Complete cohomology for extriangulated categories

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Abstract

Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category with a proper class ξ of $\mathbb{E}$-triangles. In this paper, we study complete cohomology of objects in $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ by applying ξ-projective resolutions and ξ-injective coresolutions constructed in $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$. Vanishing of complete cohomology detects objects with finite ξ-projective dimension and finite ξ-injective dimension. As a consequence, we obtain some criteria for the validity of the Wakamatsu Tilting Conjecture and give a necessary and sufficient condition for a virtually Gorenstein algebra to be Gorenstein. Moreover, we give a general technique for computing complete cohomology of objects with finite ξ-$\mathcal{G}$projective dimension. As an application, the relationships between ξ-projective dimensions and ξ-$\mathcal{G}$projective dimensions for objects in $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ are given.

Keywords: complete cohomology; extriangulated category; proper class; Gorenstein projective dimension.

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1. Introduction

Tate cohomology was established in the 1950s, based on Tate's observation that the $\mathbb{Z}G$-module $\mathbb{Z}$ with the trivial action admits a complete projective resolution [13]. It was further extended by Avramov and Martsinkovsky [6] to finitely generated modules of finite Gorenstein dimension over Noetherian rings and by Veliche [32] to complexes of finite Gorenstein projective dimension. On the other hand, Mislin [28], Benson and Carlson [10] and Vogel (first published account in [20]), independently, generalized the theory to arbitrary groups. It was shown that these theories are isomorphic, and complete cohomology is a common name for them. The theory has also been extended to the setting of unbounded complexes by Asadollahi and Salarian [4].

Beligiannis developed in [7] a relative version of homological algebra in triangulated categories in analogy to relative homological algebra in abelian categories, in which the notion of a proper class of exact sequences is replaced by a proper class of triangles. By specifying a class of triangles ξ, which is called a proper class of triangles, he introduced ξ-projective and ξ-injective objects. In an attempt to extend the theory, Asadollahi and Salarian [2] introduced

and studied ξ-Gorenstein projective, injective objects, and corresponding ξ-Gorenstein dimensions in triangulated categories by modifying what Enochs, Jenda [18] and Holm [22] had done in the category of modules. Motivated by the properties of Tate cohomology in the category of groups, Asadollahi and Salarian [3] developed a Tate cohomology theory in a triangulated category $\mathcal{C}$ with a proper class of triangles, which was done for objects of finite ξ-Gorenstein projective dimension. They showed that this theory not only shares basic properties with ordinary cohomology, but also enjoys some distinctive features. We refer to [30, 31] for a more discussion on this matter.

The notion of extriangulated categories was introduced by Nakaoka and Palu in [29] as a simultaneous generalization of exact categories and triangulated categories. Exact categories and extension closed subcategories of an extriangulated category are extriangulated categories, while there exist some other examples of extriangulated categories which are neither exact nor triangulated, see [23, 29, 38]. Hence many results on exact categories and triangulated categories can be unified in the same framework.

Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category with a proper class ξ of $\mathbb{E}$-triangles. The authors [23] studied a relative homological algebra in $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ which parallels the relative homological algebra in a triangulated category. By specifying a class of $\mathbb{E}$-triangles, which is called a proper class ξ of $\mathbb{E}$-triangles, the authors introduced ξ-projective dimensions and ξ-$\mathcal{G}$projective dimensions, and discussed their properties.

In this paper we attempt to develop a complete cohomology theory in an extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ and demonstrate that this theory shares some basic properties of complete cohomology groups in the category of modules category and Tate cohomology groups in the triangulated category.

We now outline the results of the paper. In Section 2, we summarize some preliminaries and basic facts about extriangulated categories which will be used throughout the paper.

In Section 3, for a given extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ with a proper class ξ of $\mathbb{E}$-triangles, we define ξ-complete cohomology groups based on ξ-projective resolutions and ξ-injective coresolutions of objects in $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ (see Definition 3.4). It is proved that vanishing of ξ-complete cohomology characterizes the finiteness of ξ-projective dimension and ξ-injective dimension of objects in $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ (see Theorem 3.10). As consequences, we give some criteria for the validity of the Wakamatsu Tilting Conjecture; moreover, we give a necessary and sufficient condition for a virtually Gorenstein algebra to be Gorenstein (see Corollaries 3.12 and 3.13).

In Section 4, we show first that an object M in an extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ has finite ξ-$\mathcal{G}$projective dimension if and only if M admits a (split) ξ-complete resolution (see Proposition 4.4). This lets us give a general technique for computing ξ-complete cohomology of objects with finite ξ-$\mathcal{G}$projective dimension (see Theorem 4.6). As an application, the relationships between ξ-projective dimensions and ξ-$\mathcal{G}$projective dimensions for objects in $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ are given (see Corollary 4.8).

2. Preliminaries

Throughout this paper, we always assume that $\mathcal{C} = (\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an extriangulated category and ξ is a proper class of $\mathbb{E}$-triangles in $\mathcal{C}$. We also assume that the extriangulated category $\mathcal{C}$

has enough ξ-projectives and enough ξ-injectives satisfying Condition (WIC). Next we briefly recall some definitions and basic properties of extriangulated categories from [29]. We omit some details here, but the reader can find them in [29].

Let $\mathcal{C}$ be an additive category equipped with an additive bifunctor

$$\mathbb{E} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Ab},$$

where Ab is the category of abelian groups. For any objects $A, C \in \mathcal{C}$, an element $\delta \in \mathbb{E}(C, A)$ is called an $\mathbb{E}$-*extension*. Let $\mathfrak{s}$ be a correspondence which associates an equivalence class

$$\mathfrak{s}(\delta) = \ [A \xrightarrow{\ x\ } B \xrightarrow{\ y\ } C]$$

to any $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$. This $\mathfrak{s}$ is called a *realization* of $\mathbb{E}$, if it makes the diagrams in [29, Definition 2.9] commutative. A triplet $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is called an *extriangulated category* if it satisfies the following conditions.

(1) $\mathbb{E} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Ab}$ is an additive bifunctor.

(2) $\mathfrak{s}$ is an additive realization of $\mathbb{E}$.

(3) $\mathbb{E}$ and $\mathfrak{s}$ satisfy the compatibility conditions in [29, Definition 2.12].

Remark 2.1. *Note that both exact categories and triangulated categories are extriangulated categories (see [29, Example 2.13]) and extension closed subcategories of extriangulated categories are again extriangulated (see [29, Remark 2.18]). Moreover, there exist extriangulated categories which are neither exact categories nor triangulated categories (see [29, Proposition 3.30], [38, Example 4.14] and [23, Remark 3.3]).*

We will use the following terminology.

Definition 2.2. (see [29, Definitions 2.15 and 2.19]) Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category.

(1) A sequence $\ A \xrightarrow{\ x\ } B \xrightarrow{\ y\ } C\ $ is called a *conflation* if it realizes some $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$. In this case, x is called an *inflation* and y is called a *deflation*.

(2) If a conflation $A \xrightarrow{\ x\ } B \xrightarrow{\ y\ } C\ $ realizes $\delta \in \mathbb{E}(C, A)$, we call the pair $(A \xrightarrow{\ x\ } B \xrightarrow{\ y\ } C, \delta)$ an $\mathbb{E}$-*triangle*, and write it in the following.

$$A \xrightarrow{\ x\ } B \xrightarrow{\ y\ } C \dashrightarrow^{\delta}$$

We usually do not write this "δ" if it is not used in the argument.

(3) Let $\ A \xrightarrow{\ x\ } B \xrightarrow{\ y\ } C \dashrightarrow^{\delta}\ $ and $\ A' \xrightarrow{\ x'\ } B' \xrightarrow{\ y'\ } C' \dashrightarrow^{\delta'}\ $ be any pair of $\mathbb{E}$-triangles. If a triplet (a, b, c) realizes $(a, c) : \delta \to \delta'$, then we write it as

$$
\begin{array}{ccccc}
A & \xrightarrow{\ x\ } & B & \xrightarrow{\ y\ } & C \dashrightarrow^{\delta} \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\
A' & \xrightarrow{\ x'\ } & B' & \xrightarrow{\ y'\ } & C' \dashrightarrow^{\delta'}
\end{array}
$$

and call (a, b, c) a *morphism* of $\mathbb{E}$-triangles.

The following condition is analogous to the weak idempotent completeness in exact category (see [29, Condition 5.8]).

Condition 2.3. (Condition (WIC)) *Consider the following conditions.*

    (1) Let $f \in \mathcal{C}(A, B), g \in \mathcal{C}(B, C)$ be any composable pair of morphisms. If gf is an inflation, then so is f.

    (2) Let $f \in \mathcal{C}(A, B), g \in \mathcal{C}(B, C)$ be any composable pair of morphisms. If gf is a deflation, then so is g.

Example 2.4. *(1) If $\mathcal{C}$ is an exact category, then Condition* (WIC) *is equivalent to $\mathcal{C}$ is weakly idempotent complete (see* [12, *Proposition 7.6]).*

    (2) If $\mathcal{C}$ is a triangulated category, then Condition (WIC) *is automatically satisfied.*

Lemma 2.5. (see [29, *Proposition 3.15]*) *Assume that $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an extriangulated category. Let C be any object, and let $A_1 \xrightarrow{x_1} B_1 \xrightarrow{y_1} C \overset{\delta_1}{\dashrightarrow}$ and $A_2 \xrightarrow{x_2} B_2 \xrightarrow{y_2} C \overset{\delta_2}{\dashrightarrow}$ be any pair of $\mathbb{E}$-triangles. Then there is a commutative diagram in $\mathcal{C}$*

$$
\begin{array}{ccc}
A_2 & =\!=\!= & A_2 \\
\Big\downarrow{\scriptstyle m_2} & & \Big\downarrow{\scriptstyle x_2} \\
A_1 \xrightarrow{\ m_1\ } M & \xrightarrow{\ e_1\ } & B_2 \\
\Big\| \quad \Big\downarrow{\scriptstyle e_2} & & \Big\downarrow{\scriptstyle y_2} \\
A_1 \xrightarrow{\ x_1\ } B_1 & \xrightarrow{\ y_1\ } & C
\end{array}
$$

which satisfies $\mathfrak{s}(y_2^*\delta_1) = [A_1 \xrightarrow{m_1} M \xrightarrow{e_1} B_2]$ and $\mathfrak{s}(y_1^*\delta_2) = [A_2 \xrightarrow{m_2} M \xrightarrow{e_2} B_1]$.

    The following definitions are quoted verbatim from [23, Section 3]. A class of $\mathbb{E}$-triangles ξ is *closed under base change* if for any $\mathbb{E}$-triangle

$$A \xrightarrow{\ x\ } B \xrightarrow{\ y\ } C \overset{\delta}{\dashrightarrow} \; \in \xi$$

and any morphism $c \colon C' \to C$, then any $\mathbb{E}$-triangle $A \xrightarrow{x'} B' \xrightarrow{y'} C' \overset{c^*\delta}{\dashrightarrow}$ belongs to ξ.

    Dually, a class of $\mathbb{E}$-triangles ξ is *closed under cobase change* if for any $\mathbb{E}$-triangle

$$A \xrightarrow{\ x\ } B \xrightarrow{\ y\ } C \overset{\delta}{\dashrightarrow} \; \in \xi$$

and any morphism $a \colon A \to A'$, then any $\mathbb{E}$-triangle $A' \xrightarrow{x'} B' \xrightarrow{y'} C \overset{a_*\delta}{\dashrightarrow}$ belongs to ξ.

    A class of $\mathbb{E}$-triangles ξ is called *saturated* if in the situation of Lemma 2.5, whenever

$$A_2 \xrightarrow{x_2} B_2 \xrightarrow{y_2} C \overset{\delta_2}{\dashrightarrow} \quad \text{and} \quad A_1 \xrightarrow{m_1} M \xrightarrow{e_1} B_2 \overset{y_2^*\delta_1}{\dashrightarrow} \quad \text{belong to } \xi, \text{ then the } \mathbb{E}\text{-triangle}$$

$$A_1 \xrightarrow{x_1} B_1 \xrightarrow{y_1} C \overset{\delta_1}{\dashrightarrow}$$

belongs to ξ.

    An $\mathbb{E}$-triangle $A \xrightarrow{\ x\ } B \xrightarrow{\ y\ } C \overset{\delta}{\dashrightarrow}$ is called *split* if $\delta = 0$. It is easy to see that it is split if and only if x is section or y is retraction. The full subcategory consisting of the split $\mathbb{E}$-triangles will be denoted by Δ_0.

Definition 2.6. (see [23, Definition 3.1]) Let ξ be a class of $\mathbb{E}$-triangles which is closed under isomorphisms. Then ξ is called a *proper class* of $\mathbb{E}$-triangles if the following conditions hold:

(1) ξ is closed under finite coproducts and $\Delta_0 \subseteq \xi$.

(2) ξ is closed under base change and cobase change.

(3) ξ is saturated.

Definition 2.7. (see [23, Definition 4.1]) An object $P \in \mathcal{C}$ is called ξ-*projective* if for any $\mathbb{E}$-triangle

$$A \xrightarrow{x} B \xrightarrow{y} C \xdashrightarrow{\delta}$$

in ξ, the induced sequence of abelian groups $0 \longrightarrow \mathcal{C}(P,A) \longrightarrow \mathcal{C}(P,B) \longrightarrow \mathcal{C}(P,C) \longrightarrow 0$ is exact. Dually, we have the definition of ξ-*injective*.

We denote by $\mathcal{P}(\xi)$ (resp., $\mathcal{I}(\xi)$) the class of ξ-projective (resp., ξ-injective) objects of $\mathcal{C}$. It follows from the definition that the subcategories $\mathcal{P}(\xi)$ and $\mathcal{I}(\xi)$ are full, additive, closed under isomorphisms and direct summands.

An extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is said to have *enough ξ-projectives* (resp., *enough ξ-injectives*) provided that for each object A there exists an $\mathbb{E}$-triangle $K \longrightarrow P \longrightarrow A \dashrightarrow$ (resp., $A \longrightarrow I \longrightarrow K \dashrightarrow$) in ξ with $P \in \mathcal{P}(\xi)$ (resp., $I \in \mathcal{I}(\xi)$).

The ξ-*projective dimension* ξ-pd$A$ of $A \in \mathcal{C}$ is defined inductively. If $A \in \mathcal{P}(\xi)$, then define ξ-pd$A = 0$. Next if ξ-pd$A > 0$, define ξ-pd$A \leq n$ if there exists an $\mathbb{E}$-triangle $K \to P \to A \dashrightarrow$ in ξ with $P \in \mathcal{P}(\xi)$ and ξ-pd$K \leq n-1$. Finally we define ξ-pd$A = n$ if ξ-pd$A \leq n$ and ξ-pd$A \nleq n-1$. Of course we set ξ-pd$A = \infty$, if ξ-pd$A \neq n$ for all $n \geq 0$.

Dually we can define the ξ-*injective dimension* ξ-id$A$ of an object $A \in \mathcal{C}$.

Definition 2.8. (see [23, Definition 4.4]) A ξ-*exact* complex $\mathbf{X}$ is a diagram

$$\cdots \longrightarrow X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} X_{-1} \longrightarrow \cdots$$

in $\mathcal{C}$ such that for each integer n, there exists an $\mathbb{E}$-triangle $K_{n+1} \xrightarrow{g_n} X_n \xrightarrow{f_n} K_n \xdashrightarrow{\delta_n}$ in ξ and $d_n = g_{n-1}f_n$.

Definition 2.9. (see [23, Definition 4.5]) Let $\mathcal{W}$ be a class of objects in $\mathcal{C}$. An $\mathbb{E}$-triangle

$$A \longrightarrow B \longrightarrow C \dashrightarrow$$

in ξ is called to be $\mathcal{C}(-, \mathcal{W})$-*exact* (resp., $\mathcal{C}(\mathcal{W}, -)$-*exact*) if for any $W \in \mathcal{W}$, the induced sequence of abelian groups $0 \longrightarrow \mathcal{C}(C,W) \longrightarrow \mathcal{C}(B,W) \longrightarrow \mathcal{C}(A,W) \longrightarrow 0$ (resp.,

$0 \longrightarrow \mathcal{C}(W,A) \longrightarrow \mathcal{C}(W,B) \longrightarrow \mathcal{C}(W,C) \longrightarrow 0$) is exact in Ab.

Definition 2.10. (see [23, Definition 4.6]) Let $\mathcal{W}$ be a class of objects in $\mathcal{C}$. A complex $\mathbf{X}$ is called $\mathcal{C}(-, \mathcal{W})$-*exact* (resp., $\mathcal{C}(\mathcal{W}, -)$-*exact*) if it is a ξ-exact complex

$$\cdots \longrightarrow X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} X_{-1} \longrightarrow \cdots$$

in $\mathcal{C}$ such that there is a $\mathcal{C}(-, \mathcal{W})$-exact (resp., $\mathcal{C}(\mathcal{W}, -)$-exact) $\mathbb{E}$-triangle

$$K_{n+1} \xrightarrow{g_n} X_n \xrightarrow{f_n} K_n \xdashrightarrow{\delta_n}$$

in ξ for each integer n and $d_n = g_{n-1}f_n$.

A ξ-exact complex $\mathbf{X}$ is called *complete $\mathcal{P}(\xi)$-exact* (resp., *complete $\mathcal{I}(\xi)$-exact*) if it is $\mathcal{C}(-, \mathcal{P}(\xi))$-exact (resp., $\mathcal{C}(\mathcal{I}(\xi), -)$-exact).

Definition 2.11. (see [23, Definition 4.7]) A *complete ξ-projective resolution* is a complete $\mathcal{P}(\xi)$-exact complex

$$\mathbf{P} : \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} P_{-1} \longrightarrow \cdots$$

in $\mathcal{C}$ such that P_n is ξ-projective for each integer n. Dually, a *complete ξ-injective coresolution* is a complete $\mathcal{I}(\xi)$-exact complex

$$\mathbf{I} : \cdots \longrightarrow I_1 \xrightarrow{d_1} I_0 \xrightarrow{d_0} I_{-1} \longrightarrow \cdots$$

in $\mathcal{C}$ such that I_n is ξ-injective for each integer n.

Definition 2.12. (see [23, Definition 4.8]) Let $\mathbf{P}$ be a complete ξ-projective resolution in $\mathcal{C}$. So for each integer n, there exists a $\mathcal{C}(-, \mathcal{P}(\xi))$-exact $\mathbb{E}$-triangle $\quad K_{n+1} \xrightarrow{g_n} P_n \xrightarrow{f_n} K_n \dashrightarrow^{\delta_n} \quad$ in ξ. The objects K_n are called *ξ-$\mathcal{G}$projective* for each integer n. Dually if $\mathbf{I}$ is a complete ξ-injective coresolution in $\mathcal{C}$, there exists a $\mathcal{C}(\mathcal{I}(\xi), -)$-exact $\mathbb{E}$-triangle $\quad K_{n+1} \xrightarrow{g_n} I_n \xrightarrow{f_n} K_n \dashrightarrow^{\delta_n} \quad$ in ξ for each integer n. The objects K_n are called *ξ-$\mathcal{G}$injective* for each integer n.

We denote by $\mathcal{GP}(\xi)$ (resp., $\mathcal{GI}(\xi)$) the class of ξ-$\mathcal{G}$projective (resp., ξ-$\mathcal{G}$injective) objects. It is obvious that $\mathcal{P}(\xi) \subseteq \mathcal{GP}(\xi)$ and $\mathcal{I}(\xi) \subseteq \mathcal{GI}(\xi)$.

3. ξ-complete cohomology

In this section, we study ξ-complete cohomology groups of objects in an extriangulated category $\mathcal{C}$. At first, we need the following definition.

Definition 3.1. (see [24, Definition 3.1]) Let M be an object in $\mathcal{C}$. A *ξ-projective resolution* of M is a ξ-exact complex $\mathbf{P} \to M$ such that $\mathbf{P}_n \in \mathcal{P}(\xi)$ for all $n \geq 0$. Dually, a *ξ-injective coresolution* of M is a ξ-exact complex $M \to \mathbf{I}$ such that $\mathbf{I}_n \in \mathcal{I}(\xi)$ for all $n \leq 0$.

Using standard arguments from relative homological algebra, one can prove the following version of the comparison theorem for ξ-projective resolutions (resp., ξ-injective coresolutions).

Lemma 3.2. *Let M and N be objects in $\mathcal{C}$.*

(1) *If $f_M : \mathbf{P}_M \longrightarrow M$ and $f_N : \mathbf{P}_N \longrightarrow N$ are ξ-projective resolutions of M and N, respectively, then for each morphism $\mu : M \to N$, there exists a morphism $\widetilde{\mu}$, unique up to homotopy, making the following diagram*

$$\begin{array}{ccc} \mathbf{P}_M & \xrightarrow{f_M} & M \\ {\scriptstyle \widetilde{\mu}} \downarrow & & \downarrow {\scriptstyle \mu} \\ \mathbf{P}_N & \xrightarrow{f_N} & N \end{array}$$

commute. If μ is an isomorphism, then $\widetilde{\mu}$ is a homotopy equivalence.

(2) If $g_M : M \longrightarrow \mathbf{I}_M$ and $g_N : N \longrightarrow \mathbf{I}_N$ are ξ-injective coresolutions of M and N, respectively, then for each morphism $\nu : M \to N$, there exists a morphism $\widetilde{\nu}$, unique up to homotopy, making the following diagram

$$
\begin{array}{ccc}
M & \xrightarrow{g_M} & \mathbf{I}_M \\
\nu \downarrow & & \downarrow \widetilde{\nu} \\
N & \xrightarrow{g_N} & \mathbf{I}_N
\end{array}
$$

commute. If ν is an isomorphism, then $\widetilde{\nu}$ is a homotopy equivalence.

We denote by $\mathrm{Ch}(\mathcal{C})$ the category of complexes in $\mathcal{C}$; the objects are complexes and morphisms are chain maps. We write the complexes homologically, so an object $\mathbf{X}$ of $\mathrm{Ch}(\mathcal{C})$ is of the form

$$
\mathbf{X} := \cdots \longrightarrow X_{n+1} \xrightarrow{d_{n+1}^{\mathbf{X}}} X_n \xrightarrow{d_n^{\mathbf{X}}} X_{n-1} \longrightarrow \cdots .
$$

Assume that $\mathbf{X}$ and $\mathbf{Y}$ are complexes in $\mathrm{Ch}(\mathcal{C})$. A homomorphism $\varphi : \mathbf{X} \longrightarrow \mathbf{Y}$ of degree n is a family $(\varphi_i)_{i \in \mathbb{Z}}$ of morphisms $\varphi_i : X_i \longrightarrow Y_{i+n}$ for all $i \in \mathbb{Z}$. In this case, we set $|\varphi| = n$. All such homomorphisms form an abelian group, denoted by $\mathcal{C}(\mathbf{X}, \mathbf{Y})_n$, which is identified with $\prod_{i \in \mathbb{Z}} \mathcal{C}(X_i, Y_{i+n})$. We let $\mathcal{C}(\mathbf{X}, \mathbf{Y})$ be the complex of abelian groups with nth component $\mathcal{C}(\mathbf{X}, \mathbf{Y})_n$ and differential $d(\varphi_i) = d_{i+n}^{\mathbf{Y}} \varphi_i - (-1)^n \varphi_{i-1} d_i^{\mathbf{X}}$ for $\varphi = (\varphi_i) \in \mathcal{C}(\mathbf{X}, \mathbf{Y})_n$. We refer to [5] for more details.

Remark 3.3. *Let M and N be objects in $\mathcal{C}$.*

(1) *Note that there are two ξ-projective resolutions $\mathbf{P}_M \longrightarrow M$ and $\mathbf{P}_N \longrightarrow N$ of M and N, respectively. A homomorphism $\beta \in \mathcal{C}(\mathbf{P}_M, \mathbf{P}_N)$ is bounded above if $\beta_i = 0$ for all $i \gg 0$. The subset $\overline{\mathcal{C}}(\mathbf{P}_M, \mathbf{P}_N)$, consisting of all bounded above homomorphisms, is a subcomplex with components*

$$
\overline{\mathcal{C}}(\mathbf{P}_M, \mathbf{P}_N)_n = \{(\varphi_i) \in \mathcal{C}(\mathbf{P}_M, \mathbf{P}_N)_n \mid \varphi_i = 0 \text{ for all } i \gg 0\}.
$$

We set

$$
\widetilde{\mathcal{C}}(\mathbf{P}_M, \mathbf{P}_N) = \mathcal{C}(\mathbf{P}_M, \mathbf{P}_N)/\overline{\mathcal{C}}(\mathbf{P}_M, \mathbf{P}_N).
$$

(2) *Note that there are two ξ-injective coresolutions $M \longrightarrow \mathbf{I}_M$ and $N \longrightarrow \mathbf{I}_N$ of M and N, respectively. A homomorphism $\beta \in \mathcal{C}(\mathbf{I}_M, \mathbf{I}_N)$ is bounded below if $\beta_i = 0$ for all $i \ll 0$. The subset $\underline{\mathcal{C}}(\mathbf{I}_M, \mathbf{I}_N)$, consisting of all bounded below homomorphisms, is a subcomplex with components*

$$
\underline{\mathcal{C}}(\mathbf{I}_M, \mathbf{I}_N)_n = \{(\varphi_i) \in \mathcal{C}(\mathbf{I}_M, \mathbf{I}_N)_n \mid \varphi_i = 0 \text{ for all } i \ll 0\}.
$$

We set

$$
\widetilde{\mathcal{C}}(\mathbf{I}_M, \mathbf{I}_N) = \mathcal{C}(\mathbf{I}_M, \mathbf{I}_N)/\underline{\mathcal{C}}(\mathbf{I}_M, \mathbf{I}_N).
$$

Definition 3.4. Let M and N be objects in $\mathcal{C}$, and let n be an integer.

(1) Using ξ-projective resolutions, we define the nth ξ-*complete cohomology group*, denoted by $\widetilde{\xi\mathrm{xt}}_{\mathcal{P}}^n(M, N)$, as

$$
\widetilde{\xi\mathrm{xt}}_{\mathcal{P}}^n(M, N) = H^n(\widetilde{\mathcal{C}}(\mathbf{P}_M, \mathbf{P}_N)),
$$

where $\widetilde{\mathcal{C}}(\mathbf{P}_M, \mathbf{P}_N)$ is the complex defined in Remark 3.3(1).

(2) Using ξ-injective coresolutions, we define the nth ξ-*complete cohomology group*, denoted by $\widetilde{\xi\mathrm{xt}}_{\mathcal{I}}^{n}(M, N)$, as

$$\widetilde{\xi\mathrm{xt}}_{\mathcal{I}}^{n}(M, N) = H^n(\widetilde{\mathcal{C}}(\mathbf{I}_M, \mathbf{I}_N)),$$

where $\widetilde{\mathcal{C}}(\mathbf{I}_M, \mathbf{I}_N)$ is the complex defined in Remark 3.3(2).

Remark 3.5. *By Lemma 3.2, one can see that $\widetilde{\xi\mathrm{xt}}_{\mathcal{P}}^{n}(-, -)$ and $\widetilde{\xi\mathrm{xt}}_{\mathcal{I}}^{n}(-, -)$ are cohomological functors for any integer $n \in \mathbb{Z}$, independent of the choice of ξ-projective resolutions and ξ-injective coresolutions, respectively.*

Definition 3.6. (see [24, Definition 3.2]) Let M and N be objects in $\mathcal{C}$.

(1) If we choose a ξ-projective resolution $\mathbf{P} \longrightarrow M$ of M, then for any integer $n \geq 0$, the ξ-*cohomology groups* $\xi\mathrm{xt}_{\mathcal{P}(\xi)}^{n}(M, N)$ are defined as

$$\xi\mathrm{xt}_{\mathcal{P}(\xi)}^{n}(M, N) = H^n(\mathcal{C}(\mathbf{P}, N)).$$

(2) If we choose a ξ-injective coresolution $N \longrightarrow \mathbf{I}$ of N, then for any integer $n \geq 0$, the ξ-*cohomology groups* $\xi\mathrm{xt}_{\mathcal{I}(\xi)}^{n}(M, N)$ are defined as

$$\xi\mathrm{xt}_{\mathcal{I}(\xi)}^{n}(M, N) = H^n(\mathcal{C}(M, \mathbf{I})).$$

Remark 3.7. *By Lemma 3.2, one can see that $\xi\mathrm{xt}_{\mathcal{P}(\xi)}^{n}(-, -)$ and $\xi\mathrm{xt}_{\mathcal{I}(\xi)}^{n}(-, -)$ are cohomological functors for any integer $n \geq 0$, independent of the choice of ξ-projective resolutions and ξ-injective coresolutions, respectively. In fact, with the modifications of the usual proof, one obtains the isomorphism $\xi\mathrm{xt}_{\mathcal{P}(\xi)}^{n}(M, N) \cong \xi\mathrm{xt}_{\mathcal{I}(\xi)}^{n}(M, N)$, which is denoted by $\xi\mathrm{xt}_{\xi}^{n}(M, N)$.*

Lemma 3.8. *Let M and N be objects in $\mathcal{C}$.*

(1) If $f : \mathbf{P} \longrightarrow M$ and $g : \mathbf{Q} \longrightarrow N$ are ξ-projective resolutions of M and N, respectively, then we have

$$\xi\mathrm{xt}_{\xi}^{n}(M, N) = H^n(\mathcal{C}(\mathbf{P}, \mathbf{Q}))$$

for each integer $n \geq 1$.

(2) If $f' : M \longrightarrow \mathbf{I}$ and $g' : N \longrightarrow \mathbf{J}$ are ξ-injective coresolutions of M and N, respectively, then we have

$$\xi\mathrm{xt}_{\xi}^{n}(M, N) = H^n(\mathcal{C}(\mathbf{I}, \mathbf{J}))$$

for each integer $n \geq 1$.

Proof. We only prove (1) and the proof of (2) is similar. Since $g : \mathbf{Q} \longrightarrow N$ is a ξ-projective resolution N, we have that the mapping cone

$$\mathrm{Cone}(g) := \cdots \to Q_1 \to Q_0 \to N \to 0$$

is a ξ-exact complex. Note that $\mathbf{P} := \cdots \to P_1 \to P_0 \to 0$ is a complex such that P_i is ξ-projective for any $i \geq 0$. Then $\mathcal{C}(\mathbf{P}, \mathrm{Cone}(g))$ is an exact complex and the proof is similar to [16, Lemma 2.4]. Since $\mathcal{C}(\mathbf{P}, \mathrm{Cone}(g)) \cong \mathrm{Cone}(\mathcal{C}(\mathbf{P}, g))$, we get that $\mathrm{Cone}(\mathcal{C}(\mathbf{P}, g))$ is an exact complex. So $\xi\mathrm{xt}_{\xi}^{n}(M, N) = H^n(\mathcal{C}(\mathbf{P}, N)) \cong H^n(\mathcal{C}(\mathbf{P}, \mathbf{Q}))$ for each integer $n \geq 1$. This completes the proof. $\square$

Recall from [24, Definition 4.1] that a full subcategory $\mathcal{X} \subseteq \mathcal{C}$ is called a *generating subcategory* of $\mathcal{C}$ if for all $M \in \mathcal{C}$, $\mathcal{C}(\mathcal{X}, M) = 0$ implies that $M = 0$. Dually, a full subcategory $\mathcal{Y} \subseteq \mathcal{C}$ is called a *cogenerating subcategory* of $\mathcal{C}$ if for all $N \in \mathcal{C}$, $\mathcal{C}(N, \mathcal{Y}) = 0$ implies that $N = 0$.

Lemma 3.9. *Let M be an object in $\mathcal{C}$ and n a non-negative integer.*

(1) *If $\mathcal{P}(\xi)$ is a generating subcategory of $\mathcal{C}$, then ξ-$\mathrm{pd}M \leq n$ if and only if $\xi xt_\xi^{n+1}(M, N) = 0$ for any object N in $\mathcal{C}$.*

(2) *If $\mathcal{I}(\xi)$ is a cogenerating subcategory of $\mathcal{C}$, then ξ-$\mathrm{id}M \leq n$ if and only if $\xi xt_\xi^{n+1}(N, M) = 0$ for any object N in $\mathcal{C}$.*

Proof. The proofs are similar to [7, Proposition 4.17]. $\qquad\square$

We are now in a position to prove the main result of this section.

Theorem 3.10. *Let M be an object in $\mathcal{C}$.*

(1) *If $\mathcal{P}(\xi)$ is a generating subcategory of $\mathcal{C}$, then the following are equivalent:*
   (a) *ξ-$\mathrm{pd}M < \infty$;*
   (b) *$\widetilde{\xi xt}_{\mathcal{P}}^{\,i}(M, N) = 0$ for any integer $i \in \mathbb{Z}$ and any object N in $\mathcal{C}$;*
   (c) *$\widetilde{\xi xt}_{\mathcal{P}}^{\,i}(N, M) = 0$ for any integer $i \in \mathbb{Z}$ and any object N in $\mathcal{C}$;*
   (d) *$\widetilde{\xi xt}_{\mathcal{P}}^{\,0}(M, M) = 0$.*

(2) *If $\mathcal{I}(\xi)$ is a cogenerating subcategory of $\mathcal{C}$, then the following are equivalent:*
   (a) *ξ-$\mathrm{id}M < \infty$;*
   (b) *$\widetilde{\xi xt}_{\mathcal{I}}^{\,i}(N, M) = 0$ for all integer $i \in \mathbb{Z}$ and any object N in $\mathcal{C}$;*
   (c) *$\widetilde{\xi xt}_{\mathcal{I}}^{\,i}(M, N) = 0$ for all integer $i \in \mathbb{Z}$ and any object N in $\mathcal{C}$;*
   (d) *$\widetilde{\xi xt}_{\mathcal{I}}^{\,0}(M, M) = 0$.*

Proof. We prove part (1); the proof of (2) is dual. $(a) \Rightarrow (b)$. Since ξ-$\mathrm{pd}M < \infty$, there is a ξ-projective resolution of M, say $\mathbf{P}_M$, with $(\mathbf{P}_M)_i = 0$ for all $i \gg 0$. Hence $\overline{\mathcal{C}}(\mathbf{P}_M, -) = \mathcal{C}(\mathbf{P}_M, -)$, so $\widetilde{\xi xt}_{\mathcal{P}}^{\,i}(M, N) = 0$ for any integer $i \in \mathbb{Z}$ and any object N in $\mathcal{C}$.

$(b) \Rightarrow (d)$ is trivial.

$(d) \Rightarrow (a)$. Let $\mathbf{P}_M \longrightarrow M$ be a ξ-projective resolution of M. Then we have

$$\mathrm{id}_{\mathbf{P}_M} + \overline{\mathcal{C}}(\mathbf{P}_M, \mathbf{P}_M)_0 \in \mathrm{Z}_0(\widetilde{\mathcal{C}}(\mathbf{P}_M, \mathbf{P}_M)).$$

Note that $\widetilde{\xi xt}_{\mathcal{P}}^{\,0}(M, M) = 0$ by hypothesis, then we have

$$\mathrm{id}_{\mathbf{P}_M} + \overline{\mathcal{C}}(\mathbf{P}_M, \mathbf{P}_M)_0 \in \mathrm{B}_0(\widetilde{\mathcal{C}}(\mathbf{P}_M, \mathbf{P}_M)).$$

Hence there is a $\varphi \in \mathcal{C}(\mathbf{P}_M, \mathbf{P}_M)_1$ such that

$$\mathrm{id}_{\mathbf{P}_M} - d^{\mathcal{C}(\mathbf{P}_M, \mathbf{P}_M)}(\varphi) \in \overline{\mathcal{C}}(\mathbf{P}_M, \mathbf{P}_M)_0.$$

Consequently, there is $\psi \in \overline{\mathcal{C}}(\mathbf{P}_M, \mathbf{P}_M)_0$ such that $\mathrm{id}_{\mathbf{P}_M} - \psi = d^{\mathcal{C}(\mathbf{P}_M, \mathbf{P}_M)}(\varphi)$. Since ψ is bounded above, $(\mathrm{id}_{\mathbf{P}_M} - \psi)_j$ is the identity morphism on $(\mathbf{P}_M)_j$ for all $j \gg 0$. Let m be an integer such that $d_{s+1}^{\mathbf{P}_M} \varphi_s + \varphi_{s-1} d_s^{\mathbf{P}_M} = \mathrm{id}_{(\mathbf{P}_M)_s}$ for any $s \geq m$. It is easy to check that $\xi xt_\xi^{m+1}(M, N) = 0$ for any object N in $\mathcal{C}$, which deduces ξ-$\mathrm{pd}M \leq m$ by Lemma 3.9(1).

The implications of $(a) \Rightarrow (c) \Rightarrow (d)$ can be proved similarly. $\qquad\square$

Assume that $(\mathcal{T}, \Sigma, \Delta)$ is a compactly generated triangulated category, where Σ is the suspension functor and Δ is the triangulation. It follows from Krause [26] and Beligiannis [7] that the class ξ of pure triangles (which is induced by the compact objects) is proper and $\mathcal{T}$ has enough ξ-projectives or ξ-injectives. Moreover, the object $X \in \mathcal{P}(\xi)$ is called the *pure-projective* (see [26, Definition 1.2(2)]). Following Beligiannis [7], we set ξ-gl.dim$\mathcal{T} = \sup\{\xi\text{-pd}M \mid \text{for any } M \in \mathcal{T}\}$. As a corollary of Theorem 3.10, we have the following.

Corollary 3.11. *Let $(\mathcal{T}, \Sigma, \Delta)$ be a compactly generated triangulated category. Then the following are equivalent:*

(1) *ξ-gl.dim$\mathcal{T} < \infty$;*
(2) *$\widetilde{\xi\text{xt}}^{i}_{\mathcal{P}}(M, N) = 0$ for any integer $i \in \mathbb{Z}$ and all objects M and N in $\mathcal{T}$;*
(3) *$\widetilde{\xi\text{xt}}^{0}_{\mathcal{P}}(M, M) = 0$ for any object M in $\mathcal{T}$.*

Let R be a ring. Assume that $\mathcal{X}$ is a class of left R-modules. For a left R-module M, write $M \in {}^{\perp_{\infty}}\mathcal{X}$ (resp., $M \in {}^{\perp}\mathcal{X}$) if $\text{Ext}^{\geq 1}_R(M, X) = 0$ (resp., $\text{Ext}^1_R(M, X) = 0$) for each $X \in \mathcal{X}$. Dually, we can define $M \in \mathcal{X}^{\perp_{\infty}}$ and $M \in \mathcal{X}^{\perp}$. Given a left R-module M, we denote by Add$M$ (resp., add$M$) the class of all modules that are isomorphic to direct summands of direct sums (resp., finite direct sums) of copies of M.

Recall that a left R-module G is called *tilting* [1, 17] provided that the following hold:

($T1$) G has finite projective dimension.
($T2$) $\text{Ext}^i_R(G, G^{(\lambda)}) = 0$ for each $i \geq 1$ and for every cardinal λ.
($T3$) There is a long exact sequence $0 \to R \to G_0 \to \cdots \to G_r \to 0$ with $G_i \in \text{Add}T$ for $0 \leq i \leq r$, where r is the projective dimension of G.

Moreover, a left R-module ω is said to be a *Wakamatsu tilting module* [27, 33] if it has the following properties:

($W1$) There exists an exact sequence $\cdots \to P_i \to \cdots \to P_0 \to \omega \to 0$ with P_i finitely generated and projective for each $i \geq 0$;
($W2$) $\text{Ext}^i_R(\omega, \omega) = 0$ for each $i \geq 1$;
($W3$) There exists an exact sequence $0 \to R \to \omega_0 \xrightarrow{f_0} \cdots \to \omega_i \xrightarrow{f_i} \cdots$ with $\omega_i \in \text{add}\omega$ and $\ker(f_i) \in {}^{\perp_{\infty}}\omega$ for each $i \geq 0$.

It is still an open problem whether a Wakamatsu tilting R-module of finite projective dimension must be a tilting R-module whenever R is an Artin algebra. This is known as Wakamatsu Tilting Conjecture (see [9, Chapter IV]). This conjecture is also related to many other homological conjectures and attract many algebraists, see for instance [9, 25, 27, 34, 36].

Denote by $\mathcal{FP}_{\infty}$ the class of left R-modules possessing a projective resolution consisting of finitely generated modules. Therefore, we have the following corollary which is a consequence of Theorem 3.10.

Corollary 3.12. *Let R be a ring and ω a Wakamatsu tilting left R-module with finite projective dimension. Fix an exact sequence $0 \to R \to \omega_0 \xrightarrow{f_0} \cdots \to \omega_i \xrightarrow{f_i} \cdots$ with $\omega_i \in \text{add}\omega$ and $\ker(f_i) \in {}^{\perp_{\infty}}\omega$ for $i \geq 0$. If we set $A = \bigoplus\limits_{i \geq 0} \ker(f_i)$, then the following are equivalent:*

(1) *ω is a tilting module;*

(2) A has finite projective dimension;

(3) $\widetilde{\xi\mathrm{xt}}^i_{\mathcal{P}}(A, N) = 0$ for any integer $i \in \mathbb{Z}$ and any left R-module N;

(4) $\widetilde{\xi\mathrm{xt}}^i_{\mathcal{P}}(N, A) = 0$ for any integer $i \in \mathbb{Z}$ and any left R-module N;

(5) $\widetilde{\xi\mathrm{xt}}^0_{\mathcal{P}}(A, A) = 0$.

Proof. $(1) \Rightarrow (2)$ holds by the proof of $(1) \Rightarrow (2)$ in [34, Theorem 4.4].

$(2) \Rightarrow (1)$. Note that $\ker(f_i) \in \mathcal{FP}_\infty$ for any $i \geq 0$ by [11, Theorem 1.8]. It follows from [21, Lemma 3.1.6] that $A^{\perp\infty}$ is closed under direct sums. Since A has finite projective dimension, each module M in $^{\perp}(A^{\perp\infty})$ has finite projective dimension by [34, Lemma 2.2(2)]. If we set $\mathcal{K}_A = {}^{\perp}(A^{\perp\infty}) \cap A^{\perp\infty}$, then there is a tilting R-module T such that $\mathcal{K}_A = \mathrm{Add}T$ by [34, Theorem 1.1]. So (1) holds by [34, Corollary 4.5].

$(2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5)$ follow from Theorem 3.10. $\qquad\square$

Let Λ be an Artin algebra. Denote by $\mathcal{GP}(\Lambda)$ the class of Gorenstein projective left Λ-modules and by $\mathcal{GI}(\Lambda)$ the class of Gorenstein injective left Λ-modules. Recall from Beligiannis and Krause [8] that Λ is called *virtually Gorenstein* if $\mathcal{GP}(\Lambda)^{\perp} = {}^{\perp}\mathcal{GI}(\Lambda)$. It is not hard to check that for any virtually Gorenstein algebra Λ, an exact sequence $0 \to X \to Y \to Z \to 0$ of left Λ-modules is $\mathrm{Hom}_{\Lambda}(\mathcal{GP}(\Lambda), -)$-exact if and only if it is $\mathrm{Hom}_{\Lambda}(-, \mathcal{GI}(\Lambda))$-exact. By [35, Theorem 2.2], we have an exact category $(\mathrm{Mod}\Lambda, \varepsilon)$, where $\mathrm{Mod}\Lambda$ is the class of left Λ-modules and ε is the class of $\mathrm{Hom}_{\Lambda}(\mathcal{GP}(\Lambda), -)$-exact sequences $0 \to X \to Y \to Z \to 0$ of left Λ-modules. We refer to [15, 19, 37] for a more discussion on this matter.

On the other hand, it follows from [18, Theorem 12.3.1] that Λ is Gorenstein if and only if every left Λ-module has finite Gorenstein projective dimension. Note that a virtually Gorenstein algebra is not Gorenstein in general by [8, 4.3, p.560]. By Theorem 3.10, we have the following corollary which characterizes when a virtually Gorenstein algebra is Gorenstein.

Corollary 3.13. *Let Λ be a virtually Gorenstein algebra. If we assume that $\mathcal{C}$ is the exact category $(\mathrm{Mod}\Lambda, \varepsilon)$ and $\xi = \varepsilon$ is the class of $\mathrm{Hom}_{\Lambda}(\mathcal{GP}(\Lambda), -)$-exact sequences of left Λ-modules $0 \to X \to Y \to Z \to 0$, then the following are equivalent:*

(1) Λ is Gorenstein;

(2) $\widetilde{\xi\mathrm{xt}}^i_{\mathcal{P}}(M, N) = 0$ for any integer $i \in \mathbb{Z}$ and all left Λ-modules M and N;

(3) $\widetilde{\xi\mathrm{xt}}^0_{\mathcal{P}}(M, M) = 0$ for any left Λ-module M;

(4) $\widetilde{\xi\mathrm{xt}}^i_{\mathcal{I}}(M, N) = 0$ for any integer $i \in \mathbb{Z}$ and all left Λ-modules M and N;

(5) $\widetilde{\xi\mathrm{xt}}^0_{\mathcal{I}}(M, M) = 0$ for any left Λ-module M.

4. ξ-complete cohomology for objects with finite ξ-$\mathcal{G}$projective dimension

Recall from [23] that the ξ-$\mathcal{G}$projective dimension ξ-$\mathcal{G}\mathrm{pd}M$ of an object $M \in \mathcal{C}$ is defined inductively. If $M \in \mathcal{GP}(\xi)$ then define ξ-$\mathcal{G}\mathrm{pd}M = 0$. Next by induction, for an integer $n > 0$, put ξ-$\mathcal{G}\mathrm{pd}M \leq n$ if there exists an $\mathbb{E}$-triangle $K \longrightarrow G \longrightarrow M \dashrightarrow$ in ξ with $G \in \mathcal{GP}(\xi)$ and ξ-$\mathcal{G}\mathrm{pd}K \leq n-1$.

We define ξ-$\mathcal{G}\mathrm{pd}M = n$ if ξ-$\mathcal{G}\mathrm{pd}M \leq n$ and ξ-$\mathcal{G}\mathrm{pd}M \nleq n-1$. If ξ-$\mathcal{G}\mathrm{pd}M \neq n$ for all $n \geq 0$, we set ξ-$\mathcal{G}\mathrm{pd}M = \infty$.

The following observation is useful in this section.

Lemma 4.1. *Let* $A \xrightarrow{f} B \xrightarrow{g} C \dashrightarrow{\delta}$ *be an* $\mathbb{E}$-*triangle in* ξ. *For any morphism* $\alpha : B \to D$, *there exists an* $\mathbb{E}$-*triangle in* ξ

$$A \xrightarrow{\begin{bmatrix} -\alpha f \\ f \end{bmatrix}} D \oplus B \xrightarrow{\begin{bmatrix} 1 & \alpha \\ 0 & g \end{bmatrix}} D \oplus C \dashrightarrow{\begin{bmatrix} 0 & 1 \end{bmatrix}^* \delta}.$$

Proof. It follows from [23, Lemma 3.7(2)] that there exists an $\mathbb{E}$-triangle

$$A \xrightarrow{\begin{bmatrix} 0 \\ f \end{bmatrix}} D \oplus B \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & g \end{bmatrix}} D \oplus C \dashrightarrow{\begin{bmatrix} 0 & 1 \end{bmatrix}^* \delta}.$$

Moreover, it is an $\mathbb{E}$-triangle in ξ because ξ is closed under base change. It is easy to check that the following is a commutative diagram.

Hence $A \xrightarrow{\begin{bmatrix} -\alpha f \\ f \end{bmatrix}} D \oplus B \xrightarrow{\begin{bmatrix} 1 & \alpha \\ 0 & g \end{bmatrix}} D \oplus C \dashrightarrow{\begin{bmatrix} 0 & 1 \end{bmatrix}^* \delta}$ is an $\mathbb{E}$-triangle in ξ. $\square$

Lemma 4.2. (see [23, Proposition 4.13(2)]) *Let* $f : A \to B$ *and* $g : B \to C$ *be morphisms in* $\mathcal{C}$. *If the composition* gf *is a* ξ-*deflation, then* g *is a* ξ-*deflation.*

Definition 4.3. Let $M \in \mathcal{C}$ be an object. A ξ-*complete resolution* of M is a diagram

$$\mathbf{T} \xrightarrow{\nu} \mathbf{P} \xrightarrow{\pi} M$$

of morphisms of complexes satisfying: (1) $\pi : \mathbf{P} \to M$ is a ξ-projective resolution of M; (2) $\mathbf{T}$ is a complete ξ-projective resolution; (3) $\nu : \mathbf{T} \to \mathbf{P}$ is a morphism such that $\nu_i = \mathrm{id}_{T_i}$ for all $i \gg 0$. A ξ-complete resolution is *split* if ν_i has a section (i.e., there exists a morphism $\eta_i : P_i \to T_i$ such that $\nu_i \eta_i = \mathrm{id}_{P_i}$) for all $i \in \mathbb{Z}$.

The following proposition characterizes when an object in $\mathcal{C}$ admits a ξ-complete resolution.

Proposition 4.4. *Let* M *be an object in* $\mathcal{C}$. *Then the following are equivalent for any non-negative integer* n:

  (1) ξ-$\mathcal{G}\mathrm{pd}M \leq n$;

  (2) M *has a* ξ-*complete resolution* $\mathbf{T} \xrightarrow{\nu} \mathbf{P} \xrightarrow{\pi} M$ *such that* ν_i *is an isomorphism for each* $i \geq n$;

  (3) M *has a split* ξ-*complete resolution* $\mathbf{S} \xrightarrow{\mu} \mathbf{P} \xrightarrow{\pi} M$ *such that* μ_i *is an isomorphism for each* $i \geq n$.

Proof. $(1) \Rightarrow (2)$. By hypothesis, we set $\xi\text{-}\mathcal{G}\mathrm{pd}M \leq n$ and consider a ξ-projective resolution $\pi : \mathbf{P} \to M$, where

$$\mathbf{P} : \cdots \to P_n \to \cdots \to P_1 \to P_0 \to 0.$$

For any integer $i \in \mathbb{Z}$, there are $\mathbb{E}$-triangles $K_i \longrightarrow P_i \longrightarrow K_{i-1} \dashrightarrow$ in ξ, where K_i is called an ith ξ-syzygy of M. It follows from [23, Proposition 5.2] that the nth ξ-syzygy K_n is in $\mathcal{GP}(\xi)$. So there exists a complete ξ-projective resolution of K_n

$$\mathbf{Q} : \cdots \longrightarrow Q_1 \xrightarrow{d_1} Q_0 \xrightarrow{d_0} Q_{-1} \longrightarrow \cdots$$

with $K_n \longrightarrow Q_{n-1} \longrightarrow K'_{n-1} \dashrightarrow$ in ξ. Then the ξ-projective resolution $\mathbf{P}$ of M and the complete ξ-projective resolution $\mathbf{Q}$ of K_n can be put together in a commutative diagram

$$
\begin{array}{ccccccccccccc}
\mathbf{T} := & \cdots \longrightarrow & P_n & \longrightarrow & Q_{n-1} & \longrightarrow & \cdots \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & Q_{-1} & \longrightarrow \cdots \\
& & \| & & \downarrow{\nu_{n-1}} & & & \downarrow{\nu_1} & & \downarrow{\nu_0} & & \downarrow{\nu_{-1}} & \\
\mathbf{P} := & \cdots \longrightarrow & P_n & \longrightarrow & P_{n-1} & \longrightarrow & \cdots \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 & \longrightarrow \cdots .
\end{array}
$$

Since the upper row is $\mathcal{C}(-, \mathcal{P}(\xi))$-exact, the vertical maps can be constructed inductively, starting from K_n. So $\mathbf{T} \xrightarrow{\nu} \mathbf{P} \xrightarrow{\pi} M$ is a ξ-complete resolution of M such that ν_i is an isomorphism for all $i \geq n$.

$(2) \Rightarrow (3)$. Note that M has a ξ-complete resolution $\mathbf{T} \xrightarrow{\nu} \mathbf{P} \xrightarrow{\pi} M$ such that ν_i is an isomorphism for each $i \geq n$ by (2). Then we have the following morphism of $\mathbb{E}$-triangles in ξ

$$
\begin{array}{ccccccc}
K_n & \xrightarrow{g'_{n-1}} & Q_{n-1} & \xrightarrow{f'_{n-1}} & K'_{n-1} & \xdashrightarrow{\rho_{n-1}} & \\
\| & & \downarrow{\nu_{n-1}} & & \downarrow{\omega_{n-1}} & & \\
K_n & \xrightarrow{g_{n-1}} & P_{n-1} & \xrightarrow{f_{n-1}} & K_{n-1} & \xdashrightarrow{\delta_{n-1}} &
\end{array}
$$

Moreover, for any integer $i < n$, we have the following morphism of $\mathbb{E}$-triangles in ξ

$$
\begin{array}{ccccccc}
K'_i & \xrightarrow{g'_{i-1}} & Q_{i-1} & \xrightarrow{f'_{n-1}} & K'_{n-1} & \xdashrightarrow{\rho_{i-1}} & \\
\downarrow{\omega_i} & & \downarrow{\nu_{i-1}} & & \downarrow{\omega_{i-1}} & & \\
K_i & \xrightarrow{g_{i-1}} & P_{i-1} & \xrightarrow{f_{i-1}} & K_{i-1} & \xdashrightarrow{\delta_{i-1}} &
\end{array}
$$

By Lemma 4.1, there is an $\mathbb{E}$-triangle in ξ

$$
K_n \xrightarrow{\left[\begin{smallmatrix} -g_{n-1} \\ g'_{n-1} \end{smallmatrix}\right]} P_{n-1} \oplus Q_{n-1} \xrightarrow{\left[\begin{smallmatrix} 1 & \nu_{n-1} \\ 0 & f'_{n-1} \end{smallmatrix}\right]} P_{n-1} \oplus K'_{n-1} \xdashrightarrow{[0 \; 1]^* \rho_{n-1}}
$$

Since the morphism $[1 \quad 0] : P_{n-1} \oplus K'_{n-1} \to P_{n-1}$ is a split epimorphism, and it is a ξ-deflation. Hence $[1 \quad \nu_{n-1}] = [1 \quad 0] \left[\begin{smallmatrix} 1 & \nu_{n-1} \\ 0 & f'_{n-1} \end{smallmatrix}\right] : P_{n-1} \oplus Q_{n-1} \to P_{n-1}$ is a ξ-deflation by [23, Corollary 3.5].

It follows from [23, Lemma 3.7(2)] that $K_n \xrightarrow{\left[\begin{smallmatrix} 0 \\ g'_{n-1} \end{smallmatrix}\right]} P_{n-1} \oplus Q_{n-1} \xrightarrow{\left[\begin{smallmatrix} 1 & 0 \\ 0 & f'_{n-1} \end{smallmatrix}\right]} P_{n-1} \oplus K'_{n-1} \xdashrightarrow{[0 \; 1]^* \rho_{n-1}}$

is an $\mathbb{E}$-triangle in ξ. Note that $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]^* [f_{n-1} \quad \omega_{n-1}]^* \delta_{n-1} = ([f_{n-1} \quad \omega_{n-1}] \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right])^* \delta_{n-1} = \omega_{n-1}^* \delta_{n-1} =$

$\rho_{n-1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^* \begin{bmatrix} 0 & 1 \end{bmatrix}^* \rho_{n-1}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}^* \begin{bmatrix} f_{n-1} & \omega_{n-1} \end{bmatrix}^* \delta_{n-1} = (\begin{bmatrix} f_{n-1} & \omega_{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix})^* \delta_{n-1} = f_{n-1}^* \delta_{n-1} = 0 =$
$\begin{bmatrix} 1 \\ 0 \end{bmatrix}^* \begin{bmatrix} 0 & 1 \end{bmatrix}^* \rho_{n-1}$, which implies that $\begin{bmatrix} f_{n-1} & \omega_{n-1} \end{bmatrix}^* \delta_{n-1} = \begin{bmatrix} 0 & 1 \end{bmatrix}^* \rho_{n-1}$. Hence we have the following morphism of $\mathbb{E}$-triangles in ξ

$$
\begin{array}{ccccccc}
K_n & \xrightarrow{\begin{bmatrix} 0 \\ g'_{n-1} \end{bmatrix}} & P_{n-1} \oplus Q_{n-1} & \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & f'_{n-1} \end{bmatrix}} & P_{n-1} \oplus K'_{n-1} & \dashrightarrow^{\begin{bmatrix} 0 & 1 \end{bmatrix}^* \rho_{n-1}} \\
\Big\| & & \Big\downarrow{\scriptstyle \begin{bmatrix} 1 & \nu_{n-1} \end{bmatrix}} & & \Big\downarrow{\scriptstyle \begin{bmatrix} f_{n-1} & \omega_{n-1} \end{bmatrix}} & \\
K_n & \xrightarrow{g_{n-1}} & P_{n-1} & \xrightarrow{f_{n-1}} & K_{n-1} & \dashrightarrow^{\delta_{n-1}}
\end{array}
$$

Since K'_{n-1} is ξ-$\mathcal{G}$projective, the upper row is $\mathcal{C}(-, \mathcal{P}(\xi))$-exact by [23, Lemma 4.10(2)]. Since

$$
\begin{bmatrix} f_{n-1} & \omega_{n-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & f'_{n-1} \end{bmatrix} = f_{n-1} \begin{bmatrix} 1 & \nu_{n-1} \end{bmatrix}
$$

is a ξ-deflation, it follows from Lemma 4.2 that the morphism $\begin{bmatrix} f_{n-1} & \omega_{n-1} \end{bmatrix}$ is also a ξ-deflation. Dual to [23, Lemma 3.7(2)], there exists an $\mathbb{E}$-triangle

$$
P_{n-1} \oplus K'_{n-1} \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & g'_{n-2} \end{bmatrix}} P_{n-1} \oplus Q_{n-2} \xrightarrow{\begin{bmatrix} 0 & f'_{n-2} \end{bmatrix}} K'_{n-2} \dashrightarrow^{\begin{bmatrix} 0 \\ 1 \end{bmatrix}_* \rho_{n-2}} ,
$$

which is also in ξ because ξ is closed under cobase change. Since $\begin{bmatrix} f_{n-1} & \omega_{n-1} \end{bmatrix}_* \begin{bmatrix} 0 \\ 1 \end{bmatrix}_* \rho_{n-2} =$
$(\begin{bmatrix} f_{n-1} & \omega_{n-1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix})_* \rho_{n-2} = \omega_{n-1*} \rho_{n-1} = \omega_{n-2}^* \delta_{n-2}$, we have the following morphism of $\mathbb{E}$-triangles in ξ

$$
\begin{array}{ccccccc}
P_{n-1} \oplus K'_{n-1} & \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & g'_{n-2} \end{bmatrix}} & P_{n-1} \oplus Q_{n-2} & \xrightarrow{\begin{bmatrix} 0 & f'_{n-2} \end{bmatrix}} & K'_{n-2} & \dashrightarrow^{\begin{bmatrix} 0 \\ 1 \end{bmatrix}_* \rho_{n-2}} \\
{\scriptstyle \begin{bmatrix} f_{n-1} & \omega_{n-1} \end{bmatrix}} \Big\downarrow & & \Big\downarrow{\scriptstyle \begin{bmatrix} d^{\mathbf{P}}_{n-1} & \nu_{n-2} \end{bmatrix}} & & \Big\downarrow{\scriptstyle \omega_{n-2}} & \\
K_{n-1} & \xrightarrow{g_{n-2}} & P_{n-2} & \xrightarrow{f_{n-2}} & K_{n-2} & \dashrightarrow^{\delta_{n-2}}
\end{array}
$$

Similarly, we have the following morphism of $\mathbb{E}$-triangles in ξ

$$
\begin{array}{ccccccc}
P_{n-1} \oplus K'_{n-1} & \xrightarrow{\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & g'_{n-2} \end{bmatrix}} & P_{n-2} \oplus P_{n-1} \oplus Q_{n-2} & \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & f'_{n-2} \end{bmatrix}} & P_{n-2} \oplus K'_{n-2} & \dashrightarrow^{\begin{bmatrix} 0 & 1 \end{bmatrix}^* \begin{bmatrix} 0 \\ 1 \end{bmatrix}_* \rho_{n-2}} \\
{\scriptstyle \begin{bmatrix} f_{n-1} & \omega_{n-1} \end{bmatrix}} \Big\downarrow & & \Big\downarrow{\scriptstyle \begin{bmatrix} 1 & d^{\mathbf{P}}_{n-1} & \nu_{n-2} \end{bmatrix}} & & \Big\downarrow{\scriptstyle \begin{bmatrix} f_{n-2} & \omega_{n-2} \end{bmatrix}} & \\
K_{n-1} & \xrightarrow{g_{n-2}} & P_{n-2} & \xrightarrow{f_{n-2}} & K_{n-2} & \dashrightarrow^{\delta_{n-2}}
\end{array}
$$

where the morphisms $\begin{bmatrix} 1 & d^{\mathbf{P}}_{n-1} & \nu_{n-2} \end{bmatrix}$ and $\begin{bmatrix} f_{n-2} & \omega_{n-2} \end{bmatrix}$ are ξ-deflations and the upper row is a $\mathcal{C}(-, \mathcal{P}(\xi))$-exact $\mathbb{E}$-triangle in ξ. By proceeding in this manner, we set

$$
S_i = \begin{cases} P_i & i \geq n \\ P_{n-1} \oplus Q_{n-1} & i = n-1 \\ P_i \oplus P_{i+1} \oplus Q_i & i < n-1 \end{cases}
$$

$$\mu_i = \begin{cases} 1 & i \geq n \\ \begin{bmatrix} 1 & \nu_{n-1} \end{bmatrix} & i = n - 1 \\ \begin{bmatrix} 1 & d^{\mathbf{P}}_{i+1} & \nu_i \end{bmatrix} & 0 \leq i < n - 1 \\ 0 & i < 0 \end{cases}$$

Consequently, we get a commutative diagram

$$\mathbf{S} := \qquad \cdots \longrightarrow P_n \longrightarrow S_{n-1} \longrightarrow \cdots \longrightarrow S_1 \longrightarrow S_0 \longrightarrow S_{-1} \longrightarrow \cdots$$

$$\mathbf{P} := \qquad \cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0 \longrightarrow \cdots .$$

with vertical maps μ_{n-1}, μ_1, μ_0, μ_{-1}.

Note that every S_i is ξ-projective, and $\mathbf{S}$ is obtained by pasting together those $\mathbb{E}$-triangles

$$K_n \longrightarrow P_{n-1} \oplus Q_{n-1} \longrightarrow P_{n-1} \oplus K'_{n-1} \dashrightarrow$$

and

$$P_i \oplus K'_i \longrightarrow P_{i-1} \oplus P_i \oplus Q_{i-1} \longrightarrow P_{i-1} \oplus K'_{i-1} \dashrightarrow$$

for all $i < n$, then the complex $\mathbf{S}$ is ξ-exact and $\mathcal{C}(-, \mathcal{P}(\xi))$-exact. Hence we get the desired ξ-complete resolution $\mathbf{S} \xrightarrow{\mu} \mathbf{P} \xrightarrow{\pi} M$ such that μ_i is a ξ-deflation for every $i \in \mathbb{Z}$. So each μ_i has a section and $\mathbf{S} \xrightarrow{\mu} \mathbf{P} \xrightarrow{\pi} M$ is a split ξ-complete resolution such that μ_i is an isomorphism for each $i \geq n$, as desired.

(3) $\Rightarrow$ (1) is trivial. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

As a similar argument to that of [3, Lemma 3.2], we have the following result.

Lemma 4.5. *Let* $\mathbf{T} \xrightarrow{\nu} \mathbf{P} \xrightarrow{\pi} M$ *and* $\mathbf{T}' \xrightarrow{\nu'} \mathbf{P}' \xrightarrow{\pi'} M'$ *be ξ-complete resolutions of* M *and* M', *respectively. For each morphism* $\mu : M \to M'$, *there exists a morphism* $\overline{\mu}$, *unique up to homotopy, making the right-hand square of the following diagram*

$$\begin{array}{ccccc} \mathbf{T} & \xrightarrow{\nu} & \mathbf{P} & \xrightarrow{\pi} & M \\ \downarrow{\scriptstyle\widetilde{\mu}} & & \downarrow{\scriptstyle\overline{\mu}} & & \downarrow{\scriptstyle\mu} \\ \mathbf{T}' & \xrightarrow{\nu'} & \mathbf{P}' & \xrightarrow{\pi'} & M' \end{array}$$

commutative, and for each choice of $\overline{\mu}$, *there exists a morphism* $\widetilde{\mu}$, *unique up to homotopy, making the left-hand square commute up to homotopy. In particular, if* $\mu = \mathrm{id}_M$, *then* $\widetilde{\mu}$ *and* $\overline{\mu}$ *are homotopy equivalences.*

We are now in a position to prove the following result, which gives a general technique for computing ξ-complete cohomology of objects with finite ξ-$\mathcal{G}$projective dimension.

Theorem 4.6. *Let M and N be objects in $\mathcal{C}$. If* $\mathbf{T} \xrightarrow{\nu} \mathbf{P} \xrightarrow{\pi} M$ *is a ξ-complete resolution of M, then for any integer n, there exists an isomorphism*

$$\widetilde{\xi\mathrm{xt}}^n_{\mathcal{P}}(M, N) \cong H^n(\mathcal{C}(\mathbf{T}, N)).$$

Proof. By Proposition 4.4, there exists a split ξ-complete resolution $\mathbf{S} \xrightarrow{\mu} \mathbf{P} \xrightarrow{\pi} M$ of M such that μ_i is an isomorphism for each $i \geq m$. Thus $\mathbf{T}$ and $\mathbf{S}$ are homotopy equivalences by Lemma 4.5, and hence $H^n(\mathcal{C}(\mathbf{S}, N)) \cong H^n(\mathcal{C}(\mathbf{T}, N))$ for any integer n. Next we show that $\widetilde{\xi \mathrm{xt}}_{\mathcal{P}}^n(M, N) \cong H^n(\mathcal{C}(\mathbf{S}, N))$ for any integer n.

Note that $\mathbf{S} \xrightarrow{\mu} \mathbf{P} \xrightarrow{\pi} M$ is a split ξ-complete resolution. Then we have a sequence $\mathbf{X} \xrightarrow{\lambda} \mathbf{S} \xrightarrow{\mu} \mathbf{P}$ of complexes such that $X_i \xrightarrow{\lambda_i} S_i \xrightarrow{\mu_i} P_i \dashrightarrow$ is a split $\mathbb{E}$-triangle for each $i \in \mathbb{Z}$. Let $\rho : \mathbf{Q} \longrightarrow N$ be a ξ-projective resolution of N. Applying the functor $\mathcal{C}(-, \mathbf{Q})$ to the sequence above, we have the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{\mathcal{C}}(\mathbf{P}, \mathbf{Q}) & \longrightarrow & \mathcal{C}(\mathbf{P}, \mathbf{Q}) & \longrightarrow & \widetilde{\mathcal{C}}(\mathbf{P}, \mathbf{Q}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{\mathcal{C}}(\mathbf{S}, \mathbf{Q}) & \longrightarrow & \mathcal{C}(\mathbf{S}, \mathbf{Q}) & \longrightarrow & \widetilde{\mathcal{C}}(\mathbf{S}, \mathbf{Q}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{\mathcal{C}}(\mathbf{X}, \mathbf{Q}) & \longrightarrow & \mathcal{C}(\mathbf{X}, \mathbf{Q}) & \longrightarrow & \widetilde{\mathcal{C}}(\mathbf{X}, \mathbf{Q}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}
$$

Since $\mathbf{X}$ is bounded above, $\overline{\mathcal{C}}(\mathbf{X}, \mathbf{Q}) = \mathcal{C}(\mathbf{X}, \mathbf{Q})$. For each integer $n \in \mathbb{Z}$, we have

$$H^n(\widetilde{\mathcal{C}}(\mathbf{P}, \mathbf{Q})) \cong H^n(\widetilde{\mathcal{C}}(\mathbf{S}, \mathbf{Q})).$$

Note that $\mathbf{S}$ is a ξ-exact complex such that $\mathcal{C}(\mathbf{S}, Q)$ is exact for all $Q \in \mathcal{P}(\xi)$. Then $\overline{\mathcal{C}}(\mathbf{S}, \mathbf{Q})$ is exact by [14, Proposition A.2(b)]. Thus

$$H^n(\mathcal{C}(\mathbf{S}, \mathbf{Q})) \cong H^n(\widetilde{\mathcal{C}}(\mathbf{S}, \mathbf{Q})), \forall n \in \mathbb{Z}.$$

Note that $\rho : \mathbf{Q} \longrightarrow N$ is a ξ-projective resolution of N. Then for any ξ-projective object P, $\mathcal{C}(P, \rho) : \mathcal{C}(P, \mathbf{Q}) \longrightarrow \mathcal{C}(P, N)$ is a quasi-isomorphism. It is not hard to check that the morphism $\mathcal{C}(\mathbf{S}, \rho) : \mathcal{C}(\mathbf{S}, \mathbf{Q}) \longrightarrow \mathcal{C}(\mathbf{S}, N)$ is a quasi-isomorphism and the proof is similar to that of Lemma 3.8(1). So

$$H^n(\mathcal{C}(\mathbf{S}, \mathbf{Q})) \cong H^n(\mathcal{C}(\mathbf{S}, N)), \forall n \in \mathbb{Z}.$$

Hence for each integer n, we have

$$H^n(\widetilde{\mathcal{C}}(\mathbf{P}, \mathbf{Q})) \cong H^n(\widetilde{\mathcal{C}}(\mathbf{S}, \mathbf{Q})) \cong H^n(\mathcal{C}(\mathbf{S}, \mathbf{Q})) \cong H^n(\mathcal{C}(\mathbf{S}, N)).$$

So $\widetilde{\xi \mathrm{xt}}_{\mathcal{P}}^n(M, N) \cong H^n(\mathcal{C}(\mathbf{S}, N)) \cong H^n(\mathcal{C}(\mathbf{T}, N)$ *for all* $n \in \mathbb{Z}$. This completes the proof. $\square$

Remark 4.7. *Assume that $\mathcal{C}$ is a triangulated category and ξ is a proper class of triangles which is closed under suspension (see [7, Section 2.2]). Let M be an object in $\mathcal{C}$ with finite ξ-$\mathcal{G}$projective dimension and N an object in $\mathcal{C}$. By Theorem 4.6, one can check that for any integer*

n, the complete cohomology group $\widetilde{\xi\mathrm{xt}}^n_{\mathcal{P}}(M, N)$ defined here is exactly the Tate cohomology group $\widehat{\xi\mathrm{xt}}^n_{\mathcal{P}}(M, N)$ defined by Asadollahi and Salarian in [3].

Let M be an object in $\mathcal{C}$. It is shown that $\xi\text{-}\mathcal{G}\mathrm{pd}M \leq \xi\text{-}\mathrm{pd}M$, and the equality holds if $\xi\text{-}\mathrm{pd}M < \infty$ (see [23, Proposition 5.4]). It seems natural to ask when $\xi\text{-}\mathcal{G}\mathrm{pd}M = \xi\text{-}\mathrm{pd}M$ provided that $\xi\text{-}\mathcal{G}\mathrm{pd}M < \infty$. Motivated by this, we have the following result which is a consequence of Theorems 3.10 and 4.6.

Corollary 4.8. *If $\mathcal{P}(\xi)$ is a generating subcategory of $\mathcal{C}$ and M is an object in $\mathcal{C}$ with finite $\xi\text{-}\mathcal{G}$projective dimension, then the following are equivalent:*

(1) $\xi\text{-}\mathcal{G}\mathrm{pd}M = \xi\text{-}\mathrm{pd}M$;
(2) $\widetilde{\xi\mathrm{xt}}^i_{\mathcal{P}}(M, N) = 0$ for any integer $i \in \mathbb{Z}$ and any object N in $\mathcal{C}$;
(3) $\widetilde{\xi\mathrm{xt}}^i_{\mathcal{P}}(M, N) = 0$ for some integer $i \in \mathbb{Z}$ and any object N in $\mathcal{C}$.

Proof. $(1) \Rightarrow (2)$ holds by Theorem 3.10.

$(2) \Rightarrow (3)$ is trivial.

$(3) \Rightarrow (1)$. Note that M is an object with finite $\xi\text{-}\mathcal{G}$projective dimension by hypothesis. Then there exists an integer m and a split ξ-complete resolution $\mathbf{S} \xrightarrow{\mu} \mathbf{P} \xrightarrow{\pi} M$ of M such that μ_i is an isomorphism for each $i \geq m$. In fact $\mathbf{S}$ is a ξ-exact complex such that there is a $\mathcal{C}(-, \mathcal{P}(\xi))$-exact $\mathbb{E}$-triangle in ξ

$$K_{i+1} \xrightarrow{g_i} S_i \xrightarrow{f_i} K_i \overset{\delta_i}{\dashrightarrow}$$

for each integer i and $d_i^{\mathbf{S}} = g_{i-1}f_i$. By (3), there exists an integer n such that $\widetilde{\xi\mathrm{xt}}^{n+1}_{\mathcal{P}}(M, N) = 0$ for any object N in $\mathcal{C}$. It follows from Theorem 4.6 that the following sequence

$$\mathcal{C}(S_{n-1}, K_n) \xrightarrow{\mathcal{C}(d_n^{\mathbf{S}}, K_n)} \mathcal{C}(S_n, K_n) \xrightarrow{\mathcal{C}(d_{n+1}^{\mathbf{S}}, K_n)} \mathcal{C}(S_{n+1}, K_n)$$

is exact. Note that $\mathcal{C}(d_{n+1}^{\mathbf{S}}, K_n)(f_n) = f_n d_{n+1}^{\mathbf{S}} = 0$. Then there exists a morphism $h : S_{n-1} \to K_n$ such that $f_n = h d_n^{\mathbf{S}} = h g_{n-1} f_n$. Let P be any object in $\mathcal{P}(\xi)$. Then $\mathcal{C}(P, f_n)$ is epic. It follows that $\mathcal{C}(P, h)\mathcal{C}(P, g_{n-1}) = \mathrm{id}_{\mathcal{C}(P, S_n)}$. Hence $\mathcal{C}(P, S_{n-1}) \cong \mathcal{C}(P, K_n \oplus K_{n-1})$. Since $\mathcal{P}(\xi)$ is a generating subcategory of $\mathcal{C}$ by hypothesis, we get that $S_{n-1} \cong K_n \oplus K_{n-1}$. It follows that $K_n \in \mathcal{P}(\xi)$, so $K_i \in \mathcal{P}(\xi)$ for all $i \geq n$. Let s be an integer such that $s \geq \max\{n, m\}$. Then $K_s \in \mathcal{P}(\xi)$ and $\xi\text{-}\mathrm{pd}M \leq s$. So (1) holds by [23, Proposition 5.4]. This completes the proof. $\square$

We end this paper with the following remark.

Remark 4.9. *Using arguments analogous to the ones employed in this section, one may show that an object M in an extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ has finite $\xi\text{-}\mathcal{G}$injective dimension if and only if M admits a (split) ξ-complete coresolution, which can help us to give a general technique for computing complete cohomology of objects with finite $\xi\text{-}\mathcal{G}$injective dimension. The details are left to the reader.*

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