

我们重新来证明正六边形是一个万有覆盖。

首先什么是万有覆盖，一个图形，他的直径小于等于1。那么他叫一个覆盖。直径的意思就是图形上距离最远的两个点的距离

万有覆盖是包含任意覆盖的图形。求万有覆盖的上界最小是多少。这个定理说的是上界一定比正六边形小。

6.1. Theorem (Radon's Theorem). Let $S \equiv \{x_1, x_2, \dots, x_r\}$ be any finite set of points in E^n . If $r \geq n + 2$, then S can be partitioned into two disjoint subsets S_1 and S_2 such that $\text{conv} S_1 \cap \text{conv} S_2 \neq \emptyset$.

证明: $P = \{P_1, \dots, P_{n+2}\}$

$$Q = \{P_2 - P_1, \dots, P_{n+2} - P_1\}$$

$$P_2 - P_1 = \sum_{i=3}^{n+2} \beta_i (P_i - P_1)$$

$$(\beta_3 + \beta_4 + \dots + \beta_{n+2} - 1)P_1 + (-\beta_3 P_3 + \dots + -\beta_{n+2} P_{n+2}) + P_2 = 0$$

$$\alpha_1 = \beta_3 + \beta_4 + \dots + \beta_{n+2} - 1$$

$$\alpha_2 = 1$$

$$\alpha_i = -\beta_i \quad (3 \leq i \leq n+2)$$

$$\sum \alpha_i P_i = 0 \quad \sum \alpha_i = 0 \quad \text{--- } \star$$

$$T_1 = \{i \mid \alpha_i \geq 0\} \quad T_2 = \{i \mid \alpha_i < 0\}$$

$$X = \frac{\sum_{i \in T_1} \alpha_i P_i}{\sum_{i \in T_1} \alpha_i} \quad \widetilde{T}_1 = \{P_i \mid \alpha_i \geq 0\}$$

$$\quad \quad \quad \widetilde{T}_2 = \{P_i \mid \alpha_i < 0\}$$

因为星式子我们有

$$X = \frac{-\sum_{i \in T_2} \alpha_i P_i}{-\sum_{i \in T_2} \alpha_i}$$

显然 X 属于 \widetilde{T}_1 的凸包, 也属于 \widetilde{T}_2 的凸包



6.2. Theorem (Helly's Theorem). Let $\mathcal{F} \equiv \{B_1, \dots, B_r\}$ be a family of r convex sets in E^n with $r \geq n + 1$. If every subfamily of $n + 1$ sets in \mathcal{F} has a nonempty intersection, then $\bigcap_{i=1}^r B_i \neq \emptyset$. ➤ 对r的大小做归纳

意思就是2维空间我们有4个圆，他们任意三个的交不是空，那么4个的交一定也不是空。

证明：我们可以假设 $\forall j = 1, \dots, n \exists x_j \in B_j$ 对于任意i 只要i 不等于j

并且 $x_j \notin B_j$

对于这n个点，根据radon定理我们有存在一个划分 A_1, A_2 ，和一个点p属于 $\text{conv}(A_1) \cap \text{conv}(A_2)$

对于 x_1 ，假设他在 A_1 中，我们有 x_1 一定不在 A_2 中，那么 A_2 一定是 B_1 的子集。
这是显然的因为 A_2 里面没有 x_1 ，而这些 A_2 里面的这些点集都属于 B_1 ，所以 A_2 是 B_1 的子集。
所以p这个点属于 B_1 。我们归纳下去即可。所以p这个点属于所有的 B_i



6.3. Theorem (Helly's Theorem). Let \mathcal{F} be a family of compact convex subsets of E^n containing at least $n + 1$ members. If every $n + 1$ members of \mathcal{F} have a point in common, then all the members of \mathcal{F} have a point in common.

这就是上面定理的换一种说法。

1.20. Definition. A subset A of E^n is said to be **compact** if it is closed and bounded.

1.16. Definition. If $A, B \subset E^n$ and $\lambda \in \mathbb{R}$, we define

$$A + B \equiv \{x + y : x \in A \text{ and } y \in B\}$$

$$\lambda A \equiv \{\lambda x : x \in A\}.$$

If A consists of a single point, $A \equiv \{x\}$, then we often write $x + B$ for $A + B$. The set $x + B$ is called a **translate** of B . The set λA is called a **scalar multiple** of A . If $\lambda \neq 0$, the set $x + \lambda A$ is said to be **homothetic** to A .

6.4. Theorem. Let $\mathcal{F} \equiv \{A_\alpha : \alpha \in \mathcal{Q}\}$ be a family of compact convex subsets of E^n containing at least $n + 1$ members. Suppose K is a compact convex subset of E^n such that the following holds: For each subfamily of $n + 1$ sets in \mathcal{F} , there exists a translate of K that is contained in all $n + 1$ of them. Then there exists a translate of K that is contained in all the members of \mathcal{F} .

平移

说的是拿一个凸集组成的集合 \mathcal{F} ， K 也是一个凸集合。假设， K 经过一个平移之后得到的set可以放到任何他们的 $n+1$ 个取定的子集中，那么就存在一个 K 的平移后的set可以放到全部 \mathcal{F} 的元素中。

证明:

$$A_2^* = \{p \mid (K+p) \subset A_2\}$$

显然满足Helly定理. 所以存在一个平移使得K能被

$$\bigcap A_2$$

包含



6.5. Theorem. Let \mathcal{F} be a family of compact convex subsets of E^n containing at least $n + 1$ members. Suppose K is a compact convex subset of E^n such that for each subfamily of $n + 1$ sets in \mathcal{F} , there exists a translate of K that contains all $n + 1$ of them. Then there exists a translate of K that contains all the members of \mathcal{F} .

上个定理的反包含版本.

$$\text{证: } A_2^* = \{p \mid (p + A_2) \subset K\}$$

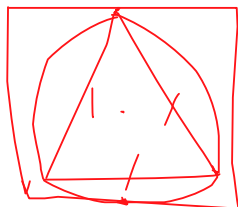
由Helly定理, 知道存在公共的平移 p 满足定义最终需要的结果.



6.5在二维空间的说法就是: 一堆单点集组成的集合, 因为单点集也是凸集, 假设K能平移覆盖其中任意的三个点. 那么K就能覆盖这个整体集合.

12.2. Theorem. The smallest square that is a universal cover in E^2 has sides of length one.

PROOF. Let S be a subset of E^2 having diameter 1 and let K be a square of side length 1. To prove that K is a universal cover, it suffices by Theorem 6.5 to show that given any three points, say x_1, x_2, x_3 , of S , there exists a translate of K that covers $\{x_1, x_2, x_3\}$. Now since the diameter of S is 1, it follows that $\{x_1, x_2, x_3\}$ is contained in a Reuleaux triangle T of width 1. But a Reuleaux triangle of width 1 can be rotated through 360° inside a square of side 1. Therefore, no matter what the orientation of T and K , there must be a translate of K that covers T and thus also covers $\{x_1, x_2, x_3\}$. Clearly, no smaller square can be a universal cover since it would not cover a circle of diameter 1. ■

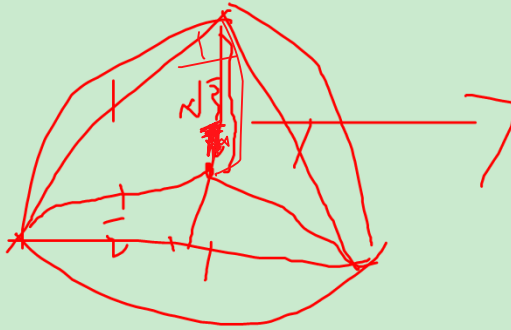


乐罗三角的上顶点到下顶点距离是1, 他能放入大小为1的正方形是显然的.

12.3. Theorem. The smallest circle that is a universal cover in E^2 has radius $1/\sqrt{3}$.

PROOF. The proof that a circle of radius $1/\sqrt{3}$ is a universal cover in E^2 is similar to that of Theorem 12.2 and is left to the reader (Exercise 12.2). No smaller circle can be a universal cover since it would not cover an equilateral triangle of side 1. ■

证明:



利用三角形的面积容易知道这个高度就是 $1/\sqrt{3}$
其余证明同上, 只要包含路罗三角形就行. 就满足万有覆盖

12.4. Theorem. The smallest regular hexagon which is a universal cover in E^2 has sides of length $1/\sqrt{3}$.

PROOF. Let S be a set of diameter 1. We begin by circumscribing a rhombus, say $abcd$, about S so that the angle at a is 60° . (See Figure 12.1.)

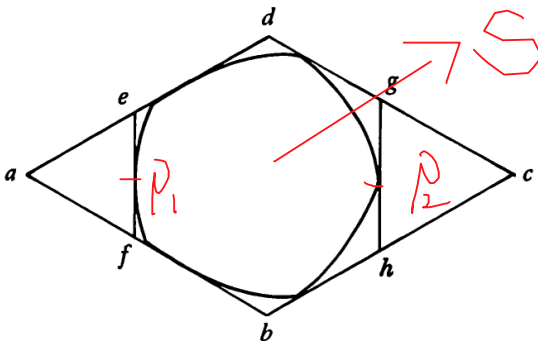


Figure 12.1.

The two support lines perpendicular to the diagonal \overline{ac} of the rhombus cut off two triangles, say $ae f$ and $cg h$, from $abcd$. If \overline{ef} has the same length as \overline{gh} , then the hexagon $efbhgd$ is regular and circumscribes S . If not, then we claim that some rotation of the rhombus about S will yield the desired regular hexagon.

我证明这个结论. 其实很显然, 注意我们 ef, gh 都是切线, 所以两个切点 $p1$ 和 $p2$ 的距离小于等于 1. 所以 ef 和 gh 的距离小于等于 1. 如果距离小于 1, 那么反例容易构造. 所以 ef, gh 距离是 1. 所以是正六边形.

If $a'b'c'd'$ is another circumscribed rhombus about S , let α be the counterclockwise angle between the diagonal \overline{ac} and the diagonal $\overline{a'c'}$. Let $m(\alpha)$ denote the length of the support line $\overline{e'f'}$ and $n(\alpha)$ the length of the support line $\overline{g'h'}$. Then $m(\alpha)$ and $n(\alpha)$ are continuous functions of α .

If \overline{ef} and \overline{gh} are not the same length, then $m(0) \neq n(0)$, and we may assume without loss of generality that $m(0) - n(0) > 0$. But the rhombus corresponding to $\alpha = \pi$ will yield the same hexagon as $efbhgd$, except that the sides \overline{ef} and \overline{gh} will be interchanged. Thus $m(\pi) - n(\pi) < 0$. Since the difference $m(\alpha) - n(\alpha)$ is a continuous function of α , it follows from the Intermediate Value Theorem that there exists an α between 0 and π such that $m(\alpha) - n(\alpha) = 0$. For this α , the corresponding circumscribing hexagon will be regular. → 这段都是为了证明他是正的六边形

Since S has diameter 1, its maximum width is 1 (Theorem 11.3), and so the regular hexagon circumscribed about S will have sides of length at most $1/\sqrt{3}$. No smaller regular hexagon can be a universal cover since it would not cover a circle of diameter 1. ■

至于他为什么是一个万有覆盖, 因为他能放下乐罗三角和单位圆即可.

定理, 下面的单位圆外接正六边形, 可以去了两个红色X部分, 依然是一个万有覆盖.

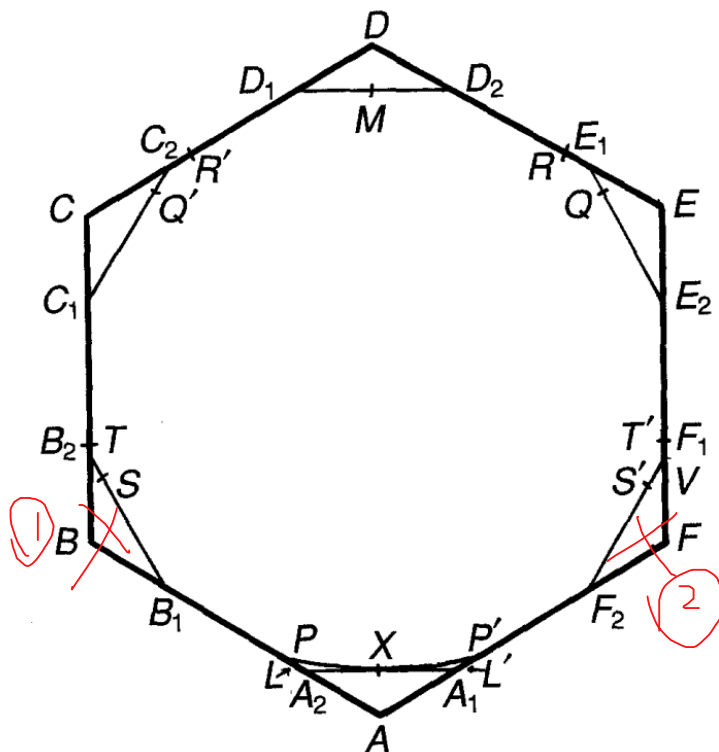


Fig. 1.

