https://books.google.com.hk/books?id=U9eOPjmaH9OC&pg=PA86&lpg=PA86&dq=why+is+regular+hexagon+a+universal+cover?&source=bl&ots=5T5__dyLmI&sig=ACfU3U3ak75_Q9L2faYDJmlYb1mgFuh-Wg&hl=en&sa=X&ved=2ahUKEwiv1cjU7Y3pAhVLyIsBHf6SCs4Q6AEwBnoECAgQAQ#v=onepage&q&f=false

这个书的第12章.

Convex Sets and Their Applications

这个书的名字是上面这个.

12.2 万有覆盖且是正方形,那么他的边长最小是1

证: 根据6.5

线性组合：$y = \sum k_i x_i$

仿射组合：$y = \sum k_i x_i$，$\sum k_i = 1$

2点的线性组合是平面，放射组合是直线

3点的线性组合是空间，放射组合是一个2维平面．所以一般来说放射组合比线性组合少一维．因为多一个约束条件．

放射独立：

2.17. Definition. A finite set of points $x_1, \ldots, x_m$ is **affinely dependent** if there exist real numbers $\lambda_1, \ldots, \lambda_m$, not all zero, such that $\lambda_1 + \cdots + \lambda_m = 0$ and $\lambda_1 x_1 + \cdots + \lambda_m x_m = \theta$. Otherwise it is **affinely independent.**

这个概念等价于这个点击中的一个点可以被其他店放射表出．

$$x = \lambda_1 x_1 + \cdots + \lambda_k x_k \qquad \text{with} \qquad \sum_{i=1}^{k} \lambda_i = 1$$

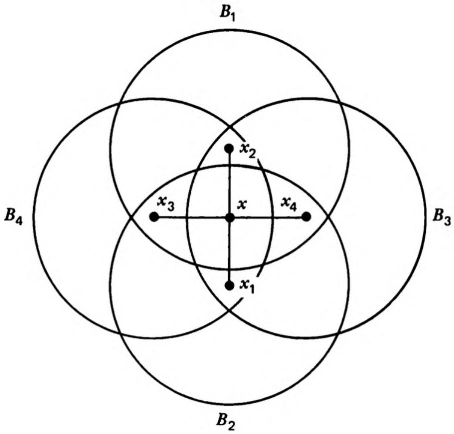
6.1. Theorem (Radon's Theorem). Let $S \equiv \{x_1, x_2, \ldots, x_r\}$ be any finite set of points in E^n. If $r \geqslant n + 2$, then S can be partitioned into two disjoint subsets S_1 and S_2 such that $\operatorname{conv} S_1 \cap \operatorname{conv} S_2 \neq \varnothing$.

只要r>=n+2，那么可以把S拆分成2个集合，里面的点互相没有重复的，但是他们的凸包的交不是空．

6.2. Theorem (Helly's Theorem). Let $\mathfrak{F} \equiv \{B_1, \ldots, B_r\}$ be a family of r convex sets in E^n with $r \geqslant n + 1$. If every subfamily of $n + 1$ sets in $\mathfrak{F}$ has a nonempty intersection, then $\bigcap_{i=1}^{r} B_i \neq \varnothing$.

对r的大小做归纳

一堆凸集集只要r>=维度+1，那么如果每一个元素个数是维度+1的子集都有非空交，那么整个点集也就有非空交．



例子:4个元,4>=2+1
且任意3个都有交,所以4个元整体也有交.

6.3. Theorem (Helly's Theorem). Let $\mathfrak{F}$ be a family of compact convex subsets of E^n containing at least $n+1$ members. If every $n+1$ members of $\mathfrak{F}$ have a point in common, then all the members of $\mathfrak{F}$ have a point in common.

1.20. Definition. A subset A of E^n is said to be **compact** if it is closed and bounded.

6.4. Theorem. Let $\mathfrak{F} \equiv \{A_\alpha : \alpha \in \mathcal{Q}\}$ be a family of compact convex subsets of E^n containing at least $n+1$ members. Suppose K is a compact convex subset of E^n such that the following holds: For each subfamily of $n+1$ sets in $\mathfrak{F}$, there exists a translate of K that is contained in all $n+1$ of them. Then there exists a translate of K that is contained in all the members of $\mathfrak{F}$.

平移

说的是拿一个凸集组成的集合F. K也是一个凸集合. 假设. K经过一个平移之后得到的set可以放到任何他们的n+1个取定的子集中.那么就存在一个k的平移后的set可以放到全部F的元素中.

1.16. Definition. If $A, B \subset \mathsf{E}^n$ and $\lambda \in \mathsf{R}$, we define

$$A + B \equiv \{x + y : x \in A \text{ and } y \in B\}$$

$$\lambda A \equiv \{\lambda x : x \in A\}.$$

If A consists of a single point, $A \equiv \{x\}$, then we often write $x + B$ for $A + B$. The set $x + B$ is called a **translate** of B. The set λA is called a **scalar multiple** of A. If $\lambda \neq 0$, the set $x + \lambda A$ is said to be **homothetic** to A.

6.5. Theorem. Let $\mathfrak{F}$ be a family of compact convex subsets of E^n containing at least $n+1$ members. Suppose K is a compact convex subset of E^n such that for each subfamily of $n+1$ sets in $\mathfrak{F}$, there exists a translate of K that contains all $n+1$ of them. Then there exists a translate of K that contains all the members of $\mathfrak{F}$.

上个定理的反包含版本.

证：$A_\alpha^* \triangleq \{p \mid (p+A_\alpha) \subset K\}$

由helly定理,知道存在公共的平移p 满足定义最终需要的结果. □

12.2. Theorem. The smallest square that is a universal cover in E^2 has sides of length one.

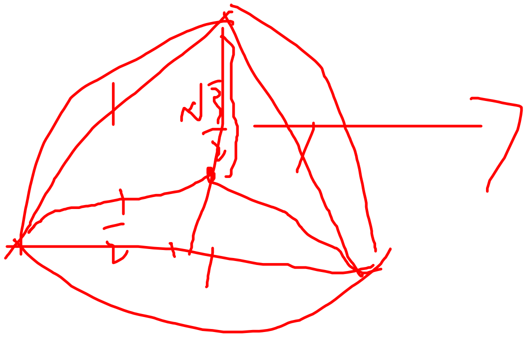
PROOF. Let S be a subset of E^2 having diameter 1 and let K be a square of side length 1. To prove that K is a universal cover, it suffices by Theorem 6.5 to show that given any three points, say x_1, x_2, x_3, of S, there exists a translate of K that covers $\{x_1, x_2, x_3\}$. Now since the diameter of S is 1, it follows that $\{x_1, x_2, x_3\}$ is contained in a Reuleaux triangle T of width 1. But a Reuleaux triangle of width 1 can be rotated through 360° inside a square of side 1. Therefore, no matter what the orientation of T and K, there must be a translate of K that covers T and thus also covers $\{x_1, x_2, x_3\}$. Clearly, no smaller square can be a universal cover since it would not cover a circle of diameter 1. ■

上面定义n=2,且单点集也是凸集，所以这个定理6.5很重要，说明只需要盖上3个点就能盖上全部的点

3点,直径为1,那么一定被一个路罗三角覆盖. 这个通过画圆即可简单看出来. 画法就是
https://tech.sina.com.cn/d/i/2020-01-14/doc-iihnzhha2304017.shtml

12.3. Theorem. The smallest circle that is a universal cover in E^2 has radius $1/\sqrt{3}$.

PROOF. The proof that a circle of radius $1/\sqrt{3}$ is a universal cover in E^2 is similar to that of Theorem 12.2 and is left to the reader (Exercise 12.2). No smaller circle can be a universal cover since it would not cover an equilateral triangle of side 1. ■



利用三角形的面积容易知道这个高度就是1/根号3
其余证明同上,只要包含路罗三角形就行.就满足万有覆盖

12.4. Theorem. The smallest regular hexagon which is a universal cover in E^2 has sides of length $1/\sqrt{3}$.

PROOF. Let S be a set of diameter 1. We begin by circumscribing a rhombus, say $abcd$, about S so that the angle at a is 60°. (See Figure 12.1.)

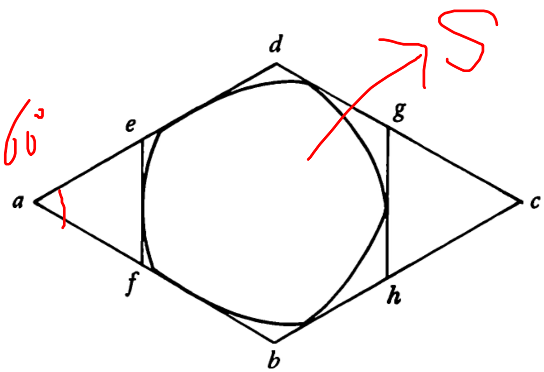


Figure 12.1.

The two support lines perpendicular to the diagonal $\overline{ac}$ of the rhombus cut off two triangles, say aef and cgh, from $abcd$. If $\overline{ef}$ has the same length as $\overline{gh}$, then the hexagon $efbhgd$ is regular and circumscribes S. If not, then we claim that some rotation of the rhombus about S will yield the desired regular hexagon.

If $a'b'c'd'$ is another circumscribed rhombus about S, let α be the counterclockwise angle beteen the diagonal $\overline{ac}$ and the diagonal $\overline{a'c'}$. Let $m(\alpha)$ denote the length of the support line $\overline{e'f'}$ and $n(\alpha)$ the length of the support line $\overline{g'h'}$. Then $m(\alpha)$ and $n(\alpha)$ are continuous functions of α. 10

If $\overline{ef}$ and $\overline{gh}$ are not the same length, then $m(0) \neq n(0)$, and we may assume without loss of generality that $m(0) - n(0) > 0$. But the rhombus corresponding to $\alpha = \pi$ will yield the same hexagon as $efbhgd$, except that the sides $\overline{ef}$ and $\overline{gh}$ will be interchanged. Thus $m(\pi) - n(\pi) < 0$. Since the difference $m(\alpha) - n(\alpha)$ is a continuous function of α, it follows from the Intermediate Value Theorem that there exists an α between 0 and π such that $m(\alpha) - n(\alpha) = 0$. For this α, the corresponding circumscribing hexagon will be regular.

Since S has diameter 1, its maximum width is 1 (Theorem 11.3), and so the regular hexagon circumscribed about S will have sides of length at most $1/\sqrt{3}$. No smaller regular hexagon can be a universal cover since it would not cover a circle of diameter 1. ■

这个定理,正六边形,因为可以包含路罗三角形,所以他就是一个万有覆盖.但是路罗三角形不能包含单位园,所以只能是外接四边形,5变形,6变形.而5变形很难把三角形放进去,因为角度问题,所以最小的是正6变形.

11.1. Definition. Let S be a nonempty compact subset of E^2 and let ℓ be a line (or line segment) in E^2. The **width of S in the direction of ℓ** is the distance between the two parallel lines of support to S that are perpendicular to ℓ and that contain S between them. (See Figure 11.1.)

11.2. Definition. Let S be a nonempty bounded subset of E^2. The **diameter** d of S is defined to be the number

$$d \equiv \sup_{\substack{x \in S \\ y \in S}} \|x - y\|.$$

11.3. Theorem. The diameter of a nonempty compact set S in E^2 is equal to the maximum width of S.

5.1. Definition. A hyperplane H is said to **support** a set S at a point $x \in S$ if $x \in H$ and if H bounds S.

12.5. Theorem. The smallest equilateral triangle that is a universal cover in E^2 has sides of length $\sqrt{3}$.

PROOF. An equilateral triangle with sides of length $\sqrt{3}$ is a universal cover in E^2 since it can cover a regular hexagon of side length $1/\sqrt{3}$, and such a hexagon is itself a universal cover by Theorem 12.4. That no smaller such triangle is a universal cover again follows from the circle of diameter 1. ■

Having found several "smallest" universal covers in E^2, we are led to pose the following question: For each fixed integer k, what is the smallest regular k-gon that is a universal cover in E^2? We have shown that for $k = 3$, 4, and 6 the answer is the k-gon circumscribed about a circle of diameter 1. The answer for k other than these is not known at this time (1982). We might be tempted to conjecture that any regular k-gon circumscribed about a circle of diameter 1

would be a universal cover in E^2. This, however, is not true for any $k > 6$. (See Exercise 12.3.) Indeed, as k increases, the circumscribed k-gons approach the circle, and the smallest circular universal cover was found to have diameter $2/\sqrt{3}$.