Midterm Review

Review Lecture 2

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Chapter 3:

- 3.1 Orthogonal Vectors and Subspaces
- 3.2 Cosines and Projection onto Lines
- 3.3 Projections and Least Squares

A Summary: You are encouraged to understand the concept of orthogonal complement and least-square algorithm. Important section: 3.3. It will take approximately 10 marks in your exam.

Midterm in Previous Years: 2019

Midterm in 2019 has 7 problems, with a total of 100 marks.

1. (12 marks) True or False. (2'$\times$6)
2. (15 marks) Fill the blanks. (3'$\times$5)
3. (24 marks) $Ax = 0$, $Ax = b$ and the dimensions, bases of four fundamental subspaces.
4. (14 marks) LDL^T factorization and find the inverse of given matrix.
5. (15 marks) Least squares and split vector into 2 orthogonal subspaces.
6. (10 marks) Linear Transformations. (Transposing)
7. (10 marks) Proof about linear independence and subspaces.

Problem 3, 4, 5 are not that difficult, and they take up 53 marks!

Midterm in Previous Years: 2020

Midterm in 2020 has 8 problems, with a total of 110 marks.

1. (15 marks) Multiple Choices. (3'$\times$5)
2. (25 marks) Fill the blanks. (5'$\times$5)
3. (10 marks) *LU* factorization.
4. (16 marks) Dimensions, bases of four fundamental subspaces.
5. (10 marks) Linear Transformations. (given basis)
6. (6 marks) Proof or counter example.
7. (10 marks) Linear Transformations. (natural basis)
8. (12 marks) Proof about linear independence.

Again, Problem 3, 4, 5 are not that difficult, and they take up 36 marks!

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Example

Let L: $\mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by

$$L(\mathbf{x}) = \begin{bmatrix} x_2 \\ x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$$

Find the matrix representations of L with respect to the ordered bases $\{\mathbf{u_1}, \mathbf{u_2}\}$ and $\{\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}\}$, where

$$\mathbf{u_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

and

$$\mathbf{b_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Linear Transformation: Given Basis

Solution:

Input coordinates $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, which is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in natural basis.

The output in natural basis will be

$$L\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1+2 \\ 1-2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

Transform to the coordinates under new basis **b**.

$$\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = a_{11} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_{21} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + a_{31} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Output coordinates $\begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}$, which is $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ in natural basis.

Solution:

Input coordinates $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ in natural basis.

The output in natural basis will be

$$L\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3+1 \\ 3-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

Output coordinates $\begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix}$, which is $\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$ in natural basis.

The transformation matrix:

$$A = \begin{bmatrix} -1 & -3 \\ 4 & 2 \\ -1 & 2 \end{bmatrix}$$

Least-Squares

Example

Let

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 1 & -1 & 2 \\ 1 & -1 & -2 \end{bmatrix}, b = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}$$

Find the least squares solution to $Ax = b$.

The system must be inconsistent or we cannot find the least squares solution, because we can find the exact solution!
The method to find least squares solution is to solve

$$A^T A\hat{x} = A^T b$$

Solve this system to get the least squares solution:

$$\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -2 & -1 & -1 \\ 0 & 1 & 2 & -2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 2 \\ 1 & -2 & 1 & -2 \\ 1 & -1 & 2 & -1 \\ 1 & -1 & -2 & 3 \end{bmatrix}$$

Multiply that:

$$\begin{bmatrix} 4 & -5 & 1 & -2 \\ -5 & 7 & -2 & 4 \\ 1 & -2 & 9 & -10 \end{bmatrix} \to \begin{bmatrix} 4 & -5 & 1 & -2 \\ 0 & 3 & -3 & 6 \\ 0 & 0 & 32 & -32 \end{bmatrix}$$

The least squares solution is

$$\hat{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

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Relations of Four Fundamental Subspaces

Always keep the 4 fundamental subspaces in mind!

- Row space is the orthogonal complement of nullspace, column space is the orthogonal complement of left nullspace.
- If $C(A)$ is the same with $C(A^T)$, then $N(A)$ is the same with $N(A^T)$.
- Dimension: $dim(C(A)) = dim(C(A^T)) = r$, $dim(N(A)) = n - r$, $dim(N(A^T)) = m - r$.
- If $AB = 0$, then $C(B) \in N(A)$, $C(A^T) \in N(B^T)$. $C(A^T)$ is orthogonal to $C(B)$, that means every vector in $C(A^T)$ is orthogonal to every vector in $C(B)$.

Properties of Linear Transformations

- T(0)=0.
- Ax is in the column space of A.
- Rank of A is the dimension of the output space $C(A)$.
- Some special linear transformations: rotation, reflection, projection.

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Proof of Rank Inequalities

The Basics: $rank(AB) \leqslant max\{rank(A), rank(B)\}$.

1. $rank(PAQ) = rank(A)$ if P, Q are invertible.

Proof:

$$rank(PAQ) \leqslant rank(A)$$

$$rank(A) = rank(P^{-1}PAQQ^{-1}) \leqslant rank(PAQ)$$

$$rank(PAQ) = rank(A)$$

It shows: Invertible (elementary) row and column operations will not change the rank.

So, we can prove a series of rank inequalities by constructing matrix and do invertible row and column operations.

Proof of Rank Inequalities

2. $rank(A+B) \leqslant rank(A) + rank(B)$.

Proof:

$$rank\,(A) + rank\,(B) = r\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right)$$

$$= r\left(\begin{bmatrix} A & A \\ 0 & B \end{bmatrix}\right) = r\left(\begin{bmatrix} A & A \\ A & A+B \end{bmatrix}\right) \geqslant rank\,(A+B)$$

3. $r\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) \leqslant r\left(\begin{bmatrix} A & 0 \\ C & B \end{bmatrix}\right)$.

Proof:

Rank equals number of pivots. Matrix C can provide additional pivots.

4. $rank(A) + rank(B) \leqslant rank(AB) + n$.

Proof:

$$rank\,(AB) + n = r\left(\begin{bmatrix} I_{n\times n} & 0 \\ 0 & AB \end{bmatrix}\right)$$

$$= r\left(\begin{bmatrix} I_{n\times n} & 0 \\ A & AB \end{bmatrix}\right) = r\left(\begin{bmatrix} I_{n\times n} & -B \\ A & 0 \end{bmatrix}\right) \geqslant rank\,(A) + rank\,(B)$$

5. $rank(AB) + rank(BC) - rank(B) \leqslant rank(ABC)$.

Proof:

$$rank\,(ABC) + rank\,(B) = r\left(\begin{bmatrix} ABC & 0 \\ 0 & B \end{bmatrix}\right)$$

$$= r\left(\begin{bmatrix} ABC & 0 \\ AB & B \end{bmatrix}\right) = r\left(\begin{bmatrix} ABC & 0 \\ AB & B \end{bmatrix}\right) = r\left(\begin{bmatrix} 0 & -BC \\ AB & B \end{bmatrix}\right)$$

$$= r\left(\begin{bmatrix} AB & B \\ 0 & -BC \end{bmatrix}\right) \geqslant rank\,(AB) + rank\,(BC)$$

A Conclusion: $rank(A^TA) = rank(A)$

Proof:

Firstly, we prove $A^TAx = 0 \to Ax = 0$.

Suppose x is in the nullspace of A^TA:

$$A^TAx = 0$$

Multiply x^T on both sides:

$$x^TA^TAx = 0 \to (Ax)^TAx = 0 \to \|Ax\| = 0 \to Ax = 0$$

Secondly, we prove $Ax = 0 \to A^TAx = 0$.

If $Ax = 0$, $A^TAx = 0$ must hold because no matrix can bring zero vector out of the origin.

So, $N(A^TA) = N(A)$, so does the row space (orthogonal complement) and rank $R(A^TA) = R(A), rank(A^TA) = rank(A)$.

Good Luck!

Hope you all can do well in the midterm exam!!!