Computations of Determinants; Intro to Eigenvalues

Lecture 9

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Last Lecture, We Discuss...

Four parts in last lecture:

1. Properties of Determinants
   10 properties of determinants; singular matrix; transposing
2. Computations of Determinant
   big formula; cofactor formula
3. Applications of Determinant
   computation of inverses, Cramer's rule
4. Topic: Techniques for Computing Determinants
   2 types of matrix

BIG FORMULA:

$$\det A = \sum_{\textit{all combinations}} (\det P)\, a_{1\alpha} a_{2\beta} \cdots a_{n\omega}$$

while P is the permutation matrix that have determinant 1 or -1 (determined by the order of chosen entries).

Another simplified expression: $P = (\alpha, \beta, \cdots, \omega)$.

Cofactor Formula

Consider 3×3 case:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

COFACTOR FORMULA:

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

Cofactors are the determinants that eliminates a row and a column, multiplying a coefficient of 1 or -1, determined by the sum of i, j.

Computation of Inverses

Cofactor matrix:

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

A formula for all square matrices (no matter singular or not):

$$AC^T = \det A \cdot I$$

Noteworthy that A^* is the same as C^T, called the adjoint matrix.

You'd better know how it comes... Referring to MIT 18.06 please!
`https://www.bilibili.com/video/BV1zx411g7gq?p=20` 07:41

If matrix A is invertible, the inverse:

$$A^{-1} = \frac{1}{\det A} A^*$$

Cramer's Rule

Consider a system of linear equations $Ax = b$:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Cramer gives

$$x_j = \frac{\det B_j}{\det A}$$

where

$$B_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & {\color{red}b_1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & {\color{red}b_2} & \cdots & a_{2n} \\ \vdots & \vdots & & {\color{red}\vdots} & & \vdots \\ a_{n1} & a_{n2} & \cdots & {\color{red}b_n} & \cdots & a_{nn} \end{bmatrix}$$

For 10×10 matrix, you need to find eleven 10×10 determinants to find the solution. Please use Gaussian Elimination to solve linear equations.

Type 1: Tri-diagonal Matrix

(2019 Fall Final, 12 marks) For each natural number $n \geqslant 3$, find the determinant:

$$D_n = \begin{vmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{vmatrix}_{n\times n}$$

Solution: cofactor expansion and find the recursion formula.

For this example, $D_n = 2D_{n-1} - D_{n-2}$.

The first few terms: $D_1 = 2, D_2 = 3, D_3 = 4$.

So, the answer is: $D_n = n + 1$.

Type 2: Arrow Form Matrix

Find the determinant:

$$D_n = \begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ b_2 & 1 & 0 & \cdots & 0 \\ b_3 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$D_n \begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ b_2 & 1 & 0 & \cdots & 0 \\ b_3 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n & 0 & 0 & \cdots & 1 \end{vmatrix} = \begin{vmatrix} a_1 - a_2 b_2 - a_3 b_3 - \cdots & 0 & 0 & \cdots & 0 \\ b_2 & 1 & 0 & \cdots & 0 \\ b_3 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n & 0 & 0 & \cdots & 1 \end{vmatrix}$$

So, the answer is $D_n = a_1 - \sum_{i=2}^{n} a_n b_n$.

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Type 3: Vandermonde Determinant and Variations

Vandermonde Determinant:

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ {x_1}^2 & {x_2}^2 & {x_3}^2 & \cdots & {x_n}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ {x_1}^{n-1} & {x_2}^{n-1} & {x_3}^{n-1} & \cdots & {x_n}^{n-1} \end{vmatrix} = \prod_{2\leqslant j<i\leqslant n} (x_i - x_j)$$

Proof omitted. Please refer to baidu or other search engines.

An example:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{vmatrix} = (2-1)(3-2)(3-1)(4-3)(4-2)(4-1) = 12$$

Variation 1: First row lost.

$$\begin{vmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ {x_1}^2 & {x_2}^2 & {x_3}^2 & \cdots & {x_n}^2 \\ {x_1}^3 & {x_2}^3 & {x_3}^3 & \cdots & {x_n}^3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ {x_1}^n & {x_2}^n & {x_3}^n & \cdots & {x_n}^n \end{vmatrix}$$

Solution: extract x_i from each column and it becomes the original.

$$\begin{vmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ {x_1}^2 & {x_2}^2 & {x_3}^2 & \cdots & {x_n}^2 \\ {x_1}^3 & {x_2}^3 & {x_3}^3 & \cdots & {x_n}^3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ {x_1}^n & {x_2}^n & {x_3}^n & \cdots & {x_n}^n \end{vmatrix} = x_1 x_2 \cdots x_n \prod_{2 \leqslant j < i \leqslant n} (x_i - x_j)$$

Variation 2: Other row lost.

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \\ a^4 & b^4 & c^4 & d^4 \end{vmatrix}$$

Solution: construct complete Vandermonde and compare coefficient.

Construct complete Vandermonde matrix A:

$$\det A = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ a & b & c & d & x \\ a^2 & b^2 & c^2 & d^2 & x^2 \\ a^3 & b^3 & c^3 & d^3 & x^3 \\ a^4 & b^4 & c^4 & d^4 & x^4 \end{vmatrix}$$

Now, we want to find the minor M_{25} of matrix A.

Type 3: Vandermonde Determinant and Variations

$$\det A = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ a & b & c & d & x \\ a^2 & b^2 & c^2 & d^2 & x^2 \\ a^3 & b^3 & c^3 & d^3 & x^3 \\ a^4 & b^4 & c^4 & d^4 & x^4 \end{vmatrix}$$

Define constant $S = (d-c)(d-b)(d-a)(c-b)(c-a)(b-a)$.

Calculate the Vandermonde determinant and cofactor expansion by column n:

$$|A| = S(x-d)(x-c)(x-b)(x-a) = C_{15} + C_{25}x + C_{35}x^2 + C_{45}x^3 + C_{55}x^4$$

Compare the coefficient of x: $C_{25} = (-abc - abd - acd - bcd)S$.

So, the original determinant is $M_{25} = -C_{25} = (abc + abd + acd + bcd)S$.

Type 4: Repeated (Similar Terms) Matrix

Find the determinant:

$$A = \begin{vmatrix} 1+a_1 & 1 & 1 & 1 \\ 1 & 1+a_2 & 1 & 1 \\ 1 & 1 & 1+a_3 & 1 \\ 1 & 1 & 1 & 1+a_4 \end{vmatrix}$$

Solution: add a row or column to eliminate repeated terms.

$$\det A = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1+a_1 & 1 & 1 & 1 \\ 1 & 1 & 1+a_2 & 1 & 1 \\ 1 & 1 & 1 & 1+a_3 & 1 \\ 1 & 1 & 1 & 1 & 1+a_4 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 & -1 & -1 \\ 1 & a_1 & & & \\ 1 & & a_2 & & \\ 1 & & & a_3 & \\ 1 & & & & a_4 \end{vmatrix}$$

Back to Type 2. The answer is $\left(1+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}\right)a_1a_2a_3a_4$.

Type 4: Repeated (Similar Terms) Matrix

Find the determinant:

$$\det A = \begin{vmatrix} 1+a_1 & a_1 & a_1 & a_1 \\ a_2 & 1+a_2 & a_2 & a_2 \\ a_3 & a_3 & 1+a_3 & a_3 \\ a_4 & a_4 & a_4 & 1+a_4 \end{vmatrix}$$

$$\det A = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ a_1 & 1+a_1 & a_1 & a_1 & a_1 \\ a_2 & a_2 & 1+a_2 & a_2 & a_2 \\ a_3 & a_3 & a_3 & 1+a_3 & a_3 \\ a_4 & a_4 & a_4 & a_4 & 1+a_4 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 & -1 & -1 \\ a_1 & 1 & & & \\ a_2 & & 1 & & \\ a_3 & & & 1 & \\ a_4 & & & & 1 \end{vmatrix}$$

Back to Type 2. The answer is $a_1 + a_2 + a_3 + a_4 + 1$.

Type 4: Repeated (Similar Terms) Matrix

Find the determinant:

$$\det A = \begin{vmatrix} a_1 & b & b & \cdots & b \\ b & a_2 & b & \cdots & b \\ b & b & a_3 & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a_n \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ b & a_1 & b & b & \cdots & b \\ b & b & a_2 & b & \cdots & b \\ b & b & b & a_3 & \cdots & b \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & b & \cdots & a_n \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 & -1 & \cdots & -1 \\ b & a_1 - b & & & & \\ b & & a_2 - b & & & \\ b & & & a_3 - b & & \\ \vdots & & & & \ddots & \\ b & & & & & a_n - b \end{vmatrix}$$

Back to Type 2. The answer is $\left[1 + b\sum_{i=1}^{n} \frac{1}{a_i - b}\right](a_1 - b)\cdots(a_n - b)$.

Type 4: Repeated (Similar Terms) Matrix

Find the determinant:

$$\det A = \begin{vmatrix} a+x_1 & a+x_2 & a+x_3 \\ a^2+{x_1}^2 & a^2+{x_2}^2 & a^2+{x_3}^2 \\ a^3+{x_1}^3 & a^3+{x_2}^3 & a^2+{x_3}^3 \end{vmatrix}$$

$$\det A = \begin{vmatrix} 1 & 0 & 0 & 0 \\ a & a+x_1 & a+x_2 & a+x_3 \\ a^2 & a^2+{x_1}^2 & a^2+{x_2}^2 & a^2+{x_3}^2 \\ a^3 & a^3+{x_1}^3 & a^3+{x_2}^3 & a^3+{x_3}^3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 & -1 \\ a & x_1 & x_2 & x_3 \\ a^2 & {x_1}^2 & {x_2}^2 & {x_3}^2 \\ a^3 & {x_1}^3 & {x_2}^3 & {x_3}^3 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & x_1 & x_2 & x_3 \\ a^2 & {x_1}^2 & {x_2}^2 & {x_3}^2 \\ a^3 & {x_1}^3 & {x_2}^3 & {x_3}^3 \end{vmatrix} + \begin{vmatrix} 2 & 0 & 0 & 0 \\ a & x_1 & x_2 & x_3 \\ a^2 & {x_1}^2 & {x_2}^2 & {x_3}^2 \\ a^3 & {x_1}^3 & {x_2}^3 & {x_3}^3 \end{vmatrix}$$

Back to Type 3.

Type 5: Circulant Matrix

Find the determinant:

$$\det A = \begin{vmatrix} 0 & b & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & b \\ b & 0 & a & 0 \end{vmatrix}$$

We can have cofactor expansion on column 1:

$$\det A = -a \begin{vmatrix} b & 0 & a \\ a & 0 & b \\ 0 & a & 0 \end{vmatrix} - b \begin{vmatrix} b & 0 & a \\ 0 & b & 0 \\ a & 0 & b \end{vmatrix} = \left(a^2 - b^2\right) \begin{vmatrix} b & 0 & a \\ 0 & 1 & 0 \\ a & 0 & b \end{vmatrix} = -\left(a^2 - b^2\right)^2$$

This problem is solved, but it is not a general method for Type 5.

Type 5: Circulant Matrix

Find the determinant:

$$\det A = \begin{vmatrix} x & y & z & w \\ y & x & w & z \\ z & w & x & y \\ w & z & y & x \end{vmatrix}$$

Solution: factor extraction.

$$\det A = \begin{vmatrix} x & y & z & w \\ y & x & w & z \\ z & w & x & y \\ w & z & y & x \end{vmatrix} = (x+y+z+w) \begin{vmatrix} 1 & 1 & 1 & 1 \\ y & x & w & z \\ z & w & x & y \\ w & z & y & x \end{vmatrix}$$

$$\det A = \begin{vmatrix} x & y & z & w \\ y & x & w & z \\ z & w & x & y \\ w & z & y & x \end{vmatrix} = (x+y-z-w) \begin{vmatrix} 1 & 1 & -1 & -1 \\ y & x & w & z \\ z & w & x & y \\ w & z & y & x \end{vmatrix}$$

Type 5: Circulant Matrix

$$\det A = \begin{vmatrix} x & y & z & w \\ y & x & w & z \\ z & w & x & y \\ w & z & y & x \end{vmatrix} = (x+z-y-w)\begin{vmatrix} 1 & -1 & 1 & -1 \\ y & x & w & z \\ z & w & x & y \\ w & z & y & x \end{vmatrix}$$

$$\det A = \begin{vmatrix} x & y & z & w \\ y & x & w & z \\ z & w & x & y \\ w & z & y & x \end{vmatrix} = (x+w-y-z)\begin{vmatrix} 1 & -1 & -1 & 1 \\ y & x & w & z \\ z & w & x & y \\ w & z & y & x \end{vmatrix}$$

So, the determinant must satisfy

$$\det A = k\,(x+y+z+w)\,(x+y-z-w)\,(x+z-y-w)\,(x+w-y-z)$$

Check coefficient of x^4 (1):

$$\det A = (x+y+z+w)\,(x+y-z-w)\,(x+z-y-w)\,(x+w-y-z)$$

Type 5: Circulant Matrix

Back to the first example:

$$\det A = \begin{vmatrix} 0 & b & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & b \\ b & 0 & a & 0 \end{vmatrix}$$

By factor extraction:

$$\det A = k\,(a+b)\,(a-b)\,(a+b)\,(a-b)$$

Check coefficient of a^4 (-1):

$$\det A = -\,(a+b)\,(a-b)\,(a+b)\,(a-b) = -\left(a^2-b^2\right)^2$$

Type 6: In-Order Matrix

Find the determinant:

$$\det A = \begin{vmatrix} 1+x_1y_1 & 1+x_1y_2 & \cdots & 1+x_1y_n \\ 1+x_2y_1 & 1+x_2y_2 & \cdots & 1+x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ 1+x_ny_1 & 1+x_ny_2 & \cdots & 1+x_ny_n \end{vmatrix}$$

Solution: Decompose to 2 matrices using matrix multiplication.

If you are familiar with col-row method of matrix multiplication, that decomposition should be easy for you!

$$\det A = \begin{vmatrix} 1 & x_1 & 0 & \cdots & 0 \\ 1 & x_2 & 0 & \cdots & 0 \\ 1 & x_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & 0 & \cdots & 0 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ y_1 & y_2 & y_3 & \cdots & y_n \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{vmatrix}$$

Type 6: In-Order Matrix

Find the determinant:

$$\det A = \begin{vmatrix} 2a_1b_1 & a_1b_2 + a_2b_1 & \cdots & a_1b_n + a_nb_1 \\ a_2b_1 + a_1b_2 & 2a_2b_2 & \cdots & a_2b_n + a_nb_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_nb_1 + a_1b_n & a_nb_2 + a_1b_2 & \cdots & 2a_nb_n \end{vmatrix}$$

$$\det A = \begin{vmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ a_2 & b_2 & 0 & \cdots & 0 \\ a_3 & b_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & b_n & 0 & \cdots & 0 \end{vmatrix} \begin{vmatrix} b_1 & b_2 & b_3 & \cdots & b_n \\ a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{vmatrix}$$

Type 6: In-Order Matrix

Compute the nth order determinant:

$$\det A = \begin{vmatrix} 1+{x_1}^2 & x_1x_2 & \cdots & x_1x_n \\ x_2x_1 & 1+{x_2}^2 & \cdots & x_2x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_nx_1 & x_nx_2 & \cdots & 1+{x_n}^2 \end{vmatrix}$$

$$\det A = \det\left(I + \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \right)$$

How about using the eigenvalue products formula for determinant... Can you find all of the eigenvalues of $I + uu^T$? If you can, all you need to do is multiply all of them.

A little difficult by now... Let's get back to this problem later.

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Something to say at the beginning:

Now, we come to a new world! After learning the boring determinant, here come the eigenvalues and eigenvectors. As far as I am concerned, the eigenvalues and eigenvectors dig out the hidden core of the whole system. The concept of eigenvalues and eigenvectors brings many brilliant applications to the world, especially in engineering subjects.

This chapter combines almost all the knowledge you learnt form chapter 1 - 4, so sometimes when you meet problems, you should consider whether you are familiar with the knowledge from the previous chapters. For example, determinants will appear everywhere, nullspace gives you the eigenvectors, and also, Gram-Schmidt will give you an orthogonal diagonalizing matrix.

Just as I repeated many times, always keep the geometric interpretations in mind, or you may be lost in the pure algebra formulas.

Definition

Let A be a square matrix with degree n. If there exist a non-zero vector x and a scalar λ such that

$$Ax = \lambda x$$

then λ is called an eigenvalue of A, and x is called an eigenvector, corresponding to the eigenvalue λ.

A very important question to ask: what is the meaning of "eigen"?

German mathematician Hilbert defined that, with the meaning of "self".

Understanding Eigenvalues in Geometry

$$A\mathbf{x} = \lambda\mathbf{x}$$

$A\mathbf{x}$: the vector $\mathbf{x}$ after linear transformation.
$\lambda\mathbf{x}$: real multiples of $\mathbf{x}$, i.e. stretching of vector $\mathbf{x}$.
The linear transformation for *eigenvector* result in stretching the vector by *eigenvalue* times.

Remark

The zero vector can not be an eigenvector even though $A0 = \lambda 0$, but $\lambda = 0$ can be an eigenvalue.

Linear Transformation again... Why not try to analyze some of common linear transformations?

Understanding Eigenvalues in Geometry

Find the eigenvalues and eigenvectors for these matrix.

1. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the identity matrix (how many eigenvectors?)
2. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, "shear" matrix
3. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, projection (onto x axis) matrix
4. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, zero matrix
5. A challenging one: $A = uu^T$, a rank 1 matrix
   Hint: you may start with projection (onto u) matrix

Calculating Eigenvalues and Eigenvectors

A little bit change from the definition, or you can use linear transformation to understand: linear transformation A for eigenvector x is equivalent to a stretching linear transformation λI.

$$A\mathbf{x} = \lambda I \mathbf{x}$$

The two linear transformations for vector $\mathbf{x}$ are equivalent, leading

$$(A - \lambda I)\,\mathbf{x} = 0.$$

Therefore, vector $\mathbf{x}$ is in the nullspace of matrix $(A - \lambda I)$. To make this linear equation have nonzero solutions (i.e. to make the dimension of nullspace not zero), matrix $(A - \lambda I)$ should not be a full rank matrix. Expressed in determinant form:

$$\det\left(A - \lambda I\right) = 0$$

Examples

Calculate the eigenvalues and eigenvectors of matrix A.

$$A = \begin{bmatrix} 4 & -5 \\ 2 & 3 \end{bmatrix}$$

Solution:

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{vmatrix} = (\lambda - 2)(\lambda + 1) = 0$$

For eigenvalue $\lambda_1 = 2$,

$$(A - \lambda I)\mathbf{x} = \begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{x} = k \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Eigenvectors corresponding to $\lambda_1 = 2$ are of the form $\mathbf{x} = k \begin{bmatrix} 5 \\ 2 \end{bmatrix}, k \neq 0$

The same process for another eigenvalue $\lambda_2 = -1$

Two Formulas: Trace and Determinant

$\det(A-\lambda I)$ is a polynomial of λ. For an $n \times n$ matrix, $\det(A-\lambda I)$ is a n degree equation with only 1 unknown.

Suppose $n=2$:

$$\begin{vmatrix} x_{11}-\lambda & x_{12} \\ x_{21} & x_{22}-\lambda \end{vmatrix} = \lambda^2 - (x_{11}+x_{22})\,\lambda + (x_{11}x_{22} - x_{12}x_{21})$$

Adopt Vieta's Theorem:

$$\lambda_1 + \lambda_2 = x_{11} + x_{22},\ \ \lambda_1\lambda_2 = x_{11}x_{22} - x_{12}x_{21}$$

Pure algebra: higher-order Vieta's Theorem gives us:

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = trace(A),\ \ \lambda_1\lambda_2\cdots\lambda_n = \det A$$

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Diagonalization of Matrix

You have found all the eigenvectors and eigenvalues of a matrix, but...

Experts in Linear Algebra will not stop here, please find matrix expression.

Condition: $n \times n$ **matrix** A **has** n **linearly independent eigenvectors.** Then we write the n linearly independent eigenvectors in the columns of a matrix P (We have done this many times in this course...). Magical matrix multiplication gives us:

$$AP = P\Lambda$$

$$A \begin{bmatrix} | & | & | & & | \\ | & | & | & & | \\ x_1 & x_2 & x_3 & \cdots & x_n \\ | & | & | & & | \\ | & | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & | & & | \\ | & | & | & & | \\ x_1 & x_2 & x_3 & \cdots & x_n \\ | & | & | & & | \\ | & | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_n \end{bmatrix}$$

If you use column method of matrix multiplication, that matrix equation is easy to verify.

$$A = P\Lambda P^{-1}$$

Examples

Diagonalize matrix A.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Solution:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (\lambda - 1)(\lambda - 3) = 0$$

For eigenvalue $\lambda_1 = 1$,

$$(A - \lambda_1 I)\,\mathbf{x}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

For eigenvalue $\lambda_2 = 3$,

$$(A - \lambda_2 I)\,\mathbf{x}_2 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example of Matrix Diagonalization

Examples

Diagonalize matrix A.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The final result:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1}$$

Notice that every column of P can be multiplied by a constant, the inverse P^{-1} will guarantee that this diagonalization is still correct.

Matrix Diagonalization: A Deeper View

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1}$$

Firstly, think about this linear transformation. Why is it so simple? (Think about do n times the same linear transformation)

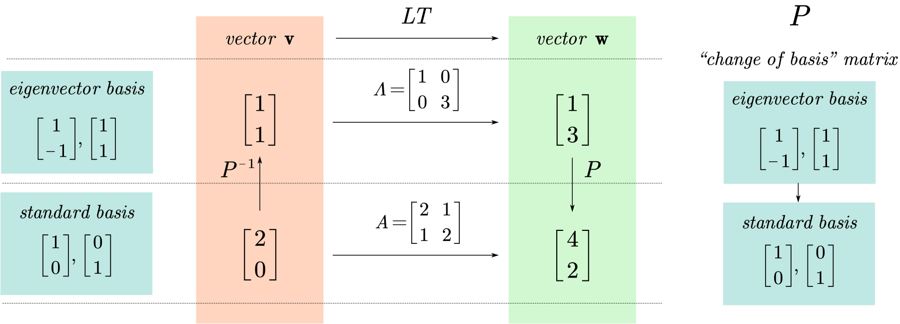
$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

We have learnt in chapter 2.6, the matrix representation of linear transformation depends on your choice of basis. In the diagonalization process, the matrix A and Λ represent the same linear transformation! The only difference is, for A we choose the standard basis, but for Λ we choose the eigenvector basis.

Matrix Diagonalization: A Deeper View

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1}$$

Suppose we have a input coordinates $(2, 0)$ in standard basis, how can we find the output?



Essence: change basis and simplify the linear transformation matrix.

Special Case: Symmetric Matrix

Suppose the matrix we want to diagonalize is a (real) symmetric matrix:

$$A = A^T$$

$$P\Lambda P^{-1} = \left(P\Lambda P^{-1}\right)^T$$

$$P\Lambda P^{-1} = \left(P^T\right)^{-1} \Lambda^T P^T$$

$$P^T P\Lambda = \Lambda P^T P$$

We can find an orthogonal matrix Q that satisfies $Q^T Q = I$ to diagonalize the matrix.

$$A = P\Lambda P^{-1} = Q\Lambda Q^T$$

In geometric perspective, you can find another rectangular coordinate system to simplify that linear transformation!

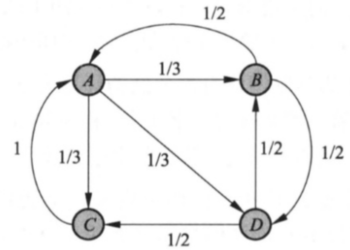
Table of Contents

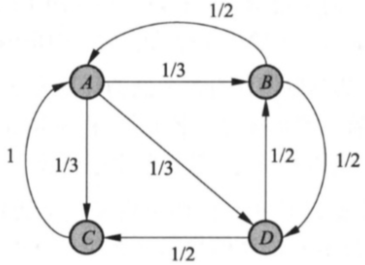
Google: PageRank Algorithm

When you want to find particular information on the search engine, why it can always put the most important web pages at first? How does the search engine work to find a rank among millions of related web pages? The answer: PageRank Algorithm proposed in 1998.

Google's success derives in large part from its PageRank algorithm, which ranks the importance of web pages according to an eigenvector of a weighted link matrix.

Here we give a brief introduction (not complete, just for fun).





Here is the mathematical model for page ranking. We all know that the web pages always link many other web pages. Just take the simplified case as example, A, B, C, D are all web pages, and the arrows indicate the link.

Assumption: When the user enter a web page, he will choose one from the links to query next. For example, if a user is currently in web page A, the user will choose one from B, C, D to query next with equal probability.

Use matrix to represent the link relations:

$$S = \begin{bmatrix} 0 & 1/2 & 1 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix}$$

Define the population score: more users enter the webpage, higher score, also higher importance.

Initialize the population score with all 1/2. First round:

$$Sx = \begin{bmatrix} 0 & 1/2 & 1 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.42 \\ 0.42 \\ 0.42 \end{bmatrix}$$

Keep going... The population score will become stable!

Google: PageRank Algorithm

$$S = \begin{bmatrix} 0 & 1/2 & 1 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix}$$

After many rounds, the population score becomes stable.

$$x = \begin{bmatrix} 0.67 \\ 0.44 \\ 0.44 \\ 0.44 \end{bmatrix}$$

The result shows us web page A is the most important, and web page B, C, D are equally important.

So, how does it related to eigenvalues and eigenvectors? The answer is, **the final score is just the eigenvector of matrix S with eigenvalue 1.**