Similarity Transformations; Exercise Problems

Lecture 11

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2022.12.13

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Last Lecture, We Discuss...

Two parts in last lecture:

1. Matrix Diagonaliztion
   simplified criteria; algebraic multiplicity and geometric multiplicity; determine a matrix is diagonalizable or not
2. Complex Matrix
   introduction; Hermitian; dot products; Hermitian matrix; unitary matrix; orthogonal diagonalization for real symmetric matrix; unitary diagonaliztion for Hermitian matrix; Gram-Schmidt

Simplified Criteria for Matrix Diagonalization

If a $n \times n$ matrix has n distinct eigenvalues, then it must be diagonalizable.

Why? I will not give you the proof, but I want you to understand. I will show from 2 sides:

1. For each eigenvalue, there must be at least 1 eigenvector (authentic to say: 1-dimensional).
2. For the same eigenvector (all vectors that in the 1-dimensional line), it cannot correspond to 2 eigenvalues.
   (Hint: Recall the knowledge in linear transformation, can a single input results in multiple outputs?)
   Geometric: One direction cannot have 2 stretching coefficient!

Notice that it is not necessary for a diagonalization matrix to have n distinct eigenvalues.

More challenging: Eigenvectors that correspond to distinct eigenvalues are linearly independent. (Hint: Can a k-dimensional space has $k+1$ stretching coefficients?)

A theorem that I don't want to prove:

$$\text{Geometric Multiplicity} \leq \text{Algebraic Multiplicity}$$

A Summary - Liar's Game:

1. All the eigenvalues are smart, they will only claim they have higher or equal multiplicity (algebraic multiplicity) than it actually has.
2. Your mission is to catch the liar. You investigate the possible liars (repeated eigenvalues) by calculating the nullspace dimension of $\det(A - \lambda I)$. If you find that the eigenvalues don't have the same number of independent eigenvectors as it claimed, it is a liar. Once you catch 1 liar, the matrix is not diagonalizable any more.

Well, that is only an analogy... That's quite similar, right?

Hermitian Matrix

For real matrix, a matrix is symmetric if $A^T = A$, so for the complex matrix, a matrix is Hermitian if $A^H = A$.

The following matrix is a Hermitian matrix:

$$A = \begin{bmatrix} 2 & 3-i \\ 3+i & 5 \end{bmatrix}$$

Recall that for real symmetric matrix, the eigenvectors corresponding to different eigenvalues are orthogonal. This property remains true for the Hermitian matrix.

Another property: Every eigenvalue of a Hermitian matrix is real. This property is definitely true for real symmetric matrix because it is a special case of Hermitian!

Unitary Matrix

For real matrix, a matrix is orthogonal if $A^T A = AA^T = I$, so for the complex matrix, a matrix is unitary if $A^H A = AA^H = I$. Unitary has orthonormal column vectors.

Properties:

- Inner products and lengths are preserved by U.

$$(Ux)^H(Uy) = x^H y, \|Ux\| = \|x\|$$

- Every eigenvector of U has absolute value $|\lambda| = 1$.
- Eigenvectors corresponding to different eigenvalues are orthogonal.

For example, consider 2×2 rotation matrix, it will have orthogonal eigenvectors. Also true for permutation matrix, which is a orthogonal matrix and of course unitary.

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Similar Matrices

Definition

Two matrices A and B are said to be similar if there is an invertible matrix M such that

$$B = M^{-1}AM$$

It is denoted by $A \backsim B$.

For similar matrices A and B, they must share the same eigenvalues. They have the same number of independent eigenvectors.

$$B - \lambda I = M^{-1}(A - \lambda I)M$$

$$\det(B - \lambda I) = \det\left(M^{-1}\right)\det(A - \lambda I)\det(M) = \det(A - \lambda I)$$

$$Ax = \lambda x \Rightarrow MBM^{-1}x = \lambda x \Rightarrow B\left(M^{-1}x\right) = \lambda\left(M^{-1}x\right)$$

Properties for Similar Matrices

Similar matrices A and B have the following properties:

- A has the same eigenvalues as B.
- A has the same number of independent eigenvectors as B. (Difference: M^{-1})
- $\det A = \det B$, $trace(A) = trace(B)$.
- $rank(A) = rank(B)$. (who can give me a translation?)
- A and B have the same characteristic polynomial.

If A and B have the same characteristic polynomial (same eigenvalues), they are not always similar.

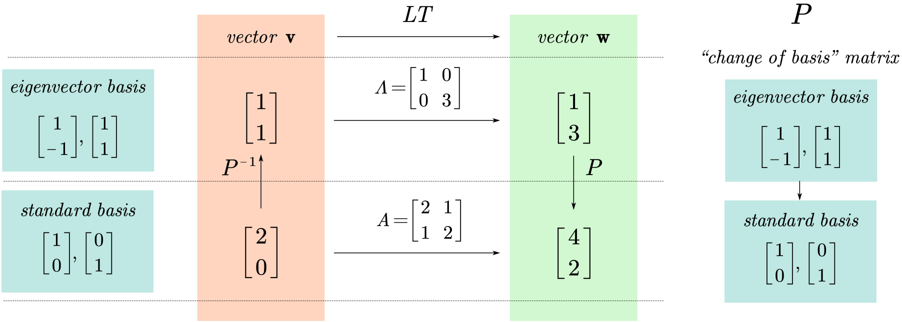
$$A = \begin{bmatrix} 5 & 3 \\ 0 & 5 \end{bmatrix}, B = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

Those properties are necessary, not sufficient (Even if you find 2 matrices that can satisfy all of them, we can not say they are similar). We will discuss later on.

Similarity Transformation: Change of Basis

Recall: matrix diagonalization $A = P\Lambda P^{-1}$, we can say $A \backsim \Lambda$.

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1}$$



Essence: change basis and simplify the linear transformation matrix.

Similarity Transformation: Change of Basis

Definition

Two matrices A and B are said to be similar if there is an invertible matrix M such that

$$B = M^{-1}AM$$

It is denoted by $A \backsim B$.

Actually, A and B represent the same linear transformation under different bases. M is the "change of basis" matrix.

- A has the same eigenvalues as B. (The strenching coefficients are the same)
- A has the same number of independent eigenvectors as B. (Difference: M^{-1}) (The same eigenvectors! we only change the basis through M^{-1})
- $rank(A) = rank(B)$. (the dimension of output space are the same)

Video: `https://www.bilibili.com/video/BV1ys411472E?p=13`

Similarity Transformation: Additional

For 2 diagonal matrices Λ_1 and Λ_2, if they have the same eigenvalues, they are similar.

$$A = \begin{bmatrix} 2020 & & & \\ & 2021 & & \\ & & 2021 & \\ & & & 2022 \end{bmatrix}, B = \begin{bmatrix} 2021 & & & \\ & 2022 & & \\ & & 2020 & \\ & & & 2021 \end{bmatrix}$$

Recall that every time you do matrix diagonalization, the result is not unique. You can change the columns in P to change the order of eigenvalues on the diagonal. They must be similar because they are all similar to original A.

If matrices A and B are diagonalizable, and they have the same eigenvalues, they are similar.

Schur's Lemma

Theorem

For a matrix of degree n, there exists a unitary matrix U of degree n such that $U^{-1}AU = T$ is triangular. The eigenvalues of A appear along the diagonal of the similar matrix T.

That is a theorem for all matrices. So every degree n matrices can be similar to a triangular matrix.

- For $A = LU$, we simplify a matrix to upper triangular without changing its nullspace.
- For $U^{-1}AU = T$, we simplify a matrix to triangular without changing eigenvalues and number of independent eigenvectors.

We are not required to calculate this by our own (99%). All you need to do is to understand this theorem.

Normal Matrices

Theorem

For a matrix of degree n, there exists a unitary matrix U of degree n such that $U^{-1}AU = T$ is triangular. The eigenvalues of A appear along the diagonal of the similar matrix T.

For some matrics, $T = \Lambda$. For that case, the matrices are called normal.

Normal matrices contain:

- Real symmetric; Hermitian. They have all real eigenvalues.
- Real skew-symmetric; skew-Hermitian. They have all imaginary eigenvalues (or zero!).
- Orthogonal; unitary. They have eigenvalues $|\lambda| = 1$.

For normal matrices, $NN^H = N^H N$. All the 3 kinds satisfy this condition.

The Jordan Form (For Advanced Students)

Although not all the matrices are diagonalizable, we can simplify all the matrix to Jordan form (that is the nearest matrix to diagonal), which has Jordan blocks on the diagonal.

Jordan block:

$$J = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}$$

While if a matrix has eigenvalues λ with algebraic multiplicity of n but geometric multiplicity of 1, its simplest similar matrix J will have a Jordan block like this.

Our shear matrix:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

is a Jordan block. It only has 1 independent eigenvectors.

The Jordan Form (For Advanced Students)

While if a matrix has eigenvalues λ with algebraic multiplicity of n but geometric multiplicity of 2. It will has 2 Jordan blocks for this eigenvlue in the Jordan block!

For example:

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

It has 2 independent eigenvectors, so it has 2 Jordan blocks in Jordan form.

$$J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The Jordan Form (For Advanced Students)

For non-diagonalizable matrices A and B, how to determine whether they are similar? Remind that the 5 properties for similar matrices are not sufficient.

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

For the example above, those matrices have the same eigenvalues ($\lambda = 2$ with algebraic multiplicity 4), the same number of independent eigenvectors (2 in this example). But they are not similar.

Equivalent Property for Similarity:

A and B share the same Jordan blocks.

If you want to know how to find the Jordan form for non-diagonalizable matrices, choose *MA109: Linear Algebra II* (but be careful!).

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Eigenvalues of Common Matrices

For common matrices, here I summarize the eigenvalues of them.

- For Hermitian matrices $A^H = A$, the eigenvalues are all real.
- For skew-Hermitian matrices $A^H = -A$, the eigenvalues are all imaginary.
- For unitary matrices $U^H U = I$, the eigenvalues should satisfy $|\lambda| = 1$.
- For rank-1 matrices $A = uv^T$, the eigenvalues are 0 with geometric multiplicity of $n-1$, and $u^T v$ with geometric multiplicity of 1. Mostly diagonalizable unless $u^T v = 0$ (orthogonal).
- For Hermitian matrices $A^H A, AA^H$, the eigenvalues are all real and satisfy $\lambda \geq 0$.
- For projection matrices $P = A(A^T A)^{-1} A^T$, the eigenvalues are 0 with geometric multiplicity of $n-r$, and 1 with geometric multiplicity of r.
- For idempotent matrices $A^2 = A$, the eigenvalues are 0 with geometric multiplicity of $n-r$, and 1 with geometric multiplicity of r.

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Problem 1 (from 5.1, 5.2)

Example

(2020 Fall, Final, 12 marks) Let $a, b \in \mathbb{R}$ and let

$$A = \begin{bmatrix} 2 & -1 & 2 \\ 5 & a & 3 \\ -1 & b & -2 \end{bmatrix}$$

Suppose that $p = (1, 1, -1)^T$ is an eigenvector of A.
(a) Find the values of a, b and find the eigenvalue λ corresponding to eigenvector p.
(b) Is A diagonalizable? Please explain your answer.

Problem 2 (from 5.1, 5.2)

Example

(a) (2018 Fall, Final, 3 marks) Suppose a 3×3 matrix has eigenvalues $0, 1, 2$, find the eigenvalues of $A(A - I)(A - 2I)$.

(b) (2019 Fall, Final, 2 marks) Let A be a real square matrix. Then a real number λ is an eigenvalue of A if and only if it is an eigenvalue of the transpose A^T. (T/F)

(c) (2020 Fall, Final, 3 marks) Let I_n be the identity matrix of order n and let α be a column vector of length 1 in $\mathbb{R}^n$, then $I_n - \alpha\alpha^T$ is not invertible. (T/F)

(d) (2018 Spring, Final, 3 marks) Suppose A has eigenvalues 0 and 1, corresponding to eigenvectors $(1, 2)^T$ and $(2, -1)^T$, find A.

(e) (2020 Fall, Final, 5 marks) Find the eigenvalues of $I_3 - uv^T$, where I_3 is the 3×3 identity matrix, and u and v are nonzero vectors in $\mathbb{R}^3$.

(f) (2020 Fall, Final, 5 marks) If $A^2 = A$ and rank(A)$= r$, find trace(A).

Properties of Idempotent Matrix

For idempotent matrices $A^2 = A$, the eigenvalues are 0 with geometric multiplicity of $n - r$, and 1 with geometric multiplicity of r.

Idempotent matrix is diagonalizable, it has n independent eigenvectors and its rank equals its trace.

Proof:

$$A\left(A - I\right) = 0$$

- If $AB = 0$, $rank(A) + rank(B) \leq n$.
- $rank(A) + rank(B) \geq rank(A + B)$.

$$n = r\left(I\right) \leqslant r\left(A\right) + r\left(A - I\right) \leqslant n$$

The sum of geometric multiplicity is n, giving n independent eigenvectors, which makes A diagonalizable. A can only have eigenvalues 0 and 1, with geometric multiplicity $n - r$, r.

Problem 3 (from 5.5)

Example

(a) (2019 Fall, Final, 2 marks) If H is a Hermitian matrix, then $I + iH$ is an invertible matrix. (T/F)

(b) (2020 Fall, Final, 3 marks) If A is a complex matrix, and $A^T = A$, then A is diagonalizable. (T/F)

(c) (2020 Fall, Final, 3 marks) Let A be a $m \times n$ complex matrix and $B = A^H A$, then the matrix $B + iI_n$ is invertible. (T/F)

Problem 4 (from 5.5)

Example

(2020 Fall, Final, 6 marks) Find a real 4×4 orthogonal matrix A such that A has no real eigenvalues but both A^2 and A^3 have real eigenvalues. Please explain why the matrix you give has the required properties.

Problem 5 (from 5.6)

Example

(a) (2020 Fall, Final, 3 marks) If A is an upper triangular (square) matrix and A is similar to a diagonal matrix, then A must be a diagonal matrix. (T/F)

(b) (2020 Fall, Final, 3 marks) Let A and B be invertible matrices. If A is similar to B, then $A + A^T$ is similar to $B + B^T$ (T/F), $A + A^{-1}$ is similar to $B + B^{-1}$ (T/F).

Answer:

1. Problem 1: (a) $a = -3, b = 0, \lambda = -1$. (b) not diagonalizable.
2. Problem 2: (a) $0, 0, 0$. (b) T. (c) T. (e) $1, 1, 1 - u^T v$. (f) r.
3. Problem 3: (a) T. (b) F. (c) T.
4. Problem 4: (for reference)

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

5. Problem 5: (a) F. (b) F, T.