Problem Set 4

Problem 1. *(2020 Fall Midterm - 10 points)Let $E = \{u_1, u_2, u_3\}$ and $F = \{v_1, v_2\}$, where*

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Define the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_2 \\ -x_1 \end{bmatrix}$$

Find the matrix A representing T with respect to the ordered bases E and F.

Solution 1.

Firstly, the matrix A is a 3×2 matrix. $(\mathbb{R}^3 \to \mathbb{R}^2)$

$$T\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + y_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Solving linear equation system to find the corresponding output coordinates:

$$\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

So when we input coordinates $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, *we can get output coordinates* $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

$$T\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ -1 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

So when we input coordinates $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, *we can get output coordinates* $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

$$T\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = -4 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

So when we input coordinates $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, *we can get output coordinates* $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$.

The Final Answer:

$$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 3 \end{bmatrix}$$

Problem 2. *(2020 Fall Midterm - 8 points)Three planes Π_1, Π_2, Π_3 are given by the equations*

$$\Pi_1 : x + y + z = 0$$

$$\Pi_2 : 2x - y + 4z = 0$$

$$\Pi_3 : -x + 2y - z = 0$$

Determine a matrix representative (in the standard basis of $\mathbb{R}^3$) of a linear transformation taking the xy plane to Π_1, the yz plane to Π_2 and the zx plane to Π_3.

Solution 2.

In order to determine the matrix, we need to find the coordinates of basis after transformation. Here, we use the standard basis, we only need to find the destination of the 3 unit vectors.

The unit vector in x axis $(\hat{i})$ goes to the intersection line of Π_1 and Π_3.

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow k_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

The unit vector in y axis $(\hat{j})$ goes to the intersection line of Π_1 and Π_2.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow k_2 \begin{bmatrix} -5 \\ 2 \\ 3 \end{bmatrix}$$

The unit vector in z axis $(\hat{k})$ goes to the intersection line of Π_2 and Π_3.

$$\begin{bmatrix} 2 & -1 & 4 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow k_3 \begin{bmatrix} -7 \\ -2 \\ 3 \end{bmatrix}$$

The Final Answer (for reference):

$$A = \begin{bmatrix} -1 & -5 & -7 \\ 0 & 2 & -2 \\ 1 & 3 & 3 \end{bmatrix}$$

Every column can be multiplied by a constant and the answer is still correct.