Linear Transformations

Lecture 4

Zhang Ce

Department of Electrical and Electronic Engineering
Southern University of Science and Technology

2022.10.25

Table of Contents

Table of Contents

Last Lecture, We Discuss...

Three parts in last lecture:

1. Linear Independence
   the definition, perspective from solving $Ax = 0$
2. Spanning, Basis and Dimension
   definition of the three concepts and their connections
3. The Four Fundamental Subspaces
   column space, nullspace, row space, left nullspace; the dimension and basis for each subspace; the whole figure for those four subspaces

We also introduced an important types of problems: given a matrix, find the basis and dimension of each fundamental subspace. Let's have a quick review.

Review Example

Consider the matrix A:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 4 & 4 & 0 \\ 2 & 2 & 0 \\ 1 & 8 & 7 \end{bmatrix}$$

Find the dimension and a basis for each of the 4 fundamental subspaces.

Review Example

Answer:
$dim(C(A)) = 2$, $dim(C(A^T)) = 2$, $dim(N(A)) = 1$, $dim(N(A^T)) = 3$.

$$C(A): \begin{bmatrix} 1 \\ 2 \\ 4 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \\ 2 \\ 8 \end{bmatrix} \quad C\left(A^T\right): \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad N(A): \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$N\left(A^T\right): \begin{bmatrix} -4/3 \\ -4/3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2/3 \\ -2/3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Table of Contents

Definition of Transformation

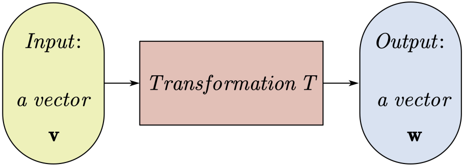
A **function** f from a set X to a set Y is a rule that assigns to each element of X a unique element of Y. Suppose set X and Y are both the set of all vectors in the space, we can treat it as a **transformation**.

Definition

Transformation is a mapping T from the vector space V to the vector space W.

$$\mathbf{v} \in V \xrightarrow{\text{Transformation}} \mathbf{w} = T(\mathbf{v}) \in W$$

Transformations? A function of vectors!



Linear Transfomations

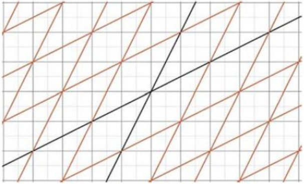
How to determine whether a transformation is linear?

Two rules:

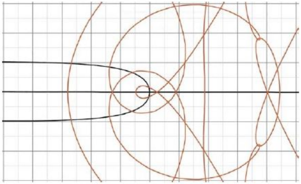
- Origin remains unchanged.
- Straight lines are still straight lines.

Generalize it and express in mathematical languages, a mapping T is a linear transformation if

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. $T(\alpha\mathbf{v}) = \alpha T(\mathbf{v})$



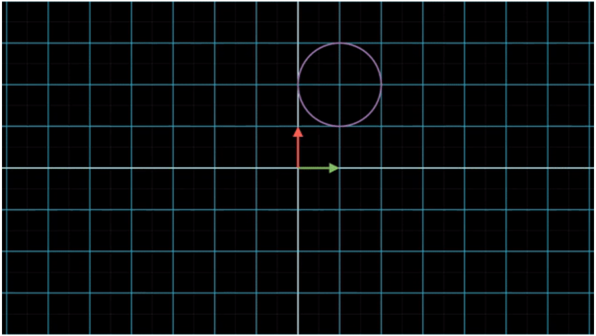
线性变换



非线性变换

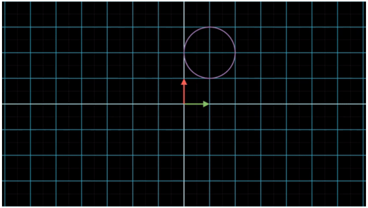
Matrix as Linear Transformation

See this example of Linear Transformation:



To fully express a Linear Transformation, we can track the movement of two random vectors. Since we know that every vector in $\mathbb{R}^2$ space can be expressed by these 2 vectors!

Matrix as Linear Transformation



Track the two unit vectors (the red one and the green one).

Before transformation:

$$\hat{\boldsymbol{i}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \hat{\boldsymbol{j}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

After transformation:

$$T(\hat{\boldsymbol{i}}) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, T(\hat{\boldsymbol{j}}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Then the linear transformation can be expressed by a matrix:

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix},$$

where the first column is the vector $\hat{\boldsymbol{i}}$ after transformation, the second column is the vector $\hat{\boldsymbol{j}}$ after transformation.

Matrix as Linear Transformation

Why we trace the movement of the unit vectors? Reasons?

Recall the definitions of linear transformations.

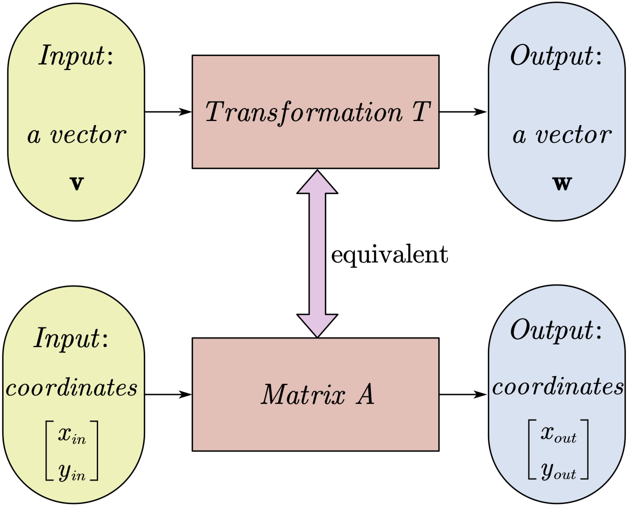
1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. $T(\alpha\mathbf{v}) = \alpha T(\mathbf{v})$

Suppose we have an input vector $\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, the vector after transformation is $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = xT\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + yT\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x_{in} \\ y_{in} \end{bmatrix} \xrightarrow[\textit{Transformation Matrix } \boldsymbol{A}]{\textit{Linear Transformation } \boldsymbol{T}} \begin{bmatrix} x_{out} \\ y_{out} \end{bmatrix}$$

A Summary by Now



Video: `https://www.bilibili.com/video/BV1ys411472E?p=4`

A New Perspective: Understanding Matrices

Now, every time you meet a matrix, try to see it as a linear transformation, imagine the transformation in your mind.

1. Identity matrix: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, do nothing.
2. Permutation matrix: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, exchange axis.
3. Rotation matrix: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, rotate the plane by 90° anti-clockwise.
4. Reflection matrix: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, reflection by x axis.
5. Projection matrix: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, projection on x axis.

Give me a matrix that can rotate the plane by $30\circ$ clockwise.

A New Perspective: Understanding Matrix Multiplication

I have once introduced you the column method of matrix multiplication without explanation, now give me a explanation from the linear transformation perspective.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x + y \\ y \end{bmatrix}$$

If I multiply 2 matrices together (and they can be multiplied), what will we get? Still a matrix! That means it is still a linear transformation!

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

The matrix multiplication process is to find the total linear transformation.

A New Perspective: Understanding Inverses

Can I find an invert transformation, get the coordinates before transformation?

Rotation matrix: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, rotate the whole plane by 90° anti-clockwise.

It is invertible. Show me the inverse of this rotation matrix (which is a matrix that can rotate the whole plane by 90° clockwise definitely).

A matrix: $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$.

We have seen this matrix before... It is not invertible. Try to explain that from the linear transformation perspective.

Many vectors result in a single vector, and I cannot find an invert transformation to transform an input vector to multiple output vectors.

A New Perspective: Understanding Column Space; Rank

A matrix: $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, think about its column space and rank.

The column space is a straight line in $\mathbb{R}^2$ plane, which is also the space consists of all the possible output vectors. The rank is the dimension of the column space, which gives us a description about the dimension of output space.

How about the non-square matrices? Consider the following matrix A.

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 4 & 3 \end{bmatrix}$$

The input is a vector in $\mathbb{R}^2$, output is a vector in $\mathbb{R}^3$. The 2 unit vectors $[1\ 0]^T, [0\ 1]^T$ are transformed to 2 vectors in $\mathbb{R}^3$. Imagine the linear transformation represented by matrix A, it transforms the $\mathbb{R}^2$ plane to a 2-dimensional plane in $\mathbb{R}^3$. The rank (2 in this example) gives us the dimension of the output space.

A New Perspective: Understanding Solvability Condition

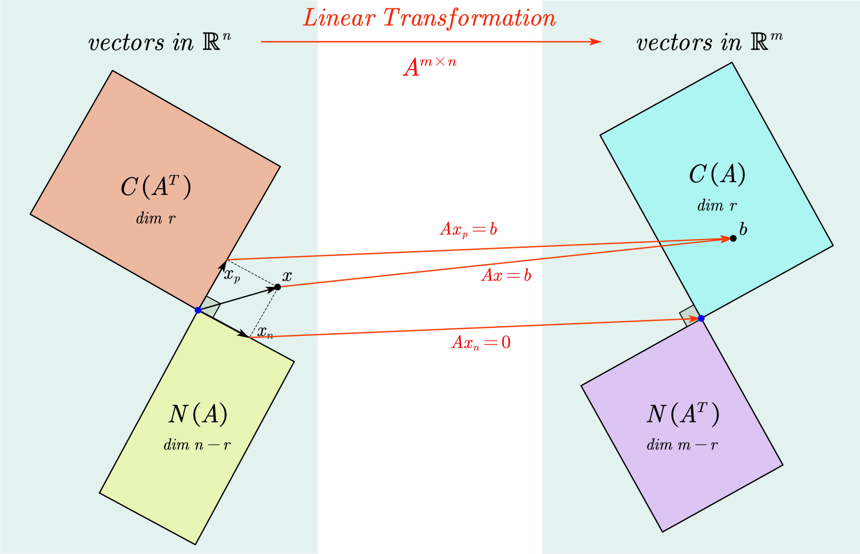
A linear system $Ax = b$ is solvable if and only if b is in column space of A.

The column space of A is the whole output space of the linear transformation represented by A. If a vector b is not in the column space, then we cannot have a solution because no input vectors will result in b after linear transformation.

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

This linear system is definitely inconsistent because transform a vector to a line will not result in a vector not in that line.



The Whole Picture of 4 Fundamental Subspaces of A

Table of Contents

Coordinates and Basis

In previous slides, we always represent vectors by coordinates such as $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}$. That is natural because we use the standard basis! For example:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

But, a space can have infinite bases, if we change the basis, the coordinates will also change. Suppose we choose $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as a basis, then the coordinates becomes $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ because:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

A summary: coordinates come from the choice of basis.

The Transformation Matrix with Other Basis

Our transformation matrix A do the easy thing: input the coordinates before transformation, and output the coordinates after transformation.

Let's now consider the new basis for $\mathbb{R}^2$: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Suppose an input coordinates $\begin{bmatrix} x \\ y \end{bmatrix} \left(x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$, the coordinates after transformation is $T\left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = xT\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) + yT\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) & T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Things get easier now. Every column in transformation matrix A is the output coordinates of an input basis vector.

Example

Let L: $\mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by

$$L(\mathbf{x}) = \begin{bmatrix} x_2 \\ x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$$

Find the matrix representations of L with respect to the ordered bases $\{\mathbf{u_1}, \mathbf{u_2}\}$ and $\{\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}\}$, where

$$\mathbf{u_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

and

$$\mathbf{b_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Example

Solution:

Input coordinates $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, which is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in natural basis.

The output in natural basis will be

$$L\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1+2 \\ 1-2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

Transform to the coordinates under new basis **b**.

$$\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = a_{11}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_{21}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + a_{31}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Output coordinates $\begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}$, which is $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ in natural basis.

Example

Solution:

Input coordinates $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ in natural basis.

The output in natural basis will be

$$L\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3+1 \\ 3-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

Transform to the coordinates under new basis $\mathbf{b}$.

$$\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = a_{12}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_{22}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + a_{32}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Output coordinates $\begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix}$, which is $\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$ in natural basis.