Notes on Grothendieck Monoids for Extriangulated Categories

Tansing Tiunn

October 11, 2025

We collect some aspects from [?]

1 Extriangulated Categories

1.1 Exact Functors

Definition 1.1 (2.2., Exact functors between extriangulated categories)**.** Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ and $(\mathcal{D}, \mathbb{F}, \mathfrak{t})$ be extriangulated categories. An exact functor is a pair (F, φ) such that

1. $F : \mathcal{C} \to \mathcal{D}$ is an additive functor,
2. $\varphi_{Z,X} : \mathbb{E}(Z, X) \to \mathbb{F}(FZ, FX)$ is a natural transformation of bifunctors,
3. for any $\mathfrak{s}$-conflation $X \xrightarrow{f} Y \xrightarrow{g} Z \overset{\delta}{\dashrightarrow}$, the sequence $FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \overset{\varphi_{Z,X}(\delta)}{\dashrightarrow}$ is a $\mathfrak{t}$-conflation.

Lemma 1.2 (2.4.)**.** *Suppose $\mathcal{D}$ is exact. The pair (F, φ) is uniquely determined by F if F is conflation preserving.*

Proof. Given a conflation-preserving functor F, we can define φ as follows: for any $\mathfrak{s}$-conflation $X \xrightarrow{f} Y \xrightarrow{g} Z \overset{\delta}{\dashrightarrow}$, we define $\varphi(\delta)$ as the extension element corresponding to the $\mathfrak{t}$-conflation $FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \overset{\varphi(\delta)}{\dashrightarrow}$. Note that the morphism of $\mathfrak{t}$-conflations is uniquely determined by two commutative squares, so $(F\gamma)^*\varphi(\delta') = (F\alpha)_*\varphi(\delta)$ in the following diagram:

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "F\alpha"] & FY \arrow[r, "Fg"] \arrow[d, "F\beta"] & FZ \arrow[r, dashed, "\varphi(\delta)"] \arrow[d, "F\gamma"] & {} \\ FX' \arrow[r, "Ff'"] & FY' \arrow[r, "Fg'"] & FZ' \arrow[r, dashed, "\varphi(\delta')"] & {} \end{tikzcd} . \tag{1}$$

□

1.2 Grothendieck Monoids

We assume all extriangulated categories are essentially small.

Notation (Monoid)**.** A monoid is an Abelian (additive) group without the requirement of inverses.

Notation. Let Iso$(-)$ denote the isomorphism classes of objects in a category. $[X]$ is the isomorphism class of an object X. Note that $[-]$ preserves direct sums, i.e. $[X \oplus Y] = [X] + [Y]$. Iso$(\mathcal{C})$ is a monoid.

Definition 1.3 (2.5. Grothendieck monoid)**.** Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. The Grothendieck monoid $\mathsf{M}(\mathcal{C})$ is characterised by a quotient of classes map

$$\pi : \mathrm{Iso}(\mathcal{C}) \twoheadrightarrow (\mathrm{Iso}(\mathcal{C})/\sim) =: \mathsf{M}(\mathcal{C}) \tag{2}$$

$\sim$: for any conflation $X \xrightarrow{f} Y \xrightarrow{g} Z \overset{\delta}{\dashrightarrow}$, one has $\pi([X]) + \pi([Z]) = \pi([Y])$.

Remark (Universal property)*.* For map $p : \mathrm{Iso}(\mathcal{C}) \to N$ respecting conflations, p factors through π uniquely.

Definition 1.4 (2.6)**.** For any conflation $X \xrightarrow{f} Y \xrightarrow{g} Z \overset{\delta}{\dashrightarrow}$, we denote $[Y] \sim_c [X \oplus Z]$. This generates an additive equivalence relation $\approx_c$.

Notation. We write $X \sim_c Y$ (resp. $X \simeq_c Y$) when $[X] \approx_c [Y]$ (resp. $[X] \approx_c [Y]$).

Proposition 1.5 (2.7.)**.** $\mathsf{M}(\mathcal{C}) \cong \mathrm{Iso}(\mathcal{C})/\approx_c$.

Proposition 1.6 (2.8.)**.** *This yields a functor*

$$\mathsf{M} : \mathsf{ETCat} \to \mathsf{Mon}, \quad (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \mapsto \mathsf{M}(\mathcal{C}), \quad (F, \varphi) \mapsto ([X] \mapsto [FX]). \tag{3}$$

Remark. The usual Grothendieck group functor K_0 is a completion of M.

Example 1.7. When $\mathcal{C}$ is Abelian length with n simple objects, then $\mathsf{M}(\mathcal{C}) \cong \mathbb{N}^n$; when $\mathcal{C}$ is triangulated, then $\mathsf{M}(\mathcal{C})$ is a group.

Remark. $D(\mathcal{A})$ and $D^{\geq 0}(\mathcal{A})$ admits the same Grothendieck monoid, since $[X] = -[\Sigma X] = [\Sigma^2 X]$. However, $D^{\geq 0}(\mathcal{A})$ is not triangulated in general.

1.3 Exmaple: Serre Subcategory

Example 1.8. There is a covariant Galois connection between the replete subcategories of $\mathcal{C}$, and the subset of $\mathsf{M}(\mathcal{C})$:

1. For a subcategory $\mathcal{A} \subseteq \mathcal{C}$, let $\mathsf{M}_{\mathcal{A}} := \mathrm{Im}(\mathrm{Iso}(\mathcal{A}) \to \mathrm{Iso}(\mathcal{C}) \xrightarrow{\pi} \mathsf{M}(\mathcal{C}))$.
2. (A monic assignment). For a subset $S \subseteq \mathsf{M}(\mathcal{C})$, let $\mathcal{C}_S := \{X \in \mathcal{C} \mid [X] \in S\}$.

Note that $\mathcal{A} \mapsto \mathsf{S} \mapsto \mathcal{A}$ is identical, if and only if $\mathcal{A}$ is closed under $\approx_c$. We obtain the closure part of the Galois connection:

$$2^{\mathsf{M}} \simeq \{\mathcal{C}_\bullet\}, \quad S \mapsto \mathcal{C}_S. \tag{4}$$

Definition 1.9 (Serre subcategory)**.** $\mathcal{A} \subseteq \mathcal{C}$ is Serre, provided that for any conflation $X \xrightarrow{f} Y \xrightarrow{g} Z \overset{\delta}{\dashrightarrow}$, $X, Z \in \mathcal{A}$ if and only if $Y \in \mathcal{A}$.

Definition 1.10 (A4)**.** Say a submonoid $N \subseteq M$ is a face, provided any $(x + y) \in N$ if and only if $x, y \in N$.

Remark. eq. (8) restricts to a bijection between Serre subcategories of $\mathcal{C}$ and faces of $\mathsf{M}(\mathcal{C})$.

We examise functorial property of eq. (3).

Example 1.11 (Inclusion)**.** Let $i : \mathcal{A} \to \mathcal{C}$ be an inclusion of extension closed subcategory. One has

$$\mathsf{M}(i) : \mathsf{M}(\mathcal{A}) \twoheadrightarrow \mathsf{M}_{\mathcal{A}} \rightarrowtail \mathsf{M}(\mathcal{C}), \quad [X] \mapsto [X]. \tag{5}$$

Proposition 1.12 (3.6, 3.7)**.** *$\mathsf{M}(\mathcal{A}) \twoheadrightarrow \mathsf{M}_{\mathcal{A}}$ in eq. (5) is an isomorphism if and only if for any $X, Y \in \mathcal{A}$, $X \sim_c Y$ in $\mathcal{C}$ iff $X \sim_c Y$ in $\mathcal{A}$. In particular, if $\mathcal{A}$ is Serre, then $\mathsf{M}(\mathcal{A}) \cong \mathsf{M}_{\mathcal{A}}$.*

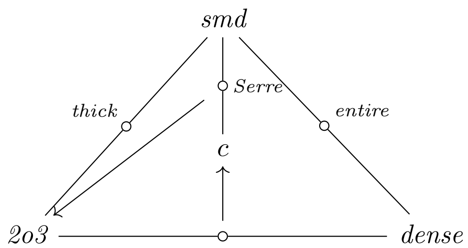
Moreover, we introduce

Definition 1.13 (3.9., subcategories)**.** Let $\mathcal{A} \subseteq \mathcal{C}$ be a subcategory.

1. Say $\mathcal{A}$ is $\sim_c$ closed, provided for any conflation $X \xrightarrow{f} Y \xrightarrow{g} Z \overset{\delta}{\dashrightarrow}$ in $\mathcal{C}$, if $X \oplus Z \in \mathcal{A}$, then so is Y;
2. Say $\mathcal{A}$ is Serre, provided for any conflation $X \xrightarrow{f} Y \xrightarrow{g} Z \overset{\delta}{\dashrightarrow}$ in $\mathcal{C}$, $X, Z \in \mathcal{A}$ if and only if $Y \in \mathcal{A}$;
3. Say $\mathcal{A}$ is dense, provided $\mathsf{add}(\mathcal{A}) = \mathcal{C}$;
4. Say $\mathcal{A}$ is 2o3, provided for any conflation $X \xrightarrow{f} Y \xrightarrow{g} Z \overset{\delta}{\dashrightarrow}$ in $\mathcal{C}$, if two of X, Y, Z are in $\mathcal{A}$, then so is the third;
5. Say $\mathcal{A}$ is thick, provided it is 2o3 and closed under direct summands.

Remark. For triangulated cases, 2o3 subcategories are precisely triangulated subcategories.

Proposition 1.14 (3.13)**.** *The relation of subcategories:*



. (6)

1.4 Example: 2o3-and-dense Subcategories

Theorem 1.15. *eq. (8) restricts to a bijection between 2o3-and-dense subcategories of $\mathcal{C}$ and submonoids $\mathsf{S} \subseteq \mathsf{M}(\mathcal{C})$ which are*

1. *(subtractive). For any $x, y \in \mathsf{M}(\mathcal{C})$, if $(x + y), x \in \mathsf{S}$, then $y \in \mathsf{S}$;*
2. *(cofinal). For any $x \in \mathsf{M}(\mathcal{C})$, there exists $s \in \mathsf{M}$ such that $(x + s) \in \mathsf{S}$.*

Remark. A substractive monoid of a group is a group. (All groups are commutative).

Cofinal subtractive submonoids comes from the following covariant Galois connection.

Example 1.16. A group completion is a funtor $\rho : \mathsf{Mon} \to \mathsf{Grp}$ sending M to a group $\rho(\mathsf{M}) = (\mathsf{M} \times \mathsf{M})/\sim$,

$$(m_1, n_1) \sim (m_2, n_2) \iff \exists x \in \mathsf{M}, m_1 + n_2 + x = m_2 + n_1 + x. \tag{7}$$

The image of (m, n) in $\rho(\mathsf{M})$ is $m - n$.

Remark. $\rho : \mathsf{Mon} \to \mathsf{Grp}$ admits a right adjoint $\iota : \mathsf{Grp} \to \mathsf{Mon}$, the inclusion. This yields the universal property that

- for any monoid morphism $f : \mathsf{M} \to G$ to a group G, there exists a unique group morphism $\tilde{f} : \rho(\mathsf{M}) \to G$ such that $f = \tilde{f} \circ \eta_{\mathsf{M}}$, where $\eta_{\mathsf{M}} : \mathsf{M} \to \rho(\mathsf{M})$ is the canonical map.

Moreover, $(\iota(-), \mathsf{M})_{\mathsf{Mon}}$ is representable when M admits a largest subgroup.

Proposition 1.17 (3.16)**.** *The assignment (natural transformation) $\eta_{\mathsf{M}} : \mathsf{M} \to \rho(\mathsf{M})$ maps m to equivalency class of $(m, 0)$. This yields a covariant Galois connection between submonoids of M and subgroups of $\rho(\mathsf{M})$:*

$$\eta_{\mathsf{M}}^{-1}(\mathsf{H}) \leftarrow\!\shortmid \mathsf{H}, \quad \{\textit{submonoids of } \mathsf{M}\} \leftrightarrows \{\textit{subgroups of } \rho(\mathsf{M})\}, \quad \mathsf{S} \mapsto \langle \eta_{\mathsf{M}}(\mathsf{S}) \rangle. \tag{8}$$

The closure part of the Galois connection restricts to the collection of cofinal subtractive submonoids on the left.

Corollary 1.18 (3.17)**.** *For theorem 1.15, there is a bijection between 2o3-and-dense subcategories of $\mathcal{C}$ and subgroups of $\rho(\mathsf{M}(\mathcal{C})) = \mathsf{K}_0(\mathcal{C})$. If $\mathcal{C}$ is triangulated, then subgroups of $\mathsf{M}(\mathcal{C}) = \mathsf{K}_0(\mathcal{C})$ corresponds to dense triangulated subcategories.*

Corollary 1.19 (3.18)**.** *When $\mathcal{A}$ is Abelian length with finite many simples, we see $\mathsf{M}(\mathcal{A}) \simeq \mathbb{N}^n$. A 2o3-and-dense subcategoy corresponds to a submonoid of $\mathbb{N}^n$ which is cofinal and subtractive, hence corresponds to a subgroup of $\mathbb{Z}^n$ containing an all positive element.*

Definition 1.20 (Generator)**.** Say an extriangulated category $\mathcal{C}$ has a generator $\mathcal{G} \subseteq \mathsf{Ob}(\mathcal{C})$, when for any $X \in \mathcal{C}$, there exists a deflation $G \twoheadrightarrow X$ with $G \in \mathcal{G}$.

Theorem 1.21. *Suppose $\mathcal{C}$ has a generator $\mathcal{G}$. Then there is a bijection between*

1. *2o3-and-dense subcategories of $\mathcal{C}$ containing $\mathcal{G}$;*
2. *substractive cofinal submonoids of $\mathsf{M}(\mathcal{C})$ containing $[\mathcal{G}] := \{[G] \mid G \in \mathcal{G}\}$;*
3. *substractive submonoids of $\mathsf{M}(\mathcal{C})$ containing $[\mathcal{G}] := \{[G] \mid G \in \mathcal{G}\}$;*
4. *subgroups of $\mathsf{K}_0(\mathcal{C})$ containing $[\mathcal{G}] := \{[G] \mid G \in \mathcal{G}\}$,*

Proof. (1) $\Leftrightarrow$ (2) by theorem 1.15. (2) $\Leftrightarrow$ (3) is clear. (3) $\Leftrightarrow$ (4) by the Galois connection (eq. (8)). $\square$

1.5 Localisations

Notation. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. Let $\mathcal{N} \subseteq \mathcal{A}$ be a thick subcategory. We set

1. $\mathcal{L}_{\mathcal{N}} := \{i \mid \exists X \xrightarrow{i} Y \to N \overset{\delta}{\dashrightarrow}, N \in \mathcal{N}\}$;
2. $\mathcal{R}_{\mathcal{N}} := \{p \mid \exists N \to X \xrightarrow{p} Y \overset{\delta}{\dashrightarrow}, N \in \mathcal{N}\}$;
3. $\mathcal{S}_{\mathcal{N}}$ the finite compositions of morphisms in $\mathcal{L}$ and $\mathcal{R}$.

Note that $\mathcal{S}_{\mathcal{N}}$ and $\mathcal{N}$ determines each other.

Remark. Both $\mathcal{L}_{\mathcal{N}}$ and $\mathcal{R}_{\mathcal{N}}$ contains all isomorphisms and are closed under composition.

Definition 1.22 (Exact localisation)**.** Suppose that S is a class of morphisms in $\mathcal{C}$, the localising functor $Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$ is exact. Say it is an exact localisation, provided that Q satisfies the following universal property:

- for any exact functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(s)$ is an isomorphism for all $s \in S$, there exists a unique exact functor $\tilde{F} : \mathcal{C}[S^{-1}] \to \mathcal{D}$ such that $F = \tilde{F} \circ Q$.

Proposition 1.23 (4.3, analogue to Serre quotient)**.** *Suppose that $\mathcal{N} \subseteq \mathcal{C}$ is thick, and the localisation $Q : \mathcal{C} \to \mathcal{C}/\mathcal{N} := \mathcal{C}[(\mathcal{S}_{\mathcal{N}})^{-1}]$ is exact. For any exact functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(\mathcal{N})$ are zero objects, there exists a unique exact functor $\tilde{F} : \mathcal{C}/\mathcal{N} \to \mathcal{D}$ such that $F = \tilde{F} \circ Q$.*

Remark. The Serre quotient of Abelian categories $Q : \mathcal{A} \twoheadrightarrow \mathcal{A} \to \mathcal{A}/\mathcal{C}$ is exact, which induces an isomorphism

$$\mathrm{Ex}_{\mathcal{C} \to 0}(\mathcal{A}/\mathcal{C}, \mathcal{B}) \simeq \mathrm{Ex}(\mathcal{A}, \mathcal{B}), \quad F \mapsto F \circ Q. \tag{9}$$

We show a technique and associated results to a two-step localisation.

Notation. We set $\mathrm{St}_{\mathcal{N}}(\mathcal{C})$ as an additive stable category.

Remark. The universal property yields $\mathcal{C} \xrightarrow{[-]} \mathrm{St}_{\mathcal{N}}(\mathcal{C}) \xrightarrow{Q} \mathcal{C}/\mathcal{N}$. We write the composition as Q for simplicity.

Notation (4.4)**.** We show a condition of $[S_{\mathcal{N}}]$ over $\mathrm{St}_{\mathcal{N}}(\mathcal{C})$ so that $\mathcal{C}/\mathcal{N}$ is extriangulated and $Q : \mathcal{C} \to \mathcal{C}/\mathcal{N}$ is exact.

1. When $[f]$ is iso for f split monic (or split epic), then $f \in S_{\mathcal{N}}$.

2. $[S_{\mathcal{N}}]$ admits two-out-of-three property of compositions in $\mathrm{St}_{\mathcal{N}}(\mathcal{C})$.

3. $[S_{\mathcal{N}}]$ is a left and right multiplicative system in $\mathrm{St}_{\mathcal{N}}(\mathcal{C})$.

4. Let Inf and Def be the class of all inflations and deflations in $\mathcal{C}$. Then $[S_{\mathcal{N}}] \circ [\mathsf{Inf}] \circ [S_{\mathcal{N}}]$ and $[S_{\mathcal{N}}] \circ [\mathsf{Def}] \circ [S_{\mathcal{N}}]$ are closed under compositions in $\mathrm{St}_{\mathcal{N}}(\mathcal{C})$.

Theorem 1.24. *With section 1.5, the calculation of fractions yields* $\mathrm{St}_{\mathcal{N}}(\mathcal{C}) \to [S_{\mathcal{N}}]^{-1}\mathrm{St}_{\mathcal{N}}(\mathcal{C})$*, which is an exact localisation of extriangulated categories, which yields an isomorphism* $[S_{\mathcal{N}}]^{-1}\mathrm{St}_{\mathcal{N}}(\mathcal{C}) \cong \mathcal{C}/\mathcal{N}$*.*

Corollary 1.25. *A morphism in* $\mathcal{C}/\mathcal{N}$ *is a left or right fraction in* $\mathrm{St}_{\mathcal{N}}(\mathcal{C})$*.*

Corollary 1.26. *An extension element in* $\mathcal{C}/\mathcal{N}$ *is a bifraction in* $\mathrm{St}_{\mathcal{N}}(\mathcal{C})$*. Note that the induced* $\mathbb{E}$*-functor exists in* $\mathcal{C}/\mathcal{N}$*, even though* $\mathrm{St}_{\mathcal{N}}(\mathcal{C})$ *is not extriangulated in general.*

Corollary 1.27. *Any inflation (or deflation) in* $\mathcal{C}/\mathcal{N}$ *is an image of an inflation (or deflation) in* $\mathcal{C}$ *composing with isomorphisms in* $\mathcal{C}/\mathcal{N}$*.*