Computations and Applications of Determinants

Lecture 8

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Four parts in last lecture:

1. Overview to Next Half Semester
   determinant; eigenvalues and eigenvectors; positive definiteness
2. Orthonormal Vectors and Orthogonal Matrices
   orthonormal vectors, orthogonal matrices, convenience of orthogonal matrices
3. Gram-Schmidt and QR Decomposition
   analysis of 2-D case, 3-D case, and deduction of QR decomposition
4. Introduction and Properties of Determinant
   geometrical view and part of the properties and computation of determinant

Gram-Schmidt

Algorithm Summary:

Gram's Part:

- Accept **a** to the orthogonal vector set.

$$\mathbf{A} = \mathbf{a}$$

- Subtract **A** component from **b** and add to the orthogonal vector set.

$$\mathbf{B} = \mathbf{b} - \frac{\mathbf{b}^T\mathbf{A}}{\mathbf{A}^T\mathbf{A}}\mathbf{A}$$

Schmidt's Part:

- Normalize the vectors in orthogonal vector set.

$$\mathbf{A} = \frac{\mathbf{A}}{||\mathbf{A}||}, \mathbf{B} = \frac{\mathbf{B}}{||\mathbf{B}||}$$

QR Decomposition

Experts in Linear Algebra will not stop here, they will go ahead to find the connection between Q and A. That is the same with LU decomposition, we find the matrix L after we know how to find U. Now, we are going to find the connection R.

$$A = QR = \left(QQ^T\right) A \Rightarrow R = Q^T A$$

$$\begin{bmatrix} | & | & | \\ a & b & c \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} {q_1}^T a & {q_1}^T b & {q_1}^T c \\ {q_2}^T a & {q_2}^T b & {q_2}^T c \\ {q_3}^T a & {q_3}^T b & {q_3}^T c \end{bmatrix}$$

In the Gram-Schmidt process, we can guarantee that ${q_2}^T a = 0$, why?

$$\begin{bmatrix} | & | & | \\ a & b & c \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} {q_1}^T a & {q_1}^T b & {q_1}^T c \\ 0 & {q_2}^T b & {q_2}^T c \\ 0 & 0 & {q_3}^T c \end{bmatrix}$$

R is upper triangular! QR decomposition complete. A little bit complex...

Properties of Determinant

Now, let's consider the following properties for determinants geometrically.

- $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$
- $\begin{vmatrix} ta & b \\ tc & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$
- Linearly dependent columns make the determinant 0
- $\det U = \begin{vmatrix} d_1 & * & * & \cdots & * \\ 0 & d_2 & * & \cdots & * \\ 0 & 0 & d_3 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & * \\ 0 & 0 & 0 & 0 & d_n \end{vmatrix} = d_1 d_2 d_3 \cdots d_n$
- $\det AB = (\det A)(\det B)$

We know all of them without any kinds of computations!

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Properties of Determinants

The order of these properties come from MIT 18.06 (Gilbert).

1. $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$
2. $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$, exchange 2 rows reverse the sign
3. $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$
4. 2 equal rows $\rightarrow$ 0 determinant, easily proved by property 2
5. Subtract k times row m from row n will not change the determinant
6. Zero row $\rightarrow$ 0 determinant

The order of these properties come from MIT 18.06 (Gilbert).

7. $\det U = \begin{vmatrix} d_1 & * & * & \cdots & * \\ 0 & d_2 & * & \cdots & * \\ 0 & 0 & d_3 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & * \\ 0 & 0 & 0 & 0 & d_n \end{vmatrix} = d_1 d_2 d_3 \cdots d_n$
8. Zero det means singular, nonzero det means invertible
9. $\det AB = (\det A)(\det B)$
10. $\det A^T = \det A$

Permutations can be classified to odd and even! That is the same as multiplying a permutation matrix, and permutation matrices have -1 or 1 determinant. Odd row exchanges reverse the sign, while even row exchanges do not change the sign.

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Big Formula

Up to now, we haven't introduce any of the computing formula for determinant. Can we find a general formula for all the determinants?

Again, start from 2×2 matrix. You all know that the formula for 2×2 determinant is like the following, but why?

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Using linearity by row:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}$$

What we have done: take an entry in each row, and only the determinants without the zero column are nonzero.

Inspiration: Choose entries from each row and column, only consider the sun of those determinants, the others are all zero if we take 2 entries from a single row or column.

Big Formula

Consider 3×3 case:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

- Taking a, then we can take e, i or f, h.
- Taking b, then we can take d, i or f, g.
- Taking c, then we can take d, h or e, g.

2×2 determinants have 2 terms, 3×3 determinants have 6 terms, what about $n \times n$ determinants?

Taking an entry from each row: for the first row, you have n choices, for the second row, you have $n-1$ choices (because you can't take the entry from the same column),...

So, $n \times n$ determinants have $n!$ terms.

BIG FORMULA:

$$\det A = \sum_{\textit{all combinations}} (\det P)\, a_{1\alpha} a_{2\beta} \cdots a_{n\omega}$$

while P is the permutation matrix that have determinant 1 or -1 (determined by the order of chosen entries).

Another simplified expression: $P = (\alpha, \beta, \cdots, \omega)$.

Example: Using Big Formula to show that

$$\det U = \begin{vmatrix} d_1 & * & * & \cdots & * \\ 0 & d_2 & * & \cdots & * \\ 0 & 0 & d_3 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & * \\ 0 & 0 & 0 & 0 & d_n \end{vmatrix} = d_1 d_2 d_3 \cdots d_n$$

Consider 3×3 case:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

COFACTOR FORMULA:

$$\det A = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n}$$

Cofactors are the determinants that eliminates a row and a column, multiplying a coefficient of 1 or -1, determined by the sum of i, j.

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Cofactor matrix:

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

A formula for all square matrices (no matter singular or not):

$$AC^T = \det A \cdot I$$

Noteworthy that A^* is the same as C^T, called the adjoint matrix.

You'd better know how it comes... Referring to MIT 18.06 please!
`https://www.bilibili.com/video/BV1zx411g7gq?p=20` 07:41

If matrix A is invertible, the inverse:

$$A^{-1} = \frac{1}{\det A} A^*$$

Cramer's Rule

Consider a system of linear equations $Ax = b$:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Cramer gives

$$x_j = \frac{\det B_j}{\det A}$$

where

$$B_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n & \cdots & a_{nn} \end{bmatrix}$$

For 10×10 matrix, you need to find eleven 10×10 determinants to find the solution. Please use Gaussian Elimination to solve linear equations.

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(2019 Fall Final, 12 marks) For each natural number $n \geqslant 3$, find the determinant:

$$D_n = \begin{vmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{vmatrix}_{n\times n}$$

Solution: cofactor expansion and find the recursion formula.

For this example, $D_n = 2D_{n-1} - D_{n-2}$.

The first few terms: $D_1 = 2, D_2 = 3, D_3 = 4$.

So, the answer is: $D_n = n - 1$.

Find the determinant:

$$D_6 = \begin{vmatrix} 1 & 1 & & & \\ -1 & 1 & 1 & & \\ & -1 & 1 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & -1 & 1 \end{vmatrix}_{6\times 6}$$

For this example, $D_n = D_{n-1} + D_{n-2}$.

The first few terms: $D_1 = 1, D_2 = 2$.

Fibonacci series, $1, 2, 3, 5, 8, 13$, so the answer is $D_6 = 13$.

Find the determinant:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 0 & 0 & 4 \end{vmatrix}$$

Solution: eliminate the first row by the diagonal entries, simplify to triangular matrix.

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 0 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 0 & 0 & 4 \end{vmatrix} = -2$$

Type 2: Arrow Form Matrix

Find the determinant:

$$D_n = \begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ b_2 & 1 & 0 & \cdots & 0 \\ b_3 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$D_n \begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ b_2 & 1 & 0 & \cdots & 0 \\ b_3 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n & 0 & 0 & \cdots & 1 \end{vmatrix} = \begin{vmatrix} a_1 - a_2 b_2 - a_3 b_3 - \cdots & 0 & 0 & \cdots & 0 \\ b_2 & 1 & 0 & \cdots & 0 \\ b_3 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n & 0 & 0 & \cdots & 1 \end{vmatrix}$$

So, the answer is $D_n = a_1 - \sum_{i=2}^{n} a_n b_n$.

Type 3: Vandermonde Determinant and Variations

Vandermonde Determinant:

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ {x_1}^2 & {x_2}^2 & {x_3}^2 & \cdots & {x_n}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ {x_1}^{n-1} & {x_2}^{n-1} & {x_3}^{n-1} & \cdots & {x_n}^{n-1} \end{vmatrix} = \prod_{1\leqslant j<i\leqslant n} (x_i - x_j)$$

Proof omitted. Please refer to baidu or other search engines.

An example:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{vmatrix} = (2-1)(3-2)(3-1)(4-3)(4-2)(4-1) = 12$$

Type 3: Vandermonde Determinant and Variations

Variation 1: First row lost.

$$\begin{vmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ {x_1}^2 & {x_2}^2 & {x_3}^2 & \cdots & {x_n}^2 \\ {x_1}^3 & {x_2}^3 & {x_3}^3 & \cdots & {x_n}^3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ {x_1}^n & {x_2}^n & {x_3}^n & \cdots & {x_n}^n \end{vmatrix}$$

Solution: extract x_i from each column and it becomes the original.

$$\begin{vmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ {x_1}^2 & {x_2}^2 & {x_3}^2 & \cdots & {x_n}^2 \\ {x_1}^3 & {x_2}^3 & {x_3}^3 & \cdots & {x_n}^3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ {x_1}^n & {x_2}^n & {x_3}^n & \cdots & {x_n}^n \end{vmatrix} = x_1 x_2 \cdots x_n \prod_{2 \leqslant j < i \leqslant n} (x_i - x_j)$$

Type 3: Vandermonde Determinant and Variations

Variation 2: Other row lost.

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \\ a^4 & b^4 & c^4 & d^4 \end{vmatrix}$$

Solution: construct complete Vandermonde and compare coefficient.

Construct complete Vandermonde matrix A:

$$\det A = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ a & b & c & d & x \\ a^2 & b^2 & c^2 & d^2 & x^2 \\ a^3 & b^3 & c^3 & d^3 & x^3 \\ a^4 & b^4 & c^4 & d^4 & x^4 \end{vmatrix}$$

Now, we want to find the minor M_{25} of matrix A.

Type 3: Vandermonde Determinant and Variations

$$\det A = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ a & b & c & d & x \\ a^2 & b^2 & c^2 & d^2 & x^2 \\ a^3 & b^3 & c^3 & d^3 & x^3 \\ a^4 & b^4 & c^4 & d^4 & x^4 \end{vmatrix}$$

Define constant $S = (d-c)(d-b)(d-a)(c-b)(c-a)(b-a)$.

Calculate the Vandermonde determinant and cofactor expansion by column n:

$$|A| = S(x-d)(x-c)(x-b)(x-a) = C_{15} + C_{25}x + C_{35}x^2 + C_{45}x^3 + C_{55}x^4$$

Compare the coefficient of x: $C_{25} = (-abc - abd - acd - bcd)\,S$.

So, the original determinant is $M_{25} = -C_{25} = (abc + abd + acd + bcd)\,S$.

Find the determinant:

$$A = \begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+a & 1 & 1 \\ 1 & 1 & 1+a & 1 \\ 1 & 1 & 1 & 1+a \end{vmatrix}$$

Add all rows to the first row, and use row of all 1s to simplify...

Other methods?

Type 4: Repeated (Similar Terms) Matrix

Find the determinant:

$$A = \begin{vmatrix} 1+a_1 & 1 & 1 & 1 \\ 1 & 1+a_2 & 1 & 1 \\ 1 & 1 & 1+a_3 & 1 \\ 1 & 1 & 1 & 1+a_4 \end{vmatrix}$$

Solution: add a row or column to eliminate repeated terms.

$$\det A = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1+a_1 & 1 & 1 & 1 \\ 1 & 1 & 1+a_2 & 1 & 1 \\ 1 & 1 & 1 & 1+a_3 & 1 \\ 1 & 1 & 1 & 1 & 1+a_4 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 & -1 & -1 \\ 1 & a_1 & & & \\ 1 & & a_2 & & \\ 1 & & & a_3 & \\ 1 & & & & a_4 \end{vmatrix}$$

Back to Type 2. The answer is $\left(1 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4}\right) a_1 a_2 a_3 a_4$.

Find the determinant:

$$\det A = \begin{vmatrix} 1+a_1 & a_1 & a_1 & a_1 \\ a_2 & 1+a_2 & a_2 & a_2 \\ a_3 & a_3 & 1+a_3 & a_3 \\ a_4 & a_4 & a_4 & 1+a_4 \end{vmatrix}$$

$$\det A = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ a_1 & 1+a_1 & a_1 & a_1 & a_1 \\ a_2 & a_2 & 1+a_2 & a_2 & a_2 \\ a_3 & a_3 & a_3 & 1+a_3 & a_3 \\ a_4 & a_4 & a_4 & a_4 & 1+a_4 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 & -1 & -1 \\ a_1 & 1 & & & \\ a_2 & & 1 & & \\ a_3 & & & 1 & \\ a_4 & & & & 1 \end{vmatrix}$$

Back to Type 2. The answer is $a_1 + a_2 + a_3 + a_4 + 1$.

Type 4: Repeated (Similar Terms) Matrix

Find the determinant:

$$\det A = \begin{vmatrix} a_1 & b & b & \cdots & b \\ b & a_2 & b & \cdots & b \\ b & b & a_3 & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a_n \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ b & a_1 & b & b & \cdots & b \\ b & b & a_2 & b & \cdots & b \\ b & b & b & a_3 & \cdots & b \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & b & \cdots & a_n \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 & -1 & \cdots & -1 \\ b & a_1 - b & & & & \\ b & & a_2 - b & & & \\ b & & & a_3 - b & & \\ \vdots & & & & \ddots & \\ b & & & & & a_n - b \end{vmatrix}$$

Back to Type 2. The answer is $\left[1 + b\sum_{i=1}^{n} \frac{1}{a_i - b}\right](a_1 - b)\cdots(a_n - b)$.

Type 4: Repeated (Similar Terms) Matrix

Find the determinant:

$$\det A = \begin{vmatrix} a+x_1 & a+x_2 & a+x_3 \\ a^2+{x_1}^2 & a^2+{x_2}^2 & a^2+{x_3}^2 \\ a^3+{x_1}^3 & a^3+{x_2}^3 & a^2+{x_3}^3 \end{vmatrix}$$

$$\det A = \begin{vmatrix} 1 & 0 & 0 & 0 \\ a & a+x_1 & a+x_2 & a+x_3 \\ a^2 & a^2+{x_1}^2 & a^2+{x_2}^2 & a^2+{x_3}^2 \\ a^3 & a^3+{x_1}^3 & a^3+{x_2}^3 & a^3+{x_3}^3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 & -1 \\ a & x_1 & x_2 & x_3 \\ a^2 & {x_1}^2 & {x_2}^2 & {x_3}^2 \\ a^3 & {x_1}^3 & {x_2}^3 & {x_3}^3 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & x_1 & x_2 & x_3 \\ a^2 & {x_1}^2 & {x_2}^2 & {x_3}^2 \\ a^3 & {x_1}^3 & {x_2}^3 & {x_3}^3 \end{vmatrix} + \begin{vmatrix} 2 & 0 & 0 & 0 \\ a & x_1 & x_2 & x_3 \\ a^2 & {x_1}^2 & {x_2}^2 & {x_3}^2 \\ a^3 & {x_1}^3 & {x_2}^3 & {x_3}^3 \end{vmatrix}$$

Back to Type 3.

Type 5: Circulant Matrix

Find the determinant:

$$\det A = \begin{vmatrix} 0 & b & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & b \\ b & 0 & a & 0 \end{vmatrix}$$

We can have cofactor expansion on column 1:

$$\det A = -a \begin{vmatrix} b & 0 & a \\ a & 0 & b \\ 0 & a & 0 \end{vmatrix} - b \begin{vmatrix} b & 0 & a \\ 0 & b & 0 \\ a & 0 & b \end{vmatrix} = \left(a^2 - b^2\right) \begin{vmatrix} b & 0 & a \\ 0 & 1 & 0 \\ a & 0 & b \end{vmatrix} = -\left(a^2 - b^2\right)^2$$

This problem is solved, but it is not a general method for Type 5.

Type 5: Circulant Matrix

Find the determinant:

$$\det A = \begin{vmatrix} x & y & z & w \\ y & x & w & z \\ z & w & x & y \\ w & z & y & x \end{vmatrix}$$

Solution: factor extraction.

$$\det A = \begin{vmatrix} x & y & z & w \\ y & x & w & z \\ z & w & x & y \\ w & z & y & x \end{vmatrix} = (x+y+z+w)\begin{vmatrix} 1 & 1 & 1 & 1 \\ y & x & w & z \\ z & w & x & y \\ w & z & y & x \end{vmatrix}$$

$$\det A = \begin{vmatrix} x & y & z & w \\ y & x & w & z \\ z & w & x & y \\ w & z & y & x \end{vmatrix} = (x+y-z-w)\begin{vmatrix} 1 & 1 & -1 & -1 \\ y & x & w & z \\ z & w & x & y \\ w & z & y & x \end{vmatrix}$$

$$\det A = \begin{vmatrix} x & y & z & w \\ y & x & w & z \\ z & w & x & y \\ w & z & y & x \end{vmatrix} = (x+z-y-w) \begin{vmatrix} 1 & -1 & 1 & -1 \\ y & x & w & z \\ z & w & x & y \\ w & z & y & x \end{vmatrix}$$

$$\det A = \begin{vmatrix} x & y & z & w \\ y & x & w & z \\ z & w & x & y \\ w & z & y & x \end{vmatrix} = (x+w-y-z) \begin{vmatrix} 1 & -1 & -1 & 1 \\ y & x & w & z \\ z & w & x & y \\ w & z & y & x \end{vmatrix}$$

So, the determinant must satisfy

$$\det A = k\,(x+y+z+w)\,(x+y-z-w)\,(x+z-y-w)\,(x+w-y-z)$$

Check coefficient of x^4 (1):

$$\det A = (x+y+z+w)\,(x+y-z-w)\,(x+z-y-w)\,(x+w-y-z)$$

Type 5: Circulant Matrix

Back to the first example:

$$\det A = \begin{vmatrix} 0 & b & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & b \\ b & 0 & a & 0 \end{vmatrix}$$

By factor extraction:

$$\det A = k\,(a+b)\,(a-b)\,(a+b)\,(a-b)$$

Check coefficient of a^4 (-1):

$$\det A = -\,(a+b)\,(a-b)\,(a+b)\,(a-b) = -\left(a^2 - b^2\right)^2$$