THE LEBESGUE PROBLEM. AN INVITATION TO INVESTIGATION*

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The Lebesgue problem (to find the least cover for all plane convex figures of unit diameter) is a challenging geometrical optimization, and, although simple, cannot be solved using conventional mathematical methods. The problem gives rise to a variety of geometrical subproblems. Besides the familiar results concerning the Lebesgue problem, the author presents some conjectures and hypotheses he encountered in his own investigation.

I started my career in mathematics with thoughts about some familiar geometrical problems, including the Lebesgue problem. It is one of many elegant conjectures in geometry, which are simple and naturally to state and so indicate a hope to find a simple solution using geometrical intuition. I would be glad if this article not only presents an interesting topic to readers but also encourages someone to make his or her own investigation.

We call a plane figure $\mathcal{F}$ such that the distance between any two of its points is less than or equal to unity a *small* figure. For example, a circle $C_{1/2}$ of radius $\frac{1}{2}$ and the equilateral triangle $\mathcal{T}_1$ with unit sides are small figures, while the unit square $\mathcal{Q}_1 = ABCD$ is not small because $AC = \sqrt{2} > 1$ (Fig. 1). Clearly each small figure $\mathcal{F}$ can be covered by a circle C_1 of unit radius with its center at some point inside the figure. A plane figure which can be moved so that it covers any small figure is called a *cover* (or a *universal cover*).

In 1914, a famous French analyst Henri Lebesgue wondered how to find the smallest cover during a discussion with the Hungarian mathematician J. Pál. This is still an open question.

Disk? Square? Hexagon?

The problem is essentially geometrical optimization. By analogy with the isoperimetric problem (i.e. which of all figures with a given perimeter has the greatest area, answer: the circle), the first logical step is to check whether a circle is the minimum cover. The smallest circle that covers the equilateral triangle $\mathcal{T}_1$ is $C_{1/\sqrt{3}}$ with radius $\frac{1}{\sqrt{3}}\,T_1$. (Prove this! What then is the smallest cover for an obtuse-angled triangle?) It turns out that the circle $C_{1/\sqrt{3}}$ is a cover. This was proved by an English mathematician H. W. E. Young in 1910.

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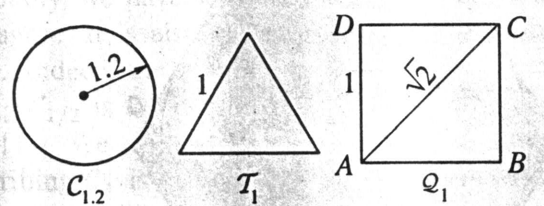

Fig.1.

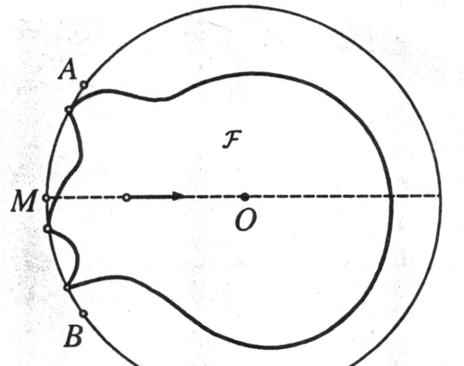

Fig.2.

Theorem. *The smallest circle that is a cover is* $C_{1/\sqrt{3}}$.

Proof. Consider the smallest circle which contains the figure $\mathcal{F}$. The figure $\mathcal{F}$ necessarily touches the circle as otherwise we could shrink the circle without changing where its center O lies (Fig. 2). Let us show that the figure $\mathcal{F}$ and the circle touch either at three points which are the vertices of an acute-angled triangle, or at two diametrically opposite points. We start by assuming the converse, i.e. that all points where $\mathcal{F}$ and the circle touch lie on an arc AB (Fig. 2). Then if we move the figure $\mathcal{F}$ along the line connecting the midpoint of the arc AB and the circle's center it will not touch the circle. However this means that the circle is the smallest possible.

If $\mathcal{F}$ touches the circle at two diametrically opposite points, the radius R of the circle cannot be greater than $\frac{1}{2}$. If $\mathcal{F}$ touches the circle at the vertices of the acute-angled triangle ABD, we get by the law of sines that $\frac{BD}{\sin A} = \frac{AB}{\sin D} = \frac{DA}{\sin B} = 2R$. Since the sum of the angles of a triangle is $180° = 60° \times 3$, at least one of the angles of the triangle ABD is at least $60°$. Let it be the angle A. Since $60° \leq A \leq 90°$, we have $R = \frac{BD}{2\sin A} \leq \frac{1}{2\sin 60°} = \frac{1}{\sqrt{3}}$. The relation becomes an equality if the triangle ABC is $\mathcal{T}_1$. QED.

The area of the circle $C_{\frac{1}{\sqrt{3}}} \approx 1.047$. However, it is easy to construct a cover with a smaller area. One such figure is the unit square $\mathcal{Q}_1$. Prove that the smallest square that is a minimal cover is $\mathcal{Q}_1$.

Now we can show how to obtain a smaller cover from $\mathcal{Q}_1$. If we draw a circle of unit diameter within the square $\mathcal{Q}_1$ and a regular octagon $\mathcal{O}$ outside this circle so that every other side lies on the side of the square $\mathcal{Q}_1$ (Fig. 3), then the octagon $\mathcal{O}$ is obtained by cutting the equilateral triangles t_1, t_2, t_3, and t_4 from the square $\mathcal{Q}_1$. Since the hypotenuses of opposite triangles t_1 and t_3 are parallel and the distance between them is equal to unity, each point x of one of these triangles which does not lie on the hypotenuse is further apart from the other triangle by more than unity. The same is true about the opposite triangles t_2 and t_4. Suppose a small figure $\mathcal{F}$ can be covered by the square $\mathcal{Q}_1$. If one point of $\mathcal{F}$ touches a point x of one of the triangles t_i, $1 \leq i \leq 4$, which does not lie on its hypotenuse, then F cannot touch the triangle opposite to t_i. Therefore $\mathcal{F}$ may intersect the interiors of no more than two adjacent triangles t_i. Consequently, any small figure can be covered by the hexagon obtained by cutting two adjacent triangles, e.g. t_1 and t_2, of the square $\mathcal{Q}_1$ (Fig. 3). Calculate the area of this hexagon.

Actually there is a hexagon cover with a much smaller area than we have found. This is the regular hexagon $\mathcal{G}$ inscribed in Young's circle $C_{1/\sqrt{3}}$ (Fig. 4). By the way, the diameter of the circle inscribed in $\mathcal{G}$ is 1. The area of $\mathcal{G}$ is $\frac{\sqrt{3}}{2} \approx 0.866$.

Exercise 1. Prove that no hexagonal cover can have an area less than that of $\mathcal{G}$.

Exercise 2. Try to prove that the hexagon $\mathcal{G}$ is a cover (if you fail you can find the proof in [1] and [2]).

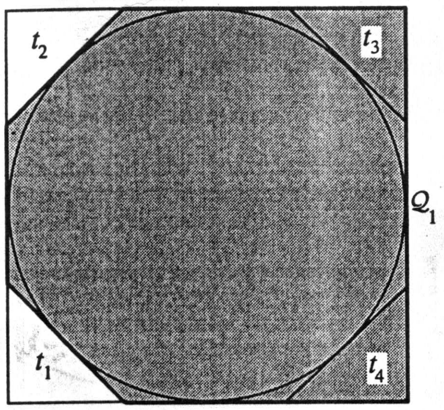

Fig.3.

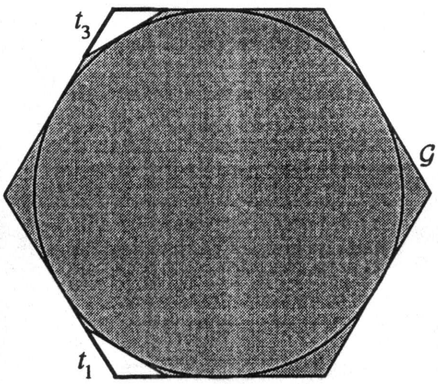

Fig.4.

It was Pál who found $\mathcal{G}$ in 1920. However, we can cut a smaller cover from $\mathcal{G}$, like we did from the square $\mathcal{Q}_1$. We draw a circle inside $\mathcal{G}$ and a regular duodecagon outside the circle (Fig. 4) so that every other side lies on a side of the hexagon $\mathcal{G}$. The duodecagon is obtained by cutting six triangles t_i, $i \le i \le 6$, from $\mathcal{G}$.

Exercise 3. Prove that if we only cut two triangles from the hexagon $\mathcal{G}$, namely t_1 and t_3, the resultant octagon $\mathcal{O}_\mathcal{G}$ will still be a cover.

Records

The octagon $\mathcal{O}_\mathcal{G}$, whose area is $2 - \frac{2}{\sqrt{3}} \approx 0.845299$, was the smallest cover found by Pál. In 1936, the German mathematician R. Sprague found an even smaller cover than Pál. Sprague's cover is obtained if two curvilinear triangles EIK and EJK are cut from the octagon $\mathcal{O}_\mathcal{G}$, where IK and JK are arcs of unit radius with centers at H and B, respectively. The area of the resultant decagon $ABCDIKJFGH$ is approximately 0.844144. Sprague thought that it was impossible to get a smaller cover by cutting his. We shall call such a cover the unimprovable. The minimal cover will be unimprovable, but an unimprovable cover need not be the minimal cover. For 40 years Sprague's cover was the smallest known. However, in 1975 H. Hansen succeeded in creating another cover by cutting two microscopic pieces out of it with total estimated area of an order of 10^{-19} square units. The Hansen cover is obtained from the Sprague cover by cutting the two corners C and G using arcs with a unit radius and centers at the points O_C and O_G which lie on the sides of the regular duodecagon inscribed inside the octagon $\mathcal{O}_\mathcal{G}$ and a distance $\approx 3.7 \cdot 10^{-7}$ from the corresponding vertices. Hansen [3] suggested a proof that his curvilinear convex duodecagon, which is symmetric about the axis AE, is unimprovable. However, this proof is based on an unproved and doubtful hypothesis.

Note that everyone from Pál onwards had found convex covers. A figure is *convex* if any two of its points are joined by a straight line that is inside the figure. The smallest known cover was constructed in 1980 by Duff. It is concave and asymmetric and has an area of 0.84413570...

Questions. Questions...

What is actually known about the minimal cover? Very little, it turns out.

Firstly, there are some estimates of its area. We already have the upper bound. The simplest estimate from below is the area of a circle with unit diameter, i.e. $\frac{\pi}{4}$=0.785. Taking account of the improved estimate of G. Elekes for the area S_c of the minimal convex cover, we can write the inequalities $0.8271\ldots < S_c < 0.8441\ldots$ Thus we know the area of the minimal convex cover quite accurately, namely $S_c \approx 0.835 \pm 1\%$.

Secondly, we have some negative results. We already know that the minimal cover is not a circle or a hexagon. It is also easy to prove that the minimal cover is not a triangle, a quadrilateral, or a pentagon. Indeed, the minimal cover contains the circle $\mathcal{C}_{1/2}$. Moreover the smallest n-sided polygon containing $\mathcal{C}_{1/2}$ is the regular n-gon circumbscribing it. Try to prove this statement (the proof can be found in [1]). We are only trying to verify that the areas of the regular triangle and the regular pentagon circumscribing $\mathcal{C}_{\frac{1}{2}}$ is greater than that of the minimal cover. We have already done this for the square $\mathcal{Q}_1$ and hexagon $\mathcal{G}$.

Exercise 4. Prove that a minimal cover cannot be bounded by an ellipse.

It seems to me that none of the significant figures of elementary geometry can be the minimal cover. Henceforth we shall start looking at the unsolved problems resulting from Lebesgue's problem. Until we can find the minimal cover, we cannot know if there is only one.

Question 1. *Is the minimal cover unique?*

This question is meaningful if we consider the minimal cover to be closed, i.e. it must contain its boundary and be unimprovable. The latter (the cover must be unimprovable) implies that there is no closed subset of it which is a cover. This requirement is necessary for the consideration of nonconvex covers because the area of a cover does not change if we add a finite number of points or segments to it.

Question 2. *Does the set of minimal covers include a polygon with a finite number of sides?*

Advanced mathematics can prove that at least one minimal cover does exist, but this is seemingly all that we know about it. So, in fact you could search for a new figure which has not been discovered. Let us imagine what it might look like. All the small covers we have discussed, except for Duff's cover, are convex.

Question 3. *Is there a convex minimal cover?*

If we can simply restrict the problem and consider only convex minimal covers (we call them c-covers), we still find no answers to any of the questions presented.

The *diameter* of a convex figure is the maximum distance between any pair of points. Prove that the diameter of a convex cover is less than two. Can you improve this bound?

The time has come to refine the formulation of the Lebesgue problem. Let us look at the definition: "A plane figure which can be MOVED so that it covers any small figure is called a *cover*. There is an ambiguity in this definition, as it is not clear which movements are permissible. Let us assume that a cover is cut from cardboard. Can we take it from the plane and turn it over? A mathematician would say that in the first case we can consider translations and reflections, while in the second case we would only accept translations. Thus, we should distinguish two types of covers. The first type includes those that can be turned over (reflected), let us call them r-covers. The second type includes covers for which only translations are possible and we call them simply covers. Most covers we have considered have symmetry axes, therefore it doesn't matter whether we admit reflections or not. The exception is Duff's cover.

Evidently, every cover is an r-cover, while a minimal cover may be larger in area than a minimal r-cover.

Question 4. *Is there a minimal r-cover which is not a cover?*

This question is associated with the next one.

Question 5. *Is there a minimal cover (r-cover) with an axis of symmetry?*

If a minimal r-cover is axisymmetric, it will also be a minimal cover. Clearly, all the above questions have to be asked distinctly for covers and r-covers.

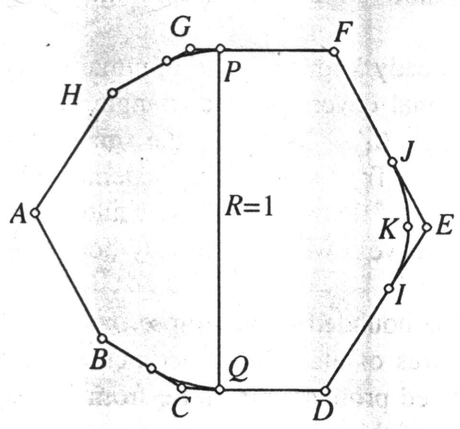

Fig.5.

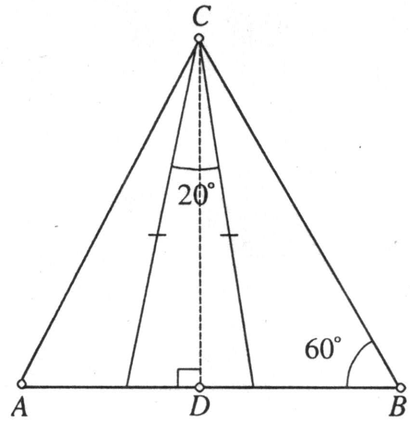

Fig.6.

Let Us Simplify the Problem

One way to handle a complicated problem is to try setting and solving similar but simpler problems. Even if this "evolutionary attack" does not lead to the final goal, it should lead to some interesting results. I simplified the Lebesgue problem by looking for a minimal cover for all small triangles, and partially proved it.

The most distant points in a triangle are two vertices (prove this!). Therefore some of the sides of a small triangle can be longer than 1. Any triangle satisfying this property can be placed in the isosceles triangle $t_\gamma = abc$ with sides $ab = ac = 1$ and the angle $0° < \gamma < 60°$ at the vertex a (Fig. 5). Prove this by yourself.

Theorem. *The minimal cover and r-cover for the family* **T** *of triangles t_γ (and hence for the family of all triangles with diameters less than or equal to 1) is the asymmetric triangle $\Delta = ABC$ with the base $AB = 1$, the angle $B = 60°$, and the altitude $CD = \cos 10°$* (Fig. 6). *The figure is unique (apart from its reflection) and its area is $\frac{1}{2}\cos 10° = 0.4924\ldots$*

The proof [4] is elementary and it is based on a happy chance. Even for the two triangles $t_{60°}$ and $t_{20°}$ in the family **T** it is impossible to find a cover with an area less than Δ.

Exercise 5. Prove that each r-cover for the family **T** of the triangles t_γ is also a cover.

Exercise 6. Prove that Δ is a cover for the family **T**.

However, the convex cover Δ is not the best. We can get a smaller nonconvex cover. Consider the set of all triangles t_γ symmetric about the x axis in a Cartesian frame $0° < \gamma < 60°$. The bases of the triangles lie on the x axis and the third vertex lies in the upper half-plane (Fig. 7). The figure $\mathcal{M}$ is the union of all three triangles in this set and is evidently a cover for the family **T**. The curvilinear part of the boundary of $\mathcal{M}$, i.e., the arc fgh in Fig. 7, is a part of a well known curve, namely the astroid $x^{2/3} + y^{2/3} = 1$. The points f and h lie on the sides of the regular triangle. The astroid can be specified as the boundary of a figure swept by a unit segment whose ends slide along the coordinate axes. The area of $\mathcal{M}$ is obtained by integration. A calculation gives it as $\frac{1}{16}\left(\pi + \frac{9\sqrt{3}}{4}\right) = 0.43992\ldots$

Starting from $\mathcal{M}$ we can construct an unimprovable cover $\mathcal{N}$ for the family **T**. Can you find a cover with the area less than $\mathcal{N}$?

Lebesgue also wondered about the cover with the smallest perimeter (let us call it the p-minimal cover). This problem, is also unsolved, and has been investigated much less extensively. We only present

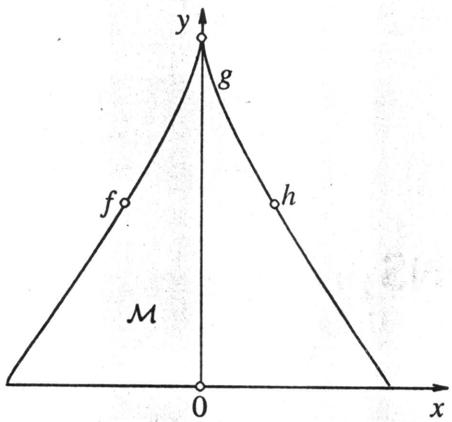

Fig.7.

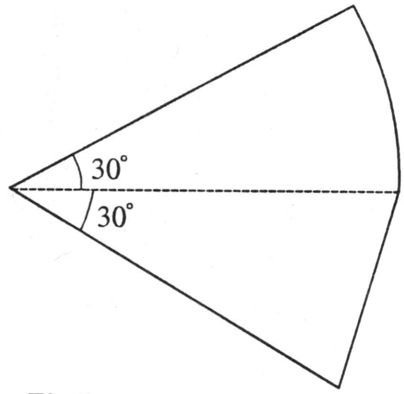

Fig.8.

the bounds of the perimeter P of a p-minimal cover found by Pál, namely $0.302\ldots = \sqrt{3}+\pi/2 \leq P < 8 - 8/\sqrt{3} = 3.382\ldots$ The only simplifying aspect is that the minimal p-cover must be convex.

Figure 8 shows the cover $\mathcal{P}$ for the family of triangles $\mathbf{T}$ which, I believe, is the p-minimal cover for this family. The figure $\mathcal{P}$ is a sector of the unit radius with the angle 60° with a segment of 30° cutout. The perimeter of $\mathcal{P}$ is $2 + \pi/6 + 2\sin 15° = 3.04123\ldots$

Exercise 7. Prove that $\mathcal{P}$ is a unimprovable cover for the family $\mathbf{T}$.

The Lebesgue problem has also been considered for multiple dimensions. Thus, small covers have been found in three-dimensional Euclidean space $\mathbf{R}^3$. They were used to find an affirmative answer (for $n = 3$) to the Borsuk problem, namely whether an arbitrary subset of $\mathbf{R}^n$ of unit diameter can be partitioned into $n+1$ parts of diameters less than 1 (see [5]). Note that the Borsuk problem was recently solved in the negative for spaces with large dimensions n. The existence of a Lebesgue cover of minimal volume in $\mathbf{R}^n$ was proved by V. V. Makeev [6] without assuming convexity. He also suggested some small covers for the multidimensional case [7].

The lower bound for the area of a Lebesgue cover for the plane case was improved by G. Elekes [8] analytically and by P. S. Pankov et al. using computer simulations [9].

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