Column Space & Nullspace; Solving Ax=0 & Ax=b

Lecture 2

Zhang Ce

Department of Electrical and Electronic Engineering
Southern University of Science and Technology

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Last Lecture, We Discuss...

Seven parts in last lecture:

1. Introduction to the Whole Course
2. The Geometry of Linear Equations
   row picture & column picture
3. Gaussian Elimination and Back-Substitution
   nonsingular & singular cases
4. Matrix Multiplication
   row-col, row, column, col-row & block method
5. LU Factorization
   elimination matrices & process of LU factorization
6. Row Exchanges and $PA = LU$ Factorization
   permutation matrix, process of $PA = LU$ factorization
7. Inverse of Matrix
   existence of inverse, calculation of inverse, block inverse

Don't forget row method and column method for matrix multiplication. Things will get easier if you know these methods!

Matrix Multiplication

Examples

Compute the matrix multiplication.

$$\begin{bmatrix} 3 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 5 \\ 2 & 2 & 2 \\ 3 & 3 & 1 \end{bmatrix}$$

1. The regular (row-col) way. (Row of A) multiply (Column of B).
2. The row way. Linear combination of (Row of B).
3. The column way. Linear combination of (Column of A).
4. The col-row way. (Column of A) multiply (Row of B).

There is also an additional method: block method.

Why $P^T = P^{-1}$?

Why $P^T = P^{-1}$? A direct explanation:

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad P^T = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- 1 at $(1, 2)$: (Row 2) $\to$ (Row 1).
- 1 at $(2, 3)$: (Row 3) $\to$ (Row 2).
- 1 at $(3, 4)$: (Row 4) $\to$ (Row 3).
- 1 at $(4, 1)$: (Row 1) $\to$ (Row 4).

- 1 at $(2, 1)$: (Row 1) $\to$ (Row 2).
- 1 at $(3, 2)$: (Row 2) $\to$ (Row 3).
- 1 at $(4, 3)$: (Row 3) $\to$ (Row 4).
- 1 at $(1, 4)$: (Row 4) $\to$ (Row 1).

Exchange A and B then exchange B and A will result in exchange nothing.

Existence of Inverse

According to definition, if it has an inverse, there exist a matrix B to let $AB = I$, thus, B is the inverse of A.

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x & \cdot \\ y & \cdot \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Recall the column method of matrix multiplication:

$$x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

It may lead you to think about column picture.

Linear combinations of the 2 column vectors are still on the straight line.

Therefore, the inverse cannot exist for this matrix A.

How about multiplying this matrix on the left side?

Gauss-Jordan Method: Another Way

Example

Find the inverse of the following matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$$

Solution:

Remember: Always do row operations.

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 7 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7 & -2 \\ 0 & 1 & -3 & 1 \end{bmatrix}$$

Or... Column operations!

$$\begin{bmatrix} 1 & 2 \\ 3 & 7 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 3 & 1 \\ 1 & -2 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 7 & -2 \\ -3 & 1 \end{bmatrix}$$

Example

Find the inverse of this matrix, given that A, B, C are all invertible matrices. Please express the result in matrix form.

$$L = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$$

Solution:

Core idea: Treat all block matrices as a single entry.

$$\begin{bmatrix} A & 0 & I & 0 \\ B & C & 0 & I \end{bmatrix} \to \begin{bmatrix} A & 0 & I & 0 \\ 0 & C & -BA^{-1} & I \end{bmatrix} \to \begin{bmatrix} I & 0 & A^{-1} & 0 \\ 0 & I & -C^{-1}BA^{-1} & C^{-1} \end{bmatrix}$$

$$L^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -C^{-1}BA^{-1} & C^{-1} \end{bmatrix}$$

A good question to ask: why not right-multiply C^{-1} in the final step?

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Vector Spaces

Once the vector in a space can have addition and scalar multiplication, and the result is still in this space, then the space can be called vector spaces.

Some standard vector spaces:

- $\mathbb{R}^1$: x axis. 1 dimensional.
- $\mathbb{R}^2$: x-y plane. 2 dimensional.
- $\mathbb{R}^3$: x-y-z space. 3 dimensional.

Example

Are these sets below vector spaces? Explain it.

1. $\mathbb{R}^3$ space without the origin.
2. $\mathbb{R}^2$ plane without the third quadrant.
3. $\mathbb{R}^{2021}$ space.
4. $\mathbb{C}^2$ space.

Subspaces

Definition

If S in a nonempty subset of a vector space V, and S satisfies the conditions: linear combinations stay in the subspace.

- $c\mathbf{u} \in S$ whenever $\mathbf{u} \in S$ for any scalar c
- $\mathbf{u} + \mathbf{v} \in S$ whenever $\mathbf{u} \in S$ and $\mathbf{v} \in S$

Then S is said to be a subspace of V.

To say it in human language, if a space S is a closed under linear combinations and it is in a vector space V, then S is a subspace of V.

Remember that the origin should exist in every subspace or vector space!

It is an abstract concept but no need to worry about that. You will understand it later.

Vectors vs Points

When we describe a vector, we always use arrows. But, when we describe multiple vectors, treat it as points! That is a very important concept in linear algebra.

Hope this video can help you establish this concept.

Source: The Essence of Linear Algebra -by *3Blue1Brown* P3 04:44

`https://www.bilibili.com/video/BV1ys411472E?p=3`

In linear algebra, lines, planes, spaces can all be treated as a collection of vectors. That is why they are called vector spaces!

By the way, this series of video is really useful and can help you get a deeper understanding in linear algebra, I recommend all of you spend some time watching the first few sections after class.

Subspaces of $\mathbb{R}^2$

Now, thing will be easier to understand the definition of vector spaces and subspaces.

According to definiton, a vector space inside $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$.

Try to decide whether the following ones are subspaces of $\mathbb{R}^2$?

1. $S_1 = \{(x, 0) \mid x \in \mathbb{R}\}$
2. $S_2 = \{(x, y) \mid 4x + y = 0 \;\&\; x, y \in \mathbb{R}\}$
3. $S_3 = \{(x, y) \mid x + y = 1 \;\&\; x, y \in \mathbb{R}\}$
4. $S_4 = \{(x, y) \mid xy = 0 \;\&\; x, y \in \mathbb{R}\}$
5. $S_5 = S_1 \cup S_2$
6. $S_6 = S_1 \cap S_2$
7. $S_7 = \mathbb{R}^2$

A conclusion: the intersection of 2 subspaces is still a subspace, but smaller. Can you find the reason? (Use the definition to verify.)

Now, try to list all the subspaces for $\mathbb{R}^2$...

Subspaces of $\mathbb{R}^2$

Subspaces of $\mathbb{R}^2$ (3 types):

1. The origin. That is the zero vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
2. Straight lines across the origin.
3. $\mathbb{R}^2$ itself.

Can I say $\mathbb{R}^1$? Why?

The vectors stored in $\mathbb{R}^1$ have only 1 component, while the vectors in $\mathbb{R}^2$ have 2. I have to say they are quite similar because they are both lines, but they are not the same.

Try to list the subspaces of $\mathbb{R}^5$.

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Vector Spaces Defined from a Matrix

Given a matrix A:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}$$

Naturally, this matrix has 2 column vectors already:

$$\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

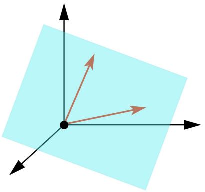
Now, find a vector space contains these 2 vectors. It is a subspace of $\mathbb{R}^3$.

Which other vectors should exist in this subspace to satisfy the definition of vector spaces?

Column Space

To satisfy the definition of vector spaces, the scalar multiplications of those vectors (2 lines) and the addition of those vectors (the vectors between the lines) are in the final subspace. We can also say linear combinations of the 2 vectors are in the final subspace.

Can you imagine the figure of this subspace? It is a 2-dimensional space across the origin in $\mathbb{R}^3$ space. Make sure you understand it, important!



Column Space of A – $C(A)$

This special subspace is the column space of matrix A.

4 vectors in $\mathbb{R}^9$ will form (or span)...

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 4 & 1 & 5 \\ 7 & 1 & 8 \end{bmatrix}$$

The column space of $A\,(4 \times 3)$ is a subspace of $\mathbb{R}^4$.

Can 3 column vectors in A span the whole $\mathbb{R}^4$ space?

We have known that every matrix represent a system of linear equations. Think about the linear equation system corresponding to matrix A. How many equations? How many unknowns?

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 4 & 1 & 5 \\ 7 & 1 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Can this equation system always have a solution?

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 4 & 1 & 5 \\ 7 & 1 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

4 equations, 3 unknowns, usually make the equation system not solvable.

Think row picture of this linear equation system. Each row represent a plane in $\mathbb{R}^3$ space, the solution is the intersection of 4 planes.

An important question is which bs allow this equation system to have solutions?

Give me some right-hand side b to make the equation system solvable.

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 4 & 1 & 5 \\ 7 & 1 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 4 & 1 & 5 \\ 7 & 1 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 7 \end{bmatrix}$$

Associate Column Space with Linear Equation

How about we write the solutions first, find which b lead to this solution?

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 4 & 1 & 5 \\ 7 & 1 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

The left side gives all possible linear combinations of the 3 column vectors.

You may already discover, the equation system is solvable exactly when b is in $C(A)$.

Are these 3 column vectors linearly independent? Or, do they all contribute to expand the dimension of column space?

No, because even if we delete a column, the column space will not change. We will call the linearly independent columns (contribute to column space) that come first pivot columns later.

Nullspace

Nullspace contains all the solutions $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 4 & 1 & 5 \\ 7 & 1 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Give me some obvious solutions.

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 4 & 1 & 5 \\ 7 & 1 & 8 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 4 & 1 & 5 \\ 7 & 1 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

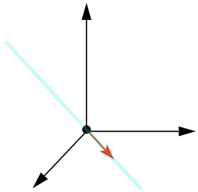
Nullspace

Well, you can write a complete solution to this equation system by observation. (I don't want to spend time proving that.)

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 4 & 1 & 5 \\ 7 & 1 & 8 \end{bmatrix} \begin{bmatrix} c \\ c \\ -c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where c is a constant.

So the nullspace looks like... A straight line across the origin!



Nullspace of A – N(A)

Nullspace

A straight line across the origin...Is it a subspace of $\mathbb{R}^3$?

Yes, it is. Are all the nullspaces subspaces? How to illustrate that?

- Firstly, let's verify addition. We choose 2 vectors u, v in the nullspace, they satisfy $Au = Av = 0$. The addition vector $u + v$ is still in the nullspace since $A(u + v) = 0$.
- Secondly, let's verify scalar multiplication. We have a vector v in the nullspace, which gives us $Av = 0$. The vector cv is still in the nullspace since $A(cv) = 0$.

The nullspace of $A\,(4 \times 3)$ is a subspace of $\mathbb{R}^3$.

Is the solutions still a subspace when the right-hand side b not equals to zero? Definitely no, because zero vector isn't in the solution space.

Now, you have known 2 important subspaces defined by a matrix. At the end of this chapter, you will know 4 subspaces. We are now approaching the core of linear algebra.

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Introduction

This section introduces an algorithm, it will take up approximately 20 marks in your midterm exam.

A good example question is important to make you understand the solving process. In this part, let's look at this matrix.

$$A = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 2 & 6 & 8 & 10 \\ 3 & 9 & 11 & 13 \end{bmatrix}$$

The question is how to find the solution of $Ax = 0$. In this example, x has 4 components.

We have learnt an algorithm for solving linear equation systems - Gauss Elimination. Gauss elimination can help us simplify the equations and find the pivots, and it will never change the nullspace.

Do Gauss elimination for matrix A.

The first step:

$$\begin{bmatrix} 1 & 3 & 3 & 3 \\ 2 & 6 & 8 & 10 \\ 3 & 9 & 11 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

It makes some difficulties for us...The second column has no pivots.

In Chapter 1 (square matrix cases), we call this situation permanent failure (or singular). But it doesn't mean $Ax = 0$ has no nonzero solution.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For the equation system above, $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$ is a solution. In Chapter 1, we avoid this case, but now we face it directly. My advice is: find the next pivot and go on.

$$A = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 2 & 6 & 8 & 10 \\ 3 & 9 & 11 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

Gauss elimination ends. The result is not a strict upper triangular matrix, we call it echelon form (U).

The matrix A has 2 pivots, that is called the rank of matrix A.

$$\textit{rank } r \; = \; \# \textit{ of pivots}$$

During this elimination process which of the following are still unchanged?

- Column Space
- Nullspace
- The subspace spanned by the rows. (Row Space)

So we simplify the equation system $Ax = 0 \Leftrightarrow Ux = 0$.

Pivot Columns and Free Columns

The most important step different with Chapter 1 comes.

The simplified equation system becomes

$$\begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Find the pivots and they determine the pivot columns, the rest are free columns. In this example, column 1 & 3 are pivot columns, column 2 & 4 are free columns.

We can take any number for the coefficient of free columns and solve the coefficient for pivot columns by back-substitution. That is we can choose x_2 & x_4 randomly.

Commonly, we choose to set a free column coefficient 1 and the others 0.

Back-Substitution

$$\begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ 1 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Write their corresponding equation systems.

$$\begin{cases} x_1 + 3x_2 + 3x_3 + 3x_4 = 0 \\ \qquad\qquad 2x_3 + 4x_4 = 0 \\ \qquad\qquad\qquad\quad 0 = 0 \end{cases}$$

Solve the coefficients of pivot columns by back-substitution now.

$$\begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Finally: Get the Solution

These solutions can be called special solutions.

$$\begin{bmatrix}1 & 3 & 3 & 3\\0 & 0 & 2 & 4\\0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}-3\\1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \qquad \begin{bmatrix}1 & 3 & 3 & 3\\0 & 0 & 2 & 4\\0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}3\\0\\-2\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

We can take any number for the coefficient of free columns x_2 & x_4.

$$\begin{bmatrix}1 & 3 & 3 & 3\\0 & 0 & 2 & 4\\0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}-3x_2\\x_2\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \qquad \begin{bmatrix}1 & 3 & 3 & 3\\0 & 0 & 2 & 4\\0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}3x_4\\0\\-2x_4\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Add them, the solution is $x_2\begin{bmatrix}-3\\1\\0\\0\end{bmatrix}+x_4\begin{bmatrix}3\\0\\-2\\1\end{bmatrix}$. 2-dimensional space!

Reduced Row Echelon Form (RREF)

Recall in Chapter 1, we can do row operations to let the matrix become an identity matirx, then the solution appears on the right side. (That is how you find the inverse of matrix.)Similarly, we can have further simplication to this linear equation system.

$$U = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & -3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

We only do row operations, $Ax = 0 \Leftrightarrow Ux = 0 \Leftrightarrow Rx = 0$.
What we have done:

- Eliminate entries on the top of pivots.
- Make the pivots all 1.

After this process, we have the Reduced Row Echelon Form (RREF) matrix R. Can you discover where the solutions appear?

Can you find the identity matrix in R?

Let's begin with the pivot columns:

$$\begin{bmatrix} 1 & 0 & 3 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Generally, the RREF matrices are like

$$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

I repeat the solution we have solved in previous slides. Correspondingly, I write the pivot column coefficients firstly.

$$x = x_2 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Now find something?

Nullspace Matrix

We define nullspace matrix N contains all special solutions in columns.

$$Rx = 0 \Leftrightarrow RN = 0$$

$$\begin{bmatrix} 1 & 0 & 3 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -3 & 3 \\ 0 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -F \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So, when we get the RREF form of matrix, we can write the nullspace matrix without calculation.

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Solvability Condition for $Ax = b$

The linear equation system is solvable exactly when b is in $C(A)$.

Another way to understand it: when we do series of row operations for the left-hand side A and get an all zero row, then the same operations should gives right-hand side b zero value.

After Gauss elimination, if we get a row with euqation $0 = c, c \neq 0$, the equation system is inconsistent(not solvable).

In our example above, still using the same matrix A. When we finish Gauss elimination, if we get

$$\begin{bmatrix} 1 & 3 & 3 & 3 & 1 \\ 0 & 0 & 2 & 4 & 4 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

The linear equation system is inconsistent since the third row gives $0 = 3$.

Particular Solution to $Ax = b$

If we slightly change the right-hand side b, the equation is solvable. How to solve that?

$$\begin{bmatrix} 1 & 3 & 3 & 3 & 1 \\ 0 & 0 & 2 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{cases} x_1 + 3x_2 + 3x_3 + 3x_4 = 1 \\ 2x_3 + 4x_4 = 4 \\ 0 = 0 \end{cases}$$

An important question: What is particular solution?

It can be any solution to the linear equation system. No matter what method you use, as long as it is a solution, it is particular solution.

I'd like to introduce you one common method: Set all free variables 0.

$$\begin{cases} x_1 + 3x_3 = 1 \\ 2x_3 = 4 \end{cases}$$

So, a particular solution is $x_{particular} = \begin{bmatrix} -5 & 0 & 2 & 0 \end{bmatrix}^T$.

Complete Solution to $Ax = b$

How to find the complete solution? Add the particular solution by nullspace vectors!

$$Ax_p = b$$

$$Ax_n = 0$$

$$A(x_p + x_n) = b$$

The complete solution is $x_{complete} = x_{particular} + x_{nullspace}$.

For our example above, the complete solution is

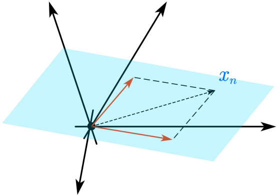
$$x_c = x_p + x_n = \begin{bmatrix} -5 \\ 0 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}, x_2, x_4 \in \mathbb{R}$$

Now try to understand any particular solution is OK.

Can the whole solution set construct a subspace of $\mathbb{R}^4$?

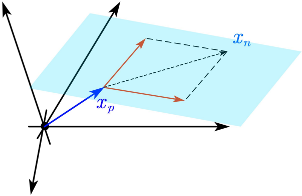
Firstly, let's consider nullspace only. It is a 2-dimensional plane through the origin in $\mathbb{R}^4$ space.

Then add a particular solution. It is now a 2-dimensional plane away form the origin in $\mathbb{R}^4$ space.



Nullspace of A

$2-$dimensional plane through the origin



Solution set of $Ax=b$

$2-$dimensional plane away from the origin

Review Example

Consider the matrix A:

$$A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & 7 & 9 & 4 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Question 1: $C(A)$ is a subspace of ...? $N(A)$ is a subspace of ...?

For a $m \times n$ matrix: column space is a subspace of $\mathbb{R}^m$, while nullspace is a subspace of $\mathbb{R}^n$. Don't try to remember that, try to understand that.

Question 2: Is $C(A)$ a line, plane or space? Find $dim(C(A))$.

Hint: (Col 1) + (Col 2) = (Col 3), 2 (Col 1) = (Col 4)

Can you sketch it geometrically? What is it like?

How about $N(A)$? You may not know it at this time, but keep this question in mind.

$$A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & 7 & 9 & 4 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Question 3: Find echelon form U, RREF matrix R, nullspace matrix N.

Do Gauss elimination and keep eliminating when failure occurs. Then we can get the echelon form. Eliminate entries on the top of pivots and make the pivots all 1 to get RREF form. Then use R to find N.

Question 4: How many pivots, pivot columns and free columns? r(A)? We already get the echelon form U, every pivot leads to a pivot column while the others are free. Rank equals the number of pivots.

Question 4*: Suppose a $m \times n$ matrix have rank r, how many pivot columns and free columns?

Review Example

$$A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & 7 & 9 & 4 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Question 5: Find $dim(N(A))$.

Every free columns bring a new dimension for the nullspace.

Question 5*: Suppose a $m \times n$ matrix have rank r, find its dimension of nullspace.

Question 5:** Suppose a $m \times n$ matrix have rank r, find its dimension of column space.

Question 6: Find all the solutions to $Ax = 0$.

Question 7: Find all the solutions to $Ax = (Col\,3)$.

How many correct answers you get in 7 questions?

An Important Reminder

$$\begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & 7 & 9 & 4 \\ 0 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

After a series of row operations, the row space and the nullspace remain unchanged, but the column space changed.

Check if this relation of columns is still unchanged?

$$(\text{Col } 1) + (\text{Col } 2) = (\text{Col } 3), \; 2\,(\text{Col } 1) = (\text{Col } 4)$$

Yes! And what is the reason behind that?

Because nullspace never changes! The nullspace shows us which combination of columns can lead us to zero and the relation is ever-lasting.

The pivot columns are the basis of $C(A)$. Try to explain that.