7-Dec-2025

HW 16 P1

$V$

**Problem 1.** 假定线性空间维数有限. 若线性映射 $f$ 在某组基下有矩阵表示 $A$, 则 $A$ 的特征值等于 $f$ 的特征值.

$\exists \ (u_1|u_2|\cdots|u_n)$ s.t.　　按定义

$$(f(u_1)|f(u_2)|\cdots|f(u_n)) = (u_1|u_2|\cdots|u_n)\cdot A$$

下证 $\lambda \in \sigma(A) \Leftrightarrow \lambda \in \sigma(f)$

"$\Rightarrow$" 若 $Av = \lambda v \quad (v \neq 0)$

$$(f(u_1)|f(u_2)|\cdots|f(u_n))v = (u_1|u_2|\cdots|u_n)\cdot Av$$

$$\| \qquad\qquad\qquad \|$$

$$f((u_1|u_2|\cdots|u_n)v) \qquad \lambda(u_1|u_2|\cdots|u_n)v$$

$$\Rightarrow (\lambda, \underbrace{(u_1|u_2|\cdots|u_n)v}_{\neq 0})$$

是 $f$ 的特征对.

"$\Leftarrow$" 若 $f(x) = \lambda x$

即：∃ 唯一的 $y$ s.t.

$$(u_1|u_2|\cdots|u_n)\cdot y = x \quad \#$$

**Problem 2.** 记 $f$ 是整系数的多项式, $p(x)=x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n$ 也是整系数的多项式. 将 $p(x)$ 在 $\mathbb{C}$ 中分解作一次因子的乘积, 即 $(x-x_1)(x-x_2)\cdots(x-x_n)$. 试证明, $(x-f(x_1))(x-f(x_2))\cdots(f-f(x_n))$ 仍是整系数多项式.

Pf. FACT 若 $(\lambda, v)$ 是 $A$ 的特征对.

则 $(f(\lambda), v)$ 是 $f(A)$ 的特征对.

$\Rightarrow$ Cor $\{A$的特征向量$\} \subseteq \{f(A)$的特征向量$\}$

$\Rightarrow$ Prop $A, B$ 可相似对角化, 且

$\exists f(A)=B,\ \exists g(B)=A$

$\Rightarrow \exists P\in GL_n(\mathbb{F})$ s.t.

$P^{-1}AP,\ P^{-1}BP$ 均是对角化阵

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

**Thm** $D$ 是半正定矩阵.

$\exists$ 唯一的半正定矩阵 $\tilde{D}$ s.t.

$$\tilde{D}^2 = D.$$

pf. $\tilde{D}$ 的存在性由 $Q^TDQ = \Lambda$（对角）给出

$= Q^T\sqrt{\Lambda}\,Q$

$\tilde{D}$ 的唯一性?

没有 $\tilde{D}^2 = D = \underset{\sim}{D}^2$

下证 $\tilde{D} = \underset{\sim}{D}$.

**FACT** $\tilde{D} = f(D)$ $(\exists f)$

pf. $\exists f$. $\begin{pmatrix} \sqrt{d_1} & & & \\ & \sqrt{d_2} & & \\ & & \ddots & \\ & & & \sqrt{d_n} \end{pmatrix} = f\left(\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}\right)$

**Thm** (Lagrange) 给"两两不同"的数组

$\{x_1, \ldots, x_n\}$ $x_i \neq x_j$

$\{y_1, \ldots, y_n\}$

$\exists$ 多项式 $f$ s.t. $f(x_k) = y_k$.

$1, 1, 2, 3, 5, 8, 114514$

**Problem 2.** 记 $f$ 是整系数的多项式, $p(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n$ 也是整系数的多项式. 将 $p(x)$ 在 $\mathbb{C}$ 中分解作一次因子的乘积, 即 $(x - x_1)(x - x_2)\cdots(x - x_n)$. 试证明, $(x - f(x_1))(x - f(x_2))\cdots(f - f(x_n))$ 仍是整系数多项式.

$= \det(xI - A)$

$= \det(xI - f(A))$

**FACT** 若 $(\lambda, v)$ 是 $A$ 的特征向量.

则 $(f(\lambda), v)$ 是 $f(A)$ 的特征向量.

Pf $\exists P$ s.t. $P^{-1}AP = \begin{pmatrix} x_1 & & * \\ & x_2 & \\ & \ddots & \\ 0 & & x_n \end{pmatrix}$

$\Rightarrow f(A) \sim f\left(\begin{pmatrix} x_1 & & * \\ & x_2 & \\ & \ddots & \\ 0 & & x_n \end{pmatrix}\right)$

$\sim \begin{pmatrix} f(x_1) & & * \\ & \ddots & \\ 0 & & f(x_n) \end{pmatrix}$

**FACT** $A \in \mathbb{R}^{n\times n}$

若 $x^TAx > 0$ $(x \neq 0)$ P

则 $\lambda \in \sigma(A)$ 有正的实部. ✓

Pf [1] 先证 $\sigma(A) \cap i\mathbb{R} = \phi$

① $0 \notin \sigma(A)$ ✓

② 若 $i \in \sigma(A)$, 则 $-i \in \sigma(A)$.

$\Downarrow$ ?

$\exists u \in \mathbb{C}^n$

$Au = iu$ $A\bar{u} = -i\bar{u}$

$\Rightarrow (Re(u))^T A (Re(u)) = 0$

$(Im(u))^T A (Im(u)) = 0$

矛盾.

[2] A 满足 P 则 $\forall B = -B^T$, $A+B$ 满足 P.

取 $B = t\cdot(A^T - A)$

$A(t) = A + t(A^T - A)$

$A(0) = A$, $A(1/2)$ 实对称且满足 P

$\Rightarrow A(1/2)$ 正定.

8. 若 $B=-B^T$ 则 $I+B$ 可逆, 且 $(I-B)(I+B)^{-1}$ 是正交矩阵.

$$\Leftrightarrow \left((I-B)(I+B)^{-1}\right)^T = \left((I-B)(I+B)^{-1}\right)^{-1}$$

$$\left((I-B)(I+B)^{-1}\right)^T = (I+B^T)^{-1}(I-B^T) \overset{B=-B^T}{=} (I-B)^{-1}\cdot(I+B)$$

$$\left((I-B)(I+B)^{-1}\right)^{-1} = (I+B)(I-B)^{-1}$$

$$A(A^T-A)=0$$

Q $A\in\mathbb{R}^{n\times n}$, $AA^T=AA$

证明 $A=A^T$.

$$A^2=AA^T=(A^T)^2$$

$$tr\left(\underbrace{(A-A^T)^T}_{B^T}\cdot\underbrace{(A-A^T)}_{B}\right)=0 \quad ✓$$

证明 $A=Q\begin{pmatrix} M & N \\ 0 & 0\end{pmatrix}Q^T$ $\quad AA^T=AA$

$$\begin{pmatrix} MM^T+NN^T & 0 \\ 0 & 0\end{pmatrix}$$

$$\begin{pmatrix} M^2 & MN \\ 0 & 0\end{pmatrix}$$

e.g. $A: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$v \longmapsto Av$

$\det(M^2) = \underbrace{\det(MM^T + NN^T)}_{\neq 0}$

$C(A)$ 为映射的结...

存在 $\underbrace{C(A)}_{v_1, \cdots, v_k}$, $\underbrace{C(A)^{\perp}}_{v_{k+1}, \cdots, v_n}$ 的标准正交基.

$$\Rightarrow (Av_1 | Av_2 | \cdots | Av_n) = \underbrace{(v_1 | v_2 | \cdots | v_n)}_{Q^T} \left(\begin{array}{c} * \\ \hline 0 \end{array}\right) \Big\} k$$

Ex $A \in \mathbb{R}^{m \times n} \Rightarrow A^T A = Q \Lambda Q^T$

$\Leftrightarrow (AQ)^T (AQ) = \Lambda$

$\boxed{\Leftrightarrow AQ \text{ 的不同列彼此正交}}$

那如 $W \cdot \begin{pmatrix} \mu_1 & & & \\ & \ddots & & 0 \\ & & \mu_k & \\ 0 & & & \ddots \end{pmatrix}$

$P^{-1}AP = B$ $M^T A M = B$

Q: $A, B$ 既相似, 又合同

是否存在正交矩阵 $Q$ s.t. $Q^T A Q = B$.

Thm $\exists Q$ 正交. $QAQ^T = B$

$\overset{\checkmark}{\Longrightarrow}$ $\Longleftrightarrow$ $\overset{?}{\Longleftarrow}$

$$\begin{array}{ccc} A & \longrightarrow & B \\ A^T & \longrightarrow & B^T \end{array}$$

可以同步地相似

$\exists P$ $\begin{cases} PAP^{-1} = B, \\ PA^TP^{-1} = B^T. \end{cases}$

正交 半正定 (极分解)

$\overset{pf}{\Longleftarrow}$ 令 $P = QS$

$\Rightarrow \begin{cases} QSAS^{-1}Q^T = B \\ QSA^TS^{-1}Q^T = B^T \end{cases}$ $QAQ^T = B$ ✓

$\Rightarrow Q(S^{-1})A^TSQ^T = QSA^TS^{-1}Q^T$

$\Rightarrow S^{-1}A^TS = SA^TS^{-1}$

$\Rightarrow AS^2 = S^2A \xRightarrow[S=f(S^2)]{} AS = SA$

$\dim V_1 = 3$, $\ker N(I-A) = 2$

$$
\begin{array}{ccc}
C(I-A) & & V_2 \\
0 \leftarrow v_1 & \leftarrow & v_2 \\
0 & \leftarrow & v_3
\end{array}
$$

$$\dim\left[C(I-A)\cap V_2\right] = 1$$

Ex $A = \begin{pmatrix} 3 & 4 & 1 \\ 1 & -4 & 3 \end{pmatrix}$, 找 $U\Sigma V^T = A$

pf. (Step 1) 找 $A^TA$ 或 $AA^T$ 的正交相对角化

$$AA^T = \begin{pmatrix} 26 & -10 \\ -10 & 26 \end{pmatrix}$$

$(\lambda-26)^2 = 100$

$\lambda = 16, 36.$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 16 & 0 \\ 0 & 36 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$$

$$= \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}}_{U}\begin{pmatrix} 16 & 0 \\ 0 & 36 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}^{-1}$$

$$\Sigma = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \end{pmatrix}$$

(Step 2) 找 $V$

回忆：$UA$ 是列正交的，即

形如 $\Sigma \cdot V^T$

$$UA=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}3&4&1\\1&-4&3\end{pmatrix}$$

$$=\frac{1}{\sqrt{2}}\underline{\begin{pmatrix}4&0&4\\2&8&-2\end{pmatrix}}$$

$$=\underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}4\sqrt{2}&0&0\\0&6\sqrt{2}&0\end{pmatrix}}_{\Sigma}\underbrace{\begin{pmatrix}\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\\\frac{1}{3\sqrt{2}}&\frac{4}{3\sqrt{2}}&\frac{-1}{3\sqrt{2}}\\\frac{2}{3}&\frac{-1}{3}&\frac{-2}{3}\end{pmatrix}}_{V^T}$$

$\underline{Q}$ $A\in\mathbb{C}^{n\times n}$ Jordan - Chevalley

$\exists$ 唯一的 $N$ 幂零

$D$ 可对角化

s.t. $\begin{cases}A=N+D\\ND=DN\end{cases}$,

$\underline{Q}$ $AX-XA=X$，证 $X^n=0$