Diagram Lemmas in Extriangulated Categories

on homotopic squares, WIC conditions, and snake lemmas

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The Axioms of Extriangulated Categories (ET1)

Definition 1.1 (Extriangulated Category)

An **extriangulated category** ([NP19]) is a triplet $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ satisfying the axioms (ET1)-(ET4). $\mathcal{C}$ is an additive category.

ET1 $\mathbb{E} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Ab}$ is an additive bifunctor sending a pair of objects (C, A) to an abelian group $\mathbb{E}(C, A)$.

- Elements in $\mathbb{E}$-groups are called **extension elements**.
- We write f_* for the natural transformation $\mathbb{E}(-, f)$ and f^* for $\mathbb{E}(f, -)$ dually. Bifunctoriality shows $f_* g^* = g^* f_*$.

$$\begin{tikzcd} C' & \mathbb{E}(C', A) \arrow[r, "\alpha_*"] \arrow[d, "\gamma^*"] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{E}(C', A') \arrow[d, "\gamma^*"] & A \arrow[d, "\alpha"] \\ C \arrow[u, "\gamma"] & \mathbb{E}(C, A) \arrow[r, "\alpha_*"] & \mathbb{E}(C, A') & A' \end{tikzcd} .$$

- A **morphism of extension elements** is a pair $(\alpha; \gamma) : \delta \to \delta'$ s.t. $\alpha_* \delta = \gamma^* \delta'$.

The Axioms of Extriangulated Categories (ET2)

ET2 $\mathfrak{s}$ is a collection of "maps" known as an **additive realisation**

$$\mathbb{E}(C,A) \to [A \xrightarrow{x} B \xrightarrow{y} C], \quad \delta \mapsto \mathfrak{s}(\delta).$$

Here $[A \xrightarrow{x} B \xrightarrow{y} C]$ is an equivalence class of sequences $A \xrightarrow{x} B \xrightarrow{y} C$ in $\mathcal{C}$ under the relation

$$(A \xrightarrow{x} B \xrightarrow{y} C) \sim (A \xrightarrow{\varphi x} B' \xrightarrow{y\varphi^{-1}} C), \quad \varphi : B \cong B'.$$

- "Additive" means that $\mathfrak{s}(0_{C,A}) = [A \xrightarrow{\binom{1}{0}} A \oplus C \xrightarrow{(0,1)} C]$, and $\mathfrak{s}$ sends direct sums in $\mathbb{E}$-groups to direct sums of sequences.
- Moreover, a morphism of extension elements $(\alpha;\gamma) : \delta \to \delta'$ is sent by $\mathfrak{s}$ to a **morphism of $\mathbb{E}$-conflations** $(\alpha;\beta;\gamma)$:

$$\begin{tikzcd}
\delta \arrow[r, mapsto, "\mathfrak{s}"] \arrow[d, "(\alpha;\gamma)"'] & \mathfrak{s}(\delta) & \ni & A \arrow[r, "f"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "g"] \arrow[d, dashed, "\beta"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, "\gamma"] \\
\delta' \arrow[r, mapsto, "\mathfrak{s}"] & \mathfrak{s}(\delta') & \ni & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C'
\end{tikzcd}.$$

The Axioms of Extriangulated Categories (ET3, ET3$^{\mathrm{op}}$)

Definition 1.2 ($\mathbb{E}$-conflations)

We write $X \xrightarrow{f} Y \xrightarrow{g} Z \overset{\delta}{\dashrightarrow}$ if $X \xrightarrow{f} Y \xrightarrow{g} Z$ belongs to $\mathfrak{s}(\delta)$.

- We call such a sequence an $\mathbb{E}$-**conflation**,
- f (or g) is called an $\mathbb{E}$-**inflation** (or $\mathbb{E}$-**deflation**).
- A morphism of $\mathbb{E}$-conflations is a triplet $(\alpha; \beta; \gamma)$ s.t.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "g"] \arrow[d, "\beta"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[r, dashed, "\delta"] \arrow[d, "\gamma"] & {} \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, dashed, "\delta'"] & {} \end{tikzcd} \quad {}_{\alpha_*\delta=\gamma^*\delta'} . \tag{1}$$

ET3 Let f and f' be $\mathbb{E}$-inflations. Any $\beta f = f'\alpha$ extends to (1).

ET3$^{\mathrm{op}}$ Let g and g' be $\mathbb{E}$-deflations. Any $\gamma g = g'\beta$ extends to (1).

Remark

ET2, ET3 and ET3$^{\mathrm{op}}$ shows that $(\alpha; -; \gamma)$, $(\alpha; \beta; -)$ and $(-; \beta; \gamma)$ complete to morphisms of $\mathbb{E}$-conflations under suitable conditions.

The Axioms of Extriangulated Categories (ET4, ET4$^{\mathrm{op}}$)

ET4 Given any two $\mathbb{E}$-conflations

$$A \xrightarrow{f} B \xrightarrow{g} C \overset{\delta}{\dashrightarrow}, \quad B \xrightarrow{u} D \xrightarrow{v} F \overset{\varepsilon}{\dashrightarrow},$$

there exists a commutative diagram of four $\mathbb{E}$-conflations

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "g"] \arrow[d, "u"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[r, dashed, "\delta"] \arrow[d, dashed, "u'"] & {} & (u')^*\delta'=\delta \\
A \arrow[r, dashed, "f'"] & D \arrow[r, dashed, "g'"] \arrow[d, "v"] \arrow[dr, phantom, "\circlearrowright"] & E \arrow[r, dashed, "\delta'"] \arrow[d, dashed, "v'"] & {} & \\
& F \arrow[r, equal] \arrow[d, dashed, "\varepsilon"] & F \arrow[d, dashed, "\varepsilon'"] & & (v')^*\varepsilon=f_*\delta' \\
& {} & {} & &
\end{tikzcd}$$

$$g_*\varepsilon=\varepsilon' \qquad .$$

To conclude, $(1_A; u; u')$, $(g; g'; 1_F)$, and $(f; 1_D; v')$ are morphisms of $\mathbb{E}$-conflations.

The Axioms of Extriangulated Categories (ET4, ET4$^{\mathrm{op}}$)

ET4$^{\mathrm{op}}$ Given any two $\mathbb{E}$-conflations

$$A \xrightarrow{f'} D \xrightarrow{g'} E \overset{\delta'}{\dashrightarrow}, \quad C \xrightarrow{u'} E \xrightarrow{v'} F \overset{\varepsilon'}{\dashrightarrow},$$

there exists a commutative diagram of four $\mathbb{E}$-conflations

$$\begin{tikzcd}
A \arrow[r, dashed, "f"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, dashed, "g"] \arrow[d, dashed, "u"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[r, dashed, "\delta"] \arrow[d, "u'"] & {} & (u')^*\delta'{=}\delta \\
A \arrow[r, "f'"] & D \arrow[r, "g'"] \arrow[d, dashed, "v"] \arrow[dr, phantom, "\circlearrowright"] & E \arrow[r, dashed, "\delta'"] \arrow[d, "v'"] & {} & \\
& F \arrow[r, equal] \arrow[d, dashed, "\varepsilon"] & F \arrow[d, dashed, "\varepsilon'"] & & (v')^*\varepsilon{=}f_*\delta' \\
& {} \arrow[r, phantom, "g_*\varepsilon{=}\varepsilon'"] & {} & &
\end{tikzcd}.$$

To conclude, $(1_A; u; u')$, $(g; g'; 1_F)$, and $(f; 1_D; v')$ are morphisms of $\mathbb{E}$-conflations.

6-term long exact sequences

We revisit some fundamental results in [NP19].

Theorem 1.3 (Corollary 3.12. in [NP19])

For any $\mathbb{E}$-conflation $X \xrightarrow{f} Y \xrightarrow{g} Z \overset{\delta}{\dashrightarrow}$, one has the following two exact sequences of functors:

$$\mathcal{C}(-,X) \xrightarrow{\mathcal{C}(-,f)} \mathcal{C}(-,Y) \xrightarrow{\mathcal{C}(-,g)} \mathcal{C}(-,Z) \xrightarrow{\delta_\sharp} \mathbb{E}(-,X) \xrightarrow{f_*} \mathbb{E}(-,Y) \xrightarrow{g_*} \mathbb{E}(-,Z),$$

$$\mathcal{C}(Z,-) \xrightarrow{\mathcal{C}(g,-)} \mathcal{C}(Y,-) \xrightarrow{\mathcal{C}(f,-)} \mathcal{C}(X,-) \xrightarrow{\delta^\sharp} \mathbb{E}(Z,-) \xrightarrow{g^*} \mathbb{E}(Y,-) \xrightarrow{f^*} \mathbb{E}(X,-)$$

Corollary 1.4

We enumerate some immediate consequences.

1. *f is a weak kernel of g. Dually, g is a weak cokernel of f.*
2. *Let $(\alpha;\beta;\gamma)$ be a morphism of $\mathbb{E}$-conflations. If two of α,β,γ are isomorphisms, then so is the third one.*
3. *If an $\mathbb{E}$-inflation is epic, then it must be a retraction. Dually, if an $\mathbb{E}$-deflation is monic, then it must be a section.*

Base change

Theorem 1.5 (Proposition 3.15. in [NP19])

Let $A_1 \xrightarrow{f_1} B_1 \xrightarrow{g_1} C \overset{\delta_1}{\dashrightarrow}$ *and* $A_2 \xrightarrow{f_2} B_2 \xrightarrow{g_2} C \overset{\delta_2}{\dashrightarrow}$ *be two* $\mathbb{E}$*-conflations. Then there exists a commutative diagram*

$$\begin{tikzcd}
 & A_2 \arrow[r, equal] \arrow[d, dashed, "n"] & A_2 \arrow[d, "f_2"] & \\
A_1 \arrow[r, dashed, "m"] \arrow[d, equal] & E \arrow[r, dashed] \arrow[d, dashed] & B_2 \arrow[r, dashed, "(g_2)^*\delta_1"] \arrow[d, "g_2"] & {} \\
A_1 \arrow[r, "f_1"] & B_1 \arrow[r, "g_1"] \arrow[d, dashed, "(g_1)^*\delta_2"] & C \arrow[r, dashed, "\delta_1"] \arrow[d, dashed, "\delta_2"] & {} \\
 & {} & {} &
\end{tikzcd}.$$

such that $m_*\delta_1 + n_*\delta_2 = 0$.

Base change

Theorem 1.6 (Proposition 3.17. in [NP19])

Given three $\mathbb{E}$-conflations (in solid arrows), there is a way to complete the diagram

$$\begin{tikzcd}
 & A_2 \arrow[r, equal] \arrow[d, "j"] & A_2 \arrow[d, dashed, "f_2"] & \\
A_1 \arrow[r, "i"] \arrow[d, equal] & E \arrow[r, "p"] \arrow[d, "q"] & B_2 \arrow[r, dashed, "\eta"] \arrow[d, dashed, "g_2"] & {} \\
A_1 \arrow[r, "f_1"] & B_1 \arrow[r, "g_1"] \arrow[d, dashed, "\varepsilon"] & C \arrow[r, dashed, "\delta_1"] \arrow[d, dashed, "\delta_2"] & {} \\
 & {} & {} &
\end{tikzcd} .$$

such that $(1_{A_1}; q; g_2)$ and $(1_{A_2}; p; g_1)$ are morphisms of $\mathbb{E}$-conflations, and $i_*(\delta_1) = j_*(\delta_2)$.

Section, Retraction, and Retract

Definition 2.1 (Section, retraction)

Let $f : A \to B$ be a morphism in any category.

- f is called a **section** if there exists a morphism $g : B \to A$ such that $gf = 1_A$. (= **split monic** in some literature).
- f is called a **retraction** if there exists a morphism $h : B \to A$ such that $fh = 1_B$. (= **split epic** in some literature).

Definition 2.2 (Retract)

A morphism $f' : A' \to B'$ is called a **retract** of a morphism $f : A \to B$, if there exists a commutative diagram

$$\begin{tikzcd} A' \arrow[r, "i"'] \arrow[d, "f'"] \arrow[rr, bend left, equal, "1_{A'}"] & A \arrow[r, "p"'] \arrow[d, "f"] & A' \arrow[d, "f'"] \\ B' \arrow[r, "j"] \arrow[rr, bend right, equal, "1_{B'}"'] & B \arrow[r, "q"] & B' \end{tikzcd} . \tag{2}$$

Definition of Homotopic Squares

This concept was from triangulated category ([BN93]), generalised to *n*-angulated ([LZ16]) and extriangulated ([He19]) cases.

- There are several variations of the names in triangulated case, e.g., homotopy Cartesian squares, homotopic pullback and pushout squares, homotopy bicatersian squares, etc. We call them **homotopic squares** for simplicity.

Definition 2.3 (Definition 3.1. in [He19])

A commutative square is called a **homotopic square**, if its induced 3-term chain map completes to an $\mathbb{E}$-conflation.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\square"] & B \arrow[d, "\beta"] \\
X \arrow[r, "u" description, "-\!\!\circ\!\!-"'] & Y
\end{tikzcd}
\qquad
A \xrightarrow{\binom{f}{\alpha}} B \oplus X \xrightarrow{(\beta,-u)} Y \overset{\delta}{\dashrightarrow} \quad . \tag{3}
$$

The left square in (3) is **homotopic with extension** δ. The arrow $-\!\!\circ\!\!\rightarrow$ indicates a minus sign in the $\mathbb{E}$-conflation.

Examples of Homotopic Squares

Lemma 2.4

A homotopic square is both a weak pullback and a weak pushout.

Proof.

Note that a square is a (weak) pullback iff its induced 3-term chain map forms a (weak) kernel. Dually for (weak) pushout. $\square$

Example 2.5

An exact category $(\mathcal{C}, \mathcal{E})$ “is” extriangulated. Then

$$\text{Homotopic Squares} \;=\; \text{Pullback-Pushout Squares.}$$

Example 2.6

Any $\mathbb{E}$-conflation forms a homotopic square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\square"] & B \arrow[d, "g"] \\ 0 \arrow[r, "\circ" marking] & C \end{tikzcd} \iff \begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] & Y \arrow[r, dashed, "\delta"] & {} \end{tikzcd} .$$

A good completion of $(f;?;1)$

Theorem 2.7 (Proposition 1.20. in [LN19])

Let $(f;1_Z):\delta\to f_*\delta$ be a morphism of extension extension elements. For any realisations of δ and $f_*\delta$, one can find a morphism of $\mathbb{E}$-conflations $(f;g;1_Z)$

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, dashed, "g"] & Z \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z \arrow[r, dashed, "f_*\delta"] & {} \end{tikzcd}, \tag{4}$$

such that $X \xrightarrow{\binom{u}{f}} Y\oplus X' \xrightarrow{(g,-u')} Y' \overset{(v')^*\delta}{\dashrightarrow}$ is an $\mathbb{E}$-conflation.

We omit the dual version of this result.

Remark

The sign rule: when v' appears in the extension element, then u' has a minus sign in the $\mathbb{E}$-conflation.

A good completion of $(?; g; 1)$

Theorem 2.8 (Theorem 3.3. in [KLW24])

For any $v'g = v$, one can find f such that $(f; g; 1_Z)$ is a morphism of $\mathbb{E}$-conflations, and $X \xrightarrow{\binom{u}{f}} Y \oplus X' \xrightarrow{(g, -u')} Y' \overset{(v')^*\delta}{\dashrightarrow}$ is an $\mathbb{E}$-conflation.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, dashed, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z \arrow[r, dashed, "\varepsilon"] & {} \end{tikzcd},$$

We omit the dual version of this result.

A good completion of ET4 axiom

Proposition 2.9

Suppose there is a diagram of $\mathbb{E}$-conflations with solid arrows

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, "u"'] \arrow[dr, phantom, "\square"] & D \arrow[r, dashed, "\delta"] \arrow[d, dashed, "w"] & {} \\
A \arrow[r, "uf"'] & C \arrow[r, "h"'] \arrow[d, "v"'] & F \arrow[r, dashed, "\theta"] \arrow[d, dashed, "q"] & {} \\
& E \arrow[r, equal] \arrow[d, dashed, "\varepsilon"] & E \arrow[d, dashed, "\eta"] & \\
& {} & {} &
\end{tikzcd} \quad . \tag{5}$$

Then there is a way to complete $D \xrightarrow{w} F \xrightarrow{q} E \overset{\eta}{\dashrightarrow}$ so that

1. *it satisfies the conditions of ET4 axiom, and*
2. *$\square$ is homotopic with extension $f_*\theta$.*

Moreover, w can be any completion which satisfies Theorem 2.8.

A good completion of ET4 axiom

Proof.

We take any w as in Theorem 2.8. There is an $\mathbb{E}$-conflation

$$B \xrightarrow{\binom{-g}{u}} D \oplus C \xrightarrow{(w,h)} F \overset{f_*\theta}{\dashrightarrow} .$$

By "base change" (1.6), there is way to complete

$$\begin{tikzcd}
 & D \arrow[r, equal] \arrow[d, "\binom{1}{0}"] & D \arrow[d, dashed, "w"] & \\
B \arrow[r, "\binom{-g}{u}"] \arrow[d, equal] & D \oplus C \arrow[r, "(w{,}h)"] \arrow[d, "(0{,}1)"] & F \arrow[r, dashed, "f_*\theta"] \arrow[d, dashed, "q"] & {} \\
B \arrow[r, "u"] & C \arrow[r, "v"] \arrow[d, dashed, "0"] & E \arrow[r, dashed, "\varepsilon"] \arrow[d, dashed, "\eta"] & {} \\
 & {} & {} &
\end{tikzcd} .$$

We show (5) satisfies ET4. We see $qh = v$, $q^*\varepsilon = f_*\theta$, and $\eta = (1,0)_*\binom{1}{0}_*\eta = -(1,0)_*\binom{-g}{u}_*\varepsilon = g_*\varepsilon$. $\square$

Good completions

Theorem 2.10

There are good completions for ET4($^{\mathrm{op}}$) and "(co-)base change".

$$
\begin{tikzcd}
 & \cdot \arrow[r, equal] \arrow[d, dashed, red] & \cdot \arrow[d] \\
\cdot \arrow[r] \arrow[d, equal] & \cdot \arrow[r] \arrow[d, dashed, blue] \arrow[dr, phantom, "\square"] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}
\quad
\begin{tikzcd}
\cdot \arrow[r, equal] \arrow[d, dashed, red] & \cdot \arrow[d] & \\
\cdot \arrow[r] \arrow[d, dashed, blue] \arrow[dr, phantom, "\square"] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[d, equal] \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}
\quad
\begin{tikzcd}
\cdot \arrow[r] \arrow[d, dashed, blue] \arrow[dr, phantom, "\square"] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[d, equal] \\
\cdot \arrow[r] \arrow[d, dashed, red] & \cdot \arrow[r] \arrow[d] & \cdot \\
\cdot \arrow[r, equal] & \cdot &
\end{tikzcd}
\quad
\begin{tikzcd}
\cdot \arrow[r] \arrow[d, equal] & \cdot \arrow[r] \arrow[d, dashed, blue] \arrow[dr, phantom, "\square"] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot \arrow[r] \arrow[d, dashed, red] & \cdot \arrow[d] \\
 & \cdot \arrow[r, equal] & \cdot
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \cdot \arrow[r, equal] \arrow[d] & \cdot \arrow[d, dashed, red] \\
\cdot \arrow[r] \arrow[d, equal] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \cdot \arrow[d, dashed, blue] \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}
\quad
\begin{tikzcd}
\cdot \arrow[r, equal] \arrow[d] & \cdot \arrow[d, dashed, red] & \\
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \cdot \arrow[r] \arrow[d, dashed, blue] & \cdot \arrow[d, equal] \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}
\quad
\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \cdot \arrow[r] \arrow[d, dashed, blue] & \cdot \arrow[d, equal] \\
\cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d, dashed, red] & \cdot \\
\cdot \arrow[r, equal] & \cdot &
\end{tikzcd}
\quad
\begin{tikzcd}
\cdot \arrow[r] \arrow[d, equal] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \cdot \arrow[d, dashed, blue] \\
\cdot \arrow[r] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[d, dashed, red] \\
 & \cdot \arrow[r, equal] & \cdot
\end{tikzcd}
$$

Moreover, the blue dashed arrows can be any completion satisfying Theorems 2.7 or 2.8.

For exact categories, any completion of ET4($^{\mathrm{op}}$) and "(co-)base change" are good, for $\square$ are pushout-pullback squares ([Büh10]).

A comparison with ker-coker sequence

Theorem 2.11 (6-term exact sequence)
Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a composable pair of morphisms in an Abelian category. Then there is a 6-term exact sequence

$$0 \to \ker f \to \ker(gf) \to \ker g \xrightarrow{\mathfrak{C}} \operatorname{cok} f \to \operatorname{cok}(gf) \to \operatorname{cok} g \to 0.$$

The connecting morphism $\mathfrak{C}$ is the composition of

$$\ker g \twoheadrightarrow \frac{\ker g}{\ker g \cap \operatorname{im} f} \simeq \frac{\ker g + \operatorname{im} f}{\operatorname{im} f} \rightarrowtail \frac{Y}{\operatorname{im} g} \simeq \operatorname{cok} f\,.$$

In diagram:

A comparison with ker-coker sequence

$$
\begin{tikzcd}
 & \ker g \arrow[rr, bend left, "\mathfrak{c}"] \arrow[dr, tail, "k"] & & \operatorname{cok} f \arrow[dr] & \\
\ker gf \arrow[ur] \arrow[dr, tail, "j"] & \square & Y \arrow[ur, two heads, "p"] \arrow[dr, "g"] & \square & \operatorname{cok} gf \arrow[dd, two heads, bend left] \\
 & X \arrow[ur, "f"] \arrow[rr, bend right, "gf"] & & Z \arrow[ur, two heads, "q"] \arrow[dr, two heads, "r"] & \\
\ker f \arrow[uu, tail, bend left] \arrow[ur, tail, "i"] & & & & \operatorname{cok} g
\end{tikzcd}
.
$$

Moreover, $\square$ are pullback-pushout squares.

Proof.

Well-known (as a corollary of snake lemma). $\square$

A comparison with ker-coker sequence

Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a composable pair of morphisms in an extriangulated category.

Corollary 2.12

When f and g are both $\mathbb{E}$-inflations, then so is gf. There is a way to complete an $\mathbb{E}$ $\mathbb{E}$-conflation $C_f \to C_{gf} \to C_g$ such that $\square$ is homotopic:

$$
\begin{tikzcd}
 & & C_g \arrow[dr, dashed] & \\
 & Y \arrow[ur, "p"] \arrow[dr, "g"] \arrow[rr, phantom, "\square"] & & C_{gf} \arrow[dd, dashed, bend left] \\
X \arrow[ur, "f"] \arrow[rr, bend right, "gf"'] & & Z \arrow[ur, "q"] \arrow[dr, "r"] & \\
 & & & C_g
\end{tikzcd}
.
$$

A comparison with ker-coker sequence

Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a composable pair of morphisms in an extriangulated category.

Corollary 2.13

If gf is an $\mathbb{E}$-inflation and g is an $\mathbb{E}$-deflation, then f is an $\mathbb{E}$-inflation (not an immediate result). There is a way to complete an $\mathbb{E}$-conflation $K_g \to C_f \to C_{gf}$ such that $\square$ is homotopic:

$$
\begin{tikzcd}
K_g \arrow[dr, "k"] \arrow[rr, dashed, bend left] & & C_g \arrow[dr, dashed] & \\
& Y \arrow[ur, "p"] \arrow[dr, "g"] & \square & C_{gf} \\
X \arrow[ur, "f"] \arrow[rr, bend right, "gf"] & & Z \arrow[ur, "q"] &
\end{tikzcd} .
$$

Composite $\mathbb{E}$-Inflations/$\mathbb{E}$-Deflations

We fix a gap in the previous page.

Proposition 2.14

If gf is an $\mathbb{E}$-inflation and g is an $\mathbb{E}$-deflation, then f is an $\mathbb{E}$-inflation.

Proof.

A good completion of ET4 yields

$$
\begin{tikzcd}
 & \cdot \arrow[r, equal] \arrow[d, dashed] & \cdot \arrow[d] & \\
\cdot \arrow[r, dashed, "s"] \arrow[rr, bend left, "f"] \arrow[dr, "1"'] & \cdot \arrow[r, dashed, "h"] \arrow[d, dashed, "p"] \arrow[dr, phantom, "\square"] & \cdot \arrow[r, dashed] \arrow[d, "g"] & \cdot \arrow[d, equal] \\
 & \cdot \arrow[r, "gf"] & \cdot \arrow[r] & \cdot
\end{tikzcd}
.
$$

Since $\square$ is a weak pullback, we see p is an $\mathbb{E}$-deflation and a retraction. In this case, p has a kernel. Now $f = hs$ is a composition of two $\mathbb{E}$-inflations. Thus, f is also an $\mathbb{E}$-inflation. $\square$

Composite $\mathbb{E}$-Inflations/$\mathbb{E}$-Deflations

We summarise some composition properties of $\mathbb{E}$-inflations and $\mathbb{E}$-deflations.

1. $\mathbb{E}$-inflations are closed under compositions.
2. If gf is an $\mathbb{E}$-inflation, and g is an $\mathbb{E}$-deflation, then f is agian an $\mathbb{E}$-inflation.
3. If f is an $\mathbb{E}$-inflation, then for any h with the same domain as f, $\binom{g}{h}$ is also an $\mathbb{E}$-inflation.
4. If f_1 is an $\mathbb{E}$-deflation, and there is an $\mathbb{E}$-conflation

$$X \xrightarrow{\binom{f_1}{f_2}} Y \oplus W \xrightarrow{(g_1, g_2)} Z \dashrightarrow .$$

   Then g_2 is also an $\mathbb{E}$-deflation.

Composite Homotopic Squares

Theorem 2.15 ([HXZ23])

The composition of two homotopic squares is also homotopic.

Proof.

We take the following homotopic squares:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\boxed{\kappa}"] & B \arrow[r, "g"] \arrow[d, "\beta"] \arrow[dr, phantom, "\boxed{\varepsilon}"] & C \arrow[d, "\gamma"] \\ D \arrow[r, "u"] & E \arrow[r, "v"] & F \end{tikzcd} .$$

By ET2, there are isomorphic $\mathbb{E}$-conflations

$$\begin{tikzcd}[ampersand replacement=\&] A \arrow[r, "\begin{pmatrix}-f\\ \alpha\\ 0\end{pmatrix}"] \arrow[d, equal] \& B \oplus D \oplus C \arrow[r, "\left(\begin{smallmatrix}\beta & u & 0\\ 0 & 0 & 1\end{smallmatrix}\right)"] \& E \oplus C \arrow[r, dashed, "\kappa\oplus 0"] \arrow[d, equal] \& {} \\ A \arrow[r, "\begin{pmatrix}-f\\ \alpha\\ gf\end{pmatrix}"] \& B \oplus D \oplus C \arrow[u, "\simeq"', "\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ g & 0 & 1\end{pmatrix}"] \arrow[r, "\left(\begin{smallmatrix}\beta & u & 0\\ g & 0 & 1\end{smallmatrix}\right)"] \& E \oplus C \arrow[r, dashed, "\kappa\oplus 0"] \& {} \end{tikzcd} .$$

Composite Homotopic Squares

By "base-change", there is a unique dashed $\mathbb{E}$-conflation

$$\begin{tikzcd}
 & B \arrow[r, equal] \arrow[d, "\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)"] & B \arrow[d, "\left(\begin{smallmatrix}\beta\\g\end{smallmatrix}\right)"] & \\
A \arrow[r, "\left(\begin{smallmatrix}-f\\\alpha\\gf\end{smallmatrix}\right)"] \arrow[d, equal] & B \oplus D \oplus C \arrow[r, "\left(\begin{smallmatrix}\beta & u & 0\\ g & 0 & 1\end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{smallmatrix}\right)"] & E \oplus C \arrow[r, dashed, "\kappa\oplus 0"] \arrow[d, "{(v,-\gamma)}"] & {} \\
A \arrow[r, dashed, "\left(\begin{smallmatrix}\alpha\\gf\end{smallmatrix}\right)"] & D \oplus C \arrow[r, dashed, "{(vu,-\gamma)}"] \arrow[d, dashed, "0"] & F \arrow[r, dashed, "\kappa\cup\varepsilon"] \arrow[d, dashed, "\varepsilon"] & {} \\
 & {} & {} &
\end{tikzcd} . \tag{6}$$

This shows $A \xrightarrow{\left(\begin{smallmatrix}\alpha\\-gf\end{smallmatrix}\right)} D \oplus X \xrightarrow{(vu,\gamma)} F \overset{\kappa\cup\varepsilon}{\dashrightarrow}$ is an $\mathbb{E}$-conflation. From (6), we also obtain that

$$v^*(\kappa \cup \varepsilon) = \kappa, \quad f_*(\kappa \cup \varepsilon) = \varepsilon.$$

Composite Homotopic Squares

Proposition 2.16

Let $\boxed{\kappa}$ be homotopic. If $\binom{\alpha}{gf}$ is an $\mathbb{E}$-inflation, then so is $\binom{g}{\beta}$. Consequently, the diagram completes to a composition of homotopic squares.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\boxed{\kappa}"] & B \arrow[r, "g"] \arrow[d, "\beta"] \arrow[dr, phantom, "\boxed{\varepsilon}"] & C \arrow[d, dashed, "\gamma"] \\
D \arrow[r, "u"] & E \arrow[r, dashed, "v"] & F
\end{tikzcd}
. \tag{7}
$$

Proof.

$$
\begin{tikzcd}[column sep=huge]
A \arrow[r, "\begin{pmatrix} -f \\ \alpha \\ 0 \end{pmatrix}"] \arrow[d, equal] & B \oplus D \oplus C \arrow[r, "\begin{pmatrix} \beta & u & 0 \\ 0 & 0 & 1 \end{pmatrix}"] & E \oplus C \arrow[r, dashed, "\kappa\oplus 0"] \arrow[d, equal] & {} \\
A \arrow[r, "\begin{pmatrix} -f \\ \alpha \\ gf \end{pmatrix}"] & B \oplus D \oplus C \arrow[u, "\simeq"', "\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g & 0 & 1 \end{pmatrix}"] \arrow[r, "\begin{pmatrix} \beta & u & 0 \\ g & 0 & 1 \end{pmatrix}"] & E \oplus C \arrow[r, dashed, "\kappa\oplus 0"] & {}
\end{tikzcd}
.
$$

Composite Homotopic Squares

Let $A \xrightarrow{\binom{\alpha}{gf}} D \oplus C \xrightarrow{(p,-\gamma)} F \overset{\delta}{\dashrightarrow}$ be any $\mathbb{E}$ $\mathbb{E}$-conflation induced by $\binom{\alpha}{gf}$. By "base-change", there is a some dashed morphism making the following diagram commute:

$$
\begin{tikzcd}
 & B \arrow[r, equal] \arrow[d, "\begin{pmatrix}1\\0\\0\end{pmatrix}"] & B \arrow[d, dashed, "\binom{\beta}{g}"] & \\
A \arrow[r, "\begin{pmatrix}-f\\ \alpha\\ gf\end{pmatrix}"] \arrow[d, equal] & B \oplus D \oplus C \arrow[r, "\begin{pmatrix}\beta & u & 0\\ g & 0 & 1\end{pmatrix}"] \arrow[d, "\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}"] & E \oplus C \arrow[r, dashed, "\kappa\oplus 0"] \arrow[d, dashed, "{(v,-\gamma)}"] & {} \\
A \arrow[r, "\binom{\alpha}{gf}"] & D \oplus C \arrow[r, "{(p,-\gamma)}"] \arrow[d, dashed, "0"] & F \arrow[r, dashed, "\delta"] \arrow[d, dashed, "\varepsilon"] & {} \\
 & {} & {} &
\end{tikzcd}
.
$$

Corollary 2.17

The completion of 7 exists if one of α, f, β, u, g is an $\mathbb{E}$-inflation.

Parallel Edges of Homotopic Squares

Theorem 2.18

We fix any homotopy square:

$$\begin{tikzcd} A_1 \arrow[r, "u"] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & B_1 \arrow[d, "g"] \\ A_2 \arrow[r, "v"] & B_2 \end{tikzcd} .$$

We have the following statements (and their duals).

1. *u is an $\mathbb{E}$-inflation if and only if v is an $\mathbb{E}$-inflation.*
2. *If v is a retraction, then so is u.*
3. *If v is a retract of some $\mathbb{E}$-inflation, then so is u.*

Proof.

The proof of the first two statements are straightforward by diagram chasing. It is worth mentioning that

$$\textit{retracts of } \mathbb{E}\textit{-inflations} \;=\; \textit{retractions} \circ \textit{inflations}.$$

Split Homotopic Squares

Theorem 2.19

Suppose the left commutative diagram is a homotopic square. If one the following conditions holds:

1. *u is an $\mathbb{E}$-inflation,*
2. *v is an $\mathbb{E}$-deflation,*
3. *γ is an $\mathbb{E}$-deflation.*

Then there is a way to write h as gf such that the right diagram is a composition of homotopic squares.

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[d, "\alpha"] & & C \arrow[d, "\gamma"] \\ D \arrow[r, "u", "\circ"'] \arrow[urr, phantom, "\boxed{\mathcal{T}}"] & E \arrow[r, "v", "\circ"'] & F \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, dashed, "f"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\boxed{\varepsilon}"] & B \arrow[r, dashed, "g"] \arrow[d, dashed, "\beta"] \arrow[dr, phantom, "\boxed{\kappa}"] & C \arrow[d, "\gamma"] \\ D \arrow[r, "u", "\circ"'] & E \arrow[r, "v", "\circ"'] & F \end{tikzcd} .$$

Morphisms are both $\mathbb{E}$-inflations and $\mathbb{E}$-deflations

Definition 2.20 (S, $\mathcal{L}$, and $\mathcal{R}$)

We set $S := (\mathbb{E}\text{-Inflations}) \cap (\mathbb{E}\text{-Deflations})$, and

$$\mathcal{L} = \{X \mid (X \to 0) \in S\}, \quad \mathcal{R} = \{X \mid (0 \to X) \in S\}.$$

S is closed under compositions and contains all isomorphisms.

Lemma 2.21

S, $\mathcal{L}$, and $\mathcal{R}$ are mutually determined.

Proof.

The key observation: any $f \in S$ gives $K \to 0 \to C \dashrightarrow$ by

$$\begin{tikzcd}
K \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\
0 \arrow[r] & Y \arrow[r, "c"] & C
\end{tikzcd}.$$

$\square$

Morphisms are both $\mathbb{E}$-inflations and $\mathbb{E}$-deflations

Lemma 2.22

$X \in \mathcal{L}$ iff any $X \xrightarrow{\varphi} T$ is an $\mathbb{E}$-inflation for all $T \in \mathcal{C}$.

Proof.

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\square"] & 0 \arrow[r] \arrow[d, dashed] & R \arrow[r, dashed] \arrow[d, equal] & {} \\
T \arrow[r, dashed] & E \arrow[r, dashed] & R \arrow[r, dashed] & {}
\end{tikzcd}.$$

$\square$

Proposition 2.23

Once we fix for $X \in \mathcal{L}$ an $\mathbb{E}$-conflation $X \to 0 \to FX \overset{\delta_X}{\dashrightarrow}$, then

1. *$F : \mathcal{L} \to \mathcal{R}$ is an equivalence of full subcategories.*
2. *Any $X \in \mathcal{L}$ yields a natural isomorphism in $\mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Ab})$:*

$$\theta^{(X)} : \mathcal{C}(-, FX) \simeq \mathbb{E}(-, X), \quad \varphi \mapsto \varphi^* \delta_X.$$

Moreover, $\theta^{(X)}$ is functorial in $X \in \mathcal{L}$.

Proposition 2.24

If $\mathcal{L} = \mathrm{Ob}(\mathcal{C})$, then the category admits a right triangulated structure with suspension functor F:

$$\underbrace{(X \xrightarrow{f} Y \xrightarrow{g} Z \overset{\delta}{\dashrightarrow})}_{\mathbb{E}\text{-conflation}} \iff \underbrace{(X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\theta^{-1}(\delta)} FX)}_{\text{Distinguished right triangle}}.$$

Proof.

We briefly explain the axioms of morphism-embedding and clockwise rotation by homotopic squares.

1. *(Morphism embedding) Any $X \xrightarrow{f} Y$ embeds to*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & 0 \arrow[r] \arrow[d] & FX \arrow[r, dashed, "\delta_X"] \arrow[d, equal] & {} \\ Y \arrow[r, "g"] & E \arrow[r, "h"] & FX \arrow[r, dashed, "f_*\delta_X"] & {} \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "-g"] & E \arrow[r, dashed, "h^*\delta_X"] & {} \\ X \arrow[r, "f"] & Y \arrow[r, "-g"] & E \arrow[r, "h"] & FX \end{tikzcd}.$$

Since θ is a natural iso, $\theta^{-1}(h^*\delta_X) = \theta^{-1}(\delta_X) \circ h = h$.

Proof.

2. (Clockwise rotation) Any distinguished right triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \overset{\theta^{-1}(\varepsilon)}{\dashrightarrow} FX$ admits a clockwise rotation

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d] \arrow[dr, phantom, "\square"] & Z \arrow[r, dashed, "\varepsilon"] \arrow[d, dashed, "\gamma"] & {} \\ X \arrow[r] & 0 \arrow[r] & FX \arrow[r, dashed, "\delta_X"] & {} \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "-v"] & Z \arrow[r, "\gamma"] & FX \arrow[r, dashed, "u_*\delta_X"] & {} \\ Y \arrow[r, "-v"] & Z \arrow[r, "\gamma"] & FX \arrow[r, "Ff"] & FY \end{tikzcd}$$

Here γ is uniquely determined. Since θ is a natural isomorphism, we have $\theta^{-1}(u_*\delta_X) = (Fu) \circ \theta^{-1}(\delta_X) = Fu$.

$\square$

Morphisms are both E-inflations and E-deflations

Remark

Clockwise rotation in diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "u"'] \arrow[dr, phantom, "\boxed{w}"] & 0 \arrow[d] & \\ Y \arrow[r, "v"] & Z \arrow[r, "w"] & \Sigma X \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r] \arrow[d, "v"'] \arrow[dr, phantom, "\boxed{\Sigma u}"] & 0 \arrow[d] & \\ Z \arrow[r, "w"] & \Sigma X \arrow[r, "\Sigma u"] & \Sigma Y \end{tikzcd} \quad . \tag{8}$$

Example 2.25

Not all right triangulated categories arise in this way. Let $\mathcal{A}$ be Abelian with distinguished right triangles $\{X \xrightarrow{f} Y \xrightarrow{\pi} \operatorname{cok} f \to 0\}$. The suspension 0 is not an isomorphism.

Happel's Theorems

Theorem 2.26

The following shows relations between extriangulated categories and triangulated categories.

1. *An extriangulated category is triangulated iff* $\mathcal{L} = \mathcal{R} = \mathrm{Ob}(\mathcal{C})$*.*
2. *An extriangulated category is triangulated iff* $S = \mathrm{Mor}(\mathcal{C})$*.*
3. *The ideal quotient* $\mathcal{C}/[\mathcal{B}]$ *is triangulated iff* $\mathcal{C}$ *is Frobenius with* $\mathcal{B}$ *the subcategory of projective-injective objects.*
4. *Let* $(\mathcal{D}, \mathbb{F}, \mathfrak{t}) \subseteq (\mathcal{C}, \mathbb{E}, \mathfrak{s})$ *be an extriangulated subcategory with* $\mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(\mathcal{D})$*. If* $(\mathcal{D}, \mathbb{F}, \mathfrak{t})$ *is triangulated, then so is* $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$*.*

Remark (A historical remark)

The third statement in 2.26 is shown for extension-closed subcategories of triangulated categories ([IY08], [Nak10]). This generalisation motivates the definition of extriangulated categories ([hp15]).

WIC v.s. Weakly Idempotent Complete

[NP19] introduced a WIC condition for extriangulated categories, serving as a strong version of being weakly idempotent completeness.

1. (weakly idempotent complete) every section has a cokernel;
2. (WIC) if gf is an $\mathbb{E}$-inflation, then so is f.

The equivalency is shown in [Kla23]. We propose a simple proof and another equivalent condition for establising 3×3-lemmas.

Theorem 3.1

An additive category is weakly idempotent complete if and only if the following condition holds: any section has a cokernel; any retraction has a kernel.

Remark

Triangulated categories are automatically WIC. For exact categories, see [Büh10].

WIC v.s. Weakly Idempotent Complete

Proposition 3.2

The following are equivalent for an extriangulated category.

1. *(weakly idempotent complete) every section has a cokernel;*
2. *(WIC) if gf is an $\mathbb{E}$-inflation, then so is f.*

Proof.

$(1 \to 2)$. $\binom{f}{0} = \begin{pmatrix} 1 & 0 \\ -g & 1 \end{pmatrix} \binom{f}{gf}$ *is an $\mathbb{E}$-inflation. We consider*

$$\begin{tikzcd}
X \arrow[r, "\binom{f}{0}"] \arrow[dr, bend right, "0"'] & Y \oplus W \arrow[r, "{(s,t)}"] \arrow[d, "\begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}"] & Z \arrow[r, dashed, "\delta"] \arrow[dl, dashed, bend left, "\binom{a}{b}"] & {} \\
& Z \oplus W & &
\end{tikzcd} .$$

Since $bt = 1_W$, (s,t) takes the form $\begin{pmatrix} s_1 & 0 \\ s_2 & 1 \end{pmatrix} : Y \oplus W \to Q \oplus W$ where $Q \simeq \operatorname{cok} t$. By "base-change", we obtain $\square$

WIC v.s. Weakly Idempotent Complete

$$
\begin{tikzcd}
 & W \arrow[r, equal] \arrow[d, "\binom{0}{1}"] & W \arrow[d, "\binom{0}{1}"] & \\
X \arrow[r, "\binom{f}{0}"] \arrow[d, equal] & Y \oplus W \arrow[r, "\begin{pmatrix} s_1 & 0 \\ s_2 & 1 \end{pmatrix}"] \arrow[d, "{(1,0)}"] & Q \oplus W \arrow[r, dashed, "\delta"] \arrow[d, "{(1,0)}"] & {} \\
X \arrow[r, dashed, "f"] & Y \arrow[r, dashed, "s_1"] \arrow[d, dashed, "0"] & Q \arrow[r, dashed, "\varepsilon"] \arrow[d, dashed, "0"] & {} \\
 & {} & {} &
\end{tikzcd}
.
$$

$(2 \to 1)$. Let p be section with right inverse i. Since $pi = 1$ is an $\mathbb{E}$-inflation, i is an $\mathbb{E}$-inflation by WIC. Hence i has a cokernel. We obtain the structure map of the direct sums.

Equivalent Conditions for WIC

Theorem 3.3

The following are equivalent for an extriangulated category.

1. *(Definition). Any section has a cokernel.*
2. *If* $\binom{i}{0}$ *is an* $\mathbb{E}$*-inflation, then so is* i*.*
3. *If* $\begin{pmatrix} i & 0 \\ 0 & j \end{pmatrix}$ *is an* $\mathbb{E}$*-inflation, then so are* i *and* j*.*
4. *(WIC). If* fi *is an* $\mathbb{E}$*-inflation, then so is* i*.*
5. $\mathbb{E}$*-inflations are closed under retracts.*
6. *Any* $\circlearrowright$ *completes to a* 3×3*-diagram*

$$\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, dashed, "i_1"] & A_2 \arrow[r, "g_A"] \arrow[d, "i_2"] \arrow[dr, phantom, "\circlearrowright"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, "i_3"] & {} \\
B_1 \arrow[r, "f_B"] \arrow[d, dashed, "p_1"] & B_2 \arrow[r, "g_B"] \arrow[d, "p_2"] & B_3 \arrow[r, dashed, "\delta_B"] \arrow[d, "p_3"] & {} \\
C_1 \arrow[r, dashed, "f_C"] \arrow[d, dashed, "\varepsilon_1"] & C_2 \arrow[r, dashed, "g_C"] \arrow[d, dashed, "\varepsilon_2"] & C_3 \arrow[r, dashed, "\delta_C"] \arrow[d, dashed, "\varepsilon_3"] & {} \\
{} & {} & {} &
\end{tikzcd}.$$

Proof.

We show $(6 \to 1)$ via (not $1 \to$ not 6). Let $X \xrightarrow{i} Y$ be a section without cokernel, with q as its right inverse. Consider

$$\begin{tikzcd}
Y \arrow[r, "\binom{q}{1-iq}"] \arrow[d, equal] & X \oplus Y \arrow[r, "{(0,q)}"] & X \arrow[d, equal] \\
Y \arrow[r, "\binom{0}{1}"] & X \oplus Y \arrow[r, "{(1,0)}"] \arrow[u, "\cong"', "\begin{pmatrix} 0 & q \\ i & 1-iq \end{pmatrix}"] & X
\end{tikzcd} \quad \begin{pmatrix} 0 & q \\ i & 1-iq \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The following fails to be a 3×3-diagram

$$\begin{tikzcd}
X \arrow[r, "1"] \arrow[d, dashed] & X \arrow[r] \arrow[d, "\binom{1}{0}"] & 0 \arrow[d] \\
Y \arrow[r, "\binom{q}{1-iq}"] \arrow[d, dashed] & X \oplus Y \arrow[r, "{(0,q)}"] \arrow[d, "{(0,1)}"] & X \arrow[d, "1"] \\
Z \arrow[r, dashed] & Y \arrow[r, dashed, "q"] & X
\end{tikzcd}.$$

q is never an $\mathbb{E}$-deflation, since it has no kernel. $\square$

Homotopic Morphisms of $\mathbb{E}$-conflations

Definition 3.4 (Homotopic morphism of $\mathbb{E}$-triangles)

Say $(\alpha;\beta;\gamma)$ is a homotopic morphism of $\mathbb{E}$-triangles, if there is a decomposition

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\boxed{b^*\delta}"] & Y \arrow[r, "g"] \arrow[d, dashed, "s"] \arrow[dd, bend left, "\beta"] & Z \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\
A \arrow[r, "a"] \arrow[d, equal] & M \arrow[r, "b"] \arrow[d, dashed, "t"] \arrow[dr, phantom, "\boxed{a_*\gamma}"] & Z \arrow[r, dashed, "\alpha_*\delta=\gamma^*\varepsilon"] \arrow[d, "\gamma"] & {} \\
A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, dashed, "\varepsilon"] & {}
\end{tikzcd}
.
$$

Completing Homotopic Morphisms

Theorem 3.5

Let $(\alpha;\beta;\gamma)$ be a morphism of $\mathbb{E}$-extensions. Then there exists modifications $(\alpha;\beta;\overline{\gamma})$, $(\alpha;\overline{\beta};\gamma)$, and $(\overline{\alpha};\beta;\gamma)$ which are homotopic morphisms of $\mathbb{E}$-triangles.

Proof.

We obtain $\overline{\beta}$ and $\overline{\gamma}$ by

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\square"] & Y \arrow[r, "g"] \arrow[d, dashed, "s"] \arrow[dd, dashed, bend left, "\overline{\beta}"] & Z \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\
A \arrow[r, dashed, "a"] \arrow[d, equal] & M \arrow[r, dashed, "b"] \arrow[d, dashed, "t"] \arrow[dr, phantom, "\square"] & Z \arrow[r, dashed] \arrow[d, "\gamma"] & {} \\
A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, dashed, "\varepsilon"] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r] \arrow[d, "\alpha"] \arrow[dr, phantom, "\square"] & Y \arrow[r] \arrow[d, "s"] \arrow[dd, bend left, "\beta"] & Z \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\
A \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, dashed, "t"] \arrow[dr, phantom, "\square"] & Z \arrow[r, dashed] \arrow[d, dashed, "\gamma"] & {} \\
A \arrow[r] & B \arrow[r] & C \arrow[r, dashed, "\varepsilon"] & {}
\end{tikzcd}
.
$$

$\square$

Completing Homotopic Morphisms

Proposition 3.6

Any morphism of $\mathbb{E}$-conflations is a composition of two homotopic morphisms of $\mathbb{E}$-conflations.

Proof.

Note that homotopic morphisms are closed under isomorphisms.

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\binom{1}{0}"] & Y \arrow[r, "g"] \arrow[d, "\binom{1}{0}"] & Z \arrow[r, dashed, "\delta"] \arrow[d, "\binom{1}{0}"] & {} \\
X \oplus A \arrow[r, "\begin{pmatrix} f & 0 \\ 0 & u \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}", "\simeq"'] & Y \oplus B \arrow[r, "\begin{pmatrix} g & 0 \\ 0 & v \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}", "\simeq"'] & Z \oplus C \arrow[r, dashed, "\delta\oplus\varepsilon"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}", "\simeq"'] & {} \\
X \oplus A \arrow[r, "\begin{pmatrix} f & 0 \\ 0 & u \end{pmatrix}"] \arrow[d, "{(0,1)}"] & Y \oplus B \arrow[r, "\begin{pmatrix} g & 0 \\ 0 & v \end{pmatrix}"] \arrow[d, "{(0,1)}"] & Z \oplus C \arrow[r, dashed, "\delta\oplus\varepsilon"] \arrow[d, "{(0,1)}"] & {} \\
A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, dashed, "\varepsilon"] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\binom{1}{0}"] \arrow[dr, phantom, "\square"] & Y \arrow[r, "g"] \arrow[d, "\binom{1}{0}"] & Z \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\
X \oplus A \arrow[r, "\begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}"] \arrow[d, equal] & Y \oplus A \arrow[r, "{(g,0)}"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}"] \arrow[dr, phantom, "\square"] & Z \arrow[r, dashed] \arrow[d, "\binom{1}{0}"] & {} \\
X \oplus A \arrow[r, "\begin{pmatrix} f & 0 \\ 0 & u \end{pmatrix}"] & Y \oplus B \arrow[r, "\begin{pmatrix} g & 0 \\ 0 & v \end{pmatrix}"] & Z \oplus C \arrow[r, dashed, "\delta\oplus\varepsilon"] & {}
\end{tikzcd}
.
$$

We claim $\left(\binom{1}{0};\binom{1}{0};\binom{1}{0}\right)$ is a morphism of $\mathbb{E}$-conflations. In fact, $\square$ *are direct sums of easy $\mathbb{E}$-conflations. We omit the details.* $\square$

Completing Homotopic Morphisms

Theorem 3.7 (Completing Homotopic Morphisms)

Let $(\alpha; \beta; \gamma)$ be homotopic morphisms of $\mathbb{E}$-triangles. If α is an $\mathbb{E}$-inflation, β is an $\mathbb{E}$-deflation, then γ is an $\mathbb{E}$-inflation.

The full table (black $\Longrightarrow$ red):

α	β	γ	α	β	γ
Infl	*Infl*	$\forall$	*Defl*	*Infl*	*Infl*^{Rtc}
Infl	*Infl*	*Infl*	*Defl*	*??*	*Infl*
Infl^{Rtc}	*Infl*	*Infl*	*Defl*^{Rtc}	*Defl*	*Infl*
Infl	*Defl*	*Defl*	*Defl*	*Defl*	*Defl*^{Rtc}
Infl	$\forall$	*Defl*	*Defl*	*Defl*	*Defl*
Infl	*Infl*	*Defl*	$\forall$	*Defl*	*Defl*

.

Completing Homotopic Morphisms

We show a detailed proof of one case.

Proposition 3.8 (3×3 lemma in the middle)

Suppose we have $\mathbb{E}$-conflations realising δ_A, δ_B, ε_1 and ε_3 with $(i_3)^*\delta_B = (i_1)_*\delta_A$ in the following diagram:

$$
\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, "i_1"] & A_2 \arrow[r, "g_A"] \arrow[d, dashed, "i_2"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, "i_3"] & {} \\
B_1 \arrow[r, "f_B"] \arrow[d, "p_1"] & B_2 \arrow[r, "g_B"] \arrow[d, dashed, "p_2"] & B_3 \arrow[r, dashed, "\delta_B"] \arrow[d, "p_3"] & {} \\
C_1 \arrow[r, dashed, "f_C"] \arrow[d, dashed, "\varepsilon_1"] & C_2 \arrow[r, dashed, "g_C"] \arrow[d, dashed, "\varepsilon_2"] & C_3 \arrow[r, dashed, "\delta_C"] \arrow[d, dashed, "\varepsilon_3"] & {} \\
{} & {} & {} &
\end{tikzcd}
\tag{9}
$$

There is a way to complete this diagram to a 3×3-diagram.

Proof.

There is a way to an $\mathbb{E}$-inflation i_2 s.t. $(i_1; i_2; i_3)$ is homotopic:

$$\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, "i_1"] \arrow[dr, phantom, "\square"] & A_2 \arrow[r, "g_A"] \arrow[d, "j_1"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, equal] & {} \\
B_1 \arrow[r, "s"] \arrow[d, equal] & E \arrow[r, "t"] \arrow[d, "j_2"] \arrow[dr, phantom, "\square"] & A_3 \arrow[r, dashed, "\kappa"] \arrow[d, "i_3"] & {} \\
B_1 \arrow[r, "f_B"] & B_2 \arrow[r, "g_B"] & B_3 \arrow[r, dashed, "\delta_B"] & {}
\end{tikzcd} . \tag{10}$$

Here j_1 and j_2 are constructed by

$$\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, "i_1"] \arrow[dr, phantom, "\square"] & A_2 \arrow[r, "g_A"] \arrow[d, dashed, "j_1"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, equal] & {} \\
B_1 \arrow[r, "s"] \arrow[d, "p_1"] & E \arrow[r, "t"] \arrow[d, dashed, "q"] & A_3 \arrow[r, dashed, "\kappa"] & {} \\
C_1 \arrow[r, equal] \arrow[d, dashed, "\varepsilon_1"] & C_1 \arrow[d, dashed, "(f_A)_*\varepsilon_1"] & & \\
{} & {} & &
\end{tikzcd}
\qquad
\begin{tikzcd}
B_1 \arrow[r, "s"] \arrow[d, equal] & E \arrow[r, "t"] \arrow[d, dashed, "j_2"] \arrow[dr, phantom, "\square"] & A_3 \arrow[r, dashed, "\kappa"] \arrow[d, "i_3"] & {} \\
B_1 \arrow[r, "f_B"] & B_2 \arrow[r, "g_B"] \arrow[d, dashed, "p_3 g_B"] & B_3 \arrow[r, dashed, "\delta_B"] \arrow[d, "p_3"] & {} \\
 & C_3 \arrow[r, equal] \arrow[d, dashed, "\theta"] & C_3 \arrow[d, dashed, "\varepsilon_3"] & \\
 & {} & {} &
\end{tikzcd} . \tag{11}$$

Proof.

We finally take $C_1 \xrightarrow{f_C} C_2 \xrightarrow{g_C} C_3 \overset{\delta_C}{\dashrightarrow}$ as follows:

$$\begin{tikzcd}
A_2 \arrow[r, equal] \arrow[d, "j_1"] & A_2 \arrow[d, dashed, "i_2"] & & \\
E \arrow[r, "j_2"] \arrow[d, "q"] & B_2 \arrow[r, "p_3 g_B"] \arrow[d, dashed, "p_2"] & C_3 \arrow[r, dashed, "\theta"] \arrow[d, equal] & {} \\
C_1 \arrow[r, dashed, "f_C"] \arrow[d, dashed, "(f_A)_*\varepsilon_1"] & C_2 \arrow[r, dashed, "g_C"] \arrow[d, dashed, "\varepsilon_2"] & C_3 \arrow[r, dashed, "\delta_C"] & {} \\
{} & {} & &
\end{tikzcd} \tag{12}$$

It remains to verify such completion satisfies the condition of a 3×3-diagram, i.e., $f_B i_1 = i_2 f_A$, $g_B i_2 = i_3 g_A$, $f_C p_1 = p_2 f_B$, $g_C p_2 = p_3 g_B$, $(i_1)_* \delta_A = (i_3)^* \delta_B$, $(p_1)_* \delta_B = (p_3)^* \delta_C$, $(f_A)_* \varepsilon_1 = (f_C)^* \varepsilon_2$, and $(g_A)_* \varepsilon_2 = (g_C)^* \varepsilon_3$. $\square$

A merged diagram:

$$
\begin{tikzcd}
 & & & A_1 \arrow[r, "f_A"] \arrow[dl, "i_1"'] & A_2 \arrow[r, "g_A"] \arrow[dl] \arrow[dl, dashed, "j_1"] \arrow[dr, equal] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[dl, equal] & {} & \\
 & & B_1 \arrow[r, shift left, "s"] \arrow[r, shift right] \arrow[dl, "p_1"'] \arrow[dr, equal, "q"] & E \arrow[r, shift left, "t"] \arrow[r, shift right] \arrow[dl] \arrow[dl, dashed] \arrow[dr, dashed, "j_2"] \arrow[dr, dashed] & A_3 \arrow[r, dashed, shift left, "\kappa"] \arrow[r, dashed, shift right] \arrow[dr, "i_3"] & A_2 \arrow[dl, "i_2"] & & \\
 & C_1 \arrow[r, equal] \arrow[dl, dashed, "\varepsilon_1"'] & C_1 \arrow[dl, dashed, "(f_A)_*\varepsilon_1"'] \arrow[dl, dashed] \arrow[dr, "f_C"] & B_1 \arrow[r, "f_B"] & B_2 \arrow[r, "g_B"] \arrow[dl, "p_2"] \arrow[dr, dashed, "p_3 g_B"] \arrow[dr, dashed] & B_3 \arrow[r, dashed, "\delta_B"] \arrow[dr, "p_3"] & {} & \\
{} & {} & & C_2 \arrow[dl, dashed, "\varepsilon_2"] \arrow[dr, "g_C"] & & C_3 \arrow[r, equal] \arrow[dl, equal] \arrow[dr, dashed, "j_2"] \arrow[dr, dashed] & C_3 \arrow[dr, dashed, "\varepsilon_3"] & \\
 & & {} & & C_3 \arrow[dr, dashed, "\delta_C"] & & {} & {} \\
 & & & & & {} & &
\end{tikzcd}
.
$$

$$
\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, "i_1"] \arrow[dr, phantom, "\square"] & A_2 \arrow[r, "g_A"] \arrow[d, "j_1"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, equal] & {} \\
B_1 \arrow[r, "s"] \arrow[d, equal] & E \arrow[r, "t"] \arrow[d, "j_2"] \arrow[dr, phantom, "\square"] & A_3 \arrow[r, dashed] \arrow[d, "i_3"] & {} \\
B_1 \arrow[r, "f_B"] & B_2 \arrow[r, "g_B"] & B_3 \arrow[r, dashed, "\delta_B"] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
A_2 \arrow[r, equal] \arrow[d, "j_1"] & A_2 \arrow[d, "i_2"] & & \\
E \arrow[r, "j_2"] \arrow[d, "q"] \arrow[dr, phantom, "\square"] & B_2 \arrow[r, "p_3 g_B"] \arrow[d, "p_2"] & C_3 \arrow[r, dashed, "\theta"] \arrow[d, equal] & {} \\
C_1 \arrow[r, "f_C"] \arrow[d, dashed, "(f_A)_* \varepsilon_1"] & C_2 \arrow[r, "g_C"] \arrow[d, dashed, "\varepsilon_2"] & C_3 \arrow[r, dashed, "\delta_C"] & {} \\
{} & {} & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, "i_1"] \arrow[dr, phantom, "\square"] & A_2 \arrow[r, "g_A"] \arrow[d, dashed, "j_1"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, equal] & {} \\
B_1 \arrow[r, "s"] \arrow[d, "p_1"] & E \arrow[r, "t"] \arrow[d, dashed, "q"] & A_3 \arrow[r, dashed, "\kappa"] & {} \\
C_1 \arrow[r, equal] \arrow[d, dashed, "\varepsilon_1"] & C_1 \arrow[d, dashed, "(f_A)_* \varepsilon_1"] & & \\
{} & {} & &
\end{tikzcd}
\qquad
\begin{tikzcd}
B_1 \arrow[r, "s"] \arrow[d, equal] & E \arrow[r, "t"] \arrow[d, dashed, "j_2"] \arrow[dr, phantom, "\square"] & A_3 \arrow[r, dashed, "\kappa"] \arrow[d, "i_3"] & {} \\
B_1 \arrow[r, "f_B"] & B_2 \arrow[r, "g_B"] \arrow[d, dashed, "p_3 g_B"] & B_3 \arrow[r, dashed, "\delta_B"] \arrow[d, "p_3"] & {} \\
& C_3 \arrow[r, equal] \arrow[d, dashed, "\theta"] & C_3 \arrow[d, dashed, "\varepsilon_3"] & \\
& {} & {} &
\end{tikzcd}
$$

$$
i_2 f_A \xlongequal{\text{Top Right}} j_2 j_1 f_A \xlongequal{\text{Top Left}} f_B i_1 .
$$

$$
\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, "i_1"] \arrow[dr, phantom, "\Box"] & A_2 \arrow[r, "g_A"] \arrow[d, "j_1"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, equal] & {} \\
B_1 \arrow[r, "s"] \arrow[d, equal] & E \arrow[r, "t"] \arrow[d, "j_2"] \arrow[dr, phantom, "\Box"] & A_3 \arrow[r, dashed] \arrow[d, "i_3"] & {} \\
B_1 \arrow[r, "f_B"] & B_2 \arrow[r, "g_B"] & B_3 \arrow[r, dashed, "\delta_B"] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
A_2 \arrow[r, equal] \arrow[d, "j_1"] & A_2 \arrow[d, "i_2"] & & \\
E \arrow[r, "j_2"] \arrow[d, "q"] \arrow[dr, phantom, "\Box"] & B_2 \arrow[r, "p_3 g_B"] \arrow[d, "p_2"] & C_3 \arrow[r, dashed, "\theta"] \arrow[d, equal] & {} \\
C_1 \arrow[r, "f_C"] \arrow[d, dashed, "(f_A)_* \varepsilon_1"] & C_2 \arrow[r, "g_C"] \arrow[d, dashed, "\varepsilon_2"] & C_3 \arrow[r, dashed, "\delta_C"] & {} \\
{} & {} & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, "i_1"] \arrow[dr, phantom, "\Box"] & A_2 \arrow[r, "g_A"] \arrow[d, dashed, "j_1"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, equal] & {} \\
B_1 \arrow[r, "s"] \arrow[d, "p_1"] & E \arrow[r, "t"] \arrow[d, dashed, "q"] & A_3 \arrow[r, dashed, "\kappa"] & {} \\
C_1 \arrow[r, equal] \arrow[d, dashed, "\varepsilon_1"] & C_1 \arrow[d, dashed, "(f_A)_* \varepsilon_1"] & & \\
{} & {} & &
\end{tikzcd}
\qquad
\begin{tikzcd}
B_1 \arrow[r, "s"] \arrow[d, equal] & E \arrow[r, "t"] \arrow[d, dashed, "j_2"] \arrow[dr, phantom, "\Box"] & A_3 \arrow[r, dashed, "\kappa"] \arrow[d, "i_3"] & {} \\
B_1 \arrow[r, "f_B"] & B_2 \arrow[r, "g_B"] \arrow[d, dashed, "p_3 g_B"] & B_3 \arrow[r, dashed, "\delta_B"] \arrow[d, "p_3"] & {} \\
& C_3 \arrow[r, equal] \arrow[d, dashed, "\theta"] & C_3 \arrow[d, dashed, "\varepsilon_3"] & \\
& {} & {} &
\end{tikzcd}
$$

$$
g_B i_2 \overset{\text{Top Right}}{=\!=\!=} g_B j_2 j_1 \overset{\text{Buttom Row}}{=\!=\!=} i_3 t j_1 \overset{\text{Top Left}}{=\!=\!=} i_3 g_A .
$$

$$
\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, "i_1"] \arrow[dr, phantom, "\square"] & A_2 \arrow[r, "g_A"] \arrow[d, "j_1"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, equal] & {} \\
B_1 \arrow[r, "s"] \arrow[d, equal] & E \arrow[r, "t"] \arrow[d, "j_2"] \arrow[dr, phantom, "\square"] & A_3 \arrow[r, dashed] \arrow[d, "i_3"] & {} \\
B_1 \arrow[r, "f_B"] & B_2 \arrow[r, "g_B"] & B_3 \arrow[r, dashed, "\delta_B"] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
A_2 \arrow[r, equal] \arrow[d, "j_1"] & A_2 \arrow[d, "i_2"] & & \\
E \arrow[r, "j_2"] \arrow[d, "q"] \arrow[dr, phantom, "\square"] & B_2 \arrow[r, "p_3 g_B"] \arrow[d, "p_2"] & C_3 \arrow[r, dashed, "\theta"] \arrow[d, equal] & {} \\
C_1 \arrow[r, "f_C"] \arrow[d, dashed, "(f_A)_*\varepsilon_1"] & C_2 \arrow[r, "g_C"] \arrow[d, dashed, "\varepsilon_2"] & C_3 \arrow[r, dashed, "\delta_C"] & {} \\
{} & {} & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, "i_1"] \arrow[dr, phantom, "\square"] & A_2 \arrow[r, "g_A"] \arrow[d, dashed, "j_1"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, equal] & {} \\
B_1 \arrow[r, "s"] \arrow[d, "p_1"] & E \arrow[r, "t"] \arrow[d, dashed, "q"] & A_3 \arrow[r, dashed, "\kappa"] & {} \\
C_1 \arrow[r, equal] \arrow[d, dashed, "\varepsilon_1"] & C_1 \arrow[d, dashed, "(f_A)_*\varepsilon_1"] & & \\
{} & {} & &
\end{tikzcd}
\qquad
\begin{tikzcd}
B_1 \arrow[r, "s"] \arrow[d, equal] & E \arrow[r, "t"] \arrow[d, dashed, "j_2"] \arrow[dr, phantom, "\square"] & A_3 \arrow[r, dashed, "\kappa"] \arrow[d, "i_3"] & {} \\
B_1 \arrow[r, "f_B"] & B_2 \arrow[r, "g_B"] \arrow[d, dashed, "p_3 g_B"] & B_3 \arrow[r, dashed, "\delta_B"] \arrow[d, "p_3"] & {} \\
& C_3 \arrow[r, equal] \arrow[d, dashed, "\theta"] & C_3 \arrow[d, dashed, "\varepsilon_3"] & \\
& {} & {} &
\end{tikzcd}
.
$$

$$
f_C p_1 \overset{\text{Buttom Row}}{=\!=\!=\!=} f_C q s \overset{\text{Top Right}}{=\!=\!=\!=} p_2 j_2 s \overset{\text{Top Left}}{=\!=\!=\!=} p_2 f_B .
$$

$$
\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, "i_1"] \arrow[dr, phantom, "\square"] & A_2 \arrow[r, "g_A"] \arrow[d, "j_1"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, equal] & {} \\
B_1 \arrow[r, "s"] \arrow[d, equal] & E \arrow[r, "t"] \arrow[d, "j_2"] \arrow[dr, phantom, "\square"] & A_3 \arrow[r, dashed] \arrow[d, "i_3"] & {} \\
B_1 \arrow[r, "f_B"] & B_2 \arrow[r, "g_B"] & B_3 \arrow[r, dashed, "\delta_B"] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
A_2 \arrow[r, equal] \arrow[d, "j_1"] & A_2 \arrow[d, "i_2"] & & \\
E \arrow[r, "j_2"] \arrow[d, "q"] \arrow[dr, phantom, "\square"] & B_2 \arrow[r, "p_3 g_B"] \arrow[d, "p_2"] & C_3 \arrow[r, dashed, "\theta"] \arrow[d, equal] & {} \\
C_1 \arrow[r, "f_C"] \arrow[d, dashed, "(f_A)_* \varepsilon_1"] & C_2 \arrow[r, "g_C"] \arrow[d, dashed, "\varepsilon_2"] & C_3 \arrow[r, dashed, "\delta_C"] & {} \\
{} & {} & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, "i_1"] \arrow[dr, phantom, "\square"] & A_2 \arrow[r, "g_A"] \arrow[d, dashed, "j_1"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, equal] & {} \\
B_1 \arrow[r, "s"] \arrow[d, "p_1"] & E \arrow[r, "t"] \arrow[d, dashed, "q"] & A_3 \arrow[r, dashed, "\kappa"] & {} \\
C_1 \arrow[r, equal] \arrow[d, dashed, "\varepsilon_1"] & C_1 \arrow[d, dashed, "(f_A)_* \varepsilon_1"] & & \\
{} & {} & &
\end{tikzcd}
\qquad
\begin{tikzcd}
B_1 \arrow[r, "s"] \arrow[d, equal] & E \arrow[r, "t"] \arrow[d, dashed, "j_2"] \arrow[dr, phantom, "\square"] & A_3 \arrow[r, dashed, "\kappa"] \arrow[d, "i_3"] & {} \\
B_1 \arrow[r, "f_B"] & B_2 \arrow[r, "g_B"] \arrow[d, dashed, "p_3 g_B"] & B_3 \arrow[r, dashed, "\delta_B"] \arrow[d, "p_3"] & {} \\
& C_3 \arrow[r, equal] \arrow[d, dashed, "\theta"] & C_3 \arrow[d, dashed, "\varepsilon_3"] & \\
& {} & {} &
\end{tikzcd}
.
$$

$$
g_C p_2 \xlongequal{\text{Top Right}} p_3 g_B .
$$

$$
\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, "i_1"] \arrow[dr, phantom, "\square"] & A_2 \arrow[r, "g_A"] \arrow[d, "j_1"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, equal] & {} \\
B_1 \arrow[r, "s"] \arrow[d, equal] & E \arrow[r, "t"] \arrow[d, "j_2"] \arrow[dr, phantom, "\square"] & A_3 \arrow[r, dashed] \arrow[d, "i_3"] & {} \\
B_1 \arrow[r, "f_B"] & B_2 \arrow[r, "g_B"] & B_3 \arrow[r, dashed, "\delta_B"] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
A_2 \arrow[r, equal] \arrow[d, "j_1"] & A_2 \arrow[d, "i_2"] & & \\
E \arrow[r, "j_2"] \arrow[d, "q"] \arrow[dr, phantom, "\square"] & B_2 \arrow[r, "p_3 g_B"] \arrow[d, "p_2"] & C_3 \arrow[r, dashed, "\theta"] \arrow[d, equal] & {} \\
C_1 \arrow[r, "f_C"] \arrow[d, dashed, "(f_A)_* \varepsilon_1"] & C_2 \arrow[r, "g_C"] \arrow[d, dashed, "\varepsilon_2"] & C_3 \arrow[r, dashed, "\delta_C"] & {} \\
{} & {} & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, "i_1"] \arrow[dr, phantom, "\square"] & A_2 \arrow[r, "g_A"] \arrow[d, dashed, "j_1"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, equal] & {} \\
B_1 \arrow[r, "s"] \arrow[d, "p_1"] & E \arrow[r, "t"] \arrow[d, dashed, "q"] & A_3 \arrow[r, dashed, "\kappa"] & {} \\
C_1 \arrow[r, equal] \arrow[d, dashed, "\varepsilon_1"] & C_1 \arrow[d, dashed, "(f_A)_* \varepsilon_1"] & & \\
{} & {} & &
\end{tikzcd}
\qquad
\begin{tikzcd}
B_1 \arrow[r, "s"] \arrow[d, equal] & E \arrow[r, "t"] \arrow[d, dashed, "j_2"] \arrow[dr, phantom, "\square"] & A_3 \arrow[r, dashed, "\kappa"] \arrow[d, "i_3"] & {} \\
B_1 \arrow[r, "f_B"] & B_2 \arrow[r, "g_B"] \arrow[d, dashed, "p_3 g_B"] & B_3 \arrow[r, dashed, "\delta_B"] \arrow[d, "p_3"] & {} \\
 & C_3 \arrow[r, equal] \arrow[d, dashed, "\theta"] & C_3 \arrow[d, dashed, "\varepsilon_3"] & \\
 & {} & {} &
\end{tikzcd}
$$

$((i_1)_* \delta_A = (i_3)^* \delta_B)$. We presuppose this identity.

$$
\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, "i_1"] \arrow[dr, phantom, "\square"] & A_2 \arrow[r, "g_A"] \arrow[d, "j_1"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, equal] & {} \\
B_1 \arrow[r, "s"] \arrow[d, equal] & E \arrow[r, "t"] \arrow[d, "j_2"] \arrow[dr, phantom, "\square"] & A_3 \arrow[r, dashed] \arrow[d, "i_3"] & {} \\
B_1 \arrow[r, "f_B"] & B_2 \arrow[r, "g_B"] & B_3 \arrow[r, dashed, "\delta_B"] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
A_2 \arrow[r, equal] \arrow[d, "j_1"] & A_2 \arrow[d, "i_2"] & & \\
E \arrow[r, "j_2"] \arrow[d, "q"] \arrow[dr, phantom, "\square"] & B_2 \arrow[r, "p_3 g_B"] \arrow[d, "p_2"] & C_3 \arrow[r, dashed, "\theta"] \arrow[d, equal] & {} \\
C_1 \arrow[r, "f_C"] \arrow[d, dashed, "(f_A)_* \varepsilon_1"] & C_2 \arrow[r, "g_C"] \arrow[d, dashed, "\varepsilon_2"] & C_3 \arrow[r, dashed, "\delta_C"] & {} \\
B_1 \arrow[r, "s"] \arrow[d, equal] & E \arrow[r, "t"] \arrow[d, dashed, "j_2"] \arrow[dr, phantom, "\square"] & A_3 \arrow[r, dashed, "\kappa"] \arrow[d, "i_3"] & {} \\
B_1 \arrow[r, "f_B"] & B_2 \arrow[r, "g_B"] \arrow[d, dashed, "p_3 g_B"] & B_3 \arrow[r, dashed, "\delta_B"] \arrow[d, "p_3"] & {} \\
& C_3 \arrow[r, equal] \arrow[d, dashed, "\theta"] & C_3 \arrow[d, dashed, "\varepsilon_3"] & \\
& {} & {} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, "i_1"] \arrow[dr, phantom, "\square"] & A_2 \arrow[r, "g_A"] \arrow[d, dashed, "j_1"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, equal] & {} \\
B_1 \arrow[r, "s"] \arrow[d, "p_1"] & E \arrow[r, "t"] \arrow[d, dashed, "q"] & A_3 \arrow[r, dashed, "\kappa"] & {} \\
C_1 \arrow[r, equal] \arrow[d, dashed, "\varepsilon_1"] & C_1 \arrow[d, dashed, "(f_A)_* \varepsilon_1"] & & \\
{} & {} & &
\end{tikzcd}
.
$$

$$
(p_1)_* \delta_B \xlongequal{\text{Buttom Row}} q_* s_* \delta_B \xlongequal{\text{Top Right}} q_* (p_3)^* \theta =
$$

$$
(p_3)^* q_* \theta \xlongequal{\text{Top Right}} (p_3)^* \delta_C .
$$

$$
\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, "i_1"] \arrow[dr, phantom, "\square"] & A_2 \arrow[r, "g_A"] \arrow[d, "j_1"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, equal] & {} \\
B_1 \arrow[r, "s"] \arrow[d, equal] & E \arrow[r, "t"] \arrow[d, "j_2"] \arrow[dr, phantom, "\square"] & A_3 \arrow[r, dashed] \arrow[d, "i_3"] & {} \\
B_1 \arrow[r, "f_B"] & B_2 \arrow[r, "g_B"] & B_3 \arrow[r, dashed, "\delta_B"] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
A_2 \arrow[r, equal] \arrow[d, "j_1"] & A_2 \arrow[d, "i_2"] & & \\
E \arrow[r, "j_2"] \arrow[d, "q"] \arrow[dr, phantom, "\square"] & B_2 \arrow[r, "p_3 g_B"] \arrow[d, "p_2"] & C_3 \arrow[r, dashed, "\theta"] \arrow[d, equal] & {} \\
C_1 \arrow[r, "f_C"] \arrow[d, dashed, "(f_A)_*\varepsilon_1"] & C_2 \arrow[r, "g_C"] \arrow[d, dashed, "\varepsilon_2"] & C_3 \arrow[r, dashed, "\delta_C"] & {} \\
{} & {} & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, "i_1"] \arrow[dr, phantom, "\square"] & A_2 \arrow[r, "g_A"] \arrow[d, dashed, "j_1"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, equal] & {} \\
B_1 \arrow[r, "s"] \arrow[d, "p_1"] & E \arrow[r, "t"] \arrow[d, dashed, "q"] & A_3 \arrow[r, dashed, "\kappa"] & {} \\
C_1 \arrow[r, equal] \arrow[d, dashed, "\varepsilon_1"] & C_1 \arrow[d, dashed, "(f_A)_*\varepsilon_1"] & & \\
{} & {} & &
\end{tikzcd}
\qquad
\begin{tikzcd}
B_1 \arrow[r, "s"] \arrow[d, equal] & E \arrow[r, "t"] \arrow[d, dashed, "j_2"] \arrow[dr, phantom, "\square"] & A_3 \arrow[r, dashed, "\kappa"] \arrow[d, "i_3"] & {} \\
B_1 \arrow[r, "f_B"] & B_2 \arrow[r, "g_B"] \arrow[d, dashed, "p_3 g_B"] & B_3 \arrow[r, dashed, "\delta_B"] \arrow[d, "p_3"] & {} \\
& C_3 \arrow[r, equal] \arrow[d, dashed, "\theta"] & C_3 \arrow[d, dashed, "\varepsilon_3"] & \\
& {} & {} &
\end{tikzcd}
$$

$$(f_A)_*\varepsilon_1 \xlongequal{\text{Top Right}} (f_C)^*\varepsilon_2.$$

$$
\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, "i_1"] \arrow[dr, phantom, "\square"] & A_2 \arrow[r, "g_A"] \arrow[d, "j_1"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, equal] & {} \\
B_1 \arrow[r, "s"] \arrow[d, equal] & E \arrow[r, "t"] \arrow[d, "j_2"] \arrow[dr, phantom, "\square"] & A_3 \arrow[r, dashed] \arrow[d, "i_3"] & {} \\
B_1 \arrow[r, "f_B"] & B_2 \arrow[r, "g_B"] & B_3 \arrow[r, dashed, "\delta_B"] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
A_2 \arrow[r, equal] \arrow[d, "j_1"] & A_2 \arrow[d, "i_2"] & & \\
E \arrow[r, "j_2"] \arrow[d, "q"] \arrow[dr, phantom, "\square"] & B_2 \arrow[r, "p_3 g_B"] \arrow[d, "p_2"] & C_3 \arrow[r, dashed, "\theta"] \arrow[d, equal] & {} \\
C_1 \arrow[r, "f_C"] \arrow[d, dashed, "(f_A)_*\varepsilon_1"] & C_2 \arrow[r, "g_C"] \arrow[d, dashed, "\varepsilon_2"] & C_3 \arrow[r, dashed, "\delta_C"] & {} \\
{} & {} & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, "i_1"] \arrow[dr, phantom, "\square"] & A_2 \arrow[r, "g_A"] \arrow[d, dashed, "j_1"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, equal] & {} \\
B_1 \arrow[r, "s"] \arrow[d, "p_1"] & E \arrow[r, "t"] \arrow[d, dashed, "q"] & A_3 \arrow[r, dashed, "\kappa"] & {} \\
C_1 \arrow[r, equal] \arrow[d, dashed, "\varepsilon_1"] & C_1 \arrow[d, dashed, "(f_A)_*\varepsilon_1"] & & \\
{} & {} & &
\end{tikzcd}
\qquad
\begin{tikzcd}
B_1 \arrow[r, "s"] \arrow[d, equal] & E \arrow[r, "t"] \arrow[d, dashed, "j_2"] \arrow[dr, phantom, "\square"] & A_3 \arrow[r, dashed, "\kappa"] \arrow[d, "i_3"] & {} \\
B_1 \arrow[r, "f_B"] & B_2 \arrow[r, "g_B"] \arrow[d, dashed, "p_3 g_B"] & B_3 \arrow[r, dashed, "\delta_B"] \arrow[d, "p_3"] & {} \\
& C_3 \arrow[r, equal] \arrow[d, dashed, "\theta"] & C_3 \arrow[d, dashed, "\varepsilon_3"] & \\
& {} & {} &
\end{tikzcd}
.
$$

$$
(g_A)_*\varepsilon_2 \xlongequal{\text{Top Left}} t_*(j_1)_*\varepsilon_2 \xlongequal{\text{Top Right}} t_*(g_C)^*\theta = (g_C)^* t_*\theta \xlongequal{\text{Buttom Row}} (g_C)^*\varepsilon_3.
$$

3×3 lemma from [NP19]

Theorem 3.9

Any commutative square consists of two parallel $\mathbb{E}$-inflations and $\mathbb{E}$-deflations completes to the following diagram

$$\begin{tikzcd}
A_1 \arrow[r, "f_A"] \arrow[d, dashed, "i_1"] & A_2 \arrow[r, "g_A"] \arrow[d, "i_2"] \arrow[dr, phantom, "\circlearrowright"] & A_3 \arrow[r, dashed, "\delta_A"] \arrow[d, "i_3"] & {} \\
B_1 \arrow[r, "f_B"] & B_2 \arrow[r, "g_B"] \arrow[d, "p_2"] & B_3 \arrow[r, dashed, "\delta_B"] \arrow[d, "p_3"] & {} \\
& C_2 \arrow[r, dashed, "g_C"] \arrow[d, dashed, "\varepsilon_2"] & C_3 \arrow[d, dashed, "\varepsilon_3"] & \\
& {} & {} &
\end{tikzcd},$$

such that $(i_1; i_2; i_3)$ and $(g_A; g_B; g_C)$ are morphisms of $\mathbb{E}$-conflations, i_1 is a retract of an $\mathbb{E}$-inflation and g_C is a retract of an $\mathbb{E}$-deflation.

- *This completion always exists iff WIC holds.*

Example 3.10 (Weak snake lemma)

Let f and g be composable morphisms that are both $\mathbb{E}$-inflations and $\mathbb{E}$-deflations. Then there is a way to find dashed arrows

$$
\begin{tikzcd}
 & K_g \arrow[dr, "k"] \arrow[rr, dashed, bend left, "c"] & & C_f \arrow[dr, dashed, "d"] & \\
K_{gf} \arrow[ur, dashed, "b"] \arrow[dr, "j"] \arrow[rr, phantom, "\square"] & & Y \arrow[ur, "p"] \arrow[dr, "g"] \arrow[rr, phantom, "\square"] & & C_{gf} \arrow[dd, dashed, bend left, "e"] \\
 & X \arrow[ur, "f"] \arrow[rr, bend right, "gf"] & & Z \arrow[ur, "q"] \arrow[dr, "r"] & \\
K_f \arrow[uu, dashed, bend left, "a"] \arrow[ur, "i"] & & & & C_g
\end{tikzcd},
$$

such that $\square$ are homotopic squares, and any three terms in $K_F \xrightarrow{a} K_{gf} \xrightarrow{b} K_g \xrightarrow{c} C_f \xrightarrow{d} C_{gf} \xrightarrow{e} C_g$ form $\mathbb{E}$-conflations.

Back to Homotopic Squares

Proposition 3.11

Suppose we have a homotopic square between two $\mathbb{E}$-conflations:

$$
\begin{tikzcd}
K \arrow[r, "i"] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, "u"] \arrow[dr, phantom, "\boxed{j_*\delta}"] & Y \arrow[r, dashed, "\delta"] \arrow[d, "v"] & {} \\
K \arrow[r, "j"] & A \arrow[r, "g", "\circ" marking] & B \arrow[r, dashed, "\varepsilon"] & {}
\end{tikzcd}
.
$$

If and only f is an $\mathbb{E}$-inflation ($\mathbb{E}$-deflation), then is g. If f is both an $\mathbb{E}$-inflation and an $\mathbb{E}$-deflation, we have

$$
\begin{tikzcd}
K \arrow[r, dashed, "i"] \arrow[d, dashed, equal] & X \arrow[r, "f"] \arrow[d, "u"] \arrow[dr, phantom, "\boxed{q^*\eta}"] & Y \arrow[r, "p"] \arrow[d, "v"] & C \arrow[r, dashed, "\eta"] \arrow[d, equal] & {} \\
K \arrow[r, dashed, "j"] & A \arrow[r, "g", "\circ" marking] & B \arrow[r, "q"] & C \arrow[r, dashed, "\kappa"] & {}
\end{tikzcd}
,
$$

such that $j_*\delta = q^*\eta$.

The proof is not straightforward, but needs a careful verification.

Let $(\alpha;\beta;\gamma)$ be a homotopic morphism of $\mathbb{E}$-conflations, where α, β, and γ are all $\mathbb{E}$-inflations and $\mathbb{E}$-deflations. We have

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "\beta_1"] \\
L \arrow[r, "s"'] & E
\end{tikzcd}
\qquad
\begin{tikzcd}
K_\alpha \arrow[r, "f'"] \arrow[d, "i_A"'] & K_\beta \arrow[r, "g'"] \arrow[d, "i_B"'] & K_\gamma \arrow[d, "i_C"'] \arrow[dddddll, dashed, bend left=60, "x" pos=0.3, "y" pos=0.7] \\
X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta_1"] & Z \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\
L \arrow[r, "s"] \arrow[d, equal] & E \arrow[r, "t"'] \arrow[d, "\beta_2"'] & Z \arrow[d, "\gamma"'] \\
L \arrow[r, "u"'] \arrow[d, "p_A"] & M \arrow[r, "v"'] \arrow[d, "p_B"] & N \arrow[r, dashed, "\varepsilon"'] \arrow[d, "p_C"] & {} \\
C_\alpha \arrow[r, "u'"'] & C_\beta \arrow[r, "v'"'] & C_\gamma
\end{tikzcd}
\qquad
\begin{tikzcd}
K_\beta \arrow[r, "g'"] \arrow[d, "i_\beta"'] \arrow[dr, phantom, "\square"] & K_\gamma \arrow[d, "i_C"] \\
Y \arrow[r, "\beta_1"'] & E
\end{tikzcd}
$$

$$
\begin{tikzcd}
E \arrow[r, "\beta_2"] \arrow[d, "y"'] \arrow[dr, phantom, "\square"] & M \arrow[d, "p_\beta"] \\
C_\alpha \arrow[r, "u'"'] & C_\beta
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, "t"] \arrow[d, "\beta_2"'] \arrow[dr, phantom, "\square"] & Z \arrow[d, "\gamma"] \\
M \arrow[r, "v"'] & N
\end{tikzcd}
$$

consisting of 4 homotopic squares and a 6-term chain map

$$K_\alpha \xrightarrow{f'} K_\beta \xrightarrow{g'} K_\gamma \xrightarrow{yx} C_\alpha \xrightarrow{u'} C_\beta \xrightarrow{v'} C_\gamma$$

where any two consecutive morphisms form an $\mathbb{E}$-conflation.

Snake Lemma: Proof Sketch

We decompose $(\alpha; \beta; \gamma)$ into

$$
\begin{tikzcd}
K_\alpha \arrow[r, equal] \arrow[d, "i_\alpha"] & K_\alpha \arrow[d, "f i_\alpha"] & & \\
X \arrow[r, "f"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\square"] & Y \arrow[r, "g"] \arrow[d, "\beta_1"] & Z \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\
L \arrow[r, "s"] \arrow[d, dashed, "l_\alpha"] & E \arrow[r, "t"] \arrow[d, dashed, "\eta_1"] & Z \arrow[r, dashed, "\kappa"] & {} \\
{} & {} & &
\end{tikzcd}
\qquad
\begin{tikzcd}
L \arrow[r, "s"] \arrow[d, equal] & E \arrow[r, "t"] \arrow[d, "\beta_2"] \arrow[dr, phantom, "\square"] & Z \arrow[r, dashed, "\kappa"] \arrow[d, "\gamma"] & {} \\
L \arrow[r, "u"] & M \arrow[r, "v"] \arrow[d, "p_\gamma v"] & N \arrow[r, dashed, "\varepsilon"] \arrow[d, "p_\gamma"] & {} \\
& C_\gamma \arrow[r, equal] \arrow[d, dashed, "\eta_2"] & C_\gamma \arrow[d, dashed, "r_\gamma"] & \\
& {} & {} &
\end{tikzcd}
.
$$

By properties of homotopic squares, β_1 and β_2 are both $\mathbb{E}$-inflations and $\mathbb{E}$-deflations.

Snake Lemma: Proof Sketch (contd.)

By Proposition 3.11, we have homotopic morphisms of $\mathbb{E}$-conflations $(1; f; s)$, $(f; s; 1)$, $(1; t; v)$ and $(t; v; 1)$ as follows (dashed arrows indicate extension elements)

$$
\begin{tikzcd}
K_\alpha \arrow[r, "i_\alpha"] \arrow[d, equal] & X \arrow[r, "\alpha"] \arrow[d, "f"] \arrow[dr, phantom, "\boxed{y^* r_\alpha}"] & L \arrow[r, "p_\alpha"] \arrow[d, "s"] \arrow[dr, dashed, "l_\alpha"] & C_\alpha \arrow[r, dashed, "r_\alpha"] \arrow[d, equal] & {} \\
K_\alpha \arrow[r, "fi_\alpha"] & Y \arrow[r, "\beta_1"] & E \arrow[r, "y"] \arrow[dr, dashed, "\eta_1"] & C_\alpha \arrow[r, dashed, "\mu_1"] & {} \\
& & & {} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
K_\gamma \arrow[r, "x"] \arrow[d, equal] & E \arrow[r, "\beta_2"] \arrow[d, "t"] \arrow[dr, phantom, "\boxed{x_* l_\gamma}"] & M \arrow[r, "p_\gamma v"] \arrow[d, "v"] \arrow[dr, dashed, "\mu_2"] & C_\gamma \arrow[r, dashed, "\eta_2"] \arrow[d, equal] & {} \\
K_\gamma \arrow[r, "i_\gamma"] & Z \arrow[r, "\gamma"] & N \arrow[r, "p_\gamma"] \arrow[dr, dashed, "l_\gamma"] & C_\gamma \arrow[r, dashed, "r_\gamma"] & {} \\
& & & {} &
\end{tikzcd}
$$

.

Snake Lemma: Proof Sketch (contd.)

The first four terms:

$$
\begin{tikzcd}
K_\alpha \arrow[r, "f'"] \arrow[d, equal] & K_\beta \arrow[r, "g'"] \arrow[d, "i_\beta"] \arrow[dr, phantom, "\square"] & K_\gamma \arrow[r, dashed, "x^*\eta_1"] \arrow[d, "x"] & {} \\
K_\alpha \arrow[r, "i_\beta"] & Y \arrow[r, "\beta_1"] \arrow[d, "\beta"] & E \arrow[r, dashed, "\eta_1"] \arrow[d, "\beta_2"] & {} \\
& M \arrow[r, equal] \arrow[d, dashed, "l_\beta"] & M \arrow[d, dashed, "\mu_2"] & \\
& {} & {} &
\end{tikzcd}
\qquad
\begin{tikzcd}
K_\beta \arrow[r, dashed, "\overline{g'}"] \arrow[d, "i_\beta"] \arrow[dr, phantom, "\square"] & K_\gamma \arrow[r, dashed, "yx"] \arrow[d, "x"] & C_\alpha \arrow[r, dashed, "\theta"] \arrow[d, equal] & {} \\
Y \arrow[r, "\beta_1"] \arrow[d, "\beta"] & E \arrow[r, "y"] \arrow[d, "\beta_2"] & C_\alpha \arrow[r, dashed, "\mu_1"] & {} \\
M \arrow[r, equal] \arrow[d, dashed, "l_\beta"] & M \arrow[d, dashed, "\mu_2"] & & \\
{} & {} & &
\end{tikzcd}
.
$$

$$
K_\beta \xrightarrow{\binom{g'}{i_\beta}} K_\gamma \oplus Y \xrightarrow{(x,-\beta_1)} E \overset{-(\beta_2)^* l_\beta}{\dashrightarrow}
\qquad
K_\beta \xrightarrow{\binom{\overline{g'}}{i_\beta}} K_\gamma \oplus Y \xrightarrow{(x,-\beta_1)} E \overset{-(\beta_2)^* l_\beta}{\dashrightarrow}
$$

By our construction of good completion, we can take $\overline{g'} = g'$ as long as the extension element of $\square$ coincides.

- $K_\gamma \xrightarrow{x} E \xrightarrow{y} C_\alpha$ is the connecting morphism.

We omit the verification of the remaining terms.

Snake Lemma (Special cases)

Proposition 3.13

Let $s(\alpha) \xrightarrow{\alpha} t(\alpha)$ and $s(\gamma) \xrightarrow{\gamma} t(\gamma)$ be both $\mathbb{E}$-inflations and $\mathbb{E}$-deflations. For any identity $\alpha_*\delta = \gamma^*$, there is a diagram

$$\begin{tikzcd}
& & & s(\gamma) \arrow[r, green] & t(\gamma) \arrow[dr, bend left, green] & \\
K_\alpha \arrow[r, blue] \arrow[dr, bend right, red] & K_\beta \arrow[r, blue] & K_\gamma \arrow[r, blue] \arrow[ur, bend left, green] & C_\alpha \arrow[r, blue] & C_\beta \arrow[r, blue] & C_\gamma \; . \\
& s(\alpha) \arrow[r, red] & t(\alpha) \arrow[ur, bend right, red] & & &
\end{tikzcd}$$

where any two consecutive morphisms with the same colour forms an $\mathbb{E}$-conflation.

Snake Lemma (Special cases)

Proposition 3.14

Suppose $\beta f = u\alpha$, where f and u are $\mathbb{E}$-inflations, and α and β are both $\mathbb{E}$-inflations and $\mathbb{E}$-deflations. Assume WIC, there exists some γ which is both an $\mathbb{E}$-inflation and an $\mathbb{E}$-deflation making the diagram below commute:

$$
\begin{tikzcd}
& K_\alpha \arrow[r, "f'"] \arrow[d] & K_\beta \arrow[r, "g'"] \arrow[d] & K_\gamma \arrow[d] \arrow[r, dash] & {} \\
& X \arrow[r, "f"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, dashed, "\delta"] \arrow[d, dashed, "\gamma"] & {} \\
& L \arrow[r, "u"] \arrow[d] & M \arrow[r, "v"] \arrow[d] & N \arrow[r, dashed, "\varepsilon"] \arrow[d] & {} \\
{} \arrow[r, "z"] & C_\alpha \arrow[r, "u'"] & C_\beta \arrow[r, "v'"] & C_\gamma &
\end{tikzcd}
.
$$

Any two consecutive morphisms as follows form an $\mathbb{E}$-conflation:

$$K_\alpha \xrightarrow{f'} K_\beta \xrightarrow{g'} K_\gamma \xrightarrow{z} C_\alpha \xrightarrow{u'} C_\beta \xrightarrow{v'} C_\gamma.$$

Snake Lemma (Special cases)

Proposition 3.15

Suppose $\beta f = u\alpha$, where f and u are $\mathbb{E}$-inflations, and α and β are both $\mathbb{E}$-inflations and $\mathbb{E}$-deflations. There exists some $\mathbb{E}$-deflation γ making the diagram below commute:

$$
\begin{tikzcd}
 & K_\alpha \arrow[r, "f'"] \arrow[d] & K_\beta \arrow[r, "g'"] \arrow[d] & K_\gamma \arrow[d] \arrow[r, dash] & {} \\
 & X \arrow[r, "f"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, dashed, "\delta"] \arrow[d, dashed, "\gamma"] & {} \\
 & L \arrow[r, "u"] \arrow[d] & M \arrow[r, "v"] \arrow[d] & N \arrow[r, dashed, "\varepsilon"] & {} \\
 {} \arrow[r, "z"] & C_\alpha \arrow[r, "u'"] & C_\beta & &
\end{tikzcd}
.
$$

Any two consecutive morphisms as follows form an $\mathbb{E}$-conflation:

$$K_\alpha \xrightarrow{f'} K_\beta \xrightarrow{g'} K_\gamma \xrightarrow{z} C_\alpha \xrightarrow{u'} C_\beta.$$

Snake Lemma (Special cases)

Proposition 3.16

Suppose $\beta f = u\alpha$, where f and u are $\mathbb{E}$-inflations. α is an $\mathbb{E}$-inflation. β is both an $\mathbb{E}$-inflation and an $\mathbb{E}$-deflation. There exists some $\mathbb{E}$-deflation γ making the diagram below commute:

$$
\begin{tikzcd}
 & & K_\beta \arrow[r, "g'"] \arrow[d] & K_\gamma \arrow[d] \arrow[r, dash] & {} \\
 & X \arrow[r, "f"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, dashed, "\delta"] \arrow[d, dashed, "\gamma"] & {} \\
 & L \arrow[r, "u"] \arrow[d] & M \arrow[r, "v"] \arrow[d] & N \arrow[r, dashed, "\varepsilon"] & {} \\
 {} \arrow[r, "z"] & C_\alpha \arrow[r, "u'"] & C_\beta & &
\end{tikzcd}
.
$$

Any two consecutive morphisms as follows form an $\mathbb{E}$-conflation:

$$K_\beta \xrightarrow{g'} K_\gamma \xrightarrow{z} C_\alpha \xrightarrow{u'} C_\beta.$$

Snake Lemma (Special cases)

Proposition 3.17

Suppose $\beta f = u\alpha$, where f and u are $\mathbb{E}$-inflations. α is an $\mathbb{E}$-deflation. β is both an $\mathbb{E}$-inflation and an $\mathbb{E}$-deflation. Assume WIC, *there exists some γ which is both an $\mathbb{E}$-deflation and an $\mathbb{E}$-deflations making the diagram below commute:*

$$
\begin{tikzcd}
 & & K_\beta \arrow[r, "g'"] \arrow[d] & K_\gamma \arrow[r, dash] \arrow[d] & {} \\
 & X \arrow[r, "f"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, dashed, "\delta"] \arrow[d, dashed, "\gamma"] & {} \\
 & L \arrow[r, "u"] \arrow[d] & M \arrow[r, "v"] \arrow[d] & N \arrow[r, dashed, "\varepsilon"] \arrow[d] & {} \\
{} \arrow[r, "z"] & C_\alpha \arrow[r, "u'"] & C_\beta \arrow[r, "v'"] & C_\gamma &
\end{tikzcd}
.
$$

Any two consecutive morphisms as follows form an $\mathbb{E}$-conflation:

$$K_\beta \xrightarrow{g'} K_\gamma \xrightarrow{z} C_\alpha \xrightarrow{u'} C_\beta \xrightarrow{v'} C_\gamma.$$

Snake Lemma (Special cases)

Proposition 3.18

Suppose $\beta f = u\alpha$, where f and u are $\mathbb{E}$-inflations. α is both an $\mathbb{E}$-inflation and an $\mathbb{E}$-deflation. β is an $\mathbb{E}$-deflation. There exists some $\mathbb{E}$-deflation γ making the diagram below commute:

$$
\begin{tikzcd}
K_\alpha \arrow[r, "f'"] \arrow[d] & K_\beta \arrow[r, "g'"] \arrow[d] & K_\gamma \arrow[d] \arrow[ddll, rounded corners, "z"] & \\
X \arrow[r, "f"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, dashed, "\delta"] \arrow[d, dashed, "\gamma"] & {} \\
L \arrow[r, "u"] \arrow[d] & M \arrow[r, "v"] & N \arrow[r, dashed, "\varepsilon"] & {} \\
C_\alpha & & &
\end{tikzcd}
.
$$

Any two consecutive morphisms as follows form an $\mathbb{E}$-conflation:

$$K_\alpha \xrightarrow{f'} K_\beta \xrightarrow{g'} K_\gamma \xrightarrow{z} C_\alpha.$$

Snake Lemma (Special cases)

Proposition 3.19

Suppose $\beta f = u\alpha$, where f and u are $\mathbb{E}$-inflations. α is an $\mathbb{E}$-deflation. β is an $\mathbb{E}$-inflation. Assume WIC, there exists some $\mathbb{E}$-inflation γ making the diagram below commute:

$$\begin{tikzcd}
 & K_\beta \arrow[r, "g'"] \arrow[d] & K_\gamma \arrow[d] \arrow[dddll, "z"] & \\
X \arrow[r, "f"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, dashed, "\delta"] \arrow[d, dashed, "\gamma"] & {} \\
L \arrow[r, "u"] \arrow[d] & M \arrow[r, "v"] & N \arrow[r, dashed, "\varepsilon"] & {} \\
C_\alpha & & &
\end{tikzcd} .$$

Moreover, there is an $\mathbb{E}$-conflation: $K_\beta \xrightarrow{g'} K_\gamma \xrightarrow{z} C_\alpha \dashrightarrow$.

Snake Lemma (Special cases)

Proposition 3.20

Suppose $\beta f = u\alpha$, where f and u are $\mathbb{E}$-inflations. α is an $\mathbb{E}$-inflation. β is an $\mathbb{E}$-deflation. There exists some $\mathbb{E}$-deflation γ making the diagram below commute:

$$\begin{tikzcd}
 & & & K_\gamma \arrow[d] \arrow[r, dash] & {} \\
 & X \arrow[r, "f"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, dashed, "\delta"] \arrow[d, dashed, "\gamma"] & {} \\
 & L \arrow[r, "u"] \arrow[d] & M \arrow[r, "v"] \arrow[d] & N \arrow[r, dashed, "\varepsilon"] & {} \\
{} \arrow[r, "z"] & C_\alpha \arrow[r, "u'"] & C_\beta & &
\end{tikzcd}.$$

Moreover, there is an $\mathbb{E}$-conflation: $K_\gamma \xrightarrow{z} C_\alpha \xrightarrow{u'} C_\beta \dashrightarrow$.

Snake Lemma (Special cases)

Proposition 3.21

Suppose $\beta f = u\alpha$, where f and u are $\mathbb{E}$-inflations. β and γ are both $\mathbb{E}$-inflations and $\mathbb{E}$-deflations. Assume WIC*, there exists some α which is both an $\mathbb{E}$-inflation and an $\mathbb{E}$-deflation making the diagram below commute:*

$$\begin{tikzcd}
& K_\alpha \arrow[r, "f'"] \arrow[d] & K_\beta \arrow[r, "g'"] \arrow[d] & K_\gamma \arrow[r, no head] \arrow[d] & {} \\
& X \arrow[r, "f"] \arrow[d, dashed, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] \arrow[dr, phantom, "\circlearrowright"] & Z \arrow[r, dashed, "\delta"] \arrow[d, "\gamma"] & {} \\
& L \arrow[r, "u"] \arrow[d] & M \arrow[r, "v"] \arrow[d] & N \arrow[r, dashed, "\varepsilon"] \arrow[d] & {} \\
{} \arrow[r, "z"] & C_\alpha \arrow[r, "u'"] & C_\beta \arrow[r, "v'"] & C_\gamma &
\end{tikzcd} .$$

Any two consecutive morphisms as follows form an $\mathbb{E}$-conflation:

$$K_\alpha \xrightarrow{f'} K_\beta \xrightarrow{g'} K_\gamma \xrightarrow{z} C_\alpha \xrightarrow{u'} C_\beta \xrightarrow{v'} C_\gamma .$$

Snake Lemma (Special cases)

Proposition 3.22

Suppose $\beta f = u\alpha$, where f and u are $\mathbb{E}$-inflations. β and γ are both $\mathbb{E}$-inflations and $\mathbb{E}$-deflations. There exists some $\mathbb{E}$-inflation α making the diagram below commute:

$$\begin{tikzcd}
 & & K_\beta \arrow[r, "g'"] \arrow[d] & K_\gamma \arrow[d] \arrow[r, dash] & {} \\
 & X \arrow[r, "f"] \arrow[d, dashed, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] \arrow[dr, phantom, "\circlearrowright"] & Z \arrow[r, dashed, "\delta"] \arrow[d, "\gamma"] & {} \\
 & L \arrow[r, "u"] \arrow[d] & M \arrow[r, "v"] \arrow[d] & N \arrow[r, dashed, "\varepsilon"] \arrow[d] & {} \\
{} \arrow[r, "z"] & C_\alpha \arrow[r, "u'"] & C_\beta \arrow[r, "v'"] & C_\gamma &
\end{tikzcd}.$$

Any two consecutive morphisms as follows form an $\mathbb{E}$-conflation:

$$K_\beta \xrightarrow{g'} K_\gamma \xrightarrow{z} C_\alpha \xrightarrow{u'} C_\beta \xrightarrow{v'} C_\gamma.$$

Snake Lemma (Special cases)

Proposition 3.23

Suppose $\beta f = u\alpha$, where f and u are $\mathbb{E}$-inflations. β is an $\mathbb{E}$-inflation. γ is both an $\mathbb{E}$-inflation and an $\mathbb{E}$-deflation. There exists some $\mathbb{E}$-inflation α making the diagram below commute:

$$\begin{tikzcd}
 & & & K_\gamma \arrow[d] \arrow[r, no head] & {} \\
 & X \arrow[r, "f"] \arrow[d, dashed, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] \arrow[dr, phantom, "\circlearrowright"] & Z \arrow[r, dashed, "\delta"] \arrow[d, "\gamma"] & {} \\
 & L \arrow[r, "u"] \arrow[d] & M \arrow[r, "v"] \arrow[d] & N \arrow[r, dashed, "\varepsilon"] \arrow[d] & {} \\
{} \arrow[r, "z"] & C_\alpha \arrow[r, "u'"] & C_\beta \arrow[r, "v'"] & C_\gamma &
\end{tikzcd}$$

Any two consecutive morphisms as follows form an $\mathbb{E}$-conflation:

$$K_\gamma \xrightarrow{z} C_\alpha \xrightarrow{u'} C_\beta \xrightarrow{v'} C_\gamma.$$

Snake Lemma (Special cases)

Proposition 3.24

Suppose $\beta f = u\alpha$, where f and u are $\mathbb{E}$-inflations. γ is both an $\mathbb{E}$-inflation and an $\mathbb{E}$-deflation. β is an $\mathbb{E}$-deflation. Assume WIC, *there exists some $\mathbb{E}$-deflation α making the diagram below commute:*

$$\begin{tikzcd}
 & K_\alpha \arrow[r, "f'"] \arrow[d] & K_\beta \arrow[r, "g'"] \arrow[d] & K_\gamma \arrow[r, dash] \arrow[d] & {} \\
 & X \arrow[r, "f"] \arrow[d, dashed, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] \arrow[dr, phantom, "\circlearrowright"] & Z \arrow[r, dashed, "\delta"] \arrow[d, "\gamma"] & {} \\
 & L \arrow[r, "u"] \arrow[d] & M \arrow[r, "v"] & N \arrow[r, dashed, "\varepsilon"] & {} \\
{} \arrow[r, "z"] & C_\alpha & & &
\end{tikzcd}.$$

Any two consecutive morphisms as follows form an $\mathbb{E}$-conflation:

$$K_\alpha \xrightarrow{f'} K_\beta \xrightarrow{g'} K_\gamma \xrightarrow{z} C_\alpha.$$

Snake Lemma (Special cases)

Proposition 3.25

Suppose $\beta f = u\alpha$, where f and u are $\mathbb{E}$-inflations. γ is an $\mathbb{E}$-inflation. β is both an $\mathbb{E}$-inflation and an $\mathbb{E}$-deflation. Assume WIC, α, β, and γ are both $\mathbb{E}$-inflations and $\mathbb{E}$-deflations making the diagram below commute:

$$
\begin{tikzcd}
 & K_\alpha \arrow[r, "f'"] \arrow[d] & K_\beta \arrow[r, "g'"] \arrow[d] & K_\gamma \arrow[r, dash] \arrow[d] & {} \\
 & X \arrow[r, "f"] \arrow[d, dashed, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] \arrow[dr, phantom, "\circlearrowright"] & Z \arrow[r, dashed, "\delta"] \arrow[d, "\gamma"] & {} \\
 & L \arrow[r, "u"] \arrow[d] & M \arrow[r, "v"] \arrow[d] & N \arrow[r, dashed, "\varepsilon"] \arrow[d] & {} \\
{} \arrow[r, "z"] & C_\alpha \arrow[r, "u'"] & C_\beta \arrow[r, "v'"] & C_\gamma &
\end{tikzcd}
.
$$

Any two consecutive morphisms as follows form an $\mathbb{E}$-conflation:

$$K_\alpha \xrightarrow{f'} K_\beta \xrightarrow{g'} K_\gamma \xrightarrow{z} C_\alpha \xrightarrow{u'} C_\beta \xrightarrow{v'} C_\gamma.$$

Snake Lemma (Special cases)

Proposition 3.26

Suppose $\beta f = u\alpha$, where f and u are $\mathbb{E}$-inflations. γ is an $\mathbb{E}$-deflation. β is both an $\mathbb{E}$-inflation and an $\mathbb{E}$-deflation. There exists some $\mathbb{E}$-inflation α making the diagram below commute:

$$\begin{tikzcd}
& & K_\beta \arrow[r, "g'"] \arrow[d] & K_\gamma \arrow[d] \arrow[r, dash] & {} \\
& X \arrow[r, "f"] \arrow[d, dashed, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] \arrow[dr, phantom, "\circlearrowright"] & Z \arrow[r, dashed, "\delta"] \arrow[d, "\gamma"] & {} \\
& L \arrow[r, "u"] \arrow[d] & M \arrow[r, "v"] \arrow[d] & N \arrow[r, dashed, "\varepsilon"] & {} \\
{} \arrow[r, "z"] & C_\alpha \arrow[r, "u'"] & C_\beta & &
\end{tikzcd}.$$

Any two consecutive morphisms as follows form an $\mathbb{E}$-conflation:

$$K_\beta \xrightarrow{g'} K_\gamma \xrightarrow{z} C_\alpha \xrightarrow{u'} C_\beta.$$

Snake Lemma (Special cases)

Proposition 3.27

Suppose $\beta f = u\alpha$, where f and u are $\mathbb{E}$-inflations. γ is an $\mathbb{E}$-inflation. β is an $\mathbb{E}$-deflation. Assume WIC, there exists some $\mathbb{E}$-deflation α making the diagram below commute:

$$
\begin{tikzcd}
 & & K_\beta \arrow[r, "g'"] \arrow[d] & K_\gamma \arrow[d] \arrow[r, dash] & {} \\
 & X \arrow[r, "f"] \arrow[d, dashed, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] \arrow[dr, phantom, "\circlearrowright"] & Z \arrow[r, dashed, "\delta"] \arrow[d, "\gamma"] & {} \\
 & L \arrow[r, "u"] \arrow[d] & M \arrow[r, "v"] & N \arrow[r, dashed, "\varepsilon"] & {} \\
{} \arrow[r, "z"] & C_\alpha & & &
\end{tikzcd}
.
$$

Moreover, there exists an $\mathbb{E}$-conflation: $K_\beta \xrightarrow{g'} K_\gamma \xrightarrow{z} C_\alpha$.

Snake Lemma (Special cases)

Proposition 3.28

Suppose $\beta f = u\alpha$, where f and u are $\mathbb{E}$-inflations. β is an $\mathbb{E}$-inflation. γ is an $\mathbb{E}$-deflation. There exists some $\mathbb{E}$-inflation α making the diagram below commute:

$$
\begin{tikzcd}
 & & & K_\gamma \arrow[d] \arrow[r, dash] & {} \\
 & X \arrow[r, "f"] \arrow[d, dashed, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] \arrow[dr, phantom, "\circlearrowright"] & Z \arrow[r, dashed, "\delta"] \arrow[d, "\gamma"] & {} \\
 & L \arrow[r, "u"] \arrow[d] & M \arrow[r, "v"] \arrow[d] & N \arrow[r, dashed, "\varepsilon"] & {} \\
{} \arrow[r, "z"] & C_\alpha \arrow[r, "u'"] & C_\beta & &
\end{tikzcd}
$$

Moreover, there exists an $\mathbb{E}$-conflation: $K_\gamma \xrightarrow{z} C_\alpha \xrightarrow{u'} C_\beta$.

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Thanks for your attention!