Matrix Diagonalization; Complex Matrix

Lecture 10

Zhang Ce

Department of Electrical and Electronic Engineering
Southern University of Science and Technology

2022.12.6

Table of Contents

Table of Contents

Last Lecture, We Discuss...

Three parts in last lecture:

1. Computing Techniques of Determinants
   type 1-6
2. Eigenvalues and Eigenvectors
   definition; meaning of "eigen"; geometrical interpretation; calculation; two formulas: trace and determinant
3. Important Applications of Eigenvalues and Eigenvectors
   Google: PageRank algorithm

Type 6: In-Order Matrix

Compute the nth order determinant:

$$\det A = \begin{vmatrix} 1+{x_1}^2 & x_1x_2 & \cdots & x_1x_n \\ x_2x_1 & 1+{x_2}^2 & \cdots & x_2x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_nx_1 & x_nx_2 & \cdots & 1+{x_n}^2 \end{vmatrix}$$

$$\det A = \det\left(I + \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \right)$$

Hint: Using eigenvalue formula for determinants:

$$\lambda_1\lambda_2\cdots\lambda_n = \det A$$

Eigenvalues are $1, 1+u^Tu$. Multiply all of them, $\det A = 1 + \sum_{i=1}^n x_i^2$.

Understanding Eigenvalues in Geometry

Find the eigenvalues and eigenvectors for these matrix.

1. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the identity matrix (how many eigenvectors?)
2. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, "shear" matrix
3. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, projection (onto x axis) matrix
4. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, zero matrix
5. A challenging one: $A = uu^T$, a rank 1 matrix
   Hint: you may start with projection (onto u) matrix

Calculating Eigenvalues and Eigenvectors

A little bit change from the definition, or you can use linear transformation to understand: linear transformation A for eigenvector x is equivalent to a stretching linear transformation λI.

$$A\mathbf{x} = \lambda I \mathbf{x}$$

The two linear transformations for vector $\mathbf{x}$ are equivalent, leading

$$(A - \lambda I)\,\mathbf{x} = 0.$$

Therefore, vector $\mathbf{x}$ is in the nullspace of matrix $(A - \lambda I)$. To make this linear equation have nonzero solutions (i.e. to make the dimension of nullspace not zero), matrix $(A - \lambda I)$ should not be a full rank matrix. Expressed in determinant form:

$$\det (A - \lambda I) = 0$$

Two Formulas: Trace and Determinant

$\det(A-\lambda I)$ is a polynomial of λ. For an $n\times n$ matrix, $\det(A-\lambda I)$ is a n degree equation with only 1 unknown.

Suppose $n=2$:

$$\begin{vmatrix} x_{11}-\lambda & x_{12} \\ x_{21} & x_{22}-\lambda \end{vmatrix} = \lambda^2-(x_{11}+x_{22})\lambda+(x_{11}x_{22}-x_{12}x_{21})$$

Adopt Vieta's Theorem:

$$\lambda_1+\lambda_2=x_{11}+x_{22},\ \ \lambda_1\lambda_2=x_{11}x_{22}-x_{12}x_{21}$$

Pure algebra: higher-order Vieta's Theorem gives us:

$$\lambda_1+\lambda_2+\cdots+\lambda_n=trace(A),\ \ \lambda_1\lambda_2\cdots\lambda_n=\det A$$

You have found all the eigenvectors and eigenvalues of a matrix, but...

Experts in Linear Algebra will not stop here, please find matrix expression.

Condition: $n \times n$ matrix A has n linearly independent eigenvectors.
Then we write the n linearly independent eigenvectors in the columns of a matrix P (We have done this many times in this course...). Magical matrix multiplication gives us:

$$AP = P\Lambda$$

$$A\begin{bmatrix} | & | & | & & | \\ | & | & | & & | \\ x_1 & x_2 & x_3 & \cdots & x_n \\ | & | & | & & | \\ | & | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & | & & | \\ | & | & | & & | \\ x_1 & x_2 & x_3 & \cdots & x_n \\ | & | & | & & | \\ | & | & | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_n \end{bmatrix}$$

If you use column method of matrix multiplication, that matrix equation is easy to verify.

$$A = P\Lambda P^{-1}$$

Examples

Diagonalize matrix A.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Solution:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (\lambda - 1)(\lambda - 3) = 0$$

For eigenvalue $\lambda_1 = 1$,

$$(A - \lambda_1 I)\,\mathbf{x}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

For eigenvalue $\lambda_2 = 3$,

$$(A - \lambda_2 I)\,\mathbf{x}_2 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example of Matrix Diagonalization

Examples

Diagonalize matrix A.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The final result:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1}$$

Notice that every column of P can be multiplied by a constant, the inverse P^{-1} will guarantee that this diagonalization is still correct.

Matrix Diagonalization: A Deeper View

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1}$$

Firstly, think about this linear transformation. Why is it so simple? (Think about do n times the same linear transformation)

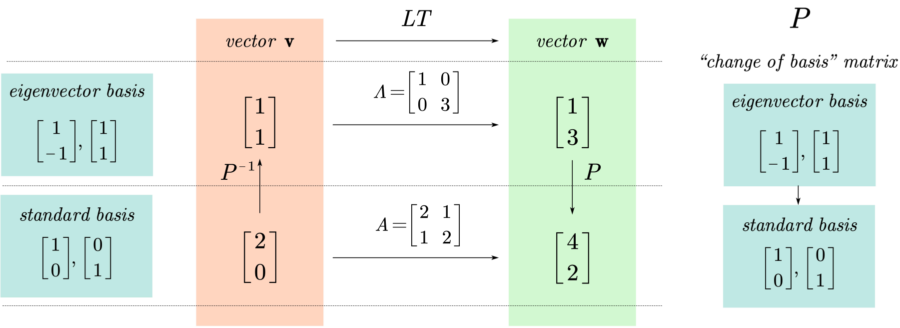
$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

We have learnt in chapter 2.6, the matrix representation of linear transformation depends on your choice of basis. In the diagonalization process, the matrix A and Λ represent the same linear transformation! The only difference is, for A we choose the standard basis, but for Λ we choose the eigenvector basis.

Matrix Diagonalization: A Deeper View

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1}$$

Suppose we have a input coordinates $(2,0)$ in standard basis, how can we find the output?



Essence: change basis and simplify the linear transformation matrix.

Special Case: Symmetric Matrix

Suppose the matrix we want to diagonalize is a (real) symmetric matrix:

$$A = A^T$$

$$P\Lambda P^{-1} = \left(P\Lambda P^{-1}\right)^T$$

$$P\Lambda P^{-1} = \left(P^T\right)^{-1} \Lambda^T P^T$$

$$P^T P\Lambda = \Lambda P^T P$$

We can find an orthogonal matrix Q that satisfies $Q^T Q = I$ to diagonalize the matrix.

$$A = P\Lambda P^{-1} = Q\Lambda Q^T$$

In geometric perspective, you can find another rectangular coordinate system to simplify that linear transformation!

Table of Contents

Core Problems of Matrix Diagonalization

$$A = P\Lambda P^{-1}$$

Problems:

1. How to tell whether a matrix is diagonalizable or not?
2. How to find a diagonalization matrix P?

Answers:

For problem 1, we know that a matrix is diagonalizable if and only if it has n linearly independent eigenvectors.

For problem 2, we need to find all the eigenvalues and eigenvectors. The eigenvalues will go into Λ and eigenvectors will go into P.

If a matrix is diagonalizable, we can easily find the diagonalization matrix! But the question is, we cannot tell if a matrix has n linearly independent eigenvectors (i.e. if it is diagonalizable).

Another Simplified Criteria

If a $n \times n$ matrix has n distinct eigenvalues, then it must be diagonalizable.

Why? I will not give you the proof, but I want you to understand. I will show from 2 sides:

1. For each eigenvalue, there must be at least 1 eigenvector (authentic to say: 1-dimensional).
2. For the same eigenvector (all vectors that in the 1-dimensional line), it cannot correspond to 2 eigenvalues.
   (Hint: Recall the knowledge in linear transformation, can a single input results in multiple outputs?)
   Geometric: One direction cannot have 2 stretching coefficient!

Notice that it is not necessary for a diagonalization matrix to have n distinct eigenvalues.

More challenging: Eigenvectors that correspond to distinct eigenvalues are linearly independent. (Hint: Can a k-dimensional space has $k + 1$ stretching coefficients?)

Algebraic Multiplicity and Geometric Multiplicity

Can we find a general method to determine whether a matrix is diagonalizable or not? The method is very simple, find all the eigenvectors and count if there are n independent eigenvectors.

Example

Decide whether the following matrix A is diagonalizable or not.

$$A = \begin{bmatrix} 5 & -1 & -1 \\ 3 & 1 & -1 \\ 4 & -2 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = -(\lambda - 2)^2 (\lambda - 3) = 0$$

For a n degree polynomial, we can find n roots.

$$\lambda_1 = \lambda_2 = 2, \lambda_3 = 3$$

We call that 2 is a repeated eigenvalue, with algebraic multiplicity of 2.

$$\lambda_1 = \lambda_2 = 2, \lambda_3 = 3$$

For those repeated eigenvalues, we need to check whether it has sufficient independent eigenvectors (for those not repeated ones, we can skip because it must have 1 independent eigenvector). Back to chapter 2, Find the nullspace dimension of $A - \lambda I$!

$$\begin{bmatrix} 3 & -1 & -1 \\ 3 & -1 & -1 \\ 4 & -2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \rightarrow \begin{bmatrix} 3 & -1 & -1 \\ 0 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

1 free column, 1 independent eigenvector for eigenvector 2. Geometric multiplicity: 1.

The independent eigenvectors are not sufficient, so the matrix is not diagonalizable. (We say that the eigenvalue 2 is a "liar", it claims that it has the multiplicity of 2 but we can only find 1 independent eigenvector.)

A theorem that I don't want to prove:

$$\text{Geometric Multiplicity} \leq \text{Algebraic Multiplicity}$$

A Summary - Liar's Game:

1. All the eigenvalues are smart, they will only claim they have higher or equal multiplicity (algebraic multiplicity) than it actually has.
2. Your mission is to catch the liar. You investigate the possible liars (repeated eigenvalues) by calculating the nullspace dimension of $\det(A - \lambda I)$. If you find that the eigenvalues don't have the same number of independent eigenvectors as it claimed, it is a liar. Once you catch 1 liar, the matrix is not diagonalizable any more.

Well, that is only an analogy... That's quite similar, right?

Algebraic Multiplicity and Geometric Multiplicity

Now, let's consider matrix $A = I + uu^T$ once again. Geometric imagination: projection onto lines plus identity!

Firstly, we have known that A has 2 distinct eigenvalues: 1 and $1 + u^T u$. An important question: multiplicity of those eigenvalues?

Consider eigenvalue 1. How about find its geometric multiplicity first? $A - \lambda I = uu^T$, find its nullspace dimension (or you can also judge from the number of free columns). Then the algebraic multiplicity must be greater or equal than it. Now. can you figure out all the multiplicity?

Matrix $A = I + uu^T$ has

repeated eigenvalue 1 with algebraic and geometric multiplicity of $n - 1$
eigenvalue $1 + u^T u$ with algebraic and geometric multiplicity of 1

You can get from that: $\det A = 1 + u^T u, trace(A) = n + u^T u$.

Table of Contents

We are going to extend linear algebra to complex region. We have learnt all the knowledge in this chapter, the only thing we do is extend the previous knowledge to adapt the complex case.

To extend, what is the major difference between real and complex region?

Consider the vector length firstly:

$$v_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, v_2 = \begin{bmatrix} 3i \\ 4 \end{bmatrix}$$

Calculate with formula $||v||^2 = v^T v$:

$${v_1}^T v_1 = 3^2 + 4^2 = 25, {v_2}^T v_2 = (3i)^2 + 4^2 = 7$$

Well, they should both be 25... The length of v_1 and v_2 are both 5.

New Operation: Hermitian

$$v_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, v_2 = \begin{bmatrix} 3i \\ 4 \end{bmatrix}$$

We should calculate by formula $||v||^2 = \bar{v}^T v$ now:

$$\bar{v_1}^T v_1 = 3^2 + 4^2 = 25, \bar{v_2}^T v_2 = (3i)(-3i) + 4^2 = 25$$

The length of v_1 and v_2 are both 5, that's correct.

To simplify the notation, we have the new Hermitian operation:

$$v^H = \bar{v}^T$$

That is the conjugate transpose. This operation will act as transpose in complex region. Remember for the real matrix, there is no difference between Hermitian and transpose.

Inner Products

For the real region, the inner product is defined as $u^T v$, so naturally in complex region, the inner product is defined as $u^H v$.

Notice: we know that $x^T y = y^T x$ in real region, which means the order of vectors are not important. But in the complex region, it is important. See the following example:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}^H \begin{bmatrix} 1-i \\ 1+i \end{bmatrix} = 1-i, \begin{bmatrix} 1-i \\ 1+i \end{bmatrix}^H \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1+i$$

In the complex region, the order of vectors will influence the final inner products.

- In real region, $x^T y = y^T x$.
- In complex region, $x^H y = \overline{y^H x}$.

For orthogonal vectors x, y, they satisfy $x^H y = 0$.

Hermitian Matrix

For real matrix, a matrix is symmetric if $A^T = A$, so for the complex matrix, a matrix is Hermitian if $A^H = A$.

The following matrix is a Hermitian matrix:

$$A = \begin{bmatrix} 2 & 3-i \\ 3+i & 5 \end{bmatrix}$$

Recall that for real symmetric matrix, the eigenvectors corresponding to different eigenvalues are orthogonal. This property remains true for the Hermitian matrix.

Another property: Every eigenvalue of a Hermitian matrix is real. This property is definitely true for real symmetric matrix because it is a special case of Hermitian!

Unitary Matrix

For real matrix, a matrix is orthogonal if $A^TA = AA^T = I$, so for the complex matrix, a matrix is unitary if $A^HA = AA^H = I$. Unitary has orthonormal column vectors.

Properties:

- Inner products and lengths are preserved by U.

$$(Ux)^H(Uy) = x^Hy, \|Ux\| = \|x\|$$

- Every eigenvector of U has absolute value $|\lambda| = 1$.
- Eigenvectors corresponding to different eigenvalues are orthogonal.

For example, consider 2×2 rotation matrix, it will have orthogonal eigenvectors. Also true for permutation matrix, which is a orthogonal matrix and of course unitary.

Unitary Diagonalization for Hermitian Matrix

For Hermitian matrix (real symmetric included), it must have a unitary (orthogonal included) matrix to diagonalize. Λ is still real diagonal matrix just the same as the case in real region.

$$A = U\Lambda U^H$$

An important problem type is to find a unitary matrix to diagonalize a given Hermitian matrix, and in the following slides, I will guide you through this problem.

Gram-Schmidt is necessary here because the eigenvectors corresponding to repeated eigenvalues will not be orthogonal in most of the time, don't forget Gram-Schmidt!

Example

Do Gram-Schmidt orthogonalization for the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Solution: Accept **a** to the orthogonal vector set:

$$a_1' = a_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Subtract a_1' component from a_2 and add to the orthogonal vector set.

$$a_2' = a_2 - \frac{a_2^T a_1'}{a_1'^T a_1'} a_1' = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

Gram-Schmidt

Subtract a_1' and a_2' component from a_3 and add to the orthogonal vector set.

$$a_3' = a_3 - \frac{a_3^T a_1'}{a_1'^T a_1'} a_1' - \frac{a_3^T a_2'}{a_2'^T a_2'} a_2'$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1/2}{3/2} \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

Finally, normalization:

$$q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, q_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, q_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Example

Find a orthogonal matrix Q to diagonalize the following matrix A.

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix}$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & -2 & 2 \\ -2 & 4-\lambda & -4 \\ 2 & -4 & 4-\lambda \end{bmatrix} = 0$$

$$\lambda_1 = \lambda_2 = 0, \lambda_3 = 9$$

Example 1: Real Symmetric Matrix

For eigenvalue $\lambda = 0$:

$$\begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & -0 & 0 \end{bmatrix}$$

Nullspace solutions:

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

For eigenvalue $\lambda = 9$:

$$\begin{bmatrix} -8 & -2 & 2 \\ -2 & -5 & -4 \\ 2 & -4 & -5 \end{bmatrix} \to \begin{bmatrix} -8 & -2 & 2 \\ 0 & -9/2 & -9/2 \\ 0 & -9/2 & -9/2 \end{bmatrix} \to \begin{bmatrix} -8 & -2 & 2 \\ 0 & -9/2 & -9/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Nullspace solutions:

$$\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

Example 1: Real Symmetric Matrix

Do Gram-Schmidt for those eigenvectors.

For eigenvalue $\lambda = 0$:

$$a_1' = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, a_2' = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + \frac{4}{5} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 4/5 \\ 1 \end{bmatrix}$$

$$q_1 = \begin{bmatrix} \frac{2\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} \\ 0 \end{bmatrix}, q_2 = \begin{bmatrix} -\frac{2\sqrt{5}}{15} \\ \frac{4\sqrt{5}}{15} \\ \frac{5\sqrt{5}}{15} \end{bmatrix}$$

Example 1: Real Symmetric Matrix

For eigenvalue $\lambda = 9$, the eigenvectors are automatically orthogonal with the previous ones, we only do normalization:

$$q_3 = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$$

So the final result is:

$$\begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix} = \begin{bmatrix} \frac{2\sqrt{5}}{5} & -\frac{2\sqrt{5}}{15} & \frac{1}{3} \\ \frac{\sqrt{5}}{5} & \frac{4\sqrt{5}}{15} & -\frac{2}{3} \\ 0 & \frac{5\sqrt{5}}{15} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} \frac{2\sqrt{5}}{5} & -\frac{2\sqrt{5}}{15} & \frac{1}{3} \\ \frac{\sqrt{5}}{5} & \frac{4\sqrt{5}}{15} & -\frac{2}{3} \\ 0 & \frac{5\sqrt{5}}{15} & \frac{2}{3} \end{bmatrix}^T$$

Example 2: Hermitian Matrix

Example

Find a unitary matrix U to diagonalize the following matrix A.

$$A = \begin{bmatrix} 2 & i & 0 \\ -i & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{bmatrix} 2-\lambda & i & 0 \\ -i & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix} = (2-\lambda)(1-\lambda)(3-\lambda)$$

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$$

Example 2: Hermitian Matrix

For eigenvalue $\lambda = 1$:

$$\begin{bmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, a_1 = \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix}$$

For eigenvalue $\lambda = 2$:

$$\begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} -i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{bmatrix}, a_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

For eigenvalue $\lambda = 3$:

$$\begin{bmatrix} -1 & i & 0 \\ -i & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} -1 & i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, a_3 = \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}$$

Example 2: Hermitian Matrix

Normalize all of them:

$$U = \begin{bmatrix} -\frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & i & 0 \\ -i & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -\frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}^H$$

You have the privilege to keep root at denominator position.

Table of Contents

Definition

Two matrices A and B are said to be similar if there is an invertible matrix M such that

$$B = M^{-1}AM$$

It is denoted by $A \backsim B$.

For similar matrices A and B, they must share the same eigenvalues. They have the same number of independent eigenvectors.

$$B - \lambda I = M^{-1}\left(A - \lambda I\right)M$$

$$\det\left(B - \lambda I\right) = \det\left(M^{-1}\right)\det\left(A - \lambda I\right)\det\left(M\right) = \det\left(A - \lambda I\right)$$

$$Ax = \lambda x \Rightarrow MBM^{-1}x = \lambda x \Rightarrow B\left(M^{-1}x\right) = \lambda\left(M^{-1}x\right)$$

Properties for Similar Matrices

Similar matrices A and B have the following properties:

- A has the same eigenvalues as B.
- A has the same number of independent eigenvectors as B. (Difference: M^{-1})
- $\det A = \det B$, $trace(A) = trace(B)$.
- $rank(A) = rank(B)$. (who can give me a translation?)
- A and B have the same characteristic polynomial.

If A and B have the same characteristic polynomial (same eigenvalues), they are not always similar.

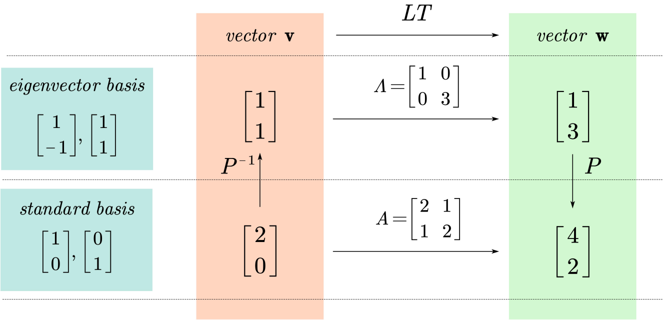
$$A = \begin{bmatrix} 5 & 3 \\ 0 & 5 \end{bmatrix}, B = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

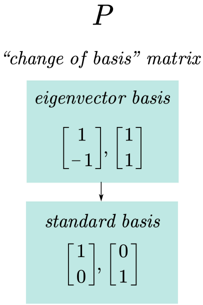
Those properties are necessary, not sufficient (Even if you find 2 matrices that can satisfy all of them, we can not say they are similar). We will discuss later on.

Similarity Transformation: Change of Basis

Recall: matrix diagonalization $A = P\Lambda P^{-1}$, we can say $A \backsim \Lambda$.

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1}$$





Essence: change basis and simplify the linear transformation matrix.

Similarity Transformation: Change of Basis

Definition

Two matrices A and B are said to be similar if there is an invertible matrix M such that

$$B = M^{-1}AM$$

It is denoted by $A \backsim B$.

Actually, A and B represent the same linear transformation under different bases. M is the "change of basis" matrix.

- A has the same eigenvalues as B. (The strenching coefficients are the same)
- A has the same number of independent eigenvectors as B. (Difference: M^{-1}) (The same eigenvectors! we only change the basis through M^{-1})
- $rank(A) = rank(B)$. (the dimension of output space are the same)

Video: `https://www.bilibili.com/video/BV1ys411472E?p=13`