Diagrams Lemmas in Extriangulated Categories

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Contents

Abstract

$\varnothing$.

1 Preliminaries

1.1 Axiom of Extriangulated Categories

Extriangulated categories were introduced by Nakaoka and Palu in [11], which simultaneously generalise exact categories and triangulated categories. We recall the basic definitions from [11].

Notation. We fix an additive category $\mathcal{C}$ and an additive bifunctor

$$\mathbb{E} : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Ab}, \quad (X, Y) \mapsto \mathbb{E}(Y, X). \tag{1.1.1}$$

We introduce some notations concerning $\mathbb{E}$.

- For any morphism $f \in \mathsf{Mor}(\mathcal{C})$, we denote the natural transformation $f^* := \mathbb{E}(f, -)$ and $g_* := \mathbb{E}(-, g)$. Note that the bifunctoriality implies $f_* g^* = g^* f_*$.
- A morphism of extension elements $\delta \to \delta'$ is a pair of morphisms $(\alpha; \gamma)$ such that $\alpha_* \delta = \gamma^* \delta'$.
- For any $\delta \in \mathbb{E}(Z, X)$ and $\delta' \in \mathbb{E}(Z', X')$, we denote $\delta \oplus \delta' \in \mathbb{E}(Z \oplus Z', X \oplus X')$ as the image of $(\delta, \delta') \in \mathbb{E}(Z, X) \oplus \mathbb{E}(Z', X')$ under the inclusion $\mathbb{E}(Z, X) \oplus \mathbb{E}(Z', X') \hookrightarrow \mathbb{E}(Z \oplus Z', X \oplus X')$.

We also fix $\mathfrak{s}$ as a collection of "mappings" sending each $\delta \in \mathbb{E}(Z, X)$ to an equivalence class of sequences $[X \xrightarrow{f} Y \xrightarrow{g} Z]$. Here two sequences $X \xrightarrow{f} Y \xrightarrow{g} Z$ and $X \xrightarrow{f'} Y' \xrightarrow{g'} Z$ are equivalent if there exists an isomorphism $\varphi : Y \to Y'$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, "\varphi", "\cong"'] & Z \arrow[d, equal] \\ X \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z \end{tikzcd} . \tag{1.1.2}$$

We begin with the axiom of extriangulated categories. An extriangulated category is characterised by a triple $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ satisfying a list of axioms, including ET1, ET2, ET3 (ET3^{op}), ET4 (ET4^{op}).

Axiom (ET1). *$\mathcal{C}$ is an additive cateogry, and $\mathbb{E} : \mathcal{C}^{op} \times \mathcal{C} \to Ab$ is an additive bifunctor.*

Axiom (ET2). *$\mathfrak{s}$ is an additive realisation, which satisfies the following conditions.*

- *(Additive). $\delta(0) = [X \xrightarrow{\binom{1}{0}} X \oplus Y \xrightarrow{(0,1)} Y]$. For any δ_1, δ_2, $\mathfrak{s}(\delta_1 \oplus \delta_2) = \mathfrak{s}(\delta_1) \oplus \mathfrak{s}(\delta_2)$, explicitly,*

$$[X_1 \xrightarrow{f_1} Y_1 \xrightarrow{g_1} Z_1] \oplus [X_2 \xrightarrow{f_2} Y_2 \xrightarrow{g_2} Z_2] = [X_1 \oplus X_2 \xrightarrow{\begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}} Y_1 \oplus Y_2 \xrightarrow{\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}} Z_1 \oplus Z_2]. \tag{1.1.3}$$

- *(Realisation). For any morphism of extension elements $(\alpha; \gamma) : \delta \to \delta'$, we take arbitrary representatives $X \xrightarrow{f} Y \xrightarrow{g} Z$ of $\mathfrak{s}(\delta)$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z'$ of $\mathfrak{s}(\delta')$. Then there exists $\beta : Y \to Y'$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "g"] \arrow[d, dashed, "\beta"] & Z \arrow[d, "\gamma"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \end{tikzcd} . \tag{1.1.4}$$

Notation. For triple $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ satisfying ET1 and ET2, we denote an element of $\mathfrak{s}(\delta)$ by $X \xrightarrow{f} Y \xrightarrow{g} Z \overset{\delta}{\dashrightarrow}$. We call it an $\mathbb{E}$-*conflation*, f an $\mathbb{E}$-*inflation*, and g an $\mathbb{E}$-*deflation*. A morphism of $\mathbb{E}$-conflations is a triple $(\alpha; \beta; \gamma)$ such that $(\alpha; \beta)$ is a morphism of extension elements, and the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, dashed, "\delta"] \arrow[d, "\gamma"] & {} \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, dashed, "\delta'"] & {} \end{tikzcd} \quad {\scriptstyle (\alpha_* \delta = \gamma^* \delta')} . \tag{1.1.5}$$

Axiom (ET3). *For $\beta \circ f = f' \circ \alpha$ where f and f' are $\mathbb{E}$-inflations, there exists γ making $(\alpha; \beta; \gamma)$ a morphism of $\mathbb{E}$-conflations.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, dashed, "\delta"] \arrow[d, dashed, "\gamma"] & {} \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, dashed, "\delta'"] & {} \end{tikzcd} . \tag{1.1.6}$$

Axiom (ET3$^{\mathrm{op}}$)**.** *For $\gamma \circ g = g' \circ \gamma$ where g and g' are $\mathbb{E}$-deflations, there exists α making $(\alpha;\beta;\gamma)$ a morphism of $\mathbb{E}$-conflations.*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, dashed, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, dashed, "\delta"] \arrow[d, "\gamma"] & {} \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, dashed, "\delta'"] & {}
\end{tikzcd} . \tag{1.1.7}$$

Axiom (ET4)**.** *Let $A \xrightarrow{f} B \xrightarrow{g} D \overset{\delta}{\dashrightarrow}$ and $B \xrightarrow{u} C \xrightarrow{v} E \overset{\varepsilon}{\dashrightarrow}$ be $\mathbb{E}$-conflations. There exists a commutative diagram*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, "u"'] & D \arrow[r, dashed, "\delta"] \arrow[d, dashed, "w"] & {} \\
A \arrow[r, dashed, "m"'] & C \arrow[r, dashed, "h"'] \arrow[d, "v"'] & F \arrow[r, dashed, "\theta"] \arrow[d, dashed, "q"] & {} \\
 & E \arrow[r, equal] \arrow[d, dashed, "\varepsilon"] & E \arrow[d, dashed, "\eta"] & \\
 & {} & {} &
\end{tikzcd} \tag{1.1.8}$$

such that $(1_A;u;w)$, $(f;1_C;q)$ and $(g;h;1_E)$ are morphisms of $\mathbb{E}$-conflations.

Axiom (ET4$^{\mathrm{op}}$)**.** *Let $A \xrightarrow{m} C \xrightarrow{h} F \overset{\theta}{\dashrightarrow}$ and $F \xrightarrow{w} F \xrightarrow{q} E \overset{\eta}{\dashrightarrow}$ be $\mathbb{E}$-conflations. There exists a commutative diagram*

$$\begin{tikzcd}
A \arrow[r, dashed, "f"] \arrow[d, equal] & B \arrow[r, dashed, "g"] \arrow[d, dashed, "u"'] & D \arrow[r, dashed, "\delta"] \arrow[d, "w"] & {} \\
A \arrow[r, "m"'] & C \arrow[r, "h"'] \arrow[d, dashed, "v"'] & F \arrow[r, dashed, "\theta"] \arrow[d, "q"] & {} \\
 & E \arrow[r, equal] \arrow[d, dashed, "\varepsilon"] & E \arrow[d, dashed, "\eta"] & \\
 & {} & {} &
\end{tikzcd} \tag{1.1.9}$$

such that $(1_A;u;w)$, $(f;1_C;q)$ and $(g;h;1_E)$ are morphisms of $\mathbb{E}$-conflations.

1.2 Corollaries of Six-term Long Exact Sequences

Lemma 1.1 (**Corollary 3.12.** [11])**.** *For any conflation $X \xrightarrow{f} Y \xrightarrow{g} Z \overset{\delta}{\dashrightarrow}$, one has the following two exact sequences of functors:*

$$\mathcal{C}(-,X) \xrightarrow{\mathcal{C}(-,f)} \mathcal{C}(-,Y) \xrightarrow{\mathcal{C}(-,g)} \mathcal{C}(-,Z) \xrightarrow{\delta_\sharp} \mathbb{E}(-,X) \xrightarrow{f_*} \mathbb{E}(-,Y) \xrightarrow{g_*} \mathbb{E}(-,Z), \tag{1.2.1}$$

$$\mathcal{C}(Z,-) \xrightarrow{\mathcal{C}(g,-)} \mathcal{C}(Y,-) \xrightarrow{\mathcal{C}(f,-)} \mathcal{C}(X,-) \xrightarrow{\delta^\sharp} \mathbb{E}(Z,-) \xrightarrow{g^*} \mathbb{E}(Y,-) \xrightarrow{f^*} \mathbb{E}(X,-). \tag{1.2.2}$$

Here $\delta_\sharp : \mathcal{C}(-,Z) \to \mathbb{E}(-,X)$ is a natural transformation sending $T \xrightarrow{\varphi} Z$ to $\varphi^*\delta$, and $\delta^\sharp : \mathcal{C}(X,-) \to \mathbb{E}(Z,-)$ is a natural transformation sending $X \xrightarrow{\psi} T$ to $\psi_*\delta$.

Corollary 1.2. *We show some corollaries of six-term long exact sequences.*

1. *(**Corollary 3.5.** [11]). Let $X \xrightarrow{f} Y \xrightarrow{g} Z \overset{\delta}{\dashrightarrow}$ be an $\mathbb{E}$-conflation. Then f is a section if and only if g is a retraction if and only if $\delta = 0$.*

2. *A monic deflation is a section, and an epic inflation is an retraction.*

3. *(**Corollary 3.6.** [11]). Let $(\alpha;\beta;\gamma)$ be a morphism of $\mathbb{E}$-conflations. If two of α,β,γ are isomorphisms, so is the third one.*

4. *Any $\mathbb{E}$-inflation ($\mathbb{E}$-deflation) fits into an $\mathbb{E}$-conflation unique up to isomorphisms.*

We only show the second statement here.

Proof. We consider an $\mathbb{E}$-conflation $X \xrightarrow{f} Y \xrightarrow{g} Z \overset{\delta}{\dashrightarrow}$ where g is monic. By eq. (1.2.1), $\mathcal{C}(-,f)$ is zero. Hence, $f = 0$. By eq. (1.2.2), $\mathcal{C}(g,-)$ is epic. Thus, for the identity morphism 1_Y, there exists $h : Y \to X$ such that $hf = 1_Y$. Therefore, f is a section. The dual argument is analogous. □

Thanks to *2.* in corollary 1.2, we obtain two strict forms of ET4 axiom.

Lemma 1.3. *Let $A \xrightarrow{f} B \xrightarrow{g} D \overset{\delta}{\dashrightarrow}$, $B \xrightarrow{u} C \xrightarrow{v} E \overset{\varepsilon}{\dashrightarrow}$, and $A \xrightarrow{m} C \xrightarrow{h} F \overset{\theta}{\dashrightarrow}$ be conflations. Then there exists a commutative diagram*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, "u"'] & D \arrow[r, dashed, "\delta"] \arrow[d, dashed, "w"] & {} \\
A \arrow[r, "m"'] & C \arrow[r, "h"'] \arrow[d, "v"'] & F \arrow[r, dashed, "\theta"] \arrow[d, dashed, "q"] & {} \\
 & E \arrow[r, equal] \arrow[d, dashed, "\varepsilon"] & E \arrow[d, dashed, "\eta"] & \\
 & {} & {} &
\end{tikzcd} . \tag{1.2.3}$$

which satisfy the condition in ET4 axiom.

Proof. We apply ET4-axiom to conflations realising from δ and ε. By *4.* in corollary 1.2, $m = uf$ fits into a conflation of the form

$$A \xrightarrow{m} C \xrightarrow{\varphi^{-1}h} F' \overset{\varphi^*\theta}{\dashrightarrow} .$$

Here $\varphi : F' \to F$ is an isomorphism. $\square$

There is another strict form of ET4 axiom.

Lemma 1.4. *Let $A \xrightarrow{f} B \xrightarrow{g} D \overset{\delta}{\dashrightarrow}$, $D \xrightarrow{w} F \xrightarrow{q} E \overset{\eta}{\dashrightarrow}$, and $A \xrightarrow{m} C \xrightarrow{h} F \overset{\theta}{\dashrightarrow}$ be conflations. Then there exists a commutative diagram*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, dashed, "u"'] & D \arrow[r, dashed, "\delta"] \arrow[d, "w"] & {} \\
A \arrow[r, "m"'] & C \arrow[r, "h"'] \arrow[d, dashed, "v"'] & F \arrow[r, dashed, "\theta"] \arrow[d, "q"] & {} \\
 & E \arrow[r, equal] \arrow[d, dashed, "\varepsilon"] & E \arrow[d, dashed, "\eta"] & \\
 & {} & {} &
\end{tikzcd} . \tag{1.2.4}$$

Proof. Note that the deflation $v = qh$ is uniquely determined. We take arbitrary realisation of ε. We apply lemma 1.3 for realisations of ε, θ and η, there is an conflation $A \xrightarrow{\varphi m} B' \xrightarrow{g\varepsilon^{-1}} D \overset{\delta}{\dashrightarrow}$. Here $\delta = w^*\theta$ is uniquely determined, and $\varphi : B' \to B$ is an isomorphism. $\square$

1.3 Pullbacks of Two $\mathbb{E}$-Deflations

ET4 shows that pulling back (pushing out) an inflation along a deflation yields four merged conflations. There is also a result for pushing out (pulling back) two inflations (deflations) along each other.

Proposition 1.5 (**Proposition 3.15.** [11])**.** *Let $A_1 \xrightarrow{f_1} B_1 \xrightarrow{g_1} C \overset{\delta_1}{\dashrightarrow}$ and $A_2 \xrightarrow{f_2} B_2 \xrightarrow{g_2} C \overset{\delta_2}{\dashrightarrow}$ be two conflations. Then there exists a commutative diagram*

$$\begin{tikzcd}
 & A_2 \arrow[r, equal] \arrow[d, dashed, "e_2"] & A_2 \arrow[d, "f_2"] & \\
A_1 \arrow[r, dashed, "e_1"] \arrow[d, equal] & E \arrow[r, dashed, "p_2"] \arrow[d, dashed, "p_1"] & B_2 \arrow[r, dashed, "(g_2)^*\delta_1"] \arrow[d, "g_2"] & {} \\
A_1 \arrow[r, "f_1"] & B_1 \arrow[r, "g_1"] \arrow[d, dashed, "(g_1)^*\delta_2"] & C \arrow[r, dashed, "\delta_1"] \arrow[d, dashed, "\delta_2"] & {} \\
 & {} & {} &
\end{tikzcd} , \tag{1.3.1}$$

such that $(1_{A_1}; p_2; g_2)$, $(1_{A_2}; p_1; g_1)$ are morphisms of $\mathbb{E}$-conflations, and $(e_1)_*\delta_1 + (e_2)_*\delta_2 = 0$.

Proposition 1.6. *We may choose $A_1 \xrightarrow{e_1} E \xrightarrow{p_2} B_2 \overset{(g_2)^*\delta_1}{\dashrightarrow}$ in proposition 1.5 to be any conflation realised from $(g_2)^*\delta_1$.*

Proof. The proof is similar to that of lemma 1.4, by *2.* in corollary 1.2. $\square$

Remark. We denote $e_1 : A_1 \to E$ and $e_2 : A_2 \to E$ in a general diagram eq. (1.3.1) consisting of four conflations, three commutative squares. There is no $e_{1*}\delta_1 + e_{2*}\delta_2 = 0$ in general. For instance, consider the following diagram in a triangulated category with shift functor Σ:

$$\begin{tikzcd}
 & X \arrow[r, equal] \arrow[d, dashed, "\varphi"] & X \arrow[d] & \\
X \arrow[r, dashed, "\psi"] \arrow[d, equal] & X \arrow[r, dashed] \arrow[d, dashed] & 0 \arrow[r, dashed, "0"] \arrow[d] & {} \\
X \arrow[r] & 0 \arrow[r] \arrow[d, dashed, "0"] & \Sigma X \arrow[r, dashed, "1_{\Sigma X}"] \arrow[d, dashed, "1_{\Sigma X}"] & {} \\
 & {} & {} &
\end{tikzcd} \tag{1.3.2}$$

φ and ψ are chosen to be arbitrary isomorphisms. We do not have $\varphi_*(1_{\Sigma X}) + \psi_*(1_{\Sigma X}) = 0$ in general.

Proposition 1.7 (**Proposition 3.17.** [11])**.** *Given three conflations (in solid arrows), there is a way to complete the diagram*

$$\begin{tikzcd}
 & A_2 \arrow[r, equal] \arrow[d, "e_2"] & A_2 \arrow[d, dashed, "f_2"] & \\
A_1 \arrow[r, "e_1"] \arrow[d, equal] & E \arrow[r, "p_2"] \arrow[d, "p_1"] & B_2 \arrow[r, dashed, "\eta"] \arrow[d, dashed, "g_2"] & {} \\
A_1 \arrow[r, "f_1"] & B_1 \arrow[r, "g_1"] \arrow[d, dashed, "\varepsilon"] & C \arrow[r, dashed, "\delta_1"] \arrow[d, dashed, "\delta_2"] & {} \\
 & {} & {} &
\end{tikzcd} . \tag{1.3.3}$$

which satisfy the condition of proposition 1.8.

1.4 Pushouts of Two $\mathbb{E}$-Inflations

We revisit the dual statements section 1.3.

Proposition 1.8 (Dual to proposition 1.5)**.** *Let $A \xrightarrow{f_1} B_1 \xrightarrow{g_1} C_1 \overset{\delta_1}{\dashrightarrow}$ and $A \xrightarrow{f_2} B_2 \xrightarrow{g_2} C_2 \overset{\delta_2}{\dashrightarrow}$ be two conflations. Then there exists a commutative diagram*

$$\begin{tikzcd}
A \arrow[r, "f_2"] \arrow[d, "f_1"] & B_2 \arrow[r, "g_2"] \arrow[d, dashed, "e_2"] & C_2 \arrow[r, dashed, "\varepsilon_2"] \arrow[d, equal] & {} \\
B_1 \arrow[r, dashed, "e_1"] \arrow[d, "g_1"] & E \arrow[r, dashed, "p_2"] \arrow[d, dashed, "p_1"] & C_2 \arrow[r, dashed, "(f_1)_*\varepsilon_2"] & {} \\
C_1 \arrow[r, equal] \arrow[d, dashed, "\varepsilon_1"] & C_1 \arrow[d, dashed, "(f_2)_*\varepsilon_1"] & & \\
{} & {} & &
\end{tikzcd} , \tag{1.4.1}$$

such that $(f_1; 1_{C_2}; p_2)$, $(f_2; 1_{C_1}; p_1)$ are morphisms of $\mathbb{E}$-conflations, and $(f_1)_*\varepsilon_2 + (f_2)_*\varepsilon_1 = 0$.

Proposition 1.9 (Dual to proposition 1.6)**.** *We may choose $B_1 \xrightarrow{e_1} E \xrightarrow{p_2} C_2 \overset{(f_1)_*\varepsilon_2}{\dashrightarrow}$ in proposition 1.8 to be any conflation realised from $(f_1)_*\varepsilon_2$.*

Proposition 1.10 (Dual to proposition 1.7)**.** *Given three conflations (in solid arrows), there is a way to complete the diagram*

$$\begin{tikzcd}
A \arrow[r, "f_2"] \arrow[d, dashed, "f_1"] & B_2 \arrow[r, "g_2"] \arrow[d, "e_2"] & C_2 \arrow[r, dashed, "\varepsilon_2"] \arrow[d, equal] & {} \\
B_1 \arrow[r, "e_1"] \arrow[d, dashed, "g_1"] & E \arrow[r, "p_2"] \arrow[d, "p_1"] & C_2 \arrow[r, dashed, "\eta"] & {} \\
C_1 \arrow[r, equal] \arrow[d, dashed, "\varepsilon_1"] & C_1 \arrow[d, dashed, "\varepsilon"] & & \\
{} & {} & &
\end{tikzcd} , \tag{1.4.2}$$

which satisfy the condition of proposition 1.8.

2 Homotopic Square

2.1 Homotopic squares and morphisms

The concept of homotopic squares originated from triangulated categories ([1]), and was generalised to n-angulated ([9]) and extriangulated ([4]) cases. This concept is a generalisation of both pullback-and-pushout squares in exact categories, and homotopic bicartesian squares in triangulated categories.

Definition 2.1 (**Definition 3.1.** [4])**.** A *homotopic square* in an extriangulated category is a commutative diagram

$$\begin{tikzcd} A_1 \arrow[r, "u"] \arrow[d, "f"] & B_1 \arrow[d, "g"] \\ A_2 \arrow[r, "v"] & B_2 \end{tikzcd}, \tag{2.1.1}$$

such that $A_1 \xrightarrow{\binom{f}{u}} B_1 \oplus A_2 \xrightarrow{(v,-g)} B_2 \dashrightarrow$ is a conflation.

Remark. There are various of names of homotopic squares in literature, e.g. homotopy bicartesian squares, homotopy pullback squares, Mayer-Vietoris squares, or distinguished weak squares. We use the name *homotopic square* for simplicity.

Notation. We use $\boxed{\varepsilon}$ to denote the extension element associated with the homotopic square as in eq. (2.1.1).

$$\begin{tikzcd} A_1 \arrow[r, "u"] \arrow[d, "f"] \arrow[dr, phantom, "\boxed{\varepsilon}"] & B_1 \arrow[d, "g"] \\ A_2 \arrow[r, "v"] & B_2 \end{tikzcd} \qquad A_1 \xrightarrow{\binom{f}{-u}} B_1 \oplus A_2 \xrightarrow{(v,g)} B_2 \overset{\varepsilon}{\dashrightarrow} . \tag{2.1.2}$$

The circled arrow indicates the morphism with a negative sign in the $\mathbb{E}$-conflation. We omit the content in $\square$ and the circled arrow when there is no confusion.

Proposition 2.2. *Homotopic squares are weak pullback and weak pushout squares.*

Proof. To show eq. (2.1.2) is a weak pullback square, it is equivalent to show that $\binom{f}{-u}$ is a weak kernel of (v,g). This is clear by long exact sequences eq. (1.2.1). The dual statement is similar. $\square$

Definition 2.3. Say a morphism of $\mathbb{E}$-conflations $(\alpha;\beta;\gamma)$ is *homotopic*, provided

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\boxed{t^*\kappa}"] & Y \arrow[r, "g"] \arrow[d, dashed, "\beta_1"] & Z \arrow[r, dashed, "\kappa"] \arrow[d, equal] & {} \\ A \arrow[r, "s"] \arrow[d, equal] & E \arrow[r, "t"] \arrow[d, dashed, "\beta_2"'] \arrow[dr, phantom, "\boxed{s_*\varepsilon}"] & Z \arrow[r, dashed] \arrow[d, "\gamma"] & {} \\ A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, dashed, "\varepsilon"] & {} \end{tikzcd} \quad (\beta=\beta_2\circ\beta_1) . \tag{2.1.3}$$

We revisit some results in completing two morphisms into a homotopic square.

Lemma 2.4 (**Proposition 1.20.** [10])**.** *Let $(f;1_Z)$ be a morphism of extensions. We can find a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\boxed{v'^*\delta}"] & Y \arrow[r, "v"] \arrow[d, dashed, "g"] & Z \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z \arrow[r, dashed, "f_*\delta"] & {} \end{tikzcd}, \tag{2.1.4}$$

such that $(f;g;1_Z)$ is a homotopic morphism of $\mathbb{E}$-conflations.

Lemma 2.5 (**Theorem 3.3.** in [8])**.** *For any $\mathbb{E}$-deflations v and v' with $v'g = v$, one can find a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, dashed, "f"'] \arrow[dr, phantom, "\boxed{v'^*\delta}"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z \arrow[r, dashed, "f_*\delta"] & {} \end{tikzcd}, \tag{2.1.5}$$

such that $(f;g;1_Z)$ is a homotopic morphism of $\mathbb{E}$-conflations.

Lemma 2.6 (Dual to lemma 2.4)**.** *Let $(1_X;h)$ be a morphism of extensions. We can find a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, dashed, "g"] \arrow[dr, phantom, "\boxed{u_*\delta}"] & Z \arrow[r, dashed, "h^*\varepsilon"] \arrow[d, "h"] & {} \\ X \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, dashed, "\varepsilon"] & {} \end{tikzcd}, \tag{2.1.6}$$

such that $(1_X;g;h)$ is a homotopic morphism of $\mathbb{E}$-conflations.

Lemma 2.7 (Dual to lemma 2.5)**.** *For any $\mathbb{E}$-inflations u and u' with $u'f = u$, one can find a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, "g"] \arrow[dr, phantom, "\boxed{u_*\delta}"] & Z \arrow[r, dashed, "h^*\varepsilon"] \arrow[d, dashed, "h"] & {} \\ X \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, dashed, "\varepsilon"] & {} \end{tikzcd}, \tag{2.1.7}$$

such that $(1_X; g; h)$ is a homootpic morphism of $\mathbb{E}$-conflations.

The above lemmas demonstrate that the completion of morphisms of $\mathbb{E}$-conflations in ET2, ET3, ET3$^{\mathrm{op}}$ can be made homotopic.

Theorem 2.8. *Let $(\alpha; \beta; \gamma)$ be a morphism of $\mathbb{E}$-conflations. Then there are modifications $(\alpha'; \beta; \gamma)$, $(\alpha; \beta'; \gamma)$, and $(\alpha; \beta; \gamma')$ which are all homotopic morphisms of $\mathbb{E}$-conflations.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, dashed, "\delta"] \arrow[d, "\gamma"] & {} \\ A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, dashed, "\varepsilon"] & {} \end{tikzcd} \quad (\alpha_*\delta = \gamma^*\varepsilon). \tag{2.1.8}$$

Proof. We show the existence of β'. We realise $\alpha_*\delta = \gamma^*\varepsilon$ by any $\mathbb{E}$-conflation, and take β_1 and β_2 by lemma 2.4 and lemma 2.6 respectively.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\square"] & Y \arrow[r, "g"] \arrow[d, dashed, "s"] \arrow[dd, bend left, "\beta"] & Z \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\ A \arrow[r, dashed, "a"] \arrow[d, equal] & M \arrow[r, dashed, "b"] \arrow[d, dashed, "t"] \arrow[dr, phantom, "\square"] & Z \arrow[r, dashed, "\alpha_*\delta=\gamma^*\varepsilon"] \arrow[d, "\gamma"] & {} \\ A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, dashed, "\varepsilon"] & {} \end{tikzcd}. \tag{2.1.9}$$

Then $\beta' = \beta_2 \circ \beta_1$ gives the desired modification.
We show the existence of α'. By lemma 2.6, we can find a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "g"] \arrow[d, dashed, "s"] \arrow[dd, bend right, "\beta"'] & Z \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\ A \arrow[r, "a"] \arrow[d, equal] & M \arrow[r, "b"] \arrow[d, "t"] \arrow[dr, phantom, "\square"] & Z \arrow[r, dashed, "\gamma^*\varepsilon"] \arrow[d, "\gamma"] & {} \\ A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, dashed, "\varepsilon"] & {} \end{tikzcd}. \tag{2.1.10}$$

Since $\square$ is a weak pullback square (proposition 2.2), there is s such that $ts = \beta$ and $bs = g$. We complete $\alpha : X \to A$ by lemma 2.5. The existence of γ' is dual to that of α'. $\square$

2.2 Morphism of $\mathbb{E}$-conflations $(f; g; 1)$ revisited

We examine how $(f; g; 1)$ fails to be a homotopic morphism of conflations. Here we fix

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z \arrow[r, dashed, "f_*\delta"] & {} \end{tikzcd}. \tag{2.2.1}$$

Lemma 2.9. *For mapping sequence $X \xrightarrow{\binom{u}{f}} Y \oplus X' \xrightarrow{(g,-u')} Y' \overset{v'^*\delta}{\dashrightarrow}$ associated to eq. (2.2.1), we have $(g, -u') \circ \binom{u}{f} = 0$, $(g, -u')^*(v'^*\delta) = 0$ and $\binom{u}{f}_*(v'^*\delta) = 0$.*

Proof. The commutative diagram shows $(g, -u') \circ \binom{u}{f} = 0$. We can also check

$$(g, -u')^*(v'^*\delta) = (v' \circ (g, -u'))^*\delta = (v, 0)^*\delta = 0. \tag{2.2.2}$$

By lemma 2.4 and long exact sequence eq. (1.2.2), we have $\binom{u}{f}_*(v'^*\delta) = 0$. $\square$

Proposition 2.10. *In comparison to eq. (1.2.2), we have the following 6-term chain complex*

$$\mathcal{C}(Y', -) \xrightarrow{\mathcal{C}((g,-u'),-)} \underset{\triangle}{\mathcal{C}(Y \oplus X', -)} \xrightarrow{\mathcal{C}(\binom{u}{f},-)} \underset{\triangle}{\mathcal{C}(X, -)} \xrightarrow{((v')^*\delta)^\sharp} \mathbb{E}(Y', -) \xrightarrow{(g,-u')^*} \underset{\triangle}{\mathbb{E}(Y \oplus X', -)} \xrightarrow{(\binom{u}{f})^*} \mathbb{E}(X, -), \tag{2.2.3}$$

which is exact at $\mathcal{C}(Y \oplus X', -)$, $\mathcal{C}(X, -)$, and $\mathbb{E}(Y \oplus X, -)$ (labelled by $\triangle$).

Proof. We show exactness at each position.

1. (Exactness at $\mathcal{C}(Y \oplus X', -)$). By lemma 2.9, $\ker \mathcal{C}(\binom{u}{f}, -) \supseteq \operatorname{im} \mathcal{C}((g, -u'), -)$. For the converse, we take (a, b) such that $(a, b)\binom{u}{f} = 0$. Since $b_*(f_*\delta) = (au)_*\delta = 0$, we find s such that $su' = b$

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] \arrow[ddr, bend left, "a"] & Z \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\ X' \arrow[r, "u'"] \arrow[drr, bend right, "b"'] & Y' \arrow[r, "v'"] \arrow[dr, dashed, "s"] & Z \arrow[r, dashed, "f_*\delta"] \arrow[d, dashed, bend left, "t"] & {} \\ & & T & \end{tikzcd} . \tag{2.2.4}$$

Since $(sg - a)u = (su'f - au) = 0$, there is t such that $tv = (sg - a)$. We can verify that

$$(s - tv')u' = su' = b, \quad (s - tv')g = sg - tv'gsg - (sg - a) = a. \tag{2.2.5}$$

Hence, (a, b) is in the image of $\mathcal{C}((g, -u'), -)$. It also shows that the left square is a weak poshout.

2. (Exactness at $\mathcal{C}(X, -)$). There exists a homotopic square $X \xrightarrow{\binom{u}{f}} Y \oplus X' \xrightarrow{(\overline{g}, -u')} Y' \overset{(v')^*\delta}{\dashrightarrow}$ for some $\overline{g}$ (lemma 2.4). By eq. (1.2.2), the exactness holds.

3. (At $\mathbb{E}(Y', -)$). We show $\operatorname{im}((v')^*\delta)^\sharp \subseteq \ker(g, -u')^*$. For any $X \xrightarrow{\varphi} \cdot$, we have $(g, -u')^*(v')^*\delta(\varphi) = \varphi_*(v, 0)^*\delta = 0$.

4. (Exactness at $\mathbb{E}(Y \oplus X', -)$). $(\overline{g}, -u')\binom{u}{f} = 0$ is clear. Conversely, we take any $\varphi \in \mathbb{E}(Y \oplus X', T)$ such that $\binom{u}{f}^* \varepsilon = 0$. Note that there is a conflation $X \xrightarrow{\binom{u}{f}} Y \oplus X' \xrightarrow{(\overline{g}, -u')} Y' \overset{(v')^*\delta}{\dashrightarrow}$, hence $\varepsilon = (\overline{g}, -u')^*\eta$ for some $\eta \in \mathbb{E}(Y', T)$. By weak pushout square, we obtain s such that $s(g, -u') = (\overline{g}, -u')$:

$$\begin{tikzcd} & & X \arrow[dll, dashed, bend right] \arrow[d, "\binom{u}{f}"] & \\ T \arrow[r, "i"] \arrow[d, equal] & M \arrow[r, "p"] \arrow[d] & Y \oplus X' \arrow[r, dashed, "\varepsilon"] \arrow[d, "(\overline{g}, -u')"] & {} \\ T \arrow[r] & N \arrow[r] & Y' \arrow[r, dashed, "\eta"] \arrow[d, dashed] & {} \\ & & {} & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \arrow[ddr, bend left, "\overline{g}"] & \\ X' \arrow[r, "u'"'] \arrow[drr, bend right, "u'"'] & Y' \arrow[dr, dashed, "s"] & \\ & & Y' \end{tikzcd} . \tag{2.2.6}$$

Hence $\varepsilon = (g, -u')^*(s^*\eta)$.

□

We show that the suffient criterion for in [3] is also valid in extriangulated categories.

Condition (Condition C). *Say a unital ring R satisfies condition **C**, if it satisfies **C1** and **C2**.*

C1 *For any $r \in R$, there exists $a \in R$ such that $1 + r + ar^2$ is a unit in R, and*

C2 *For any $r \in R$, there exists $b \in R$ such that $1 + r + r^2b$ is a unit in R.*

*For instance, a finite dimensional algebra over a field satisfies **C**.*

The next theorem is sligtly different from **Proposition 2.1** in [3].

Proposition 2.11. *If Y' in eq. (2.2.1) satisfies condition **C1**, then □ is a homotopic square:*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\ X' \arrow[r, circ, "u'"] & Y' \arrow[r, "v'"] & Z \arrow[r, dashed, "f_*\delta"] & {} \end{tikzcd} . \tag{2.2.7}$$

The extension element associated to □ is $\theta^*(v'^*\delta)$ for some automorphism $\theta \in \operatorname{Aut}(Y')$.

Proof. By lemma 2.4, there is $\overline{g} : Y \to Y'$ such that $X \xrightarrow{\binom{u}{f}} Y \oplus X' \xrightarrow{(\overline{g}, -u')} Y' \overset{(v')^*\delta}{\dashrightarrow}$ is an $\mathbb{E}$-conflation. Since $(g - \overline{g}) \circ u = 0$, there is $\varphi : Z \to Y'$ such that $\varphi \circ v = (\overline{g} - g)$:

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, shift right, "g"'] \arrow[d, shift left, "\overline{g}"] & Z \arrow[r, dashed, "\delta"] \arrow[d, equal] \arrow[dl, dashed, "\varphi"'] & {} \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z \arrow[r, dashed, "f_*\delta"] & {} \end{tikzcd} . \tag{2.2.8}$$

By assumption **C1**, there is some a such that $1+(\varphi v')+a(\varphi v')^2$ is a unit. We can verify

$$(1+(\varphi v')+a(\varphi v')^2)u' = u' + (\varphi + a\varphi v'\varphi)\circ(v'u') = u', \tag{2.2.9}$$

and

$$(1+(\varphi v')+a(\varphi v')^2)g = \overline{g} - \varphi\circ v + \varphi(v'g) + a\varphi v'\varphi v' g = \overline{g} + a\varphi v'(\overline{g}-g) = \overline{g}. \tag{2.2.10}$$

$\square$

Proposition 2.12. *If Y in eq. (2.2.11) satisfies condition **C2**, then $\square$ is a homotopic square:*

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, "g"] \arrow[dr, phantom, "\square"] & Z \arrow[r, dashed, "h^*\varepsilon"] \arrow[d, "h"] & {} \\
X \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, dashed, "\varepsilon"] & {}
\end{tikzcd}. \tag{2.2.11}$$

The extension element associated to $\square$ is $\theta_*(h^*\varepsilon)$ for some automorphism $\theta\in \mathrm{Aut}(Y)$.

Proposition 2.13. *The left square in eq. (2.2.1) is not always homotopic, see **Section 3** in [3].*

2.3 More examples of homotopic morphisms

We show more examples of homotopic morphisms.

Example 2.14. Let $(\mathcal{A},\mathcal{E})$ be an Ext^1-small exact category. It has a natural extriangulated structure (**Example 2.13.** [11]). In this case,

1. all $\mathbb{E}$-conflations are exactly short exact sequences in $\mathcal{E}$,
2. any homotopic square is both a pushout and a pullback square, and
3. any morphisms of conflations are homotopic.

Lemma 2.15. *Let $(f;g;h)$ be a homotopic morphism of $\mathbb{E}$-conflations. Suppose there is an isomorphism of $\mathbb{E}$-conflations $(\alpha;\beta;\gamma)$ such that $(f\circ\alpha; g\circ\beta; h\circ\gamma)$ is composable. Then $(f\circ\alpha; g\circ\beta; h\circ\gamma)$ is also a homotopic morphism of $\mathbb{E}$-conflations.*

Proof. We consider the following diagram:

$$\begin{tikzcd}
X' \arrow[r, "\alpha", "\simeq"'] \arrow[d, "u'"] & X \arrow[r, "f"] \arrow[d, "u"] \arrow[dr, phantom, "\boxed{t^*\kappa}"] & A \arrow[r, equal] \arrow[d, "s"] & A \arrow[d, "m"] \\
Y' \arrow[r, "\beta", "\simeq"'] \arrow[d, "v'"] & Y \arrow[r, "g_1"] \arrow[d, "v"] & E \arrow[r, "g_2"] \arrow[d, "t"] \arrow[dr, phantom, "\boxed{s_*\varepsilon}"] & B \arrow[d, "n"] \\
Z' \arrow[r, "\gamma", "\simeq"'] \arrow[d, dashed, "\kappa'"] & Z \arrow[r, equal] \arrow[d, dashed, "\kappa"] & Z \arrow[r, "h"] \arrow[d, dashed] & C \arrow[d, dashed, "\varepsilon"] \\
{} & {} & {} & {}
\end{tikzcd}. \tag{2.3.1}$$

It suffices to show the following diagram is a homotopic morphism of $\mathbb{E}$-conflations:

$$\begin{tikzcd}
X' \arrow[r, "f\alpha"] \arrow[d, "u'"'] \arrow[dr, phantom, "\boxed{(\gamma^{-1})t^*\kappa}"] & A \arrow[r, equal] \arrow[d, "s"] & A \arrow[d, "m"'] \\
Y' \arrow[r, "g_1\beta"'] \arrow[d, "v'"] & E \arrow[r, "g_2"] \arrow[d, "\gamma^{-1}t"'] \arrow[dr, phantom, "\boxed{s_*\varepsilon}"] & B \arrow[d, "n"] \\
Z' \arrow[r, equal] \arrow[d, dashed, "\kappa'"] & Z' \arrow[r, "h\gamma"'] \arrow[d, dashed] & C \arrow[d, dashed, "\varepsilon"] \\
{} & {} & {}
\end{tikzcd}. \tag{2.3.2}$$

The verification of $\boxed{s_*\varepsilon}$ is clear. Note that

$$\begin{tikzcd}
X' \arrow[r, "\binom{f\alpha}{u'}"] \arrow[d, "\alpha"] & A\oplus Y' \arrow[r, "{(-s,g_1\beta)}"] \arrow[d, "1\oplus\beta"] & E \arrow[r, dashed, "(\gamma^{-1}t)^*\kappa"] \arrow[d, equal] & {} \\
X \arrow[r, "\binom{f}{u}"] & A\oplus Y \arrow[r, "{(-s,g_1)}"] & E \arrow[r, dashed, "t^*\kappa"] & {}
\end{tikzcd}. \tag{2.3.3}$$

The diagram is commutative and $\alpha_*(\gamma^{-1}t)^*\kappa = (\gamma^*)^{-1}(\alpha_*)\kappa = t^*\kappa$. Hence, $\boxed{(\gamma^{-1})t^*\kappa}$ is verified. $\square$

Proposition 2.16. *Homotopic morphisms are not closed under composition. Indeed, any morphism of $\mathbb{E}$-conflations is a compostion of two homotopic morphisms.*

Proof. Let $(\alpha;\beta;\gamma)$ be a morphism of $\mathbb{E}$-conflations. We consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\binom{1}{0}"] & Y \arrow[r, "g"] \arrow[d, "\binom{1}{0}"] & Z \arrow[r, dashed, "\delta"] \arrow[d, "\binom{1}{0}"] & {} \\
X \oplus A \arrow[r, "\begin{pmatrix} f & 0 \\ 0 & u \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}", "\simeq"'] & Y \oplus B \arrow[r, "\begin{pmatrix} g & 0 \\ 0 & v \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}", "\simeq"'] & Z \oplus C \arrow[r, dashed, "\delta\oplus\varepsilon"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}", "\simeq"'] & {} \\
X \oplus A \arrow[r, "\begin{pmatrix} f & 0 \\ 0 & u \end{pmatrix}"] \arrow[d, "{(0,1)}"] & Y \oplus B \arrow[r, "\begin{pmatrix} g & 0 \\ 0 & v \end{pmatrix}"] \arrow[d, "{(0,1)}"] & Z \oplus C \arrow[r, dashed, "\delta\oplus\varepsilon"] \arrow[d, "{(0,1)}"] & {} \\
A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, dashed, "\varepsilon"] & {}
\end{tikzcd} . \tag{2.3.4}$$

Here $(\binom{1}{0};\binom{1}{0};\binom{1}{0})$ and $((0,1);(0,1);(0,1))$ are morphisms of $\mathbb{E}$-conflations. We show $(\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix};\begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix};\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix})$ is an automorphism of $\mathbb{E}$-conflations. The commutativity of the left square is due to

$$\begin{pmatrix} f & 0 \\ 0 & u \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \circ \begin{pmatrix} f & 0 \\ 0 & u \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ u\alpha - \beta f & 0 \end{pmatrix} = 0. \tag{2.3.5}$$

The commutativity of the right square is dually verified. We show that the extension elements are equal:

$$\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}^* (\delta\oplus\varepsilon) - \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}_* (\delta\oplus\varepsilon) = (\delta, \gamma^*\varepsilon, 0, \varepsilon) - (\delta, \alpha_*\delta, 0, \varepsilon) = 0. \tag{2.3.6}$$

Here the elements are identified in $\mathbb{E}(Z\oplus C, X\oplus A) \cong \mathbb{E}(Z,X)\oplus\mathbb{E}(Z,A)\oplus\mathbb{E}(C,X)\oplus\mathbb{E}(C,A)$.
By lemma 2.15, it remains to verify that both $(\binom{1}{0};\binom{1}{0};\binom{1}{0})$ and $((0,1);(0,1);(0,1))$ are homotopic morphisms of $\mathbb{E}$-conflations. We only verify $(\binom{1}{0};\binom{1}{0};\binom{1}{0})$. Consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\binom{1}{0}"] \arrow[dr, phantom, "\boxed{(g,0)^*\delta}"] & Y \arrow[r, "g"] \arrow[d, "\binom{1}{0}"] & Z \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\
X \oplus A \arrow[r, "f\oplus 1_A"'] \arrow[d, equal] & Y \oplus A \arrow[r, "{(g,0)}"] \arrow[d, "1_Y\oplus u"'] \arrow[dr, phantom, "\boxed{(f\oplus 1)_*(\delta\oplus\varepsilon)}"] & Z \arrow[r, dashed, "\delta\oplus 0_A"] \arrow[d, "\binom{1}{0}"] & {} \\
X \oplus A \arrow[r, "\begin{pmatrix} f & 0 \\ 0 & u \end{pmatrix}"'] & Y \oplus B \arrow[r, "\begin{pmatrix} g & 0 \\ 0 & v \end{pmatrix}"'] & Z \oplus C \arrow[r, dashed, "\delta\oplus\varepsilon"] & {}
\end{tikzcd} . \tag{2.3.7}$$

We verify the homotopy element $\boxed{(g,0)^*\delta}$. Note that the following is a split $\mathbb{E}$-conflation:

$$X \xrightarrow{\begin{pmatrix} f \\ -1 \\ 0 \end{pmatrix}} Y\oplus X\oplus A \xrightarrow{\begin{pmatrix} 1 & f & 0 \\ 0 & 0 & 1 \end{pmatrix}} Y\oplus A \overset{(g,0)^*\delta=0}{\dashrightarrow} . \tag{2.3.8}$$

The extension elment in the right buttom is $(f_*\delta)\oplus\varepsilon$. Note that

$$Y\oplus A \xrightarrow{\begin{pmatrix} g & 0 \\ 1 & 0 \\ 0 & u \end{pmatrix}} Z\oplus Y\oplus B \xrightarrow{\begin{pmatrix} 1 & -g & 0 \\ 0 & 0 & -v \end{pmatrix}} Z\oplus C \overset{\varepsilon\oplus 0}{\dashrightarrow} \tag{2.3.9}$$

is a direct sum of $\mathbb{E}$-conflations, which is again an $\mathbb{E}$-conflation. We complete the proof. $\square$

Proposition 2.17. *In lemma 1.3, we may choose w to be any morphism constructed from lemma 2.7. Then there is q such that the following diagram satisfy the condition in ET4 axiom.*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, "u"'] \arrow[dr, phantom, "\boxed{f_*\theta}"] & D \arrow[r, dashed, "\delta"] \arrow[d, dashed, "w"] & {} \\
A \arrow[r, "m"'] & C \arrow[r, "h"'] \arrow[d, "v"'] & F \arrow[r, dashed, "\theta"] \arrow[d, dashed, "q"] & {} \\
 & E \arrow[r, equal] \arrow[d, dashed, "\varepsilon"] & E \arrow[d, dashed, "\eta"] & \\
 & {} & {} &
\end{tikzcd} . \tag{2.3.10}$$

Proof. We take w as in lemma 2.5. By proposition 1.7, we take the conflation realising η in the following commutative diagram:

$$\begin{tikzcd}
& D \arrow[r, equal] \arrow[d, "\binom{1}{0}"] & D \arrow[d, dashed, "w"] & \\
B \arrow[r, "\binom{-g}{u}"] \arrow[d, equal] & D \oplus C \arrow[r, "{(w,h)}"] \arrow[d, "{(0,1)}"] & F \arrow[r, dashed, "f_*\theta"] \arrow[d, dashed, "q"] & {} \\
B \arrow[r, "u"] & C \arrow[r, "v"] \arrow[d, dashed, "0"] & E \arrow[r, dashed, "\varepsilon"] \arrow[d, dashed, "\eta"] & {} \\
& {} & {} &
\end{tikzcd} . \tag{2.3.11}$$

We verify such construction satisfies ET4 axiom. It is straightforward to obtain $qh = v$ and $q^*\varepsilon = f_*\theta$ from the above diagram. Moreover, since $\binom{1}{0}_*\eta + \binom{-g}{u}_*\varepsilon = 0$, we have $g_*\varepsilon = \eta$. This complete the verification. ☐

Proposition 2.18. *In lemma 1.4, we may choose u to be any morphism constructed from lemma 2.6. Then there is a way to complete the diagram which satisfies ET4 axiom.*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, dashed, "u"'] \arrow[dr, phantom, "\boxed{f_*\theta}"] & D \arrow[r, dashed, "\delta"] \arrow[d, "w"] & {} \\
A \arrow[r, "m"'] & C \arrow[r, "h"'] \arrow[d, dashed, "v"'] & F \arrow[r, dashed, "\theta"] \arrow[d, "q"] & {} \\
& E \arrow[r, equal] \arrow[d, dashed, "\varepsilon"] & E \arrow[d, dashed, "\eta"] & \\
& {} & {} &
\end{tikzcd} . \tag{2.3.12}$$

Proof. We take u as in lemma 2.6. By proposition 1.10, we take the conflation realising ϵ in the following commutative diagram:

$$\begin{tikzcd}
& D \arrow[r, equal] \arrow[d, "\binom{1}{0}"] & D \arrow[d, "w"] & \\
B \arrow[r, "\binom{-g}{u}"] \arrow[d, equal] & D \oplus C \arrow[r, "{(w,h)}"] \arrow[d, "{(0,1)}"] & F \arrow[r, dashed, "f_*\theta"] \arrow[d, "q"] & {} \\
B \arrow[r, dashed, "u"] & C \arrow[r, dashed, "v"] \arrow[d, dashed, "0"] & E \arrow[r, dashed, "\varepsilon"] \arrow[d, dashed, "\eta"] & {} \\
& {} & {} &
\end{tikzcd} . \tag{2.3.13}$$

The verification of ET4 axiom is the same as in proposition 2.17. ☐

Proposition 2.19. *In proposition 1.9, we may choose e_2 to be any morphism constructed from lemma 2.4. Then there is a way to complete the diagram which satisfies the condition in proposition 1.8.*

$$\begin{tikzcd}
A \arrow[r, "f_2"] \arrow[d, "f_1"'] \arrow[dr, phantom, "\boxed{p_2^*\varepsilon_2}"] & B_2 \arrow[r, "g_2"] \arrow[d, dashed, "e_2"] & C_2 \arrow[r, dashed, "\varepsilon_2"] \arrow[d, equal] & {} \\
B_1 \arrow[r, "e_1"] \arrow[d, "g_1"'] & E \arrow[r, "p_2"] \arrow[d, dashed, "p_1"] & C_2 \arrow[r, dashed, "(f_1)_*\varepsilon_2"] & {} \\
C_1 \arrow[r, equal] \arrow[d, dashed, "\varepsilon_1"] & C_1 \arrow[d, dashed, "(f_2)_*\varepsilon_1"] & & \\
{} & {} & &
\end{tikzcd} . \tag{2.3.14}$$

Proof. We take e_2 as in lemma 2.4. By proposition 1.10, we take the conflation realising $-\kappa$ in the following commutative diagram:

$$\begin{tikzcd}
& B_2 \arrow[r, equal] \arrow[d, "\binom{0}{1}"] & B_2 \arrow[d, dashed, "e_2"] & \\
A \arrow[r, "\binom{f_1}{f_2}"] \arrow[d, equal] & B_1 \oplus B_2 \arrow[r, "{(-e_1,e_2)}"] \arrow[d, "{(1,0)}"] & E \arrow[r, dashed, "p_2^*\varepsilon_2"] \arrow[d, dashed, "-p_1"] & {} \\
A \arrow[r, "f_1"] & B_1 \arrow[r, "g_1"] \arrow[d, dashed, "0"] & C_1 \arrow[r, dashed, "\varepsilon_1"] \arrow[d, dashed, "-\kappa"] & {} \\
& {} & {} &
\end{tikzcd} . \tag{2.3.15}$$

The identity $\binom{0}{1}_*(-\kappa) + \binom{f_1}{f_2}_*\varepsilon_1 = 0$ yields $(f_2)_*\varepsilon_1 = \kappa$. We show such construction satisfies the condition in proposition 1.8. It is straightforward to obtain $p_1e_1 = g_1$ and $p_1^*\varepsilon_1 + p_2^*\varepsilon_2 = 0$ from the above diagram. This complete the verification. $\square$

Proposition 2.20. *In proposition 1.10, we may choose f_1 to be any morphism constructed from lemma 2.5. Then there is a way to complete the diagram which satisfies the condition in proposition 1.8.*

$$\begin{tikzcd}
A \arrow[r, "f_2"] \arrow[d, dashed, "f_1"'] \arrow[dr, phantom, "\boxed{p_2^*\varepsilon_2}"] & B_2 \arrow[r, "g_2"] \arrow[d, "e_2"] & C_2 \arrow[r, dashed, "\varepsilon_2"] \arrow[d, equal] & {} \\
B_1 \arrow[r, "e_1"] \arrow[d, dashed, "g_1"'] & E \arrow[r, "p_2"] \arrow[d, "p_1"] & C_2 \arrow[r, dashed, "\eta"] & {} \\
C_1 \arrow[r, equal] \arrow[d, dashed, "\varepsilon_1"] & C_1 \arrow[d, dashed, "\varepsilon"] & & \\
{} & {} & &
\end{tikzcd} . \tag{2.3.16}$$

Proof. We take f_1 as in lemma 2.5. By proposition 1.9, we take the conflation realising ε_1 in the following commutative diagram:

$$\begin{tikzcd}
 & B_2 \arrow[r, equal] \arrow[d, "\binom{0}{1}"] & B_2 \arrow[d, "e_2"] & \\
A \arrow[r, "\binom{f_1}{f_2}"] \arrow[d, equal] & B_1 \oplus B_2 \arrow[r, "{(-e_1,e_2)}"] \arrow[d, "{(1,0)}"] & E \arrow[r, dashed, "p_2^*\varepsilon_2"] \arrow[d, "-p_1"] & {} \\
A \arrow[r, dashed, "f_1"] & B_1 \arrow[r, dashed, "g_1"] \arrow[d, dashed, "0"] & C_1 \arrow[r, dashed, "\varepsilon_1"] \arrow[d, dashed, "-\kappa"] & {} \\
 & {} & {} &
\end{tikzcd} . \tag{2.3.17}$$

The identity $\binom{0}{1}_*(-\kappa) + \binom{f_1}{f_2}_*\varepsilon_2 = 0$ yields $(f_2)_*\varepsilon_1 = \kappa$. We show such construction satisfies the condition in proposition 1.8. It is straightforward to obtain $p_1e_1 = g_1$ and $p_1^*\varepsilon_1 + p_2^*\varepsilon_2 = 0$ from the above diagram. This complete the verification. $\square$

Propositions 2.17 and 2.18 show the good completions for ET4, while propositions 2.19 and 2.20 show the good completions for proposition 1.8 (pushout of two $\mathbb{E}$-inflations). There are dual results for ET4$^{\mathrm{op}}$ and the pullback of two $\mathbb{E}$-deflations. We omit them here.

3 Diagram Lemmas

3.1 Composites of Morphisms

We summarise some useful diagram lemmas involving morphism compositions and homotopic squares.

Definition 3.1. A morphism φ called a *section* if there exists a morphism ψ such that $\psi \circ \varphi = \mathrm{id}$, and called a *retraction* if there exists a morphism θ such that $\varphi \circ \theta = \mathrm{id}$. Say f' is a retract of f if there exists a commutative diagram

$$\begin{tikzcd}
A' \arrow[r, "i"'] \arrow[d, "f'"] \arrow[rr, bend left, "1_{A'}"] & A \arrow[r, "p"'] \arrow[d, "f"] & A' \arrow[d, "f'"] \\
B' \arrow[r, "j"] \arrow[rr, bend right, "1_{B'}"'] & B \arrow[r, "q"] & B'
\end{tikzcd} . \tag{3.1.1}$$

Proposition 3.2. *Let $g \circ f : X \xrightarrow{f} Y \xrightarrow{g} Z$ be a composition of morphisms in an extriangulated category.*

1. *If f and g are $\mathbb{E}$-inflations, then so is $g \circ f$.*
2. *If gf is an $\mathbb{E}$-inflation and g is an $\mathbb{E}$-deflation, then f is an $\mathbb{E}$-inflation.*
3. *If gf is an $\mathbb{E}$-inflations, then f is a retract of an $\mathbb{E}$-inflation.*
4. *If gf is an $\mathbb{E}$-inflation and f is an $\mathbb{E}$-deflation, then g is a retract of an $\mathbb{E}$-inflation.*

Proof. *1.* By ET4.
2. We apply proposition 2.17 to gf and g, and obtain

$$\begin{tikzcd}
& K \arrow[r, equal] \arrow[d, dashed] & K \arrow[d] & \\
X \arrow[r, dashed, "s"] \arrow[rr, bend left, "f"] \arrow[dr, "1"'] & S \arrow[r, dashed, "h"] \arrow[d, dashed, "p"] \arrow[dr, phantom, "\square"] & Y \arrow[r, dashed] \arrow[d, "g"] & C \arrow[d, equal] \\
& X \arrow[r, "gf"] & Z \arrow[r] & C
\end{tikzcd} . \tag{3.1.2}$$

The homotopic square $\square$ is a weak pullback (proposition 2.2). Hence, there is s such that $ps = 1_X$ and $hs = f$. Since p is both an $\mathbb{E}$-deflation and a retraction, it has a kernel K. Therefore, s is a split $\mathbb{E}$-inflation. $f = hs$ is again an $\mathbb{E}$-inflation.
3. Since $gf = (1, 0) \circ \binom{gf}{f}$ is an inflation, $\binom{gf}{f}$ is also an $\mathbb{E}$-inflation by *2.* The composition $\binom{0}{f} = \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}^{-1} \circ \binom{gf}{f}$ is again an $\mathbb{E}$-inflation. Note that f is a retract of the inflation $\binom{0}{f}$.
4. We apply proposition 2.17 to gf and f, and obtain

$$\begin{tikzcd}
K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\
X \arrow[r, "gf"] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & Z \arrow[r] \arrow[d, "h"] \arrow[dd, bend left, "1_Z"] & C \arrow[d, equal] \\
Y \arrow[r, dashed, "i"] \arrow[dr, "g"'] & E \arrow[r, dashed] \arrow[d, dashed, "s"] & C \\
& Z &
\end{tikzcd} . \tag{3.1.3}$$

The homotopic square $\square$ is a weak pushout proposition 2.2. Hence, there is s such that $sh = 1_Z$ and $si = g$. We see g is a retract of an $\mathbb{E}$-inflation. $\square$

The next lemma shows structure of retract of $\mathbb{E}$-inflations ($\mathbb{E}$-deflations).

Lemma 3.3. *Any retract of an $\mathbb{E}$-inflation take the form $p \circ u$, where u is an $\mathbb{E}$-inflation and p is a retraction. Dually, any retract of an $\mathbb{E}$-deflation take the form $v \circ i$, where v is an $\mathbb{E}$-deflation and i is a section.*

Proof. We show the first statement only. Let $f' : A' \to B'$ be a retract of an $\mathbb{E}$-inflation $f : A \to B$. We fix an $\mathbb{E}$-conflation $A \xrightarrow{f} B \xrightarrow{g} C \overset{\delta}{\dashrightarrow}$ and a realisation of $p_*\delta$ as $A' \xrightarrow{\overline{f}} E \xrightarrow{v} C \overset{p_*\delta}{\dashrightarrow}$. By lemma 2.4, there is a homotopic morphism of $\mathbb{E}$-conflations $(p; m; 1_C)$:

$$\begin{tikzcd}
A' \arrow[r, "i"] \arrow[d, "f'"] & A \arrow[r, "p"] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & A' \arrow[r, equal] \arrow[d, "\overline{f}"] & A' \arrow[d, "f'"] \\
B' \arrow[r, "j"] & B \arrow[r, "m"] \arrow[d, "g"] \arrow[rr, bend right, "q"] & E \arrow[r, dashed, "s"] \arrow[d, "\overline{g}"] & B' \\
& C \arrow[r, equal] \arrow[d, dashed, "\delta"] & C \arrow[d, dashed, "p_*\delta"] & \\
& {} & {} &
\end{tikzcd} . \tag{3.1.4}$$

$\square$ is a weak pushout by proposition 2.2. There is s such that $q = sm$ and $f' = s\overline{f}$. Since $smj = 1_{b'}$, s is a retraction. $\square$

3.2 On Homotopic Squares

We examine the properties traversing parallel edges of homotopic squares. Furthermore, we discuss the composition and cancellation properties of homotopic squares.

Proposition 3.4. *If u is an inflation in the following homotopic square*

$$\begin{tikzcd}
A_1 \arrow[r, "u"] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & B_1 \arrow[d, "g"] \\
A_2 \arrow[r, "v"] & B_2
\end{tikzcd} , \tag{3.2.1}$$

then v is also an $\mathbb{E}$-inflation. Conversely, if v is an $\mathbb{E}$-inflation, then so is u. In this case, $(f; g; 1)$ is a homotopic morphism of $\mathbb{E}$-conflations.

Proof. We assume u to be an $\mathbb{E}$-inflation. Let $A_1 \xrightarrow{u} B_1 \xrightarrow{p} C \overset{\delta_1}{\dashrightarrow}$ be an $\mathbb{E}$-conflation. We complete the following commutative diagram by proposition 1.7

$$\begin{tikzcd}
& A_2 \arrow[r, equal] \arrow[d, "\binom{0}{1}"] & A_2 \arrow[d, dashed, "v"] & \\
A_1 \arrow[r, "\binom{u}{f}"] \arrow[d, equal] & B_1 \oplus A_2 \arrow[r, "{(-g,v)}"] \arrow[d, "{(1,0)}"] & B_2 \arrow[r, dashed, "\varepsilon"] \arrow[d, dashed, "-q"] & {} \\
A_1 \arrow[r, "u"] & B_1 \arrow[r, "p"] \arrow[d, dashed, "0"] & C \arrow[r, dashed, "\delta_1"] \arrow[d, dashed, "-\delta_2"] & {} \\
& {} & {} &
\end{tikzcd} . \tag{3.2.2}$$

This diagram gives a conflation $A_2 \xrightarrow{v} B_2 \xrightarrow{q} C \overset{\delta_2}{\dashrightarrow}$, showing that v is an inflation. Since $\binom{0}{1}_*(-\delta_2) + \binom{u}{f}_*\delta_1 = 0$, we see $f_*\delta_1 = \delta_2$. Hence, we have $qg = p$ and $f_*\delta_1 = \delta_2$, yielding that $(f; g; 1)$ is a homotopic morphism of conflations.

Conversely, when u is an $\mathbb{E}$-inflation in $A_2 \xrightarrow{u} B_2 \xrightarrow{q} C \overset{\delta_2}{\dashrightarrow}$. We complete the following commutative diagram by proposition 1.10:

$$\begin{tikzcd}
& A_2 \arrow[r, equal] \arrow[d, "\binom{0}{1}"] & A_2 \arrow[d, "v"] & \\
A_1 \arrow[r, "\binom{u}{f}"] \arrow[d, equal] & B_1 \oplus A_2 \arrow[r, "{(-g,v)}"] \arrow[d, "{(1,0)}"] & B_2 \arrow[r, dashed, "\varepsilon"] \arrow[d, "-q"] & {} \\
A_1 \arrow[r, dashed, "u"] & B_1 \arrow[r, dashed, "p"] \arrow[d, dashed, "0"] & C \arrow[r, dashed, "\delta_1"] \arrow[d, dashed, "-\delta_2"] & {} \\
& {} & {} &
\end{tikzcd} . \tag{3.2.3}$$

Hence, u is an $\mathbb{E}$-inflation. The rest of the verification is the same as the previous case. $\square$

Theorem 3.5. *We consider the following homotopic square:*

$$\begin{tikzcd}
A_1 \arrow[r, "u"] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & B_1 \arrow[d, "g"] \\
A_2 \arrow[r, "v"] & B_2
\end{tikzcd} . \tag{3.2.4}$$

Then u is an $\mathbb{E}$-inflation (resp. $\mathbb{E}$-deflation) if and only if v is an $\mathbb{E}$-inflation (resp. $\mathbb{E}$-deflation).

Proof. The $\mathbb{E}$-inflation case follows from proposition 3.4. The $\mathbb{E}$-deflation case is dual. $\square$

Lemma 3.6. *We take arbitrary homotopic square:*

$$\begin{tikzcd} A_1 \arrow[r, "u"] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & B_1 \arrow[d, "g"] \\ A_2 \arrow[r, "v"] & B_2 \end{tikzcd} . \tag{3.2.5}$$

When v is a retraction, then so is u. Dually, when u is a section, then so is v.

Proof. We show the first statement only. Assume v is a retraction with right inverse i. By proposition 2.2, there is s such that the following diagram commutes:

$$\begin{tikzcd} B_1 \arrow[drr, bend left, "1_{B_1}"] \arrow[dr, dashed, "s"] \arrow[ddr, bend right, "ig"'] & & \\ & A_1 \arrow[r, "u"] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & B_1 \arrow[d, "g"] \\ & A_2 \arrow[r, "v"] & B_2 \end{tikzcd} . \tag{3.2.6}$$

Hence, we have $su = 1_{B_1}$, showing that u is a retraction. $\square$

Theorem 3.7 (**Theorem 3.2.**, [5]). *Homotopic squares are closed under horizontal and vertical compositions.*

Proof. We consider horizontal compotisions only. Let $\square$ be homotopic:

$$\begin{tikzcd} A \arrow[r, "f", circ] \arrow[d, "\alpha"] \arrow[dr, phantom, "\boxed{\kappa}"] & B \arrow[r, "g", circ] \arrow[d, "\beta"] \arrow[dr, phantom, "\boxed{\epsilon}"] & C \arrow[d, "\gamma"] \\ D \arrow[r, "u"] & E \arrow[r, "v"] & F \end{tikzcd} . \tag{3.2.7}$$

We take the direct sum of the $\mathbb{E}$-conflation realising the left square and $0 \to C \xrightarrow{1} C \overset{0_{C0}}{\dashrightarrow}$, and obtain

$$\begin{tikzcd} A \arrow[r, "\begin{pmatrix} -f \\ \alpha \\ 0 \end{pmatrix}"] \arrow[d, equal] & B \oplus D \oplus C \arrow[r, "\begin{pmatrix} \beta & u & 0 \\ 0 & 0 & 1 \end{pmatrix}"] & E \oplus C \arrow[r, dashed, "\kappa \oplus 0_{C0}"] \arrow[d, equal] & {} \\ A \arrow[r, "\begin{pmatrix} -f \\ \alpha \\ gf \end{pmatrix}"] & B \oplus D \oplus C \arrow[u, "\simeq"', "\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g & 0 & 1 \end{pmatrix}"] \arrow[r, "\begin{pmatrix} \beta & u & 0 \\ g & 0 & 1 \end{pmatrix}"] & E \oplus C \arrow[r, dashed, "\kappa \oplus 0_{C0}"] & {} \end{tikzcd} . \tag{3.2.8}$$

By proposition 1.7, there exists a completion of the following diagram

$$\begin{tikzcd} & B \arrow[rr, equal] \arrow[d, "\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}"] & & B \arrow[d, "\begin{pmatrix} \beta \\ g \end{pmatrix}"] & \\ A \arrow[r, "\begin{pmatrix} -f \\ \alpha \\ gf \end{pmatrix}"] \arrow[d, equal] & B \oplus D \oplus C \arrow[rr, "\begin{pmatrix} \beta & u & 0 \\ g & 0 & 1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}"] & & E \oplus C \arrow[r, dashed, "\kappa \oplus 0_{C0}"] \arrow[d, "(v, -\gamma)"] & {} \\ A \arrow[r, dashed, "\begin{pmatrix} \alpha \\ gf \end{pmatrix}"] & D \oplus C \arrow[rr, dashed, "(vu, -\gamma)"] \arrow[d, dashed, "0"] & & F \arrow[r, dashed, "\delta"] \arrow[d, dashed, "\varepsilon"] & {} \\ & {} & & {} & \end{tikzcd} . \tag{3.2.9}$$

Such completion is unique, as the buttom conflation is solved to be unique. $\square$

Corollary 3.8. *Following eq. (3.2.9), we see $(v, -\gamma)^*\delta = (\delta \oplus 0_{C0})$. Hence $v^*\delta = \kappa$. The identity $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_* \varepsilon + \begin{pmatrix} -f \\ \alpha \\ gf \end{pmatrix}_* \delta$ yields $f_*\delta = \varepsilon$.*

Theorem 3.9. *Let $\boxed{\kappa}$ be a homotopic square. If $\begin{pmatrix} \alpha \\ gf \end{pmatrix}$ is an $\mathbb{E}$-inflation, then so is $\begin{pmatrix} g \\ \beta \end{pmatrix}$. Consequently, the diagram completes to a composition of homotopic squares.*

$$\begin{tikzcd} A \arrow[r, "f", circ] \arrow[d, "\alpha"] \arrow[dr, phantom, "\boxed{\kappa}"] & B \arrow[r, "g", circ] \arrow[d, "\beta"] \arrow[dr, phantom, "\boxed{\varepsilon}"] & C \arrow[d, dashed, "\gamma"] \\ D \arrow[r, "u"] & E \arrow[r, dashed, "v"] & F \end{tikzcd} . \tag{3.2.10}$$

Proof. We take the direct sum of the $\mathbb{E}$-conflation realising the left square and $0 \to C \xrightarrow{1} C \overset{0_{C0}}{\dashrightarrow}$, and obtain

$$\begin{tikzcd}
A \arrow[r, "\begin{pmatrix} -f \\ \alpha \\ 0 \end{pmatrix}"] \arrow[d, equal] & B \oplus D \oplus C \arrow[r, "\begin{pmatrix} \beta & u & 0 \\ 0 & 0 & 1 \end{pmatrix}"] & E \oplus C \arrow[r, dashed, "\kappa \oplus 0_{C0}"] \arrow[d, equal] & {} \\
A \arrow[r, "\begin{pmatrix} -f \\ \alpha \\ gf \end{pmatrix}"] & B \oplus D \oplus C \arrow[r, "\begin{pmatrix} \beta & u & 0 \\ g & 0 & 1 \end{pmatrix}"] \arrow[u, "\simeq", "\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g & 0 & 1 \end{pmatrix}"'] & E \oplus C \arrow[r, dashed, "\kappa \oplus 0_{C0}"] & {}
\end{tikzcd} . \tag{3.2.11}$$

Let $A \xrightarrow{\binom{\alpha}{gf}} D \oplus C \xrightarrow{(p,-\gamma)} F \overset{\delta}{\dashrightarrow}$ be any $\mathbb{E}$-conflation. By proposition 1.7, we obtain:

$$\begin{tikzcd}
 & B \arrow[r, equal] \arrow[d, "\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}"] & B \arrow[d, dashed, "\binom{\beta}{g}"] & \\
A \arrow[r, "\begin{pmatrix} -f \\ \alpha \\ gf \end{pmatrix}"] \arrow[d, equal] & B \oplus D \oplus C \arrow[r, "\begin{pmatrix} \beta & u & 0 \\ g & 0 & 1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}"] & E \oplus C \arrow[r, dashed, "\kappa \oplus 0"] \arrow[d, dashed, "(v,-\gamma)"] & {} \\
A \arrow[r, "\binom{\alpha}{gf}"] & D \oplus C \arrow[r, "(p,-\gamma)"] \arrow[d, dashed, "0"] & F \arrow[r, dashed, "\delta"] \arrow[d, dashed, "\varepsilon"] & {} \\
 & {} & {} &
\end{tikzcd} .$$

Hence, $\binom{g}{\beta}$ is an $\mathbb{E}$-inflation. $\square$

Theorem 3.10. *Let $\boxed{\varepsilon}$ be a homotopic square. If (γ, vu) is an $\mathbb{E}$-inflation, then so is (γ, v). Consequently, the diagram completes to a composition of homotopic squares.*

$$\begin{tikzcd}
A \arrow[r, dashed, "f"] \arrow[d, dashed, "\alpha"] \arrow[dr, phantom, "\boxed{\kappa}"] & B \arrow[r, "g"] \arrow[d, "\beta"] \arrow[dr, phantom, "\boxed{\varepsilon}"] & C \arrow[d, "\gamma"] \\
D \arrow[r, "u"] & E \arrow[r, "v"] & F
\end{tikzcd} . \tag{3.2.12}$$

Proof. Dual to theorem 3.9. $\square$

Corollary 3.11. *Equation (3.2.10) completes to a composition of homotopic squares if one of α, β, g is an $\mathbb{E}$-inflation.*

Proof. When g or β is an $\mathbb{E}$-inflation, then so is $\binom{g}{\beta}$. α is an $\mathbb{E}$-inflation if any only if β is so, by theorem 3.5. $\square$

Proposition 3.12 (Splitting condition)**.** *Suppose the left commutative diagram is a homotopic square. If one the following conditions holds: (1). u is an inflation, (2). v is a deflation, (3). γ is a deflation. Then there is a way to write h as gf such that the right commutative diagram is a composite of homotopic squares.*

$$\begin{tikzcd}
A \arrow[rr, "h"] \arrow[d, "\alpha"] & \square & C \arrow[d, "\gamma"] \\
D \arrow[r, "u"] & E \arrow[r, "v"] & F
\end{tikzcd} \qquad
\begin{tikzcd}
A \arrow[r, dashed, "f"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\square"] & B \arrow[r, dashed, "g"] \arrow[d, dashed, "\beta"] \arrow[dr, phantom, "\square"] & C \arrow[d, "\gamma"] \\
D \arrow[r, "u"] & E \arrow[r, "v"] & F
\end{tikzcd} . \tag{3.2.13}$$

Proof. It suffices to show that (v, γ) is a deflation in each of the three cases.
(Case 1). Since $(v, -\gamma)\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} = (vu, -\gamma)$ is a deflation and $\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$ is an inflation, we see $(v, -\gamma)$ is also a deflation by proposition 3.2. (Case 2 and 3). By proposition 3.2, (v, γ) is a deflation.
By theorem 3.10, we obtain two homotopic squares:

$$\begin{tikzcd}
A \arrow[drr, dashed, "f"] \arrow[drrr, bend left, "h"] \arrow[dr, dashed, "\varphi"] \arrow[ddr, bend right, "\alpha"'] & & & \\
 & \overline{A} \arrow[r, dashed, "\overline{f}"] \arrow[d, dashed, "\overline{\alpha}"] \arrow[dr, phantom, "\square"] & B \arrow[r, "g"] \arrow[d, "\beta"] \arrow[dr, phantom, "\square"] & C \arrow[d, "\gamma"] \\
 & D \arrow[r, "u"] & E \arrow[r, "v"] & F
\end{tikzcd} . \tag{3.2.14}$$

The morphism f is constructed by proposition 2.2. The composition of the two homotopic squares is also homotopic (theorem 3.7), and φ is constructed by applying ET3$^{\mathrm{op}}$ to the following diagram:

$$\begin{tikzcd}
A \arrow[r, "\binom{h}{\alpha}"] \arrow[d, dashed, "\varphi"] & B \oplus C \arrow[r, "(\gamma,-vu)"] \arrow[d, equal] & F \arrow[r, dashed] \arrow[d, equal] & {} \\
\overline{A} \arrow[r, "\binom{g\overline{f}}{\overline{\alpha}}"] & B \oplus C \arrow[r, "(\gamma,-vu)"] & F \arrow[r, dashed] & {}
\end{tikzcd} . \tag{3.2.15}$$

φ is an isomorphism by corollary 1.2. This completes the proof. $\square$

Lemma 3.13. *We take arbitrary homotopic square:*

$$\begin{tikzcd} A_1 \arrow[r, "u"] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & B_1 \arrow[d, "g"] \\ A_2 \arrow[r, "v"] & B_2 \end{tikzcd} . \tag{3.2.16}$$

When v is a retract of some $\mathbb{E}$-inflation, then so is u.

Proof. By lemma 3.3, v takes the form pw for some inflation w and retraction p. Consider the following diagram. By theorem 3.10, the diagram splits into two homotopic squares. It yields that u is a composition of an $\mathbb{E}$-inflation and a retraction. Hence, u a retract of an $\mathbb{E}$-inflation. $\square$

Lemma 3.14. *We take arbitrary homotopic square:*

$$\begin{tikzcd} A_1 \arrow[r, "u"] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & B_1 \arrow[d, "g"] \\ A_2 \arrow[r, "v"] & B_2 \end{tikzcd} . \tag{3.2.17}$$

When u is a retract of some $\mathbb{E}$-inflation, then so is v.

Proof. We take $u = ri$ such that i is an $\mathbb{E}$-inflation and r is a retraction. We construct the left homotopic square by lemma 2.4. Since $\binom{r}{f'}$ is an $\mathbb{E}$-inflation by theorem 3.9, we complete the right homotopic square. The composite of the two homotopic squares is again homotopic by theorem 3.7.

$$\begin{tikzcd} A_1 \arrow[r, "i"] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & E \arrow[r, "r"] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & B_1 \arrow[d, dashed, "g'"] \\ A_2 \arrow[r, "i'"] & F \arrow[r, dashed, "r'"] & B_2' \end{tikzcd} . \tag{3.2.18}$$

By corollary 1.2, there is an isomorphism φ such that the following diagram is a morphism of $\mathbb{E}$-conflations

$$\begin{tikzcd} A \arrow[r, "\binom{ri}{f}"] \arrow[d, equal] & B_1 \oplus A_2 \arrow[r, "{(g', -i')}"] \arrow[d, equal] & B_2' \arrow[r, dashed] \arrow[d, dashed, "\varphi"] & {} \\ A \arrow[r, "\binom{u}{f}"] & B_1 \oplus A_2 \arrow[r, "{(g, -v)}"] & B_2 \arrow[r, dashed] & {} \end{tikzcd} . \tag{3.2.19}$$

$\square$

Theorem 3.15. *We consider the following homotopic square:*

$$\begin{tikzcd} A_1 \arrow[r, "u"] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & B_1 \arrow[d, "g"] \\ A_2 \arrow[r, "v"] & B_2 \end{tikzcd} . \tag{3.2.20}$$

Then u is a retract of an $\mathbb{E}$-inflation (resp. retract of an $\mathbb{E}$-deflation) if and only if v is a retract of an $\mathbb{E}$-inflation (resp. retract of an $\mathbb{E}$-deflation).

Proof. By lemmas 3.13 and 3.14 and their duals. $\square$

3.3 An Application: Happel's Theorem

Definition 3.16 (S, $\mathcal{L}$, and $\mathcal{R}$)**.** We define the following classes of morphisms and objects in an extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$:

- S is the class of morphisms which are both $\mathbb{E}$-inflations and $\mathbb{E}$-deflations.
- $\mathcal{L}$ is the class of objects L such that $L \to 0$ is an $\mathbb{E}$-deflation.
- $\mathcal{R}$ is the class of objects R such that $0 \to R$ is an $\mathbb{E}$-inflation.

Note that $\mathcal{L}$ and $\mathcal{R}$ are additive full subcategories.

Proposition 3.17. *Either one of S, $\mathcal{L}$, and $\mathcal{R}$ determines the other two.*

Proof. It suffices to show $\mathcal{L}$ and S are mutually determined. The dual argument works for $\mathcal{R}$ and S (S determines $\mathcal{L}$). Any $f \in S$ admits two $\mathbb{E}$-conflations:

$$K \xrightarrow{k} A \xrightarrow{f} B \overset{\kappa}{\dashrightarrow}, \quad A \xrightarrow{f} B \xrightarrow{c} C \overset{\varepsilon}{\dashrightarrow}. \tag{3.3.1}$$

In homotopis squares, we have

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d] \arrow[dr, phantom, "\boxed{\kappa}"] & A \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\boxed{\varepsilon}"] & 0 \arrow[d] \\ 0 \arrow[r] & B \arrow[r, "c"] & C \end{tikzcd}. \tag{3.3.2}$$

By theorem 3.7, we see $K \in \mathcal{L}$. Indeed, any $L \in \mathcal{L}$ is determined in this way. We take the conflation $L \to 0 \to R \dashrightarrow$ and find that $(0 \to R) \in S$. We do the same construction for $0 \to R$ and complete the proof.

($\mathcal{L}$ determines S). We claim that $f \in S$ iff there is a $\mathbb{E}$-conflation $K \to X \xrightarrow{f} Y \dashrightarrow$ for some $K \in \mathcal{L}$. The "only if ($\to$)" part is clear. For the "if ($\leftarrow$)" part, we consider the homotopic square

$$\begin{tikzcd} K \arrow[r] \arrow[d] & 0 \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}. \tag{3.3.3}$$

By theorem 3.5, f is both an $\mathbb{E}$-inflation and an $\mathbb{E}$-deflation since $(K \to 0)$ is so. This completes the proof. ☐

Proposition 3.18. *S is closed under composition and contains all isomorphisms. Moreover, it satisfies the 2-out-of-3 property when $\mathbb{E}$-inflations and $\mathbb{E}$-deflations are closed under retracts.*

Proof. Isomorphisms are both $\mathbb{E}$-inflations and $\mathbb{E}$-deflations. By ET4 and ET4$^{\mathrm{op}}$, S is closed under composition. Now we suppose $g \circ f$ and g are in S. By proposition 3.2, f is both an $\mathbb{E}$-inflation and a retract of an $\mathbb{E}$-deflation. Hence, $f \in S$ by assumption. ☐

We then show a direct connection of $\mathcal{L}$ and $\mathcal{R}$.

Theorem 3.19. *For each $X \in \mathcal{L}$, we fix $X \to 0 \to FX \overset{\delta_X}{\dashrightarrow}$. Then the assignment of objects $X \mapsto FX$ induces an equivalence of categories. Moreover, there is a collection of natural isomorphisms functorial in X:*

$$\ell_{-,X} : \mathbb{E}(-,X) \cong \mathcal{C}(-,FX) \quad (X \in \mathcal{L}). \tag{3.3.4}$$

Proof. We show functoriality of F. For any morphism f in the category $\mathcal{L}$, there is g such that $(f;0;g)$ is a morphism of $\mathbb{E}$-conflations:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & FX \arrow[r, dashed, "\delta_X"] \arrow[d, "g"] & {} \\ Y \arrow[r] & 0 \arrow[r] & FY \arrow[r, dashed, "\delta_Y"] & {} \end{tikzcd}. \tag{3.3.5}$$

We claim g is unique. If not, then there is another g' such that $g'^*\delta_Y = f_*\delta_X = g^*\delta_Y$. Since $(g-g')^*\delta_Y = 0$, $(g-g')$ passes through $0 \to FY$ by eq. (1.2.1). Thus, $g = g'$.
It remains to show F is an equivalence. The above analysis (and its dual) shows the isomorphism $\mathcal{C}(X,Y) \cong \mathcal{C}(FX,FY)$. To see that F is dense, for any $R \in \mathcal{R}$, we take the conflation $K \to 0 \to R \dashrightarrow$. Note that $K \in \mathcal{L}$ and $FK \cong R$ by corollary 1.2. This completes the proof.
Finally we define ℓ by ET3 as follows:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[r, dashed, "\varepsilon"] \arrow[d, dashed, "\ell_{Z,X}(\varepsilon)"] & {} \\ X \arrow[r] & 0 \arrow[r] & FX \arrow[r, dashed, "\delta_X"] & {} \end{tikzcd}. \tag{3.3.6}$$

This assignment is unique; otherwise, the minus of two candicate morphism $(g-g') : Z \to FX$ factors through $0 \to FX$, which implies $g = g'$. Conversely, any $\gamma : Z \to FX$ determines $\gamma^*\delta_X \in \mathbb{E}(Z,X)$. Since $\ell(\gamma^*\delta_X) = \gamma$, and $\ell(\varepsilon)^*\delta_X = \varepsilon$, we find the inverse map of ℓ. To see the naturality, it suffices to show $\ell(a_*c^*\varepsilon) = (Fa) \circ \ell(\varepsilon) \circ c$. Consider

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d] & X \arrow[r, equal] \arrow[d, "f"] & X \arrow[r, "a"] \arrow[d] & X' \arrow[r, equal] \arrow[d] & X' \arrow[d] & X \arrow[l, "a"'] \arrow[d] \\ 0 & Y \arrow[l] \arrow[d, "g"] & E \arrow[l] \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \arrow[d] & 0 \arrow[l] \arrow[d] \\ FX \arrow[d, dashed, "\delta_X"] & Z \arrow[l, dashed, "\ell(\varepsilon)"'] \arrow[d, dashed, "\varepsilon"] & Z' \arrow[l, "c"'] \arrow[r, equal] \arrow[d, dashed, "c^*\varepsilon"] & Z' \arrow[r, "\ell(a_*c^*\varepsilon)"] \arrow[d, dashed, "a_*c^*\varepsilon"] & FX' \arrow[d, dashed, "\delta_{X'}"] & FX \arrow[l, "Fa"'] \arrow[d, dashed, "\delta_X"] \\ {} & {} & {} & {} & {} & {} \end{tikzcd}. \tag{3.3.7}$$

We see $\ell(a_*c^*\varepsilon)^*\delta_{X'} = a_*c^*\ell(\varepsilon)^*\delta_X = c^*\ell(\varepsilon)^*(Fa)^*\delta_{X'}$. Hence, $\Big(\ell(a_*c^*\varepsilon) - (Fa)\ell(\varepsilon)c\Big)^*\delta_{X'} = 0$. The above analysis shows $\varphi^*\delta_{X'} = 0$ iff $\varphi = 0$. This completes the proof. ☐

Theorem 3.20. *In theorem 3.19, there is a collection of natural isomorphisms functorial in X:*

$$\rho_{-,X} : \mathbb{E}(FX,-) \cong \mathcal{C}(X,-) \quad (X \in \mathcal{L}). \tag{3.3.8}$$

Proof. We define ρ by ET3$^{\mathrm{op}}$ as follows:

$$\begin{tikzcd} X \arrow[r] \arrow[d, dashed, "\rho(\varepsilon)"] & 0 \arrow[r] \arrow[d] & FX \arrow[r, dashed, "\delta_X"] \arrow[d, equal] & {} \\ Y \arrow[r] & Z \arrow[r] & FX \arrow[r, dashed, "\varepsilon"] & {} \end{tikzcd} . \tag{3.3.9}$$

Such completion by ET3$^{\mathrm{op}}$ is unique. If there is another α such that $(\alpha; 0_{Z0}; 1_{FX})$ and $(\rho(\varepsilon); 0_{Y0}; 1_{FX})$ are both morphisms of $\mathbb{E}$-conflations, then $(\alpha - \rho(\varepsilon))_*\delta_X = 0$. Hence $(\alpha - \rho(\varepsilon))$ factors through $X \to 0$, which yields that $\alpha = \rho(\varepsilon)$.
We show ρ is an isomorphism by finding its inverse. $f \mapsto f_*\delta_X$. Note that $\rho(f_*\delta_X) = f$ by unique completion of ET3$^{\mathrm{op}}$. $\rho(\varepsilon)_*\delta_X = \varepsilon$ is clearly shown in eq. (3.3.9).
We finally show the naturality. It suffices to show $\rho((F\gamma)^*\alpha_*\varepsilon) = \alpha \circ \rho(\varepsilon) \circ \gamma$ for any $\gamma : X' \to X$ and $\alpha : Y \to Y'$. Consider

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[d] & X' \arrow[r, "\rho(\alpha_*(F\gamma)^*\varepsilon)"] \arrow[d] & Y' \arrow[d] & Y \arrow[l, "\alpha"'] \arrow[r, equal] \arrow[d] & Y \arrow[d] & X \arrow[l, dashed, "\rho(\varepsilon)"'] \arrow[d] \\ 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & F \arrow[d] & E \arrow[l] \arrow[r] \arrow[d] & Z \arrow[d] & 0 \arrow[l] \arrow[d] \\ FX \arrow[r, "F\gamma"] \arrow[d, dashed, "\delta_X"] & FX' \arrow[r, equal] \arrow[d, dashed, "\delta_{X'}"] & FX' \arrow[r, equal] \arrow[d, dashed, "\alpha_*(F\gamma)^*\varepsilon"] & FX' \arrow[r, "F\gamma"] \arrow[d, dashed, "(F\gamma)^*\varepsilon"] & FX \arrow[r, equal] \arrow[d, dashed, "\varepsilon"] & FX \arrow[d, dashed, "\delta_X"] \\ {} & {} & {} & {} & {} & {} \end{tikzcd} . \tag{3.3.10}$$

We see $\rho(\alpha_*(F\gamma)^*\varepsilon)_*\delta_{X'} = \alpha_*(F\gamma)^*\rho(\varepsilon)_*\delta_X = \alpha_*\rho(\varepsilon)_*\gamma_*\delta_{X'}$. Hence, $\Big(\rho(\alpha_*(F\gamma)^*\varepsilon) - \alpha \circ \rho(\varepsilon) \circ \gamma\Big)_* \delta_{X'} = 0$. The above analysis shows $\varphi_*\delta_{X'} = 0$ iff $\varphi = 0$. This completes the proof. $\square$

We then examine some conditions for any extriangulated category to be (right) triangulated.

Definition 3.21. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. Say such extriangulated category admits a (right) triangulated structure, if there is an auto-equivalence $F : \mathcal{C} \to \mathcal{C}$ and a natural isomorphism $\ell : \mathbb{E}(\cdot, -) \cong \mathcal{C}(\cdot, F(-))$ such that

$$\triangle := \{(X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\ell(\delta)} FX) \mid (X \xrightarrow{f} Y \xrightarrow{g} Z \overset{\delta}{\dashrightarrow}) \text{ is an } \mathbb{E}\text{-conflation}\} \tag{3.3.11}$$

is a (right) triangulated structure on $\mathcal{C}$. That is, $(\mathcal{C}, F, \triangle)$ is a (right) triangulated category.

Theorem 3.22. *An extriangulated cateogry $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ admits a right triangulated structure iff the following equivalent conditions are satisfied.*

1. *(**Theorem 3.12**, [12]). $\{0\}$ provides enough injective objects.*
2. *All $X \to 0$ are $\mathbb{E}$-inflations, in other words, $\mathcal{L} = \mathcal{C}$.*
3. *All morphisms are $\mathbb{E}$-inflations.*
4. *All $\mathbb{E}$-deflations are $\mathbb{E}$-inflations, in other words, S is the class of all $\mathbb{E}$-deflations.*

Proof. We show the equivalence of the above four conditions. $(1 \to 2)$. Clear. $(2 \to 3)$. Since $X \to 0$ an $\mathbb{E}$-inflation, any $[X \xrightarrow{f} Y] = [X \xrightarrow{\binom{f}{0}} Y \oplus 0 \cong Y]$ is also an $\mathbb{E}$-inflation by proposition 3.2. $(3 \to 4)$. Clear. $(4 \to 1)$. For any object X, the $\mathbb{E}$-deflation $X \to 0$ is also an $\mathbb{E}$-inflation by assumption. Hence, $\{0\}$ provides enough injective objects.
We then show that $\mathcal{C}$ admits an extriangulated structure if at least one of the following equivalent conditions is satisfied. When $\mathcal{C}$ admits a right triangulated structure, all morphisms are $\mathbb{E}$-inflations. Conversely, if $\mathcal{L} = \mathcal{C}$, then there is an natural isomorphism $\ell : \mathbb{E}(-, X) \cong \mathcal{C}(-, FX)$ and a equivalence $F : \mathcal{C} \simeq \mathcal{R}$ by theorem 3.19. We show that $(\mathcal{C}, F, \triangle)$ is a right triangulated category by verifying the SP-axioms in [6].

1. (Verificaiton of SP0 and SP1). $\mathbb{E}$-conflations are closed under isomorphisms and contain all $[0 \to X \xrightarrow{1_X} X \dashrightarrow]$ by definition. By *3.*, any morphism $f : X \to Y$ is an $\mathbb{E}$-inflation.

2. (Verificaiton of SP2). For any $\mathbb{E}$-conflation $X \xrightarrow{f} Y \xrightarrow{g} Z \overset{\delta}{\dashrightarrow}$, we show $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\ell(\delta)} FX$ is closed under clockwise rotation. Consider

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\boxed{f_*\delta_X}"] & Z \arrow[r, dashed, "\varepsilon"] \arrow[d, dashed, "\ell(\varepsilon)"] & {} \\ X \arrow[r] & 0 \arrow[r] & FX \arrow[r, dashed, "\delta_X"] & {} \end{tikzcd} . \tag{3.3.12}$$

Hence, $Y \xrightarrow{-g} Z \xrightarrow{\ell(\varepsilon)} FX \xrightarrow{\ell(f_*\delta_X)} FY$ is also a right triangle. Note that $\ell(f_*\delta_X) = (Ff) \circ \ell(\delta_X) = Ff$ by naturality of ℓ. This completes the verification.

3. (Verificaiton of SP3 and SP4). It follows from ET3 and ET4 directly.

□

Remark. Not all right triangulated categories are obtained in this way. For example, we choose Ab as our ambient category, and $\{X \xrightarrow{f} Y \xrightarrow{\pi} \operatorname{cok} f \to 0 \mid f \in \mathsf{Mor}(\mathrm{Ab})\}$ as the class of right triangles. This gives a right triangulated structure on Ab, but it does not arise from an extriangulated category since the suspension functor is not an equivalence.

Theorem 3.23. *We show some equivalent conditions for an extriangulated category to be triangulated.*

1. *(**Proposition 3.2**, [11]). There is an auto-equivalence $F : \mathcal{C} \to \mathcal{C}$ and a natural isomorphism $\ell : \mathbb{E}(\cdot, -) \cong \mathcal{C}(\cdot, F(-))$.*

2. *$\mathcal{L} = \mathcal{R} = \mathcal{C}$, that is, $0 \to X$ and $X \to 0$ are both $\mathbb{E}$-inflations and $\mathbb{E}$-deflations for any X.*

3. *$S = \mathsf{Mor}(\mathcal{C})$, that is, all morphisms are both $\mathbb{E}$-inflations and $\mathbb{E}$-deflations.*

Proof. If *1.* holds, then $(\mathcal{C}, F, \triangle)$ is triangulated. The verification is similar to that of theorem 3.22. A triangulated satisfies both *2.* and *3.*. The equivalence of *2.* and *3.* is clear by proposition 3.17. If *3.* holds, then we have *1.* by theorem 3.19. □

Corollary 3.24 (Happel's theorem and its converse)**.** *Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be extriangulated. If any only if $\mathcal{C}$ is Frobenius exact, there exists an additive full subcategory $\mathcal{B} \subseteq (\mathit{Proj} \cap \mathit{Inj})$ such that the ideal quotient (**Proposition 3.30.**, [11]) $\mathcal{C}/\mathcal{B}$ is triangulated. In this case, the class of projective-injective objects are precisely the summands of objects in $\mathcal{B}$.*

Proof. $(\leftarrow)$. If $\mathcal{C}$ is Frobenius exact, then we take $\mathcal{B}$ are the class of projective-injective objects. Any $X \in \mathcal{C}$ admits two types of conflations

$$K \to P \to X \dashrightarrow, \quad X \to I \to Q \dashrightarrow \quad P, I \in \mathcal{B}. \tag{3.3.13}$$

Hence, any $X \to 0$ and $0 \to X$ are both $\mathbb{E}$-inflations and $\mathbb{E}$-deflations in $\mathcal{C}/\mathcal{B}$. By theorem 3.23, $\mathcal{C}/\mathcal{B}$ is triangulated.
$(\to)$. If there is $\mathcal{B} \subseteq \mathrm{Proj} \cap \mathrm{Inj}$ such that $\mathcal{C}/\mathcal{B}$ is triangulated, then any $X \to 0$ and $0 \to X$ are both $\mathbb{E}$-inflations and $\mathbb{E}$-deflations in $\mathcal{C}/\mathcal{B}$ (by theorem 3.23). Hence, any $X \in \mathcal{C}$ admits two types of conflations as described in eq. (3.3.13). This shows that $\mathcal{B}$ provides enough projective-injective objects. We embed all projective (injective) objects in $\mathcal{C}$ into eq. (3.3.13), and find that all projective (injection) objects in $\mathcal{C}$ are a summands of objects in $\mathcal{B}$. □

Corollary 3.25. *Let $(\mathcal{C}', \mathbb{E}', \mathfrak{s}') \subseteq (\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated subcategory with $\mathsf{Ob}(\mathcal{C}) = \mathsf{Ob}(\mathcal{C}')$. If $(\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ admits a (right) triangulated structure, then so is $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$.*

Proof. Note that an extriangulated category admits a right triangulated structure iff all $X \to 0$ are $\mathbb{E}$-inflations (theorem 3.22). If $(\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ admits a (right) triangulated structure, then all $X \to 0$ are $\mathbb{E}'$-inflations, which are also $\mathbb{E}$-inflations. This completes the proof. The proof for triangulated case is similar by theorem 3.23. □

3.4 Remarks on WIC Condition

Anaglous to exact categories (**Proposition 7.6.**, [2]), [11] introduced a WIC condition for extriangulated categories, serving as a strong version of being weakly idempotent completeness.

1. (Weakly idempotent complete) every section has a cokernel;

2. (**Condition 5.8.**, [11] WIC) if gf is an $\mathbb{E}$-inflation, then so is f.

The equivalency of these two conditions are shown in [7]. We propose a simple proof and another equivalent condition inspired by Heller's axiom (**Appendix B.**, [2]).

Lemma 3.26. *An additive category $\mathcal{C}$ is weakly idempotent complete, if and only if the following condition holds: 1. any section has a cokernel; 2. any retraction has a kernel.*

Proof. We show *1.* implies *2.* only. Let $X \xrightarrow{q} C$ be a retraction, with section $C \xrightarrow{i} X$ as its right inverse. We denote by $X \xrightarrow{p} K$ the cokernel of i. Since $(1 - iq)i = 0$, we find j such that $jp = (1 - iq)$.

$$\begin{tikzcd} C \arrow[r, bend left, "i"] & X \arrow[l, bend left, "q"] \arrow[r, bend left, "p"] \arrow[d, "1-iq"] & K \arrow[dl, dashed, bend left, "j"] \\ & X & \end{tikzcd} . \tag{3.4.1}$$

We see $pj = 1_K$ as $pjp = p(1 - iq) = p$, and $qj = 0$ as $qjp = q(1 - iq) = 0$. We find structure maps of this direct sum. □

Remark. Triangulated categories are automatically WIC. For exact categories, see [2].

Definition 3.27. A 3×3 *diagram* consists of 6 conflations arranged in a commutative diagram

$$\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, "i_1"] & A_2 \arrow[r, "g_A"] \arrow[d, "i_2"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, "i_3"] & {} \\
B_1 \arrow[r, "f_B"] \arrow[d, "p_1"] & B_2 \arrow[r, "g_B"] \arrow[d, "p_2"] & B_3 \arrow[r, dashed, "\delta_B"] \arrow[d, "p_3"] & {} \\
C_1 \arrow[r, "f_C"] \arrow[d, dashed, "\varepsilon_1"] & C_2 \arrow[r, "g_C"] \arrow[d, dashed, "\varepsilon_2"] & C_3 \arrow[r, dashed, "\delta_C"] \arrow[d, dashed, "\varepsilon_3"] & {} \\
{} & {} & {} &
\end{tikzcd}, \tag{3.4.2}$$

such that $(i_1; i_2; i_3)$, $(p_1; p_2; p_3)$, $(f_A; f_B; f_C)$ and $(g_A; g_B; g_C)$ are morphisms of conflations.

Theorem 3.28. *An extriangulated category is weakly idempotent complete, if and only if the following equivalent statements holds.*

1. *(The definition). Any section has a cokernel.*
2. *When there is an inflation takes the form $\binom{i}{0}$, then i is an inflation.*
3. *When there is an inflation takes the form $\begin{pmatrix} i & 0 \\ 0 & j \end{pmatrix}$, then i and j are inflations.*
4. *(WIC condition). When there is an inflation takes the form fi, then i is an inflation.*
5. *Inflations are closed under retracts.*
6. *Let g_A, g_B be $\mathbb{E}$-deflations and i_2, i_3 be $\mathbb{E}$-inflations, such that $g_B \circ i_2 = i_3 \circ g_A$. One can complete this commutative square into a 3×3 diagram:*

$$\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, dashed, "i_1"] & A_2 \arrow[r, "g_A"] \arrow[d, "i_2"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, "i_3"] & {} \\
B_1 \arrow[r, "f_B"] \arrow[d, dashed, "p_1"] & B_2 \arrow[r, "g_B"] \arrow[d, "p_2"] & B_3 \arrow[r, dashed, "\delta_B"] \arrow[d, "p_3"] & {} \\
C_1 \arrow[r, dashed, "f_C"] \arrow[d, dashed, "\varepsilon_1"] & C_2 \arrow[r, dashed, "g_C"] \arrow[d, dashed, "\varepsilon_2"] & C_3 \arrow[r, dashed, "\delta_C"] \arrow[d, dashed, "\varepsilon_3"] & {} \\
{} & {} & {} &
\end{tikzcd}. \tag{3.4.3}$$

We omit the dual statements for 1. to 5..

Proof. (*1.* $\to$ *2.*). Let $X \xrightarrow{\binom{i}{0}} Y \oplus W \xrightarrow{(s,t)} Z \overset{\delta}{\dashrightarrow}$ be a conflation. Note that $\begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}\binom{i}{0} = 0$. By eq. (1.2.2), we can find $\binom{a}{b}$ such that $\binom{a}{b}(s,t) = \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}$:

$$\begin{tikzcd}
X \arrow[r, "\binom{i}{0}"] \arrow[dr, "0"'] & Y \oplus W \arrow[r, "{(s,t)}"] \arrow[d, "\begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}"] & Z \arrow[r, dashed, "\delta"] \arrow[dl, dashed, "\binom{a}{b}"] & {} \\
& Z \oplus W & &
\end{tikzcd}. \tag{3.4.4}$$

This shows that t a section. By assumption, $Z \simeq Q \oplus W$. Hence $X \xrightarrow{\binom{i}{0}} Y \oplus W \xrightarrow{\begin{pmatrix} s_1 & 0 \\ s_2 & 1 \end{pmatrix}} Q \oplus W \overset{\delta}{\dashrightarrow}$ is a conflation. By proposition 1.7, there is a way to complete the following diamgram:

$$\begin{tikzcd}
& W \arrow[r, equal] \arrow[d, "\binom{0}{1}"] & W \arrow[d, "\binom{0}{1}"] & \\
X \arrow[r, "\binom{i}{0}"] \arrow[d, equal] & Y \oplus W \arrow[r, "\begin{pmatrix} s_1 & 0 \\ s_2 & 1 \end{pmatrix}"] \arrow[d, "{(1,0)}"] & Q \oplus W \arrow[r, dashed, "\delta"] \arrow[d, "{(1,0)}"] & {} \\
X \arrow[r, dashed, "i"] & Y \arrow[r, dashed, "s_1"] \arrow[d, dashed, "0"] & Q \arrow[r, dashed, "\varepsilon"] \arrow[d, dashed, "0"] & {} \\
& {} & {} &
\end{tikzcd}. \tag{3.4.5}$$

The morphism i and s_1 at the bottom row is uniquely determined by a straightforward calculation. Hence, i is an inflation.
(*2.* $\to$ *4.*). When fi is an inflation, then so is $\binom{i}{fi}$ by proposition 3.2. Here $\binom{i}{0} = \begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix}^{-1} \circ \binom{i}{fi}$ is again an $\mathbb{E}$-inflation. By assumption, i is an inflation.
(*4.* $\to$ *1.*). Since isomorphisms are inflations, sections are inflations. Thus they have cokernels.

(*5.* $\to$ *3.* $\to$ *2.*). This is straightforward.
(*1.* and *4.* implies *5.*). Let f' be a retract of some inflation f, i.e.

$$\begin{tikzcd} X' \arrow[r, "i"] \arrow[d, "f'"] & X \arrow[r, "p"] \arrow[d, "f"] & X' \arrow[d, "f'"] \\ Y' \arrow[r, "j"] & Y \arrow[r, "q"] & Y' \end{tikzcd} . \tag{3.4.6}$$

By *1.*, fi is a composite of inflations. By *4.*, f' is an inflation.
(*4.* $\to$ *6.*). This is **Lemma 5.9.** in [11].
(*6.* $\to$ *1.*). For sake of contradiction, we prove the contrapositive statement. Let $X \xrightarrow{i} Y$ be a section which does not have a cokernel. We denote q as its right inverse. Consider

$$\begin{pmatrix} 0 & q \\ i & 1-iq \end{pmatrix} \begin{pmatrix} 0 & q \\ i & 1-iq \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.4.7}$$

We obtain isomorphic split $\mathbb{E}$-conflations:

$$\begin{tikzcd} Y \arrow[r, "\binom{0}{1}"] \arrow[d, equal] & X \oplus Y \arrow[r, "{(1,0)}"] \arrow[d, "\begin{pmatrix} 0 & q \\ i & 1-iq \end{pmatrix}"] & X \arrow[r, dashed, "0"] \arrow[d, equal] & {} \\ Y \arrow[r, "\binom{q}{1-iq}"] & X \oplus Y \arrow[r, "{(0,q)}"] & X \arrow[r, dashed, "0"] & {} \end{tikzcd} . \tag{3.4.8}$$

It remains to show the following diagram fails to be completed to a 3×3-diagram:

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, dashed] & X \arrow[r] \arrow[d, "\binom{1}{0}"] & 0 \arrow[d] \\ Y \arrow[r, "\binom{q}{1-iq}"] \arrow[d, dashed] & X \oplus Y \arrow[r, "{(0,q)}"] \arrow[d, "{(0,1)}"] & X \arrow[d, equal] \\ Z \arrow[r, dashed] & Y \arrow[r, dashed, "q"] & X \end{tikzcd} . \tag{3.4.9}$$

If such completion exists, then q is both an $\mathbb{E}$-deflation and a retraction, thus it has a kernel. This contradicts our assumption. $\square$

4 Snake Lemmas

4.1 3×3 Lemmas

Our 3×3-lemmas begin with two conflations with either two of the morphisms in (i_1, i_2, i_3)

$$\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, dashed, "i_1"] & A_2 \arrow[r, "g_A"] \arrow[d, dashed, "i_2"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, dashed, "i_3"] & {} \\
B_1 \arrow[r, "f_B"] & B_2 \arrow[r, "g_B"] & B_3 \arrow[r, dashed, "\delta_B"] & {}
\end{tikzcd} . \tag{4.1.1}$$

Example 4.1. Let f be any morphism in the category. Let $i_1 = 0$ and $i_2 = \binom{1}{f}$ be $\mathbb{E}$-inflations in the following diagram.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "0"'] & X \arrow[r, "1_X"] \arrow[d, "\binom{1}{f}"'] & X \arrow[r, dashed, "0"] \arrow[d, dashed, "f"'] & {} \\
X \arrow[r, "\binom{1}{0}"] & X \oplus Y \arrow[r, "{(0,1)}"] & Y \arrow[r, dashed, "0"] & {}
\end{tikzcd} . \tag{4.1.2}$$

One must have $i_3 = f$. This diagram fails to be a 3×3-diagram.

Theorem 4.2. *Suppose we have conflations realising δ_A, δ_B, ε_1 and ε_3 in the following diagram:*

$$\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, "i_1"] & A_2 \arrow[r, "g_A"] \arrow[d, dashed, "i_2"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, "i_3"] & {} \\
B_1 \arrow[r, "f_B"] \arrow[d, "p_1"] & B_2 \arrow[r, "g_B"] \arrow[d, dashed, "p_2"] & B_3 \arrow[r, dashed, "\delta_B"] \arrow[d, "p_3"] & {} \\
C_1 \arrow[r, dashed, "f_C"] \arrow[d, dashed, "\varepsilon_1"] & C_2 \arrow[r, dashed, "g_C"] \arrow[d, dashed, "\varepsilon_2"] & C_3 \arrow[r, dashed, "\delta_C"] \arrow[d, dashed, "\varepsilon_3"] & {} \\
{} & {} & {} &
\end{tikzcd} , \tag{4.1.3}$$

such that $(i_3)^*\delta_B = (i_1)_*\delta_A$. Then there is a way to complete the diagram to a 3×3-diagram.

Proof. By theorem 2.8, we take i_2 such that $(i_1; i_2; i_3)$ is homotopic:

$$\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, "i_1"] \arrow[dr, phantom, "\square"] & A_2 \arrow[r, "g_A"] \arrow[d, "j_1"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, equal] & {} \\
B_1 \arrow[r, "s"] \arrow[d, equal] & E \arrow[r, "t"] \arrow[d, "j_2"] \arrow[dr, phantom, "\square"] & A_3 \arrow[r, dashed, "\kappa"] \arrow[d, "i_3"] & {} \\
B_1 \arrow[r, "f_B"] & B_2 \arrow[r, "g_B"] & B_3 \arrow[r, dashed, "\delta_B"] & {}
\end{tikzcd} , \tag{4.1.4}$$

We denote $\kappa = (i_1)_*\delta_A = (i_3)^*\delta_B$. The construction of j_1 and j_2 are due to propositions 2.18 and 2.19:

$$\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, "i_1"] \arrow[dr, phantom, "\square"] & A_2 \arrow[r, "g_A"] \arrow[d, dashed, "j_1"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, equal] & {} \\
B_1 \arrow[r, "s"] \arrow[d, "p_1"] & E \arrow[r, "t"] \arrow[d, dashed, "q"] & A_3 \arrow[r, dashed, "\kappa"] & {} \\
C_1 \arrow[r, equal] \arrow[d, dashed, "\varepsilon_1"] & C_1 \arrow[d, dashed, "(f_A)_*\varepsilon_1"] & & \\
{} & {} & &
\end{tikzcd}
\qquad
\begin{tikzcd}
B_1 \arrow[r, "s"] \arrow[d, equal] & E \arrow[r, "t"] \arrow[d, dashed, "j_2"] \arrow[dr, phantom, "\square"] & A_3 \arrow[r, dashed, "\kappa"] \arrow[d, "i_3"] & {} \\
B_1 \arrow[r, "f_B"] & B_2 \arrow[r, "g_B"] \arrow[d, dashed, "p_3 g_B"] & B_3 \arrow[r, dashed, "\delta_B"] \arrow[d, "p_3"] & {} \\
& C_3 \arrow[r, equal] \arrow[d, dashed, "\theta"] & C_3 \arrow[d, dashed, "\varepsilon_3"] & \\
& {} & {} &
\end{tikzcd} . \tag{4.1.5}$$

We construct ε_2 and δ_C by ET4,

$$\begin{tikzcd}
A_2 \arrow[r, equal] \arrow[d, "j_1"] & A_2 \arrow[d, dashed, "i_2"] & & \\
E \arrow[r, "j_2"] \arrow[d, "q"] & B_2 \arrow[r, "p_3 g_B"] \arrow[d, dashed, "p_2"] & C_3 \arrow[r, dashed, "\theta"] \arrow[d, equal] & {} \\
C_1 \arrow[r, dashed, "f_C"] \arrow[d, dashed, "(f_A)_*\varepsilon_1"] & C_2 \arrow[r, dashed, "g_C"] \arrow[d, dashed, "\varepsilon_2"] & C_3 \arrow[r, dashed, "\delta_C"] & {} \\
{} & {} & &
\end{tikzcd} . \tag{4.1.6}$$

It remains to verify eq. (4.1.3) is a 3×3-diagram under the above construction.

1. $(f_Bi_1 = i_2f_A)$. $i_2f_A \xlongequal{\text{eq. (4.1.6)}} j_2j_1f_A \xlongequal{\text{eq. (4.1.4)}} f_Bi_1$.

2. $(g_Bi_2 = i_3g_A)$. $g_Bi_2 \xlongequal{\text{eq. (4.1.6)}} g_Bj_2j_1 \xlongequal{\text{eq. (4.1.5)}} i_3tj_1 \xlongequal{\text{eq. (4.1.4)}} i_3g_A$.

3. $(f_Cp_1 = p_2f_B)$. $f_Cp_1 \xlongequal{\text{eq. (4.1.5)}} f_Cqs \xlongequal{\text{eq. (4.1.6)}} p_2j_2s \xlongequal{\text{eq. (4.1.4)}} p_2f_B$.

4. $(g_Cp_2 = p_3g_B)$. $g_Cp_2 \xlongequal{\text{eq. (4.1.6)}} p_3g_B$.

5. $((i_1)_*\delta_A = (i_3)^*\delta_B)$. We presuppose this identity.

6. $((p_1)_*\delta_B = (p_3)^*\delta_C)$. $(p_1)_*\delta_B \xlongequal{\text{eq. (4.1.5)}} q_*s_*\delta_B \xlongequal{\text{eq. (4.1.6)}} q_*(p_3)^*\theta = (p_3)^*q_*\theta \xlongequal{\text{eq. (4.1.6)}} (p_3)^*\delta_C$.

7. $((f_A)_*\varepsilon_1 = (f_C)^*\varepsilon_2)$. $(f_A)_*\varepsilon_1 \xlongequal{\text{eq. (4.1.6)}} (f_C)^*\varepsilon_2$.

8. $((g_A)_*\varepsilon_2 = (g_C)^*\varepsilon_3)$. $(g_A)_*\varepsilon_2 \xlongequal{\text{eq. (4.1.4)}} t_*(j_1)_*\varepsilon_2 \xlongequal{\text{eq. (4.1.6)}} t_*(g_C)^*\theta = (g_C)^*t_*\theta \xlongequal{\text{eq. (4.1.5)}} (g_C)^*\varepsilon_3$.

$\square$

Corollary 4.3. *Let α, β and γ be all inflations (or dually, all deflations) such that $(\alpha;\beta;\gamma)$ is a homotopic morphism of conflations. Then it extends to a 3×3-diagram.*

Proof. There is a way to construct eq. (4.1.4). Now theorem 4.2 completes the proof. $\square$

Corollary 4.4. *Let α, β and γ be all inflations (or dually, all deflations) such that $(\alpha;\beta;\gamma)$ is a morphism of conflations. There is a way to find $(\alpha';\beta;\gamma)$, $(\alpha;\beta';\gamma)$ and $(\alpha;\beta;\gamma')$ which completes to 3×3 diagram.*

Proof. By theorem 2.8 and corollary 4.3, we are done. $\square$

Proposition 4.5. *Suppose we have the commutative diagram of conflations:*

$$\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, dashed, "i_1"] & A_2 \arrow[r, "g_A"] \arrow[d, "i_2"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, "i_3"] & {} \\
B_1 \arrow[r, "f_B"] & B_2 \arrow[r, "g_B"] \arrow[d, "p_2"] & B_3 \arrow[r, dashed, "\delta_B"] \arrow[d, "p_3"] & {} \\
& C_2 \arrow[r, dashed, "g_C"] \arrow[d, dashed, "\varepsilon_2"] & C_3 \arrow[d, dashed, "\varepsilon_3"] & \\
& {} & {} &
\end{tikzcd} . \tag{4.1.7}$$

Then there exists i_1 and p_3 such that the above diagram commutes, and

1. *i_1 is a retract of some inflation, p_3 is a retract of some deflation,*

2. *$(i_1;i_2;i_3)$ and $(g_A;g_B;g_C)$ are homotopic morphisms of conflations.*

Proof. We construct i_1 as follows. By lemma 2.7, we can find an inflation j_2 such that $(1_{B_1};j_2;i_3)$ is a homotopy morphism of conflations. We then construct j_1 by proposition 2.2, and i_1 by lemma 2.5.

$$\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, dashed, "i_1"] \arrow[dr, phantom, "\square"] & A_2 \arrow[r, "g_A"] \arrow[d, dashed, "j_1"] \arrow[dd, bend right, "i_2"'] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, equal] & {} \\
B_1 \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "j_2"] \arrow[dr, phantom, "\square"] & A_3 \arrow[r, "(i_3)^*\delta_B"] \arrow[d, "i_3"] & {} \\
B_1 \arrow[r, "f_B"] & B_2 \arrow[r, "g_B"] & B_3 \arrow[r, dashed, "\delta_B"] & {}
\end{tikzcd} . \tag{4.1.8}$$

Here j_1 is a retract of some inflation by proposition 3.2, and i_1 is also a retract of some inflation by lemma 3.13. Dually, we can construct p_3 which is a retract of some deflation. The rest is clear. $\square$

Remark. Under WIC condition, proposition 4.5 completes to a 3×3-diagram.

Theorem 4.6. *We use $(\alpha;\beta;\gamma)$ to denote a morphism of $\mathbb{E}$-conflations.*

1. *For any morphism γ, there are $\mathbb{E}$-inflations α and β such that $(\alpha;\beta;\gamma)$ is a homotopic morphism of $\mathbb{E}$-conflations.*

2. *If α and γ are $\mathbb{E}$-inflations, then there is a way to find some $\mathbb{E}$-inflation β such that $(\alpha;\beta;\gamma)$ is a homotopic morphism of $\mathbb{E}$-conflations.*

3. *If β and γ are $\mathbb{E}$-inflations, then there is a way to find some retract of $\mathbb{E}$-inflation α such that $(\alpha;\beta;\gamma)$ is a homotopic morphism of $\mathbb{E}$-conflations.*
4. *If α is an $\mathbb{E}$-inflation and β is an $\mathbb{E}$-deflation, then there is a way to find some $\mathbb{E}$-deflation γ such that $(\alpha;\beta;\gamma)$ is a homotopic morphism of $\mathbb{E}$-conflations.*
5. *For any morphism β, there is a way to find some $\mathbb{E}$-inflation α and some $\mathbb{E}$-deflation γ such that $(\alpha;\beta;\gamma)$ is a homotopic morphism of $\mathbb{E}$-conflations.*
6. *If β is an $\mathbb{E}$-inflation and γ is an $\mathbb{E}$-deflation, then there is a way to find some $\mathbb{E}$-inflation α such that $(\alpha;\beta;\gamma)$ is a homotopic morphism of $\mathbb{E}$-conflations.*
7. *If α is an $\mathbb{E}$-deflation and β is an $\mathbb{E}$-inflation, then there is a way to find some retract of $\mathbb{E}$-inflation γ such that $(\alpha;\beta;\gamma)$ is a homotopic morphism of $\mathbb{E}$-conflations.*
8. *If β is an $\mathbb{E}$-deflation and γ is an $\mathbb{E}$-inflation, then there is a way to find some retract of $\mathbb{E}$-deflation γ such that $(\alpha;\beta;\gamma)$ is a homotopic morphism of $\mathbb{E}$-conflations.*
9. *If α and γ are $\mathbb{E}$-deflations, then there is a way to find some retract of $\mathbb{E}$-deflation γ such that $(\alpha;\beta;\gamma)$ is a homotopic morphism of $\mathbb{E}$-conflations.*
10. *If α and γ are $\mathbb{E}$-deflations, then there is a way to find some $\mathbb{E}$-deflation β such that $(\alpha;\beta;\gamma)$ is a homotopic morphism of $\mathbb{E}$-conflations.*
11. *For any morphism α, there are $\mathbb{E}$-deflations β and γ such that $(\alpha;\beta;\gamma)$ is a homotopic morphism of $\mathbb{E}$-conflations.*

Note that the final three statements are duals of the first three statements.

Proof. *1.* See example 4.1. *2.* See theorem 4.2. *3.* See proposition 4.5.
4. We construct $\overline{\alpha}$ by lemma 2.4, and $\overline{\gamma}$ by proposition 2.2. Here $\overline{\gamma}$ is an $\mathbb{E}$-deflation by proposition 3.2. We then construct γ by lemma 2.7, which is an $\mathbb{E}$-deflation by theorem 3.5.

$$\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\square"] & A_2 \arrow[r, "g_A"] \arrow[d, "\overline{\alpha}"'] \arrow[dd, bend left, "\beta"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, equal] & {} \\
B_1 \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, dashed, "\overline{\gamma}"'] \arrow[dr, phantom, "\square"] & A_3 \arrow[r, dashed] \arrow[d, dashed, "\gamma"] & {} \\
B_1 \arrow[r, "f_B"] & B_2 \arrow[r, "g_B"] & B_3 \arrow[r, dashed, "\delta_B"] & {}
\end{tikzcd} . \tag{4.1.9}$$

This complete the proof.
5. For any $\beta : M \to N$, we can find a homotopic morphism of $\mathbb{E}$-conflations:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & M \arrow[r, "1_M"] \arrow[d, "\binom{1}{0}"] & M \arrow[r, dashed, "0"] \arrow[d, equal] & {} \\
N \arrow[r, "\binom{0}{1}"'] \arrow[d, equal] & M \oplus N \arrow[r, "{(1,0)}"] \arrow[d, "{(\beta,1)}"'] \arrow[dr, phantom, "\square"] & M \arrow[r, dashed, "0"] \arrow[d] & {} \\
N \arrow[r, "1_N"] & N \arrow[r] & 0 \arrow[r, dashed, "0"] & {}
\end{tikzcd} . \tag{4.1.10}$$

Note that all $\square$ correspond to extension element 0.
6. We construct $\overline{\gamma}$ by lemma 2.6, and $\overline{\alpha}$ by proposition 2.2. Here $\overline{\alpha}$ is an $\mathbb{E}$-inflation by proposition 3.2. We then construct α by lemma 2.5, which is an $\mathbb{E}$-inflation by theorem 3.5.

$$\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, dashed, "\alpha"] \arrow[dr, phantom, "\square"] & A_2 \arrow[r, "g_A"] \arrow[d, dashed, "\overline{\alpha}"'] \arrow[dd, bend left, "\beta"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, equal] & {} \\
B_1 \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\overline{\gamma}"'] \arrow[dr, phantom, "\square"] & A_3 \arrow[r, dashed] \arrow[d, "\gamma"] & {} \\
B_1 \arrow[r, "f_B"] & B_2 \arrow[r, "g_B"] & B_3 \arrow[r, dashed, "\delta_B"] & {}
\end{tikzcd} . \tag{4.1.11}$$

This complete the proof.

7. We construct $\overline{\alpha}$ by lemma 2.6, and $\overline{\gamma}$ by proposition 2.2. Here $\overline{\gamma}$ is a retract of some $\mathbb{E}$-inflation by proposition 3.2. We then construct γ by lemma 2.5, which is a retract of some $\mathbb{E}$-inflation by theorem 3.15.

$$\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\square"] & A_2 \arrow[r, "g_A"] \arrow[d, "\overline{\alpha}"'] \arrow[dd, bend left, "\beta"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, equal] & {} \\
B_1 \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, dashed, "\overline{\gamma}"'] \arrow[dr, phantom, "\square"] & A_3 \arrow[r, dashed] \arrow[d, dashed, "\gamma"] & {} \\
B_1 \arrow[r, "f_B"] & B_2 \arrow[r, "g_B"] & B_3 \arrow[r, dashed, "\delta_B"] & {}
\end{tikzcd}. \tag{4.1.12}$$

This complete the proof.
8. We construct $\overline{\gamma}$ by lemma 2.6, and $\overline{\alpha}$ by proposition 2.2. Here $\overline{\alpha}$ is a retract of some $\mathbb{E}$-deflation by proposition 3.2. We then construct α by lemma 2.5, which is a retract of some $\mathbb{E}$-deflation by theorem 3.15.

$$\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, dashed, "\alpha"] \arrow[dr, phantom, "\square"] & A_2 \arrow[r, "g_A"] \arrow[d, dashed, "\overline{\alpha}"'] \arrow[dd, bend left, "\beta"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, equal] & {} \\
B_1 \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\overline{\gamma}"'] \arrow[dr, phantom, "\square"] & A_3 \arrow[r, dashed] \arrow[d, "\gamma"] & {} \\
B_1 \arrow[r, "f_B"] & B_2 \arrow[r, "g_B"] & B_3 \arrow[r, dashed, "\delta_B"] & {}
\end{tikzcd}. \tag{4.1.13}$$

This complete the proof.
9., *10.* and *11.* are dual to *1.*, *2.* and *3.* respectively. $\square$

4.2 Morphisms that are both $\mathbb{E}$-inflations and $\mathbb{E}$-deflations

We construct various of snake lemmas in the forthcoming sections. The main obstacle is to extend $\mathbb{E}$-conflations to ≥ 3 terms. Unlike admissible morphisms in exact categories (**Definition 8.1.**, [2]), it is somehow difficult to decompose a morphism into an $\mathbb{E}$-inflation followed by an $\mathbb{E}$-deflation in extriangulated categories in a unique way. Thus we focus on morphisms that are both inflations and deflations (definition 3.16).

Notation. Let S be the collection of morphisms that are both $\mathbb{E}$-inflations and $\mathbb{E}$-deflations.

Remark. In an exact category, S is the collection of isomorphisms. In a triangulated category, S is the collection of all morphisms. By proposition 3.18, S contains all isomorphisms and closed under compositions.

Proposition 4.7. *Suppose we have a homotopic square between two $\mathbb{E}$-conflations:*

$$\begin{tikzcd}
K \arrow[r, "i"] \arrow[d, equal] & X \arrow[r, "f", "\circ" marking] \arrow[d, "u"] \arrow[dr, phantom, "\boxed{i_*\varepsilon}"] & Y \arrow[r, dashed, "\delta"] \arrow[d, "v"] & {} \\
K \arrow[r, "j"] & A \arrow[r, "g"] & B \arrow[r, dashed, "\varepsilon"] & {}
\end{tikzcd}. \tag{4.2.1}$$

If and only f is an $\mathbb{E}$-inflation ($\mathbb{E}$-deflation), then is g.
If f is both an $\mathbb{E}$-inflation and an $\mathbb{E}$-deflation, there is some $\mathbb{E}$-conflation $(u; v; 1_C)$

$$\begin{tikzcd}
K \arrow[r, dashed, "i"] \arrow[d, dotted, equal] & X \arrow[r, "f"] \arrow[d, "u"] \arrow[dr, phantom, "\boxed{q^*\eta}"] & Y \arrow[r, "p"] \arrow[d, "v"] & C \arrow[r, dashed, "\eta"] \arrow[d, equal] & {} \\
K \arrow[r, dashed, "j"] & A \arrow[r, "g", "\circ" marking] & B \arrow[r, "q"] & C \arrow[r, dashed, "\kappa"] & {}
\end{tikzcd}, \tag{4.2.2}$$

such that $i_*\varepsilon = -q^*\eta$.

Proof. By theorem 3.5, if and only f is an $\mathbb{E}$-inflation ($\mathbb{E}$-deflation), then is g. When f and g are $\mathbb{E}$-inflations, we take an $\mathbb{E}$-conflation $K \to 0 \to FK \overset{\delta_K}{\dashrightarrow}$ as in theorem 3.19 and obtain

$$\begin{tikzcd}
K \arrow[r, "i"] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, dashed] \arrow[dr, phantom, "\boxed{-i_*\delta_K}"] & Y \arrow[r, dashed, "\delta"] \arrow[d, "p"] & {} \\
K \arrow[r] & 0 \arrow[r, "\circ" marking] & FK \arrow[r, dashed, "\delta_K"] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
K \arrow[r, "j"] \arrow[d, equal] & A \arrow[r, "g"] \arrow[d, dashed] \arrow[dr, phantom, "\boxed{-j_*\delta_K}"] & B \arrow[r, dashed, "\varepsilon"] \arrow[d, "q"] & {} \\
K \arrow[r] & 0 \arrow[r, "\circ" marking] & FK \arrow[r, dashed, "\delta_K"] & {}
\end{tikzcd}. \tag{4.2.3}$$

Here the natural isomorphism $\ell : \mathbb{E}(-, K) \simeq (-, FK)$ sends δ and ε to p and q respectively. We show eq. (4.2.2).
We claim that $qv = p$. Note that

$$(qv)^*\delta_K = v^*(q^*\delta_K) = v^*\varepsilon = \delta = p^*\delta_K. \tag{4.2.4}$$

Hence, $(qv - p)^*\delta_K = 0$. By theorem 3.19, we see $(qv - p) = 0$.
We then show that $q^*\eta = -i_*\varepsilon$. Here $\eta = i_*\delta_K$. A straightforward computation shows

$$q^*\eta = q^*(-i_*\delta_K) = -i_*(q^*\delta_K) = -i_*\varepsilon. \tag{4.2.5}$$

We take $FK = C$. $\square$

Theorem 4.8. *Given a homotopic square with a pair of parallel edges (f, g) in S, there is a way to complete the diagram*

$$
\begin{tikzcd}
K \arrow[r, "i"] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, "u"] \arrow[dr, phantom, "\square"] & Y \arrow[r, "p"] \arrow[d, "v"] \arrow[dr, dashed, "l_f"] & C \arrow[r, dashed, "r_f"] \arrow[d, equal] & {} \\
K \arrow[r, "j"] & A \arrow[r, "g"] \arrow[d, dotted] & B \arrow[r, "q"] \arrow[drr, dashed, "l_g"] & C \arrow[r, dashed, "r_g"] & {} \\
& \boxed{i_* l_g} \arrow[r, equal] & \boxed{-q^* r_f} & & {}
\end{tikzcd}, \tag{4.2.6}
$$

such that $(1_K; u; v)$ and $(u; v; 1_C)$ are homotopic morphisms of conflations. Moreover, we can choose $F : \mathcal{L} \simeq \mathcal{R}$ as in theorems 3.19 and 3.20 so that

$$
\ell(l_f) = p, \quad \ell(l_g) = q, \quad \rho(r_f) = -i, \quad \rho(r_g) = -j. \tag{4.2.7}
$$

Proof. We complete the following diagram such that $(1_K; u; v)$ is a homotopic morphism of $\mathbb{E}$-conflations by proposition 4.7:

$$
\begin{tikzcd}
K \arrow[r, "i"] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, "u"] \arrow[dr, phantom, "\boxed{i_* l_g}"] & Y \arrow[r, dashed, "l_f"] \arrow[d, "v"] & {} \\
K \arrow[r, "j"] & A \arrow[r, "g"] & B \arrow[r, dashed, "l_g"] & {}
\end{tikzcd}. \tag{4.2.8}
$$

We define $p := \ell^{-1}(l_f)$ and $q := \ell^{-1}(l_g)$. Since $v^* l_g = l_f$, we see $qv = p$ by theorem 3.19. By construction, $r_f = -i_* \delta_K$ and $r_g = -j_* \delta_K$. Hence $\rho(r_f) = -i$ and $\rho(r_g) = -j$. Finally, we see

$$
i_* l_g = i_* q^* \delta_K = q^* i_* \delta_K = -q^* r_f. \tag{4.2.9}
$$

□

4.3 Snake lemmas

Theorem 4.9. *Let α, β, and γ be both inflations and deflations, and $(\alpha; \beta; \gamma)$ be a homotopic morphism of conflations. Then there is a commutative diagram with dashed z*

$$
\begin{tikzcd}
K_\alpha \arrow[r, "f'"] \arrow[d] & K_\beta \arrow[r, "g'"] \arrow[d] & K_\gamma \arrow[d] \arrow[dddll, dashed, rounded corners, "z"'] \\
X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, dashed, "\delta"] \arrow[d, "\gamma"] & {} \\
L \arrow[r, "u"] \arrow[d] & M \arrow[r, "v"] \arrow[d] & N \arrow[r, dashed, "\varepsilon"] \arrow[d] & {} \\
C_\alpha \arrow[r, "u'"] & C_\beta \arrow[r, "v'"] & C_\gamma.
\end{tikzcd} \tag{4.3.1}
$$

such that any three terms in the mapping sequence

$$
K_\alpha \xrightarrow{f'} K_\beta \xrightarrow{g'} K_\gamma \xrightarrow{z} C_\alpha \xrightarrow{u'} C_\beta \xrightarrow{v'} C_\gamma \tag{4.3.2}
$$

is an $\mathbb{E}$-conflation. Moreover, there is a way to take morphisms such that $K_\alpha \xrightarrow{f'} K_\beta \xrightarrow{g'} K_\gamma \overset{\ell^{-1}(z)}{\dashrightarrow}$, $K_\beta \xrightarrow{g'} K_\gamma \xrightarrow{z} C_\alpha \overset{-\ell^{-1}(u')}{\dashrightarrow}$, and $K_\gamma \xrightarrow{z} C_\alpha \xrightarrow{u'} C_\beta \overset{\ell^{-1}(v')}{\dashrightarrow}$ are $\mathbb{E}$-conflations.

Proof. We decompose the homotopic morphism of $\mathbb{E}$-conflations $(\alpha; \beta; \gamma)$ into following diagrams:

$$
\begin{tikzcd}
K_\alpha \arrow[r, equal] \arrow[d, "i_\alpha"'] & K_\alpha \arrow[d, "f i_\alpha"] \\
X \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\boxed{(i_\alpha)_* \eta_1}"] & Y \arrow[r, "g"] \arrow[d, "\beta_1"] & Z \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\
L \arrow[r, "s"] \arrow[d, dashed, "l_\alpha"'] & E \arrow[r, "t"] \arrow[d, dashed, "\eta_1"] & Z \arrow[r, dashed, "\kappa"] & {} \\
{} & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
L \arrow[r, "s"] \arrow[d, equal] & E \arrow[r, "t"] \arrow[d, "\beta_2"'] \arrow[dr, phantom, "\boxed{(p_\gamma)^* \eta_2}"] & Z \arrow[r, dashed, "\kappa"] \arrow[d, "\gamma"] & {} \\
L \arrow[r, "u"] & M \arrow[r, "v"] \arrow[d, "p_\gamma v"'] & N \arrow[r, dashed, "\varepsilon"] \arrow[d, "p_\gamma"] & {} \\
& C_\gamma \arrow[r, equal] \arrow[d, dashed, "\eta_2"'] & C_\gamma \arrow[d, dashed, "r_\gamma"] \\
& {} & {}
\end{tikzcd}. \tag{4.3.3}
$$

Then β_1 and β_2 are both inflations and deflations by theorem 3.5. By theorem 4.8, we homotopic morphisms of conflations $(1; f; s)$, $(f; s; 1)$, $(1; t; v)$ and $(t; v; 1)$ as follows (dashed arrows indicate extension elements):

$$
\begin{tikzcd}
K_\alpha \arrow[r, "i_\alpha"] \arrow[d, equal] & X \arrow[r, tail, "\alpha"] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & L \arrow[r, "p_\alpha"] \arrow[d, "s"] \arrow[dr, dashed, "l_\alpha"] & C_\alpha \arrow[r, dashed, "r_\alpha"] \arrow[d, equal] & {} \\
K_\alpha \arrow[r, "f i_\alpha"] & Y \arrow[r, "\beta_1"] & E \arrow[r, "y"] \arrow[drr, dashed, "\eta_1"] & C_\alpha \arrow[r, dashed, "\mu_1"] & {} \\
& \boxed{(i_\alpha)_* \eta_1} \arrow[r, equal] & \boxed{-y^* r_\alpha} & & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
K_\gamma \arrow[r, "x"] \arrow[d, equal] & E \arrow[r, "\beta_2"] \arrow[d, "t"] \arrow[dr, phantom, "\square"] & M \arrow[r, "p_\gamma v"] \arrow[d, "v"] \arrow[dr, dashed, "\mu_2"] & C_\gamma \arrow[r, dashed, "\eta_2"] \arrow[d, equal] & {} \\
K_\gamma \arrow[r, "i_\gamma"] & Z \arrow[r, tail, "\gamma"] & N \arrow[r, "p_\gamma"] \arrow[drr, dashed, "l_\gamma"] & C_\gamma \arrow[r, dashed, "r_\gamma"] & {} \\
& \boxed{x_* l_\gamma} \arrow[r, equal] & \boxed{(p_\gamma)^* \eta_2} & & {}
\end{tikzcd}
. \tag{4.3.4}
$$

We obtain $K_\alpha \xrightarrow{f'} K_\beta \xrightarrow{g'} K_\gamma \overset{x^*\eta_1}{\dashrightarrow}$ by proposition 2.17

$$
\begin{tikzcd}
K_\alpha \arrow[r, dashed, "f'"] \arrow[d, equal] & K_\beta \arrow[r, dashed, "g'"] \arrow[d, "i_\beta"] \arrow[dr, phantom, "\square"] & K_\gamma \arrow[r, dashed, "x^*\eta_1"] \arrow[d, tail, "x"] & {} & \boxed{(f')_* \eta_1} \arrow[d, equal] \\
K_\alpha \arrow[r, "f i_\alpha"] & Y \arrow[r, "\beta_1"] \arrow[d, "\beta"] & E \arrow[r, dashed, "\eta_1"] \arrow[d, "\beta_2"] & {} & \boxed{(\beta_2)^* l_\beta} \\
& M \arrow[r, equal] \arrow[d, dashed, "l_\beta"] & M \arrow[d, dashed, "\mu_2"] & & \\
& {} & {} & &
\end{tikzcd}
. \tag{4.3.5}
$$

We then construct $K_\beta \xrightarrow{\overline{g'}} K_\gamma \xrightarrow{z} C_\alpha \overset{\theta}{\dashrightarrow}$ by proposition 2.20

$$
\begin{tikzcd}
K_\beta \arrow[r, dashed, "\overline{g'}"] \arrow[d, "i_\beta"] \arrow[dr, phantom, "\square"] & K_\gamma \arrow[r, dashed, "yx"] \arrow[d, tail, "x"] & C_\alpha \arrow[r, dashed, "\theta"] \arrow[d, equal] & {} & \boxed{(\beta_2)^* l_\beta} \arrow[d, equal] \\
Y \arrow[r, "\beta_1"] \arrow[d, "\beta"] & E \arrow[r, "y"] \arrow[d, "\beta_2"] & C_\alpha \arrow[r, dashed, "\mu_1"] & {} & \boxed{-y^* \mu_1} \\
M \arrow[r, equal] \arrow[d, dashed, "l_\beta"] & M \arrow[d, dashed, "\mu_2"] & & & \\
{} & {} & & &
\end{tikzcd}
. \tag{4.3.6}
$$

Recall that in our construction of proposition 2.20, $\overline{g'}$ can be any morphism such that $\boxed{\beta_2^* l_\beta}$ is a homotopic square. Hence, we take $\overline{g'} = g'$. It remains to show $\ell(x^*\eta_1) = z$ here. Note that the construction in theorem 4.8 shows $\ell(\eta_1) = y$. Hence,

$$
\ell(x^*\eta_1) = \ell(\eta_1)x = yx = z. \tag{4.3.7}
$$

We next construct $K_\gamma \xrightarrow{z} C_\alpha \xrightarrow{u'} C_\beta \overset{\xi}{\dashrightarrow}$ by dual of proposition 2.20:

$$
\begin{tikzcd}
& Y \arrow[r, equal] \arrow[d, "\beta_1"] & Y \arrow[d, "\beta"] & & \\
K_\gamma \arrow[r, "x"] \arrow[d, equal] & E \arrow[r, "\beta_2"] \arrow[d, tail, "y"] \arrow[dr, phantom, "\square"] & M \arrow[r, dashed, "\mu_2"] \arrow[d, "p_\beta"] & {} & \boxed{(\beta_1)_* r_\beta} \arrow[d, equal] \\
K_\gamma \arrow[r, dashed, "yx"] & C_\alpha \arrow[r, dashed, "u'"] \arrow[d, dashed, "\mu_1"] & C_\beta \arrow[r, dashed, "\xi"] \arrow[d, dashed, "r_\beta"] & {} & \boxed{-x_* \xi} \\
& {} & {} & &
\end{tikzcd}
. \tag{4.3.8}
$$

Recall that in our construction of proposition 2.20, u' can be any morphism such that $\boxed{-x_*\xi}$ is a homotopic square. We take $u' = \ell(\theta)$. Note that

1. $u'y = \ell(y^*\theta) \overset{\text{eq. (4.3.6)}}{=\!=\!=} -\ell(\beta_2^* l_\beta) = -\ell(l_\beta)\beta_2 \overset{\text{eq. (4.3.4)}}{=\!=\!=} p_\beta \beta_2$.
2. $\mu_1 \overset{\text{eq. (4.3.6)}}{=\!=\!=} (i_\beta)_*\theta = (i_\beta)_*\ell^{-1}(u') = \ell(F(i_\beta)u') = \ell(\ell^{-1}(r_\beta)u') = \ell\ell^{-1}((u')^* r_\beta) = (u')^* r_\beta$.

3. We claim that $E \xrightarrow{\binom{\beta_2}{-y}} M \oplus C_\alpha \xrightarrow{(p_\beta, u')} C_\beta \overset{(\beta_1)_* r_\beta}{\dashrightarrow}$ is an $\mathbb{E}$-conflation. Note that there are some $\mathbb{E}$-conflation $E \xrightarrow{\binom{\beta_2}{-y}} M \oplus C_\alpha \to C_\beta \dashrightarrow$, we see $\binom{\beta_2}{-y} \in S$ by proposition 3.17. Hence, $\binom{\beta_2}{-y}$ is an $\mathbb{E}$-deflation. By proposition 2.20, there is a dashed $\mathbb{E}$-conflation

$$\begin{tikzcd}
 & K_\beta \arrow[r, equal] \arrow[d, dashed, "?"] & K_\beta \arrow[d, "\binom{i_\beta}{g'}"] & & \\
 & K_\beta \oplus K_\beta \arrow[r, "i_\beta \oplus g'"] \arrow[d, dashed, "?"] & Y \oplus K_\gamma \arrow[r, "\beta \oplus yx"] \arrow[d, "{(\beta_1, -x)}"] & M \oplus C_\alpha \arrow[r, dashed, "l_\beta \oplus \theta"] \arrow[d, equal] & {} \\
 & ? \arrow[r, "?"] \arrow[d, dashed, "?"] & E \arrow[r, "\binom{\beta_2}{-y}"] \arrow[d, dashed, "(f')_* \eta_1"] & M \oplus C_\alpha \arrow[r, dashed, "?"] & {} \\
 & {} & {} & &
\end{tikzcd} \quad . \tag{4.3.9}$$

Note that the top left morphism must be $\binom{1}{1}$, thus the left dashed $\mathbb{E}$-conflation splits. Note that

$$\begin{tikzcd}
K_\beta \arrow[r, equal] \arrow[d, "\binom{1}{1}"] & K_\beta \arrow[r, equal] \arrow[d, dashed, "\binom{1}{1}"] & K_\beta \arrow[d, "\binom{i_\beta}{g'}"] & & \\
K_\beta \oplus K_\beta \arrow[r, equal] \arrow[d, "{(1,-1)}"] & K_\beta \oplus K_\beta \arrow[r, "i_\beta \oplus g'"] \arrow[d, dashed, "?"] & Y \oplus K_\gamma \arrow[r, "\beta \oplus yx"] \arrow[d, "{(\beta_1, -x)}"] & M \oplus C_\alpha \arrow[r, dashed, "l_\beta \oplus \theta"] \arrow[d, equal] & {} \\
K_\beta \arrow[r, "\cong", "\varphi"'] \arrow[d, dashed, "0"] & K_\beta \arrow[r, "?"] \arrow[d, dashed, "0"] & E \arrow[r, "\binom{\beta_2}{-y}"] \arrow[d, dashed, "(f')_* \eta_1"] & M \oplus C_\alpha \arrow[r, dashed, "?"] & {} \\
{} & {} & {} & &
\end{tikzcd} \quad . \tag{4.3.10}$$

By corollary 1.2, there is some isomorphism φ making $(1_{K_\beta}; 1_{K_\beta \oplus K_\beta}; \varphi)$ an isomorphism of $\mathbb{E}$-conflations. By diagram, the buttom row is the $\mathbb{E}$-conflation $K_\beta \xrightarrow{\beta_1 i_\beta} E \xrightarrow{\binom{\beta_2}{-y}} M \oplus C_\alpha \overset{(1,-1)_*(l_\beta \oplus \theta)}{\dashrightarrow}$. Note that $\ell((1,-1)_*(l_\beta \oplus \theta)) = (\ell(l_\beta), -\ell(\theta)) = (p_\beta, u')$, and $\ell(\beta_1 i_\beta) = (\beta_1)_* \ell(i_\beta) = (\beta_1)_* r_\beta$. This proves the claim.

We finally take the following diagram by proposition 2.17

$$\begin{tikzcd}
Y \arrow[r, equal] \arrow[d, "\beta_1"] & Y \arrow[d, "\beta"] & & & \\
E \arrow[r, "\beta_2"] \arrow[d, "y"] \arrow[dr, phantom, "\square"] & M \arrow[r, "p_\gamma v"] \arrow[d, "p_\beta"] & C_\gamma \arrow[r, dashed, "\eta_2"] \arrow[d, equal] & {} & \boxed{(\beta_1)_* r_\beta} \arrow[d, equal] \\
C_\alpha \arrow[r, dashed, "\overline{u'}"] \arrow[d, dashed, "\mu_1"] & C_\beta \arrow[r, dashed, "v'"] \arrow[d, dashed, "r_\beta"] & C_\gamma \arrow[r, dashed, "y_* \eta_2"] & {} & \boxed{(v')^* \eta_2} \\
{} & {} & & &
\end{tikzcd} \quad . \tag{4.3.11}$$

Recall that in our construction of proposition 2.17, $\overline{u'}$ can be any morphism such that $\boxed{(\beta_1)_* r_\beta}$ is homotopic. Hence, we can take $\overline{u'} = u'$. A comparison of eqs. (4.3.8) and (4.3.11) show that $v' = \ell(\xi)$.
This complete our verification. ☐

We show some degenerate cases of theorem 4.9.

Proposition 4.10 (**Case: S-?-S**)**.** *For $\alpha, \gamma \in S$ such that $(\alpha; \gamma)$ is a morphism of extensions, we can find $\beta \in S$ such that $(\alpha; \beta; \gamma)$ is a homotopic morphism of $\mathbb{E}$-conflations. Moreover, we have the following commutative diagram with outer 6-term $\mathbb{E}$-conflation*

$$\begin{tikzcd}
K_\alpha \arrow[r, dashed, "f'"] \arrow[d] & K_\beta \arrow[r, dashed, "g'"] \arrow[d, dashed] & K_\gamma \arrow[d] \arrow[dddll, dashed, rounded corners, "z"'] & \\
X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "g"] \arrow[d, dashed, "\beta"] & Z \arrow[r, dashed, "\delta"] \arrow[d, "\gamma"] & {} \\
L \arrow[r, "u"] \arrow[d] & M \arrow[r, "v"] \arrow[d, dashed] & N \arrow[r, dashed, "\varepsilon"] \arrow[d] & {} \\
C_\alpha \arrow[r, dashed, "u'"] & C_\beta \arrow[r, dashed, "v'"] & C_\gamma &
\end{tikzcd} \quad . \tag{4.3.12}$$

Proposition 4.11 (**Case: S-S-?**)**.** *For $\alpha, \beta \in S$ such that $(\alpha;\beta;?)$ is a morphism of some $\mathbb{E}$-conflations, we can find γ such that $(\alpha;\beta;\gamma)$ is a homotopic morphism of $\mathbb{E}$-conflations. Moreover,*

1. *(Without WIC) γ is an $\mathbb{E}$-deflation, and we have the following commutative diagram with outer 5-term $\mathbb{E}$-conflation:*
2. *(Assume WIC) $\gamma \in S$, and we have the following commutative diagram with outer 6-term $\mathbb{E}$-conflation:*

$$\begin{tikzcd}
K_\alpha \arrow[r, dashed, "f'"] \arrow[d] & K_\beta \arrow[r, dashed, "g'"] \arrow[d] & K_\gamma \arrow[d, dashed] \arrow[dddll, dashed, "z"'] & \\
X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, dashed, "\delta"] \arrow[d, dashed, "\gamma"] & {} \\
L \arrow[r, "u"] \arrow[d] & M \arrow[r, "v"] \arrow[d] & N \arrow[r, dashed, "\varepsilon"] \arrow[d, dashed] & {} \\
C_\alpha \arrow[r, dashed, "u'"] & C_\beta \arrow[r, dashed, "v'"] & C_\gamma &
\end{tikzcd} \tag{4.3.13}$$

Remark. **I-?-I** and **D-?-D** cases can be shown without WIC condition. **?-I-I** and **D-D-?** cases can be shown assuming WIC condition (in fact, they are equivalent coditions of WIC, theorem 3.28).

Proposition 4.12 (**Case: I-S-?**)**.** *For α an $\mathbb{E}$-inflation and $\beta \in S$ such that such that $(\alpha;\beta;?)$ is a morphism of some $\mathbb{E}$-conflations, we can find γ such that $(\alpha;\beta;\gamma)$ is a homotopic morphism of $\mathbb{E}$-conflations. Moreover,*

1. *(Without WIC) γ is an $\mathbb{E}$-deflation, and we have the following commutative diagram with outer 5-term $\mathbb{E}$-conflation:*
2. *(Assume WIC) $\gamma \in S$, and we have the following commutative diagram with outer 5-term $\mathbb{E}$-conflation:*

$$\begin{tikzcd}
 & K_\beta \arrow[r, dashed, "g'"] \arrow[d] & K_\gamma \arrow[d, dashed] \arrow[dddll, dashed, "z"'] & \\
X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, dashed, "\delta"] \arrow[d, dashed, "\gamma"] & {} \\
L \arrow[r, "u"] \arrow[d] & M \arrow[r, "v"] \arrow[d] & N \arrow[r, dashed, "\varepsilon"] \arrow[d, dashed] & {} \\
C_\alpha \arrow[r, dashed, "u'"] & C_\beta \arrow[r, dashed, "v'"] & C_\gamma &
\end{tikzcd} \tag{4.3.14}$$

Proposition 4.13 (**Case: S-D-?**)**.** *For $\alpha \in S$ and β an $\mathbb{E}$-deflation such that such that $(\alpha;\beta;?)$ is a morphism of some $\mathbb{E}$-conflations, we can find some $\mathbb{E}$-deflation β such that $(\alpha;\beta;\gamma)$ is a homotopic morphism of $\mathbb{E}$-conflations. Moreover, we have the following commutative diagram with outer 4-term $\mathbb{E}$-conflation:*

$$\begin{tikzcd}
K_\alpha \arrow[r, dashed, "f'"] \arrow[d] & K_\beta \arrow[r, dashed, "g'"] \arrow[d] & K_\gamma \arrow[d, dashed] \arrow[dddll, dashed, "z"'] & \\
X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, dashed, "\delta"] \arrow[d, dashed, "\gamma"] & {} \\
L \arrow[r, "u"] \arrow[d] & M \arrow[r, "v"] & N \arrow[r, dashed, "\varepsilon"] & {} \\
C_\alpha & & &
\end{tikzcd} \tag{4.3.15}$$

There is also a twist case for 3×3 lemma.

Proposition 4.14 (**Case: I-D-?**)**.** *For α and $\mathbb{E}$-inflation and β an $\mathbb{E}$-deflation such that such that $(\alpha;\beta;?)$ is a morphism of some $\mathbb{E}$-conflations, we can find retract of some $\mathbb{E}$-deflation β such that $(\alpha;\beta;\gamma)$ is a homotopic morphism of $\mathbb{E}$-*

conflations. Moreover, we have the following commutative diagram with outer $\mathbb{E}$-conflation:

$$
\begin{tikzcd}
 & K_\beta \arrow[r, dashed, "g'"] \arrow[d] & K_\gamma \arrow[d, dashed] \arrow[dddll, dashed, rounded corners, "z"'] & \\
X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, dashed, "\delta"] \arrow[d, dashed, "\gamma"] & {} \\
L \arrow[r, "u"] \arrow[d] & M \arrow[r, "v"] & N \arrow[r, dashed, "\varepsilon"] & {} \\
C_\alpha & & &
\end{tikzcd}
\tag{4.3.16}
$$

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