Problem Set 12

Problem 1. *(2020 Fall Final Exam, 12 marks)Consider the quadratic form*

$$f(x_1, x_2, x_3) = (x_1 - x_2 + x_3)^2 + (x_2 + x_3)^2 + (x_1 + ax_3)^2$$

where $a \in \mathbb{R}$ is a parameter.
(a) What are the possible values of a if the quadratic form f is positive definite?
(b) What are the possible values of a if the equation $f(x_1, x_2, x_3) = 0$ has infinitely many solutions.
(c) Let y be a new system of variables and equation $f(x_1, x_2, x_3) = 0$ has infinitely many solutions. Find an invertible linear transformation $y = Px$, such that the quadratic form f has a diagonal form.

Solution 1.
(a) If the quadratic form f is positive definite, then $f(x_1, x_2, x_3) = 0$ if and only if x_1, x_2, x_3 all equal to 0. So the following equation system can only have zero solution.

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So the matrix must be a full-rank matrix, leading $a \neq 2$.
(b) The above equation system should have infinitely many solutions, which means $a = 2$.
(c) Let $y_1 = x_1 - x_2 + x_3$, $y_2 = x_2 + x_3$, $y_3 = x_1 + 2x_3$, the quadratic form will be transformed to a standard form. The required matrix P:

$$P = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

Problem 2. *(2019 Fall Final Exam, 15 marks) Consider the quadratic form*

$$f(x_1, x_2, x_3) = -2{x_1}^2 - 4{x_2}^2 - 5{x_3}^2 + 4x_1x_3$$

(a) Find the matrix A for the quadratic form $f(x_1, x_2, x_3)$.
(b) Decide for or against the positive definiteness of A.
(c) Find an orthogonal matrix Q to diagonalize A.
(d) Is there a real solution to the quadratic form $f(x_1, x_2, x_3) = 1$? Explain why.

Solution 2.
(a)

$$A = \begin{bmatrix} -2 & 0 & 2 \\ 0 & -4 & 0 \\ 2 & 0 & -5 \end{bmatrix}$$

(b) Adding Row 1 to Row 3, giving 3 negative pivots, the matrix is negative definite.
(c)

$$\det(A - \lambda I) = \begin{bmatrix} -2-\lambda & 0 & 2 \\ 0 & -4-\lambda & 0 \\ 2 & 0 & -5-\lambda \end{bmatrix} = -\lambda^3 - 11\lambda^2 - 34\lambda - 24 = -(\lambda+6)(\lambda+4)(\lambda+1)$$

For eigenvalue $\lambda = -1$:

$$\begin{bmatrix} -1 & 0 & 2 \\ 0 & -3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \Rightarrow a_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \Rightarrow q_1 = \begin{bmatrix} 2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix}$$

For eigenvalue $\lambda = -4$:

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix} \Rightarrow a_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow q_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

For eigenvalue $\lambda = -6$:

$$\begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} \Rightarrow a_3 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \Rightarrow q_3 = \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ -2/\sqrt{5} \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & -4 & 0 \\ 2 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \\ 1/\sqrt{5} & 0 & -2/\sqrt{5} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \\ 1/\sqrt{5} & 0 & -2/\sqrt{5} \end{bmatrix}$$

(d) No. Negative definite, $f(x_1, x_2, x_3) \leqslant 0$.

Problem 3. *(2020 Fall Final Exam, 4 marks) Let* $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$. *Find all the singular values of A.*

Solution 3. *The singular values are $\sqrt{2}$ and 2.*