Problem Set 10

Problem 1. *Suppose that a 3×3 real symmetric matrix A has the eigenvalues $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 0$. The eigenvectors corresponding to λ_1, λ_2 are $p_1 = (1, 2, 2)^T, p_2 = (2, 1, -2)^T$. Find the matrix A.*

Solution 1. *Real symmetric matrix can be written as $A = Q\Lambda Q^T$.*
We can set the diagonalizing matrix to:

$$Q = \begin{bmatrix} 1/3 & 2/3 & a \\ 2/3 & 1/3 & b \\ 2/3 & -2/3 & c \end{bmatrix}$$

As Q is an orthogonal matrix, we can get:

$$\begin{cases} a + 2b + 2c = 0 \\ 2a + b - 2c = 0 \\ a^2 + b^2 + c^2 = 1 \end{cases}, \begin{cases} a = 2/3 \\ b = -2/3 \\ c = 1/3 \end{cases}$$

Q and λ are all known, so

$$A = Q\Lambda Q^T = \begin{bmatrix} -1/3 & 0 & 2/3 \\ 0 & 1/3 & 2/3 \\ 2/3 & 2/3 & 0 \end{bmatrix}$$

Problem 2. *Find an orthogonal diagonalizing matrix for the following matrix:*

$$A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{bmatrix}$$

Solution 2.

$$\det(A - \lambda I) = (1 - \lambda)(1 - \lambda)(10 - \lambda) = 0, \lambda_1 = \lambda_2 = 1, \lambda_3 = 10$$

For $\lambda = 1$:

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, a_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, a_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda = 10$:

$$\begin{bmatrix} -8 & 2 & -2 \\ 2 & -5 & -4 \\ -2 & -4 & -5 \end{bmatrix} \to \begin{bmatrix} -8 & 2 & -2 \\ 0 & -9/2 & -9.2 \\ 0 & 0 & 0 \end{bmatrix}, a_3 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$$

Do Gram-Schmidt:

$$a_2' = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{5} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 1 \\ 4/5 \end{bmatrix}$$

So, the diagonalizing matrix is:

$$Q = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{2}{\sqrt{45}} & -\frac{1}{3} \\ 0 & \frac{5}{\sqrt{45}} & -\frac{2}{3} \\ \frac{1}{\sqrt{5}} & \frac{4}{\sqrt{45}} & \frac{2}{3} \end{bmatrix}$$

(Note that the first 2 columns can be exchanged and the vector in every column can be reversed.)

Problem 3. *Find a unitary diagonalizing matrix for the following matrix:*

$$A = \begin{bmatrix} 0 & 1 - i \\ 1 + i & 1 \end{bmatrix}$$

Solution 3.

$$\det(A-\lambda I)=\begin{vmatrix}-\lambda & 1-i\\ 1+i & 1-\lambda\end{vmatrix}=\lambda^2-\lambda+2=0,\lambda_1=-1,\lambda_2=2$$

For eigenvalue $\lambda=-1$*:*

$$\begin{bmatrix}1 & 1-i\\ 1+i & 2\end{bmatrix}\to\begin{bmatrix}1 & 1-i\\ 0 & 0\end{bmatrix},x_1=\begin{bmatrix}-1+i\\ 1\end{bmatrix},q_1=\begin{bmatrix}\frac{-1+i}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{bmatrix}$$

For eigenvalue $\lambda=2$*:*

$$\begin{bmatrix}-2 & 1-i\\ 1+i & -1\end{bmatrix}\to\begin{bmatrix}-2 & 1-i\\ 0 & 0\end{bmatrix},x_2=\begin{bmatrix}\frac{1-i}{2}\\ 1\end{bmatrix},q_2=\begin{bmatrix}\frac{1-i}{2\sqrt{3/2}}\\ \frac{1}{\sqrt{3/2}}\end{bmatrix}=\begin{bmatrix}\frac{1-i}{\sqrt{6}}\\ \frac{2}{\sqrt{6}}\end{bmatrix}$$

$$A=\begin{bmatrix}0 & 1-i\\ 1+i & -1\end{bmatrix}=\begin{bmatrix}\frac{-1+i}{\sqrt{3}} & \frac{1-i}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}}\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}\frac{-1+i}{\sqrt{3}} & \frac{1-i}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}}\end{bmatrix}^H=U\Lambda U^H$$