CS712/812 – Stochastic Modeling

Spring 2018

Professor Stephan Olariu
Department of Computer Science
Old Dominion University

Programming Project: Modeling time-dependent parking lot occupancy

We are interested in the following problem. Consider a parking lot with finite capacity N. At time $t = 0$, the parking lot contains n_0, $(0 \le n_0 \le N)$, cars. After that, cars arrive and depart at time-dependent rates as described next. If the parking lot contains k, $(0 \le k \le N)$, cars at time t, the car arrival rate $\alpha_k(t)$ is

$$\alpha_k(t) = \frac{N-k}{N}\lambda(t) \tag{1}$$

and the car departure rate $\beta_k(t)$ is

$$\beta_k(t) = k\mu(t), \tag{2}$$

where for all $t \ge 0$, $\lambda(t)$ and $\mu(t)$ are *integrable* on $[0, t]$. It is worth noting that both $\alpha_k(t)$ and $\beta_k(t)$ are functions of t and k. In particular, it may well be the case that for $t_1 \ne t_2$, $\alpha_k(t_1) \ne \alpha_k(t_2)$ and similarly for $\beta_k(t_1)$ and $\beta_k(t_2)$, giving mathematical expression to the fact that the arrival and departure rates depend not only on the parking lot occupancy, but also on time-dependent factors such as time of the day or day of the week.

Consider the counting process $\{X(t) \mid t \ge 0\}$ of continuous parameter t, where for every positive integer k, $(1 \le k \le N)$, the event $\{X(t) = k\}$ occurs if the parking lot contains k cars at time t. We let $P_k(t)$ denote the probability of the event $\{X(t) = k\}$ and write

$$P_k(t) = \Pr[\{X(t) = k\}].$$

In addition to $P_k(t)$, of interest is the expected value, $E[X(t)]$ of the parking lot occupancy at time $t > 0$, as well as the limiting behavior of $E[X(t)]$ as $t \to \infty$ and/or $N \to \infty$, whenever such limits exist.

0.1 Deriving a closed form for $P_k(t)$

To make the mathematical derivations more manageable, we set $P_k(t) = 0$ for $k < 0$ and $k > N$. Thus, $P_k(t)$ is well defined for all integers $k \in (-\infty, \infty)$ and for all $t \ge 0$. In particular, the assumption about the parking lot containing n_0 cars at $t = 0$ translates into $P_k(0) = 1$ if $k = n_0$ and 0 otherwise.

Let t, $(t \geq 0)$, be arbitrary and let h be sufficiently small that in the time interval $[t, t+h]$ the probability of two or more arrivals or departures, or of a simultaneous arrival and departure, is $o(h)$. With h chosen as stated, the probability $P_k(t+h)$ that the parking lot contains k, $(0 \leq k \leq N)$, cars at time $t+h$ has the following components:

- $P_k(t)\left[1 - h\frac{N-k}{N}\lambda(t) - kh\mu(t) + o(h)\right]$
- $P_{k-1}(t)\left[h\frac{N-k+1}{N}\lambda(t) + o(h)\right]$
- $P_{k+1}(t)[(k+1)h\mu(t) + o(h)]$.

Visibly,

$$P_k(t+h) \quad = \quad P_k(t)\left[1 - h\frac{N-k}{N}\lambda(t) - kh\mu(t)\right] + P_{k-1}(t)h\frac{N-k+1}{N}\lambda(t) + P_{k+1}(t)(k+1)h\mu(t) + o(h).$$

After transposing $P_k(t)$ and dividing by h we have

$$\frac{P_k(t+h) - P_k(t)}{(t+h) - t} = -\left[\frac{N-k}{N}\lambda(t) + k\mu(t)\right]P_k(t) + \frac{N-k+1}{N}\lambda(t)P_{k-1}(t) + P_{k+1}(t)(k+1)\mu(t) + \frac{o(h)}{h}.$$

Taking limits on both sides as $h \to 0$ yields the differential equation

$$\frac{\mathrm{d}P_k(t)}{\mathrm{d}t} = -\left[\frac{N-k}{N}\lambda(t) + k\mu(t)\right]P_k(t) + \frac{N-k+1}{N}\lambda(t)P_{k-1}(t) + (k+1)\mu(t)P_{k+1}(t) \tag{3}$$

with the initial condition $P_k(0) = 1$ for $k = n_0$ and 0 otherwise.

Let

$$G(z,t) = \sum_k P_k(t)z^k \tag{4}$$

be the *probability generating function* of $X(t)$. Recall that since $P_k = 0$ for $k < 0$ and $k > N$, there is no harm working with $k \in (-\infty, \infty)$. Upon multiplying (3) by z^k and upon summing over $k \in (-\infty, \infty)$ we obtain

$$\begin{aligned}
\frac{\partial G(z,t)}{\partial t} &= \sum_k \frac{dP_k(t)}{dt}z^k \\
&= -\sum_k \left[\frac{N-k}{N}\lambda(t) + k\mu(t)\right]P_k(t)z^k + \lambda(t)\sum_k \frac{N-k+1}{N}P_{k-1}(t)z^k + \mu(t)\sum_k (k+1)P_{k+1}(t)z^k \\
&= -\lambda(t)G(z,t) + \left[\frac{\lambda(t)}{N} - \mu(t)\right]z\frac{\partial G(z,t)}{\partial z} + \lambda(t)zG(z,t) - \frac{\lambda(t)}{N}z^2\frac{\partial G(z,t)}{\partial z} + \mu(t)\frac{\partial G(z,t)}{\partial z} \\
&= -(z-1)\left[\frac{\lambda(t)}{N}z + \mu(t)\right]\frac{\partial G(z,t)}{\partial z} + \lambda(t)(z-1)G(z,t).
\end{aligned}$$

Thus, we have obtained the following partial differential equation

$$\frac{\partial G(z,t)}{\partial t} + (z-1)\left[\frac{\lambda(t)}{N}z + \mu(t)\right]\frac{\partial G(z,t)}{\partial z} = \lambda(t)(z-1)G(z,t) \tag{5}$$

with $G(z,0) = z^{n_0}$ and auxiliary equations

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}z}{(z-1)\left[\frac{\lambda(t)}{N}z + \mu(t)\right]} = \frac{\mathrm{d}G}{(z-1)\lambda(t)G}. \tag{6}$$

After the change of variable $z-1=\frac{1}{y}$, the differential equation

$$\frac{\mathrm{d}t}{1}=\frac{\mathrm{d}z}{(z-1)\left[\frac{\lambda(t)}{N}z+\mu(t)\right]}$$

becomes

$$\frac{\mathrm{d}y}{\mathrm{d}t}+\left[\frac{\lambda(t)}{N}+\mu(t)\right]y=-\frac{\lambda(t)}{N}. \tag{7}$$

Using standard techniques, equation (7) yields

$$e^{h(t)}y+\int_0^t\frac{\lambda(u)}{N}e^{h(u)}\,\mathrm{d}u=\text{constant}$$

or, equivalently,

$$\frac{e^{h(t)}}{z-1}+\int_0^t\frac{\lambda(u)}{N}e^{h(u)}\,\mathrm{d}u=c_1 \tag{8}$$

where c_1 is an arbitrary constant and the function $h:[0,\infty)\to[0,\infty)$ is such that for all non-negative x,

$$h(x)=\int_0^x\left[\frac{\lambda(s)}{N}+\mu(s)\right]\mathrm{d}s. \tag{9}$$

For later reference, we now prove the following

$$c_1=-1+\frac{z}{z-1}e^{h(t)}-\int_0^t\mu(u)e^{h(u)}\,\mathrm{d}u. \tag{10}$$

To see that (10) holds, observe that by simple manipulations (8) yields

$$\begin{aligned}
c_1 &= \frac{e^{h(t)}}{z-1}+\int_0^t\frac{\lambda(u)}{N}e^{h(u)}\,\mathrm{d}u\\
&= \frac{e^{h(t)}}{z-1}+\int_0^t\left[\frac{\lambda(u)}{N}+\mu(u)\right]e^{h(u)}\,\mathrm{d}u-\int_0^t\mu(u)e^{h(u)}\,\mathrm{d}u\\
&= \frac{e^{h(t)}}{z-1}+e^{h(t)}-1-\int_0^t\mu(u)e^{h(u)}\,\mathrm{d}u\\
&= -1+\frac{z}{z-1}e^{h(t)}-\int_0^t\mu(u)e^{h(u)}\,\mathrm{d}u
\end{aligned}$$

where we have used the fact that

$$\int_0^t\left[\frac{\lambda(u)}{N}+\mu(u)\right]e^{h(u)}\,\mathrm{d}u=e^{h(t)}-1 \tag{11}$$

which is implied by the Fundamental Theorem of Calculus. Thus (10) holds, as claimed.

Returning to the auxiliary equations (6), we observe that by selecting the multiplicands x_1, x_2, x_3 as

$$\begin{aligned}
x_1 &= -(z-1)G\left[\frac{\lambda(t)}{N}+\mu(t)\right]\\
x_2 &= G\\
x_3 &= -\frac{z-1}{N}
\end{aligned}$$

we can write

$$\frac{x_1\,\mathrm{d}t+x_2\,\mathrm{d}z+x_3\,\mathrm{d}G}{x_1+x_2(z-1)\left[\frac{\lambda(t)}{N}z+\mu(t)\right]+x_3(z-1)\lambda(t)G} = \frac{-\left[\frac{\lambda(t)}{N}+\mu(t)\right]\mathrm{d}t+\frac{\mathrm{d}z}{z-1}-\frac{\mathrm{d}G}{NG}}{0}$$

implying that

$$-\int_0^t \left[\frac{\lambda(u)}{N} + \mu(u)\right] \mathrm{d}u + \ln(z-1) - \ln G^{\frac{1}{N}} = \text{constant}$$

which, in turn, yields

$$-h(t) + \ln \frac{z-1}{G(z,t)^{\frac{1}{N}}} = \text{constant}$$

whereupon, by exponentiation, we obtain

$$\frac{z-1}{G(z,t)^{\frac{1}{N}}} e^{-h(t)} \quad = \quad c_2, \tag{12}$$

for some constant c_2. The two constants c_1 and c_2 are related by

$$c_2 = \Psi[c_1] \tag{13}$$

where Ψ is an arbitrary function.

As it turns out, (12), (13), along with condition $G(z,0) = z^{n_0}$ can be used to determine Ψ. For this purpose, we first find an explicit closed form for Ψ. It is easy to confirm that for an arbitrary real x,

$$\Psi[x] = \frac{1}{x}\left[\frac{x}{x+1}\right]^{\frac{n_0}{N}}. \tag{14}$$

Now, (8), (10), (12) and (14), combined, allow us to write

$$\begin{aligned}
G(z,t) \quad &= \quad (z-1)^N e^{-Nh(t)} (1+c_1)^{n_0} c_1^{N-n_0} \\
&= \quad \left[e^{-h(t)}(z-1)(1+c_1)\right]^{n_0} \left[e^{-h(t)}(z-1)c_1\right]^{N-n_0} \\
&= \quad \left[z\left(1 - e^{-h(t)}\int_0^t \mu(u)e^{h(u)}\,\mathrm{d}u\right) + e^{-h(t)}\int_0^t \mu(u)e^{h(u)}\,\mathrm{d}u\right]^{n_0} \\
&\quad\quad \left[ze^{-h(t)}\int_0^t \frac{\lambda(u)}{N}e^{h(u)}\,\mathrm{d}u + \left(1 - e^{-h(t)}\int_0^t \frac{\lambda(u)}{N}e^{h(u)}\,\mathrm{d}u\right)\right]^{N-n_0}
\end{aligned} \tag{15}$$

In spite of its complexity, (15) reveals a whole lot about the structure of the process $\{X(t) \mid t \geq 0\}$. To see this, observe that $G(z,t)$ is the product of the following two factors:

- $\left[z\left(1 - e^{-h(t)}\int_0^t \mu(u)e^{h(u)}\,\mathrm{d}u\right) + e^{-h(t)}\int_0^t \mu(u)e^{h(u)}\,\mathrm{d}u\right]^{n_0}$ which is the probability generating function of a binomial random variable with parameter n_0 and success probability

$$p(t) = 1 - e^{-h(t)}\int_0^t \mu(u)e^{h(u)}\,\mathrm{d}u; \tag{16}$$

- $\left[ze^{-h(t)}\int_0^t \frac{\lambda(u)}{N}e^{h(u)}\,\mathrm{d}u + \left(1 - e^{-h(t)}\int_0^t \frac{\lambda(u)}{N}e^{h(u)}\,\mathrm{d}u\right)\right]^{N-n_0}$ which is the probability generating function of a binomial random variable with parameter $N - n_0$ and success probability

$$q(t) = e^{-h(t)}\int_0^t \frac{\lambda(u)}{N}e^{h(u)}\,\mathrm{d}u. \tag{17}$$

This observation motivates us to define two additional counting processes:

- $\{R(t) \mid t \geq 0\}$ that keeps track of the number of the n_0 cars present at time $t = 0$ that are still in the parking lot at time t; it is clear that the success probability $p(t) = 1 - e^{-h(t)}\int_0^t \mu(u)e^{h(u)}\,\mathrm{d}u$ is precisely the probability that a generic such car is still in the parking lot at time t;

- $\{S(t) \mid t \geq 0\}$ that keeps track of the number of cars in the parking lot at time t that were not in the parking lot at time $t = 0$; this is also a binomial process with parameters $N - n_0$ and success probability $q(t) = e^{-h(t)} \int_0^t \frac{\lambda(u)}{N} e^{h(u)}\, \mathrm{d}u$.

It is immediate that for all t, $R(t)$ and $S(t)$ are independent random variables. Further, the expression of $G(z,t)$ as a product implies that for all $t \geq 0$, $X(t)$ is the convolution of $R(t)$ and $R(t)$ and so

$$X(t) = R(t) + S(t). \tag{18}$$

Recall that the probability $P_k(t)$ of having exactly k cars in the parking lot at time t is the coefficient of z^k in the probability generating function $G(z,t)$ whose expression if given in (15). Now, it is an immediate consequence of (18) that $P_k(t)$ can be written as

$$\begin{aligned} P_k(t) &= \Pr[\{X(t) = k\}] \\ &= \sum_{i=0}^{k} \binom{n_0}{i} \binom{N-n_0}{k-i} [p(t)]^i \,[1-p(t)]^{n_0-i} \,[q(t)]^{k-i} \,[1-q(t)]^{N-n_0-k+i}. \end{aligned} \tag{19}$$

As it stands, the closed form for $P_k(t)$ is quite unwieldy and uninspiring. Nonetheless, it is a matter of simple algebra to confirm that $P_k(0) = 1$ for $k = n_0$ and 0 otherwise. Of some interest is also the question of "extinction", namely, determining the probability that the parking lot will be empty at some point, as well as that of "saturation" namely determining the probability that the parking lot is full.

We begin by evaluating $P_0(t)$ for $t \neq 0$. To simplify the notation we write $I = \int_0^t \mu(u) e^{h(u)}\, \mathrm{d}u$ and $J = \int_0^t \frac{\lambda(u)}{N} e^{h(u)}\, \mathrm{d}u$.

$$\begin{aligned} P_0(t) &= [1-p(t)]^{n_0} \,[1-q(t)]^{N-n_0} \\ &= \left[\frac{1-p(t)}{1-q(t)}\right]^{n_0} [1-q(t)]^N \\ &= \left[\frac{e^{-h(t)} I}{1-e^{-h(t)} J}\right]^{n_0} \left[1-e^{-h(t)} J\right]^N \\ &= \left[\frac{I}{e^{h(t)} - J}\right]^{n_0} \left[1-e^{-h(t)} J\right]^N \\ &= \left[\frac{I}{1+I}\right]^{n_0} \left[\frac{1+I}{e^{h(t)}}\right]^N \\ &= \left[1 - \frac{1}{I+1}\right]^{n_0} \left[\frac{1+I}{e^{h(t)}}\right]^N. \end{aligned} \tag{20}$$

Using the same notation as above, we evaluate $P_N(t)$ for $t \neq 0$ as follows

$$\begin{aligned} P_N(t) &= [p(t)]^{n_0} \,[q(t)]^{N-n_0} \\ &= \left[\frac{p(t)}{q(t)}\right]^{n_0} [q(t)]^N \\ &= \left[\frac{1-e^{-h(t)} I}{e^{-h(t)} J}\right]^{n_0} \left[e^{-h(t)} J\right]^N \\ &= \left[\frac{e^{h(t)} - I}{J}\right]^{n_0} \left[\frac{J}{e^{h(t)}}\right]^N \\ &= \left[1 + \frac{1}{J}\right]^{n_0} \left[\frac{J}{e^{h(t)}}\right]^N. \end{aligned} \tag{21}$$

Next, we turn to the task of computing a closed form for the expected number, $E[X(t)]$, of cars in the parking lot at time t. Observe that by (11), (18) and the linearity of expectation we can write

$$
\begin{aligned}
E[X(t)] &= E[R(t)] + E[S(t)] \\
&= n_0 \left(1 - e^{-h(t)} \int_0^t \mu(u) e^{h(u)} \, \mathrm{d}u\right) + (N - n_0) e^{-h(t)} \int_0^t \frac{\lambda(u)}{N} e^{h(u)} \, \mathrm{d}u \\
&= n_0 + N e^{-h(t)} \int_0^t \frac{\lambda(u)}{N} e^{h(u)} \, \mathrm{d}u - n_0 e^{-h(t)} \int_0^t \left[\frac{\lambda(u)}{N} + \mu(u)\right] e^{h(u)} \, \mathrm{d}u \\
&= n_0 e^{-h(t)} + e^{-h(t)} \int_0^t \lambda(u) e^{h(u)} \, \mathrm{d}u \\
&= e^{-h(t)} \left[n_0 + \int_0^t \lambda(u) e^{h(u)} \, \mathrm{d}u\right]. \qquad (22)
\end{aligned}
$$