# Final Exam Review

## Lecture 13

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# We Have Learnt...

**Chapter 3, 4:**

- 3.3 Projections and Least Squares
- 3.4 Orthogonal Bases and Gram-Schmidt
- 4.1 Introduction to Determinants
- 4.2 Properties of Determinants
- 4.3 Formulas of Determinants
- 4.4 Applications of Determinants

**A Summary:** Least squares is still important in the final exam. Section 3.4 is the first section in the second half semester and $QR$ decomposition is must-know! Chapter 4 is about determinant, which is not the most important, but you should be familiar with tri-diagonal determinants!

# We Have Learnt...

**Chapter 5:**

- 5.1 Introduction to Eigenvalues and Eigenvectors
- 5.2 Diagonalization of Matric
- 5.5 Complex Matrix
- 5.6 Similarity Transformation

**A Summary:** Chapter 5 is the most important chapter in the final exam. You should try your best to review this chapter, all of the sections are important, they will take nearly 40 marks in midterm exam.

**Chapter 6:**

- 6.1 Minima, Maxima, and Saddle Points
- 6.2 Tests for Positive Definiteness
- 6.3 Singular Value Decomposition

**A Summary:** Section 6.1 is kind of useless, section 6.2 will have 20 marks in the exam, and SVD will definitely exist (5 marks as blank-filling or 15 marks as solving).

# Final Exam in Previous Years: 2019

Final exam in 2019 has 10 problems, with a total of 110 marks.

1. (10 marks) True or False. (2'$\times$5)
2. (15 marks) Fill the blanks. (3'$\times$5)
3. (12 marks) Find determinants of tri-diagonal matrix.
4. (8 marks) Least-squares (line fitting).
5. (15 marks) Quadratic form and positive definiteness.
6. (10 marks) Four fundamental subspaces, properties of $A^TA$.
7. (8 marks) Proof about linear independence.
8. (12 marks) Eigenvalues of idempotent matrix.
9. (10 marks) Eigenvalues, polynomial expression, diagonalization.
10. (10 marks) Properties about $I + uu^T$.

Problem 3, 4, 5, 6, 8, 9, 10 are all fundamental problems, and they take up 77 marks!!!!

Final exam in 2020 has 8 problems, with a total of 100 marks.

1. (15 marks) Multiple Choices. (3'$\times$5)
2. (25 marks) Fill the blanks. (5'$\times$5)
3. (12 marks) Gram-Schmidt and $QR$ decomposition.
4. (10 marks) Find determinants of tri-diagonal matrix.
5. (12 marks) Eigenvalues, diagonalization.
6. (12 marks) Quadratic form and positive definiteness.
7. (8 marks) Singular values and SVD.
8. (6 marks) Eigenvalues of orthogonal matrix.

Again, Problem 3, 4, 5, 6, 7 are not that difficult, and they take up 52 marks!

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# A Summary

**High-Frequency Problems:**

1. Least-Squares
2. Orthogonal Bases and QR Decomposition
3. Determinant of Tri-Diagonal Matrix
4. Eigenvalues; Matrix Diagonalization
5. Quadratic Form and Positive Definiteness
6. Singular Value Decomposition

They will all definitely appear on your final exam paper! Please review these problems with highest priority.

# Least-Squares

**Example**

Let

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 1 & -1 & 2 \\ 1 & -1 & -2 \end{bmatrix}, b = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}$$

Find the least squares solution to $Ax = b$.

The system must be inconsistent or we cannot find the least squares solution, because we can find the exact solution!
The method to find least squares solution is to solve

$$A^T A\hat{x} = A^T b$$

# Least-Squares

Solve this system to get the least squares solution:

$$\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -2 & -1 & -1 \\ 0 & 1 & 2 & -2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 2 \\ 1 & -2 & 1 & -2 \\ 1 & -1 & 2 & -1 \\ 1 & -1 & -2 & 3 \end{bmatrix}$$

Multiply that:

$$\begin{bmatrix} 4 & -5 & 1 & -2 \\ -5 & 7 & -2 & 4 \\ 1 & -2 & 9 & -10 \end{bmatrix} \to \begin{bmatrix} 4 & -5 & 1 & -2 \\ 0 & 3 & -3 & 6 \\ 0 & 0 & 32 & -32 \end{bmatrix}$$

The least squares solution is

$$\hat{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

# Least-Squares (Fitting)

**Example**

Suppose that a dataset consists of points $(-6, -1), (-2, 2), (1, 1), (7, 6)$ on the xy-plane. Find an equation for the line that best models the relation between the $x$ and $y$ coordinates of these sample values in the sense of least-squares.

The core problem is how you can transform the problem to a least-square problem! Remember: all the fitting problems are least-square problems, even if they are polynomial function.

# Least-Squares (Fitting)

Set the equation of the line as:

$$y = kx + b$$

Substituting the data points in:

$$\begin{bmatrix} -6 & 1 \\ -2 & 1 \\ 1 & 1 \\ 7 & 1 \end{bmatrix} \begin{bmatrix} k \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 6 \end{bmatrix}$$

Why this equation system is inconsistent? Write the $A^T Ax = A^T b$ form:

$$\begin{bmatrix} -6 & -2 & 1 & 7 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -6 & 1 & -1 \\ -2 & 1 & 2 \\ 1 & 1 & 1 \\ 7 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 90 & 0 & 45 \\ 0 & 4 & 8 \end{bmatrix}$$

$$\hat{k} = 1/2, \hat{b} = 2$$

So the best fitting line is $\hat{y} = \frac{1}{2}x + 2$. You'd better check your answer!

**Example**

(Final Exam, Fall 2020, 12 marks)
(a) Find an orthonormal basis for the column space of

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 0 & 1 \\ 2 & -4 & 2 \\ 4 & 0 & 0 \end{bmatrix}$$

(b) Write $A$ as $QR$, where $Q$ has orthonormal columns and $R$ is upper triangular.

$QR$ decomposition is must-know knowledge. Make sure you verify the orthogonality at every step and thus you will get it correct!

Accept column 1:

$$a_1' = a_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \end{bmatrix}$$

For column 2:

$$a_2' = a_2 - \frac{{a_2}^T a_1'}{{a_1'}^T a_1'} a_1' = \begin{bmatrix} -2 \\ 0 \\ -4 \\ 0 \end{bmatrix} - \frac{-10}{25} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -8/5 \\ 4/5 \\ -16/5 \\ 8/5 \end{bmatrix} \to \begin{bmatrix} -2 \\ 1 \\ -4 \\ 2 \end{bmatrix}$$

For column 3:

$$a_3' = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix} - \frac{5}{25} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \end{bmatrix} - \frac{-5}{25} \begin{bmatrix} -2 \\ 1 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} -8/5 \\ 4/5 \\ 4/5 \\ -2/5 \end{bmatrix} \to \begin{bmatrix} -4 \\ 2 \\ 2 \\ -1 \end{bmatrix}$$

Finally, normalize them:

$$Q = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} & -\frac{4}{5} \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{4}{5} & \frac{2}{5} \\ \frac{4}{5} & \frac{2}{5} & -\frac{1}{5} \end{bmatrix}$$

How to find $R$? What's the size of $R$. Does $R$ have some great properties? $A = QR$, $R = Q^T A$ by $Q^T Q = I$. By Gram-Schmidt, $R$ is upper triangular.

$$R = \begin{bmatrix} 5 & -2 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

You can check by column method of matrix multiplication since the matrix $R$ is definitely a upper triangular matrix.

# Determinant of Tri-Diagonal Matrix

## Example

(Final Exam, Fall 2020, 10 marks) Consider the nth order determinant:

$$D_n(x,y) = \begin{vmatrix} x+y & xy & & & & \\ 1 & x+y & xy & & & \\ & 1 & x+y & xy & & \\ & & 1 & \ddots & \ddots & \\ & & & \ddots & x+y & xy \\ & & & & 1 & x+y \end{vmatrix}, n \geqslant 2$$

(a) Find a recurrence relation relating $D_n(x,y)$ to $D_{n-1}(x,y)$ and $D_{n-2}(x,y)$ for $n \geq 4$.
(b) Compute the determinant $D_n(x,y)$ for all $n \geq 2$.

You should be sensitive, cofactor expansion 2 times and you can get the recurrence relation!

# Determinant of Tri-Diagonal Matrix

Cofactor expansion on row 1, following by cofactor expansion on column 1:

$$D_n(x,y) = (x+y)\,D_{n-1} - xy\begin{vmatrix} 1 & xy & & & \\ 0 & x+y & xy & & \\ 0 & 1 & x+y & \ddots & \\ 0 & & \ddots & \ddots & xy \\ 0 & & & 1 & x+y \end{vmatrix}$$

$$D_n(x,y) = (x+y)\,D_{n-1} - xyD_{n-2}$$

Check $D_1(x,y) = x+y$ and $D_2(x,y) = x^2+xy+y^2$.

By mathematical induction, $D_n(x,y) = x^n + x^{n-1}y + \ldots + xy^{n-1} + y^n$.

Mathematical induction needs 2 parts:

1. Case $n=1$ satisfies.
2. If case $n=i$ satisfies, then case $n=i+1$ also satisfies.

Please indicate you are using mathematical induction at first.

## Example

Decide whether the following matrix $A$ is diagonalizable or not.

$$A = \begin{bmatrix} 5 & -1 & -1 \\ 3 & 1 & -1 \\ 4 & -2 & 1 \end{bmatrix}$$

Calculating eigenvalues is important! I will show my methods here. You can take or still using your own as long as you can get it correct.

$$\det(A - \lambda I) = -(\lambda - 2)^2 (\lambda - 3) = 0$$

For a $n$ degree polynomial, we can find $n$ roots.

$$\lambda_1 = \lambda_2 = 2, \lambda_3 = 3$$

2 is a repeated eigenvalue, with algebraic multiplicity of 2, so we need to check the geometric multiplicity.

**Example**

Decide whether the following matrix $A$ is diagonalizable or not.

$$A = \begin{bmatrix} 5 & -1 & -1 \\ 3 & 1 & -1 \\ 4 & -2 & 1 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} 3 & -1 & -1 \\ 3 & -1 & -1 \\ 4 & -2 & -1 \end{bmatrix} \to \begin{bmatrix} 3 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \to \begin{bmatrix} 3 & -1 & -1 \\ 0 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

1 free column, 1 independent eigenvector for eigenvector 2. Geometric multiplicity: 1.

So, we have no enough number of independent eigenvectors. The matrix is not diagonalizable.

**Example**

Find an orthogonal diagonalizing matrix for the following matrix:

$$A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{bmatrix}$$

$$\det(A - \lambda I) = (1-\lambda)(1-\lambda)(10-\lambda) = 0, \lambda_1 = \lambda_2 = 1, \lambda_3 = 10$$

Here I give the eigenvalues here, in the exam, please make sure the sum of your eigenvalues equals to the trace!

For $\lambda = 1$:

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, a_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, a_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda = 10$:

$$\begin{bmatrix} -8 & 2 & -2 \\ 2 & -5 & -4 \\ -2 & -4 & -5 \end{bmatrix} \to \begin{bmatrix} -8 & 2 & -2 \\ 0 & -9/2 & -9/2 \\ 0 & 0 & 0 \end{bmatrix}, a_3 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$$

Do Gram-Schmidt:

$$a_2' = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 1 \\ 4/5 \end{bmatrix} \rightarrow \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix}$$

So, the diagonalizing matrix is:

$$Q = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{2}{\sqrt{45}} & -\frac{1}{3} \\ 0 & \frac{5}{\sqrt{45}} & -\frac{2}{3} \\ \frac{1}{\sqrt{5}} & \frac{4}{\sqrt{45}} & \frac{2}{3} \end{bmatrix}$$

(Note that the first 2 columns can be exchanged because they have the same eigenvalues and the vector in every column can be reversed.)

**Example**

Find a unitary diagonalizing matrix for the following matrix:

$$A = \begin{bmatrix} 0 & 1-i \\ 1+i & 1 \end{bmatrix}$$

Note that every Hermitian matrix can have unitary diagonalization matrix. Gram-Schmidt for complex matrix is so difficult, so in most of cases, we should get $n$ distinct eigenvalues. Final reminder: when you verify your answer, make sure you use $v^H v$ to calculate inner product.

$$\det(A-\lambda I)=\begin{vmatrix}-\lambda & 1-i\\ 1+i & 1-\lambda\end{vmatrix}=\lambda^2-\lambda+2=0,\lambda_1=-1,\lambda_2=2$$

For eigenvalue $\lambda=-1$:

$$\begin{bmatrix}1 & 1-i\\ 1+i & 2\end{bmatrix}\to\begin{bmatrix}1 & 1-i\\ 0 & 0\end{bmatrix},x_1=\begin{bmatrix}-1+i\\ 1\end{bmatrix},q_1=\begin{bmatrix}\frac{-1+i}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{bmatrix}$$

For eigenvalue $\lambda=2$:

$$\begin{bmatrix}-2 & 1-i\\ 1+i & -1\end{bmatrix}\to\begin{bmatrix}-2 & 1-i\\ 0 & 0\end{bmatrix},x_2=\begin{bmatrix}\frac{1-i}{2}\\ 1\end{bmatrix},q_2=\begin{bmatrix}\frac{1-i}{2\sqrt{3/2}}\\ \frac{1}{\sqrt{3/2}}\end{bmatrix}=\begin{bmatrix}\frac{1-i}{\sqrt{6}}\\ \frac{2}{\sqrt{6}}\end{bmatrix}$$

$$A=\begin{bmatrix}0 & 1-i\\ 1+i & -1\end{bmatrix}=\begin{bmatrix}\frac{-1+i}{\sqrt{3}} & \frac{1-i}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}}\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}\frac{-1+i}{\sqrt{3}} & \frac{1-i}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}}\end{bmatrix}^H=U\Lambda U^H$$

## Example

Consider the quadratic form

$$f(x_1, x_2, x_3) = -2{x_1}^2 - 4{x_2}^2 - 5{x_3}^2 + 4x_1x_3$$

(a) Find the matrix $A$ for the quadratic form $f(x_1, x_2, x_3)$.
(b) Decide for or against the positive definiteness of $A$.
(c) Find an orthogonal matrix $Q$ to change the variable $y = Qx$ such that the quadratic form can be converted into diagonal form. Write the final standard quadratic form down.
(d) (by myself) Find an upper triangular matrix $U$ to change the variable $y = Ux$ such that the quadratic form can be converted into diagonal form. Write the final standard quadratic form down.

For the problems about quadratic form, always remember $LDL^T$ and $Q\Lambda Q^T$ are your weapons. Hope I can use this single problem to help you review whole section 6.2.

(a)

$$A = \begin{bmatrix} -2 & 0 & 2 \\ 0 & -4 & 0 \\ 2 & 0 & -5 \end{bmatrix}$$

(b) Giving 3 negative pivots, the matrix is negative definite.

(c)

$$\det(A - \lambda I) = \begin{bmatrix} -2-\lambda & 0 & 2 \\ 0 & -4-\lambda & 0 \\ 2 & 0 & -5-\lambda \end{bmatrix} = -\lambda^3 - 11\lambda^2 - 34\lambda - 24$$

$$\det(A - \lambda I) = -(\lambda + 6)(\lambda + 4)(\lambda + 1)$$

For eigenvalue $\lambda = -1$:

$$\begin{bmatrix} -1 & 0 & 2 \\ 0 & -3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \Rightarrow a_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \Rightarrow q_1 = \begin{bmatrix} 2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix}$$

For eigenvalue $\lambda = -4$:

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix} \Rightarrow a_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow q_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

For eigenvalue $\lambda = -6$:

$$\begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} \Rightarrow a_3 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \Rightarrow q_3 = \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ -2/\sqrt{5} \end{bmatrix}$$

$$A = \begin{bmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \\ 1/\sqrt{5} & 0 & -2/\sqrt{5} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \\ 1/\sqrt{5} & 0 & -2/\sqrt{5} \end{bmatrix}$$

$$f(x_1, x_2, x_3) = -\left(\frac{2}{\sqrt{5}}x_1 + \frac{1}{\sqrt{5}}x_3\right)^2 - 4{x_2}^2 - 6\left(\frac{1}{\sqrt{5}}x_1 - \frac{2}{\sqrt{5}}x_3\right)^2$$

(d) Find $LDL^T$ factorization for matrix $A$:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$f(x_1, x_2, x_3) = -2\left(x_1 - x_3\right)^2 - 4{x_2}^2 - 3{x_3}^2$$

The required $U$:

$$U = L^T = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The required $Q$ in (c):

$$Q = \begin{bmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \\ 1/\sqrt{5} & 0 & -2/\sqrt{5} \end{bmatrix}$$

## Example

(2020 Fall Final Exam, 8 marks) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$.

(a) Find all the singular values of $A$.

(b) Find the singular value decomposition of $A$, in other words. find 2 orthogonal matrices $U$ and $V$ (of suitable size) such that $A = U\Sigma V^T$.

$U$ is the orthogonal eigenvector matrix of $AA^T$, $V$ is the orthogonal eigenvector matrix of $A^TA$.

Process of SVD:

1. Find the eigenvalues and eigenvectors (orthonormal) $A^TA$ to get $V$.
2. By $Av_i = \sigma_i u_i$, find $u_1$ to $u_r$.
3. Find an orthonormal basis for $N(A^T)$ for the rest column vectors in $U$.

# Singular Value Decomposition

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Eigenvalues for $A^TA$: 2, 4, thus singular value: $\sqrt{2}, 2$.

You should at least get this answer correctly. And try to follow me to complete the SVD process step by step.

We can also get the "diagonal" non-square matrix $\Sigma$, which is as same size with $A$, with singular values on the "diagonal":

$$\Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

# Singular Value Decomposition

For $A^TA$, $\lambda = 2$:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \to a_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \to q_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

For $A^TA$, $\lambda = 4$:

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \to a_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \to q_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

The orthogonal eigenvector matrix of $A^TA$ is:

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

## Singular Value Decomposition

Find the corresponding column vectors in $U$:

$$Av_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sqrt{2} \end{bmatrix} \to u_1 = \frac{1}{\sigma_1} Av_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$Av_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{bmatrix} \to u_2 = \frac{1}{\sigma_2} Av_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

Finally, add the left nullspace basis $N(A^T)$ in (make it orthonormal).

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}, u_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

Write them together and verify your answer. This solution is only for reference, exchange columns for all matrices or reverse the direction of any column vector is still correct.

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# Relations Between $A, B, A+B, AB, A^T, A^{-1}$

We have learnt some of the relations in midterm review, how are these matrices related in Chapter 4, 5, 6?

In Chapter 4:

- $\det(A+B) \neq \det(A) + \det(B)$, $\det(AB) = \det(A)\det(B)$.
- $\det(A^{-1}) = 1/\det(A)$, $\det(A^T) = \det(A)$.
- $\det(I_m - AB) = \det(I_n - BA)$, A is $m \times n$, B is $n \times m$.

In Chapter 5:

- $A$ and $A^T$ share the same eigenvalues, not same eigenvectors.
- $A$ has eigenvalue $\lambda$, then $A^{-1}$ has eigenvalue $1/\lambda$, same eigenvectors.
- $A$ has eigenvalue $\lambda$, then $f(A)$ has eigenvalue $f(\lambda)$, same eigenvectors.
- $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$, $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.
- $A$ and $B$ are diagonalizable, $AB$ is diagonalizable when $AB = BA$.
- Eigenvalues of $A+B \neq$ eigenvalues of $A$ + eigenvalues of $B$.
- If $A \sim B$, then $A^T \sim B^T$, $A^{-1} \sim B^{-1}$, $A + A^{-1} \sim B + B^{-1}$, $A^k \sim B^k$, $A + I \sim B + I$ but $A + A^T$ is not similar to $B + B^T$.

# Relations Between $A, B, A+B, AB, A^T, A^{-1}$

In Chapter 6:

- If $A$ and $B$ are positive definite, then $A+B$ is positive definite.
- If $A$ and $B$ are positive definite, $AB$ is not positive definite (unless $AB = BA$).
- If $A$ is positive definite, then $A^{-1}$ is positive definite.
- If $A$ is positive semidefinite, then $A + kI (k > 0)$ is positive definite.

Be confident when meeting those problems in the exam, you have learnt all of them. No proof is given here, try to understand instead of memorize the conclusion.

# Eigenvalues of Common Matrices

For common matrices, here I summarize the eigenvalues of them.

- For Hermitian matrices $A^H = A$, the eigenvalues are all real.
- For skew-Hermitian matrices $A^H = -A$, the eigenvalues are all imaginary.
- For unitary matrices $U^H U = I$, the eigenvalues should satisfy $|\lambda| = 1$.
- For rank-1 matrices $A = uv^T$, the eigenvalues are 0 with geometric multiplicity of $n-1$, and $u^T v$ with geometric multiplicity of 1. Mostly diagonalizable unless $u^T v = 0$ (orthogonal).
- For Hermitian matrices $A^H A, AA^H$, the eigenvalues are all real and satisfy $\lambda \geq 0$.
- For projection matrices $P = A(A^T A)^{-1} A^T$, the eigenvalues are 0 with geometric multiplicity of $n-r$, and 1 with geometric multiplicity of $r$.
- For idempotent matrices $A^2 = A$, the eigenvalues are 0 with geometric multiplicity of $n-r$, and 1 with geometric multiplicity of $r$.

# Properties of Idempotent Matrix

For idempotent matrices $A^2 = A$, the eigenvalues are 0 with geometric multiplicity of $n - r$, and 1 with geometric multiplicity of $r$.

Idempotent matrix is diagonalizable, it has $n$ independent eigenvectors and its rank equals its trace.

**Proof:**

$$A(A - I) = 0$$

- If $AB = 0$, $rank(A) + rank(B) \leq n$.
- $rank(A) + rank(B) \geq rank(A + B)$.

$$n = r(I) \leqslant r(A) + r(A - I) \leqslant n$$

The sum of geometric multiplicity is $n$, giving $n$ independent eigenvectors, which makes $A$ diagonalizable. $A$ can only have eigenvalues 0 and 1, with geometric multiplicity $n - r$, $r$.

# Geometry with Positive Definiteness

You are familiar with 2-dimensional cases:

1. For positive definite quadratic form $f(x, y)$, $f(x, y) = 1$ produces an ellipse.
2. For indefinite quadratic form $f(x, y)$, $f(x, y) = 1$ produces a hyperbola.

Make sure you can understand the 2-D cases, or you may feel uneasy with 3-D cases.

3-dimensional cases:

1. For positive definite quadratic form $f(x, y, z)$, $f(x, y, z) = 1$ produces an ellipsoid.
2. For indefinite quadratic form $f(x, y, z)$ that has 2 positive eigenvalues and 1 negative eigenvalues, $f(x, y, z) = 1$ produces a hyperboloid of one sheet.
3. For indefinite quadratic form $f(x, y, z)$ that has 1 positive eigenvalues and 2 negative eigenvalues, $f(x, y, z) = 1$ produces a hyperboloid of two sheets.

# A Brief Proof for $\det(I_m - AB) = \det(I_n - BA)$

Recall the rank relations we learnt in Lecture 6 Midterm Review. Proof about determinants can also use similar method to solve. Row and column operations (without exchanging them) will not change the determinants.

And we have the following formula:

$$\det \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \det A \cdot \det B$$

That is by Big Formula! Then we do row exchanges in 2 ways:

$$\begin{bmatrix} I_m & A \\ B & I_n \end{bmatrix} \xrightarrow{c2=c2-Ac1} \begin{bmatrix} I_m & 0 \\ B & I_n - BA \end{bmatrix}$$
$$\begin{bmatrix} I_m & A \\ B & I_n \end{bmatrix} \xrightarrow{r1=r1-Ar2} \begin{bmatrix} I_m - AB & 0 \\ B & I_n \end{bmatrix}$$

All these matrices have the same determinants. Then we can get:

$$\det(I_m)\det(I_n - BA) = \det(I_n)\det(I_m - AB)$$

Hope you can all do well in the final exam!

If you have any questions, please feel free to ask me or discuss in the QQ group. Good luck!