# Problem Set 8



**Problem 1.** *(Final Exam, Fall 2020, Version A, 16 marks) Compute the nth order determinant:*

$$\det A = \begin{vmatrix} a & 0 & \cdots & \cdots & 0 & 1 \\ 0 & a & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & a & 0 \\ 1 & 0 & \cdots & \cdots & 0 & a \end{vmatrix}, n \geqslant 2$$

**Solution 1.**
*Cofactor expansion on row 2, row 3, ..., row n-1:*

$$\det A = \begin{vmatrix} a & 0 & \cdots & \cdots & 0 & 1 \\ 0 & a & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & a & 0 \\ 1 & 0 & \cdots & \cdots & 0 & a \end{vmatrix} = a^{n-2} \begin{vmatrix} a & 1 \\ 1 & a \end{vmatrix} = a^n - a^{n-2}$$

**Problem 2.** *(Final Exam, Fall 2020, Version B, 10 marks) Compute the nth order determinant:*

$$D_n(x, y) = \begin{vmatrix} x+y & xy & & & & \\ 1 & x+y & xy & & & \\ & 1 & x+y & xy & & \\ & & 1 & \ddots & \ddots & \\ & & & \ddots & x+y & xy \\ & & & & 1 & x+y \end{vmatrix}, n \geqslant 2$$

**Solution 2.**
*Cofactor expansion on row 1, following by cofactor expansion on column 1:*

$$D_n(x, y) = (x+y) D_{n-1} - xy \begin{vmatrix} 1 & xy & & & \\ 0 & x+y & xy & & \\ 0 & 1 & x+y & \ddots & \\ 0 & & \ddots & \ddots & xy \\ 0 & & & 1 & x+y \end{vmatrix} = (x+y) D_{n-1} - xy D_{n-2}$$

*Check* $D_1(x, y) = x + y$ *and* $D_2(x, y) = x^2 + xy + y^2$.
*By mathematical induction,* $D_n(x, y) = x^n + x^{n-1}y + \ldots + xy^{n-1} + y^n$. *(Process omitted here.)*

**Problem 3.** *Compute the nth order determinant:*

$$\begin{vmatrix} 1 + {x_1}^2 & x_1x_2 & \cdots & x_1x_n \\ x_2x_1 & 1 + {x_2}^2 & \cdots & x_2x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_nx_1 & x_nx_2 & \cdots & 1 + {x_n}^2 \end{vmatrix}$$

**Solution 3.**
*Add column 1:*

$$\det A = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ x_1 & 1 + {x_1}^2 & x_1x_2 & \cdots & x_1x_n \\ x_2 & x_2x_1 & 1 + {x_2}^2 & \cdots & x_2x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n & x_nx_1 & x_nx_2 & \cdots & 1 + {x_n}^2 \end{vmatrix} = \begin{vmatrix} 1 & -x_1 & -x_2 & \cdots & -x_n \\ x_1 & 1 & & & \\ x_2 & & 1 & & \\ \vdots & & & \ddots & \\ x_n & & & & 1 \end{vmatrix} = 1 + \sum_{i=1}^{n} {x_i}^2$$

**Problem 4.** *Compute the nth order determinant:*

$$\det A = \begin{vmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 0 \end{vmatrix}_{n\times n}$$

**Solution 4.**
*Add column 1:*

$$\det A = \begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 & -1 & \cdots & -1 \\ 1 & -1 & & & & \\ 1 & & -1 & & & \\ 1 & & & -1 & & \\ \vdots & & & & \ddots & \\ 1 & & & & & -1 \end{vmatrix} = \begin{vmatrix} 1-n & & & & & \\ 1 & -1 & & & & \\ 1 & & -1 & & & \\ 1 & & & -1 & & \\ \vdots & & & & \ddots & \\ 1 & & & & & -1 \end{vmatrix} = (-1)^n (1-n)$$

**Problem 5.** *Compute the determinant:*

$$\begin{vmatrix} a & a^2 & a^3 & a^5 \\ b & b^2 & b^3 & b^5 \\ c & c^2 & c^3 & c^5 \\ d & d^2 & d^3 & d^5 \end{vmatrix}$$

**Solution 5.**

$$\begin{vmatrix} a & a^2 & a^3 & a^5 \\ b & b^2 & b^3 & b^5 \\ c & c^2 & c^3 & c^5 \\ d & d^2 & d^3 & d^5 \end{vmatrix} = abcd \begin{vmatrix} 1 & a & a^2 & a^4 \\ 1 & b & b^2 & b^4 \\ 1 & c & c^2 & c^4 \\ 1 & d & d^2 & d^4 \end{vmatrix}$$

*Let* $S = (d-c)(d-b)(d-a)(c-b)(c-a)(b-a)$:

$$\det A = \begin{vmatrix} 1 & a & a^2 & a^3 & a^4 \\ 1 & b & b^2 & b^3 & b^4 \\ 1 & c & c^2 & c^3 & c^4 \\ 1 & d & d^2 & d^3 & d^4 \\ 1 & x & x^2 & x^3 & x^4 \end{vmatrix} = S(x-a)(x-b)(x-c)(x-d) = C_{51} + C_{52}x + C_{53}x^2 + C_{54}x^3 + C_{55}x^4$$

*Compare the coefficient of* $x^3$:

$$C_{54} = (-a-b-c-d)\,S$$

$$M_{54} = (-1)^{5+4}(-a-b-c-d)\,S = (a+b+c+d)S$$

*Therefore the final result is:*

$$\begin{vmatrix} a & a^2 & a^3 & a^5 \\ b & b^2 & b^3 & b^5 \\ c & c^2 & c^3 & c^5 \\ d & d^2 & d^3 & d^5 \end{vmatrix} = abcd(a+b+c+d)(d-c)(d-b)(d-a)(c-b)(c-a)(b-a)$$

**Problem 6.** *Compute the determinant:*

$$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix}$$

**Solution 6.**
*Add column 1:*

$$\det A = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1+x & 1 & 1 & 1 \\ 1 & 1 & 1-x & 1 & 1 \\ 1 & 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1 & 1-y \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 & -1 & -1 \\ 1 & x & & & \\ 1 & & -x & & \\ 1 & & & y & \\ 1 & & & & -y \end{vmatrix} = \begin{vmatrix} 1 & & & & \\ 1 & x & & & \\ 1 & & -x & & \\ 1 & & & y & \\ 1 & & & & -y \end{vmatrix} = x^2y^2$$