1} \circ T(f, f)$ verify

$$T'^k_{ij} = \sum_{l,s,r} a_{il} a_{js} T^r_{ls} b_{rk} ,$$

where (b_{ij}) *is the matrix of* f^{-1}.

I.3. Rigid tensors. Open orbits

We provide the vector space $\mathcal{T}^n_{2,1}$ with a euclidean topology.

Definition 2. *A tensor* $T \in \mathcal{T}^n_{2,1}$ *is said to be rigid if its orbit* $\mathcal{O}(T)$ *is open in* $\mathcal{T}^n_{2,1}$.

Example. Let T_0 be the tensor of $\mathcal{T}^2_{2,1}$ given by $T_0(e_1,e_2) = e_1$. We show that T_0 is rigid. Every tensor of T of $\mathcal{T}^2_{2,1}$ is expressed by

$$T(e_1,e_2) = ae_1 + be_2 .$$

If f is the isomorphism with matrix (a_{ij}), then the tensor $T' = f^{-1} \circ T(f, f)$ is written as

$$T'(e_1,e_2) = (aa_{22} - ba_{12})\, e_1 + (ba_{11} - aa_{21})\, e_2 .$$

If $(a,b) \neq (0,0)$, we can always find an isomorphism (a_{ij}) so that we have $T'(e_1,e_2) = T_0(e_1,e_2) = e_1$. Particularly, if T is in a neighborhood of T_0, i.e. is if (a,b) is near(1,0), then $(a,b) \neq (0,0)$ and T is isomorphic to T_0, which shows the rigidity of this tensor.

Consequence. *Every tensor* T *of* $\mathcal{T}^2_{2,1}$ *is either isomorphic to the nul tensor, or isomorphic to the tensor* T_0.

II. THE VARIETY $\mathcal{L}^n$ OF THE LIE ALGEBRAIC LAWS

II.1. Lie algebraic laws on $\mathbb{C}^n$

Definition 3. *A Lie algebraic law* μ *on* $\mathbb{C}^n$ *is a tensor of* $\mathcal{T}^n_{2,1}$ *verifying the Jacobi identity.*

$$\mu(\mu(x,y),z) + \mu(\mu(y,z),x) + \mu(\mu(z,x),y) = 0$$

for every x, y *and* z *in* $\mathbb{C}^n$.

We fix, once for ever, a basis (e_i) of $\mathbb{C}^n$. We will note C^k_{ij} as the structure constants of

μ relative to this basis. Jacobi's identity can be reduced to the polynomial equations system:

$$(S)\ :\ \sum_{l=1}^{n} C^l_{ij} C^s_{lk} + C^l_{jk} C^s_{li} + C^l_{ki} C^s_{lj} = 0\,,\quad 1 \le s \le n\,,\quad 1 \le i < j < k \le n\ .$$

So we can consider a Lie algebraic law μ as a tensor $\mu = (C^k_{ij})$, the C^k_{ij} of which verifies (S).

Remark. Until now, we have only spoken of Lie algebras and a Lie algebra has been defined like a space vector $\mathfrak{g}$, provided with a square brackets [,] verifying the Jacobi identity. If one identifies the vectorial $\mathfrak{g}$ to $\mathbb{C}^n$ and the brackets to a Lie algebraic law, the Lie algebra is nothing but a pair $\mathfrak{g} = (\mu,\mathbb{C}^n)$, where μ is a Lie algebraic law. It is clear that most of the time, we will identify $\mathfrak{g}$ to its law.

II.2. The algebraic variety $\mathcal{L}^n$

One identifies the law μ with its structure constants. The set $\mathcal{L}^n$ of the Lie algebraic laws is then the subset of $\mathbb{C}^{\frac{n^3-n^2}{2}} = \mathcal{T}^n_{2,1}$ defined by the system of polynomial equations (S). So $\mathcal{L}^n$ is provided with a structure of the algebraic variety imbedded in $\mathbb{C}^{\frac{n^3-n^2}{2}}$.

The topology of $\mathcal{L}^n$.

As a subset of $\mathbb{C}^{\frac{n^3-n^2}{2}}$, $\mathcal{L}^n$ can be provided with the classical topology. It is almost natural to provide $\mathcal{L}^n$ with the Zariski topology which is less fine than the preceding topology. We recall that a Zariski closed set is defined by a finite number of polynomial equations. In this chapter, somewhat contrary to the uses, we will consider $\mathcal{L}^n$ with a metric topology. When it will be necessary to restrict oneself to the Zariski topology, we will precise it.

Examples.

1. The variety $\mathcal{L}^2$

Since every tensor $T = (T^1_{12}, T^2_{12})$ verifies (S), we immediately deduce that $\mathcal{L}^2 = \mathcal{T}^2_{2,1}$. Here the algebraic variety $\mathcal{L}^2$ is nothing but a complex plane of dimension 2.

2. The variety $\mathcal{L}^3$

A tensor $\mu = \left(C^k_{ij}\right)$ $1 \le i \le j \le 3$ and $1 \le k \le 3$ verifies the Jacobi conditions, if and only if we have

$$(S_3) \;:\; \begin{aligned} &C^1_{12}\,C^2_{23} + C^1_{13}\,C^3_{23} - C^1_{23}\left(C^3_{13} + C^2_{12}\right) = 0\,, \\ &C^2_{13}\left(C^3_{23} - C^1_{12}\right) + C^1_{13}\,C^2_{12} - C^2_{23}\,C^3_{23} = 0\,, \\ &C^3_{12}\left(C^2_{23} + C^1_{13}\right) - C^1_{12}\,C^3_{13} - C^2_{12}\,C^3_{23} = 0\,. \end{aligned}$$

The variety $\mathcal{L}^3$ is an algebraic variety of $\mathbb{C}^9$ defined by the system S_3. We will see later that this variety is a reunion of two irreducible algebraic components. The tangent geometry of $\mathcal{L}^3$ is more difficult to describe, because of the existence of singular points. For example, the point μ_0 defined by $C^3_{12} = 1$, $C^k_{ij} = 0$ for the other indices i, j, k (μ_0 corresponds to the Heisenberg law H_3) is a singular point.

To interpret better some geometric or algebraic proprieties of the variety $\mathcal{L}^n$, it is convenient to consider $\mathcal{L}^n$ as an affine scheme. In the next paragraph, we will present briefly these notions of algebraic geometry.

II.3. The scheme $\mathcal{L}^n$

1. Spectrum and scheme

Let be R a commutative ring. Its spectrum is the set

$$\mathrm{Spec}(R) = \left\{ I \subset R \text{ where } I \text{ is a prime ideal } I \text{ of } R \text{ different to } R \right\}.$$

(A prime ideal is an ideal which verifies $pq \in I \Rightarrow p \in I$ or $q \in I$.) One provides Spec(R) with a topology whose basis of open sets is given by

$$\mathrm{Spec}(R)_f = \left\{ I \in \mathrm{Spec}(R) \,/\, f \notin I \right\}.$$

On this topological space one constructs a sheaf of functions. Let I be a prime ideal of R and R_I the localized of R in I :

$$R_I = \left\{ \frac{a}{s} / a \in R, s \in R - I \right\} \text{ modulo the equivalence relation :}$$

$$\frac{a}{s} \sim \frac{a'}{s'} \Leftrightarrow \exists\, s'' \in R - I \quad \text{such that} \quad s''(as' - a's) = 0 \ .$$

Let U be an open set of Spec(R). We define

$$\Gamma(U) = \left| s : U \to \prod_{I \in U} R_I \quad \text{such that} \quad s(I) \in R_I \right| .$$

One suppose, moreover, that s is locally a quotient of elements of R :

$\forall I \in U \ , \ \exists V(I) \subset U$ neighborhood in I and a and b in R such that

$\forall J \in V(I), \ b \notin J$ so $s(J) = \frac{a}{b}$ in R_J .

Approximately, it remains to define the regular functions on Spec(R). The fibre $\Gamma(I)$ of the corresponding sheaf above the point I corresponds to R_I.

The affine scheme is the datum of Spec(R) and its sheaf.

2. The affine scheme $\mathcal{L}^n$

We consider the ring $R = \mathbb{C}[X_1, ..., X_N]$ of complex polynomials with $N = (n^3\text{-}n^2)/2$. Let J be the ideal of R generated by the Jacobi polynomials of the system (S) and let $A = R/I$. The scheme $\mathcal{L}^n$ is an affine scheme constructed from Spec A. We will not distinguish, at least in the notations, the scheme $\mathcal{L}^n$ with the variety $\mathcal{L}^n$.

It is interesting to consider $\mathcal{L}^n$ as a scheme which allows us to guess some of the properties of the set of Lie algebra laws which are expressed very clearly in the language of the schemes.

We recall the notion of reduced scheme.

Definition 4. *A scheme* (Spec(R),Γ) *is reduced if the local ring* R(I) *has no nilpotent ideal for all* I *in* Spec(R).

This means that doesn't exist non trivial polynomial f such that $(f)^p$ is in the ideal of the Jacobi .

3. The tangent space to the scheme $\mathcal{L}^n$

Let $\mu_0 = (a^k_{ij})$ be a point of the space $\mathcal{L}^n$. The tangent space in μ_0 to the scheme $\mathcal{L}^n$ (we can identify μ_0 with a point of the scheme) is defined by the linear equations which correspond to the homogeneous part with degree 1 of the Jacobi equations centred on the point μ_0.

Examples.

(1) If $\mu_0 = (0)$ is the Abelian law, so the tangent space is $\mathbb{C}^N$, where $N=(n^3-n^2)/2$ because there is no component of degree 1 in the Jacobi equations.

(2) $\mu_0 = (a^3_{12} = 1\ ,\ a^k_{ij} = 0$ otherwise$) \in \mathcal{L}^3$. The equations translated on this point are written

$$C^1_{12}\, C^2_{23} + C^1_{13}\, C^3_{23} - C^1_{23}\left(C^3_{13} + C^2_{12}\right) = 0\,,$$
$$C^2_{13}\left(C^3_{23} - C^1_{12}\right) + C^1_{13}\, C^2_{12} - C^2_{23}\, C^3_{13} = 0\,,$$
$$-\left(C^3_{23} + C^1_{13}\right) + C^2_{12}\left(C^2_{23} + C^1_{13}\right) + \left(C^1_{12} - C^3_{23}\right) C^2_{12} = 0\,,$$

and the equations of the plan tangent to the scheme $\mathcal{L}^3$ on μ_0 are

$$-\left(C^2_{23} + C^1_{13}\right) = 0 .$$

It is an 8-plane.

Theorem 1. *The tangent plane at the point μ_0 to the scheme L^n coincides with the space $Z^2(\mu_0,\mu_0)$ of the 2-cocycles for the cohomology of μ_0 with values in the adjoint module.*

In the notation $Z^2(\mu_0,\mu_0)$ we identify, to agree with the preceding chapters, the law μ_0 with the Lie algebra $\mathfrak{g}_0 = (\mu_0,\mathbb{C}^n)$.

Proof. Let $\varphi \in Z^2(\mu_0,\mu_0)$. Then

$$\delta\varphi(e_i,e_j,e_k) = 0 , \quad \forall\, i , j , k .$$

We put $\varphi(e_i,e_j) = \Sigma\, \alpha^k_{ij}\, e_k$. The identity hereunder is equivalent to the system

$$\left\{\Sigma\, C^k_{ij}\, a^s_{kk} + C^s_{lk}\, a^l_{ij} + C^l_{jk}\, a^s_{li} + C^s_{li}\, a^l_{jk} + C^l_{ki}\, a^s_{lj} + C^s_{lj}\, a^l_{ki} = 0\right\} ,$$

which is nothing else than the linearization of (S) translated on the point $\mu_0 = (C^k_{ij})$.

II.4. The action of $GL(n,\mathbb{C})$. Fibration by orbits

The action of $GL(n,\mathbb{C})$ on $\mathcal{T}^n_{2,1}$ restricted to L^n induces an action on the variety of the Lie algebra laws : two laws μ_1 and μ_2 are isomorphic, if it exists, $f \in GL(n,\mathbb{C})$, such that $\mu_2 = f^{-1} \circ \mu(f,f)$.

We denote by $O(\mu)$ the orbit of μ corresponding to this action.

Lemma 1. *For every $\mu \in L^n$ (or in $\mathcal{T}^n_{2,1}$) the orbit $O(\mu)$ is provided with the structure of a differentiable manifold.*

Effectively, $O(\mu)$ is the image through the action of the Lie group $GL(n,\mathbb{C})$ of the point

μ (considered here as a point of the vector space $\mathcal{T}^n_{2,1}$). The isotropy subgroup of μ, i.e. the subgroup of GL(n,$\mathbb{C}$) defined by

$$\mathrm{Iso}(\mu) = \{f \in GL(n,\mathbb{C}) \ / \ \mu = f^{-1} \circ \mu(f,f)\},$$

coincides with the group of automorphisms of μ. Then it is a closed subgroup and the orbit is isomorphic with the homogeneous space GL(n,$\mathbb{C}$) / Iso(μ). This proves the lemma.

Theorem 2. *Let* μ be in L^n. *Then the tangent space at the point* μ *on the orbit* $O(\mu)$ *coincides with the space of coboundaries* $B^2(\mu,\mu)$.

Recall that the elements of $B^2(\mu,\mu)$ are coboundaries of the form $\varphi = \delta_\mu f$, where f is in gl(n,$\mathbb{C}$) with

$$\delta_\mu f(X, Y,) = \mu(f(X), Y) + \mu(X, f(Y)) - f(\mu(X, Y)).$$

Proof. Let μ_1 be a point close to μ in $O(\mu)$ (for the topology of the manifold $O(\mu)$ which coincides with the topology induced by $\mathcal{T}^n_{2,1}$) in $O(\mu)$. Then $\mu_1 = f^{-1} \circ \mu(f,f)$ with f near to the identity in gl(n,$\mathbb{C}$). We put $f = Id + g$, where g is a linear mapping given by a matrix which belongs to the neighborhood of 0 in gl(n,$\mathbb{C}$). Knowing that, one can write

$$f^{-1} = Id + h \quad \text{with} \quad h \in V(0) \subset gl(n,\mathbb{C}),$$

we have

$$\mu_1(x,y) = \mu(x,y) + \mu(g(x),y) + \mu(x,g(y)) + \mu(g(x),g(y)) + h\mu(x,y) + \\ + h\mu(g(x),y) + h(x,g(y)) + h\mu(g(x),y) \ .$$

As $(Id + h) \circ (Id + g) = Id + h + g + h \circ g = Id$, one deduces $h = -g - h \circ g$, so

$$\mu_1(x,y) - \mu(x,y) = \delta_\mu g(x,y) + \varepsilon,$$

where ε is an "infinitesimal" of order 2.

This shows that the tangent vectors in μ to $O(\mu)$ have the form $\delta_\mu g$. Hence this proves the theorem.

II.5. Open Orbits. Laws of rigid Lie algebras

The algebraic variety L^n can be provided with two structures of topological spaces by considering either the metric topology or the Zariski topology which, recall, is less fine than the first. We can also consider the orbits, either like differentiable manifolds provided with the metric topology or like topological spaces provided with the Zariski topology.

Proposition 2. *The two topologic spaces $O(\mu)$ (metric topology) and $O(\mu)$ (Zariski topology) are homeomorphic.*

This results from the fact that $O(\mu)$ is differentiable linear (see, for example, Mumford "the Red Book").

Definition 5. *A Lie algebra law μ of L^n is said to be rigid if its orbit is a Zariski open set of L^n.*

According to the preceding proposition, it means that $O(\mu)$ is open for the metric topology.

Proposition 3. *Only a finited number of open orbits (or of isomorphic classes of rigid laws) exist in L^n.*

Effectively, all algebraic varieties are decomposed to a finite number of irreducible algebraic components. Now, if $O(\mu)$ is a Zariski open set, its closure, for the Zariski topology, is an algebraic component of L^n. From which we obtain the proposition.

Examples.

1. Only one open orbit exists in L^2, which is defined in μ by

$$\mu(e_1,e_2) = 0$$

(Recall the implicit convention : the undefined brackets are *nul*).
In this case, L^2 is irreducible and coincides with $\overline{O(\mu)}$.

2. We consider in $\mathcal{L}^3$ the law defined by

$$\mu(e_1,e_2) = e_3 ,$$
$$\mu(e_3,e_1) = e_2 ,$$
$$\mu(e_2,e_3) = e_1 .$$

It is the simple Lie algebra law sl(2,$\mathbb{C}$). We will see later that all simple laws are rigid. So $\overline{O(\mu)}$ is a component of $\mathcal{L}^3$. The following law

$$\mu_1(e_1,e_2) = e_2 ,$$
$$\mu_1(e_1,e_3) = e_3 ,$$

is not in $\overline{O(\mu)}$. Hence, $\mathcal{L}^3$ is not irreducible (see §. V).

II.6. The dimension of orbits

Let μ_0 be in $\mathcal{L}^n$. The orbit $O(\mu_0)$ identifies itself with $GL(n,\mathbb{C})/Aut(\mu_0)$, where $Aut(\mu_0)$ is the group of isomorphisms of μ_0. We have

$$\dim O(\mu_0) = n^2 - \dim Aut(\mu_0) .$$

We note that the Lie algebra of the Lie group $Aut(\mu_0)$ is nothing more than the algebra of derivations $Der(\mu_0)$.

Proposition 4. *For all every $\mu_0 \in \mathcal{L}^n$, we have*

$$\dim O(\mu_0) = n^2 - \dim Der(\mu_0) .$$

Examples.

(1) If μ_0 is Abelian, $Der(\mu_0) = gl(n,\mathbb{C})$. Then

$$\dim O(\mu_0) = 0 .$$

(2) If μ_0 is the law of the Heisenberg algebra H_1 of dimension 3, then

$$\dim Der(\mu_0) = 6 ,$$
$$\dim O(\mu_0) = 3 .$$

(3) If μ_0 is the law of simple algebra $sl(2,\mathbb{C})$, then $\dim Der(\mu_0) = 3$ and $\dim O(\mu_0) = 6$.

II.7. The components of $\mathcal{L}^n$

We can see very easily that, after dimension 3, the variety $\mathcal{L}^n$ is not yet irreducible. The problem of determining the components is also naturally presented.
Unfortunately, this problem is a little difficult to tackle. We will see at the end of this chapter that this determination is only explicit for dimensions less than 7. The problem is displaced by an estimation of this number of components. But here also, we need refined techniques, and the approaches are difficult. Nevertheless, the next proposition allows us to envisage an estimation of the number of components.

Proposition 5. *Let* μ_0 *be a rigid law in* $\mathcal{L}^n$. *Then the closure in the Zariski sense* $\overline{O(\mu_0)}$ *of the open orbit* $O(\mu_0)$ *is an irreducible component of* $\mathcal{L}^n$.

But there exist component given by nonrigid laws. We can consider some "rigid" families parametrized by one, two or several parameters. In theses caces, the orbits of theses families give components. The following proposition precise such orbits.

Proposition 6. *Let* Λ *be an irreducible algebraic set in* $\mathbb{C}^k$ *and we consider a subset* $\{\mu_\alpha\}_{\alpha\in\Lambda}$ *of* $\mathcal{L}^n$ *formed by the laws of two pairwise nonisomorphics. If this family is rigid (i.e. the union* Γ *of the orbits of the laws* μ_α *is an open set), then the closure of* Γ *is an irreducible component of* $\mathcal{L}^n$.

III. CONTRACTIONS. DEFORMATIONS

III.1. Contractions in $\mathcal{L}^n$.

Let μ_0 be a point of $\mathcal{L}^n$ and $(f_\rho)_{\rho\in\mathbb{N}}$ a sequence of isomorphisms of $\mathbb{C}^n$ (with, to simplify the notations, $f_0 = \mathrm{Id}$). For all $\rho \in \mathbb{N}$, the law $\mu_\rho = f^{-1}_\rho \circ \mu_0\ (f_\rho \times f_\rho)$ is in the orbit of μ_0. But the limit $\left(\lim\limits_{\rho\to\infty} \mu_\rho\right)$ perhaps does not exist in $\mathcal{T}^n_{2,1}$. If this limit does exist, it is inside $\mathcal{L}^n$ and is called *a contraction of* μ_0. We will note it, for example as μ_∞. So

$$\mu_\infty = \lim_{\rho\to\infty} \left(f_\rho^{-1} \circ \mu_\rho(f_\rho,f_\rho)\right) .$$

Proposition 7. *Let μ_∞ be a contraction of μ_0, then μ_∞ is in the closure of the orbit of μ_0.*

This proceeds directly from the definition of μ_∞.

Corollary. *Let $\mathcal{C}$ be an irreducible component containing μ_0. If μ_∞ is a contraction of μ_0, then $\mu_\infty \in \mathcal{C}$.*

In fact, if $\mu_0 \in \mathcal{C}$, $\mathcal{O}(\mu_0) \subset \mathcal{C}$ and $\overline{\mathcal{O}(\mu_0)} \in \mathcal{C}$.

Consequence. If μ_∞ is a contraction of μ_0, then

$$\mathcal{O}(\mu_\infty) \subset \overline{\mathcal{O}(\mu_0)} .$$

Remark. The notion of contraction (sometimes called degeneration) has been introduced by theorician physicians, more precisely by Segal, for the purpose of understanding the relationship between relativistic and classical mechanics. Effectively, if we consider the Lie algebra of a Lorentz group as an algebra parametriezed by the speed of light, this contracts itself by stretching the speed to infinity, to the Galileo' algebra. This notion of contraction was developed sometimes later by Inonu and Wigner with the particularization : in their works, they suppose that one subalgebra remains fixed throughout the contraction. As this notion is a little bit restrictive, it has been generalized by Saletan, Levi-Nahas, and some others. If we carry to extremes the generalizations of Inonu and Wigner, we will probably discover the original notion of contractions !

III.2. The deformations

The notion of deformation is supposed to be the "dual" of the notion of contraction. It was also introduced to dispose of a topological tool. Unfortunately, the definition of a deformation, as was initially introduced by Gerstenhaber and then developed by Nijenhuis and Richardson, lays in a formal concept. However, as its use has shown, iteself to be most effective, we will conserve it and try, in the next chapters, to study some proprieties in the case of nilpotent laws. But the problem of the contraction duality-deformation stays entire. We will tackle it and perhaps produce a convincing solution about it, by introducing a notion which is closer to the reality than that of deformation. This we will call *perturbations.* But returning to the deformations, we have the following definition.

Definition 6. *A deformation of a law* μ_0 *of* L^n *is a formal sequence with a parameter* t $\mu_t = \mu_0 + \sum_{1\le i\le\infty} t^i \varphi_i$, *where the* φ_i *are in* $T^n_{2,1}$ *and verifying the formal identity of Jacobi.*

$$\mu_t \circ \mu_t = \sum_{1\le i+j\le\infty} t^{i+j} \left[(\varphi_i\circ\varphi_j + \varphi_j\circ\varphi_i) + \varphi_{\mu_0}\,\varphi_{i+j}\right] + \sum_{1\le i\le\infty} t^{2i}\, \varphi_i\circ\varphi_i = 0 \ .$$

We define the notations which intervene in this definition.

If φ and ψ are in $T^n_{2,1}$, we note $\varphi \circ \psi$ the trilinear application alternated with values in $\mathbb{C}^n$ defined by

$$\varphi \circ \psi (X,Y,Z) = \varphi(\psi(X,Y),Z) + \varphi(\psi(Y,Z),X) + \varphi(\psi(Z,X),Y) +$$
$$+ \psi(\varphi(X,Y),Z) + \psi(\varphi(Y,Z),X) + \psi(\varphi(Z,X),Y) \ .$$

In particular, we have

$$\delta_{\mu_0}\varphi = \mu_0 \circ \varphi = \varphi \circ \mu_0 \text{ as soon as } \mu_0 \in \mathcal{L}^n \text{ and } \varphi \in \mathcal{T}^n_{2,1} \text{ and } \mu \circ \mu = 0$$

is nothing more than the Jacobi condition.

Example. A linear deformation is given by

$$\mu_t = \mu_0 + t\varphi_1$$

(i.e. $\varphi_i = 0$ if $i \geq 2$).
In this case, the formal Jacobi condition is written as

$$t\delta_{\mu_0}\varphi_1 + t^2 \varphi_1 \circ \varphi_1 = 0 ,$$

which implies

$$\delta_{\mu_0}\varphi_1 = 0 \ , \quad \varphi_1 \circ \varphi_1 = 0$$

Proposition 8. *A 2-cocycle φ_1 for a given law μ_0 defines a linear deformation if and only if φ_1 is a law of $\mathcal{L}^n$.*

Proposition 9. *Let $\mu_t = \mu_0 + \Sigma\, t^i \varphi_i$ be a deformation of μ_0. Then the first term φ_1 is a 2-cocycle for μ_0, i.e. $\delta_{\mu_0} \varphi_1 = 0$*

Effectively, $\delta_{\mu_0}\varphi_1$ corresponds to the coefficient of the linear part of the formal Jacobi identity.

Remark. We can always consider a deformation μ_t of μ_0 as a Lie algebra on the field $\mathbb{C}((t))$ of the formal series in t. This,moreover, permits us to generalize this notion in considering not $\mathbb{C}((t))$ but the ring of formal series with some variables as the space of the scalars. So one is led to the very general notion of deformations parametrized by $\mathbb{C}$-complete local algebras A, so $A/m_A{}^n$ is of finite dimension, where m_A is the maximal ideal of A.

This generalization allows us to foresee a tie with the notion of contraction. But we prefer to present here a more direct aspect and most topologic than the perturbation for tackling this problem.

Proposition 10. *So* μ_t *is a deformation of* μ_0. *Then there exists a neighborhood* V *of* 0 *in* $\mathbb{C}$ *such that* $\forall u \in V$, *the series*

$$\mu_u = \mu_0 + \Sigma u^i \varphi_i$$

converges in $\mathcal{T}^n{}_{2,1}$.

This results from the Artin theorem relative to the polynomial developments with coefficients in integer series.

This proposition justifies, at least to a certain extent, our desire to conserve as a definition of deformation, the original notion. We have seen that the coefficient of the linear part of a deformation of μ_0 belongs to the space $Z^2(\mu_0,\mu_0)$. Now this space coincides with the tangent plan in μ_0 to the scheme $\mathcal{L}^n$. The formal Jacobi conditions relative to μ_t are equivalent to the infinite system :

$$\text{(N.R)} \qquad \begin{aligned} &\delta_{\mu_0}\varphi_1 = 0 , \\ &\sum_{i+j=2k+1} \varphi_i \circ \varphi_j + \varphi_j \circ \varphi_i + \delta_{\mu_0}\varphi_{2k+1} = 0 , \\ &\sum_{i+j=2k} \varphi_i \circ \varphi_j + \varphi_j \circ \varphi_i + \varphi_k \circ \varphi_k + \delta_{\mu_0}\varphi_{2k} = 0 . \end{aligned}$$

The system (N.R) can be interpreted as the necessary and sufficient conditions for a vector φ_1 of the tangent plan of Zariski in μ_0 to the scheme $\mathcal{L}^n$ being the first term of a deformation.

If this system admits a solution, we will say that the *linear development* $\mu_0 + t\,\varphi_1$ is *integrable*. We also will say that φ_1 is *integrable.*

We suppose that there exists a neighborhood U of μ_0 in L^n (for the metric topology for example) such that every point μ of U is the sum of a convergent deformation. In this case, the system (N.R) describes the sufficient and necessary conditions for a vector tangent to the scheme in μ_0 to be a tangent vector in μ_0 to the variety L^n (this means a vector of the cone of the tangents).

Theorem 3. *Let* $\mu_0 \in L^n$ *such that there exists an open* U *containing* μ_0 *whose elements are sums of convergent deformations of* μ_0. *Then a vector tangent* φ_1 *to the scheme* L^n *in* μ_0 *is tangent to the variety* L^n *if and only if there is a sequence of tensors* $(\varphi_2, ..., \varphi_n, ...)$ *of* $T^n{}_{2,1}$ *as* (N.R) *is satisfied.*

Remark. We will see, by studing the perturbations of μ_0, that the hypothesis relative to the existence of this is always satisfied and that the system (N.R) is equivalent to a system which contains a finite number of equations.

Consequence. *Let* T_2 *be the smallest vector subspace containing the cone of the tangents to the variety* L^n *in* μ_0. *If* T_2 *is strictly contained in* $Z^2(\mu_0,\mu_0)$ *, then the scheme* L^n *is not reduced in* μ_0.

III.3. Equivalent deformations

Let f(t) be a $\mathbb{C}[[t]]$ automorphism of a set of formal series (which is a $\mathbb{C}[[t]]$-module).

This automorphism is written

$$f_t = Id + tf_1 + t^2f_2 + ... + t^nf_n + ... ,$$

where every f_i is an endomorphism of $\mathbb{C}^n$.
The deformation μ'_t defined by

$$\mu'_t(X,Y) = f_t^{-1}(\mu_t(f_t(X),f_t(Y))) ,$$

where f_t^{-1} is the formal inverse of f_t, is called a deformation equivalent to the deformation μ_t of the law μ_0.

A deformation equivalent to the null deformation $\mu_t = \mu_0$ of μ_0 is usually called a trivial deformation.

IV. PERTURBATIONS. NOTIONS OF INFINITESIMAL ALGEBRA

To give a precise sense to the object which represents a law close to a given law, and to turn the formal obstacle of the deformations, we will introduce the notion of perturbation, which is nothing more than a law infinitely close to a given law (then it is a concept marvellously adapted to the metric topology of L^n). This notion is based on the notion of infinitesimals. We will briefly recall the few basic notions of the theory of the complex infinitesimal.

IV.1. The infinitesimals in $\mathbb{C}^n$

1) *A short survey of the Nelson axioms*

The infinitesimals are introduced in an axiomatic way into the classical axiom of the set theory. First, we will describe this by summing up the consequences of this theory on complex space vectors.

In the field of complex numbers $\mathbb{C}$, there exist both standard and nonstandard elements. Every scalar constructed by formulas ofclassical mathematics is standard ; for example 1, 10, 10^{-10}, i are standards (usually every set classically defined is standard ; and also the field $\mathbb{C}$ is standard).

Axiom I : (idealization). *There exists an element of $\mathbb{C}$ which is nonstandard.*

If a nonstandard element exists, there is an infinite number of them, otherwise these nonstandard elements would be standard and Axiom I would be at fault. On the other hand, there does not exist any subset of $\mathbb{C}$ constituted by all the nonstandard elements of $\mathbb{C}$, because these elements are not, by nature, constructed by the formulas of the classical set theory.

Definition 7. *An element* ε *of* $\mathbb{C}$ *is said to be infinitesimal if it verifies* $|\varepsilon| \leq |z|$ *for all standard* $z \in \mathbb{C}$. *We will write* $\varepsilon \cong 0$.
An element ω *in* $\mathbb{C}$ *is infinitely large if* $\omega^{-1} \cong 0$.
An element a is limited if it is not infinitely large, although if it is not infinitesimal, one will say that it is appreciable.

Axiom S (standardization). *Every limited element* a *of* $\mathbb{C}$ *is infinitely close to a unique standard element noted by* $^{\circ}a$ *and called the shadow of* a.

Saying that the two elements a and b of $\mathbb{C}$ are infinitely close, means that their difference is infinitesimal ; we will write $a \cong b$.

Axiom T (transference). *About every theorem of classical mathematic, one can deduce a nonstandard theorem supposing that every variable is standard and vice versa.*

Example. Let (T) be the theorem : "every complex polynomial admits a root".

Of course, this affirmation keeps a theorem with in the nonstandard frame, because this extension is conservative : every theorem keeps a theorem in the extension and every theorem of the extension expressible in the language of the initial theory, is a theorem of this. If we suppose every variable to be standard, we will write : "every complex standard polynomial admits a standard root".

The axiom of transfer permits us to affirm that this sentence is also a theorem (nonexpressible within the classical theory of the sets) ; without this axiom, we only would write "every complex standard polynomial admits a root", without we can pretend that the root is standard.

We will currently use the transfer in "the opposite sense". For example, we suppose that we want to prove (T) ; we can prove within the (nonstandard) frame, that every standard polynomial admits a standard root. With the transfer principle, we deduce that (T) is true. The rewards of this approach reflect perfectly the quintescence of the nonstandard method : to demonstrate the transference of (T), we can use all the tools that are produced by nonstandard analysis, which are the classical tools (within the nonstandard framework we can directly demonstration (T)), to which are added the specific Nonstandard tools. As we are richer the proof of (T) transfered can only be more simple.

2) *The algebra of the infinitesimals*

We easily prove the natural rules of calculus concerning the infinitesimal and infinitely large elements :

$$\Phi + \Phi = \Phi ,$$
$$\Phi \times \Phi = \Phi ,$$
$$\mathcal{L} \times \Phi = \Phi ,$$
$$\mathcal{L} + \mathcal{L} = \mathcal{L} ,$$
$$\omega + \Phi = \omega ,$$
$$\omega \times \omega = \omega ,$$

where Φ designates every infinitesimal, $\mathcal{L}$ designates every limited and ω an infinitely large. On the other hand, one can not pretend about the result of the operations of the type :

$$\Phi / \Phi ,$$
$$\omega + \omega ,$$
$$\Phi \times \omega ,$$

which correspond to the famous undetermined forms of elementary analysis.

3) *Infinitesimal vectorial algebra*

We consider a standard integer n. Then the vector space $\mathbb{C}^n$ is standard. According to the transfer axiom, there exists a standard basis ; all the vectorial calculus in $\mathbb{C}^n$ will be established with respect to this standard basis. So a standard vector will have standard components with respect to this basis.

Definition 8. *A vector* V *of* $\mathbb{C}^n$ *is called infinitesimal if its components* $(v_1, v_2, ..., v_n)$ *are infinitesimal. It is said to be limited if the components are limited, and infinitely large if one of the components is infinitely large.*

We consider a standard norm in $\mathbb{C}^n$ (for example, that associated to the usual Hermitian product). If V is infinitesimal (resp. limited, resp. infinitely large), its norm is infinitesimal (resp. limited, resp. infinitely large).

Shadow of a limited vector.

We suppose V is limited. From the axiom S, each of its components v_1 admits a shadow $^0v_1$ and the standard vector $(^0 v_1, ..., ^0 v_n)$ is called the shadow of V ; it is noted ^oV.

Usually, the vectors V and ^oV are linearly independent. The aim of the next section is to determine the relations between these two vectors.

IV.2. Theorem of decomposition of a limited vector of $\mathbb{C}^n$

This theorem is the basis of all calculus relative to the perturbations of a point in a vectorial plane.

Theorem 4. *Let* V *be a limited vector of* $\mathbb{C}^n$ *and* ^oV *its shadow. We suppose these two vectors to be independent. We have the following decomposition :*

$$V = {}^{o}V + \varepsilon_1 V_1 + \varepsilon_1 \varepsilon_2 V_2 + ... + \varepsilon_1 \varepsilon_2 ... \varepsilon_k V_k$$

with (i) $\varepsilon_1, \varepsilon_2, ..., \varepsilon_k$ *infinitesimal in* $\mathbb{C}$,

(ii) $V_1, V_2, ..., V_k$ *standard and linearly independent in* $\mathbb{C}^n$.

This decomposition is unique up to an equivalence :

If

$$V = {}^{o}V + \varepsilon_1 V_1 + \varepsilon_1\varepsilon_2 V_2 + ... + \varepsilon_1\varepsilon_2 ... \varepsilon_k V_k =$$
$$= {}^{o}V + \alpha_1 W_1 + \alpha_1\alpha_2 W_2 + ... + \alpha_1\alpha_2 ... \alpha_s W_s$$

are two decompositions of V, *then we have* :

(i) $k = s$,

(ii) $W_i = \sum_{j=1}^{i} a^j_i V_j$ *with* a^j_i *standard and* $a^i_i \neq 0$,

(iii) $\varepsilon_1\varepsilon_2 ... \varepsilon_i = \sum_{j\geq 1} a_j^i \alpha_1\alpha_2 ... \alpha_i\alpha_{i+1} ... \alpha_j$.

Comments

1) The integer k determined by a decomposition of V and which does not depend on the choice of the decomposition, is called the **length** of V. If V is standard, it has as its length 0. Further, the length is always one standard smaller or equal to n, which is the dimension of the vectorial frame.

2) The vector V defines in a single way the flag $(V_1,(V_1,V_2), ..., (V_1,V_2, ..., V_k))$. This proceeds directly from relations (ii). The most esthetic illustration of this geometry linked to a limited vector is based on the Euclidean plan. If we take a limited point M situated on a standard smooth curve, the flag linked to this point is nothing other than the flag given by the tangent vector, the osculator plane, the second osculator plane, etc.

3) *Geometric interpretation of the decomposition*

The flag $(V_1,(V_1,V_2), ..., (V_1,V_2, ..., V_k))$ which is canonically linked to the vector V, is defined as follows : we consider a (nonstandard vectorial) straight line with a director vector $V-{}^oV$. It is infinitely close to a unique standard straight line whose director vector is V_1. We consider now a vector plane containing these straight lines. Let us note that if these two straight lines are the same, the length of the decomposition is 1. Otherwise, the plan is well determined. It is standard or not. If it is (standard), the length of the decomposition is 2 and (V_1, V_2) is a standard basis of this plane. If it is nonstandard, it is infinitely close to a unique standard plane (its shadow) and (V_1, V_2) is a basis of this standard plane, etc.

This permits us to interpret geometrically the integer k linked to V.

Proposition 11. *The length of* V *coincides with the dimension of the smallest standard vectorial subspace containing the infinitesimal vector* $V-{}^oV$.

Proof of the theorem.

1) *Existence of the decomposition*

We can suppose V to be infinitesimal, otherwise we replace it by $V-{}^oV$. Its components $(v_1, v_2, ..., v_n)$ are all infinitesimal. Let ε_1 be the largest module of the components v_1. The quantities v_1/ε_1 are all limited and we can write :

$$v_1 = a_1\varepsilon_1 + \varepsilon_1 w_1$$

with a_1 standard (eventually null) and $w_1 \equiv 0$. Let us note V_1 as the standard vector $(a_1, a_2, ..., a_n)$ and W_1 the infinitesimal vector $(w_1, w_2, ..., w_n)$. By construction, the vector V_1 is not zero, the component corresponding to ε_1 is equal to 1. If W_1 is zero, decomposition is achieved, it has a length equal to 1. Otherwise we repeat the above mentioned procedure to the vector W_1 whose component corresponding to ε_1 is zero, that reduces by 1 the dimension of the space in which we operate. By construction, the vectors V_1 and W_1 are lineary independents. As the vectorial frame is of finite dimension, the proceedings stop before n steps (at least).

2) *Equivalence of two decompositions*

Let us consider two decompositions of V :

$$\varepsilon_1 V_1 + \varepsilon_1\varepsilon_2 V_2 + \dots + \varepsilon_1\varepsilon_2 \dots \varepsilon_k V_k = \alpha_1 W_1 + \alpha_1\alpha_2 W_2 + \dots + \alpha_1\alpha_2 \dots \alpha_s W_s \quad (1)$$

and let us suppose $k \geq 1$. The vector V_1 is nonzero.

Lemma. *The infinitesimals ε_1 and α_1 are equivalent, which means α_1 / ε_1 is appreciable (this implies that α_1 is nonnull).*

Proof of the lemma. If α_1 / ε_1 is not appreciable, it is infinitely large or infinitesimal. In the first case, we divide the identity (1) by α_1 and then take the shadow of the two members. We get $W_1 = 0$, which is the contrary to the hypothesis made on the vectors of the decomposition. In the second case, it is the vector V_1 which is zero. So the lemma is proved.

As ε_1 and α_1 are equivalent, we can write

$$\alpha_1 = \varepsilon_1(a_1^1 + \alpha_1^1)$$

with a_1^1 standard nonzero and $\alpha_1^1 \cong 0$.
We get

$$\varepsilon_1 V_1 + \varepsilon_1\varepsilon_2 V_2 + \dots + \varepsilon_1\varepsilon_2 \dots \varepsilon_k V_k = \varepsilon_1 \left(a_1^1 + \alpha_1^1\right) (W_1 + \alpha_2 W_2 + \dots + \alpha_2 \dots \alpha_s W_s)$$

and, after division by ε_1,

$$V_1 + \varepsilon_2 V_2 + \dots + \varepsilon_2 \dots \varepsilon_k V_k = \left(a_1^1 + \alpha_1^1\right) (W_1 + \alpha_2 W_2 + \dots + \alpha_2 \dots \alpha_s W_s) \ . \quad (2)$$

So, the two vectors are equal and limited and their shadows are equal, this gives :

$$V_1 = a_1^1 + W_1 \quad \text{and} \quad a_1^1 \text{ nonnull standard.}$$

Relation (2) is reduced to

$$\varepsilon_2 V_2 + \dots + \varepsilon_2 \dots \varepsilon_k V_k = \alpha_1^1 W_1 + \left(a_1^1 + \alpha_1^1\right) (\alpha_2 W_2 + \dots + \alpha_2 \dots \alpha_s W_s) \ . \ (3)$$

If $\varepsilon_2 = 0$, we recover a trivial linear relation of the independant vectors W_1 , W_2 ,..., W_n. This gives $\alpha^1{}_1 = 0$ and $\alpha_2 = 0$. In this case, the equivalence is proved.
Let us suppose $\varepsilon_2 \neq 0$. We pose $\alpha^1{}_2 = (a^1{}_1 + \alpha^1{}_1)\alpha_2$. Thi infinitesimal equivalent to α_2. The identity is written :

$$\varepsilon_2 V_2 + \dots + \varepsilon_2 \dots \varepsilon_k V_k = \alpha_1^1 W_1 + \alpha_2^1 W_2 + \dots + \alpha_2^1 \alpha_3 \dots \alpha_s W_s \ . (4)$$

We are led to comparing the infinitesimals ε_2, $\alpha^1{}_1$ and $\alpha^2{}_1$. The different situations are condensed in the next scheme :

$\alpha_1^1 \varepsilon_2 \cong 0$ and $\alpha_2^1/\varepsilon_2 \cong 0 \Rightarrow V_2 = 0$,

$\alpha_1^1 \varepsilon_2$ appreciable and $\alpha_2^1/\varepsilon_2 \cong 0 \Rightarrow (W_1, V_2)$ dependent ,

$\alpha_1^1 \varepsilon_2$ infinitely large and $\alpha_2^1/\varepsilon_2 \cong 0 \Rightarrow W_1 = 0$,

$\alpha_1^1 \varepsilon_2$ infinitely large and α_2^1/ε_2 appreciable $\Rightarrow W_1 = 0$,

$\alpha_1^1 \varepsilon_2 \cong 0$ and α_2^1/ε_2 infinitely large $\Rightarrow W_2 = 0$,

$\alpha_1^1 \varepsilon_2$ appreciable and α_2^1/ε_2 infinitely large $\Rightarrow W_2 = 0$,

$\alpha_1^1 \varepsilon_2$ infinitely large and α_2^1/ε_2 infinitely large $\Rightarrow (W_1, V_2)$ dependent .

All these cases disprove the hypothesis. The only possibilities are

$\alpha_1^1 \varepsilon_2$ limited and α_2^1/ε_2 appreciable .

We can put

$\alpha_2^1 = \varepsilon_2\left(a_2^2 + \alpha_2^2\right)$ with $\alpha_2^2 \cong 0$ and a_2^2 standard nonnul,

$\alpha_1^1 = \varepsilon_2\left(a_2^1 + \alpha_1^2\right)$ with $\alpha_2^1 \cong 0$ and a_2^1 standard .

This gives the relations :

$$V_2 = a_2^1 W_1 + a_2^2 W_2 ,$$

$$\alpha_1 = \varepsilon_1 a_1^1 + \varepsilon_1\varepsilon_2 a_2^1 + \varepsilon_1\varepsilon_2\alpha_1^2 .$$

$$\varepsilon_3 V_3 + \ldots + \varepsilon_3 \ldots \varepsilon_k V_k = \alpha_2^1 W_1 + \alpha_2^2 W_2 + \alpha_3^2 W_3 + \alpha_3^2\alpha_4 W_4 + \ldots + \alpha_3^2\alpha_4 \ldots \alpha_s W_s \quad (5)$$

We are now led to comparing the infinitesimals ε_3, $\alpha^2{}_1$, $\alpha^2{}_2$ and $\alpha^2{}_3$. The hypothesis relative to the vectors V_1 and W_1 implies that the quantities $\alpha^2{}_i/\varepsilon_3$ are not infinitely large. So ε_3 is equivalent to one of the $\alpha^2{}_1$, this gives the linear relations between V_3 and W_1 for $i \leq 3$. This process decreases the number of terms in the left part of the identity (5). Before r steps $(r \leq k)$, we get :

$$V_{r-1} = \sum_{l=1}^{k-1} a^l{}_{r-1} W_l \quad \text{with} \quad a^{r-1}_{r-1} \neq 0 \text{ and standard}$$

$$\alpha_1 \ldots \alpha_{r-1} = a_{r-1}^{r-1}\varepsilon_1\varepsilon_2 \ldots \varepsilon_{r-1} + \varepsilon_1\varepsilon_2 \ldots \varepsilon_{r-1}\varepsilon_{r-1}^{r-1} .$$

In particular, if $r = k$, then we have

$$\varepsilon_1 V_k = \alpha_1^{k-1} W_1 + \alpha_2^{k-1} W_2 + \ldots + \alpha_{k-1}^{k-1} W_{k-1} + \alpha_k\left(a_{k-1}^{k-1} + \alpha_{k-1}^{k-1}\right) W_k + \ldots +$$
$$+ \alpha_k\left(a_{k-1}^{k-1} + \alpha_{k-1}^{k-1}\right) \alpha_{k+1} \ldots \alpha_s W_s \qquad \text{(we suppose that } k \leq s)$$

We put

$$\alpha_k^{k-1} = \alpha_k\left(a_{k-1}^{k-1} + \alpha_{k-1}^{k-1}\right) .$$

The above mentioned identity is written

$$\varepsilon_k V_k = \alpha_1^{k-1} W_1 + \alpha_2^{k-1} W_2 + \ldots + \alpha_{k-1}^{k-1} W_{k-1} + \alpha_k^{k-1} W_k + \alpha_k^{k-1}\alpha_{k+1} W'$$

with

$$W' = W_{k+1} + \alpha_{k+2} W_{k+2} + \ldots + \alpha_s W_s .$$

After division by ε_k, we find

$$V_k = \lambda_1 W_1 + \lambda_2 W_2 + ... + \lambda_{k-1} W_{k-1} + \lambda_k W_k + \lambda_k \alpha_{k+1} W_{k+1} + ... + \lambda_k \alpha_{k+1} ... \alpha_s W_s .$$

But the vectors $W_1, ..., W_s$ are standard and lineary independent. As V_k is also standard, its decomposition in the frame $W_1, ..., W_s$ has standard coefficients. Every scalar $\lambda_i, i = 1, ..., k$ and $\lambda_k \alpha_{k+1}, ..., \lambda_k \alpha_{k+1} ... \alpha_s$ is standard. As the α_i are infinitesimals, we must have $\alpha_{k+1} = 0$, this gives the equivalence between the two decompositions.

IV.3. About the compactness of the Grassmannian manifold

1) *A little bit more of Non-Standard : the I.S.T. Theory*

In the first section, we exposed just enough for an infinitesimal study of the perturbation problems of Lie algebra laws or, more generally, of the perturbation problems in linear algebra and in algebraic geometry. Of course, nonstandard analysis does not confine itself only to these simple studies. The axiomatic presentation, established by E. Nelson and called I.S.T. (for Internal Set Theory), is a universal presentation of nonstandard analysis. But it can be good in precise application of A.N.S., like here in the study of the perturbations of algebraic laws, to group and to understand what, in the Nelson axiomatic, play a fundamental role.

Let us take, for example, the properties of elementary classical analysis ; the Nelson axiomatic of course permits a wonderful use of infinitesimals ; but somebody will reproach us for using too rich an axiom for the use we want, hence the temptation to find a minimal model (but with risks limited to its own preoccupations). The infinitesimals in the algebraic problems which interest us, in fact intervene only in situations of comparison, and the essentialness of the Nelson axiomatic consists in the existence of infinitesimals and into taking the shadow of limited elements. Perhaps the most characteristic example of this situation concerns the interpretations of the famous theorem of Hironaka : that we can summarize as follows : let us suppose that we give a standard number p of infinitesimal complex which are solutions of a standard number q of polynomial standard relations ; then all of these infinitesimals can be compared to an infinitesimal ε. This means that they are written like a standard sum of the powers of ε. This example stresses the role which the infinitesimals take in algebra : they have only a minor interest in the computational field, but they are very important in the comparative approximations : the theorem of decomposition is made in this perspective.

We will go on to briefly formulate Nelson's axiom to gratify the lower of the quantificators.

AXIOM I (Idealization)

So $\mathcal{R}$ *is an internal (or classical) relation* :

$$\forall^{sf} F, \exists x = x_F \left[\mathcal{R}(x,y) \ \forall y \in F\right] \Leftrightarrow \exists x\left[\mathcal{R}(x,y), \forall^{s} y\right]$$

where $\forall^{s}$ *resumes 'for all standard' and* $\forall^{sf}$ *'for all finite standards'*
This is translated by : let $\mathcal{R}$ *be a classical formula; for finding an element* x *satisfying* $\mathcal{R}(x,y)$ *for all standard* y *it is necessary and sufficient that for all finite and standard parts* F *we can find* $x = x_F$ *such that* $\mathcal{R}(x,F)$.

As applications, we deduce from (I) the existence of the infinitesimals in $\mathbb{C}$ (that was announced as an axiom in the first part).

Consequence. *There is a finite set containing all the standards.*

This most surprising consequence permits us to show that all infinite standard sets contain at least a standard element or that in a standard finite set, all elements are standard. This will be the key for the study of rigid algebras by infinitesimal methods. Now let us give the reason for this : every algebraic variety admits a finite number of irreducible components ; if this variety is standard, then the components are standard.

AXIOM S (Standardization)

Let E *be a standard set and* $\mathcal{P}$ *a standard or nonstandard property; then*

$$\exists^{s} A , \forall^{s} x \left(x \in A \iff x \in E \text{ and } \mathcal{P}(x)\right)$$

This axiom permits us to construct standard elements from an internal propriety (i.e. is expressible in a classical language, which does not use either the standard predicate or its derivative) or external propriety (which is not internal). From this axiom, we deduce the existence of the shadow of a limited complex element.

AXIOM T (Transference)

This is nothing but the axiom previously enunciated.

2) *Shadow of a vector subspace*

Theorem 5. *Let* V *be a vector subspace of* $\mathbb{C}^n$ (n standard). *We suppose that* V *is of dimension* p. *Then the shadow of* V, *which is the only standard set whose standard elements are shadows of the limited elements of* V, *is also a vector subspace of* $\mathbb{C}^n$ *with the same dimensions as* V.

Proof. As the sum and the external product are continuous operations, we can write for limited vectors of V : ${}^{o}(x+y) = {}^{o}x + {}^{o}y$ and if α is a limited scalar, ${}^{o}(\alpha x) = {}^{o}\alpha {}^{o}x$. This permits us to provide ${}^{o}V$ with the structure of a vector space. To look for its dimension, one considers a standard scalar product in $\mathbb{C}^n$ and an orthonormal basis of V. Every vector of this basis is limited, admits a shadow, and its orthogonality implies that these shadows are independent and form a basis of ${}^{o}V$.

Remark. This theorem does not translate anything other than the compactness of the Grassmannian manifold, although here no topology of this variety has been defined.

IV.4. Some problems of perturbations

1) *Perturbations of polynomials*

We consider the standard vector space $\mathbb{C}^n[X]$ of complex polynomials of standard degree n.

Definition 9. *Let* P(X) *be a standard polynomial of* $\mathbb{C}^n[X]$. *We call perturbation of* P(X) *for every polynomial of* $\mathbb{C}^n[X]$ *infinitely close to* P(X), *i.e. its coefficients are infinitely close of those to* P(X).

We choose a standard basis of $\mathbb{C}^n[X]$. If Q(X) is a perturbation of P(X), it admits a decomposition. As an application, we immediatly deduce the theorem of the continuity of the roots with respect to the coefficients, which is enunciated here as follows :

Theorem 6. *The roots of* Q(X) *are infinitely close to the roots of* P(X) *and the multiplicity of the (standard) root of* P(X) *is the sum of the multiplicities of the roots of* Q(X) *which are infinitely close to the given root of* P(X).

This theorem is the basis of complex algebraic geometry, which one easily understand by its easy formulation and by the evident proof in the nonstandard frame that the approach of some problems of algebraic geometry, via the N.S.A., can be easily made (see, for example, [Go4]).

2) Perturbations of linear operators

Let E be a complex (or real) standard vector space with a finite and standard dimension. The space End(E) of the endomorphisms of E is also standard and we can define, as previously, a natural notion of perturbation in this space. Then we can observe the role of the decomposition for the limited vectors of End(E).

Definition 10. *Let* f *be a standard endomorphism of* E *A perturbation of* f *is an endomorphisme* g *in* End(E) *verifying* $g(X) \cong f(X)$ *for every* X *standard in* E.

Some proprieties of perturbations

(i) If g is a perturbation of f, it verifies $g(X) \cong f(X)$ for all X limited in E. In fact, a linear mapping being continuous, we have for all limited vectors X : $f({}^{o}X) \cong f(X)$. From this we deduce :

$$g(X) \cong g({}^{o}X) \cong f({}^{o}X) \cong f(X) .$$

(ii) We consider a standard basis (e_i) of E. Then, if g is a perturbation of f, the matrix of g relative to (e_i) is infinitely close to the standard matrix of f. In particular, the characteristic polynomials of g and f are infinitely close. From the preceding remarks, we can affirm

Proposition 12. *The eigenvalues of* g *are infinitely close to the eigenvalues of* f. *If* v *is a limited eigenvector of* g *associated to the eigenvalue* λ, *then* ${}^{o}v$ *is an eigenvector of* f *associated to the eigenvalue* ${}^{o}\lambda$.

Within the framework of the nonstandard analysis, we are led to call standard a Lie algebra whose constants of structure relative to a standard basis of $\mathbb{C}^n$ (n standard) are standard.

This can induce some confusion with the notion of standard nilpotent Lie algebra introduced in Chapter 2. It is difficult to modify the terminology of one or other of these notions without loosing the relation with the existing literature. As the risk of confusion between standard algebras (about the viewpoint of N.S.A.) and standard algebras (like the nilpotent algebras of a parabolic algebra), absolutely excluded in this treatise (we will neither talk about standard algebra, nor of standard nonstandard algebra !!!), we will conserve for the two notions the word standard when necessary, we explain what it means.

Definition 11. *Let* n *be standard. A Lie algebra law* μ_0 *on* $\mathbb{C}^n$ *is standard, if its constants of structure relative to a standard basis of* $\mathbb{C}^n$ *are standard. A perturbation* μ *of* μ_0 *is a Lie algebra on* $\mathbb{C}^n$ *verifying :*

$$\mu(X,Y) \cong \mu_0(X,Y) \qquad \forall\ X, Y \ \textit{ standard in } \mathbb{C}^n .$$

2) *Decomposition of a perturbation*

Let μ be a perturbation of the standard law μ_0 on $\mathbb{C}^n$. As these laws are elements of the standard vector space $T^n{}_{2,1}$ of tensors (2,1) on $\mathbb{C}^n$, we can apply the theorem of the decomposition of limited vectors.

Theorem 7. *Let* μ *be a perturbation of the standard Lie algebra* μ_o *on* $\mathbb{C}^n$. *Then there are independent standard mappings* $\varphi_1, \ldots, \varphi_k$ *in* $T^n{}_{2,1}$ *and there are infinitesimals* $\varepsilon_1, \ldots, \varepsilon_k$ *such that*

$$\mu = \mu_0 + \varepsilon_1\varphi_1 + \varepsilon_1\varepsilon_2\varphi_2 + \ldots + \varepsilon_1 \ldots \varepsilon_k\varphi_k \ .$$

We will see in the next section the relations which must verify the tensors φ_i , so the Jacobi conditions are satisfied for μ. We note now the difference between the deformations and the perturbations, not from the conceptual viewpoint (these two objects must represent a close law), but from the viewpoint of the presentation : a perturbation is a law of L^n and is written as a finite sum of independent tensors and a deformation is an infinite formal series.

3) *Duality between deformation and contraction*

Let f be an isomorphism (not standard of $\mathbb{C}^n$) and μ_0 a standard law of L^n. The law $\mu = f^{-1} \circ \mu_0(f,f)$ is certainly nonstandard and is isomorphic to μ_0. If this law is limited (i.e. admits constants of structure limited with respect to a standard basis of $\mathbb{C}^n$), then it admits a (standard) shadow that we will denote μ_∞ ; it is a law of L^n.

Proposition 13. *The law* μ_∞ *is a contraction of* μ_0.

This results from the definition of the contraction. So, conserving the above-mentioned notations, we note that if μ_∞ (which is standard) is a contraction of the standard law μ_0, then there is a perturbation μ of μ_∞ isomorphic to the standard law μ_0

Theorem 8. *Let* μ *be a nonstandard limited law of* $\mathcal{L}^n$. *Then the shadow of* μ *is a contraction of a law* μ_0 *if and only if* μ *is isomorphic to a standard law.*

V. THE TANGENT GEOMETRY TO $\mathcal{L}^n$

V.1. Cohomological spaces and tangent spaces

Let μ be a law of $\mathcal{L}^n$ and $O(\mu)$ its orbit. We have seen that the tangent space $T_\mu O(\mu)$ to the variety $O(\mu)$ at the point μ, is identified with the space $B^2(\mu,\mu)$.

$$T_\mu O(\mu) = B^2(\mu,\mu)$$

The space of cocycles $Z^2(\mu,\mu)$ is identified with the tangent space at the point μ to the scheme $\mathcal{L}^n$, otherwise $Z^2(\mu,\mu)$ is the formal tangent space at μ to $\mathcal{L}^n$.

$$T_\mu^{\mathrm{Zariski}}\mathcal{L}^n = Z^2(\mu,\mu)$$

We must now determine the equations of the tangent plane to the variety $\mathcal{L}^n$ at a regular point and the equations of the tangent cone in a singular point.

V.2. Resolution of the equation of deformations

Let $\mu_t = \mu + t\varphi_1 + t^2\varphi_2 + \dots + t^n\varphi_n + \dots$ be a deformation of the law μ of $\mathcal{L}^n$. We write that

μ_t formally verifies the Jacobi identity. First, we recall the notations introduced previously. If φ and ψ are in $\mathcal{T}^n_{2,1}$, we note by $\varphi \circ \psi$ the trilinear mapping with values in $\mathbb{C}^n$ defined by

$$\varphi\circ\psi(X,Y,Z) = \varphi(\psi(X,Y),Z) + \varphi(\psi(Y,Z),X) + \varphi(\psi(Z,X),Y) + \\ + \psi(\varphi(X,Y),Z) + \psi(\varphi(Y,Z),X) + \psi(\varphi(Z,X),Y).$$

In particular, $\varphi \in \mathcal{L}^n$ if and only if $\varphi \circ \varphi = 0$. More, if $\mu \in \mathcal{L}^n$ and if $\varphi \in \mathcal{T}^n_{2,1}$, then

$$\mu \circ \varphi = \delta_\mu \varphi .$$

Now the condition $\mu_t \circ \mu_t = 0$ is equivalent to

$$\mu\circ\mu + t\delta_\mu\varphi_1 + t^2\left(\frac{1}{2}\varphi_1\circ\varphi_1 + \delta_\mu\varphi_2\right) + \ldots + t^{2p}\left(\sum_{\substack{i+j=2p\\ i<j}} \varphi_i\circ\varphi_j + \frac{1}{2}\varphi_p\circ\varphi_p\right) + \\ + t^{2p+1}\left(\sum_{\substack{i+j=2p+1\\ i<j}} \varphi_i\circ\varphi_j\right) + \ldots = 0 ,$$

which leads to the next system :

$$\text{(E.D.)}\quad \begin{cases} \mu\circ\mu = 0 , \\ \delta_\mu\varphi_1 = 0 , \\ \frac{1}{2}\varphi_1\circ\varphi_1 + \delta_\mu\varphi_2 , \\ \vdots \\ \sum_{\substack{i+j=2p\\ i<p}} \varphi_i\circ\varphi_j + \frac{1}{2}\varphi_p\circ\varphi_p + \delta_\mu\varphi_{2p} = 0 , \\ \sum_{i+j=2p+1} \varphi_i\circ\varphi_j + \delta_\mu\varphi_{2p+1} = 0 . \\ \cdot \end{cases}$$

The system (E.D.) , which has an infinity of equations, can be interpreted as the necessary and sufficiant conditions that an element $\varphi_1 \in Z^2(\mu,\mu)$ is the first term of a one parameter family, i.e. a deformation of the law μ.

For example, the first obstruction is

$$\frac{1}{2}\varphi_1\circ\varphi_1 + \delta_\mu\varphi_2 = 0\ ,$$

i.e. $[\varphi_1 \circ \varphi_1]_3 = 0$, where $[\ \ ]_3$ designates the cohomology class in the space $H^3(\mu,\mu)$. We suppose the first obstruction to be cleared up. Then the second obstruction is written

$$\varphi_1\circ\varphi_2 + \delta_\mu\varphi_3 = 0\ .$$

This is interpreted in the following manner : we give an element $\varphi_1 \in \mathcal{T}^n{}_{2,1}$ such that $\delta_\mu\varphi_1 = 0$. The first obstruction is :

$$[\varphi_1\circ\varphi_1]_3 = 0\ \ .$$

Let us choose φ_2 such that $\varphi_1\circ\varphi_1 + \delta_\mu\varphi_2 = 0$.

In order that φ_1 is the first term of a deformation, it is necessary that $[\varphi_1 \circ \varphi_2]_3 = 0$ for every representative φ_2 of the coboundary $\varphi_1 \circ \varphi_1$. It is clear that this does not depend on the choice of φ_2.

Applications.

1. φ_1 defines an infinitesimal deformation of μ if and only if

$$\delta_\mu\varphi_1 = 0\ ,$$
$$\varphi_1\circ\varphi_1 = 0\ .$$

2. φ_1 is the first term of a deformation of degree 2 (i.e. all the coefficients of t^i are null for $i \geq 3$) if and only if

$$\delta_\mu\varphi_1 = 0\ ,$$
$$[\varphi_1\circ\varphi_1] = 0\ ,$$
$$[\varphi_1\circ\varphi_2] = 0 \quad \forall\ \varphi_2\ \text{, such that}\ \ \varphi_1\circ\varphi_1 = \delta\varphi_2\ ,$$
$$\varphi_2\circ\varphi_2 = 0\ .$$

Theorem 9. *Let* φ_1 *be in* $Z_2(\mu,\mu)$*. Then the obstructions for* φ_1*, being the first term of a deformation, are in the space* $H^3(\mu,\mu)$.

Proof. Let $\mu_t = \mu + t\varphi_1 + \dots + t^p\varphi_p + \dots$. Let us suppose that all obstructions up to order k are cleared up, that is

$$\mu_t \circ \mu_t = t^{k+1}\psi_{k+1} + t^{k+2}\psi_{k+2} + \dots ,.$$

where

$$\psi_p = \sum_{\substack{i+j=p \\ i<p}} \varphi_i \circ \varphi_j + \delta_\mu \varphi_p \left(+ \frac{1}{2}\varphi_p \circ \varphi_p \ \text{ if p is even} \right)$$

and we consider the polynomial development

$$\mu'_t = \mu + t\varphi_1 + \dots + t^k\varphi_k + t^{k+1}\varphi_{k+1}$$

Then $\mu'_t \circ \mu'_t = 0$ is equivalent to

$$0 = \delta_\mu \varphi_{k+1} + \varphi_1 \circ \varphi_k + \varphi_2 \circ \varphi_{k-1} + \dots + \varphi_i \circ \varphi_j + \dots + \varphi_s \circ \varphi_{s+1}$$

with $i + j = k + 1$ and $k = 2s$ or to

$$0 = \delta_\mu \varphi_{k+1} + \varphi_1 \circ \varphi_k + \dots + \varphi_{s-1} \circ \varphi_{s+1} + \frac{1}{2}\varphi_s \circ \varphi_s$$

if $k = 2s-1$.

In order that these equations be satisfied, it is necessary that

$$\sum_{\substack{i+j=2k+1 \\ i<j}} \varphi_i \circ \varphi_j \quad \left(\text{or } \ \Sigma\, \varphi_i \circ \varphi_j + \frac{1}{2}\varphi_s \circ \varphi_s\right)$$

is a coboundary $B^3(\mu,\mu)$.

We can show directly that, if the obstructions are satisfied up to the order k, then $\Sigma\, \varphi_i \circ \varphi_j \left(\text{or } \ \Sigma\, \varphi_i \circ \varphi_j + \frac{1}{2}\varphi_s \circ \varphi_s\right)$ is a cocycle of $Z^3(\mu,\mu)$.

This proves the theorem.

Corollary. *If* $H^3(\mu,\mu) = 0$, *every element* φ_1 *in* $Z^2(\mu,\mu)$ *is the first term of a deformation of* μ.

V.3. The equations of perturbations

Let μ' be a perturbation of a Lie standard algebra μ_0 which is decomposed in

$$\mu' = \mu + \varepsilon_1\varphi_1 + \dots + \varepsilon_1 \dots \varepsilon_k\varphi_k ,$$

where the φ_i are standard and linearly independent in $\mathcal{T}^n_{2,1}$. Let us write that μ' verifies the Jacobi identities

$$\mu' = \mu' \circ \mu' = \mu \circ \mu + \varepsilon_1\delta_\mu\varphi_1 + \dots + \varepsilon_1 \dots \varepsilon_k\delta_\mu\varphi_k + \frac{1}{2}\varepsilon^2{}_1\varphi_1 \circ \varphi_1 + \dots +$$

$$+ \varepsilon^2{}_1 \dots \varepsilon_k\varphi_1 \circ \varphi_k + \frac{1}{2}\varepsilon^2{}_1\,\varepsilon^2{}_2\,\varphi_2 \circ \varphi_2 + \dots + \varepsilon^2{}_1\,\varepsilon^2{}_2 \dots \varepsilon_k\,\varphi_2 \circ \varphi_k + \dots +$$

$$+ \dots + \frac{1}{2}\varepsilon^2{}_1 \dots \varepsilon^2{}_k\,\varphi_k \circ \varphi_k \ .$$

As every mapping is standard, this equation represents a lineary combination with nonstandard coefficients of standard vectors in the space of the trilinear mapping (or of the 3 cochains). This equation is reduced to $\mu \circ \mu = 0$ and

$$\delta_\mu\varphi_1 = 0 ,$$

$$\text{(E.P.)} \quad \varepsilon_2\delta_\mu\,\varphi_2 + \dots + \varepsilon_2 \dots \varepsilon_k\,\delta_\mu\,\varphi_k + \frac{1}{2}\varepsilon_1\,\varphi_1 \circ \varphi_1 + \dots + \frac{1}{2}\varepsilon_1\,\varepsilon^2{}_2 \dots \varepsilon^2{}_k\,\varphi_k \circ \varphi_k = 0 .$$

So the first term φ_1 of a perturbation is an element of $Z^2(\mu,\mu)$. The system (E.P.) is interpreted like this.

Theorem 10. *An element* φ_1 *of* $Z^2(\mu,\mu)$ *is the first term of a perturbation* μ' *of* μ *if and only if there are bilinear standard independent forms* $\varphi_2, \dots, \varphi_k$ *and infinitesimals* $\varepsilon_1, \dots, \varepsilon_k$ *such that* (E.P.) *is satisfied.*

Remark. One will note that the resolution of the obstructions to the prolongation of a cocycle into a perturbation, makes to occur only a finite number of mapping, but the prolongation of this cocycle in a deformation is traduced by an infinity of conditions.

Theorem 11. *Nontandard linear equation* (E.P.) *equivalent to a standard finite linear system.*

Proof. We will solve the equation (E.P.) by playing on the index k of the decomposition of μ'. This integer is called the length of μ'.

1st case. *The length of* μ' *is* 1.

$$\mu' = \mu + \varepsilon\varphi_1 \quad .$$

In this case, we have

$$(\mathrm{E.P.}) \qquad \begin{cases} \delta_\mu \varphi_1 = 0 \\ \varphi_1 \circ \varphi_1 = 0 \end{cases}$$

So an element φ_1 *of* $Z^2(\mu,\mu)$ *is the first term of a perturbation with length* 1 *if and only if* $\varphi_1 \in L^n$.
(We will compare this result with that of the infinitesimal deformation.)

2nd case. μ' *has a length* 2 :

We have

$$\mu' = \mu + \varepsilon_1\varphi_1 + \varepsilon_1\varepsilon_2\varphi_2 \quad \text{and}$$

$$(\mathrm{E.P.}) \qquad \begin{cases} \delta_\mu \varphi_1 = 0 \\ \varepsilon_2\, \delta_\mu\, \varphi_2 + \frac{1}{2}\varepsilon_1\, \varphi_1 \circ \varphi_1 + \varepsilon_1\, \varepsilon_2\, \varphi_1 \circ \varphi_2 + \frac{1}{2}\varepsilon_1\, \varepsilon_2^2 \varphi_2 \circ \varphi_2 = 0 \quad . \end{cases}$$

Let us suppose $\varepsilon_2 \neq 0$. Then $\delta_\mu \varphi_2 = 0$ implies $\varphi_1 \circ \varphi_1 = \varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_2 = 0$ and $\mu + \varepsilon_1 \varphi_1$ is a perturbation of μ.

Suppose $\delta_\mu \varphi_2 \neq 0$. Then the vectors $\left(\varphi_1 \circ \varphi_1 , \varphi_1 \circ \varphi_2 , \varphi_2 \circ \varphi_2 , \delta_\mu \varphi_2\right)$ form a system of rank 1. Effectively, if the rank of these vectors is equal to 3, there is a linear relation with standard coefficients between these vectors :

$$a_1 \varphi_1 \circ \varphi_1 + a_2 \varphi_1 \circ \varphi_2 + a_3 \varphi_2 \circ \varphi_2 + a_4 \delta_\mu \varphi_2 = 0 \ , \ a_i \text{ standard.}$$

This gives $a_1 = \frac{1}{2} \alpha_1 \varepsilon_1$, $a_2 = \alpha_1 \varepsilon_1 \varepsilon_2$, $a_3 = \frac{1}{2} \alpha_1 \varepsilon_1 \varepsilon_2^2$, $a_4 = \alpha_1 \varepsilon_2$.

As $\varepsilon_i \cong 0$ and a_j is standard, this is impossible.
So the rank is at most 2. By the same arguments, one shows that the rank cannot be equal to 2. Then the rank is 1 and we have :

$$\varphi_1 \circ \varphi_1 = a \delta_\mu \varphi_2 \ ,$$

$$\varphi_1 \circ \varphi_2 = b \delta_\mu \varphi_2 \ ,$$

$$\varphi_2 \circ \varphi_2 = c \delta_\mu \varphi_2 \ ,$$

and $\varepsilon_2 + a\varepsilon_1 + b\varepsilon_1\varepsilon_2 + c\varepsilon_1\varepsilon_2^2 = 0$.
In this case, (E.P) is equivalent to

$$\delta_\mu \varphi_1 = 0 \ ,$$

$$\varphi_1 \circ \varphi_1 = a \delta_\mu \varphi_2 \ ,$$

$$\varphi_1 \circ \varphi_2 = b \delta_\mu \varphi_2 \ ,$$

$$\varphi_2 \circ \varphi_2 = c \delta_\mu \varphi_2 \ ,$$

$$\varepsilon_1 \left(a + b\varepsilon_2 + c\varepsilon_2^2\right) = \varepsilon_2 \ .$$

Before we interpret this system, let us introduce some notations relative to the products of cohomology classes.

Lemma 1. *Let* $\varphi \in Z^2(\mu,\mu)$. *Then* $\varphi \circ \varphi \in Z^3(\mu,\mu)$.

Let us recall that a 3-cochain ϕ is closed if

$$
\begin{aligned}
0 = \delta_\mu \Phi(X_1,X_2,X_3,X_4) = \\
&= \mu\big(X_1,\Phi(X_2,X_3,X_4)\big) - \mu\big(X_2,\Phi(X_1,X_3,X_4)\big) + \\
&+ \mu\big(X_3,\Phi(X_1,X_2,X_4)\big) - \mu\big(X_4,\Phi(X_1,X_2,X_3)\big) - \Phi(\mu(X_1,X_2),X_3,X_4) + \\
&+ \Phi(\mu(X_1,X_3),X_2,X_4) - \Phi(\mu(X_1,X_4),X_2,X_4) - \Phi(\mu(X_2,X_3),X_1,X_4) + \\
&+ \Phi(\mu(X_2,X_4),X_1,X_3) - \Phi(\mu(X_3,X_4),X_1,X_2) ,
\end{aligned}
$$

for all vectors X_1, X_2, X_3, X_4.
The lemma is proved by replacing ϕ with $\varphi \circ \varphi$ and by using the hypothesis $\delta_\mu \varphi = 0$. We will note $[\varphi^2]_3$ as the class of cohomologis of $\varphi \circ \varphi$. This class exists only if $[\varphi]_2$ (the class of φ in $H^2(\mu,\mu)$) exists.

Lemma 2. *Let* $[\varphi^2]_3 = 0$ *and let* ψ *be a representative of* $\varphi \circ \varphi$ *in* $B^3(\mu,\mu)$, i.e. $\delta_\mu \psi = \varphi \circ \varphi$. *Then* $\varphi \circ \psi \in Z^3(\mu,\mu)$.

The proof is long and technical. It is not reproduced here.

As the class of $\varphi \circ \psi$ only depends on φ , we will denote it $[\varphi^3]_3$. This class exists only if $[\varphi^2]_3 = 0$. Let us suppose that $[\varphi^3]_3 = 0$. Let ρ be such that $\varphi \circ \varphi = \delta\rho$. Then we have $\delta(\varphi \circ \rho .+ \varphi \circ \psi \circ \psi) = 0$. The class of cohomology does not depend on the choice of ρ (nor of ψ). One will denote it $[\varphi^4]_3$. The system (E.P) for the perturbations of length 2 is also interpreted as :

Proposition 14. *Let* $\varphi_1 \in Z^2(\mu, \mu)$. *Then* φ_1 *is the first term of a perturbation of length 2 if and only if*

(1) $\left[\varphi^2{}_1\right] = \left[\varphi^3{}_1\right] = \left[\varphi^4{}_1\right] = 0$,

(2) *the rank of the representatives of these classes in* $B^3(\mu,\mu)$ *is equal to* 1.

Remarks

1. The first terms of the perturbations of length 1 are directing vectors of the generative lines to L^n in μ. One can so write again the results relative to the perturbations of length 1 :

Let be $\varphi_1 \in Z^2(\mu,\mu)$. *Then* φ_1 *is a generative line in* μ *to* L^n *if and only if* $\varphi_1 \circ \varphi_1 = 0$.

2. In the case of the perturbations of length 2, the term φ_1 represents the tangent vector to the curve described by the laws $\mu = \mu_0 + \varepsilon_1 \varphi_1 + \varepsilon_1 \varepsilon_2 \varphi_2$, where φ_1 and φ_2 are fixed and $c_1 = \varepsilon_2 / \left(a+b\varepsilon_2+c\varepsilon_2^2\right)$ (from the equation E.P.). *Such a curve is a rational curve* (it is cubic), *the tangent vector of which in* μ *is given by* φ_1.

2. case. *General case*

Let us consider the equation of perturbations

$$\varepsilon_2\, \delta_\mu\, \varphi_2 + \dots + \varepsilon_2 \dots \varepsilon_k \delta_\mu \varphi_k + \frac{1}{2}\varepsilon_1 \varphi_1 \circ \varphi_1 + \dots + \frac{1}{2}\varepsilon_1 \varepsilon_2^2 \dots \varepsilon_k^2 \varphi_k \circ \varphi_k = 0 .$$

This equation represents a linear combination of the standard vectors $\varphi_i \circ \varphi_j$ with infinitesimal coefficients.

Definition 12. *The rank of* μ *is the rank of vectors* $\{\delta_\mu \varphi_2, \dots, \delta_\mu \varphi_k, \varphi_1 \circ \varphi_1, \varphi_1 \circ \varphi_2, \dots, \varphi_{k-1} \circ \varphi_k, \varphi_k \circ \varphi_k\}$.

Theorem 12. *Let* μ *be a perturbation of length* k. *Then the rank of* μ *is equal to the rank of the vectors* $(\varphi_i \circ \varphi_j)$ $1 \leq i \leq j \leq k-1$.

Proof. Let ω be a linear form whose kernel contains the vectors $(\varphi_i \circ \varphi_j)$, $1 \leq i \leq j \leq k-1$. Let us apply ω to the linear combination defined by (E.D.) :

$$0 = \varepsilon_1 \dots \varepsilon_k \omega(\varphi_1 \circ \varphi_k) + \varepsilon_1 \varepsilon_2^2 \varepsilon_3 \dots \varepsilon_k \omega(\varphi_2 \circ \varphi_k) + \dots + \varepsilon_1 \varepsilon_2^2 \dots \varepsilon_k^2 \omega(\varphi_k \circ \varphi_k) +$$
$$+ 2\varepsilon_2 \omega\left(\delta_{\mu_0} \varphi_2\right) + 2\varepsilon_2 \varepsilon_3 \omega\left(\delta_{\mu_0} \varphi_3\right) + \dots + 2\varepsilon_2 \dots \varepsilon_k \omega\left(\delta_{\mu_0} \varphi_k\right).$$

After division by the infinitesimal having the smallest module and by taking the shadow, one obtains

$$0 = \omega(\delta_{\mu_0}\varphi_2) = \dots = \omega(\delta_{\mu_0}\varphi_k) = \omega(\varphi_1 \circ \varphi_k) = \dots = \omega(\varphi_k \circ \varphi_k).$$

Then the vectors $\delta_{\mu_0}\varphi_2, \dots, \varphi_k \circ \varphi_k$ are in the kernel of ω for every form ω whose kernel contains the vectors $(\varphi_i \circ \varphi_j)$ $1 \le i \le j \le k-1$. Then the vectors $(\delta_{\mu_0}\varphi_i)$ $i = 2,\dots,k$ and $(\varphi_i \circ \varphi_k)$ $i = 1,\dots,k$, are in the space generated by $(\varphi_i \circ \varphi_j)$ $1 \le i \le j \le k-1$. This proves the theorem.

The relations of dependence which are consequences of this theorem are :

$$\varphi_1 \circ \varphi_1 = a_1 \delta_\mu \varphi_2 ,$$
$$a_{12}\varphi_1 \circ \varphi_2 + a_{22}\varphi_2 \circ \varphi_2 + a_{32}\varphi_1 \circ \varphi_1 = a_2 \delta_\mu \varphi_3 ,$$
$$\vdots$$
$$a_{1+k-1}\varphi_1 \circ \varphi_k + \dots + a_{2+k-1}\varphi_{k-1} \circ \varphi_{k-1} + \sum_{\substack{i<k-1 \\ j<k-1}} b_{ij}\varphi_i \circ \varphi_j = a_{k-1}\delta_\mu \varphi_{k-1} .$$

Consequence. Let φ_1 be a vector of the tangent formal plane in μ to $\mathcal{L}^n$. Then if φ_1 is a tangent vector to a curve in $\mathcal{L}^n$ passing through μ (or φ_1 is a vector of the tangent line), we have :

$$\varphi_1 \circ \varphi_1 \in B^3(\mu,\mu) .$$

Theorem 13. *Let* $\varphi_1 \in Z^2(\mu,\mu)$ *and let* φ_1 *be a tangent vector at* μ *to* $\mathcal{L}^n$. *Then we have :*

$$\varphi_1 \circ \varphi_1 \in B^3(\mu,\mu) .$$

Example. Perturbations and contractions

Let μ_0 be a standard law and f an isomorphism of $GL(n)$. The law $\mu = f^{-1} \circ \mu_0 \circ f{\times}f$ defined by $\mu(X,Y) = f^{-1}(\mu_0(f(X), f(Y))$, is isomorphic to the standard law μ_0. If this law has a shadow μ_∞ then, by definition, μ_∞ is a contraction of μ_0 and μ is a perturbation of μ_∞ isomorphic to the standard law μ_0. Thus, *A perturbation μ of a standard law μ_∞ is related with a contraction if and only if μ is isomorphic to a standard law μ_0. In this case, μ_∞ is a contraction of μ_0.*

Example : The Heisenberg case

Let us consider the Heisenberg algebra of dimension 2p+1, denoted by H_p and defined by

$$[X_2,X_3] = \dots = [X_{2k},X_{2k+1}] = \dots = [X_{2p},X_{2p+1}] = X_1 ,$$

A Lie algebra $\mathfrak{g}$ of dimension 2p+1 is provided with a contact form if there is a basis $(X_1, \dots, X_{2p+1})$ such that

$$\mu(X_2, X_3) = \dots = \mu(X_{2p}, X_{2p+1}) = X_1 .$$

$$\mu(X_i,X_j) = \sum_{k>1} C_{ij}^k X_k .$$

Let us consider the isomorphism f of $\mathbb{C}^{2p+1}$ defined by

$$f(X_1) = \varepsilon^2 X_1 ,$$

$$f(X_i) = \varepsilon X_i \quad \text{with} \quad \varepsilon \equiv 0 , \quad i = 2, \dots, 2p+1 .$$

The law μ' defined by $\mu' = f^{-1} \circ \mu \circ f \times f$ verifies

$$\mu'(X_2,X_3) = \dots = \mu'(X_{2p},X_{2p+1}) = X_1 ,$$

$$\mu'(X_i,X_j) = \sum \varepsilon_k C_{ij}^k X_k , \quad \text{where } \varepsilon_k \text{ is a positive power of } \varepsilon .$$

The shadow of μ' is the Heisenberg algebra. We will deduce

Theorem 14. *Every Lie algebra of dimension* 2p+1 *provided with a contact form can be contracted on the Heisenberg algebra* H_p.

The perturbation μ' of the law of Heisenberg algebra has for its first term the cocycle φ_1 defined by

$$\varphi_1(X_2,X_3) = \dots = \varphi(X_{2p},X_{2p+1}) = 0\,,$$

$$\varphi_1(X_1,X_i) = 0\,,$$

$$\varphi_1(X_i,X_j) = \sum_{k>1} C^k_{ij}X_k\,, \text{ if one of } C^k_{ij} \text{ is not zero}\,.$$

So we have a description of vectors tangent to the corresponding point to the Heisenberg algebra.

VI. THE COMPONENTS OF $\mathcal{L}^n$

VI.1. The components of $\mathcal{L}^n$ for $n \leq 7$

As $\mathcal{L}^n$ is an algebraic variety, it is the union of a finite number of algebraic irreducible components. One proposes to determine these components for the dimension n less than 7. The expressed results essentially lay in the next proposition :

Proposition 15. *Let* F *be an irreducible closed set of* $\mathcal{L}^n$ *and* μ *a point of* F *such that* $\dim F = \dim Z^2(\mu,\mu)$. *Then*

(1) μ *is a simple point of the scheme* $\mathcal{L}^n$

(2) F *is the only irreducible component containing* μ .

Proof. Let $\mu \in F$ and let $\mathcal{C}$ be an irreducible component containing F. As $Z^2(\mu,\mu)$ is identified to the Zariski tangent space of $\mathcal{L}^n$ (that is, the tangent space in μ to the scheme $\mathcal{L}^n$), the Zariski tangent space in $\mathcal{C}$ at the point μ is contained in $Z^2(\mu,\mu)$. Then $\dim \mathcal{C} \leq \dim Z^2(\mu,\mu)$. But

$$\dim F = \dim Z^2(\mu,\mu) \leq \dim \mathcal{C} \leq \dim Z^2(\mu,\mu)$$

which gives $\dim F = \dim \mathcal{C}$ and $F = \mathcal{C}$. QED.

In the following classification we have need of some notation:

Notation. $sl(2,\mathbb{C}) \oplus D_i$ designates the irreducible representation of $sl(2,\mathbb{C})$ on the module D_i.

Theorem 15. *List of the components of* L^n

(1) $n = 2$ $\quad L^2$ *is irreductible* : $L^2 = \overline{O(a_2)}$,

where a_2 is the solvable algebra defined by $[X_1,X_2] = X_1$. The dimension of the component is 2.

(2) $n = 3$ $\quad L^3$ *is the union of 2 components.*

(i) $\overline{O(sl(2,\mathbb{C}))}$ (the Zariski closure of the orbit of $sl(2,\mathbb{C})$ (dim 6),

(ii) the set R_3 of solvable algebras (dim 6).

$n = 4$ $\quad L^4$ *is the union of 4 irreducible components*

(i) $\overline{O(sl(2,\mathbb{C}) \oplus \mathbb{C})}$ (dim 12),

(ii) $\overline{O(r_2 \oplus r_2)}$ (dim 12),

(iii) $\overline{F_4}$, where F_4 is the set of Lie algebras containing one Abelian ideal of dimension 3 (dim 12),

(iv) $\overline{G_4}$ where G_4 is the set of Lie algebras containing one noncommutative nilpotent ideal of dimension 3 (dim 12).

$n = 5$ $\quad L^5$ *allows 7 irreducible components*

(i) $\overline{O(sl(2,\mathbb{C}) \oplus r_2)}$ (direct sum) (dim 20),

(ii) $\overline{O(sl(2,\mathbb{C}) \oplus D_2)}$, where D_2 is the irreducible $sl(2,\mathbb{C})$-module of dimension 2 (dim 19),

(iii) $\overline{O(\mu_5)}$, where μ_5 is defined by

$$\mu_5(e_2,e_3) = e_1 \ ; \ \mu_5(e_1,e_4) = e_1 \ ; \ \mu_5(e_2,e_5) = e_2 ,$$

$\mu_5(e_3,e_4) = e_3$; $\mu_2(e_3,e_5) = -e_5$ (dim 20).

(iv) $\overline{K_5}$, where K_5 is the set of solvable Lie algebras whose nilradical is commutative and is of dimension 3 (dim 21),

(v) $\overline{F_5}$, where F_5 is the set of Lie algebras containing one Abelian ideal dimension 4 (dim 20),

(vi) $\overline{G_5}$, where G_5 is the set of Lie algebras whose radical is filiform and of dimension 4 (dim 20),

(vii) $\overline{M_5}$, where M_5 is the set of Lie algebras whose nilradical is of dimension 4 and s isomorphic to the direct sum $H_1 \oplus \mathbb{C}$, where H_1 is the 3-dimensional Heisenberg algebras.

$n = 6$ *L^6 admits 17 irreducible components*

(i) $\overline{O(sl(2,\mathbb{C}) \oplus sl(2,\mathbb{C}))}$ (dim 30),

(ii) $\overline{O(sl(2,\mathbb{C}) \oplus H_1)}$ (semidirect sum obtained from the action of $sl(2,\mathbb{C})$ on H_1) (dim 30),

(iii) $\overline{L^0}(sl(2,\mathbb{C}) \oplus \mathbb{C}^2)$ (dim 30),

(iv) $\overline{L^0}(sl(2,\mathbb{C}) \oplus D_2)$ (dim 30) (D_2 is the irreducible $sl(2,\mathbb{C})$ module of dimension 2),

(v) $\overline{R^0}(\mathbb{C}^3)$ (dim 30),

(vi) $\overline{R^0}(\mathbb{C}^4)$ (dim 32),

(vii) $\overline{R^0}(H_1 \oplus \mathbb{C})$ (dim 31),

(viii) $\overline{R^0}(n_4)$, where n_4 is the 4-dimensional filiform algebra (dim 30),

(ix) $\overline{R^0}(\mathbb{C}^5)$ (dim 30),

(x) $\overline{R^0(H_1 \oplus \mathbb{C}^2)}$ (direct sum) (dim 30),

(xi) $\overline{R^0(\mathfrak{n}_4 \oplus \mathbb{C})}$ (direct sum) (dim 30),

(xii) $\overline{R^0(\mathfrak{n}^5)}$ for every 5-dimensional nilpotent algebra $\mathfrak{n}^5_k$ (k = 1,...,6) (dim 30),

where $R^0(\mathfrak{a})$ $($*resp.* $L^0(\mathfrak{a}))$ *is the set of solvable Lie algebras* (*resp. of Lie algebras*) $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t}$, *where* $\mathfrak{n}$ *is the nilradical,* $\mathfrak{t}$ *is Abelian and reductive on* $\mathfrak{g}$ (*resp.* $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t} \oplus \mathfrak{s}$, *where* $\mathfrak{s}$ *is a Levi subalgebra) whose maximal ideal with a nilpotent radical is isomorphic to* $\mathfrak{a}$.

n = 7 L^7 *admits 49 irredutible components*

(i) $\overline{O(sl(2,\mathbb{C}) \oplus G_2 \oplus G_2)}$, where G_2 is the solvable law of dim 2 (dim 42),

(ii) $\overline{O(sl(2,\mathbb{C}) \oplus sl(2,\mathbb{C}) \oplus \mathbb{C})}$ (dim 42),

(iii) $\overline{O(sl(2,\mathbb{C}) \oplus D_4)}$, where D_4 is the irreducible $sl(2,\mathbb{C})$-module of dimension 4 (dim 41),

(iv) $\overline{O(sl(2,\mathbb{C}) \oplus D_2 \oplus D_2)}$ (dim 37) (D_2 is the irreducible $sl(2,\mathbb{C})$-module),

(v) $\overline{L^0(sl(2,\mathbb{C}) \oplus \mathbb{C}^3)}$ (dim 42),

(vi) $\overline{L^0(sl(2,\mathbb{C}) \oplus H_1)}$ (direct sum) (dim 42),

(vii) $\overline{L^0(sl(2,\mathbb{C}) \oplus D_2 \oplus \mathbb{C})}$ (dim 42),

(viii) $\overline{L^0(sl(2,\mathbb{C}) \oplus D_3)}$ (dim 42),

(ix) $\overline{L^0(sl(2,\mathbb{C}) \oplus H_1)}$ (semidirect sum) (dim 42),

(x) $\overline{R^0\left(\mathbb{C}^4\right)}$ (dim 44),

(xi) $\overline{R^0\left(H_1 \oplus \mathbb{C}\right)}$ (direct sum) (dim 42),

(xii) $\overline{R^0\left(\mathbb{C}^5\right)}$ (dim 45),

(xiii) $\overline{R^0\left(H_1 \oplus \mathbb{C}^2\right)}$ (direct sum) (dim 44),

(xiv) $\overline{R^0\left(\mathfrak{n}_4 \oplus \mathbb{C}\right)}$ (dim 43),

(xv) $\overline{R^0\left(\mathfrak{n}^5_i\right)}$ $i = 1, \ldots, 5$ (see the classification of 5-dimensional nilpotent algebras 5) (dim 42),

(xvi) $\overline{R^0\left(\mathfrak{n}^6_i\right)}$ for every 6-dimensional nilpotent Lie algebra (dim 42).

Sketch of the proof. One begins to determine the components of the variety $\mathcal{R}^n$ of solvable Lie algebras. Let us again take the notations introduced in this theorem. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a} = \mathfrak{D}^1\mathfrak{g} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$. It is clear that $\mathfrak{a}$ is the greatest ideal of $\mathfrak{g}$ whose radical is nilpotent. We denote by $L^n(\mathfrak{a})$ (resp. $R^n(\mathfrak{a})$) the set of the laws of $\mathcal{L}^n$ (resp. $\mathcal{R}^n$) which have $\mathfrak{a}$ as the greatest ideal with a nilpotent radical.

Definition 13. *A Lie algebra* $\mathfrak{g}$ *is decomposable if it can be written* $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{n} \oplus s$ *where* s *is a semisimple subalgebra,* $\mathfrak{g}_0 \oplus \mathfrak{n}$ *the radical,* $\mathfrak{n}$ *the nilradical, and* $\mathfrak{g}_0$ *Abelian, reductive in* $\mathfrak{g}$ *and verifying* $[\mathfrak{g}_0, s] = 0$.

In the next section, we will see the part which these algebras play in the determination of rigid Lie algebras.

Now, we note that if $n \leq 7$, then all Lie algebras $\mathfrak{g} \in L^n(\mathfrak{a})$ are close (in the Zariski sense) to a decomposable algebra of $L^n(\mathfrak{a})$. This is verified by using the classification of nilpotent Lie algebras of dimensions less than 6. We are led to introduce the subset $L^0{}_n(\mathfrak{a})$ (resp. $R^0{}_n(\mathfrak{a})$) of $L^n(\mathfrak{a})$ (resp. $R^0(\mathfrak{a})$) formed by decomposable algebras. The previous remark mentions the density of $L^0(\mathfrak{a})$ in $L^n(\mathfrak{a})$.

Lemma 1. *The components of* $R^n(\mathfrak{a})$ *for* $n \leq 7$ *are the closed subset* $R^0(\mathfrak{a})$ *with* $\dim \mathfrak{a} < n$.

Proof. One directly verifies that there does not exist, for $n \leq 7$, any component which contains only nilpotent laws.

The fact that the $\overline{R_0^n(a)}$ are component results from proposition (VI.1).

As for the determination of components of L^n, this is made by studying the spaces $L^0{}_n(\mathfrak{a})$ and the determination of the orbits of the algebra $\mathfrak{g}$ containing $sl(2,\mathbb{C})$ as Lie subalgebras, and constructed from the irreductible representations of $sl(2,\mathbb{C})$.

VI.2. Rigid Lie algebras

VI.2.1. **Definition 14.** *Let be* $\mu \in L^n$. *One says that* μ *is rigid if its orbit* $O(\mu)$ *is a Zariski open set.*

Equivalent definition. *Let* μ *be a standard law of* L^n. *Then* μ *is rigid if every perturbation is isomorphic to it (i.e.* $O(\mu)$ *is open for the metric topology).*

We have previously shown the equivalence of these definitions. The interest of rigid Lie algebras in the determination of the components is revealed in the next proposition.

Proposition 16. *If* μ *is rigid, then the Zariski closure of its orbit is an irreductible component of the variety* L^n.

Corollary

1. *There does exist only a finite number (for a fixed dimension) of isomorphic classes of rigid laws.*
2. *Every rigid law is isomorphic to a standard law.*

We now propose to determine the rigid laws. As there is only a finite number of isomorphic classes, we have here a good family to classify.

VI.2.2. The Theorem of **Nijenhuis and Richardson.**

Theorem 16. *Let* $\mu \in \mathcal{L}^n$ *such that* $H^2(\mu,\mu) = 0$. *Then* μ *is rigid.*

Proof. We have seen that the formal tangent space in μ to $\mathcal{L}^n$ is $Z^2(\mu,\mu)$ and the tangent space in μ to $O(\mu)$ is $B^2(\mu,\mu)$. If $H^2(\mu,\mu) = 0$, then $Z^2(\mu,\mu)$ is the tangent space to $\mathcal{L}^n$. As it is equal to $B^2(\mu,\mu)$, so the theorem of the inverse functions shows that $O(\mu)$ is open in $\mathcal{L}^n$. From where we have proved the theorem.

Remark. *The converse is false.*

Indeed, consider the law μ_0 on $\mathcal{L}^{11}$ defined by

$$
\begin{array}{lll}
\mu_0 (X,X_i) = iX_i\,, & i = 0, ..., 9\,, & \\
\mu_0 (X_0,X_i) = X_i\,, & i = 4, 5, ..., 9\,, & \\
\mu_0 (X_1,X_i) = X_{i+1}\,, & i = 2, 4, 5, ..., 8 & (\mu_0 (X_1,X_3) = 0)\,, \\
\mu_0 (X_2,X_i) = X_{i+2}\,, & i = 4, 5, ..., 7\,, &
\end{array}
$$

the other brackets are null $\left(\text{here } (X,X_0,X_1, ..., X_9) \text{ is a basis of } \mathbb{C}^{11}\right)$. This law is rigid. Effectively, let us consider a perturbation μ of μ_0. The linear operator $ad_\mu X$ is infinitely close to the operator $ad_{\mu_0}X$. As this last one is diagonalizable and has for eigenvalues $(0, 0, 1, ..., 9)$, then $ad_\mu X$ admits for eigenvalues $(0, \lambda_0, \lambda_1, ..., \lambda_9)$, where λ_i is infinitely close to i. But every nontrivial standard vector $Y = aX_0 + bX$ verifies $ad_{\mu_0}Y(X_i) = c_iX_i$, $i = 1,...,9$ and $c_i \neq 0$. The Jacobi conditions imply that $\lambda_0 = 0$ and $ad_\mu X$ is diagonalizable, the eigenvalues being $(0, 0, \lambda_1, ..., \lambda_9)$ with $\lambda_i \cong i$. One chooses a basis $(X, U_0, U_1,...,U_9)$ of eigenvectors of $ad_\mu X$ verifying $U_i \cong X_i$, $i = 0, ..., 9$. Jacobi's identities show that the vector $\mu(U_i, U_j)$ is an eigenvector for the eigenvalues $\lambda_i + \lambda_j$. If this vector is nonnull, then $\lambda_i + \lambda_j$ is an eigenvalue infinitely close to $(i + j)$. One deduces the next system :

$$\lambda_0 = 0\,,$$

$$\lambda_1 + \lambda_2 = \lambda_3\,, \qquad \lambda_2 + \lambda_4 = \lambda_6\,,$$

$$\lambda_1 + \lambda_4 = \lambda_5\,, \qquad \lambda_2 + \lambda_5 = \lambda_7\,,$$

$$\vdots \qquad\qquad \vdots$$

$$\lambda_1 + \lambda_8 = \lambda_9\,, \qquad \lambda_2 + \lambda_7 = \lambda_9\,.$$

Note that $\mu(U_i,U_j) \neq 0$ as soon as $\mu_0(X_i,X_j) \neq 0$. The resolution of this linear system is written

$$\lambda_2 = 2\lambda_1\,,$$

$$\lambda_3 = 3\lambda_1\,,$$

$$\lambda_i = \lambda_4 + (i-1)\lambda_1\,, \qquad i = 5, ..., 9\,.$$

One can choose a new base $(Y, Y_0, U_1, ..., U_9)$ so $Y \equiv X$, $Y_0 \equiv U_0$ and

$$\mu(Y, U_i) = iU_i\,, \qquad i = 0, 1, ..., 9\,,$$

$$\mu(Y_0, U_i) = 0\,, \qquad i = 1, 2, 3\,,$$

$$\mu(Y_0, U_i) = U_i\,, \qquad i = 4, 5, ..., 9\,,$$

$$\mu(U_i, U_j) = a_{ij}\, U_{i+j}\,, \qquad \text{with } a_{1i} \equiv 1\,, \quad i = 2, 4, 5, ..., 8\,,$$
$$a_{2i} \equiv 1\,, \quad i = 4, 5, 6, 7\,.$$

One considere a new change of basis

$$Y_i = b_i U_i \quad \text{with} \quad \left\{ \begin{array}{ll} b_i = \dfrac{b_{i+1}}{a_{1i}\text{-}b_1} & i = 2,4,5,...,n-2 \\ b_2 b_4 = \dfrac{b_6}{a_{24}} & \end{array} \right\}.$$

The constants of the structure of μ verify $a_{24} = a_{1i} = 1$ for $i = 2, 4, 5, 6, 7, 8$. The Jacobi conditions imply $a_{25} = a_{26} = a_{27} = 1$ and $a_{13} = a_{23} = a_{ij} = 0$ for $3 \leq i \leq j$. So μ is isomorphic to μ_0 and the last one is rigid.

Now we can calculate the dimension of $H^2(\mu_0,\mu_0)$ by using either a program of formal calculus, or a reduction via the Hochschild-Serre series, to the calculus of $H^2(\mathfrak{n},\mathfrak{n})$, where $\mathfrak{n}$ is the nilradical. One show

$$\dim H^2(\mu_0,\mu_0) = 1 .$$

Conclusion. *There are rigid Lie algebras whose second group of cohomology is not null.*

Let us note that the above-mentioned example is the most simple example known of rigid algebra with nonnull cohomology. We could show directly that all rigid Lie algebra of a dimension less than 8 have a trivial group of cohomologies.

VI.2.3. The rigidity of a semisimple complex Lie algebra. Rigidity of Borel subalgebras

The rigidity of a semisimple complex Lie algebra is a consequence of the theorem of Nijenhuis and Richardson. Indeed, one has the next result :

Proposition 17. *If* $\mathfrak{g}$ *is a semisimple complex Lie algebra, then* $\dim H^2(\mathfrak{g},\mathfrak{g}) = 0$.

We don't expose the classical proof of this theorem. We prefer to prove directly the rigidity of $\mathfrak{g}$ to understand better the phenomen on which implies the rigidity of simple and semisimple Lie algebras.

Theorem 17. *Every semisimple complex Lie algebra is rigid.*

Proof. Let μ_0 be a law of a semisimple complex Lie algebra. Then every perturbation μ of μ_0 is also semisimple. Effectively, if K_0 (resp. K) designates a Killing Cartan form of μ_0 (resp. K), the form K verifies $K \cong K_0$. Then it is nondegenerated and verifies ${}^0K = K_0$. As μ is semisimple, it admits a Weyl base $(U_\alpha, Y_a, Z_\alpha)_{\alpha\in D}$ verifying :

$$\mu(U_\alpha, Y_\alpha) = \alpha(H_\alpha) Z_\alpha, \qquad (H_\alpha = -iU_\alpha),$$
$$\mu(U_\alpha, Z_\alpha) = -\alpha(H_\alpha) Y_\alpha,$$
$$\mu(Y_\alpha, Z_\alpha) = 2\,U_\alpha,$$

and the scalars $\alpha(H_\alpha)$, and other constants of structure only depend on the weights of the roots. One knows that $\alpha(H_\alpha) \neq 0$ and that

$$\begin{array}{lll} K(Y_\alpha, Z_\beta) = 0, & K(Y_\alpha, Y_\beta) = 0, & \alpha \neq \pm\beta, \\ K(U_\alpha, Z_\beta) = 0, & K(Y_\alpha, Y_\alpha) = -2, & \\ K(U_\alpha, Y_\beta) = 0, & K(Z_\alpha, Z_\beta) = 0, & \\ K(U_\alpha, U_\alpha) = -\alpha(H_\alpha), & K(Z_\alpha, Z_\alpha) = -2. & \end{array}$$

Let $(\alpha_1, \ldots, \alpha_r)$ be a basis of roots. Then $K(U_{\alpha_i}, U_{\alpha_j}) = 0$ as soon as $i \neq j$.

The vectors U_{α_i} $(i = 1, \ldots, r)$, Y_α $(\alpha \in \Delta^+)$ and Z_α $(\alpha \in \Delta^+)$ generate the Lie algebra μ. This basis is orthogonal for K. It also generates the real compact form $\mu^{\mathbb{R}}$ of μ and the Killing form $K^{\mathbb{R}}$ of $\mu^{\mathbb{R}}$ correspond to the restriction of μ to $\mu^{\mathbb{R}}$, the real law corresponding to μ. Remember that $K^{\mathbb{R}}$ is a scalar product.

Lemma. *The constants of structure of a simple Lie algebra are standard.*

For example we can refer to the preceding chapters. The constants of structure of simple algebras only depend on the roots, and on the lengths of these roots.

Lemma. *The shadow of the Weyl basis of μ is a Weyl basis of μ_0.*

We show, firstly, that ${}^0(\mu^{\mathbb{R}})$ is the real compact form of μ_0. Effectively, if $\mathfrak{g}^{\mathbb{R}}$ (resp. $\mathfrak{g}_0^{\mathbb{R}}$) designates the real compact subalgebra of the algebra $\mathfrak{g}$ (resp. $\mathfrak{g}_0$), we have :

$$\mathbb{C}^n = \mathfrak{g}^{\mathbb{R}} \oplus i\mathfrak{g}^{\mathbb{R}} = {}^0(\mathfrak{g}^{\mathbb{R}}) \oplus i\,{}^0(\mathfrak{g}^{\mathbb{R}})$$

and ${}^{0}(K^{\mathbb{R}})$ is nondegenerated on ${}^{0}\mathfrak{g}^{\mathbb{R}}$. Let us denote this form by $K_0^{\mathbb{R}}$. As $K_0^{\mathbb{R}}(x,x) \cong K(x,x)$ for all standard vector, we deduce that $K_0^{\mathbb{R}}$ is defined and negative on ${}^{0}(\mathfrak{g}^{\mathbb{R}})$. Then ${}^{0}(\mathfrak{g}^{\mathbb{R}})$ is the real compact form of $\mathfrak{g}_0$.

Let us show now that the vectors $(U_\alpha, Y_\alpha, Z_\alpha)$ are limited and noninfinitesimals.

We consider a standard norm on $\mathbb{C}^n$. We put

$$a = |U_\alpha| \quad , \quad b = |Y_\alpha| \quad , \quad c = |Z_\alpha| \ .$$

The vectors

$$U'_\alpha = \frac{U_\alpha}{a} \, , \; Y'_\alpha = \frac{Y_\alpha}{b} \, , \quad Z'_\alpha = \frac{Z_\alpha}{c}$$

have a nonnull shadow in the algebra ${}^{0}(\mathfrak{g}^{\mathbb{R}})$. But

$$\mu_0({}^0U'_\alpha, {}^0Y'_\alpha) \cong \mu(U'_\alpha, Y'_\alpha) = \frac{c}{ab}\, \alpha(H_\alpha)\, Z'_\alpha \ .$$

As $\alpha(H_\alpha)$ is standard and strictly positive, c/ab is limited. We show also that b/ac and c/ab are limited. Let us suppose that c/ab is infinitesimal. Then

$$\mu_0({}^0U'_\alpha, {}^0Y'_\alpha) = 0 \, .$$

But

$$K_0^{\mathbb{R}}(\mu_0({}^0U'_\alpha, {}^0Y'_\alpha), {}^0Z'_\alpha) = K_0^{\mathbb{R}}({}^0U'_\alpha, \mu_0({}^0Y'_\alpha, {}^0Z'_\alpha)) = 2\,{}^{0}\left(\frac{a}{bc}\right) K_0^{\mathbb{R}}({}^0U'_\alpha, {}^0U'_\alpha) \ .$$

As $K_0^{\mathbb{R}}({}^0U'_\alpha, {}^0U'_\alpha) < 0$, then ${}^{0}\left(\frac{a}{bc}\right) = 0$.

As $0 = K_0^{\mathbb{R}}({}^0Y'_\alpha, \mu_0({}^0U'_\alpha, {}^0Z'_\alpha)) = {}^{0}\left(\frac{b}{ac}\right) K_0^{\mathbb{R}}({}^0Y'_\alpha, {}^0Y'_\alpha)\alpha(H_\alpha)$

then ${}^{0}\left(\frac{b}{ac}\right) = 0$.

In fact, we can show that every scalar $\frac{a}{bc}$, $\frac{b}{ac}$, $\frac{c}{ab}$ is infinitesimal as soon as one of them is.

We deduce, by doing the products two by two, that the scalars $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are infinitesimall. But $K(U'_\alpha, U'_\alpha) = \frac{1}{a^2} K(U_\alpha, U_\alpha) = -\frac{1}{a^2}\alpha(H_\alpha) \cong 0$. This implies that $K_0^{\mathbb{R}}(U'_\alpha, U'_\alpha) = 0$ and $U'_\alpha = 0$; that is impossible. So the numbers $\frac{c}{ab}, \frac{b}{ac}, \frac{a}{bc}$ are limited and not infinitesimal. It is the same for $1/a$, $1/b$, and $1/c$. Then the vectors U_α, Y_α, Z_α have nonnul shadows in $\mathfrak{g}_0^{\mathbb{R}}$ - ${}^0\mathfrak{g}^{\mathbb{R}}$. As the shadows of the vectors of the Weyl basis U_α, Y_α, Z_α , $\alpha > 0$ are orthogonal for the scalar product $(- K_0^{\mathbb{R}})$, they are also independent. As dim $\mathfrak{g}_0^{\mathbb{R}}$ = dim $\mathfrak{g}^{\mathbb{R}}$, then we have a basis of $\mathfrak{g}_0^{\mathbb{R}}$ and a complex basis for $\mathfrak{g}_0$. So the constants of structure of $\mathfrak{g}$ with respect to the Weyl basis are standard. They stay by placing their shadows. We recover, therefore, a Weyl basis of $\mathfrak{g}_0$. This proves the lemma.

The theorem is deduced directly from this proof. The constants of structure of $\mathfrak{g}$ with respect to the Weyl basis are equal to the constants of structure of $\mathfrak{g}_0$ with respect to the Weyl basis, given by the shadows of the Weyl basis of $\mathfrak{g}$. Therefore, these two algebras are isomorphic.

Corollary. *Let $\mathfrak{g}_0$ be a semisimple complex Lie algebra. Every Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}_0$ is rigid.*

In fact, $\mathfrak{b}$ is defined from the roots system. As this is rigid (from the precedent lemma), the rigidity of $\mathfrak{b}$ will be deduced from this.

Remark. The rigidity of the Borel subalgebras is shown classically by the Nijenhuis-Richardson Theorem. But the nullity of H^2 is, to our mind, a consequence of a relatively long and hard calculus.

VII. CONSTRUCTION OF SOLVABLE RIGID LIE ALGEBRAS

In the preceding section, we have shown the importance of the rigid algebras in the study of the components of a variety of Lie algebra laws. We propose to approach here the classification of solvable rigid Lie algebras and to develop a technique for the construction of these algebras.

VII.1. The decomposability of rigid algebras

Let $\mathfrak{g}$ be a solvable nonnilpotent Lie algebra.

Definition 15. *The Lie algebra $\mathfrak{g}$ is called decomposable if there is a decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$ and $\mathfrak{t}$ an Abelian subalgebra whose elements are $\mathrm{ad}\,\mathfrak{g}$-semisimple.*

Examples.

1. Every algebraic Lie algebra is decomposable.
2. Let $\mathfrak{g}$ be the Lie algebra defined by the next law :

$$\left\{\begin{array}{l} \mu(X_1, X_2) = \mu(X_3, X_4) = \ldots = \mu(X_{2p-1}, X_{2p}) = X_1 \\ \mu(X_2, X_3) = -\frac{1}{2} X_3 \\ \mu(X_2, X_4) = -\frac{1}{2} X_4 + X_3 \\ \mu(X_2, X_{2p-1}) = -\frac{1}{2} X_{2p-1} \\ \mu(X_2, X_{2p}) = -\frac{1}{2} X_{2p} + X_{2p-1} \quad . \end{array}\right\}$$

The nilradical of $\mathfrak{g}$ is generated by $(X_1, X_3, \ldots, X_{2p})$. It is isomorphic to the abelian algebra $\mathbb{C}^{2p-1}$. Every supplementary subalgebra of $\mathfrak{n}$ is one dimensional. It is abelian and possesses a basis formed by a vector of the form $X_2 + U$ with $U \in \mathfrak{n}$. As $\mathrm{ad}(X_2 + U)$ is never diagonalizable, this algebra is not decomposable.

Let us remark that this algebra is not rigid. In fact, consider the perturbation μ' of μ defined by

$$\begin{array}{l} \mu'(X_1, X_2) = \mu'(X_3, X_4) = \ldots = \mu'(X_{2p-1}, X_{2p}) = X_1 \\ \mu'(X_2, X_3) = \left(-\frac{1}{2} + \varepsilon_1\right) X_3, \\ \mu'(X_2, X_4) = \left(-\frac{1}{2} - \varepsilon_1\right) X_4 + X_3, \\ \mu'(X_2, X_{2p-1}) = \left(-\frac{1}{2} + \varepsilon_p\right) X_{2p-1}, \\ \mu'(X_2, X_{2p}) = \left(-\frac{1}{2} - \varepsilon_p\right) X_{2p} + X_{2p-1}. \end{array}$$

This algebra μ' is decomposable ($\mathrm{ad}_{\mu} X_2$ is diagonalizable) and is nonisomorphic to μ.

Theorem 18. *If $\mathfrak{g}$ is a solvable rigid Lie algebra, it is decomposable.*

We will admit this structure theorem, the proof given by Carles essentially is technical and unfortunately does not permit us to understand part of the decomposability in the rigidity propriety.

We will use this result for the construction of rigid solvable (nonnilpotent) Lie algebras.

VII.2. Linear systems of roots associated to a rigid Lie algebra

Let μ be a law of $\mathcal{L}^n$ whose associated Lie algebra $\mathfrak{g}$ is solvable, non nilpotent and decomposable.

Definition 16. *A vector* X *of* $\mathfrak{t}$ *is said to be regular, if the dimension of the linear space* $V_0(X) = \mathrm{Ker}(\mathrm{ad}\, X)$ *is minimal*:

$$\dim V_0(X) = \min\{\dim \mathrm{Ker}(\mathrm{ad}\, Y),\ Y \in \mathfrak{t}\}.$$

As $\mathrm{ad}\, X$ is diagonalizable and $\mathfrak{n}$ is ad X-invariant, one can find a basis $(X_1, X_2, \dots, X_{p+q}, \dots, X_n = X)$ of eigenvectors of ad X such that $(X_1, X_2, \dots, X_{p+q})$ is a basis of $\mathfrak{n}$, $(X_{p+q+1}, \dots, X_n)$ a basis of $\mathfrak{t}$ and $(X_{p+1}, \dots, X_n)$ a basis of $V_0(X)$.

Definition 17. *Let* X *be a regular vector of the decomposable Lie algebra* $\mathfrak{g}$. *The linear system of roots associated to* $(X_1, \dots, X_n)$ *is the linear system to* $n-1$ *variables* x_i *whose equations are* $x_i + x_j = x_k$ *if the component of the vector* $\mu(X_i, X_j)$ *on* X_k *is nonnull.*

We will note this system by $S(X)$ or, more simply, S, when the regular vector X is fixed.

Theorem 19 (of the rank). *If the Lie algebra $\mathfrak{g}$ is rigid, then for all regular vector* X, *one has* $\mathrm{rank}(S(X)) = \dim \mathfrak{n} - 1$.

Proof. In fact, the eigenvalues λ_i of ad X are solutions of $S(X)$; this results in the Jacobi equations :

$$\mu(X, \mu(X_i, X_j)) = (\lambda_i + \lambda_j)\, \mu(X_i, X_j) .$$

As the Lie algebra $\mathfrak{g}$ is decomposable, none of the operators ad Y for $Y \in \mathfrak{t}$ is trivial. As these operators are simultaneously diagonalizable, their eigenvalues determine solutions of the system $S(X)$. It results from this, that the rank of $S(X)$ is less than $\dim \mathfrak{n} - 1$.

Now let us suppose that we have $rg(S(X)) < \dim \mathfrak{n} - 1 = p+q-1$.

1st case . $S(X)$ admits a solution $(\alpha_1, \alpha_2, ..., \alpha_{p+1}, ..., \alpha_{n-1})$ such that the vector $(\alpha_{p+1}, ..., \alpha_{n-1})$ is nonnull. Then there is a nonisomorphic perturbation of μ In fact, let φ be a bilinear alternated mapping given by $\varphi(X, X_i) = \alpha_i X_i$ $i = 1, ..., n-1$; the perturbation $\mu + \varepsilon\varphi$ where ε is an infinitesimal complex, is not isomorphic to μ. So $\mathfrak{g}$ is not rigid.

2° case . The solutions of $S(X)$ verifies $\alpha_i = 0$ for $i = p+1, ..., n-1$. The constants of structure of μ can be written like this :

$\mu(X_i, X_j) = a_i^j X$ for $i = p+q+1, ..., n$ and $j = 1, ..., p$ with $a_n^j \neq 0$ when $i \leq j \leq p$,

$$\mu(X_i, X_j) = \sum_{1 \leq k \leq p} a_{ij}^k X_k \quad \text{for } i = p+1, ..., p+q \text{ and } j = 1, ..., p,$$

$$\mu(X_i, X_j) = \sum_{p+1 \leq k \leq p+q} a_{ij}^k X_k \quad \text{for } p+1 \leq i < j \leq p+q,$$

$$\mu(X_i, X_j) = 0 \quad \text{for } p+q+1 \leq i \leq n \text{ and } p+1 \leq j \leq n,$$

$$\mu(X_i, X_j) = \sum_{1 \leq k \leq p+q} a_{ij}^k X_k \quad \text{for } 1 \leq i < j \leq p \text{ with } a_{ij}^k = 0 \text{ if } a_n^i + a_n^j \neq a_n^k .$$

As rg(S) < dim $\mathfrak{n}$–1, there exits a solution of S in the form of

$$V = \left(\beta_1, \ldots, \beta_p, 0, \ldots, 0\right) \quad \text{which is independent of the vectors :}$$

$$V_{p+q+1} = \left(a^1_{p+q+1}, \ldots, a^p_{p+q+1}, 0, 0, \ldots, 0\right), \ldots, V_n = \left(a^1_n, \ldots, a^p_n, 0, \ldots, 0\right).$$

Let φ be the bilinear alternative mapping defined by $\varphi(X, X_i) = \beta_i X_i$ $i = 1, \ldots, p$, the other products being null. We will show that the perturbation $\mu + \varepsilon\varphi$ is not isomorphic to μ as soon as ε is a not zero infinitesimal. Suppose that $\mathfrak{g}$ is rigid; the law $\mu + \varepsilon\varphi$ is in the orbit of μ and φ represents a tangent vector to this orbit at the point μ. As the tangent space to the orbit coincides with the space $B^2(\mu,\mu)$ of 2-coboundaries for the Chevalley cohomology of $\mathfrak{g}$, we must have $\varphi = \delta_\mu h$ where δ_μ is the coboundary operator of this cohomology and h an endomorphism of $\mathbb{C}^n$. We put

$$h(X) = \sum h^i X_i \text{ (recall that } X = X_n)$$

As $\delta_\mu h(X, X_i) = \beta_i X_i$, $1 \le i \le p$ and $\delta_\mu h(X, X_i) = 0$, $p+1 \le i \le n$, we obtain

$$\beta_i = \sum_{p+1}^{p+q} h^j a^i_{ji} + \sum_{p+q+1}^{n} h^j a^i_j \qquad 1 \le i \le p,$$

$$0 = \sum_{p+1}^{p+q} h^k a^i_{kj} \qquad 1 \le i, j \le p,$$

$$0 = \sum_{p+1}^{p+q} h^k a^i_{kj} \qquad p+1 \le i, j \le p+q$$

This shows that the vector $Y = \sum_{p+1}^{n} h^i X_i$ verifies $\mu(Y, X_i) = \beta_i X_i$ for $i = 1, \ldots, p$

and $\mu(Y, X_i)$ for $i \ge p+1$. As for the vector $Z = Y - \sum_{p+q+1}^{n} h^i X_i$, it is in the nilradical and verifies

$$\mu(Z, X_i) = \left(\beta_i - \sum_{p+q+1}^{n} h^j a^i_j\right) X_i \text{ for } 1 \le i \le p \text{ and } \mu(Z, X_j) = 0 \text{ for } j \ge p+1.$$

This is impossible and the vectors V, $V_{p+q+1}, \dots, V_n$ are not linearly independent.

Corollary 1. *If* $\mathfrak{g}$ *is rigid, the rank of the linear system of roots does not depend of the choice of the vectors* $(X_1, \dots, X_n)$.

Corollary 2. *If* $\mathfrak{g}$ *is rigid, there is a basis* $(X_{p+q+1}, \dots, X_n)$ *of* $\mathfrak{t}$ *such that the eigenvalues of the operators* ad X_i *are integers.*

In fact, if we give a basis of $\mathfrak{t}$, the eigenvalues of the operators ad X, for every X in this basis, are solutions of the system $S(X)$. As this has integer coefficients, we deduce the corollary about this.

Remark. The above-mentioned corollary does not imply the rationality of the solvable rigid Lie algebras. In the next section, we will give an example of *nonrational* rigid algebra.

Corollary 3. *If* $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t}$ *is rigid, then* $\mathfrak{t}$ *is the maximal torus of derivations of* $\mathfrak{n}$.

VII.3 Rigid Lie algebras whose nilradical is filiform

In this section, we will consider the consequences of the previous theorem on the classification of solvable rigid Lie algebras whose nilradical is filiform. The choice of this propriety concerning the nilradical is not haphazard : a good approach of the classification of the nilpotent algebras is studying of the characteristic sequence which is an invariant related to the problem of the perturbations. Then the characteristic sequence is classified by a natural order relation and the filiform nilpotent Lie algebras correspond to the nilpotent algebras whose characteristic sequence is maximal. So the classification of the rigids begins naturally by considering by filiform nilradical.

Definition 18. *The nilradical $\mathfrak{n}$ of $\mathfrak{g}$ is filiform, if there is a vector* U *in the cone* $\mathfrak{n} - [\mathfrak{n}, \mathfrak{n}]$ *such that the invariants of similitude of the nilpotent operator* $ad_{\mathfrak{n}}U$ *are equal to* $(\dim \mathfrak{n} - 1, 1)$.

In this whole section, we will suppose that $\mathfrak{g}$ is solvable, not nilpotent, decomposable and the nilradical $\mathfrak{n}$ is filiform. Let us suppose that there is a basis $(X_1, ..., X_n)$ of $\mathfrak{g}$ such that

(a) $X = X_n$ is a regular vector of $\mathfrak{g}$.

(b) $(X_1, ..., X_k)$ is a basis of $\mathfrak{n}$ and $(X_{k+1}, ..., X_n) = X$ is a basis of $\mathfrak{t}$

(c) $\mu(X_1, ..., X_i) = X_{i+1}$ for $2 \leq i \leq k-1$.

Such a basis always exists, if the nilradical $\mathfrak{n}$ is a deformation of the filiform graded algebra L_n defined by a cocycle whose minimal homogeneity degree is at least equal to 1 ($\mathfrak{n}$ is not a deformation of the filiform algebra Q_n).

Note that, recently , we have examinated all rigid Lie algebras whose nilradical is filiform and eliminated this previous hypothesis concerning the basis ([G.A]). But the method is the same. This general case is based on the classification of filiform Lie algebras having a nonnull rank. We prefer to present here the particular case correponding to the existence of an operator ad X which is diagonalible on $\mathfrak{t}$ and jordanizable on $\mathfrak{n}$.

Proposition 18. *Let $\mathfrak{g}$ be a solvable rigid Lie algebra whose nilradical is filiform and which is provided with an adapted basis. Then the external torus is of dimension* 1 *or* 2.

Proof. Consider the linear system of roots S corresponding to the basis of the previous lemma. Its rank is greater or equal to n-3. Indeed, from the definition of this system, the equations $x_1 + x_i = x_{i+1}$, $2 \leq i \leq k-1$, form a subsystem of S. More, S contains the equations $X_{k+j} = 0$, $1 \leq j \leq n-k-1$ (see the proof of the theorem of rank). So the rank is greater than by n-3.

VII.4 Classification when the torus is of dimension 2

Theorem 20. *Every solvable rigid Lie algebra of dimension* n, $n \geq 4$, *whose nilradical is filiform and of codimension 2, is isomorphic to the Lie algebra defined by*

$$\mu(X_n, X_i) = iX_i \quad 1 \leq i \leq n-2 \;\; ; \;\; \mu(X_{n-1}, X_i) = X_i \quad 2 \leq i \leq n-2 \;\; ;$$
$$\mu(X_1, X_i) = X_{i+1} \quad 2 \leq i \leq n-3$$

the undefined brackets are null.

As the rank of every system of roots is equal to $n-3$, the nontrivial brackets are the square brackets resulting from the filiformity of the nilradical.

The proof of the rigidity of the above-mentioned law, just as the other laws which appear in this work, is direct : one studies a perturbation, we compute the rank of the associated linear system and one concludes that the perturbation is isomorphic to the given standard law.

VII.5 Classification when the torus is 1-dimensional

Theorem 21. *Let* $\mathfrak{g}$ *be a rigid Lie algebra whose nilradical is filiform and of codimension* 1. *There is a regular vector* $X \in \mathfrak{t}$ *such that the eigenvalues of the operator* ad X *are* $(1, k, k+1, ..., n+k-3, 0)$ *where* k *is a positive integer.*

Proof. Effectively, let X be a regular vector and $(X_1, ..., X_{n-1})$ a basis of eigenvectors of ad X given by Lemma 1. Let $(\lambda_1, ..., \lambda_{n-1}, 0)$ be the corresponding eigenvalues. If $\lambda_1 = 0$, the Jacobi identities imply $\lambda_i = \lambda_j$ for $2 \leq i, j \leq n-1$. As $\mathfrak{g}$ is nonnilpotent, one of the eigenvalues is nonnull. This shows that $\lambda_2 \neq 0$ and we have $\mu(X_i, X_j) = 0$ $2 \leq i < j$. The rank of the associated linear system is equal to $n-2$ and the Lie algebra $\mathfrak{g}$ cannot be rigid. So $\lambda_1 \neq 0$ and the eigenvalues λ_i are, two by two distinct.

From the theorem of the rank, we deduce the existence of two integers i and j such that $\mu(X_i, X_j) \neq 0$. As the product of two eigenvectors is also an eigenvector, there is an index i_0 such that $\mu(X_2, X_{j0})$ is a nonnull eigenvector of ad X. The index h of the corresponding eigenvalue λ_h verifies $h \geq j_0$.

Corollary. *If* $\mathfrak{g}$ *is rigid, then* $k \leq n-4$ ($n = \dim \mathfrak{g}$).

VII.6. On the classification of rigid Lie algebras

VII.6.1. A parametrization of these algebras

Let us remember the hypothesis (H) : $\mathfrak{g}$ is *rigid* solvable and its nilradical $\mathfrak{n}$ is of codimension 1 (other cases have been studied). We note X as a regular vector and suppose that its eigenvectors $(1, k, k+1, \ldots, n+k-3, 0)$ with $k \in \mathbb{N}^*$. We suppose that $k \leq n-4$.

Theorem 22.

1) *If* $n = k+4$ and $n \geq 6$ *then* $\mathfrak{g}$ *is isomorphic to the Lie algebra defined by:*
$[X_n, X_1] = X_1$; $[X_n, X_i] = (k+i-2)X_i$, $2 \leq i \leq n-1$; $[X_1, X_i] = X_{i+1}$, $2 \leq i \leq n-2$; $[X_2, X_3] = X_{n-1}$.

2) *If* $n = k+5$ *and* $n \geq 7$ *then* $\mathfrak{g}$ *is isomorphic to the next Lie algebra* :
$[X_n, X_1] = X_1$; $[X_n, X_i] = (k+i-2)X_i$, $2 \leq i \leq n-1$; $[X_1, X_i] = X_{i+1}$. $2 \leq i \leq n-2$; $[X_2, X_3] = X_{n-2}$; $[X_2, X_4] = X_{n-1}$.

Remarks

(i) The proof of this theorem only goes through the resolution of the equations applied to the λ_i deduced from the Jacobi equations.

(ii) This theorem gives a start to the classification of rigid Lie algebras whose nilradical is filiform and of codimension 1. The case of a codimension greater or equal to 2 has been deduced in the previous section. The case of the codimension 0 (of rigid filiform algebras) has been studied by Carles. In an unpublished work, Carles demonstrated that such algebras do not exist (see also [GA]).

(iii) As for the general classification of rigid Lie algebras whose nilradicals are of codimension 1, it is necessary to examine only the roots system $(1, k, k+1, ..., n+k-2, 0)$. As k is an integer, the most natural process is to vary this parameter k in the finite interval determinated by the above-mentioned theorem.

VII.6.2. Case $k = 1$

A direct calculation permits us to demonstrate that rigid Lie algebras associated to the system $(1, 1, 2, ..., n-2)$ do not exist.

VII.6.3. Case $k = 2$

These algebras have been partially studied by Bratzlavsky and Carles. In [BR2], Bratzlavsky considers the nilpotent Lie algebras $\mathfrak{n}$ defined by $[X_i, X_j] = a_{ij} X_{i+j}$ with $i+j < n-1 = \dim \mathfrak{n}$ and supposes the following conditions : $a_{1j} \neq 0$ for $2 \leq j \leq n-3$ and $a_{23} \neq 0$. Then he constructs the solvable algebras of dimension n whose nilradical is $\mathfrak{n}$ by putting $[X, X_i] = iX_i$ for $i = 1, ..., n-1$. These algebras are decomposable by construction ($\mathfrak{g} = \mathfrak{n} \oplus \mathbb{C}X$) and are complete : $H^0(\mathfrak{g}, \mathfrak{g}) = H^1(\mathfrak{g}, \mathfrak{g}) = \{0\}$. Carles shows that, for $n \geq 12$, rigids exists with $H^2(\mathfrak{g}, \mathfrak{g}) \neq \{0\}$. These algebras, in the terminology of the previous section, correspond to the system $(1, 2, 2, ..., n-1, 0)$. We want to generalize this work.

Let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t}$ be a solvable decomposable Lie algebra of dimension n such that its nilradical is of 1-dimensional, and admitting a vector X whose eigenvalues of ad X are $(1, 2, 3, ..., n-1, 0)$.

Theorem 23.

(i) *If* $n \leq 5$, $\mathfrak{g}$ *is not rigid.*

(ii) *If* $n = 6$ *and if* $\mathfrak{g}$ *is rigid, so it is isomorphic to the next Lie algebra*

$$[X, X_i] = iX_i \quad i = 1, ..., 5,$$
$$[X_1, X_i] = X_{i+1} \quad i = 2, 3, 4,$$
$$[X_2, X_3] = X_5 .$$

(iii) *If* $n = 7$ *and if* $\mathfrak{g}$ *is rigid, it is isomorphic to the next algebra*

$$[X, X_i] = iX_i, \quad 1 \leq i \leq 6,$$
$$[X_1, X_i] = X_{i+1}, \quad 2 \leq i \leq 5,$$
$$[X_2, X_i] = X_{i+2}, \quad i = 3, 4.$$

iv) *If* $8 \leq n \leq 10$, *the algebra* $\mathfrak{g}$ *is not rigid.*

Proof. We know that there is a basis $\{X_1, ..., X_{n-1}, X\}$ such that $[X, X_i] = i X_i$ for $1 \leq i \leq n-1$ and $[X_1, X_i] = X_{i+1}$ for $2 \leq i \leq n-2$. This theorem is a consequence of the theorem of the rank.

Now let us suppose $n \geq 11$.

We put $\lambda_i = a_{i+1 i+2}$ where a_{ij} is defined by $[X_i, X_j] = a_{ij} X_{i+j}$. As we have

$$a_{ij} = a_{ij+1} + a_{i+1j},$$

all the constants of structure are defined from λ_i, for $i = 1, ..., p-1$, as soon as $n = 2p+2$ or $n = 2p+1$ about its parity. In particular, all the Jacobi identities are only expressed with λ_i. And these identities $\Sigma [X_i, [X_j, X_k]] = 0$ are trivial as soon as $i+j+k > \dim \mathfrak{n}$. In the next section, we will resolve the system of polynomial Jacobi identities by considering the integer $i+j+k$ as a parameter. From this, we will say that the integer $\rho = i+j+k$ is the *weight* of the identity $\Sigma [X_i, [X_j, X_k]] = 0$.

Lemma. *The Jacobi identities of weight* ρ *are the same for all the Lie algebras verifying the given hypothesis as soon as* $\dim \mathfrak{n} \geq p$.

Although its demonstration is trivial, this lemma is very important. It will permit us to classify the rigid laws by extension. Resolution of the system of the Jacobi equations in dimension n will depend on the resolution in a dimension inferior to n.

Jacobi equations with weight less than 12

$\rho \leq 8$: no relations

$\rho = 9$: $-2\lambda_1\lambda_3 + 3\lambda_2^2 - \lambda_2\lambda_3 = 0$

$\rho = 10$: mo more relations

$\rho = 11$: $2\lambda_1\lambda_4 + \lambda_2\left(-4\lambda_3 - \lambda_4\right) + \lambda_3\left(6\lambda_3 - \lambda_4\right) = 0$

$\rho = 12$: $-4\lambda_3^2 + 3\lambda_3\lambda_4 + 3\lambda_2\lambda_4 = 0$.

Resolutions. There is a trivial solution corresponding to $\lambda_i = 0$. From the rank theorem, the corresponding Lie algebras are not rigid. So we suppose that one of the constants λ_i is nonnull and by a change of basis, this constant can be taken as being equal to 1. If n = 11 or 12, the Jacobi system is only reduced to equations with weight less or equal to 11. It is an algebraic system with 3 variables (one of the λ_i is equal to 1) and 2 equations. There is an undetermined parameter λ_i. As these λ_i are parameters of perturbation, the corresponding laws are not rigid.

The resolution of the system n = 13 (ρ = 12) gives the next result:

Theorem 24. *The only rigid Lie algebras of dimension 13 verifying the hypothesis (H) are isomorphic to one of the following algebras*

i) $\mathfrak{g}^1_{13}$: $[X, X_i] = iX_i \quad 1 \leq i \leq 12$; $[X_1, X_i] = X_{i+1} \quad 2 \leq i \leq 11$;

$[X_2, X_i] = X_{i+2} \quad 3 \leq i \leq 10$.

ii) $\mathfrak{g}^2_{13}$: $[X, X_i] = iX_i \quad 1 \le i \le 12 \;;\; [X_1, X_i] = X_{i+1} \quad 2 \le i \le 11 \;;$

$[X_2, X_3] = X_5 \;;\; [X_2, X_4] = X_6 \;;\; [X_2, X_5] = 9/10X_7 \;;\; [X_2, X_6] = 4/5X_8 \;;$

$[X_2, X_7] = 5/7X_9 \;;\; [X_2, X_8] = 9/14X_{10} \;;\; [X_2, X_9] = 7/12X_{11} \;;$

$[X_2, X_{10}] = 8/15X_{12} \;;\; [X_3, X_4] = 1/10X_7 \;;\; [X_3, X_5] = 1/10X_8 \;;$

$[X_3, X_6] = 3/35X_9 \;;\; [X_3, X_7] = 1/14X_{10} \;;\; [X_3, X_8] = 5/84X_{11} \;;$

$[X_3, X_9] = 1/20X_{12} \;;\; [X_4, X_5] = 1/70X_9 \;;\; [X_4, X_6] = 1/70X_{10} \;;$

$[X_4, X_7] = 1/84X_{11} \;;\; [X_4, X_8] = 1/105X_{12} \;;\; [X_5, X_6] = 1/420X_{11} \;;$

$[X_5, X_7] = 1/420X_{12}$.

iii) $\mathfrak{g}^3_{13}$: $[X, X_i] = iX_i \quad 1 \le i \le 12 \;;\; [X_1, X_i] = X_{i+1} \quad 2 \le i \le 11 \;;$

$[X_2, X_9] = -X_{11} \;;\; [X_2, X_{10}] = -4X_{12} \;;\; [X_3, X_8] = X_{11} \;;\; [X_3, X_9] = 3X_{12} \;;$

$[X_4, X_7] = -X_{11} \;;\; [X_4, X_8] = -2X_{12} \;;\; [X_5, X_6] = X_{11} \;;\; [X_5, X_7] = X_{12}$.

These three Lie algebras are two by two nonisomorphic and verify :

$$H^2(\mathfrak{g}^1_{13}, \mathfrak{g}^1_{13}) = H^2(\mathfrak{g}^2_{13}, \mathfrak{g}^2_{13}) = \mathbb{C} \quad \text{and} \quad H^2(\mathfrak{g}^3_{13}, \mathfrak{g}^3_{13}) = \{0\} .$$

Proof. If $\lambda_1 \neq 0$ then we can take $\lambda_1 = 1$ and the solutions of the system are :
$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (1,0,0,0)$ which corresponds to the algebra $\mathfrak{g}^1_{13}$
$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (1,1/10,1/70,1/420)$ which corresponds to the algebra $\mathfrak{g}^2_{13}$
If $\lambda_1 = 0$, then $\lambda_2 = \lambda_3 = 0$ and the solutions of the system are reduced to :
$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0, 0, 0, 0)$ and $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0, 0, 0, 1)$. The first system of solution does not correspond to any one rigid algebra, the second one to the algebra $\mathfrak{g}^3_{13}$.
The second cohomology group is determined from the equations in λ_i. The tangent plane on a given point of the variety imbedded in $\mathbb{C}^4$, parametrized by λ_i and whose equations are given by the Jacobi polynomials, is defined by the linear part of these equations after a translation of the origin to given point. For example, the tangent plane to the point $(1, 0, 0, 0)$ is determined by putting $\alpha_1 = \lambda_1 - 1$, $\alpha_i = \lambda_i$, $i = 2, 3, 4$.

The equations of Jacobi are written :

$$-2\alpha_3 + (2^\circ \text{ degree}) = 0\,,$$
$$2\alpha_4 + (2^\circ \text{ degree}) = 0\,,$$
$$-2\alpha_4 + (2^\circ \text{ degree}) = 0\,.$$

The Zariski tangent plane is of codimension 2, and the orbit of the law associated to (1, 0, 0, 0) is of dimension 1. This corresponds with the straight line (a, 0, 0, 0). We deduce $\dim(H^2(\mathfrak{g}^1{}_{13}, \mathfrak{g}^1{}_{13})) = 1$.

Jacobi equations with weight less than 14. Preparation of the induction

Jacobi equations with weight $\rho \leq 14$ are parametrized by $(\lambda_1, ..., \lambda_5)$, ; also these parameters are solutions of the equations of weight $\rho \leq 12$. These last ones being solved, we will write only the simplified equations of weight 13 and 14 by considering the solutions found.

If $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (1, 0, 0, 0)$, so $\lambda_5 = 0$.
If $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (1, 1/10, 1/90, 1/420)$, then $\lambda_5 = 1/42$.
If $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0, 0, 0, \lambda_4)$, the equations of weight $\rho = 13$ are reduced to :

$$-\lambda_4(-10\lambda_4 + \lambda_5) = 0\,.$$

Then the solutions are $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (0, 0, 0, 0, 0)$ which corresponds with the nonrigid algebra
$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (0, 0, 0, 0, 1)$ which corresponds to another rigid nonisomorphic algebra which is noted $\mathfrak{g}^3{}_{14}$,

Remark. The hypotheses made by Carles and Bratzlavsky correspond in the above-mentioned study to the case $\lambda_1 = 1$.

The study of Jacobi equations of weight 14, by the previous remark, is reduced to the study of the case $\lambda_1 = 0$; this implies $\lambda_2 = \lambda_3 = 0$. These equations are written :

$$\lambda_4 \left(70\lambda_4 - 19\lambda_5\right) = 0 ,$$
$$\lambda_4 \left(- 20\lambda_4 + 5\lambda_5\right) = 0 ,$$
$$\lambda_4 \left(10\lambda_4 - 4\lambda_5\right) = 0 .$$

Then $\lambda_4 = 0$ and the only solution $\left(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\right) = (0, 0, 0, 0, 1)$ defines the rigid Lie algebra $\mathfrak{g}^3{}_{15}$.

Conclusion. *The 14-dimensional rigid Lie algebras are isomorphic to one of the four following algebras* :

$$\mathfrak{g}^1{}_{14} = t \oplus f_{13} \ , \ \mathfrak{g}^2{}_{14} = t \oplus w_{13} \ , \ \mathfrak{g}^3{}_{14} \ \textit{and} \ \mathfrak{g}^4{}_{15} \ .$$

The 15-dimensional rigid Lie algebras are isomorphic to one of the three following algebras :

$$\mathfrak{g}^1{}_{15} = t \oplus f_{14} \ , \ \mathfrak{g}^2{}_{15} = t \oplus w_{14} \ \textit{and} \ \mathfrak{g}^3{}_{15} \ .$$

All these algebras are two by two nonisomorphic.

The theorem of classification

The previous process shows how to pass from classification in dimension n to classification in dimension n+1 by extension of the Jacobi system. The initialization of this induction is made in dimension 13 and 14.

Theorem 25. *Every solvable rigid Lie algebra* $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t}$ *such that the nilradical* $\mathfrak{n}$ *is filiform and of codimension* 1 *and admitting a regular vector* X *whose roots are equal to* $(1, 2, 3, ..., n, 0)$, $n \geq 12$, *is isomorphic to one of the following Lie algebras* :

(i) $\mathfrak{g}^1{}_{n+1} = t \oplus f_n \quad [X , X_i] = iX_i \quad 1 \leq i \leq n \ ; \ [X_1 , X_i] = X_{i+1} \quad 2 \leq i \leq n-1$,

$$[X_2, X_i] = X_{i+2} \quad 3 \leq i \leq n-2 \ ;$$

(ii) $\mathfrak{g}^2_{n+1} = t \oplus w_n \quad [X, X_i] = iX_i \quad 1 \leq i \leq n \ ; \ [X_i, X_j] = (i-j)X_{i+j} \quad i+j \leq n \ ;$

(iii) *case* $n = 2p + 1$

$\mathfrak{g}^3_{2p+2}$ *associated to* $(\lambda_1, ..., \lambda_{p-1}) = (0, ..., 0, 1)$,

$\mathfrak{g}^4_{2p+2}$ *associated to* $(\lambda_1, ..., \lambda_{p-1}) = (0, ..., 0, 1, (p-1)(p-2)/2)$;

(iii) *case* $n = 2p + 2$,

$\mathfrak{g}^3_{2p+2}$ *associated to* $(\lambda_1, ..., \lambda_{p-1}) = (0, 0, ..., 0, 1)$.

All of these algebras are two by two nonisomorphic.

Proof. We suppose that $\lambda_1 = 0$ and make an induction on $n = \dim \mathfrak{n}$.

1st case: $n = 2p$. We suppose that the Jacobi system only admits as a solution $(\lambda_1,..., \lambda_{p-2}) = (0,..., 0, 1)$ or $(\lambda_1,..., \lambda_{p-2}) = (0,..., 0, 0)$. This is the only rigid Lie algebra corresponding to $(\lambda_1,..., \lambda_{p-2}) = (0,..., 0, 1)$. Then, if $n=2p+1$, the solutions $(\lambda_1,...,\lambda_{p-2},\lambda_{p-1})$ of the Jacobi identities must verify $\lambda_1 = ... = \lambda_{p-3} = 0$. The equations of weight $\rho = 2p+1$ are reduced to $\lambda_{p-2} [(p-1)(p-2) \lambda_{p-2} - 2 \lambda_{p-1}] = 0$ and this implies $(\lambda_1,..., \lambda_{p-2}, \lambda_{p-1}) = (0,..., 0, 0)$ or $(\lambda_1,..., \lambda_{p-2}, \lambda_{p-1}) = (0,..., 0, 1, (p-2)(p-1)/2)$. The algebras corresponding to the two last solutions are rigid.

2° case : $n = 2p + 1$

The hypothesis of induction is : the only solutions of the Jacobi equations are $(\lambda_1,..., \lambda_{p-2}, \lambda_{p-1}) = (0,..., 0, 0)$ or $(0, 0,..., 1)$ or $(\lambda_1,..., \lambda_{p-2}, \lambda_{p-1}) = (0,..., 0, 1, (p-2)(p-1)/2)$. Only the algebras corresponding to the solutions $(\lambda_1,..., \lambda_{p-2}, \lambda_{p-1}) = (0, 0,..., 1)$ and $(\lambda_1,..., \lambda_{p-2}, \lambda_{p-1}) = (0,..., 0, 1, (p-2)(p-1)/2)$ are rigid. Let us take $n = 2p+2$. The equations of weight $\rho = 2p+1$ are reduced to :

$$\lambda_{p-2}\big((p-1)(p-2)\lambda_{p-2} - 2\lambda_{p-1}\big) = 0$$

$$\lambda_{p-2}\left(\sum_{i=1}^{2p-1} (p-i)(p-i-1)\lambda_{p-2} - 2(p-1)\lambda_{p-1}\right) = 0 ,$$

which gives $\lambda_{p-2} = 0$.

VII.7 Classification of solvable rigid laws in small dimension

We can directly verify that there is no nilpotent rigid Lie algebra of dimension less or equal to 8. The approach developed in the previous section permits us to construct all of the solvable rigid Lie algebras, as soon as the dimension is no greater than 8. In fact, we can determine the other rigid Lie algebras ; the only obstacle we can meet comes from the large number of nonisomorphic rigid algebras which appear.

Notation : In the next list, the regular vector is noted X and the eigenvectors associated to the eigenvalue $\lambda_i = i$ are noted Y_i , Y'_i , ...

Dimension 2

$$\mu_2^1(X , Y_1) = Y_1$$

Dimension 4

$$\mu_4^1 = (\mu_2^1)^2$$

Dimension 5

$\mu_5^1 (X , Y_0) = 0 ,$

$\mu_5^1 (X , Y_1) = Y_1 ,$ $\mu_5^1 (Y_0 , Y'_1) = Y'_1 ,$ $\mu_5^1 (Y_1 , Y'_1) = Y_2$,

$\mu_5^1 (X , Y'_1) = Y'_1 ,$ $\mu_5^1(Y_0 , Y_2) = Y_2 ,$ $\mu_5^1(X , Y_2) = 2Y_2 .$

Dimension 6

$\mu_6^1 = (\mu_2^1)^3 ,$

$\mu_6^2(X, Y_0) = 0 ,\quad \mu_6^2(Y_0, Y_1) = 0 ,\quad \mu_6^2(Y_1, Y'_1) = Y_2 ,$

$\mu_6^2(X, Y_1) = Y_1 ,\quad \mu_6^2(Y_0, Y'_1) = Y'_1 ,\quad \mu_6^2(Y_1, Y_2) = Y_3 ,$

$\mu_6^2(X, Y'_1) = Y'_1 ,\quad \mu_6^2(Y_0, Y_2) = Y_2 ,$

$\mu_6^2(X, Y_2) = 2Y_2 ,\quad \mu_6^2(Y_0, Y_3) = Y_3 ,$

$\mu_6^2(X, Y_3) = 3Y_3 ,$

$\mu_6^3(X, Y_1) = Y_2 ,\quad \mu_6^3(Y_1, Y_2) = Y_3 ,\quad \mu_6^3(Y_2, Y_3) = Y_5 ,$

$\mu_6^3(X, Y_2) = 2Y_2 ,\quad \mu_6^3(Y_1, Y_3) = Y_4 ,$

$\mu_6^3(X, Y_3) = 3Y_3 ,\quad \mu_6^3(Y_1, Y_4) = Y_5 ,$

$\mu_6^3(X, Y_4) = 4Y_4 ,$

$\mu_6^3(X, Y_5) = 5Y_5 ,$

Dimension 7

$\mu_7^1 = \mu_5^1 \times \mu_2^1 ,$

$\mu_7^2(X, Y_0) = 0 ,\quad \mu_7^2(Y_0, Y'_1) = Y'_1 ,\quad \mu_7^2(Y_1, Y'_1) = Y'_2 ,$

$\mu_7^2(X, Y_1) = Y_1 ,\quad \mu_7^2(Y_0, Y'_2) = Y'_2 ,\quad \mu_7^2(Y_1, Y'_2) = Y_3 ,$

$\mu_7^2(X, Y'_1) = Y'_1 ,\quad \mu_7^2(Y_0, Y_3) = Y_3 ,\quad \mu_7^2(Y'_1, Y_2) = Y_3 ,$

$\mu_7^2(X, Y_2) = 2Y_2 ,$

$\mu_7^2(X, Y'_2) = 2Y'_2 ,$

$\mu_7^2(X, Y_3) = 3Y_3 ,$

$\mu_7^3(X, Y_0) = 0,$

$\mu_7^3(X, Y_1) = Y_1,$

$\mu_7^3(X, Y'_1) = Y'_1,$

$\mu_7^3(X, Y''_1) = Y''_1,$

$\mu_7^3(X, Y_2) = 2Y_2,$

$\mu_7^3(Y_0, Y'_0) = Y'_0,$

$\mu_7^3(Y_0, Y'_1) = Y'_1,$

$\mu_7^3(Y_0, Y''_1) = 2Y''_1,$

$\mu_7^3(Y_0, Y_2) = Y_2,$

$\mu_7^3(Y_1, Y'_0) = Y'_1,$

$\mu_7^3(Y_1, Y'_1) = Y_2,$

$\mu_7^3(Y'_0, Y'_1) = Y''_1,$

$\mu_7^4(X, Y_0) = Y_0,$

$\mu_7^4(X, Y_1) = Y_1,$

$\mu_7^4(X, Y'_1) = Y'_1,$

$\mu_7^4(X, Y_2) = 2Y_2,$

$\mu_7^4(X, Y_3) = 3Y_3,$

$\mu_7^4(X, Y_4) = 4Y_4$

$\mu_7^4(Y_0, Y'_1) = Y'_1,$

$\mu_7^4(Y_0, Y_2) = Y_2,$

$\mu_7^4(Y_0, Y_3) = Y_3,$

$\mu_7^4(Y_0, Y_4) = Y_4,$

$\mu_7^4(Y_1, Y'_1) = Y_2,$

$\mu_7^4(Y_1, Y_2) = Y_3,$

$\mu_7^4(Y_1, Y_3) = Y_4,$

$\mu_7^5(X, Y_1) = Y_1,$

$\mu_7^5(X, Y_1) = Y_1,$

$\mu_7^5(X, Y_3) = 3Y_3,$

$\mu_7^5(X, Y'_3) = 3Y'_3,$

$\mu_7^5(X, Y_4) = 4Y_4,$

$\mu_7^5(X, Y_5) = 5Y_5,$

$\mu_7^5(Y_1, Y_2) = Y_3,$

$\mu_7^5(Y_1, Y_3) = Y_3,$

$\mu_7^5(Y_1, Y_4) = Y_5,$

$\mu_7^5(Y_2, Y_3) = Y_7,$

$\mu_7^5(Y_2, Y'_3) = Y_7,$

$\mu_7^6(X, Y_1) = Y_1,$

$\mu_7^6(X, Y_3) = 3Y_3,$

$\mu_7^6(X, Y_4) = 4Y_4,$

$\mu_7^6(X, Y_5) = 5Y_5,$

$\mu_7^6(X, Y_6) = 6Y_6,$

$\mu_7^6(X, Y_7) = 7Y_7.$

$\mu_7^6(Y_1, Y_3) = Y_4,$

$\mu_7^6(Y_1, Y_4) = Y_5,$

$\mu_7^6(Y_1, Y_5) = Y_6,$

$\mu_7^6(Y_1, Y_6) = Y_7,$

$\mu_7^6(Y_3, Y_4) = Y_7,$

VIII. STUDY OF THE VARIETY $\mathcal{L}^n$ IN THE NEIGHBORHOOD OF THE NILRADICAL OF A PARABOLIC SUBALGEBRA OF A SAMPLE COMPLEX LIE ALGEBRA

We use here the notations of Chapter 4, where we have given a description of the space $H^2(\mathfrak{n},\mathfrak{n})$ in decomposing it in to the sum of the two subspaces $H^1(\mathfrak{n},\mathfrak{g}/\mathfrak{n})$ and $H^2_{fund}(\mathfrak{n},\mathfrak{n})$.

Let us first study the integrability of certain infinitesimal deformations which belong to $H^1(\mathfrak{n},\mathfrak{g}/\mathfrak{n})$.

Let $Gr(\mathfrak{g},\mathfrak{n})$ be the Grasmannian of the subspaces of $\mathfrak{g}$ of dimension n, where $n = \dim \mathfrak{n}$. The subset of $Gr(\mathfrak{g},\mathfrak{n})$ formed by the Lie subalgebras of $\mathfrak{g}$ is an algebraic variety. Let us denote it by M. The tangent space to the variety M at the point $\mathfrak{n}$ is identified by a subspace of the space $Z^1(\mathfrak{n},\mathfrak{g}/\mathfrak{n})$. This gives us an idea for finding the prolongation of an infinitesimal deformation $\varphi \in Z^1(\mathfrak{n},\mathfrak{g}/\mathfrak{n})$ as a family of Lie subalgebras of $\mathfrak{g}$.

1. Let p be the nilindex of $\mathfrak{n}$ and $n = \dim \mathfrak{n}$.

Lemma 1. *Let* $\alpha \in \Delta_1$, *such that* $R_\alpha = \{ \gamma \in \Delta_0 \mid \alpha + \gamma \in \Delta \} = \emptyset$ *and let* $\mathfrak{m} = \Sigma\, \mathbb{C}\, X_\varphi$ *for all roots of* $\Delta' - \{\alpha\}$ *(it is clear that* $\mathfrak{m}$ *is an ideal of codimension* 1 *of* $\mathfrak{n}$*). Then the root vector* $X_{-\alpha}$ *normalizes* $\mathfrak{m}$, *that is* $[X_{-\alpha}, \mathfrak{m}] \subset \mathfrak{m}$.

Proof. Let X_γ be a root vector of $\mathfrak{m}$. Let us consider the difference $\gamma-\alpha$. As $R_\alpha = \emptyset$ and $\gamma - \alpha \notin \Delta_0$, two cases are possible :

(a) $\gamma - \alpha \notin \Delta$,

(b) $\gamma - \alpha \in \Delta_i$ where $i \geq 1$.

In the case (a), we have $[X_{-\alpha},X_\gamma] = 0 \in \mathfrak{m}$.

In the case (b), we have $[X_{-\alpha},X_\gamma] = c_{-\alpha,\gamma} X_{-\alpha+\gamma} \in \mathfrak{m}$ because $\gamma-\alpha \neq \alpha$. This gives the following lemma.

Lemma 2. *Every infinitesimal deformation* φ *of the form* $df_{\omega i}$ *with* $\omega_i \in \Omega_i$, $i = 0, 1$ *(see Chapter 4), is linearly integrable (it means* $\mu_0 + \psi$ *verifies the Jacobi conditions, where* μ_0 *is the law of the Lie algebra* $\mathfrak{n}$*).*

Proof. Let $\varphi = df_{\omega_0}$, where $\omega_0 = (\alpha, h_\theta) \in \Omega_0$, and we consider the ideal $\mathfrak{m}$ of codimension 1 of the Lie algebra $\mathfrak{n}$ (see lemma 1). We put

$$Y_t = X_\alpha - t[X_\beta, X_{-\beta}],$$

where $t \in \mathbb{C}$. It is clear that Y_t, normalizes $\mathfrak{m}$. Then the endomorphism ad $Y_t|_{\mathfrak{m}}$ is a derivation of $\mathfrak{m}$. This derivation defines an extension $\mathfrak{n}_t$ of the Lie algebra $\mathfrak{m}$, which is the semidirect product $\mathbb{C}.Y_t \oplus \mathfrak{m}$ with

$$[Y_t, x] = \text{ad}\, Y_t(x).$$

Consider a law μ_t of Lie algebra on the space $\mathfrak{n}$ defined by

$$\mu_t{}^1 = \mu_0 + t\, df_{\omega_0}.$$

We denote $\mathfrak{n}^1{}_t$ as the obtained algebra.

The mapping, which associates to $(b.\varphi_t, X)$ the vector $b.X_\alpha + X$, where $b \in \mathbb{C}$, is an isomorphism of $\mathfrak{n}_t$ on $\mathfrak{n}^1{}_t$ and $\mu^1{}_t$ verifies the Jacobi identities. The family of Lie algebra $\mathfrak{n}^1{}_t$ constructed from this is a linear deformation whose differential is df_{ω_0}.

Of the same manner, we show that all of the infinitesimal deformation df_{ω_1}, where $\omega_1 \in \Omega_1$, is also a linear deformation. This proves the lemma.

Lemma 3. *Every infinitesimal deformation* $\varphi = df_{\omega_2}$, *where* $\omega_2 \in \Omega_2$ *verifies* $\varphi \circ \varphi = 0$. *The proof of this lemma is based on Lemma 1 and is analogous to the proof of Lemma 2.*

Lemma 4. *Let* $\alpha, \beta \in S$, *with* $\alpha + \beta \in \Delta$, $R_\beta = \emptyset$ *and let* $\Delta(\alpha,\beta)$ *be a subset of* Δ' *formed by the roots, which can be expressed by* α *and* β. *Then the subspace*

$$\mathfrak{m} = \sum_{\alpha \in \Delta' - \Delta(\alpha,\beta)} \mathfrak{g}_\alpha$$

of the Lie algebra $\mathfrak{n}$ *is an ideal of* $\mathfrak{n}$. *Further, the vector* $X_{-\beta}$ *of the Lie algebra* $\mathfrak{g}$ *normalizes* $\mathfrak{m}$.

Proof. It is clear that the subset $\Delta' - \Delta(\alpha,\beta)$ of Δ' is closed with respect to the addition of the roots. It is also clear that $\tau + \nu \in \Delta' - \Delta(\alpha,\beta)$, if $\tau \in \Delta'$, $\nu \in \Delta' - \Delta(\alpha,\beta)$. Then $\mathfrak{m}$ is an ideal of $\mathfrak{n}$. Suppose that $X_{-\beta}$ does not normalize the ideal $\mathfrak{m}$. Then there is a root vector X_ν so $\nu - \beta \in \Delta_0 \cup \Delta(\alpha,\beta)$. We remark that the case $\nu - \beta \in \Delta_0$ is excluded, because $R_\beta = \emptyset$. The case $\nu - \beta \in \Delta(\alpha,\beta)$ is also excluded, since $\nu \in \Delta' - \Delta(\alpha,\beta)$.

Lemma 5. *Every infinitesimal deformation* $\varphi = df_{\omega_4}$ *with* $\alpha, \beta \in S_1$, $\alpha + 2\beta \notin \Delta$ *is linearly integrable, i.e.* $\varphi \circ \varphi = 0$.

Proof. Let $\varphi = df_{\omega_4}$, where $\omega_4 = (\alpha,-\beta)$, and $\alpha, \beta \in S_1$, $\alpha + \beta \in \Delta$, $\alpha + 2\beta \notin \Delta$, and let $\Delta(\alpha, \beta)$ be the set of positive roots which can be expressed using only α and β. For $t \in \mathbb{C}$ fixed, let us consider the subspace $\mathfrak{a}_t \subset \mathfrak{g}$ formed by elements of the form

$$\sum_{\gamma \in \Delta(\alpha,\beta)} a_\gamma X_\gamma - t.a_\alpha X_{-\beta} + \frac{t.a_{\alpha+\beta}}{N_{\alpha,\beta}} . [X_{-\beta}, X_\beta],$$

where $a_\gamma \in \mathbb{C}$ for $\gamma \in \Delta$. We can easily show, that $\mathfrak{a}_t$ is a subalgebra of $\mathfrak{g}$. Let

$$\mathfrak{m} = \sum_{\alpha \in \Delta' - \Delta(\alpha,\beta)} \mathfrak{g}_\alpha .$$

Then $\mathfrak{m}$ is an ideal of $\mathfrak{n}$ and from Lemma 4, an arbitrary element $X \in \mathfrak{a}_t$ normalizes the ideal $\mathfrak{m}$. So $\mathrm{ad}X|_{\mathfrak{m}}$ is a derivation of $\mathfrak{m}$. Denote $\mathfrak{n}_t$ the semi-direct sum $\mathfrak{m} \oplus \mathfrak{a}_t$ defined by $[X,Y] = \mathrm{ad}\,X(Y)$, where $X \in \mathfrak{a}_t$, $Y \in \mathfrak{m}$. Consider an algebraic law on the space $\mathfrak{n}$ defined by :

$$\mu^4{}_t = \mu_0 + t\, df_{\omega_1}.$$

We designate the obtained algebra by $\mathfrak{n}^4{}_t$.

Let X be an element of $\mathfrak{a}_t$ and let $Y \in \mathfrak{m}$. Consider the mapping f of $\mathfrak{n}_t$ on $\mathfrak{n}^4{}_t$, defined by :

$$f(x,y) = \sum_{y \in \Delta(\alpha,\beta)} a_\gamma . e_\gamma + y .$$

It is an isomorphism and the law $\mu^4{}_t$ verifies the Jacobi identities ; so $\mathfrak{n}^4{}_t$ is a linear deformation of $\mathfrak{n}$. This proves the lemma.

2. Let W_2 be the subset of the Weyl group W defined in Chapter 4. An arbitrary element w of W_2 is writen $w = s_\alpha . s_\beta$, where α, $\beta \in S$, s_α and s_β are the reflexions (symmetries) with respect to the elements α and β.
Let be $w(\Delta^-) \cap \Delta^+ = \{\gamma_1,\gamma_2\}$. Let us denote f_w the cocycle definited by :

$$f_w(X_{\gamma_1},X_{\gamma_2}) = X_{w(\delta)} ,$$

where δ is the maximal root.

Lemma 6. *Let* $S_1 = S$ *(*$\mathfrak{n}$ *is the nilradical of a Borel subalgebra of* $\mathfrak{g}$*). Then, every cocycle (infinitesimal deformation)* $f_w \in Z^2(\mathfrak{n},\mathfrak{n})$, *where* $w \in W_2$ *defines a linear deformation of* $\mathfrak{n}$ *(that is, it verifies the condition* $f_w \circ f_w = 0$).

Proof. Let $w = s_\alpha . s_\beta$, where α, $\beta \in S$. Let us consider the ideal

$$m = \sum_{\gamma \in \Delta^+ \setminus \{\alpha\}} \mathfrak{g}_\gamma$$

It is of codimension 1 ideal of the Lie algebra n. Two cases are possible :

(a) $\alpha + \beta \in \Delta$,

(b) $\alpha + \beta \notin \Delta$.

Suppose that $\alpha + \beta \in \Delta$. It is easy to see that the endomorphism h of m defined by

$$h(X_{\alpha+\beta}) = X_{w(\delta)},$$

is a derivation of the Lie algebra m (m is the nilradical of the parabolic subalgebra defined by $S_1 = S \setminus \{\alpha\}$ and we can use the description of Der m studied in Chapter 4). Let t be a fixed number of $\mathbb{C}$. We put

$$D = t.h + \mathrm{ad}\, X_\alpha$$

We have $D \in$ Der m. The semidirect product $\mathbb{C}.D \oplus m$, defined by

$$[D,X] = D(X) = t.h(X) + \mathrm{ad}\, X_\alpha(X),$$

where $X \in m$defines a Lie algebra denoted n_t. Consider an algebra structure, noted n^ω_t , on the linear space underlying the n associated to the product

$$[X,Y]_t = [X,Y]_n + t.f_\omega(X,Y).$$

The mapping which associates to (b.D,X) the vector $b.X_\alpha + X$, is an isomorphism of n_t on n^w_t. Then the law $[X,Y]_t$ verifies the Jacobi identities. The family of Lie algebras n^w_t obtaines as this gives the sought-after linear deformation.

Now suppose that : $\alpha + \beta \notin \Delta$. It is clear that at least one of the simple roots α, β, for example α, is not singular. We consider the mapping h, defined by $h(X_\beta) = X_{\omega(\delta)}$. We have $h \in$ Der m (see the description of the algebra of derivation of a nilradical of a parabolic subalgebra given in Chapter 4). In the same manner as the previous case, we can show that the deformation is linear.

This gives the lemma.

Proposition 19. *Let* $\mathfrak{n}$ *be the nilradical of a Borel subalgebra of a simple Lie algebra* $\mathfrak{g}$ *of rank* > 1 *(it means* $S_1 = S$, $\mathrm{card}(S) > 1$*). Then, there is a system of cocycles (infinitesimal deformations)*

$$\{ \varphi_i \mid i = 1, ..., \dim H^2(\mathfrak{n},\mathfrak{n}) \}$$

with $\varphi_i \circ \varphi_i = 0$ *such that the system formed by the cohomology classes of these cocycles forms a basis of the space* $H^2(\mathfrak{n},\mathfrak{n})$.

Proof. We choose a basis of the space $H^1(\mathfrak{n},\mathfrak{g}/\mathfrak{n})$ formed by the classes of cohomology of the next cocycles (this is possible from the description of the space $H^1(\mathfrak{n},\mathfrak{g}/\mathfrak{n})$ given in Chapter 4) :

$$df_{\omega_0}\ , \text{ where } \omega_0 = (\alpha,h_\theta)\ , \ \alpha,\theta \in S\ ,$$

$$df_{\omega_2}\ , \text{ where } \omega_2 = (\alpha,-\alpha)\ , \ \alpha \in S\ ,$$

$$df_{\omega_4}\ , \text{ where } \omega_4 = (\alpha,-\beta)\ , \ \alpha,\beta \in S\ , \ \alpha+\beta \in \Delta\ , \ \alpha+2\beta \notin \Delta\ .$$

We can suppose that the condition $\alpha + 2\beta \notin \Delta$ is fulfilled, because the element f_{ω_4} of $H^1(\mathfrak{n},\mathfrak{g}/\mathfrak{n})$ is cohomologous to $f_{\omega'_4}$, where $\omega'_4 = (\beta, -\alpha)$. Lemmas 2, 3, and 5 show that all of these cocycles (infinitesimal deformations) are linearly integrable. From the description of the space $H^2_{\mathrm{fund}}(\mathfrak{n},\mathfrak{n})$ (see Chapter 4), every element z of this space is expressed by

$$z = \sum_{w \in W_2} a_w f_w\ ,$$

where $a_w \in \mathbb{C}$. From Lemma 6, every cocycle f_w verifies $f_w \circ f_w = 0$. This means that it is linearly integrable. Then we can choose a basis of the space $H^2_{\mathrm{fund}}(\mathfrak{n},\mathfrak{n})$ formed by the linearly integrable cocycles. This gives the proposition.

Theorem 26. *Let* $\mathfrak{n}$ *the nilradical of a Borel subalgebra of* $\mathfrak{g}$*. Then the tangent space to the scheme* L_n *coincides to the tangent space to the corresponding reduced scheme at the point* $\mathfrak{n}$.

The theorem proceeds from Proposition 19.

3. Here we consider (nonlinearly) a deformation of the nilradical $\mathfrak{n}$ of a Borel subalgebra of $\mathfrak{g} = sl(r+1,\mathbb{C})$ on a simple Lie subalgebra of $\mathfrak{g}$. Let $\mathfrak{g} = sl(r+1,\mathbb{C})$ and let $S = \{\alpha_1, ..., \alpha_r\}$, where $r > 1$, the corresponding system of the simple roots. We put

$$z = d\,(f_{(\alpha_1,-\alpha_1)} + f_{(\alpha_2,-\alpha_2)} + ... + f_{(\alpha_r,-\alpha_r)})\ ,$$

where $(\alpha_i,-\alpha_i) \in \Omega_2$ (see Chapter 4) .

Let $t \in \mathbb{C}$. We consider the subspace $\mathfrak{n}_t$ of the space $\mathfrak{g}$, formed by all the matrix (a_{ij}) with $a_{ji} = -\ t^{j-i}\ .\ a_{ij}$ for $1 \le i < j \le r+1$. Then $\mathfrak{n}_t$ is a Lie subalgebra of $\mathfrak{g}$ isomorphic to the simple algebra $so(r+1,\mathbb{C}) \subset \mathfrak{g}$, formed by the antisymetric matrix. Then we obtain a family $\mathfrak{n}_t$ of the subalgebras of $\mathfrak{g}$ verifying $\mathfrak{n}_0 = \mathfrak{n}$, $\mathfrak{n}_t \cong so(r+1,\mathbb{C})$ for $t \neq 0$. This family gives us a deformation of $\mathfrak{n}$ such that its linear part is z.

CHAPTER 6

VARIETY OF NILPOTENT LIE ALGEBRAS

I. THE TANGENT SPACE OF THE VARIETY OF NILPOTENT LIE ALGEBRAS

Let $\mathcal{L}^n$ be the variety of n-dimensional complex Lie algebras (see Chapter 5). We denote by $\mathcal{N}^n_p$ the subset of $\mathcal{L}^n$ constituted from the nilpotent Lie algebras whose nilindex is less than p. This subset is defined by

$$\mathcal{N}^n_p = \left\{\mu \in \mathcal{L}^n \text{ such that } \mu(x_1,\mu(x_2 ,..., \mu(x_p,x))...) = 0,\ \forall\, x_1 ,..., x_p, x \in \mathbb{C}^n\right\} .$$

It is an algebraic subset of $\mathcal{L}^n$. If we fix a basis of $\mathbb{C}^n$ the laws of Lie algebras are identified with their constants of structure which verify some polynomial relations (see Chapter 5). The nilpotent laws whose nilindex is less or equal to p are also given by polynomial relations. Then $\mathcal{N}^n_p$ is a Zariski closed subset of $\mathcal{L}^n$. We also designate the corresponding affine schema by $\mathcal{N}^n_p$.

The nilindex of a n-dimensional nilpotent Lie algebra does not exceed n−1. Then $\mathcal{N}^n_{n-1}$ contains all the n-dimensional complex nilpotent Lie algebra laws. For simplicity, we note $\mathcal{N}^n_{n-1}$ by $\mathcal{N}^n$.

Let $\mathfrak{g} = (\mathbb{C}^n,\mu_0)$ be a nilpotent Lie algebra. We want to determine the tangent space to the point $\mu_0 \in \mathcal{N}^n$. Suppose that the nilindex of $\mathfrak{g}$ is equal to p. We consider the filtration of $\mathfrak{g}$ given by the descending central sequence : $F_i\mathfrak{g} = C^{i-1}\mathfrak{g}$ and the filtration of $\mathfrak{g}$, viewed as an adjoint $\mathfrak{g}$-module, given by $T_i\mathfrak{g} = C_{p-i+1}\mathfrak{g}$. These filtrations induce filtrations on the spaces of cochains, cocycles, coboundaries and cohomology spaces (we have studied all these spaces in Chapter 3). We follow the notations of this chapter. Let

$$\mu(t) = \mu_0 + t\varphi_1 + t^2\varphi_2 + \ldots$$

be a deformation of μ_0 in the variety $\mathcal{N}^n_q$ with $q \geq p$. We know that $\varphi_1 \in Z^2(\mu_0,\mu_0)$. As $\mu(t)$ is a law with a nilindex equal to q, we have

$$\varphi_1\big(x_1,\mu_0(x_2,\mu_0(\ldots\mu_0(x_{q-1},x_q)\ldots) + \mu_0\big(x_1,\varphi_1\big(x_2,\mu_0\big(x_3,\mu_0(\ldots x_q)\big)\ldots\big)\big)\big)\ldots\big)$$
$$+ \ldots + \mu_0(x_1,\mu_0(x_2,\ldots\mu_0(x_{q-2},\varphi_1(x_{q-1},x_q)\ldots))) = 0 .$$

From this equality, we deduce the existence of a coboundary $df \in B^2(\mu,\mu_0)$ such that the cocycle $\psi_1 = \varphi_1 - df$ verifies $\psi_1(F_i\mathfrak{g},F_j\mathfrak{g}) \subset F_{i+j+p-q}\mathfrak{g}$, that is $\varphi_1 \in F_{p-q}Z^2(\mu_0,\mu_0)$ (in this notation $\mathfrak{g}$ is the Lie algebra associated to μ_0). Then, we have the following proposition

Proposition 1. *Let* $\mathfrak{g} = (\mathbb{C}^n,\mu_0)$ *be a nilpotent Lie algebra of nilindex* p *in the variety* $\mathcal{N}^n_q$ *and let* φ_1 *be in* $Z^2(\mu_0,\mu_0)$. *Then there is a coboundary* $df \in B^2(\mu_0,\mu_0)$ *such that* $\varphi_1 - df = \psi_1$, *where* ψ_1 *is a cocycle satisfying*

$$\psi_1 \in F_{p-q}Z^2(\mu_0,\mu_0) .$$

Corollary. *Let* $\mathfrak{g} = (\mathbb{C}^n,\mu_0)$ *be a nilpotent Lie algebra with a nilindex equal to p. Then the Zariski tangent space in* $\mathfrak{g}$ *to* $\mathcal{N}^n_q$ *is the subspace* W *whose elements are the cocycles* $\varphi \in Z^2(\mu_0,\mu_0)$ *such that their cohomology classes are in* $F_{p-q}H^2(\mu_0,\mu_0)$.

Remark. The space W is the tangent space in μ_0 to the schema $\mathcal{N}^n_q$. If this schema is reduced at the point μ_0, then W coincides with the tangent space in μ_0 to the variety $\mathcal{N}^n{}_q$.

II. ON THE FILIFORM COMPONENTS OF THE VARIETY $\mathcal{N}^n$

In this section, we study some components of the variety $\mathcal{N}^n$. Let $\mathcal{F}^n$ be the subset of $\mathcal{N}^n$ of filiform Lie algebras. As $\mathcal{F}^n = \mathcal{N}^n - \mathcal{N}^n{}_{n-2}$, this subset is a Zariski open subset of $\mathcal{N}^n$. Each component of $\mathcal{F}^n$ determines a component of $\mathcal{N}^n$.

Let $L = (\mathbb{C}^n,\mu_0)$ be a n-dimensional Lie algebra. Recall that the elements of $Z^2(\mu_0,\mu_0)$ can be interpreted as infinitesimal deformations of μ_0. The simplest case corresponds to the case where the deformation is linear, i.e. the deformation has the form $\mu = \mu_0 + \psi$ with $\psi \in Z^2(\mu_0,\mu_0)$. As μ verifies the Jacobi conditions, the cocycle ψ also verifies these conditions, and $\psi \in \mathcal{L}^n$. We will denote by L_ψ the Lie algebra corresponding to $\mu = \mu_0 + \psi$ (that is $L_\psi = (\mathbb{C}^n, \mu_0 + \psi)$).

Now consider the filiform Lie algebra $L_n = (\mathbb{C}^{n+1}, \mu_0)$ defined in the basis $(e_0,e_1,\dots,e_n)$ by

$$\mu_0(e_0, e_i) = e_{i+1}, \ 1 \le i \le n-1 .$$

This Lie algebra has been studied more precisely in Chapter 2.

Let T be the tangent space at the point L_n to the schema $\mathcal{N}^{n+1}$. From Proposition 1 and Corollary 2 (Chapter 4, § II), we can affirm that this space is generated by the cocycles $\psi_{k,3} + \varphi$ with $4 \le s \le k$, $2k+1 \le s$, $\varphi \in B^2(L_n,L_n)$. The study of the variety $\mathcal{N}^{n+1}$ in a neighborhood of the point L_n is traduced in terms of integrability of the elements of T ; we can discover if each infinitesimal deformation of L_n can be prolonged in a deformation.

Lemma 1. *Let* $\psi = \Sigma\, a_{k,s}\psi_{k,s}$ *be in* $F_0H^2(L_n,L_n)$ *with* $a_{k,s} \neq 0$ *for a pair* (k, s) *satisfying* $s = 2k+1 < n$. *Then the infinitesimal deformation* ψ *is not integrable.*

Proof. A cocycle ψ is integrable if and only if the 3-cocycle $\psi \circ \psi$ belongs to $B^3(L_n,L_n)$. Let k_0 be the smallest integer satisfying the hypothesis of the lemma. The description of the space $F_0H^2(L_n,L_n)$ gives us $k_0 \geq 2$. Let us determine the coefficient of e_{2k_0+2} in the expression $(\psi.\psi)(e_1,e_{k_0},e_{k_0+1})$. From the choice of k_0, this coefficient is equal to $(-1)^{k_0-1}\, k_0 a^2_{k_0,3}$ and is different to 0. As the coefficient of e_{2k_0+2} in $\varphi \circ \varphi(e_1,e_{k_0},e_{k_0+1})$ is equal to 0 for all $\varphi \in B^3(L_n,L_n)$, we have $\psi \circ \psi \notin B^3(L_n,L_n)$. Then the infinitesimal deformation ψ is not integrable.

Corollary. *The schema* $\mathcal{N}^{n+1}$ *is not reduced at the point* L_n.

From Lemma 1, the study of $\mathcal{N}^{n+1}$ around the point L_n is brought back to problems of the integrability of infinitesimal deformation φ . These cocycles are combinations of the cocycles $\psi_{k,s}$ with $2k+1 < s < n$ and $\psi_{r,n}$ with $n = 2r+1$ (if n is odd). We denote by X the linear space generated by these cocycles. As these cocycles are linearly independents, every $\psi \in X$ is written

$$\psi = \Sigma\, a_{k-s},\, \psi_{k-s} \ .$$

Let M be the affine algebraic variety in X defined by the relation $\psi \circ \psi = 0$. The decomposition

$$F_0H^2\,(L_n,L_n) = \bigoplus_{i\geq 0} H^2_i\,(L_n,L_n)$$

of the space $F_0H^2(L_n,L_n)$ considered in Chapter 3, gives the decomposition of X :

$$X = \bigoplus_{i\geq 0} X_i$$

with

$X_i = H^2_i(L_n,L_n)$ if $i \geq 1$,

$X_0 = 0$ if n is even,

$X_0 = \mathbb{C}\{\psi_{r,n}\}$ if $n = 2r+1$ is odd.

Let μ_0 be the law of L_n. A filiform law $\mu \in \mathcal{F}^{n+1}$ always admits an adapted basis (see Chapter 2) and then can be described as being of the form $\mu = \mu_0 + \beta$ with $\beta \in F_0C^2(L_n,L_n)$, $\beta(e_0,e_i) = 0$. The equation $(\mu_0+\beta)\circ(\mu_0+\beta) = 0$ gives :

$$\beta \circ \mu_0 + \mu_0 \circ \beta + \beta \circ \beta = 0$$

and $(\beta \circ \mu_0 + \mu_0 \circ \beta + \beta \circ \beta)(e_0, e_i, e_j) = (\beta \circ \mu_0 + \mu_0 \circ \beta)(e_0,e_i,e_j) = 0$.
Then $\beta \circ \mu_0 + \mu_0 \circ \beta = 0$ and $\beta \circ \beta = 0$. In consequence, we have $\beta \in F_0Z^2(L_n,L_n)$ and $(\mathbb{C}^n,\mu) = (L_n)_\beta$. As $\dim B^2(L_n,L_n) = n^2$, a basis of this space is given by the cocycles φ_{ij} defined by $\varphi_{i,j}(e_0,e_i) = e_j$. Then $\beta \in M$.

Lemma 2. *Every filiform Lie algebra law* $\mu \in \mathcal{F}^{n+1}$ *is isomorphic to a law* $\mu_0 + \beta$ *obtained by a linear deformation of* μ_0 *associated to an element* $\beta \in M$.

The following proposition reduces the study of filiform components of $\mathcal{N}^n$ to the study of the components of M.

Proposition 2. *Let* C *be an irreducible component of* M. *Then* $GL(\mathbb{C}^{n+1})(\mu_0+C)$ *is an irreducible component of* $\mathcal{N}^{n+1}$. *Its dimension is equal to* $n^2 + \dim C$, *and the mapping which associates* C *to its image in* $\mathcal{N}^{n+1}$ *is bijective in the set of irreducible components of* $\mathcal{N}^{n+1}$ *meeting the open set* $\mathcal{F}^{n+1}$.

Proof. Let C be an irreducible component of M and C_1 an irreducible component of $\mathcal{N}^{n+1}$ containing the irreducible set $GL(\mathbb{C}^{n+1})(\mu_0+C)$. Then $\mu_0 \in C_1$. Consider the open set U of $\mathcal{N}^{n+1}$ containing μ_0 and the laws μ such that the vectors $\{e_0, e_1, x_{i+1}(\mu) = (\mathrm{ad}_\mu e_0)^i(e_1), 1 \leq i \leq n-1\}$ are linearly independent. Then, we have

$$\mu(e_1, x_2(\mu)) = \sum_{i \geq 3} \alpha_i(\mu) x_i(\mu) ,$$

where the functions $x_i(\mu)$ are rational and defined on U. We put

$$e_1(\mu) = \alpha_3(\mu) e_0 - e_1 ,$$
$$e_{i+1}(\mu) = (ad_\mu(e_0))^i \, e_1(\mu) .$$

Then the elements e_0, $e_1(\mu)$, $e_2(\mu)$, ..., $e_n(\mu)$ determine an adapted basis of the algebra μ and the mapping $\mu \to (e_0, e_1(\mu), \ldots, e_n(\mu))$ is a rational mapping. Then, we have $\mu \circ \mu = \varphi(\mu) \circ (\mu_0 + \beta(\mu))$ where $\varphi(\mu) \in GL(\mathbb{C}^{n+1})$ and $\beta(\mu) \in M$. The mapping $\mu \to \beta(\mu)$ is rational. Thus, $\beta(U \cap C_1)$ is an irreducible set containing C. This implies

$$\beta\big(U \cap C_1\big) = C \quad \text{and} \quad U \cap C_1 \subset GL(\mathbb{C}^{n+1}) \circ (\mu_0 + C),$$

then

$$C_1 = GL(\mathbb{C}^{n+1}) \circ (\mu_0 + C)$$

Conversely, if C_1 is an irreducible component of $\mathcal{N}^{n+1}$ cutting the open set of filiform laws, then C_1 contains μ_0. The same arguments show that if C is an irreducible component containing $\beta(C_1 \cap U)$, then

$$GL(\mathbb{C}^{n+1}) \circ (\mu_0 + C) = C_1$$

Finally, if C is an irreducible component of M, we have

$$\dim\big(GL(\mathbb{C}^{n+1}) \circ (\mu_0 + C)\big) = \dim\big(GL(\mathbb{C}^{n+1}) \circ \mu_0\big) + \dim C =$$
$$= (n+1)^2 - \dim Der(L_n) + \dim C = n^2 + \dim C \ .$$

This gives the proposition.

III. ON THE REDUCIBILITY OF THE VARIETY $\mathcal{N}^n$, $n \geq 12$

In this section, we consider only the case $n \geq 12$. The study of the variety $\mathcal{N}^n$ for $n \leq 11$ will be presented in section VII.

Let $n \geq 12$ and consider the following closed subsets of M :

(1) M_1 is the closed subset of M defined by the relations

$$a_{2,6} = a_{3,8} = \dots = a_{\frac{n-2}{2},n} = 0$$

if n is even, and by the relations

$$a_{2,6} = a_{3,8} = \dots = a_{\frac{n-3}{2},n-1} = a_{\frac{n-1}{2},n} = 0$$

if n is odd.

(2) M_2 is the closed subset of M defined by the relations

$$a_{k,2k+2} = \left(4+\frac{6}{k}\right)a_{k+1,2k+4} \quad , \quad k = 1, 2, \dots, \left[\frac{n-4}{2}\right].$$

If n is odd, we add the relations concerning $a_{\frac{n-1}{2},n}$.

(3) M_3 is the closed subset of M defined by

$$a_{1,4} = a_{2,6} = \dots = a_{\frac{n-4}{2},n-2} = 0 \ ,$$

if n is even, and by

$$a_{1,4} = a_{2,6} = \dots = a_{\frac{n-3}{2},n-1} = 0 \ ,$$

if n is odd.

(4) M_4, which is defined only if n is odd, is the closed set of M given by

$$a_{1,4} = a_{2,6} = \dots = a_{\frac{n-5}{2},n-2} = 0 \ , \quad a_{\frac{n-1}{2},n} = 0 \ .$$

Lemma 1. *The closed subsets* M_1, M_2, M_3 *(and* M_4, *if* n *is odd) are nontrivial subsets of* M, *strictly contained in* M.

Proof. We put

$$\varphi_1 = \psi_{1,4} \ ,$$

$$\varphi_2 = \sum_{k=1}^{(n-2)/2} a_{k,2k+2}\,\psi_{k,2k+2} \quad \text{with} \quad a_{1,4} = 1 \ ,$$

$$\varphi_3 = \psi_{\frac{n-1}{2},n} \ ,$$

$$\varphi_4 = \psi_{\frac{n-3}{2},n-1} \quad \text{(if n is odd)} \ .$$

The existence of the filiform algebras R_n and W_n (Chapter 2) shows that the cocycles φ_1 and φ_2 are linearly integrable, i.e. $\varphi_1 \circ \varphi_1 = 0$ and $\varphi_2 \circ \varphi_2 = 0$. It is obvious that also we have $\varphi_3 \circ \varphi_3 = 0$ and $\varphi_4 \circ \varphi_4 = 0$. Then, $\varphi_i \in M_i$ for $i = 1, 2, 3, 4$. Moreover, $\varphi_i \notin M_j$ if $i \neq j$. This proves the lemma.

Lemma 2. *We have* $M = M_1 \cup M_2 \cup M_3$ *if* n *is even, and* $M = M_1 \cup M_2 \cup M_3 \cup M_4$ *if* n *is odd.*

Proof. Let $\psi \in M$ and consider the equalities

$$(\psi\circ\psi)(e_1,e_2,e_3) = 0, \quad (\psi\circ\psi)(e_1,e_3,e_4) = 0 \ , \quad (\psi\circ\psi)(e_2,e_3,e_4) = 0 \ .$$

By writing that the coefficients of e_8, e_{10} and e_{11} are zero, we obtain the following relations :

$$-3a^2_{2,6} + a_{3,8}\, a_{2,6} + 2\, a_{1,4}\, a_{3,8} = 0 \ ,$$

$$6a^2_{3,8} - 4\, a_{2,6}\, a_{3,8} - a_{3,8}\, a_{4,10} + 2\, a_{1,4}\, a_{4,10} - a_{2,6}\, a_{4,10} = 0 \ ,$$

$$-4\, a^2_{3,8} + 3\, a_{3,8}\, a_{4,10} + 3\, a_{2,6}\, a_{4,10} = 0 \ .$$

The solution of this system is the union of the three straight-lines in the 4-dimensional spaces parametrized by $(a_{1,4}, a_{2,6}, a_{3,8}, a_{4,10})$ whose equations are $(t,0,0,0)$, $(0,0,0,t)$ and $\left(t, \frac{t}{10}, \frac{t}{70}, \frac{t}{420}\right)$. We note that the cocycle $\psi = \Sigma\, a_{k,s}\, \psi_{k,s} \in F_0H^2(L_n,L_n)$ with $a_{1,4} = -a_{2,6} = a_{3,8} = 0$ and $a_{4,10} \neq 0$ is not integrable if $n > 11$. For verifying this, it is sufficient to consider the coefficient of e_{12} in $(\psi o \psi)\ (e_1,e_4,e_5)$. This coefficient is equal to $(a_{4,10})^2$ and is not equal to zero. But an element of $B^3(L_n,L_n)$ doesn't possess this property.
By successively using the following relations

$$(\psi.\psi)\,(e_1,e_4,e_5) = 0 \ ,$$

$$(\psi.\psi)\,(e_1,e_5,e_6) = 0 \ ,$$

$$(\psi.\psi)\,(e_1,e_m,e_{m+1}) = 0 \ ,$$

where $m = \frac{n-4}{2}$ and using an induction, we find the following assertions :

(a) if n is even, the vector $\left(a_{1,4}, a_{2,6},\dots, a_{\frac{n-2}{2},n}\right)$ is equal to one of the following vectors :

$$a_{1,4}(1, 0, \dots, 0) \ ,$$

$$a_{\frac{n-2}{2},n}\,(0, 0, \dots, 1) \ ,$$

$$a_{1,4}\left(\alpha_1, \alpha_2, \dots, \alpha_{\frac{n-2}{2}}\right) \ ,$$

where $\alpha_1 = 1 \ , \ \alpha_{k+1} = \frac{k}{4k+6}\,\alpha_k \ , \ 1 \leq k \leq \frac{n-4}{2}$.

(b) if n is odd, the vector $\left(a_{1,4}, a_{2,6},\dots, a_{\frac{n-3}{2},n-1}, a_{\frac{n-1}{2},n}\right)$ is equal to one of the following :

$$a_{1,4}(1, 0, \dots, 0) \ ,$$

$$a_{\frac{n-1}{2},n}(0, 0, \ldots, 0, 1) ,$$

$$a_{\frac{n-3}{2},n-1}(0, 0, \ldots, 0, 1, 0) ,$$

$$a_{1,4}\left(\alpha_1, \alpha_2, \ldots, \alpha_{\frac{n-3}{2}}, 0\right) ,$$

where $\alpha_1 = 1$, $\alpha_{k+1} = \frac{k}{4k+6}\alpha_k$, $1 \le k \le \frac{n-4}{2}$.
The lemma is proved.

Theorem 1. *The variety* $\mathfrak{F}^m$ *(and then also the variety* $\mathfrak{N}^m$*), with* $m \geq 12$, *contains at least three irreducible components if* m *is odd, and at least four irreducible components if* m *is even.*

This theorem is a consequence of Lemmas 1 and 2.

IV. DESCRIPTION OF AN IRREDUCIBLE COMPONENT OF $\mathfrak{N}^{n+1}$ CONTAINING R_n

Suppose $n \geq 11$. We denote U_1 as the Zariski open subset of M_1, defined by the inequation $a_{1,4} \neq 0$ (we use the previous notations). It is clear that $R_n \in U_1$.

Definition 1. *A cocycle* $\psi = \Sigma\, a_{k,s}\, \psi_{k,s}$ *belonging to* $F_0H^2(L_n,L_n)$ *is called nondegenerated in the layer* k_0 *if* $a_{k_0,s} \neq 0$ *for some integer* s.

Lemma. *Let* $\psi = \Sigma\, a_{ks}, \psi_{k,s} \in U_1$ *be a nondegenerated cocycle in layer* 2 *and let* s_0 *be the minimal value of the index* s *such that* $a_{2,s_0} \neq 0$. *Then*

(a) $a_{3,l} = 0$, *if* $l < 2s_0 - 4$;

b) $a_{3,2s_0-4} = \dfrac{(\varphi_0-3)a^2_{2,s_0}}{2a_{1,4}}$

(c) $a_{k,2k+1+r} = 0$, *if* $k > 3$, $1 \le r \le 2s_0 - 11$.

Proof. First, we show the following assertion :

$a_{k,s} = 0$ *if* $k \ge 3$ *and if the degree of homogeneousness of the cocycle* $\psi_{k,s}$ *is less than* $2s_0-11$ (*that is if* $s - 2k - 1 < 2s_0 - 11$).

We suppose the converse and let ψ be a cocycle such that $a_{k,s} \neq 0$, where $r = s - 2k - 1 < 2s_0 - 11$. We can suppose that r is the minimal value satisfying this hypothesis and that k is the maximal value such that $s - 2k - 1 = r$. By writting that the coefficient of e_{2k+1+r} in $\psi o \psi(e_1, e_{k-1}, e_k)$ is equal to 0, we obtain $a_{1,4}\, a_{k,2k+1+r} = 0$. As $a_{1,4} \neq 0$, then $a_{k,2k+1+r} = 0$. This is impossible. We deduce the case (a). To verify point (b), it is sufficient to consider the relation $(\psi o \psi)(e_1, e_2, e_3) = 0$ and to write that the coefficient of e_{2s_0-4} is zero. To end the proof, we can show that $a_{k,2k+1+r} = 0$ if if $k > 3$ and $r = 2s_0 - 11$. Let $k > 3$ be and $r = 2s_0 - 11$. We consider the relation $(\psi \circ \psi)(e_1, e_{k-1}, e_k) = 0$. The previous assertion permits us to write $a_{k,2k+1+r} = 0$.

Lemma 2. *Let* $\psi = \Sigma\, a_{k,s}\, \psi_{k,s} \in U_1$ *be a nondegenerated cocycle in layer 2 and layer 3 and let* s_0, t_0 *be the minimal values of the index* s *and* t *with* $a_{2,s_0} \neq 0, a_{3,t_0} \neq 0$ *(from Lemma 1 , we have* $t_0 = 2s_0 - 4$*). Then*

(a) $a_{4,l} = 0$, *if* $l < 3s_0 - 8$;

(b) $$a_{4,3s_0-8} = \frac{(2s_0-7)(s_0-3)a^3_{2,s_0}}{4a^2_{1,4}} \quad , \text{ if } 3s_0 - 8 \le n \ .$$

The proof of this lemma is analogous to the previous lemma.

Lemma 3. *Let* $\psi = \Sigma\, a_{k,s}\, \psi_{k,s}$ *be a cocycle of* U_1. *Then* $a_{2,s} = 0$, *if* $s \le \frac{n+7}{3}$.

Proof. If $n \ge 14$, the lemma is obvious. We suppose now that the index s satisfies $s \le \frac{n+7}{3}$ with $a_{2,s} \neq 0$ and denote by s_0 the minimal value of such index. The relation

$(\psi \circ \psi)(e_1,e_3,e_5) = 0$ gives, by writing that the coefficient of e_{3s_0-7} is zero,

$$(-t_0+3)\,a_{2,s_0}\,a_{3,t_0}=0 \quad \text{if} \quad s>7$$

and

$$(-t_0+3)\,a_{2,s_0}\,a_{3,t_0}+2a_{1,4}a_{4,3s_0-8}=0 \quad \text{if} \quad s=7\ .$$

From Lemmas 1 and 2, we have $a_{2,s_0} = 0$.

Corollary. *If* $n \equiv 1 \bmod 3$, *then* $a_{4,l} = 0$.

This depends of the inequalities $s_0 > \frac{n+7}{3}$ and $3s_0 - 7 < n$ which can be verified simultaneously if $s_0 = \frac{n+8}{3}$.

Lemma 4. *We consider the arbitrary scalars* $\alpha_{1,4}, \alpha_{1,5}, \ldots, \alpha_{1,n}, \alpha_{2,s_0}, \ldots, \alpha_{2,n}$ *satisfying* $s_0 = \left[\frac{n+10}{3}\right]$ *and* $\alpha_{1,4} \neq 0$. *Then there is only one cocycle* $\psi = \Sigma\, a_{k,s}\,\psi_{k,s}$ *of* U_1 *such that* $a_{1,j} = \alpha_{1,j}$ *and* $a_{2,j} = \alpha_{2,j}$.

Proof. We put

$$\psi = \sum_{s=4}^{n} \alpha_{1,s}\,\psi_{1,s} + \sum_{s=s_0}^{n} \alpha_{2,s}\,\psi_{2,s} + \sum_{s=2s_0-4}^{n} x_{3,s}\,\psi_{3,s} + x_{4,n}\,\psi_{4,n}\ .$$

We choose the values $x_{3,s}$ in order that the condition $(\psi\circ\psi)(e_1,e_2,e_3) = 0$ is verified. By writting that the coefficients of vectors $e_{2s_0-4}, e_{2s_0-3}, \ldots, e_n$ are zero, we obtain the values of the variables $x_{3,s}$, where $s = 2s_0-4, \ldots, n$. They have the desired form. In particular, we have :

$$x_{3,2s_0-4} = \frac{(s_0-3)\,\alpha^2_{2,s_0}}{2\alpha_{1,4}}\ .$$

If $n \equiv 1 \bmod 3$, then $s_0 = \frac{n+8}{3}$ and we put

$$x_{4,n} = \frac{(2s_0-7)(s_0-3)\,\alpha^2_{2,s_0}}{4\alpha^2_{1,4}} .$$

In other cases, we put $x_{4,n} = 0$. Lemmas 1, 2 and 3 show that the equalities

$$(\psi o \psi)(e_1,e_3,e_4) = 0, \quad (\psi o \psi)(e_2,e_3,e_4) = 0, \quad (\psi o \psi)(e_1,e_3,e_5) = 0$$

also are verified. As for as equalities

$$(\psi o \psi)(e_1,e_2,e_4) = 0 \quad , \quad (\psi o \psi)(e_1,e_2,e_5) = 0 \quad , \quad (\psi o \psi)(e_1,e_2,e_6) = 0 \quad ,$$

ey depend on the previous relations and of $(\psi \circ \psi)(e_1,e_2,e_3) = 0$. The others relations $(\psi \circ \psi)(e_i,e_j,e_r)$ can be verified directly. This gives the lemma.

Corollary 1. *The variety* $\overline{U_1}$ *is irreducible and its dimension is equal to* $2n - \left[\frac{n+16}{3}\right]$.

Corollary 2. *The dimension of the tangent space at the point* 0 *to the variety* $\overline{U_1}$ *is equal to* $3\left(n - \left[\frac{n+7}{3}\right]\right)$ *if* $n \equiv 0$ *or* 2 mod 3 *and is equal to* $3\left(n - \left[\frac{n+7}{3}\right]\right) + 1$ *if* $n \equiv 1 \bmod 3$.

Corollary 3. *The dimension of the tangent space at an arbitrary point of* U_1 *to the variety* $\overline{U_1}$ *is equal to* $2n - \left[\frac{n+16}{3}\right]$.

Finally we have the following theorem :

Theorem 2. *Let* $n \geq 11$. *The point* R_n *of the variety* $\mathcal{N}^{n+1}$ *is a simple point. The dimension of the irreducible component* C *of* $\mathcal{N}^{n+1}$ *passing through* R_n *is equal to* $n^2 + 2n - \frac{n+16}{3}$. *Moreover, all the points* $(L_n)_\psi$ *of* C *with* $\psi \in U_1$ *are simple points.*

V. DESCRIPTION OF AN IRREDUCIBLE COMPONENT OF $\mathcal{N}^{n+1}$ CONTAINING W_n

We denote U_2 as the Zariski open set of M_2 defined by $a_{1,4} \neq 0$.

Lemma 1. *Let be* $2 \leq r \leq n-3$. *Then there is a cocycle* $\psi = \psi_1 + \psi_r + \psi_{r+1} + \ldots + \psi_{n-3}$ *in* U_2 *such that* $\psi_r \neq 0$ *and* $\psi_1 \in H^2(L_n,L_n)$.

Proof. We put

$$\psi_1 = \sum_{1 \leq i \leq k} a_i \psi_{i,2i+2} ,$$

where

$$k = \frac{n-2}{2} \ , \ a_1 = 1 \ , \ a_j = \left(4 + \frac{6}{j}\right) a_{j+1} \ , \ j = 1,\ldots,k-1 \ .$$

$$k = \frac{n-2}{2} \ , \ a_1 = 1 \ , \ a_j = \left(4 + \frac{6}{j}\right) a_{j+1} \ , \ j = 1,\ldots,k-1 \ .$$

Then $\psi_1 \circ \psi_1 = 0$ and $(L_n)_{\psi_1} = W_n$.

Let $r > 2$ and consider $g \in G = GL(\mathbb{C}_{n+1})$ defined by

$$g(e_0) = e_0 \ ,$$
$$g(e_i) = e_i + e_{i+r} \ , \ i = 1,\ldots, n-r \ ;$$
$$g(e_j) = e_j \ , \ j = n-r+1,\ldots, n \ .$$

We have $g((L_n)_{\psi_1}) = (L_n)_\psi$, where ψ is a cocycle verifying the hypothesis of the previous lemma. Now suppose that $r = 2$. In this case, it is sufficient to consider the linear transformation $h \in G$ defined by the following relations

$$h(e_0) = e_0 + e_1 \ , \ h(e_1) = e_1 \ ,$$
$$h(e_i) = [e_0, h(e_{i+r})] + \psi_1(e_1, h(e_{i-1}))$$

to deduce the lemma.

Lemma 2. *Let* $2 \leq r \leq n-10$ *and* $n \geq 12$ *and consider the following cocycle given by* $\varphi_1 = \Sigma\, a_{i,2i+2}\, \psi_{i,2i+2}$ *with* $a_{1,4} = 1$, $a_{j,2j+2} = \left(4 + \frac{6}{j}\right) a_{j+1,2j+4}$, $j = 1, \ldots, \frac{n-4}{2}$. *Then there is a unique nonnull solution (up a constant factor) of the equation* $\varphi_1 \circ x + x \circ \varphi_1 = 0$ *belonging to* $H^2_r(L_n, L_n)$.

Proof. Let x be a cocycle

$$x = \sum_{1 \leq i \leq m} x_i\, \psi_{i,2i+1+r} \in H^2_r\,(L_n, L_n)$$

with $m = \frac{1}{2}(n-r-1)$ and verifying $\varphi_1 \circ x + x \circ \varphi_1 = 0$.

By verifying this equation for the vectors (e_1,e_2,e_3), (e_1,e_3,e_4), (e_2,e_3,e_4), we obtain the following homogeneous system :

$$\alpha_{1,1}(r)x_1 + \alpha_{1,2}(r)x_2 + \alpha_{1,3}(r)x_3 + \alpha_{1,4}(r)x_4 = 0\,,$$

$$\alpha_{2,1}(r)x_1 + \alpha_{2,2}(r)x_2 + \alpha_{2,3}(r)x_3 + \alpha_{2,4}(r)x_4 = 0\,,$$

$$\alpha_{3,1}(r)x_1 + \alpha_{3,2}(r)x_2 + \alpha_{3,3}(r)x_3 + \alpha_{3,4}(r)x_4 = 0\,.$$

The numbers $\alpha_{i,j}(r)$ are defined by the formulas :

$$\alpha_{1,1}(r) = \frac{1}{60} - \frac{r+2}{(r+4)(r+5)(r+6)} + \frac{2r}{(r+3)(r+4)(r+5)(r+6)}\,,$$

$$\alpha_{1,2}(r) = -\frac{13}{60} + \frac{r+4}{(r+5)(r+6)}\,,$$

$$\alpha_{1,3}(r) = \frac{7}{20}\,,$$

$$\alpha_{1,4}(r) = 0\,,$$

$$\alpha_{2,1}(r) = \frac{1}{420} - \frac{2\,(r+2)}{(r+5)(r+6)(r+7)(r+8)}\,,+\frac{6\,(r-1)}{(r+3)(r+4)(r+5)(r+6)(r+7)(r+8)}\,,$$

$$\alpha_{2,2}(r) = \frac{5}{420} + \frac{2\,(r+2)}{(r+5)(r+6)(r+7)(r+8)}\,,$$

$$\alpha_{2,3}(r) = -\frac{4}{35} + \frac{r+6}{(r+7)(r+8)}\,,$$

$$\alpha_{2,4}(r) = \frac{11}{35}$$

$$\alpha_{3,1}(r) = 0 \ ,$$

$$\alpha_{3,2}(r) = \frac{1}{420} - \frac{2(r+3)}{(r+6)(r+7)(r+8)(r+9)} + \frac{6(r+1)}{(r+5)(r+6)(r+7)(r+8)(r+9)} \ ,$$

$$\alpha_{3,3}(r) = -\frac{11}{420} + \frac{r+5}{(r+7)(r+8)(r+9)} \ ,$$

$$\alpha_{3,4}(r) = \frac{2}{35}$$

We have

$$\begin{vmatrix} \alpha_{1,2}(r) & \alpha_{1,3}(r) & \alpha_{1,4}(r) \\ \alpha_{2,2}(r) & \alpha_{2,3}(r) & \alpha_{2,4}(r) \\ \alpha_{3,2}(r) & \alpha_{3,3}(r) & \alpha_{3,4}(r) \end{vmatrix} \neq 0$$

for all values $r \geq 2$. The verification of this assertion is reduced to the verification of the nonexistence of integer roots greater than 2 of the following polynomial:

$$\begin{aligned} f(r) = {} & 1392\,(r+5)(r+6)(r+7)(r+8)(r+9) - 131040\,(r+5)(r+6)^2(r+9) + \\ & + 720\,720\,(r+5)^2(r+6) + 18\,000(r+4)(r+7)(r+8)(r+9) + 604\,800\,(r+4)(r+6)(r+9) - \\ & - 423\,360\,(r+2)(r+9) - 2\,328\,480(r+3)(r+5) + 15\,301\,440\,(r+6) \ . \end{aligned}$$

The supremum of the positive roots of this polynomial is equal to 50. An easy calculation shows that $f(r) \neq 0$ for all intergers r, $2 \leq r \leq 50$. Then the solutions of this system generate a 1-dimensional vector space in the 4-dimension vector space. The relations

$$(\varphi_1 \circ x + x \circ \varphi_1)(e_1, e_k, e_{k+1}) = 0 \ , \ k = 4, 5, \ldots$$

permits us to compute $x_5, x_6, \ldots, x_m$ from the parameters x_1, x_2, x_3, x_4. This means that the space of the solutions of the homogeneous system

$$(\varphi_1 \circ x + x \circ \varphi_1)\,(e_i,e_j,e_k) = 0$$

is, at most, of dimension 1. The previous lemmas have shown that this dimension is exactly one.

Let be $2 \leq r \leq n-10$. We designate by φ_r the cocycle of $H^2_r(L_n,L_n)$, satisfying the equation : $\varphi_1 \circ x + x \circ \varphi_1 = 0$ and such that $\varphi_r(e_1,e_2) = e_{3+r}$ Lemma 2 assures the existence of such a cocycle.

Lemma 3. *Let* $n-9 \leq r \leq n-7$, $n \geq 11$ *and* φ_1 *the cocycle defined in Lemma 2. Then the dimension of the space of solutions of the equation* $\varphi_1 \circ x + x \circ \varphi_1 = 0$ *belonging to* $H^2_r(L_n,L_n)$ *is equal to* 2.

Proof. Let

$$x = \sum_{1 \leq i \leq m} x_i\,\psi_{i,2i+1+r} \in H^2_r(L_n,L_n)$$

be a cocycle satisfying the equation $\varphi_1 \circ x + x \circ \varphi_1 = 0\left(m = \frac{n-r-1}{2}\right)$. In the case $r = n-9$, this equation gives a system of 2 linear homogeneous equations at 4 indeterminates (see the proof of Lemma 2). In the cases $r = n-8$ and $r = n-7$, we obtain an homogeneous linear equation at 3 indeterminates. In all cases, the dimension of the space of the solutions is equal to 2. Moreover, we can take x_1 and x_2 as free parameters. This gives the lemma.

Let $n-9 \leq r \leq n-7$ and $n \geq 11$. We denote by φ_r and φ'_r the cocycles of $H^2_r(L_n,L_n)$ satisfying the equation $\varphi_1 \circ x + x \circ \varphi_1 = 0$ and such that

$$\varphi_r(e_1,e_2) = e_{r+3}\ ,\quad \varphi_r(e_2,e_3) = 0\ ,$$
$$\varphi'_r(e_1,e_2) = 0\ ,\quad \varphi'_r(e_2,e_3) = e_{r+5}\ .$$

The existence of such cocycles is deduced from Lemma 3. We put

$$\varphi_{n-6} = \psi_{1,n-3} \ , \quad \varphi_{n-5} = \psi_{1,n-2} \ , \quad \varphi_{n-4} = \psi_{1,n-1} \ ,$$
$$\varphi_{n-3} = \psi_{1,n} \ , \quad \varphi'_{n-6} = \psi_{2,n-1} \ , \quad \varphi'_{n-5} = \psi_{2,n} \ .$$

Lemma 4. *We suppose that the hypothesis of Lemma 2 is satisfied. Let* γ_{r+1} *be a cocycle of* $Z^3{}_{r+1}(L_n,L_n)$. *Then there is a unique solution of the equation* $\varphi_1 \circ x + x \circ \varphi_1 = \gamma_{r+1}$ *belonging to* $C^2{}_r(L_n,L_n)$ *and such that* $x(e_1,e_2) = 0$.

The proof of Lemma 4 is analogous to that of Lemma 3.

Lemma 5. *Suppose that the hypothesis of Lemma 3 is satisfied and let* γ_{r+1} *be a cocycle of* $Z^3{}_{r+1}(L_n,L_n)$. *Then there is a unique solution of the equation* $\varphi_1 \circ x + x \circ \varphi_1 = \gamma_{r+1}$ *belonging to* $C^2{}_r(L_n,L_n)$ *and such that* $x(e_1,e_2) = x(e_2,e_3) = 0$.

See Lemma 3.

Let $\psi = \psi_1 + \psi_2 + ... + \psi_{n-3}$ be a cocycle belonging to U_2 with $\psi_i \in H^2_i(L_n,L_n)$. Lemmas 2, 3, 4, and 5 show that the cocycle ψ can be uniquely written $\psi = \psi_0 + \psi_*$, where

$$\psi_0 = (\psi_0)_1 + ... + (\psi_0)_{n-3} \ ,$$
$$(\psi_0)_r = \lambda_r \varphi_r \ , \quad 1 \leq r \leq n-3 \ ,$$
$$\psi_* = (\psi_*)_3 + ... + (\psi_*)_{n-7} \ ,$$

and $(\psi_*)_r$ is the unique solution satisfying Lemma 4 or 5 if $n-9 \leq r \leq n-7$ with

$$\gamma_{r+1} = \sum_{i+j=r+1} (\psi_i \cdot \psi_j + \psi_j \cdot \psi_i) \ .$$

The cocycle ψ_0 is called the homogeneous part of the cocycle ψ. It is clear that every cocycle is determined by its homogeneous part, i.e., if we have a cocycle ψ_0 which is a linear combination of the cocycles

$$\varphi_1 , \varphi_2 , ..., \varphi_{n-3} ,$$
$$\varphi'_{n-5} , \varphi'_{n-6} , \varphi'_{n-7} , \varphi'_{n-8} , \varphi'_{n-9} ,$$

and if the coefficient of φ_1 is different of zero, then there is one and only one element ψ of U_2 whose homogeneous part is ψ_0.
This reasoning shows that the variety U_2 can be defined by the following relations :

$$a_{1,2i+r} = \alpha_{i,r}\, a_{1,3+r} + \gamma_{i,r} \quad , \quad i = 2, ..., \frac{n-2}{2} \quad , \quad 1 \le r \le n-10 \quad ;$$

$$a_{3,7+r} = \alpha_{3,r}\, a_{1,3+r} + \alpha'_{3,r}\, a_{2,5+r} + \gamma_{3,r} \, , \quad r = n-9 \, , \quad n-8 \, , n-7 \quad ;$$

$$a_{4,9+r} = \alpha_{4,r}\, a_{1,3+r} + \alpha'_{4,r}\, a_{2,5+r} + \gamma_{4,r} \, , \quad r = n-9 \, , \quad n \equiv 1 \bmod 3 \quad .$$

Here the elements $\gamma_{i,r}$, $\gamma_{3,r}$, $\gamma_{4,r}$ are polynomials whose variables are

$$a_{1,3+r'} \quad (2 \le r' \le r-1) \quad ,$$
$$a_{2,5+r'} \quad (n-9 \le r' \le n-7 \quad , \quad r' \le r) \quad ;$$

These polynomials have a degree greater than 2. The linear combination of cocycles $\varphi_1 , \varphi_2 , ..., \varphi_{n-3} , \varphi'_{n-5} , ..., \varphi'_{n-9}$ generates the tangent space at the point L_n to the variety $\overline{U_2}$. As $L_n \in \overline{G(A)}$ for every filiform algebra $A \in \mathcal{F}^{n+1}$, we have the following theorem :

Theorem 3. *Let* $n \ge 11$. *There exists a unique irreducible component of the variety* $\mathcal{N}^{n+1}$ *containing* W_n. *This component is smooth (each point is a simple point) and its dimension is equal to* $n^2 + n + 2$.

Corollary. *Let* $n \ge 11$ *and* C *be an irreducible component of the variety* $\mathcal{N}^{n+1}$ *containing* W_n. *Then* $\dim C - \dim G(W_n) = 5$.

VI. STUDY OF THE VARIETY $\mathcal{N}^n{}_p$ IN A NEIGHBORHOOD OF A NILRADICAL OF A PARABOLIC SUBALGEBRA

Let $\mathfrak{g}$ be a complex simple Lie algebra, $\mathfrak{n}$ the nilradical of a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ defined by a subset S_1 of the set of simple roots S (see Chapters 2 and 4). We denote ρ as the nilindex of $\mathfrak{n}$ and n its dimension.

VI.1. On the orbit of $\mathfrak{n}$

Theorem 4. *Let* $\Delta^+{}_0 = \Delta_0 \cap \Delta^+$ *and let* ρ *be the half sum of positive roots lying in* $\Delta^+{}_0$ *Then*

(a) *If* $\mathfrak{g} = A_r$ *or* C_r *and if* $S_1 = \{\alpha\}$, *where* α *is a singular root, we have*

$$\dim G(\mathfrak{n}) = r^2 + n + \dim Z(\mathfrak{n}) - \dim \mathfrak{g} .$$

(b) *In other cases,*

$$\dim G(\mathfrak{n}) = n^2 + n + \dim Z(\mathfrak{n}) - \dim \mathfrak{g} - \sum_{\alpha \in S_1} \bigcap_{\gamma \in \Delta_0^+} \left(1 + \frac{s_\alpha(\delta)\angle\text{-}\alpha,\gamma}{(\rho,\gamma)}\right) ,$$

where δ *is the maximal root and* s_α *the reflexion (symmetry) with respect to* α.

Proof. We define a homomorphism

$$f : \mathfrak{p} \to \operatorname{Der} \mathfrak{n}$$

by putting $f(x) = \operatorname{ad} x / \mathfrak{n}$. This mapping is surjective in case (a) from the description of $\operatorname{Der} \mathfrak{n}$ (see Chapter 1). Now assertion (a) results in the equality
$\dim \mathfrak{p} = \dim \mathfrak{g} - \dim \mathfrak{n}$ and $\operatorname{Ker} f = Z(\mathfrak{n})$. For case (b), we put

$$I_p = \{\operatorname{ad} x / \mathfrak{n} \ , \ x \in \mathfrak{p}\} .$$

For $\alpha \in S_1$, we denote d_α as the endomorphism of $\mathfrak{n}$ defined by

$$d_\alpha(X_\alpha) = X_{s_\alpha(\delta)} \cdot$$

We know that d_α is a primitive element of the dominating weight $s_\alpha(\delta) - \alpha$ of the s-module Der $\mathfrak{n}$ (see Chapter 4). We note W_α as the simple s-module generated by this element. From the description of Der $\mathfrak{n}$, we have

$$\mathrm{Der}\,\mathfrak{n} = I_{\mathfrak{g}} \oplus \left(\bigoplus_{\alpha \in S_1} W_\alpha \right).$$

The centralizer of the Lie algebra $\mathfrak{n}$ in $\mathfrak{p}$ coincides with the center $Z(\mathfrak{n})$. Then we have

$$\dim I_{\mathfrak{p}} = \dim \mathfrak{p} - \dim Z(\mathfrak{n}) = \dim \mathfrak{g} - \dim \mathfrak{n} - \dim Z(\mathfrak{n}).$$

The dimension of the module W_α can be calculated using the Weyl formula [SE1] :

$$\dim W_\alpha = \bigcap_{\gamma \in \Delta_0^+} \left(1 + \frac{(s_\alpha(\delta) - \alpha, \gamma)}{(\rho, \gamma)} \right).$$

As $\dim G(\mathfrak{n}) = n^2 - \dim \mathrm{Der}\,\mathfrak{n}$, we have the required formula.

Corollary 1. *If* $\mathfrak{n}$ *is the nilradical of a Borel subalgebra, then*

$$\dim G(\mathfrak{n}) = n^2 - n - 2r + 1,$$

where r *is the rank of* $\mathfrak{g}$.

Corollary 2. *Let* H_k *be the Heisenberg algebra of dimension* $2k+1$. *Then*

$$\dim G(H_k) = 2k^2 + k.$$

The corollary can be proved as this : consider the inclusion of H_k in the algebra $\mathfrak{g} = C_r$, where $r = k + 1$ ($\mathfrak{n} = H_k$ if $S_1 = \{\alpha\}$, where α is a singular root). From the theorem under consideration, we have :

$$\dim G(H_k) = (2k+1)^2 + (2k+1) + 1 - (2r^2+r) - 2k^2 + k \ .$$

VI.2. On the Zariski tangent space at the point $\mathfrak{n}$ to the variety $\mathcal{N}^n$

Consider the filtration of $H^2(\mathfrak{n},\mathfrak{n})$ defined by the filtrations of the nilpotent Lie algebra $\mathfrak{n}$ and the adjoint module studied in Chapters 3 and 4. From the description of $H^2(\mathfrak{n},\mathfrak{n})$ given in Chapter 4, we have :

$$F_{-2}H^2(\mathfrak{n},\mathfrak{n}) = H^2(\mathfrak{n},\mathfrak{n})\,.$$

From Proposition 1, we have the following theorem.

Theorem 5. *Let* $\mathfrak{n}$ *be the nilradical of a parabolic subalgebra of a complex simple Lie algebra,* p *be the nilindex of* $\mathfrak{n}$*, and* $n = \dim \mathfrak{n}$*, with* $p \leq n-3$*. Then the Zariski tangent space at the point* $\mathfrak{n}$ *to the schema* $\mathcal{L}^n$ *coincides with the Zariski tangent space at the same point* $\mathfrak{n}$ *to the subschema* $\mathcal{N}^n{}_{p+2}$.

Remarks.

1. For some Lie algebras $\mathfrak{n}$, we can accurately state this theorem. For example, if $\mathfrak{n} = H_k$ is the Heisenberg algebra with $k > 1$, then

$$F_{-1}H^2(\mathfrak{n},\mathfrak{n}) = H^2(\mathfrak{n},\mathfrak{n})$$

and the tangent space $\mathcal{L}^n$ coincides with the tangent space to $\mathcal{N}^n{}_{p+1}$ at the point $\mathfrak{n}$.

2. In the general case where $\mathfrak{n}$ is an arbitrary nilpotent Lie algebra with a nilindex p, we can establish a weaker result :

If $2p + 1 \leq n$*, then the tangent space to* $\mathcal{L}^n$ *coincides with the tangent space to* $\mathcal{N}^n{}_{2p}$ *at the point* $\mathfrak{n}$.

This result is deduced directly from the relation $F_{-p}H^2(\mathfrak{n},\mathfrak{n}) = H^2(\mathfrak{n},\mathfrak{n})$ (see Chapter 3).

VI.3. On the irreducible components of $\mathcal{N}^n{}_p$ containing $\mathfrak{n}$

Theorem 6. *Let $\mathfrak{n}$ the nilradical of a parabolic subalgebra of a simple Lie algebra $\mathfrak{g}$, defined by a subset S_1 of the set S of simple roots ; let p be the nilindex of $\mathfrak{n}$, and $n = \dim \mathfrak{n}$, $\mathrm{card}(S_1) \geq 5$. Then the schema $\mathcal{N}^n{}_p$ is smooth at the point $\mathfrak{n}$. The dimension of the variety $\mathcal{N}^n{}_p$ at the point $\mathfrak{n}$ is equal to*

$$m + k + n^2 + n + \dim Q_p - \dim \mathfrak{g} \ ,$$

where

$$m = \dim F_0H^2(\mathfrak{n},\mathfrak{n}) = \frac{\Sigma_{\sigma\in W_2}\Pi_{\gamma\in\Delta_0^+}\left(\sigma(\delta+\rho)\,,\gamma\right)}{\Pi_{\gamma\in\Delta_0^+}(\rho+\gamma)} \ ,$$

$$k = \sum_{\alpha\in S_1}\prod_{\gamma\in\Delta_0^+}\left(1+\frac{(\varphi_\alpha(\delta)-\alpha,\gamma)}{(\rho_1,\gamma)}\right) ,$$

W_2 being the subset constituted of the elements ω of the Weyl group W such that $\omega(\Delta^-)\cap\Delta^+\subset\Delta'$ and $\mathrm{card}(\omega(\Delta^-)\cap\Delta^+)=2$, ρ being the half sum of positive roots, ρ_1 the half sum of the roots of $\Delta^+{}_0$, and Q_p is the last ideal of the descending central sequence of $\mathfrak{n}$ ($Q_p = \mathcal{C}^{p-1}\mathfrak{n}$).

Proof. Let $\psi \in F_0H^2(\mathfrak{n},\mathfrak{n})$ be an infinitesimal deformation. It is integrable because $\mathrm{card}(S_1) \geq 5$ and $\psi \circ \psi = 0$. Then the bilinear mapping $\mu_0 + \psi$ defines a point of $\mathcal{N}^n{}_p$, where μ_0 is the law of the Lie algebra $\mathfrak{n}$. The mapping

$$f : G \times F_0H^2(\mathfrak{n},\mathfrak{n}) \to \mathcal{N}^n{}_p$$

defined by $f(g,\psi) = g(\mu_0+\psi)$ is a morphism of affine algebraic varieties. A classical calculation shows that for the differential of this morphism at the point $M_0 = (Id,0)$, the following formula is satisfied :

$$(df)_{M_0}(X,\psi)(x,y) = \psi(x,y) + X(\mu_0(x,y)) - \mu_0(X(x),y) - \mu_0(x,X(x)) \ ,$$

where $x, y \in \boldsymbol{n}$, $X \in \mathrm{End}\,\boldsymbol{n}$. This implies that the image of $(df)_{M_0}$ coincides with the space constituted from the cocycles $\varphi \in Z^2(\boldsymbol{n},\boldsymbol{n})$ whose cohomologic class is in $F_0H^2(\boldsymbol{n},\boldsymbol{n})$. This space is the tangent space to the schema $\mathcal{N}^n_p$ at the point $\boldsymbol{n}$. As the image of $(df)_{M_0}$ belongs to the tangent space $T(\mathcal{N}^n_p)$ of the variety $\mathcal{N}^n_p$ at the point $\boldsymbol{n}$, we have

$$T(\mathcal{N}^n_p) = T((\mathcal{N}^n_p)^{red}) \ ,$$

where $(\mathcal{N}^n_p)^{red}$ is the reduced schema. Moreover, the mapping $(df)_{M_0}$ is surjective. Let $M = (g_1, \psi)$ be a point of the variety $G \times F_0H^2(\boldsymbol{n},\boldsymbol{n})$ sufficiently near to the point $(Id,0)$ for the metric topology. It is clear that

$$\dim(\mathrm{Im}(df)_M) \geq \dim(\mathrm{Im}(df)_{M_0}) \ .$$

Moreover, we have

$$\mathrm{Im}(df)_M \subset T_{f(M)}(\mathcal{N}^n_p) \ .$$

If we consider another point sufficiently close to $(Id,0)$, the dimensions of the corresponding tangent spaces cannot increase. Then

$$\dim(\mathrm{Im}(df)_M) \leq \dim T_{f(M)}(\mathcal{N}^n_p) \leq \dim T_{f(M_0)}(\mathcal{N}^n_p) = \dim(\mathrm{Im}(df)_{M_0}) \ .$$

Then we have

$$\dim(\mathrm{Im}(df)_M) = \dim(\mathrm{Im}(df)_{M_0}) \ .$$

The dimension of $\mathrm{Im}(df)$ is stable in a neighborhood of the point M_0. The rank of the smooth mapping

$$f : G \times F_0H^2(\mathfrak{n},\mathfrak{n}) \to \mathcal{N}^n_p \subset \mathbb{C}^q$$

then is stable in a neighborhood of the point M_0. In this neighborhood, the image of the mapping f is a smooth subvariety whose dimension is equal to the dimension of the tangent space of $\mathcal{N}^n_p$ at the point $f(M_0)$. As the dimension of the variety $\mathcal{N}^n_p$ at a point M is less than or equal to the dimension of the tangent space at the same point, and as this dimension is stable in a neighborhood of M_0, we have

$$\dim (\mathcal{N}^n_p)_{\mathfrak{n}} = \dim T_{\mathfrak{n}} (\mathcal{N}^n_p).$$

This means that the point $\mathfrak{n}$ of the variety $\mathcal{N}^n_p$ is simple. Then, as the scheme $\mathcal{N}^n_p$ is reduced, it is smooth at the point $\mathfrak{n}$. To determinate the dimension of the variety $\mathcal{N}^n_p$, we note that

$$\dim T(\mathcal{N}^n_p) = \dim G(\mathfrak{n}) + \dim F_0H^2(\mathfrak{n},\mathfrak{n}) .$$

The condition $\operatorname{card}(S_1) \geq 5$ implies the following isomorphisms :

$$F_0H^2(\mathfrak{n},\mathfrak{n}) \cong H^2_{fond}(\mathfrak{n},\mathfrak{n}) \cong H^2(\mathfrak{n},\mathfrak{g})$$

(see Chapter 4). The description of the space $H^2(\mathfrak{n},\mathfrak{g})$ and the formula for the dimension of $G(\mathfrak{n})$ imply the required result.

VI.4. Particular case: $\mathfrak{n}$ is the nilradical of a Borel subalgebra

In this particular case, Theorem 6 can be exactly stated as :

Theorem 7. *Let $\mathfrak{n}$ be the nilradical of a Borel subalgebra of a complex simple Lie algebra $\mathfrak{g}$ of rank $r > 1$ and let p be the nilindex of $\mathfrak{n}$, $n = \dim \mathfrak{n}$. Then $\mathcal{N}^n{}_p$ is a smooth scheme at the point $\mathfrak{n}$. If the type of $\mathfrak{g}$ is A_2, A_3 or B_2, then the orbit $G(\mathfrak{n})$ is an open set on the variety $\mathcal{N}^n{}_p$. The dimension of this orbit is equal, respectively, to one of the numbers 3, 25, 9. If the type of $\mathfrak{g}$ is A_4, C_3, B_3, D_4 or G_2, then the dimension of the variety $\mathcal{N}^n{}_p$ at the point $\mathfrak{n}$ is equal to $m + n^2 - n - 2r + 1$, where the number m is one of the numbers 7, 4, 4, 6, 1 respectively. In the other cases, the dimension of the variety $\mathcal{N}^n{}_p$ at the point $\mathfrak{n}$ is equal to :*

$$n^2 - n - 2r + 1 + \frac{1}{2}(r^2 + r - 2).$$

VII. ON THE COMPONENTS OF THE VARIETY $\mathcal{N}^n$

The number of irreducible algebraic components of the variety $\mathcal{L}^n$ of n-dimensional complex Lie algebras is greater than $e^{n/4}$. This boundary relies on the study of rigid Lie algebras : the Zariski closure of the orbit of a rigid Lie algebra is an irreducible component of $\mathcal{L}^n$ and two nonisomorphic rigid laws determine two distinct components. The same problem, i.e. the determination of the number of components also appears for $\mathcal{N}^n$ because this variety is reducible when $n \geq 7$. But, in this case, the approach in terms of rigid laws cannot be applied : currently, we know of no nilpotent Lie algebras in $\mathcal{L}^n$ (and in $\mathcal{N}^n$). Then we can find another way to study this problem; we propose constructing some nilpotent laws which separate the components and to reveal some components associated to a given characteristic sequence.

VII.1 A nonfiliform component

The subset of filiform laws is an open set (irreducible or not) in $\mathcal{N}^n$. From the previous paragraph, we have :

If $n \leq 6$, the variety $\mathcal{N}^n$ is irreducible.

If $n = 7$, the variety $\mathcal{N}^7$ is union component such that one of them contains the open set of filiform laws. In particular, this shows that this open set is irreducible.

If $n \geq 7$, the variety $\mathcal{N}^n$ is reducible.
This last property has been proved by revealing at least two components in the open set of filiform laws for $n \geq 11$, $n = 8$, and $n = 10$. For the 9-dimensional case, this open set is reducible. This implies the research of another component which does not cut the set of filiform laws so that n is greater than 8.

Theorem 8. *If $n \geq 8$, then there is in $\mathcal{N}^n$ an irreducible component which does not cut the open set of filiform laws.*

The proof is supported by the results which follows. Let $(X_1, X_2, ..., X_n)$ be a basis of $\mathbb{C}^n$ and μ the n-dimensional Lie algebra law defined by :

$$\begin{gathered}\mu(X_1,X_i) = X_{i-1} \ , \quad 4 \leq i \leq n \text{ and } 8 \leq n \ , \\ \mu(X_2,X_n) = X_3 \ , \\ \mu(X_{n-3},X_{n-1}) = X_3 \ , \\ \mu(X_{n-3},X_n) = X_4 \ , \\ \mu(X_{n-2},X_{n-1}) = X_4 \ , \\ \mu(X_{n-2},X_n) = 2X_5 \ , \\ \mu(X_{n-1},X_n) = 2X_6 + X_2 \quad ,\end{gathered}$$

the nondefined brackets being null.

Proposition 3. *The irreducible algebraic component of $\mathcal{N}^n$ passing through this law μ does not cut the open set of filiform laws.*

Proof. Let us consider a nilpotent perturbation μ' of μ (see Chapter 5). As the characteristic sequence of μ is $(n-2, 1, 1)$, its μ' is equal to $(n-2, 1, 1)$ or $(n-1, 1)$. In this last case, μ' is filiform. We want to prove that μ' is filiform. It is sufficient to prove that if its center is one-dimensional, then μ' is not filiform. Let us suppose that the center of μ' is one-dimensional. Consider a basis of this center given by a vector Y_3 with $Y_3 \cong X_3$. The derived subalgebra of μ is generated by $(X_{n-1}, ..., X_3, X_2)$. As this subalgebra is also of dimension n–2, we can find a basis $(Y_{n-1}, ..., Y_3, Y_2)$ for this satisfying $Y_i \cong X_i$. As the vector X_1 is a characteristic vector for μ, there exists a characteristic vector Y_1 for μ' such that $Y_1 \cong X_1$ and $\mu'(Y_1,Y_i) = Y_{i-4}$ for $i \geq 4$. This gives $\mu'(Y_1, Y_3) = \sum_{i=2}^{n} \varepsilon_i Y_i$ and $\mu'(Y_1, Y_2) = \sum_{i=2}^{n} a_i Y_i$ with $a_i \cong 0$ and $\varepsilon_i \cong 0$. The vector space generated by the vector Y_3 is the center of μ. This gives $\varepsilon_i/\varepsilon_3 \cong 0$ for $i \neq 3$ and $\mu'(Y_2,Y_3)$ is infinitely large. Then μ is not filiform and the component does not cut the open set of these filiform laws.

VII.2. An estimation of the number of components

Theorem 9. *For* n *sufficiently large, the number of components of* $\mathcal{N}^n$ *is at least of order n.*

Let $\mathfrak{g}_n$ be the n-dimensional complex Lie algebra whose law μ is defined in the basis $(e_0, ..., e_{m-1}, f_1, ..., f_k, g_1, ..., g_k, h_1, ..., h_k)$, where $n = 3k + m$ by :

$$\mu(e_0, e_i) = e_{i-1} \ , \quad 1 \leq i \leq m-1 \quad \text{and} \quad \mu(f_i, g_i) = h_i \ , \quad i = 1, ..., k \ .$$

This Lie algebra is a direct sum of the model filiform Lie algebra L_n and of k patterns of the 3-dimensional Heisenberg algebra.

Lemma 1. $\dim \mathrm{Der}(\mathfrak{g}_n) = 2k^2 + 8k + 2m - 1$.

Corollary 1. *The dimension of the orbit of* $\mathfrak{g}_n$ *is equal to :*

$$7k^2 + 6km + m^2 - 8k - 2m + 1 \ .$$

Consider the following 2-cocycles in $Z^2(\mathfrak{g}_n, \mathfrak{g}_n)$ given by :

$$\varphi_{r,s}(e_i,e_j) = (-1)^{r-i}\, C^{k-i}_{j-k-1}\, e_{i+j+s-2r-1}$$

$$\text{for } 1 \le r \le m-2\ ,\ s \ge 4 \text{ and } 2r+1\,(m-6)/2 \le s \le m-1\ ;$$

$$\psi_{r,s,t}(f_r,g_s) = h_t$$

$$\text{for } 1 \le r,s,t \le k \text{ and } r \ne s \ne t\ ;$$

$$\eta_{r,s,t}(f_r,f_s) = h_t$$

$$\text{for } 1 \le r,s,t \le k \text{ and } r \ne s \ne t\ ;$$

$$\xi_{r,s,t}(g_r,g_s) = h_t$$

$$\text{for } 1 \le r,s,t \le k \text{ and } r \ne s \ne t\ ;$$

$$\rho_{t,u}(e_0,f_t) = h_u\ ,\quad \rho'_{t,u}(e_0,g_t) = h_u$$

$$\text{for } 1 \le t\ ,\ u \le n \text{ and } t \ne u\ ;$$

$$\theta_{t,u}(e_1,f_t) = h_u\ ,\quad \theta'_{t,u}(e_1,g_t) = h_u$$

$$\text{for } 1 \le t\ ,\ u \le n \text{ and } t \ne u\ ;$$

$$v_{t,u}(f_t,g_u) = e_{m-1}\ ,\quad v'_{t,u}(f_t,f_u) = e_{m-1}\ ,\quad v''_{t,u}(g_r,g_s) = e_{m-1}$$

$$\text{for } 1 \le t\ ,\ u \le n \text{ and } t \ne u\ ;$$

the unspecified products being null.

Lemma 2. *Let Ω be the vector subspace of $Z^2(\mathfrak{g}_n,\mathfrak{g}_n)$ generated by the previous cocycles. Then for all $\psi \in \Omega$, $\psi \ne 0$, the law $\mu + \psi$ of $\mathcal{N}^n$ is not isomorphic to μ.*

Indeed, if $\mu + \psi$ is in the orbit of μ, then ψ is a coboundary, i.e. $\psi = \delta f$ with $f \in gl(n)$. As, no cocycle of Ω is a coboundary, the laws $\mu + \psi$ are not isomorphic to μ.

Now, let C_0 be an algebraic component passing through $\mathfrak{g}_n$ and containing the family $\mu + \psi$, $\forall\ \psi \in \Omega$.

Proposition 4. *We have* :

$$\dim C_0 \geq M(k,m) = 2\,k^3 + 7\,k^2 + \frac{17}{16}\,m^2 + 6\,km - 10\,k - \frac{15}{8}\,m + \frac{13}{16}\ .$$

Indeed, $\dim \Omega = M(k, m)$. From the previous lemma, this number is a minimum of the number of nonorbital parameters of C_0.

Lemma 3. *Let* $\mathfrak{g}$ *be a Lie algebra which is a direct sum of 3 ideals* : $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$. *Let* $X_i \neq 0 \in \mathfrak{g}_i$ *and suppose that* $X_3 \notin Z(\mathfrak{g}_3)$, *where* $Z(\mathfrak{g}_i)$ *is the center of* $\mathfrak{g}_i$. *Then bilinear alternated mapping* φ *with values on* $\mathbb{C}^n$ *satisfying* $\varphi(X_1,X_2) = X_3$ *does not belong to the space* $Z^2(\mathfrak{g},\mathfrak{g})$.

In effect, consider $Y \in \mathfrak{g}_3$ such that $\mu(X_3,Y) \neq 0$, where μ is the law of $\mathfrak{g}$. Then $\delta\varphi(X_1,X_2,Y) \neq 0$.

We know that if L_m is the model filiform algebra, we have $\dim Z^2(L_m,L_m) \leq 2m^2 - 2m$. By looking at the structure of $\mathfrak{g}_n$, we see that exists an ideal I_m isomorphic to L_m. On the other hand, the dimension of the components passing through $\mathfrak{g}_n$ is smaller than $\dim Z^2(\mathfrak{g}_n,\mathfrak{g}_n)$. Now Lemma 3 implies :

Proposition 5. *Let* C *be an irreducible component of* $\mathcal{N}^n$ *containing* $\mathfrak{g}_n$. *Then* :

$$\dim C \leq N(k,m) = 2\,k^3 + 36\,k^2 + 2\,m^2 + 14\,km - 31\,k - 2\,m\ .$$

Proof of the theorem 9. From Propositions 4 and 5, we can give the boundaries of the dimensions of the component C_0. Suppose now that $k \geq m$. Then, for n sufficiently large, we have :

$$N\,(k-7,\ m+21) < M\,(k,n).$$

If k varies between n/3 and n/4, we reveal a minoration of the number of (not filiform) components of $\mathcal{N}^n$. This boundary is of the order n/74. This gives the theorem.

CHAPTER 7

CHARACTERISTICALLY NILPOTENT LIE ALGEBRAS

The definitions, the first properties, and some examples of characteristically nilpotent Lie algebras have been given in Chapter 2 (see Section III in this chapter). Here we study these algebras from a geometrical point of view, by considering them as points of the variety $\mathcal{N}^n$. This study leads us to a stupendous result : almost all nilpotent Lie algebras are characteristically nilpotents. Then the determination of the noncharacteristically nilpotents seems natural. We conclude this chapter by giving and describing this family of noncharacteristically nilpotent filiform Lie algebras.

I. CHARACTERISTICALLY NILPOTENT FILIFORM LIE ALGEBRAS

In this section, we describe a necessary and sufficient condition for a filiform Lie algebra $A = (L_n)_\psi$ with $\psi \in F_1H^2(L_n, L_n)$, $n \geq 7$, to be characteristically nilpotent (recall that the notations are those of Chapter 4). We have seen that every filiform Lie algebra is isomorphic to a Lie algebra $(L_n)_\psi$ for a cocycle $\psi \in F_1H^2(L_n,L_n)$ if n is even, and $\psi \in \mathbb{C}\, \psi_{m,n} + F_1H^2(L_n, L_n)$, $m = \frac{n-1}{2}$ if n is odd.

Let ψ be a nonnull cocycle belonging to

$$F_1H^2(L_n,L_n) = \bigoplus_{i\geq 1} H_i^2(L_n,L_n) .$$

Then $\psi = \psi_r + \psi_{r+1} + \dots + \psi_1$ with $\psi_i \in H_i{}^2(L_n,L_n)$, $\psi_r \neq 0$ and $r \geq 1$. We have called the component ψ_r the ***sill cocycle*** of ψ. It verifies $\psi_r \, . \, \psi_r = 0$ Then

Lemma 1. *Let ψ be a nonnull cocycle, linearly integrable and belonging to* $F_1H^2(L_n,L_n)$. *Then the sill cocycle ψ_r of ψ also is linearly integrable.*

Let $A = (L_n)_\psi$ be the filiform Lie algebra defined from $\psi \in F_1H^2(L_n,L_n)$ and let ψ_r be the sill cocycle of ψ. The Lie algebra $(L_n)_{\psi_r}$ is called the sill algebra of $A = (L_n)_\psi$. The main result of this section is the following theorem :

Theorem 1. *Let ψ be a nonnull cocycle linearly integrable and belonging to* $F_1H^2(L_n,L_n)$. *The Lie algebra $A = (L_n)_\psi$ is characteristically nilpotent if and only if A is not isomorphic to its sill algebra $(L_n)_{\psi_r}$.*

A large part of the proof of this theorem is based on the following lemmas :

First, we prove some of the lemmas intervening in the proof of this theorem.

Lemma 2. *Let ψ be a nonnull linearly integrable cocycle belonging to* $F_1H^2(L_n,L_n)$, *and ψ_r its sill cocycle. Consider a derivation d of the sill algebra and d_0 its homogeneous component of degree* 0, *d being considered as an endomorphism of the graded space*

$$V = \bigoplus_{1\leq i\leq n} V_i \ , \ V_1 = \mathbb{C}\, e_0 + \mathbb{C}\, e_1 , V_i = \mathbb{C}\, e_i \, , i = 2 ,\dots, n.$$

Then $d_0 = \alpha_1\theta_1 + \alpha_2\theta_2 \in \mathrm{Der}\, L_n$, *where the endomorphisms* θ_1 *and* θ_2 *are defined by*

$$\theta_1(e_i) = \left\{ \begin{array}{ll} e_1 \ , & \text{if } i = 0 \ , \\ 0 \ , & \text{if } 1 \leq i \leq n \ , \end{array} \right\} \quad \theta_2(e_i) = \left\{ \begin{array}{ll} e_0 \ , & \text{if } i = 0 \ , \\ (i+r)\, e_i \ , & \text{if } 1 \leq i \leq n \ , \end{array} \right\}$$

Proof. Let $a \in V_k$, $b \in V_m$. One has

$$d_0([a,b]) - [d_0(a),b] - [a,d_0(b)] = \left[\sum_{i\geq 1} d_i(a),b\right] + \left[a,\sum_{i\geq 1} d_i(b)\right] - \sum_{i\geq 1} d_i([a,b]) +$$
$$+ \psi_r(d(a),b) + \psi_r(a,d(b)) - d(\psi_r(a,b)) \ .$$

The left part of this equality belongs to the space V_{k+m} as soon as the right part belongs to the space $\bigoplus_{i>k+m} V_i$. This means that $d_0 \in \operatorname{Der} L_n$. The description of the Lie algebra $\operatorname{Der} L_n$ (see Chapter 4) permits us to write

$$d_0 = \lambda_1 t_1 + \lambda_2 t_2 + \lambda_3 t_3 \ ,$$

where the endomorphisms t_1, t_2 and t_3 are defined by

$$t_1(e_i) = e_i \ , \quad 1 \leq i \leq n \ , \quad t_1(e_0) = 0 \ ,$$
$$t_2(e_0) = e_0 \ , \quad t_2(e_i) = (i-1)e_i \ , \quad 2 \leq i \leq n \ ,$$
$$t_3(e_0) = e_1 \ , \quad t_3(e_i) = 0 \ , \quad i \geq 1 \ ,$$

where (e_i) is the basis corresponding to the graduation of V.
Let $\psi_r(e_k, e_{k+1}) = a_k e_{2k+1+r}$ with $a_k \neq 0$. As $d_0 \in \operatorname{Der} L_n$, we have

$$d_0(\psi_r(e_k, e_{k+1})) - \psi_r(d_0(e_k), e_{k+1}) - \psi_r(e_k, d_0(e_{k+1})) =$$
$$= \psi_r\left(\sum_{i\geq 1} d_i(e_k), e_{k+1}\right) + \psi_r\left(e_k, \sum_{i\geq 1} d_i(e_{k+1})\right) - \sum_{i\geq 1} d_i(\psi_r(e_k, e_{k+1})) \ .$$

By comparing the coefficients to e_{2k+1+r} in the two parts of this identity, we see that $\lambda_1 = (r+1)\lambda_2$ and therefore $d_0 = \lambda_2\theta_2 + \lambda_3\theta_1$. Q.E.D.

Let $A = (L_n)_\psi$ be a filiform Lie algebra, with $\psi \neq 0$ and $\psi \in F_1H^2(L_n,L_n)$. Consider a derivation $d \in \operatorname{Der} A$. By a similar reasoning as the one of Lemma 2, we can affirm that $d_0 \in \operatorname{Der} L_n$ (also here we consider d as an endomorphism of the graded space $V = \oplus V_i$). Then we have

$$d_0 = \alpha_1 t_1 + \alpha_2 t_2 + \alpha_3 t_3 \ .$$

We put $d'_0 = \alpha_1 t_1 + \alpha_2 t_2$

Lemma 3. *Let* $A' = (L_n)_{\psi_r}$ *be the sill algebra of the Lie algebra* $A = (L_n)_\psi$ *. Then we have* $d'_0 \in \mathrm{Der}\, A'$.

Proof. We have

$$d'_0\,([a,b] + \psi_r\,(a,b)) - [d'_0\,(a),b] - \psi_r\,(d'_0\,(a),b) - [a,d'_0\,(b)] - \psi_r\,(a,d'_0\,(b)) =$$

$$= (-d+d'_0)\,([a,b] + \psi(a,b)) + [(d-d'_0)\,(a),b] + \psi((d-d'_0)\,(a),b) + [a,(d-d'_0)\,(b)] +$$
$$+ \psi(a,(d-d'_0)\,(b)) + \sum_{i>r} (d'_0\,(-\psi_i\,(a,b)) + \psi_i\,(d'_0\,(a),b) + \psi_i\,(a,d'_0\,(b)))\ .$$

Consider this identity for $a = e_s$, $s \geq 1$ and $b = e_m$, $m \geq 1$. Then the left hand is in V_{s+m+r}, as soon as the right hand belongs to $\bigoplus_{i\geq 1} V_{s+m+r+1}$. This implies that the left part is equal to zero. Now consider this identity for $a = e_0$ and $b = e_m$. Then we have $\psi_r(a,b) = 0$. As $d'_0 \in \mathrm{Der}\, L_n$, also in this case, the left part is equal to zero. This proves that $d'_0 \in \mathrm{Der}\, A'$.

•

Lemma 4. *Let* $A = (L_n)_\psi$ *be a filiform Lie algebra with* $\psi \in F_1H^2(L_n,L_n)$. *Consider a derivation* $d \in \mathrm{Der}\, A$. *Then*

$$d = \lambda d'_0 + d_1 + \dots + d_n \ , \quad \textit{with}$$
$$d'_0(e_0) = e_0 \ , \quad d'_0(e_i) = (i+r)e_i \ , \quad i = 1, \dots, n \ ,$$
$$d_s(e_i) = \lambda_{s,i} e_{i+s} \ , \quad i = 0, 1, \dots, n-s \quad \text{and} \quad 1 \leq s \leq n \ .$$

This lemma is a direct consequence of Lemmas 2 and 3.

Lemma 5. *Let* $A = (L_n)_\psi$ *be a filiform Lie algebra with* $\psi \in F_1H^2(L_n,L_n)$ *and let* $d = \lambda d'_0 + d_1 + \dots + d_n$ *be a derivation of* A *with* $\lambda \neq 0$ *(we use the notations of Lemma 4). Then, there is a basis* $f_0, f_1, \dots, f_n$ *of the space* V *such that*

$$[f_0\,, f_i] = f_{i+1} \ , \quad 1 \leq i \leq n-1 \ ;$$
$$d(f_0) = \lambda f_0 \ ; \ d(f_i) = \lambda(i+r)f_i + \lambda_1 f_{i+1} + \dots + \lambda_{n-i} f_n \ , \quad 1 \leq i \leq n-1 \ .$$

Proof. We have

$$d(e_0) = \lambda e_0 + \lambda_{1,0} e_1 + \dots + \lambda_{n,0} e_n \ .$$

We put $f_0 = e_0 + \alpha_1 e_1 + \dots + \alpha_n e_n$ and we choose the scalars $\alpha_1, \dots, \alpha_n$ for having $d(f_0) = \lambda f_0$. This is possible because the numbers r and λ are different to 0. By putting

$$f_1 = e_1 \ , \ f_k = [f_0, f_{k-1}] + \psi(f_0, f_{k-1}) \ , \ r \le k \le n \ ,$$

we find the required basis.

Lemma 6. *Suppose that the hypotheses of Lemma 5 are satisfied. Then there is a basis* $z_0, z_1, \dots, z_n$ *of the space* V *such that*

$$[z_0, z_i] = z_{i+1} \ , \ i = 1, \dots, n-1 \ ;$$

$$d(z_0) = \lambda z_0 \ , \ d(z_i) = \lambda(i+r) z_i \ , \ i = 1, \dots, n-1 \ .$$

Proof. We choose a basis $f_0, f_1, \dots, f_n$ given by Lemma 5 and we put

$$z_0 = f_0 \, ,$$
$$z_1 = f_1 + \alpha_1 f_2 + \dots + \alpha_{n-1} f_n \ ,$$
$$z_2 = f_2 + \alpha_1 f_3 + \dots + \alpha_{n-2} f_n \ ,$$

$$z_n = f_n \ .$$

It is easy to choose $\alpha_1, \dots, \alpha_{n-1}$ as such, as the basis (z_i) satisfies the required conditions.

Now we can return to the proof of the theorem.

Proof of the theorem. Let $A = (L_n)_\psi$ be a filiform Lie algebra with $\psi \in F_1H^2(L_n, L_n)$ and consider $A' = (L_n)_{\psi_r}$ its sill algebra. Suppose that A' is isomorphic to A. There is a derivation d of A' defined by

$$d(e_0) = e_0\ ,\quad d(e_i) = (i+r)e_i\ ,\ 1 \le i \le n\ .$$

This derivation is no nilpotent. Then A' and thus A, is not characteristically nilpotent. As $A = (L_n)_\psi$ is not characteristically nilpotent, there is a derivation d of A which is written $d = \lambda d'_0 + d_1 + ... + d_n$ (notations of Lemma 4). We choose a basis $z_0, z_1, ..., z_n$ satisfying the conclusion of Lemma 6. In this basis, the Lie algebra A is written as $(L_n)_{\psi'}$ i.e. $g(A) = (L_n)_{\psi'}$, where g is in GL(V) and is defined by $z_i = g(e_i)$, $i = 0, 1, ..., n$). The cocycle ψ' also belongs to $F_1H^2(L_n,L_n)$ and its sill cocycle is the same as the sill cocycle ψ_r of ψ. Show that $\psi' = \psi_r$. If it is not the case, we have

$$\psi' = \psi_r + \psi_s + ... + \psi_t\ ,\ \text{with}\ \psi_s \ne 0\ \text{and}\ s > r\ .$$

Then

$$d(\psi_s(a,b)) - \psi_s(d(a),b) - \psi_s(a,d(b)) = \lambda(s-r)\psi_s(a,b) = 0$$

for all $a, b \in (L_n)_{\psi'}$. This is impossible and the theorem is proved.

II. CHARACTERISTICALLY NILPOTENT LIE ALGEBRAS IN THE VARIETY $\mathcal{N}^n$

Let $A = (L_n)_\psi$ be a filiform Lie algebra, where ψ is a nonnull cocycle of $F_1H^2(L_n,L_n)$ and let ψ_r be the sill cocycle of ψ.

Lemma 1. *We consider* $s > r$ with $s \ne 2r$. *If there is a nonnull cocycle of* $H_s^2(L_n,L_n) \cap B^2((L_n)_{\psi_r}, (L_n)_{\psi_r})$ *, then this cocycle is unique up a nontrivial factor.*

Proof. Let

$$\psi_s \in H^2(L_n,L_n) \cap B^2((L_n)_{\psi_r}, (L_n)_{\psi_s})\ ,\quad \psi_s \ne 0\ .$$

There is $f \in End(L_n)_{\psi_r}$ such that

$$\psi_s(a,b) = f(\psi_r(a,b)) - \psi_r(f(a),b) - \psi_r(a,f(b)) + f([a,b]) - [f(a),b] - [a,f(b)]$$

We can suppose that

$$f(e_i) \in \mathbb{C}e_{i+s-r} \ , \ 1 \le i \le n-s+r \ ,$$

where (e_i) is the given basis of V (Section I), because this basis satisfies

$$[e_k,e_s] = 0 \quad \text{for} \quad 1 \le k,s \le n \ .$$

More, by adding, if necessary, the derivation $\lambda_k \,\mathrm{ad}\, e_k$ of the Lie algebra $(L_n)_{\psi_r}$ to the derivation f, we can suppose that this last satisfies $f(e_0) = \alpha e_1$ for some $\alpha \in \mathbb{C}$. We have

$$-\alpha\psi_r(e_1,e_i) + f([e_0,e_i]) - [e_0,f(e_i)] = 0 \ , \ i = 1 ,..., n.$$

As $\varphi \neq 2r$, we have $f(e_i) = \lambda e_{i+s+r}$, $1 \le i \le n-S+r$ and λ does not depend of the index i. The lemma is proved.

Lemma 2. *Let* $n \ge 7$ *and let* $\mathcal{U}$ *a open set in the variety* $\mathcal{N}^n$ *with* $A \in \mathcal{U}$ *Then there always there exists a characteristically nilpotent Lie algebra belonging to* $\mathcal{U}$.

Proof. If A is characteristically nilpotent, the lemma is proved. Suppose that A is not. Then from Theorem 1 (Section I), we can suppose that $\psi = \psi_r$ with $\psi_r \in H^2{}_r(L_n,L_n)$. We examine each of the following cases :

(1) $r < n-6$, $n-6 \neq 2r$;

(2) $r < n-6$, $n-5 \neq 2r$;

(3) $r = n-6$;

(4) $r > n-6$.

(1). The dimension of the space $H^2{}_{n-6}(L_n,L_n)$ is equal to 2 ; moreover, every cocycle of this space also is a cocycle of the Lie algebra $(L_n)_{\psi_r}$. From Lemma 1, there is a cocycle $\psi_{n-6} \in H^2{}_{n-6}(L_n,L_n)$ which does not belong to $B^2((L_n)_{\psi_r},(L_n)_{\psi_r})$. Every linear

combination of cocycles τ and ψ_{n-6} where τ is a cocycle linearly integrable belonging to $F_1H^2(L_n, L_n)$, is also linearly integrable. Then every Lie algebra of the form $(L_n)_{\psi(t)}$ with $\psi(t) = \psi_r + t\psi_{n-6}$, $t \neq 0$, is not isomorphic to the sill algebra $(L_n)_{\psi_r}$.
From Theorem 1, the Lie algebra $(L_n)_{\psi(t)}$ is characteristically nilpotent. By considering the nonnull values of t, when t tends to 0, $t \neq 0$, we obtain characteristically nilpotent Lie algebras belonging to a full neighborhood of A.

(2) The proof is similar to case (1).

(3) We consider a cocycle $\varphi \in H^2_{n-5}(L_n,L_n)$ and not cohomologous to zero, φ being considered as a cocycle of the Lie algebra $(L_n)_{\psi_r}$ (this is possible from Lemma 1). Then the Lie algebra $(L_n)_{\psi_r+t\varphi}$ is characteristically nilpotent for all values of t, $t \neq 0$.

(4) We can always find a cocycle φ of $H^2_{n-2}(L_n,L_n)$ which does not belong to $B^2((L_n)_\varphi, (L_n)_\varphi)$. For example, if $r > n-5$ and $\psi_r = \alpha\psi_{1,n-2} + \beta\psi_{2,n}$, $\alpha \neq 0$, then we take $\varphi = \psi_{2,n-1}$. If $r = n-5$ with $\alpha = 0$, then we take $\varphi = \psi_{1,n-3}$. The Lie algebra $(L_n)_{\psi_r+t\varphi}$ is characteristically nilpotent for all values of t different to 0. Q.E.D.

Theorem 2. *Let* $n \geq 7$ *and* C *be an irreducible component of the variety* $\mathcal{F}^{n+1}$ *(if* n *is odd moreover we suppose that* $Q_n \notin C$*). Then there is a nonempty Zariski open set of* C *whose elements are characteristically nilpotent Lie algebras.*

Remark. Finallye, by considering the classification of noncharacteristically nilpotent Lie algebras, we have could remove the hypothesis $Q_n \notin C$ and prove the theorem for any component [GH2]. We have carried this result to the end of this chapter.

Proof of Theorem 2. Recall that an external torus of the derivation of a Lie algebra L is an Abelian subalgebra of Der L whose elements are semisimple. We know that there is only a finite number of conjugation classes of maximal torus for a nilpotent Lie algebra L (see Chapter 1). Let T_i, $0 \leq i \leq k$, be representatives of these conjugation classes, T_0 being the null torus. Consider the closed subset $\mathcal{N}_{T_i}^{n+1}$ of $\mathcal{N}^{n+1}$ whose elements are the T_i-invariant laws ; the union of the orbits $GL_{n+1}(\mathbb{C}) \times \mathcal{N}_{T_i}^{n+1}$ for $1 \leq i \leq k$ is a constructible subset, i.e. a finite union of locally closed subsets. It is complementary to a subsetof $\mathcal{N}^{n+1}$ formed by characteristically nilpotent laws. Then this last subset is also a constructible subset of $\mathcal{N}^{n+1}$. Now if the theorem is not true, there is a nonempty open set of C formed by non characteristically nilpotent Lie algebras. From the lemma 2, this is impossible. Q.E.D.

Corollary. *There always exists a nonempty Zariski open set of* $\mathcal{N}^m$, $m \geq 7$. *whose every element is characteristically nilpotent.*

This corollary is a direct consequence of Theorem 2 when $m \geq 8$. For $m = 7$, we can construct directly a nonempty Zariski open set whose elements are characteristically nilpotent. For example, the orbit of the familly $\mathfrak{n}_7{}^{13a}$ (see Chapter 2).

III. ON THE NON FILIFORM CHARACTERISTICALLY NILPOTENT LIE ALGEBRAS

III.1. Construction of some families of nonfiliform characteristically nilpotent Lie algebras

Let $\mathfrak{g}$ be a simple Lie algebra such that its type is different to A_1, A_2, A_3, A_4, A_5, B_2, B_3, C_3, C_4, D_4, and G_2 . Let $\mathfrak{n}$ be the radical of a Borel subalgebra of $\mathfrak{g}$ (we use here the notations of Chapters 4 and 6). Recall that, in this case, the spaces $F_0H^2(\mathfrak{n},\mathfrak{n})$ and $F_1H^2(\mathfrak{n}, \mathfrak{n})$ coincide. The following cocycles $f_{\alpha,\beta}$, $(\alpha , \beta) \in E$, defined by

$$f_{\alpha,\beta}(e_\gamma , e_\nu) = \begin{cases} e_{s_\alpha s_\beta(\delta)} , & \text{if } (\gamma,\nu) = (\alpha , s_\alpha(\beta)) , \\ 0 , & \text{if } (\gamma,\nu) \neq (\alpha , s_\alpha(\beta)) , \end{cases}$$

where E is the set formed of pairs (α, β) , $\alpha, \beta \in S$, give a basis of $F_1H^2(\mathfrak{n} , \mathfrak{n})$. In the definition of E, we suppose that the pair (α,β) and (β,α) are identified when α and β are are not tied by a row in the Dynkin diagram and in this case $f_{\alpha,\beta} = f_{\beta,\alpha}$. Moreover, we have

$$\dim F_0H^2(\mathfrak{n},\mathfrak{n}) - \dim F_1H^2(\mathfrak{n},\mathfrak{n}) = \frac{r^2+r-2}{2}\ , r = \operatorname{rank}\mathfrak{g}\ .$$

In the next Chapter (as in the previous chapter), we will identify the cocycles with their cohomology classes.

Let $\varphi \in F_0H^2(\mathfrak{n},\mathfrak{n})$. It is easy to verify that $\varphi \circ \varphi = 0$, i.e. the cocycle φ is linearly integrable. The Lie algebra defined in the underlying space to $\mathfrak{n}$ by the bracket

$$[a,b]_\varphi = [a,b]_\mathfrak{n} + \varphi(a,b)$$

is denoted by $\mathfrak{n}_\varphi$.

Theorem 3. *Let $\mathfrak{n}$ be the nilradical of a Borel subalgebra of a simple algebra $\mathfrak{g}$ whose type is different to A_r (r = 1,..., 5), B_2, B_3, C_3, C_4 and D_4*

Let $\varphi = \sum_{\omega\in E} \lambda_\omega f_\omega$ be an element of $F_0H^2(\mathfrak{n},\mathfrak{n})$ such that $\lambda_\omega \neq 0$ for all $\omega \in E$. Then $\mathfrak{n}_\varphi$ is a characteristically nilpotent Lie algebra.

Let $d \in \operatorname{Der}\mathfrak{n}_\varphi$ and $d = d_0 + d_1 + ... + d_{p-1}$ be the representation of d as a sum of homogeneous endomorphisms of the graded space $C^1(\mathfrak{n},\mathfrak{n})$. Here, we consider the graduation of $\mathfrak{n}$ associated to the filtration of $\mathfrak{n}$ given by the descending central sequence (see Section III of Chapter 4). It is clear that $d_0 \in \operatorname{Der}\mathfrak{n}$. From the description of Der $\mathfrak{n}$ (see Chapter 4), we have $d_0 = \operatorname{ad} h$, with $h \in \mathfrak{h}$. Let

$$g = d - d_0 = \sum_{i=1}^{p-1} d_i$$

Then

$$g([a,b]) - [g(a),b] - [a,g(b)] + g[\varphi(a,b)] - \varphi[g(a),b] - \varphi(a,g(b)) \\ + d_0(\varphi(a,b)) - \varphi(d_0(a),b) - \varphi(a,d_0(b)) = 0\ .$$

First, consider the case $\mathfrak{g}$ of type A_r (r > 5). For the pairs

$(a,b) = (e_{\alpha_k}, e_{\alpha_k+\alpha_{k+1}})$ and $(a,b) = (e_{\alpha_{k+1}}, e_{\alpha_k+\alpha_{k+1}})$

$1 < k < n-1$, we have :

(i) The projections of the elements $[a,g(b)]$ and $[g(a), b]$ on $\mathfrak{g}_\delta$, where δ is a maximal root, are null because $\delta - \alpha_k$ and $\delta - \alpha_{k+1}$ are not in Δ.

(ii) From the description of $F_0H^2(\mathfrak{n},\mathfrak{n})$, we have $\varphi(a, b) = \lambda e_\delta$ for some $\lambda \in \mathbb{C}$. Thus $g(\varphi(a),b)) = 0$, because $g = d - d_0 = d_1 + \dots + d_{p-1}$.

(iii) $\varphi(g(a),b) = 0$, because $g(a)$ and $b \in [\mathfrak{n},\mathfrak{n}]$.

(iv) $\varphi(a,g(b)) = 0$, because $g(b) \in [\mathfrak{n},[\mathfrak{n},\mathfrak{n}]]$.

By writing that the coefficient of e_δ is in the left part of the above identity, we obtain

$$\lambda_{\alpha_k,\alpha_{k+1}} \left(\delta(h) - 2\alpha_k(h) - \alpha_{k+1}(h)\right) = 0 \ ,$$
$$\lambda_{\alpha_{k+1},\alpha_k} \left(\delta(h) - \alpha_k(h) - 2\alpha_{k+1}(h)\right) = 0 \ .$$

Then $\alpha_k(h) = \alpha_{k+1}(h)$ for all $1 < k < l-1$, because $\lambda_\omega \neq 0$. Moreover, we have

$$\alpha_1(h) + \alpha_r(h) = (r-5)\alpha_k(h) \text{ for all } k,\ 1 < k < r \ .$$

Now consider the pairs $(a,b) = (e_{\alpha_1}, e_{\alpha_3})$ and $(a,b) = (e_{\alpha_{r-2}}, e_{\alpha_r})$.

The same reasoning gives :

$$\left(\delta - \alpha_1\right)(h) - \alpha_1(h) - \alpha_3(h) = 0$$
$$\left(\delta - \alpha_r\right)(h) - \alpha_{r-2}(h) - \alpha_r(h) = 0 \ .$$

This implies

$$\alpha_r(h) - \alpha_1(h) - (r-3)\alpha_2(h) = 0 \ ,$$
$$\alpha_1(h) - \alpha_r(h) + (r-3)\alpha_2(h) = 0 \ ,$$

and

$$\alpha_2(h) = 0 \ , \quad \alpha_1(h) = 0 \ . \text{ Then } \alpha_1(h) = \alpha_2(h) = \dots = \alpha_r(h) = 0$$

and, therefore, $h = 0$. As we have supposed that $d_0 = \operatorname{ad} h$, $d_0 = 0$ and any derivation d of the Lie algebra $\mathfrak{n}_\varphi$ is nilpotent.

For other Lie algebras $\mathfrak{g}$ of a type different to A_r, we can show, using the same arguments, that $\mathfrak{n}_\varphi$ is characteristically nilpotent. We will restrict ourselves to the case $\mathfrak{g} = E_6$.

We consider always the above identity but for the pairs

$$(e_{\alpha_1}, e_{\alpha_1+\alpha_3}), (e_{\alpha_3}, e_{\alpha_1+\alpha_3}), (e_{\alpha_3}, e_{\alpha_3+\alpha_4}),$$
$$(e_{\alpha_4}, e_{\alpha_4+\alpha_5}), (e_{\alpha_5}, e_{\alpha_5+\alpha_6}), (e_{\alpha_1}, e_{\alpha_2})$$

This gives

$$\delta(h) - 2\alpha_1(h) - \alpha_3(h) = \delta(h) - \alpha_1(h) - 2\alpha_3(h) = \delta(h) - 2\alpha_3(h) - \alpha_4(h) =$$
$$= \delta(h) - 2\alpha_4(h) - \alpha_5(h) = \delta(h) - 2\alpha_5(h) - \alpha_6(h) = \delta(h) - \alpha_1(h) - 2\alpha_2(h) = 0 \ ,$$

Thus

$$\delta(h) = \alpha_1(h) + 2\alpha_2(h) + 2\alpha_3(h) + 3\alpha_4(h) + 2\alpha_5(h) + \alpha_6(h) \ .$$

Then $\alpha_i(h) = 0$, $1 \le i \le 6$, this implies that $h = 0$ and $d_0 = 0$.

Q.E.D.

III.2. The study of special cases

If $\mathfrak{n}$ is the nilradical of a Borel subalgebra of simple algebras of type A_i, $i = 1, 2, 3, 4$, B_2, B_3, C_3, C_4 or D_4, the proof of Theorem 3 is not correct. In fact, in these cases, we have

$$F_0H^2(\mathfrak{n},\mathfrak{n}) \neq F_1H^2(\mathfrak{n},\mathfrak{n}) \ .$$

Moreover, the description of the cocycles of the space $F_0H^2(\mathfrak{n},\mathfrak{n})$ are, in these cases, different to the descriptions of the other cases (see Chapter 1). However, we can extend the results of Theorem 3 by studying each one of these particular cases.

Theorem 4. *Let $\mathfrak{g}$ be a simple Lie algebra of the following types* : A_4, A_5, B_3, C_3, C_4, D_4, *and let $\mathfrak{n}$ be the nilradical of a Borel subalgebra of $\mathfrak{g}$. Let* $\{f_\omega, \omega \in E\}$ *be a basis of* $F_0H^2(\mathfrak{n}, \mathfrak{n})$ *described in Chapter 4 (see Table 3). Consider*

$$\varphi_1 = \sum_{\omega \in E} \lambda_\omega f_\omega \quad \text{with} \quad f_\omega \neq 0 \ \ \forall\, \omega \in E \ ,$$

and φ_2 *given in Table 4, and characterized by the fact that* $\mu + t\,\varphi_1$ *does not satisfy the Jacobi conditions* (μ *is the law of* $\mathfrak{n}$). *Then the Lie algebra* $\mathfrak{m}$ *whose law* τ is *the deformation* $\tau = \mu + t\varphi_1 + t^2\varphi_2$ *with* $t \neq 0$, *is characteristically nilpotent.*

Remark. If $\mathfrak{g}$ is a Lie algebra of type A_1, A_2, A_3, B_2, G_2, then we have $F_0H^2(\mathfrak{n},\mathfrak{n}) = 0$.

Table 4

$\mathfrak{g}$	φ_1	φ_2	φ_i, $i \geq 3$
A_4	$a_1v_1 + a_2v_2 + a_3v_3 +$ $+ b_1u_1 + b_2u_2 + b_3u_3 + b_4u_4$ $a_1, b_3 \neq 0 \vee a_1b_4 \neq 0$	$e_{\alpha_4} \wedge e_{\alpha_2} \to \dfrac{b_3a_1}{N_{\alpha_1,\, \delta-\alpha_1}} \cdot e_{\delta-\alpha_1}$ $e_{\alpha_4} \wedge e_{\alpha_3} \to \dfrac{b_4a_1}{N_{\alpha_1,\, \delta-\alpha_1}} \cdot e_{\delta-\alpha_1}$	0
C_3	$b_1u_1 + b_2u_2 + b_3u_3$ $b_1, b_3 \neq 0$	$e_{\alpha_2} \wedge e_{\alpha_3} \to \dfrac{b_1b_3}{N_{\alpha_1,\, \delta-\alpha_1}} \cdot e_{\delta-\alpha_1}$	0
C_3	$a_1v_1 + b_1u_1 + b_2u_2 + b_3u_3$ $a_1, b_3 \neq 0$	$e_{\alpha_1+\alpha_2} \wedge e_{\alpha_2} \to \dfrac{a_1b_3}{N_{\alpha_1,\, \delta-\alpha_1}} \cdot e_{\delta-\alpha_1}$ $e_{\alpha_2} \wedge e_{\alpha_3} \to \dfrac{b_1b_3}{N_{\alpha_1,\, \delta-\alpha_1}} \cdot e_{\delta-\alpha_1}$	0
C_4	$a_1v_1 + \sum_{i=1}^{8} b_iu_i$ $a_1, b_7 \neq 0$	$e_{\alpha_3} \wedge e_{\alpha_2} \to \dfrac{a_1b_7}{N_{\alpha_1+\alpha_2,\, \delta-\alpha_1-\alpha_2}} \cdot e_{\delta-\alpha_1-\alpha_2}$	0

III.3. Characteristically nilpotent Lie algebras in the variety $\mathcal{N}_n{}^p$

The following theorem is a direct consequence of Theorems 3 and 4 of this chapter and 3 and 4 of Section VI, in Chapter 4.

Theorem 5. *Let* $\mathfrak{n}$ *be the nilradical of a Borel subalgebra of a simple algebra* $\mathfrak{g}$ *of rank* r, *of a type different to* A_1, A_2, A_3, B_2 *and* G_2 ; *let* p *be the nilindex of* $\mathfrak{n}$, $n = \dim \mathfrak{n}$ *and we consider* F *as the subset of* $\mathcal{N}_n{}^p$ *constituted from the characteristically nilpotent Lie algebras described in Theorems* 3 *and* 4. *Then* $\overline{G(F)}$ *is a component of the variety* $\mathcal{N}_n{}^p$ *and the subset of characteristically nilpotent Lie algebras of* $\overline{G(F)}$ *contains a nonempty Zariski open. If* $\mathfrak{g}$ *is of type* A_4, B_3, C_3 *or* D_4 , *then the dimension of* $\overline{G(F)}$ *is, respectively ,equal to* 90, 71, 71, 131. *In the other cases, we have*

$$\dim \overline{G(F)} = n^2 - n - 2r + 1 + \frac{r^2 + r - 2}{2} .$$

(Recall that G(F) is the orbit of F with respect to the action of GL($\mathfrak{n}$)).

CHAPTER 8

APPLICATIONS TO DIFFERENTIAL GEOMETRY : THE NILMANIFOLDS

Introduction

Of course, Lie algebras arise in a natural way in the study of transformation groups in differential geometry. But, nilpotent Lie algebras play an essential role in this field : they permit the construction of many examples of concrete differential manifolds and, more precisely, compact differential manifolds. These manifolds are called nilmanifolds and their differential calculus is, usually, nothing more than a linear calculus on the corresponding nilpotent Lie algebra. For example, the classical algebraic and topological invariants of a nilmanifold, such as the De Rham cohomological classes, are entirely defined in the Lie algebra. This explains why the exceptional examples of affine manifolds, of symplectic compact non-kahlerian manifolds, of naturally reductive Riemaniann manifolds, and the counter examples of Lichnerowicz's conjecture on the harmonic spaces are described as nilmanifolds.

I. A SHORT INTRODUCTION TO THE THEORY OF LIE GROUPS

I.1. Lie groups

A Lie group is, roughly speaking, an analytic manifold with a group structure such that the group operations

$$(x,y) \to xy^{-1}$$

are analytic.
If G is a Lie group, then the connected component passing through the identity of G is a connected Lie subgroup of G.

We note by $L_a : x \to ax$ the left translation by a in G and R_a the right translation by a. A vector field X on G is called left invariant if it satisfies

$$(L_a)^* X_x = X_{ax} ,$$

where L_a^* is the Jacobian (or the tangent mapping) of f. As

$$L_a^* [X,Y] = [L_a^* X , L_a^* Y] ,$$

the set of left invariant vector fields of G is a Lie algebra, denoted $\mathfrak{g}$, called the Lie algebra of G. As a left invariant vector field X on G is known as soon as its value X_e in the identity e of G of X is given, we can identify $\mathfrak{g}$ to the vector tangent space T_eG of G in e, the bracket on T_eG is deduced to the bracket of $\mathfrak{g}$.

$$[X_e, Y_e] = [X, Y]_e \quad . \quad X, Y \in \mathfrak{g} .$$

I.2. Correspondence between Lie groups and Lie algebras

The exponential mapping

$$\exp : \mathfrak{g} \to G$$

of the Lie algebra $\mathfrak{g}$ of G in the Lie group G sends straight lines throught the origin in $\mathfrak{g}$ into one-parameter subgroups of G. This mapping is defined as this : let $X \in \mathfrak{g}$. There exists a unique analytic homomorphism θ of $\mathbb{R}$ into G such that $\theta(0) = X$. We put $\exp X = \theta(1)$. So we have $\exp t X = \theta(t)$ and it is a one-parameter subgroup of G. We have the formula

$$\exp(t+s)X = \exp tX \exp sX$$

for all $s, t \in \mathbb{R}$ and all X in $\mathfrak{g}$.

Let us illustrate this theorem for the matrix group G. It is well know that a matrix A sufficiently close to the matrix identity I, is expressible as a matrix exponential functions

$$A = \exp X = I + X + \frac{X^2}{2!} + \dots + \frac{X^n}{n!} + \dots$$

We have

$$\exp tX = I + tX + t^2\frac{X^2}{2!} + \dots + t^n\frac{X^n}{n!} + \dots \quad \text{and} \quad \exp 1X = A \quad , \quad \exp 0X = I \ .$$

This explains the choice of the word "exponential" for the mapping between $\mathfrak{g}$ and its Lie group G.

Now let $x = \exp X$ and $y = \exp Y$ be two arbitrary elements of the Lie group G, where X and Y are in the Lie algebra $\mathfrak{g}$. The Campbell-Hausdorff series :

$$xy = \exp X \exp Y = X + Y + [X,Y] + \sum_{p_i,q_i} \frac{(-1)^{m-1}}{m} \cdot \frac{[X^{p_1}Y^{q_1} \dots X^{p_m}Y^{q_m}]}{\Sigma\,(p_i+q_i)\,\pi_{p_i!q_i!}},$$

where the summation extends over all nonnegative integers p_i, q_i, $i = 1, \dots, m$, $m = 1, 2, \dots$ and $p_i + q_i \neq 0$ (except m = 0 and 1 which have been written explicitly) defines the multiplication in a neighborhood of the groug G. Consequently, an arbitrary Lie algebra $\mathfrak{g}$ (in finite dimension) determines a unique Lie group, up to local isomorphism, such that its Lie algebra is $\mathfrak{g}$.

In particular there is a one-to-one correspondence between a finite-dimensional Lie algebra and a connected, simply connected Lie group.
An interesting problem consists in determining the class $\Omega(\mathfrak{g})$ of the Lie groups having $\mathfrak{g}$ as the Lie algebra. The class $\Omega(\mathfrak{g})$ contains the unique simply connected Lie group which appears as the universal covering group of $\Omega(\mathfrak{g})$.

Example. Let $\mathfrak{g}$ be the one-dimensional (Abelian) Lie algebra. Then $\Omega(\mathfrak{g})$ contains the additive group of reals (it is the universal covering) and the multiplicative group of complex numbers (group of rotations of a circles).

Remark. Consider a Lie algebra $\mathfrak{g}$ and G the corresponding simply connected Lie group. Then every Lie group H whose Lie algebra is $\mathfrak{g}$, can be writen as $H = G / \Gamma$, where Γ is a discrete subgroup of the center of G.

I.3. The left invariant geometry. Interpretation of $\mathfrak{g}^*$

The Lie algebra $\mathfrak{g}$ of a Lie group G has been defined as the vector space of left invariant vector fields on G. The dual space $\mathfrak{g}^*$ can be identified as the left invariant Pfaffian forms on G. Let ω be a differential form on G. It is left invariant if

$$^*(L_x)\, \omega_x = \omega_e ,$$

where $^*f$ is the transposed mapping of f^*.

This permits us to define an exterior differential d on the vector space $\mathfrak{g}^*$ by putting

$$d\omega\,(X,Y) = -\,\omega[X,Y]$$

where X, Y are in $\mathfrak{g}$ and $\omega \in \mathfrak{g}$. We can also extend this differentiation for every p-left invariant form on G (i.e. for the elements of $\Lambda^p\mathfrak{g}^*$).

So we define a complex of cochains and cohomological spaces, denoted $H^i(\mathfrak{g})$ (see Chapter 3). In particular, we have

$$H^0(\mathfrak{g}) = \mathfrak{g} \ ,$$
$$H^1(\mathfrak{g}) = \frac{Z^1(\mathfrak{g})}{B^1(\mathfrak{g})} \ ,$$

where $Z^1(\mathfrak{g}) = \left\{\omega \in \mathfrak{g}^* : d\omega = 0\right\}$,

$$B^1(\mathfrak{g}) = \{df : f \text{ invariant on } \mathfrak{g}\} = \{0\} \ .$$

This gives $H^1(\mathfrak{g}) = Z^1(\mathfrak{g})$.

Often this cohomology is called the cohomology of left invariant forms on the Lie group G. In general, this cohomology doesn't coincides with the De Rham cohomology of G. But, if G is a compact Lie group, then these cohomologies coincide. The nilpotent case is examined in the next section.

Remark. The Maurer Cartan equations. We can read the constants of the structure of a Lie algebra $\mathfrak{g}$ by writing the Maurer Cartan equations on $\mathfrak{g}^*$.
Let $(X_1, ..., X_n)$ be a basis of $\mathfrak{g}$ such that

$$[X_i , X_j] = \Sigma \, C^k_{ij} X_k \ .$$

Consider $(\omega_1, ..., \omega_n)$ as the dual basis of $\mathfrak{g}^*$. Then, we have

$$d\omega_k = - \Sigma \, C^k_{ij} \, \omega_i \wedge \omega_j \, , \qquad k = 1, ..., n \, ,$$

and these equations again givethe constants of the structure of $\mathfrak{g}$. Note that Jacobi's identities correspond to $d(d\omega_k) = 0 \quad k = 1, ..., n$.

II. THE NILMANIFOLDS

Let G be a nilpotent Lie group; then its Lie algebra $\mathfrak{g}$ is nilpotent. In this section, we suppose that G is connected and simply connected.

II.1. Definition of nilmanifolds

Definition 1. *Let Γ be a discrete closed subgroup of* G. *Then the quotient space* G/Γ *is provided with the structure of a differential manifold (of dimension equal to the dimension G). It is called a nilmanifold.*

An interesting problem consists in determining the discrete closed subgroups Γ of G such that the nilmanifolds G/Γ are compact. A problem in the classification of these manifolds naturally occurs. The answer is well known in a few cases only (Euclidean case, Heisenberg case).

II.2. Uniform subgroups

Definition 2. *A subgroup Γ of* G *is called a uniform subgroup if* G/H *is compact.*

Usually, the research of uniform subgroups passes through the research of lattice, i.e discrete, subgroups such that G/Γ carries a finite invariant measure. If every discrete uniform subgroup is a lattice, the converse is not usually true (see, for example, "Discrete subgroups of Lie groups" by Raghunatan [RAG]).

The situation is simpler in the nilpotent case.

Theorem 1. *Let* G *be a simply connected nilpotent Lie groups and Γ a closed subgroup of* G . *Then* G/Γ *is compact if and only if Γ is a lattice.*

We can also characterize the uniformity of a subgroup in terms of representation. A representation $\rho : G \to GL(n,\mathbb{R})$ is nilpotent if $\rho(x)$ is unipotent for all $x \in G$. As G is connected and simply connected, such a representation always exists. It can be also supposed free. If Γ is a closed subgroup of G, then G/Γ is compact if and only if for any representation $\rho : G \to GL(n, \mathbb{R})$, $\rho(G)$ and $\rho(\Gamma)$ have the same closure for the Zariski topology, in the algebraic group $GL(n, \mathbb{R})$.

So the existence of uniform groups of G is read through the matricial representation of the nilpotent group G.

II.3. Existence of discrete uniform subgroups

Theorem 2. *Let G a simply connected nilpotent Lie group and let $\mathfrak{g}$ be its Lie algebra. Then G admits a discrete uniform subgroup if and only if $\mathfrak{g}$ is a rational Lie algebra, i.e. $\mathfrak{g}$ admits a basis with respect to which the constants of structure are rational.*

The proof of this theorem can be read in [RAG].

This theorem is very useful for constructing nilmanifolds. The rationality of a nilpotent Lie algebra is not very difficult to see. Nevertheless, the determination of the discrete uniform subgroup of a rational nilpotent Lie group is a hard problem, just as the problem of the classification of nilmanifolds.

II.4. Rational nilpotent Lie algebras

From the classifications of nilpotent Lie algebras of dimension less than seven, we can deduce the following result.

Proposition 1. *Every complex (and real) nilpotent Lie algebra of dimension less than 6 is rational. Every complex nilpotent nonrational Lie algebra of dimension 7 belongs to a parametrized family of nonisomorphic Lie algebras.*

III. CONTACT AND SYMPLECTIC GEOMETRY ON NILPOTENT LIE ALGEBRAS

III.1. The Cartan class of an element of $\mathfrak{g}^*$

We have identified the elements of $\mathfrak{g}^*$ with the left invariant forms on a Lie group G whose Lie algebra is $\mathfrak{g}$. Then we can see how, within this framework, the classical differential invariants be have themselves. The most important of them is probably the Cartan class of a Pfaffian form.

Definition 3. *Let* ω *be in* $\mathfrak{g}^*$. *The characteristic subalgebra of* ω *is the subalgebra of* $\mathfrak{g}$ *defined by*

$$\mathfrak{h}_\omega = \{X \in \mathfrak{g} \ , \ i(X)\, d\omega = 0\},$$

where $i(X)\ d\omega$ *is the element of* $\mathfrak{g}^*$ *given by*

$$i(X)\, d\omega\,(Y) = d\omega\,(X, Y) = -\,\omega[X, Y].$$

The class (or Cartan class) of ω *is the codimension in* $\mathfrak{g}$ *of the vector space* $\operatorname{Ker}\omega \cap \mathfrak{h}_\omega$

Proposition 2. *If* $\mathfrak{g}$ *is a nilpotent Lie algebra, then every form* $\omega, \omega \in \mathfrak{g}^*, \omega \neq 0$ *has an odd class.*

Proof. Consider the 2-exterior form $d\omega$ on $\mathfrak{g}$. It is a bilinear alternated form on $\mathfrak{g}$ defined by

$$d\omega\,(X, Y) = -\,\omega\,[X, Y].$$

Lemma. *If the class of* ω *is odd, there is a basis* $(\omega = \omega_1, \omega_2, \ldots, \omega_n)$ *of* $\mathfrak{g}^*$ *such that*

$$d\omega = \omega_2 \wedge \omega_3 + \ldots + \omega_{2p} \wedge \omega_{2p+1}.$$

If the class of ω *is even and equal to* $2p$*, there is a basis* $(\omega = \omega_1, \ldots, \omega_n)$ *of* $\mathfrak{g}^*$ *such that*

$$d\omega = \omega \wedge \omega_2 + \ldots + \omega_{2p-1} \wedge \omega_{2p}.$$

This lemma is a direct consequence of the classification of the alternated bilinear form on a real or complex vector space. From this lemma, we can easily see that the dual space $(\operatorname{Ker} \omega \cap \mathfrak{h}_\omega)^*$ is generated by $(\omega_{2p+2}, \ldots, \omega_n)$ if the class is 2p+1, or by $(\omega_{2p+1}, \ldots, \omega_n)$ if the class is 2p.

Proof of the proposition. Let $(\omega = \omega_1, \ldots, \omega_n)$ be a basis of $\mathfrak{g}$ reducing to the canonical form $d\omega$ (Lemma 3). Suppose, first, that $\omega \in Z(\mathfrak{g})^*$. If $(X_1, X_2, \ldots, X_n)$ is the basis of $\mathfrak{g}$ such that $(\omega_1, \ldots, \omega_n)$ is the dual basis, then $X_1 \in Z(\mathfrak{g})$ and $[X, Y] = 0$, $\forall Y$. Thus, $cl(\omega) = 2p$ implies $d\omega(X, X_2) = 1$, i.e. $\omega[X, X_2] \neq 0$. This is impossible and $cl(\omega) = 2p+1$. Suppose now that $\omega(X) = 0$, $\forall X \in Z(\mathfrak{g})$, the matrix of the operator ad X_1 is nilpotent and nontrivial (we suppose $\omega \neq 0$). Jordan's reduction of this operator shows that the equation $[X_1, Y] = X_1 + U$, where U is independent of X_1, has no solution. Then $d\omega(X_1, Y) = 0$ for all Y and ω has an odd class.

Remark. Suppose that $\mathfrak{g}$ is rational. If G is the simply connected nilpotent Lie group associated to $\mathfrak{g}$, then there is a closed discrete uniform subgroup Γ such that G/Γ is a compact manifold. As ω corresponds to a left invariant form on G, it passes through the quotient G/Γ and defines a Pfaffian form ω' on G/Γ. It is clear that ω' has an even class. Let H_ω be the Lie group corresponding to the Lie algebra h_ω associated to ω. It is closed in G and $\pi(H_\omega) \cap G/\Gamma$ is a compact manifold whose dimension is equal to the class of ω as soon as ω is supposed to have an even class equal to 2p (π is the projection on G/Γ). The Stokes theorem implies

$$\int_{\pi(H_\omega)\cap G/\Gamma} d\omega = \int_{\partial(\pi(H_\omega)\cap G/\Gamma)} \omega = 0 .$$

This denies the fact that $(d\omega)^p$ is a volume form on $\pi(H_\omega) \cap G/\Gamma$ but explains the previous proposition.

III.2. Contact Lie algebras. A characterization of the Heisenberg algebra

Definition 4. *A* $(2p+1)$*-dimensional Lie algebra* $\mathfrak{g}$ *is called a contact Lie algebra if there is a contact form* ω *on* $\mathfrak{g}$*, i.e. is a form* $\omega \in \mathfrak{g}^*$ *having a class equal to* $2p+1$.

Proposition 3. *If* $\mathfrak{g}$ *is a contact Lie algebra, then* $\dim Z(\mathfrak{g}) \leq 1$.

Proof. Suppose that $\dim Z(\mathfrak{g}) \geq 2$. Then there are two independent vectors X_1 and X_2 such that $[X_j\, Y] = 0$, $\forall\, Y \in \mathfrak{g}$, $j = 1$ and 2. If $\omega \in \mathfrak{g}^*$, then $i(X_j)d\omega = 0$ for $j = 1$ and 2 and $cl(\omega) \leq 2p$.

Consequence. *If* $\mathfrak{g}$ *is a nilpotent contact Lie algebra, then* $\dim Z(\mathfrak{g}) = 1$.

Example. The Heisenberg algebra H_p is a contact Lie algebra. Indeed, contacts of the structure of H_p are given by

$$[X_{2i}, X_{2i+1}] = X_1, \qquad i = 1, \dots, p ,$$

and if $(\omega_1, \dots, \omega_{2p+1})$ is the dual basis, we have

$$d\omega_1 = -\omega_2 \wedge \omega_3 - \dots - \omega_{2p} \wedge \omega_{2p+1} \quad .$$

Theorem 3. *In the variety of Lie algebras* L^{2p+1}*, there exists a neighborhood of the Heisenberg algebra* H_p *meeting all the orbits of contact Lie algebras of* L^{2p+1} *and meeting only these orbits. This property characterizes the Heisenberg algebra.*

Proof. Let $\mathfrak{g}$ be a contact Lie algebra. Let $(X_1, \dots, X_{2p+1})$ be a basis of $\mathfrak{g}$ such that the dual basis $(\omega_1, \dots, \omega_{2p+1})$ satisfies

$$d\omega_1 = -\omega_2 \wedge \omega_3 + \dots + \omega_{2p} \wedge \omega_{2p+1} \quad .$$

The constants of structure corresponding are given by

$$\left\{ \begin{array}{l} [X_2, X_3] = X_1 + \sum_{i \geq 2} C^i_{23} X_i \\ \\ [X_{2p}, X_{2p+1}] = X_1 + \sum_{i \geq 2} C^i_{2p\ 2p+1} X_i \\ \\ [X_i, X_j] = \sum_{k \geq 2} C^k_{ij} X_k \qquad (i,j) \neq (2,3), \dots, (2p,2p+1 \end{array} \right\} .$$

Consider the contraction given by the following sequence of isomorphisms

$$\left\{ \begin{array}{l} f_n(X_i) = \frac{1}{n} X_i \qquad i \geq 2 \\ f_n(X_1) = \left(\frac{1}{n}\right)^2 X_1 \end{array} \right\}$$

The bracket of the corresponding Lie algebra $\mathfrak{g}_n$ defined by f_n satisfies:

$$\left\{ \begin{array}{l} [X_2, X_3] = X_1 + \frac{1}{n} \sum_{i \geq 2} C^i_{23} X_i \\ \\ [X_{2p}, X_{2p+1}] = X_1 + \frac{1}{n} \sum_{i \geq 2} C^i_{2p\ 2p+1} X_i \\ \\ [X_i, X_j] = \frac{1}{n} \sum C^k_{ij} X_k \end{array} \right\} .$$

The limit-point $\mathfrak{g}_\infty$, which is a contraction of $\mathfrak{g}$, is the Heisenberg algebra. Then every contact Lie algebra can be contracted on the Heisenberg algebra. This proves the first part of the theorem. As the existence of a contact form is an "open" property, every perturbation of the Heisenberg algebra also is a contact Lie algebra. Then we can characterize a neighborhood of H_p in L^{2p+1}. For proving that this property characterizes the Heisenberg algebra, it is sufficient to see that another "model" can be contracted on H_p and which is also a contraction of H_p. If this model is a contraction of H_p , it is nilpotent and its characteristic sequence is less than the characteristic sequence of H_p and then it is equal to (2, 1, ..., 1) or (1, 1, ..., 1). But every nilpotent Lie algebra having (2, 1, ..., 1) as a characteristic sequence is isomorphic to H_p or $H_i \oplus \mathfrak{a}_{2(p-i)}$ where $\mathfrak{a}_j$ is the j-dimensional Abelian algebra. As this model is a contact algebra, we have $i = p$ and we again find the Heisenberg algebra. The case (1, 1, ..., 1) is impossible because it corresponds to the Abelian algebra.

III.3. Lie algebras with a symplectic structure

Frst, it is important not to blend a Lie algebra with a symplectic structure and the symplectic algebra sp(n).

III.3.1 Definition

Definition 5 . *An alternated bilinear form* θ *on* $\mathfrak{g}$ *is called symplectic if it satisfies*

(1) $d\theta = 0$,

(2) $\theta^p \neq 0$, *where* $\dim \mathfrak{g} = 2p$.

Here $d\theta$ corresponds to the differential of the left invariant 2-form θ. It is given by

$$d\theta(X , Y , Z) = \theta([X , Y] , Z) + \theta([Y , Z] , X) + \theta([Z , X] , Y) .$$

Example. If there is a linear form ω on $\mathfrak{g}$ of maximal class equal to 2p , then $\theta = d\omega$ is a symplectic form.

A Lie algebra provided with such a symplectic form is called **Frobeniusian**.

Proposition 4. *There is no Frobeniusian nilpotent Lie algebra.*

Indeed, the class of a linear form on a nilpotent Lie algebra is always odd.

III.3.2. Classification of complex nilpotent Lie algebras with a symplectic structure

- dimension 2 The algebra is Abelian

$\theta = \omega_1 \wedge \omega_2$ is symplectic

- dimension 4 (we describe the constants of structure from the Maurer Cartan équations)

$\mathfrak{n}_{4,1}$

$d\omega_1 = \omega_2 \wedge \omega_3$

$d\omega_i = 0 \quad i = 2, 3, 4$ $\qquad \theta = \omega_2 \wedge \omega_3 + \omega_1 \wedge \omega_4$ is symplectic

$\mathfrak{n}_{4,2}$

$d\omega_3 = \omega_1 \wedge \omega_2$

$d\omega_4 = \omega_1 \wedge \omega_3$ $\qquad \theta = \omega_1 \wedge \omega_4 + \omega_2 \wedge \omega_3$

(the trivial equations are not written).

- dimension 6

$\mathfrak{n}_{6,3}$

$d\omega_3 = \omega_1 \wedge \omega_2$

$d\omega_4 = \omega_1 \wedge \omega_3$ $\qquad \theta = \omega_1 \wedge \omega_6 - 2\omega_3 \wedge \omega_4 + \omega_2 \wedge \omega_5$

$d\omega_5 = \omega_1 \wedge \omega_4 + \omega_2 \wedge \omega_3$

$d\omega_5 = \omega_1 \wedge \omega_4 + \omega_2 \wedge \omega_3$

$\mathfrak{n}_{6,4}$

$d\omega_3 = \omega_1 \wedge \omega_2$

$d\omega_4 = \omega_1 \wedge \omega_3$ $\qquad \theta = \omega_1 \wedge \omega_6 + \omega_2 \wedge (\omega_4 + \omega_5) - \omega_3 \wedge \omega_4$

$d\omega_5 = \omega_1 \wedge \omega_4$

$d\omega_6 = \omega_1 \wedge \omega_5 + \omega_2 \wedge \omega_3$

$\mathfrak{n}_{6,5}$

$$d\omega_3 = \omega_1 \wedge \omega_2$$
$$d\omega_4 = \omega_1 \wedge \omega_3 \qquad \theta = \omega_1 \wedge \omega_6 + \omega_2 \wedge \omega_5 - \omega_3 \wedge \omega_4$$
$$d\omega_5 = \omega_1 \wedge \omega_4$$
$$d\omega_6 = \omega_1 \wedge \omega_5$$

$\mathfrak{n}_{6,6}$

$$d\omega_3 = \omega_1 \wedge \omega_2$$
$$d\omega_4 = \omega_1 \wedge \omega_3 \qquad \theta = \omega_1 \wedge (\omega_5 + \omega_6) + \omega_2 \wedge \omega_4 + \omega_3 \wedge \omega_5$$
$$d\omega_5 = \omega_2 \wedge \omega_3$$
$$d\omega_6 = \omega_1 \wedge \omega_4 + \omega_2 \wedge \omega_5$$

$\mathfrak{n}_{6,7}$

$$d\omega_3 = \omega_1 \wedge \omega_2$$
$$d\omega_4 = \omega_1 \wedge \omega_3 \qquad \theta = \omega_1 \wedge \omega_5 + \omega_2 \wedge \omega_4 + \omega_3 \wedge (\omega_6 - \omega_5)$$
$$d\omega_5 = \omega_2 \wedge \omega_3$$
$$d\omega_6 = \omega_1 \wedge \omega_4 - \omega_2 \wedge \omega_5$$

$\mathfrak{n}_{6,8}$

$$d\omega_3 = \omega_1 \wedge \omega_2$$
$$d\omega_4 = \omega_1 \wedge \omega_3 \qquad \theta = \omega_3 \wedge \omega_4 - \omega_2 \wedge \omega_5 + \omega_1 \wedge \omega_6 + \omega_2 \wedge \omega_4$$
$$d\omega_5 = \omega_1 \wedge \omega_4$$
$$d\omega_6 = \omega_2 \wedge \omega_3$$

$\mathfrak{n}_{6,9}$

$$d\omega_4 = \omega_1 \wedge \omega_2$$
$$d\omega_5 = \omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_4 \qquad \theta = \omega_1 \wedge \omega_4 + \omega_2 \wedge \omega_6 + \omega_3 \wedge \omega_5$$
$$d\omega_6 = \omega_2 \wedge \omega_5 + \omega_3 \wedge \omega_4$$

$\mathfrak{n}_{6,10}$

$$d\omega_4 = \omega_1 \wedge \omega_2$$
$$d\omega_5 = \omega_1 \wedge \omega_4 \qquad \theta = \omega_1 \wedge \omega_6 + \omega_2 \wedge \omega_5 - \omega_3 \wedge \omega_4$$
$$d\omega_6 = \omega_1 \wedge \omega_5 + \omega_2 \wedge \omega_3 + \omega_2 \wedge \omega_4$$

$\mathfrak{n}_{6,11}$

$$d\omega_4 = \omega_1 \wedge \omega_2$$
$$d\omega_5 = \omega_1 \wedge \omega_4 \qquad \theta = \omega_1 \wedge \omega_3 + \omega_4 \wedge \omega_5 - \omega_2 \wedge \omega_6$$
$$d\omega_6 = \omega_1 \wedge \omega_5 + \omega_2 \wedge \omega_3$$

$\mathfrak{n}_{6,12}$

$$d\omega_4 = \omega_1 \wedge \omega_2$$
$$d\omega_5 = \omega_1 \wedge \omega_3 + \omega_1 \wedge \omega_4 \qquad \theta = \omega_1 \wedge \omega_6 + \omega_2 \wedge \omega_5 + \omega_3 \wedge \omega_4$$
$$d\omega_6 = \omega_2 \wedge \omega_4$$

$\mathfrak{n}_{6,13}$

$$d\omega_4 = \omega_1 \wedge \omega_2$$
$$d\omega_5 = \omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_4 \qquad \theta = \omega_1 \wedge \omega_5 + \omega_2 \wedge \omega_6 + \omega_3 \wedge \wedge \omega_4 + \omega_2 \wedge \omega_5$$
$$d\omega_6 = \omega_1 \wedge \omega_4$$

$\mathfrak{n}_{6,14}$

$$d\omega_4 = \omega_1 \wedge \omega_2$$
$$d\omega_5 = \omega_1 \wedge \omega_3 - \omega_2 \wedge \omega_4 \qquad \theta = \omega_1 \wedge \omega_6 + 2\omega_2 \wedge \omega_5 - \omega_3 \wedge \omega_4$$
$$d\omega_6 = \omega_1 \wedge \omega_4 + \omega_2 \wedge \omega_3$$

$\mathfrak{n}_{6,15}$

$$d\omega_4 = \omega_1 \wedge \omega_2$$
$$d\omega_5 = \omega_1 \wedge \omega_3 \qquad \theta = \omega_2 \wedge \omega_6 + \omega_3 \wedge \omega_5 + \omega_1 \wedge \omega_4$$
$$d\omega_6 = \omega_2 \wedge \omega_4$$

$\mathfrak{n}_{6,19}$

$$d\omega_4 = \omega_1 \wedge \omega_2$$
$$d\omega_5 = \omega_1 \wedge \omega_3 \qquad \theta = \omega_1 \wedge \omega_6 + \omega_2 \wedge \omega_4 + \omega_3 \wedge \omega_5 + \omega_2 \wedge \omega_5$$
$$d\omega_6 = \omega_1 \wedge \omega_4 + \omega_2 \wedge \omega_3$$

$\mathfrak{n}_{6,20}$

$$d\omega_4 = \omega_1 \wedge \omega_2$$
$$d\omega_5 = \omega_1 \wedge \omega_3 \qquad \theta = \omega_1 \wedge \omega_6 + \omega_3 \wedge \omega_5 + \omega_2 \wedge \omega_4$$
$$d\omega_6 = \omega_1 \wedge \omega_4$$

$\mathfrak{n}_{6,21}$

$$d\omega_4 = \omega_1 \wedge \omega_2$$
$$d\omega_5 = \omega_1 \wedge \omega_3 \qquad \theta = \omega_1 \wedge \omega_4 + \omega_2 \wedge \omega_6 + \omega_3 \wedge \omega_5$$
$$d\omega_6 = \omega_2 \wedge \omega_3$$

$\mathfrak{n}_{6,23}$

$$d\omega_5 = \omega_1 \wedge \omega_2 - \omega_3 \wedge \omega_4$$
$$\theta = \omega_1 \wedge \omega_6 + \omega_2 \wedge \omega_3 - \omega_4 \wedge \omega_5$$
$$d\omega_6 = \omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_4$$

$\mathfrak{n}_{6,24}$

$$d\omega_5 = \omega_1 \wedge \omega_2$$
$$\theta = \omega_1 \wedge \omega_6 + \omega_2 \wedge \omega_5 + \omega_3 \wedge \omega_4 - \omega_4 \wedge \omega_5$$
$$d\omega_6 = \omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_4$$

$\mathfrak{n}_{5,1} \oplus \mathbb{R}$

$$d\omega_3 = \omega_1 \wedge \omega_2$$
$$d\omega_4 = \omega_1 \wedge \omega_3 \qquad \theta = \omega_1 \wedge \omega_6 + \omega_2 \wedge \omega_5 - \omega_3 \wedge \omega_4$$
$$d\omega_5 = \omega_1 \wedge \omega_4 + \omega_2 \wedge \omega_3$$

$\mathfrak{n}_{5,2} \oplus \mathbb{R}$

$$d\omega_3 = \omega_1 \wedge \omega_2$$
$$d\omega_4 = \omega_1 \wedge \omega_3 \qquad \theta = \omega_1 \wedge \omega_6 + \omega_2 \wedge \omega_5 - \omega_3 \wedge \omega_4$$
$$d\omega_5 = \omega_1 \wedge \omega_4$$

$\mathfrak{n}_{5,4} \oplus \mathbb{R}$

$$d\omega_4 = \omega_1 \wedge \omega_2$$
$$\theta = \omega_1 \wedge \omega_5 + \omega_3 \wedge \omega_4 + \omega_2 \wedge \omega_6$$
$$d\omega_5 = \omega_2 \wedge \omega_3 + \omega_1 \wedge \omega_4$$

$\mathfrak{n}_{5,5} \oplus \mathbb{R}$

$$d\omega_4 = \omega_1 \wedge \omega_2$$
$$\theta = \omega_2 \wedge \omega_4 + \omega_3 \wedge \omega_5 + \omega_1 \wedge \omega_6$$
$$d\omega_5 = \omega_1 \wedge \omega_3$$

$H_3 \oplus H_3$

$$d\omega_3 = \omega_1 \wedge \omega_2$$
$$\theta = \omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_4 + \omega_5 \wedge \omega_6$$
$$d\omega_6 = \omega_4 \wedge \omega_5$$

$H_3 \oplus \mathbb{R}^3$

$$d\omega_3 = \omega_1 \wedge \omega_2 \qquad \theta = \omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_4 + \omega_5 \wedge \omega_6$$

$\mathbb{R}^6$ $\qquad$ $\theta = \omega_1 \wedge \omega_2 + \omega_3 \wedge \omega_4 + \omega_5 \wedge \omega_6$

III.3.3. **Construction of a Lie algebra with a symplectic structure**

The notion of double extension has been defined by Medina and Revoy [M-R] for the construction of Lie groups provided with invariant metrics and also with a symplectic structure.

Let $(\mathfrak{g}_1, \omega_1)$ be a Lie algebra with a symplectic structure. Consider f_1 as a derivation of $\mathfrak{g}_1$. It induces a bilinear mapping Φ_1 given by

$$\Phi_1(X, Y) = \omega_1(f_1(x), Y), \qquad X, Y \in \mathfrak{g}_1 .$$

This mapping Φ_1 corresponds to a 2-cocycle for the scalar cohomology of the left symmetric algebra $\mathfrak{g}^S{}_1$, whose product satisfies

$$(X . Y) . Z - X.(Y . Z) = (Y . X) . Z - Y.(X . Z)$$

and the Lie algebra structure $\mathfrak{g}_1$ is related with the algebra structure on $\mathfrak{g}^S{}_1$ by

$$[X, Y] = X . Y - Y . X$$

($\mathfrak{g}_1$ and $\mathfrak{g}_1{}^S$ have the same underlying vector space). Consider the adjoint operator $f^*{}_1$ of f_1 with respect to ω_1 :

$$\omega_1(X, f_1(Y)) = \omega_1(f^*{}_1(X), Y) .$$

We can find a 2-cocycle Φ of the Lie algebra $\mathfrak{g}_1$ for the cohomology $H^*(\mathfrak{g}_1, \mathfrak{g}_1)$ by putting

$$\Phi(X, Y) = \omega_1((f_1 + f^*{}_1)(X), Y) , \qquad X, Y \in \mathfrak{g}_1 .$$

Let be $I^\perp = \mathbb{K}U \oplus \mathfrak{g}_1$ the central extension of $\mathfrak{g}_1$ by $\mathbb{K}U$ ($\mathbb{K}$ is the scalar field of $\mathfrak{g}_1$) defined by Φ and let $\mathfrak{g}$ be the semidirect product

$$\mathfrak{g} = \mathbb{K}V \oplus I^\perp$$

given by the bracket :

$$[V, U] = 0$$
$$[V, X] = -\omega'(W, X)U - f_1(X), \qquad X \in \mathfrak{g}_1,$$

where W is defined in the left symmetric algebra $\mathfrak{g}_1{}^S$ by

$$W = V^2 .$$

Such a Lie algebra $\mathfrak{g}$ is called *a double extension of* $\mathfrak{g}_1$ following (W, f_1).

Theorem 4. *Let* $(\mathfrak{g}, \omega)$ *be a Lie algebra with a symplectic structure of dimension* 2n *such that the center* $Z(\mathfrak{g})$ *is not trivial. Then* $\mathfrak{g}$ *is a double extension of a Lie algebra* $\mathfrak{g}_1$ *with a symplectic structure of dimension* 2n–2 *following a pair* (ω, f_1) *where* $\omega \in \mathfrak{g}_1$ and $f_1 \in \mathrm{Der}\, \mathfrak{g}_1$.

Every nilpotent symplectic Lie algebra can be obtained this way from the trivial Lie algebra of dimension 0.

IV. LEFT INVARIANT METRICS ON NILPOTENT LIE GROUPS

IV.1. Left invariant metrics

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra.

Definition 6. *A left invariant Riemannian metric g on* G *is a Riemannian metric on* G *such that the mapping*

$$L_x : \mathfrak{g} \to T_r G$$

is an isometry.

Then the metric g verifies

$$\left((L_x)^* g\right)_y (X, Y) = g_{xy}\left((L_x)^*{}_y X, (L_x)^*{}_y Y\right), \quad X, Y \in T_x G .$$

The metric g defines a scalar product, also noted g, on the Lie algebra $\mathfrak{g}$. We can find a basis $(\omega_1, \ldots, \omega_n)$ of $\mathfrak{g}^*$ such that:

$$g = \omega_1{}^2 + \omega_2{}^2 + \ldots + \omega_n{}^2 \qquad (n = \dim \mathfrak{g}) .$$

Infinitesimal version

A vector field X is a Killing field of g if

$$(L_X g)(Y, Z) = X(g(Y, Z)) - g([X, Y], Z) - g(Y, [X, Z]) = 0 .$$

This can be written as:

$$(L_X g)(Y, Z) = g\left(\nabla_X Y, Z\right) + g\left(Y, \nabla_Z X\right) = 0 ,$$

where ∇ is the covariant derivative associated to the Levi-Civita connection.
The set of Killing vector fields is provided with a Lie algebra structure, called the algebra of infinitesimal isometries. It is associated to the Lie group ISO(G, g) of isometries of g which has a finite dimension.

IV.2. Classification of left invariant metrics on a 3-dimensional Heisenberg group

Let $\mathcal{H}_1$ be a 3-dimensional Heisenberg group. It is the group of nilpotent matrices

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} .$$

Proposition 5. *Every left invariant metric on* $\mathcal{H}_1$ *can be written:*

$$g_\lambda = (dx)^2 + (dy)^2 + \left(dz - \lambda x dy\right)^2 .$$

For the proof, see the following section.

Now, we study the Lie algebra of Killing vector fields for the metric g_λ.
Let

$$X = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial y} + \xi_3 \frac{\partial}{\partial z}$$

be a Killing field for the metric g_λ.

The components ξ_i satisfy

$$\frac{\partial \xi_1}{\partial x} = 0 ,$$

$$-\lambda x \frac{\partial \xi_2}{\partial z} + \frac{\partial \xi_3}{\partial z} = 0 ,$$

$$\frac{\partial \xi_1}{\partial y} + \left(1 + \lambda^2 x^2\right) \frac{\partial \xi_2}{\partial x} - \lambda x \frac{\partial \xi_3}{\partial x} = 0 ,$$

$$-\lambda \xi_1 + \left(1 + \lambda^2 x^2\right) \frac{\partial \xi_2}{\partial z} - \lambda x \frac{\partial \xi_3}{\partial z} + \frac{\partial \xi_3}{\partial y} - \lambda x \frac{\partial \xi_2}{\partial y} = 0 ,$$

$$\lambda^2 x \xi_1 + \left(1 + \lambda^2 x^2\right) \frac{\partial \xi_2}{\partial y} - \lambda x \frac{\partial \xi_3}{\partial y} = 0 ,$$

$$\frac{\partial \xi_1}{\partial z} - \lambda x \frac{\partial \xi_2}{\partial x} + \frac{\partial \xi_3}{\partial x} = 0 .$$

This system implies

$$\frac{\partial \xi_3}{\partial x} = \lambda x \frac{\partial \xi_2}{\partial x} - \frac{\partial \xi_1}{\partial z} ,$$

$$\frac{\partial \xi_2}{\partial x} = -\frac{\partial \xi_1}{\partial y} - \lambda x \frac{\partial \xi_1}{\partial z} .$$

Then, by integrating :

$$\xi_2 = -x\frac{\partial \xi_1}{\partial y} - \lambda^2 \frac{x^2}{2}\frac{\partial \xi_1}{\partial z} + \psi(y , z) ,$$

$$\xi_3 = -\left(x + \frac{\lambda^2 x^2}{3}\right)\frac{\partial \xi_1}{\partial z} - \lambda \frac{x^2}{2}\frac{\partial \xi_1}{\partial y} + \theta(y , z) .$$

We deduce :

$$\frac{\lambda^2 x^3}{6}\frac{\partial^2 \xi_1}{\partial z^2} + \frac{\lambda x^2}{2}\frac{\partial^2 \xi_1}{\partial y \partial z} - x\left(\frac{\partial^2 \xi_1}{\partial z^2} + \lambda\frac{\partial \psi}{\partial z}\right) + \frac{\partial \theta}{\partial z} = 0 .$$

As this polynomial is identically nul, we have :

$$\frac{\partial^2 \xi_1}{\partial z^2} = 0 \ ; \ \frac{\partial^2 \xi_1}{\partial y \partial z} = 0 \ \text{ and } \ \frac{\partial \psi}{\partial z} = \frac{\partial \theta}{\partial z} = 0 .$$

Likewise, the substitution of the expressions of ξ_2 and ξ_3 in the system give

$$\frac{\partial^2 \xi_1}{\partial y \partial z} = 0 \ ; \ \frac{\partial^2 \xi_1}{\partial y^2} = 0 \ ; \ \frac{\partial \psi}{\partial y} = 0 \ ; \ \frac{\partial \theta}{\partial y} = \lambda \xi_1 .$$

Then

$$\xi_1 = ay + b ,$$

$$\psi = d ,$$

$$\theta = \frac{\lambda a}{2} y^2 + \lambda b y + c ,$$

$$\xi_2 = -ax + d$$

$$\xi_3 = -\frac{\lambda a}{2}x^2 + \frac{\lambda a}{2}y^2 + \lambda by + c \ .$$

Conclusion. Every Killing vector field has the following form

$$X = (ay+b)\frac{\partial}{\partial x} + (-ax+d)\frac{\partial}{\partial y} + \left(-\frac{\lambda a}{2}x^2 + \frac{\lambda a}{2}y^2 + \lambda by + c\right)\frac{\partial}{\partial z} \ .$$

The Lie algebra of infinitesimal isometries is generated by the following 4 vector fields :

$$X = \frac{\partial}{\partial x} + \lambda y\frac{\partial}{\partial z} \ , \quad Y = \frac{\partial}{\partial y} \ , \quad Z = \frac{\partial}{\partial z} \ ,$$

$$R = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + \left(-\lambda\frac{x^2}{2} + \lambda\frac{y^2}{2}\right)\frac{\partial}{\partial z} \ .$$

Note that the fields (X, Y, Z) form basis of $\mathfrak{g} = H_1$ and correspond to the left translations.

IV.3. Some formulas of Riemannian geometry

Let X and Y two vector fields on G. We define $\nabla_X Y$ as the unique vector field on G satisfying

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - 2g(X, Y) + g([X, Y], Z) + \\ + g([Z, X], Y) + g([Z, Y], X) , \qquad \forall Z \in \mathfrak{X}(G) .$$

If we suppose g to be left invariant, this formula can be reduced to

$$2g(\nabla_X Y, Z) = g([X, Y], Z) + g([Z, X], Y) + g([Z, Y], X) \ .$$

These form of the Levi Civita connection ∇ are given by

$$\omega_i^j(X) = -g(\nabla_X e_i, e_j),$$

where $(e_1, ..., e_n)$ is an orthonormal frame of G for the metric g. They verify :

$$\omega^j{}_i + \omega^i{}_j = 0 .$$

If $(\omega_1, ..., \omega_n)$ is the dual basis, we define the symbol $\Gamma^j{}_{ki}$ by : ,

$$\omega_i^j = \sum_k \Gamma^j{}_{ki}\, \omega^k ,$$

and the Cartan equations are :

$$d\omega^j = \sum \omega^j{}_k \wedge \omega^k .$$

The Riemann curvature tensor of the connection ∇ associates to each pair of vector fields X and Y the linear transformation

$$\begin{aligned} R(X, Y), (Z) &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z \\ &= \left([\nabla_X, \nabla_Y] - \nabla_{[X, Y]}\right)(Z) \end{aligned}$$

from vector fields to vector fields (of course, here all vector fields are supposed smooth).

The transformation $R(X, Y)$ is skew-adjoint. It defines an (0,4)-tensor given by

$$R_{XYZV} = g(R(X, Y)V, Z) .$$

This tensor verifies

$$R_{XYZV} = -R_{YXZV} = R_{ZVXY}$$

and Bianchi's identities

$$R_{XYZV} + R_{YZXV} + R_{ZXYV} = 0 \quad \text{(1st identity)},$$

$$\nabla_X R_{YZVU} + \nabla_Y R_{ZXVU} + \nabla_Z R_{XYVU} = 0 \quad \text{(2nd identity)} .$$

Remark. If g is a left invariant metric on G, then the first Bianchi identity corresponds to the Jacobi identities on the Lie algebra $\mathfrak{g}$. Indeed, these relations, written on a othonormal left invariant basis (X_i), are relations concerning the constants of the structure of g related to (X_i).

Finally, recall the expressions of the curvature forms and of the Ricci tensor. The curvature forms of G are the 2-forms

$$\Omega^i{}_j = \frac{1}{2} R^i{}_{jpq}\, \omega^p \wedge \omega^q$$

They satisfy

$$d\omega^i{}_j = \omega^h{}_j \wedge \omega^i{}_h + \Omega^i{}_j .$$

The Ricci curvature is the trace of the linear transformation

$$Z \to R(X, Y)\, Z .$$

The Ricci tensor is given by the components :

$$\rho_{ij} = \rho(e_i, e_j) = \sum_{k=1}^{n} R(e_k, e_i, e_k, e_j) ,$$

where (e_i) is an orthonormal basis.

IV.4. Some formulas for left invariant metrics

Let g be a left invariant metric on G, $\mathfrak{g}$ the Lie algebra of G, $\{e_i\}$ an orthonormal basis of $\mathfrak{g}$, and $C^i{}_{jk}$ the corresponding constants of structure. Then

$$g = \sum_{i=1}^{n} \omega_i^2 ,$$

$$\Gamma^k_{ij} = \frac{1}{2}\left(C^k_{ij} + C^j_{ki} + C^i_{kj}\right),$$

$$R^i_{jpq} = \Gamma^s_{pq}\Gamma^i_{js} - \Gamma^s_{jq}\Gamma^i_{ps} - \Gamma^s_{jp}\Gamma^i_{sp} ,$$

$$R_{ijkl} = \Gamma^s_{kl}\Gamma^i_{js} - \Gamma^s_{jl}\Gamma^i_{ks} - \Gamma^s_{jk}\Gamma^i_{sl} ,$$

$$\rho_{ij} = \frac{1}{2} T_r\, C_i C_j - \frac{1}{2} T_r\, {}^tC_i C_j + \frac{1}{4}\sum_{k,r=1}^{n}\left(C^i_{kr} - C^j_{kr}\right) + \frac{1}{2}\sum_{h=1}^{n}\left(C^i_{jk} + C^j_{ik}\right) T_r\, C_k ,$$

where T_rC designates the trace of the operator C and where C_i is the operator whose matrix in the basis $\{e_i\}$ is $(C^k_{ij})_{k,j}$.

IV.5. Left invariant metrics on nilpotent Lie groups

First, we will look at the behavior of the Ricci curvature.

Recall that a nilpotent Lie algebra is a unimodular Lie algebra that satisfie $Tr(adX) = 0$ $\forall X \in \mathfrak{g}$. Then, the Ricci tensor is reduced to :

$$\rho_{ij} = -\frac{1}{2} T_r\, C_i C_j - \frac{1}{2} T_r\, {}^tC_i C_j + \frac{1}{4}\sum_{k,r=1}^{n}\left(C^i_{kr}\, C^j_{kr}\right)$$

and the scalar curvature $\tau = \sum_{i=1}^{n} \rho_{ii}$ is given by

$$\tau = -\frac{1}{2}\sum_{p=1}^{n} T_r\, C_p C_p - \frac{1}{4}\sum_{p=1}^{n} T_r\, {}^tC_p C_p .$$

Now we note by L^* the adjoint (with respect to g) of a linear mapping L on $\mathfrak{g}$. Of course, we have identified g with the associated scalar product on $\mathfrak{g}$. The operator L is antisymmetric if $L^* = -L$. In this case, we have $g(L(X), Y) = -g(X, L(Y))$.

Lemma. *If the operator* ad X *is antisymmetric, then*

$$K(X, Y) \geq 0, \ \forall Y \in \mathfrak{g}.$$

and $K(X, Y) = 0$ *if and only if* $g(X, [X, \mathfrak{g}]) = 0$.

The proof is obvious. We take $X = e_1$, $Y = e_2$ where $\{e_1, \ldots e_n\}$ is an orthonormal basis. If ad e_1 is antisymmetric, then

$$C^j_{1i} = -C^i_{1j} \text{ and } K(e_1, e_2) = \Sigma \left(\frac{C^1_{2k}}{2}\right)^2 \geq 0 .$$

Corollary. *If* $X \in Z(\mathfrak{g})$, *then* $K(X, Y) \geq 0 \ \forall Y \in \mathfrak{g}$.

Lemma. *If* ad X *is antisymmetric, then* $\rho(X, Y) \geq 0$, *and* $\rho(X, Y) = 0$ *is equivalent to* $g(X, \mathcal{D}(\mathfrak{g})) = 0$.

Lemma. *If* $\rho(X, \mathcal{D}(\mathfrak{g})) = 0$, *then* $\rho(X, X) \leq 0$. *The equality is satisfied if and only if* ad X *is antisymmetric.*

From these corollaries, we deduce the following theorem :

Theorem 5. *Suppose that the Lie algebra* $\mathfrak{g}$ *is nilpotent and not Abelian. For every left invariant metric, there exist a direction* X *such that* $\rho(X, X) > 0$ *and a direction* Y *such that* $\rho(Y, Y) < 0$.

Indeed, let $X \neq 0$ be in $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$. Then $\rho(X) > 0$. For finding a vector Y giving a negative curvature, we can take a vector orthogonal to $[\mathfrak{g}, \mathfrak{g}]$ but not belonging to $Z(\mathfrak{g})$. Such a vector exists because $\mathfrak{g}$ is nilpotent.

The group of isometries

The determination of the Lie group of isometries for a left invariant metric is easy enough. The computations result in a theorem of Wolf generalized by Wilson [Wl]. Let $\{e_i\}$ be an orthonormal basis of $\mathfrak{g}$. For every derivation f of $\mathfrak{g}$, we consider the matrix

$M(f)$ with respect to the basis $\{e_i\}$. Then f corresponds to an infinitesimal isometry if $M(f)$ is an orthogonal matrix.
Wilson's theorem affirms that every isometry for a left invariant metric g on the nilpotent group G is in Aut($\mathfrak{g}$) the group of automorphisms of $\mathfrak{g}$. This implies that the infinitesimal isometries are derivations f of $\mathfrak{g}$ such that a matrix of f related to an orthonormal basis is orthogonal.

In the following section, we compute this group for the left invariant metrics on the Heisenberg group.

Another interesting example consists in considering a nilpotent Lie group such that every derivation is orthogonal with respect to a given basis of $\mathfrak{g}$. A partial answer is given by considering the quaternionian Heisenberg algebra. This Lie algebra is given by

$$d\omega_1 = \alpha_1 \wedge \alpha_2 + \alpha_3 \wedge \alpha_4 ,$$
$$d\omega_2 = \alpha_1 \wedge \alpha_3 - \alpha_2 \wedge \alpha_4 ,$$
$$d\omega_3 = \alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_3 ,$$

where $(\omega_1, \omega_2, \omega_3 , \alpha_1 , \alpha_2 , \alpha_3 , \alpha_4)$ is the dual basis of $\mathfrak{g}^*$. Consider the corresponding basis $(X_1 , X_2 , X_3 , Y_1 , Y_2 , Y_3)$ in $\mathfrak{g}$ and the metric

$$g = \omega^2{}_1 + \omega^2{}_2 , \omega^2{}_3 + \alpha^2{}_1 + \alpha^2{}_2 + \alpha^2{}_3 + \alpha^2{}_4 .$$

Let f be a derivation of $\mathfrak{g}$. The matrix of f related to the basis $(X_1 , X_2 , X_3 , Y_1 , Y_2 , Y_3, Y_4)$ is

$$\begin{pmatrix} \alpha & -a & -b & & & & \\ a & 2\alpha & -c & & & & \\ b & -c & 2\alpha & & & & \\ 0 & 0 & 0 & \alpha & -d & -e & -f \\ 0 & 0 & 0 & d & \alpha & -h & -i \\ 0 & 0 & 0 & e & h & \alpha & -j \\ 0 & 0 & 0 & f & i & j & \alpha \end{pmatrix}$$

and the corresponding local isometry is

$$\begin{pmatrix} 0 & -a & -b & & & & \\ a & 0 & -c & & & & \\ b & -c & 0 & & & & \\ 0 & 0 & 0 & 0 & -d & -e & - \\ 0 & 0 & 0 & d & 0 & -h & -i \\ 0 & 0 & 0 & e & h & 0 & -j \\ 0 & 0 & 0 & f & i & j & 0 \end{pmatrix}$$

Then the isometry group is isomorphic to SO(3) x SO(4).

V. CLASSIFICATION OF LEFT INVARIANT METRICS ON THE HEISENBERG GROUP

The Heisenberg group $\mathcal{H}_p$ is the matricial Lie group

$$\begin{pmatrix} 1 & x_1 & & \dots & & x_p & z \\ 0 & 1 & 0 & \dots & & 0 & y_1 \\ & \cdot & \cdot & & \dots & \cdot & & \cdot \\ & \cdot & \cdot & & \dots & \cdot & & \cdot \\ 0 & 0 & 0 & \dots & & 1 & y_p \\ 0 & 0 & 0 & \dots & & 0 & 1 \end{pmatrix}$$

There is a basis $(X_1, \dots, X_p, Y_1, \dots, Y_p, Z)$ of the Heisenberg algebra H_p whose corresponding left invariant vector fields are

$$X_i = \frac{\partial}{\partial x_i} \quad , \quad Y_i = \frac{\partial}{\partial y_i} - x_i \frac{\partial}{\partial z} \quad , \quad Z = \frac{\partial}{\partial z} ,$$

where (x_i, y_i, z) are the global coordinates of $\mathcal{H}_p$ defined above. These fields define the structure of the Lie algebra H_p because they verify

$$[X_i, Y_i] = -Z \quad , \quad i = 1, ..., p .$$

The left invariant form $(\alpha_1, ..., \alpha_p, \beta_1, ..., \beta_p, \omega)$ which define the dual basis of H_p are writen

$$\alpha_i = dx_i \quad , \quad \beta_i = dy_i \quad , \quad \omega = dz + x_1 dy_1 + ... + x_p dy_p \ .$$

Proposition 6. *The Lie algebra* $\mathrm{Der}(H_p)$ *of derivations of* H_p *is isomorphic to* $sp(p, \mathbb{R}) \oplus \mathbb{R}^{2p+1}$.

We have seen this in the previous chapter but from an other point of view. We again give the proof using here the matricial representation.

Let f be a derivation of H_p. A matrix of f with respect to the basis (X_i, Y_i, Z) has the following form

$$M = \begin{pmatrix} A & C & 0 \\ B & D & 0 \\ \rho & \mu & \alpha \end{pmatrix}$$

where A, B, C, D are square matrices of the order p, B and C being symmetric, and D satisfying the relation $D = -{}^tA + \alpha I$.

The decomposition

$$M = \begin{pmatrix} A & C & 0 \\ B & -{}^tA & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha I & 0 \\ \rho & \mu & \alpha \end{pmatrix}$$

shows that $\mathrm{Der}(H_p) = sp(p, \mathbb{R}) \oplus \mathbb{R}^{2p+1}$.

This proves, in particular, that the group of automorphisms of H_p is isomorphic to $Sp(p, \mathbb{R}) \times A^{2p+1}$ where A^{2p+1} is Abelian. Now we can be interested to the classification of the left invariant metrics on $\mathcal{H}_p$ up to an automorphism. Following Wilson, this corresponds to the classification up an to isometry.

Theorem 6. *Every left invariant invariant metric on $\mathcal{H}_p$ is equivalent up to an automorphism to one of the metrics*

$$g_{\lambda_1,\dots,\lambda_p} = \lambda_1^2\left(\alpha_1^2+\beta_1^2\right) + \lambda_2^2\left(\alpha_2^2+\beta_2^2\right) + \dots + \lambda_p^2\left(\alpha_p^2+\beta_p^2\right) + \omega^2$$

with $\lambda_1, \dots, \lambda_p \in \mathbb{R} - \{0\}$. The family $g_{\lambda_1,\dots,\lambda_p}$ is irreducible.

Let g be a left invariant metric on $\mathcal{H}_p$. It is written

$$g = \Sigma g^1_{ij}\alpha_i\alpha_j + \Sigma g^2_{ij}\alpha_i\beta_j + \Sigma g^3_{ij}\beta_i\beta_j + \theta\omega,$$

where θ is a linear combination of the α_i, β_i and ω.

The action of the subgroup of $\mathrm{Aut}(H_p)$ corresponding to the matrices of the form

$$\begin{pmatrix} I & 0 \\ X & a \end{pmatrix} \quad \text{with } a \neq 0 \ ,$$

reduces the definite positive quadratic form g to

$$\begin{aligned} g &= \Sigma g^1_{ij}\alpha_i\alpha_j + \Sigma g^2_{ij}\alpha_i\beta_j + \Sigma g^3_{ij}\beta_i\beta_j + \omega^2 \\ &= g_1 + \omega^2 \end{aligned}$$

and the quadratic form g_1 is positive in restriction to a complementary of the center. As $Der(H_p)$ is isomorphic to $sp(p, \mathbb{R}) \oplus \mathbb{R}^{2p+1}$, the problem is reduced to the reduction of the quadratic form g_1 modulo the symplectic group. Let A be the matrix associated to g_1 with respect to the basis (α_i, β_i). Its eigenvalues are strictly positive. So if $E = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, the eigenvalues of the matrix AE are complex without a real part.

From the classification of Williamson (see, for example, the book of Arnold on the classical mechanic), we can reduce g to the form

$$\left(\lambda_1^2\alpha_1^2+\beta_1^2\right)+\left(\lambda_2^2\alpha_2^2+\beta_2^2\right)+\ldots+\left(\lambda_p^2\alpha_p^2+\beta_p^2\right).$$

This gives the theorem.

Application : A Riemannian characterization of the Heisenberg group

We begin by recalling the notion of a homogeneous naturally reductive space.
Consider a Riemannian space (M, g) and I_g its Lie algebra of infinitesimal isometries. Then M is a homogeneous naturally reductive space, $M = G/K$ where G is a group of isometries and K is the isotropy subgroup at a point p of M if the Lie algebra $\mathfrak{g}$ of G (which is a subalgebra of I_g) satisfies

$$\mathfrak{g} = \mathrm{m} \oplus \mathfrak{k}, \quad [\mathrm{m}, \mathfrak{k}] \subseteq \mathrm{m}$$

$$<[X, Y]_{\mathrm{m}}, Z> + <[X, Z]_{\mathrm{m}}, Y> = 0, \quad \forall X, Y, Z \in \mathrm{m},$$

(here $<\,,\,>$ denotes the product induced on m from $\mathfrak{g}$ by identification of m with T_pM and $\mathfrak{k}$ the Lie algebra of K).
Suppose now that $M = \mathcal{H}_p$ and $g = g_{\lambda_1,\ldots,\lambda_p}$.

Proposition 7. *For every left invariant metric $g_{\lambda_1,\ldots,\lambda_p}$, the Riemannian space $\left(\mathcal{H}_p, g_{\lambda_1,\ldots,\lambda_p}\right)$ is homogeneous naturally reductive.*

We consider the fields

$$\xi_i = X_i + y_i Z \ , \ \eta_i = Y_i - x_i Z \ , \ Z \ , \ i = 1, \dots, p \ ,$$

$$R = x_i Y_i - y_i Y_i - \frac{x_i^2 + y_i^2}{2} Z \quad , \quad i = 1, \dots, p \, .$$

If $\mathfrak{l}$ is the subalgebra of $Der(H_p)$ generated by these vectors, the bracket of $\mathfrak{l}$ verifies

$$[\xi_i , \eta_i] = -Z \quad , \quad [\xi_i , R_i] = \eta_i \quad , \quad [\eta_i , R_i] = -\xi_i \, .$$

Other products are zero. Then $\mathfrak{l} = \mathfrak{k} \oplus m$, where $\mathfrak{k}$ is the Abelian algebra generated by the R_i gives the naturally reductive decomposition.

Theorem 7. *Let* G *be a nilpotent Lie group such that for every left invariant metric* g, *the Riemannian space* (G, g) *is a homogeneous naturally reductive space. Then the Lie algebra* $\mathfrak{g}$ *of* G *is a direct sum of Heisenberg and Abelian algebras.*

The proof is based on the notion of Riemannian contractions. We study some models of nilpotent Riemannian Lie groups and, by deformation, we can look all the other nilpotent groups (see [GP2].

This theorem shows the importance of the riemannian Heisenberg group, because the behavior of the isometry group is analogous to the group of isometries of the Riemannian Abelian group which corresponds to the Euclidean geometry.

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