我们重新来证明正六边形是一个万有覆盖.

首先什么是万有覆盖，一个图形,他的直径小于等于1.那么他叫一个覆盖.直径的意思就是图形上距离最远的两个点的距离

万有覆盖是包含任意覆盖的图形.求万有覆盖的上界最小是多少.这个定理说的是上界一定比正六边形小.

6.1. Theorem (Radon's Theorem). Let $S \equiv \{x_1, x_2, \ldots, x_r\}$ be any finite set of points in E^n. If $r \geqslant n+2$, then S can be partitioned into two disjoint subsets S_1 and S_2 such that $\operatorname{conv} S_1 \cap \operatorname{conv} S_2 \neq \varnothing$.

证明: $P = \{P_1 \cdots P_{n+2}\}$

$$Q = \{P_2 - P_1, \cdots\cdots, P_{n+2} - P_1\}$$

$$P_2 - P_1 = \sum_{i=3}^{n+2} \beta_i (P_i - P_1)$$

$$(\beta_3 + \beta_4 + \cdots \beta_{n+2} - 1) P_1 + (-\beta_3 P_3 + \cdots + -\beta_{n+2} P_{n+2}) + P_2 = 0$$

$$\alpha_1 = \beta_3 + \beta_4 + \cdots \beta_{n+2} - 1$$

$$\alpha_2 = 1$$

$$\alpha_i = -\beta_i \quad (3 \le i \le n+2)$$

$$\sum \alpha_i P_i = 0 \quad \sum \alpha_i = 0 \quad —— \quad \star$$

$$T_1 = \{ i \mid \alpha_i \geq 0 \} \quad T_2 = \{ i \mid \alpha_i < 0 \}$$

$$x = \frac{\sum_{i \in T_1} \alpha_i P_i}{\sum_{i \in T_1} \alpha_i} \qquad \widetilde{T}_1 = \{ P_i \mid \alpha_i > 0 \}$$

$$\widetilde{T}_2 = \{ P_i \mid \alpha_i < 0 \}$$

因为星式子我们有

$$x = \frac{-\sum_{i \in T_2} \alpha_i P_i}{-\sum_{i \in T_2} \alpha_i}$$

显然x属于 $\widetilde{T}_1$ 的凸包,也属于 $\widetilde{T}_2$ 的凸包 $\square$

6.2. Theorem (Helly's Theorem). Let $\mathfrak{F} \equiv \{B_1, \ldots, B_r\}$ be a family of r convex sets in E^n with $r \geqslant n+1$. If every subfamily of $n+1$ sets in $\mathfrak{F}$ has a nonempty intersection, then $\bigcap_{i=1}^{r} B_i \neq \varnothing$.

对r的大小做归纳

意思就是2维空间我们有4个园,他们任意三个的交不是空.那么4个的交一定也不是空.

证明：我们可以假设 $\forall j=1,\dots,n\ \exists x_j \in B_i$ 对于任意i只要i不等于j

并且 $x_j \notin B_j$

对于这n个点,根据radon定理我们有存在一个划分A1,A2，和一个点p属于conv(A1)交conv(A2)

对于x1,假设他在A1中.我们有x1一定不在A2中. 那么A2一定是B1的子集.
这是显然的因为A2里面没有x1,而这些A2里面的这些点集都属于B1.所以A2是B1的子集.
所以p这个点属于B1.我们归纳下去即可.所以p这个点属于所有的Bi

□

6.3. Theorem (Helly's Theorem). Let $\mathfrak{F}$ be a family of compact convex subsets of E^n containing at least $n+1$ members. If every $n+1$ members of $\mathfrak{F}$ have a point in common, then all the members of $\mathfrak{F}$ have a point in common.

这就是上面定理的换一种说法.

1.20. Definition. A subset A of E^n is said to be **compact** if it is closed and bounded.

1.16. Definition. If $A, B \subset \mathsf{E}^n$ and $\lambda \in \mathsf{R}$, we define

$$A + B \equiv \{x + y \colon x \in A \text{ and } y \in B\}$$

$$\lambda A \equiv \{\lambda x \colon x \in A\}.$$

If A consists of a single point, $A \equiv \{x\}$, then we often write $x + B$ for $A + B$. The set $x + B$ is called a **translate** of B. The set λA is called a **scalar multiple** of A. If $\lambda \neq 0$, the set $x + \lambda A$ is said to be **homothetic** to A.

6.4. Theorem. Let $\mathfrak{F} \equiv \{A_\alpha \colon \alpha \in \mathcal{Q}\}$ be a family of compact convex subsets of E^n containing at least $n+1$ members. Suppose K is a compact convex subset of E^n such that the following holds: For each subfamily of $n+1$ sets in $\mathfrak{F}$, there exists a translate of K that is contained in all $n+1$ of them. Then there exists a translate of K that is contained in all the members of $\mathfrak{F}$.

平移

说的是拿一个凸集组成的集合F. K也是一个凸集合. 假设. K经过一个平移之后得到的set可以放到任何他们的n+1个取定的子集中.那么就存在一个k的平移后的set可以放到全部F的元素中.

证明:

$$A_\alpha^* = \{p \mid (K+p) \subset A_\alpha\}$$

显然满足Helly定理.所以存在一个平移使得K能被 $\forall A_\alpha$ 包含 $\square$

6.5. Theorem. Let $\mathfrak{F}$ be a family of compact convex subsets of E^n containing at least $n+1$ members. Suppose K is a compact convex subset of E^n such that for each subfamily of $n+1$ sets in $\mathfrak{F}$, there exists a translate of K that contains all $n+1$ of them. Then there exists a translate of K that contains all the members of $\mathfrak{F}$.

上个定理的反包含版本.

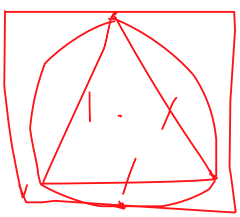
证: $A_\alpha^* \triangleq \{p \mid (p+A_\alpha) \subset K\}$

由helly定理,知道存在公共的平移p 满足定义最终需要的结果. $\square$

6.5在二维空间的说法就是: 一堆单点集组成的集合,因为单点集也是凸集, 假设K能平移覆盖其中任意的三个点.那么K就能覆盖这个整体集合.

12.2. Theorem. The smallest square that is a universal cover in E^2 has sides of length one.

PROOF. Let S be a subset of E^2 having diameter 1 and let K be a square of side length 1. To prove that K is a universal cover, it suffices by Theorem 6.5 to show that given any three points, say x_1, x_2, x_3, of S, there exists a translate of K that covers $\{x_1, x_2, x_3\}$. Now since the diameter of S is 1, it follows that $\{x_1, x_2, x_3\}$ is contained in a Reuleaux triangle T of width 1. But a Reuleaux triangle of width 1 can be rotated through 360° inside a square of side 1. Therefore, no matter what the orientation of T and K, there must be a translate of K that covers T and thus also covers $\{x_1, x_2, x_3\}$. Clearly, no smaller square can be a universal cover since it would not cover a circle of diameter 1. ■

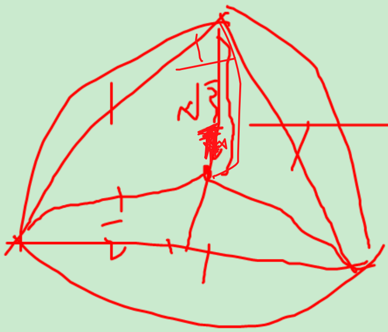


乐罗三角的上顶点到下顶点距离是1,他能放入大小为1的正方形是显然的.

12.3. Theorem. The smallest circle that is a universal cover in E^2 has radius $1/\sqrt{3}$.

PROOF. The proof that a circle of radius $1/\sqrt{3}$ is a universal cover in E^2 is similar to that of Theorem 12.2 and is left to the reader (Exercise 12.2). No smaller circle can be a universal cover since it would not cover an equilateral triangle of side 1. ■

证明:



利用三角形的面积容易知道这个高度就是1/根号3
其余证明同上，只要包含路罗三角形就行．就满足万有覆盖

下面面积比12.3小.

12.4. Theorem. The smallest regular hexagon which is a universal cover in E^2 has sides of length $1/\sqrt{3}$.

PROOF. Let S be a set of diameter 1. We begin by circumscribing a rhombus, say $abcd$, about S so that the angle at a is 60°. (See Figure 12.1.)

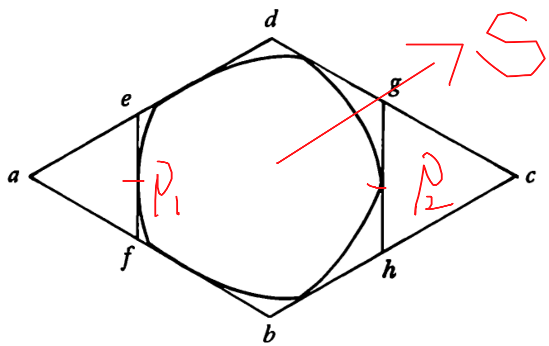


Figure 12.1.

The two support lines perpendicular to the diagonal $\overline{ac}$ of the rhombus cut off two triangles, say aef and cgh, from $abcd$. If $\overline{ef}$ has the same length as $\overline{gh}$, then the hexagon $efbhgd$ is regular and circumscribes S. If not, then we claim that some rotation of the rhombus about S will yield the desired regular hexagon.

我证明这个结论.其实很显然，注意我们ef gh都是切线.所以两个切点p1和p2的距离小于等于1. 所以ef 和gh的距离小于等于1. 如果距离小于1,那么反例容易构造.所以ef,gh距离是1.所以是正六边形.

If $a'b'c'd'$ is another circumscribed rhombus about S, let α be the counterclockwise angle beteen the diagonal $\overline{ac}$ and the diagonal $\overline{a'c'}$. Let $m(\alpha)$ denote the length of the support line $\overline{e'f'}$ and $n(\alpha)$ the length of the support line $\overline{g'h'}$. Then $m(\alpha)$ and $n(\alpha)$ are continuous functions of α.

If $\overline{ef}$ and $\overline{gh}$ are not the same length, then $m(0) \neq n(0)$, and we may assume without loss of generality that $m(0) - n(0) > 0$. But the rhombus corresponding to $\alpha = \pi$ will yield the same hexagon as $efbhgd$, except that the sides $\overline{ef}$ and $\overline{gh}$ will be interchanged. Thus $m(\pi) - n(\pi) < 0$. Since the difference $m(\alpha) - n(\alpha)$ is a continuous function of α, it follows from the Intermediate Value Theorem that there exists an α between 0 and π such that $m(\alpha) - n(\alpha) = 0$. For this α, the corresponding circumscribing hexagon will be regular. 这段都是为了证明他是正的六边形

Since S has diameter 1, its maximum width is 1 (Theorem 11.3), and so the regular hexagon circumscribed about S will have sides of length at most $1/\sqrt{3}$. No smaller regular hexagon can be a universal cover since it would not cover a circle of diameter 1. ■

至于他为什么是一个万有覆盖，因为他能放下乐罗三角和单位圆即可.

定理，下面的单位圆外接正六边形，可以去了两个红色X部分.依然是一个万有覆盖.

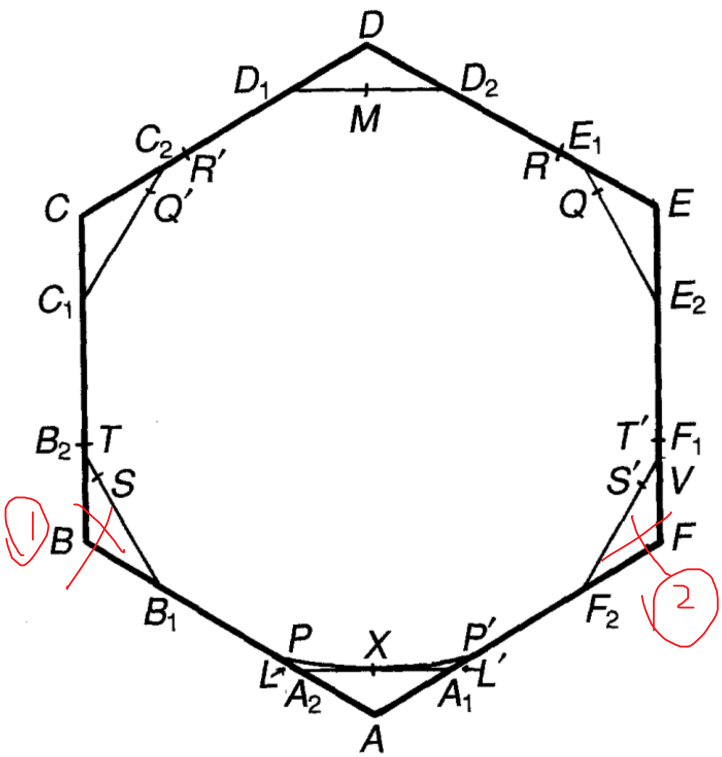


Fig. 1.

这个证明在2015-baez论文中，就在library文件夹里面.论文的第七页.