Homework 10

Problem 1.

(a) Let X be a random variable taking values in $[0,1]$. Prove that if $\mathbb{E}(X)=\varepsilon$, then
$$\Pr\Big(X \geq \frac{\varepsilon}{2}\Big) \geq \frac{\varepsilon}{2}.$$

(b) Let $X \geq 0$ be a random variable. Prove that
$$\Pr(X=0) \leq \frac{\mathrm{Var}(X)}{\big(\mathbb{E}(X)\big)^2}.$$

Solution. (a) Since $\mathbb{E}(X)=\varepsilon$, say the PDF of X is $p(x)$, where $p(x)\geq 0$ and $\int_0^1 p(x)dx = 1$, we have

$$\begin{aligned}
\varepsilon = \mathbb{E}(X) &= \int_0^1 xp(x)dx = \int_0^{\frac{\varepsilon}{2}} xp(x)dx + \int_{\frac{\varepsilon}{2}}^1 xp(x)dx \\
&\leq \frac{\varepsilon}{2}\int_0^{\frac{\varepsilon}{2}} p(x)dx + 1\times\int_{\frac{\varepsilon}{2}}^1 p(x)dx \\
&\leq \frac{\varepsilon}{2}\int_0^{1} p(x)dx + \int_{\frac{\varepsilon}{2}}^1 p(x)dx \\
&= \frac{\varepsilon}{2} + \Pr\Big(X \geq \frac{\varepsilon}{2}\Big).
\end{aligned}$$

So
$$\Pr\Big(X \geq \frac{\varepsilon}{2}\Big) \geq \frac{\varepsilon}{2}.$$

(b) By Chebychev's inequality, we have
$$\Pr(|X-\mathbb{E}(x)|\geq t) \leq \frac{\mathrm{Var}(X)}{t^2}.$$

By letting $t=\mathbb{E}(x)>0$ (since $X\geq 0$, and $\mathbb{E}(x)$ is placed in the denominator so it's none-zero), we have:
$$\Pr(X=0) \leq \Pr(X\leq 0 \text{ or } X\geq 2\,\mathbb{E}(x)) \leq \frac{\mathrm{Var}(X)}{\big(\mathbb{E}(X)\big)^2}.$$

Problem 2. Let RandomSign(n) be the distribution of vectors of n entries where each entry is independently chosen to be ± 1 with probability $\frac{1}{2}$. Sample m vectors $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(m)} \sim \text{RandomSign}(n)$. Define the normalized vectors $\mathbf{w}^{(i)} = \mathbf{v}^{(i)}/\sqrt{n}$ so that $\left\|\mathbf{w}^{(i)}\right\| = 1$ for all $i = 1, 2, \ldots, m$. Prove the following claims:

(a) For all $i \neq j$, the inner product $\left\langle \mathbf{w}^{(i)}, \mathbf{w}^{(j)} \right\rangle = \sum_k \mathbf{w}_k^i \mathbf{w}_k^j$ is small with high probability. That is,

$$\Pr\Big(\Big|\Big\langle \mathbf{w}^{(i)}, \mathbf{w}^{(j)} \Big\rangle\Big| \geq \delta\Big) \leq \exp\Big(-\Omega\Big(\delta^2 n\Big)\Big).$$

(b) There exists some $m = \exp(\Omega(\delta^2 n))$ such that the m vectors are pairwise almost-orthogonal with high probability. More precisely,

$$\Pr\left(\left|\left\langle \mathbf{w}^{(i)}, \mathbf{w}^{(j)} \right\rangle\right| \leq \delta \text{ for all pairs } i \neq j\right) \geq 0.99.$$

(Note: By probabilistic method, this proves that there are exponentially many almost-orthogonal unit vectors in $\mathbb{R}^n$ even though there are at most n exactly orthogonal vectors.)

Solution. (a) As $\left\langle \mathbf{w}^{(i)}, \mathbf{w}^{(j)} \right\rangle = \sum_k \mathbf{w}_k^i \mathbf{w}_k^j = \frac{1}{n} \sum_k \mathbf{v}_k^i \mathbf{v}_k^j$, We use random variable $X_k \in \{-1, 1\}$ to denote $\mathbf{v}_k^i \mathbf{v}_k^j$. Then the inner product became the average of X_k's.

To make $\left\langle \mathbf{w}^{(i)}, \mathbf{w}^{(j)} \right\rangle \geq \delta$ we need $\sum_k X_k \geq \delta n$. By Chernoff's inequality, we have:

$$\begin{aligned}
\Pr(\sum_k X_k \geq t) &= \Pr\Big(\exp(\lambda \sum_k X_k) \geq \exp(\lambda t)\Big) = \Pr\Big(\prod_k \exp(\lambda X_k) \geq \exp(\lambda t)\Big) \\
&\leq \frac{\mathbb{E}(\prod_k \exp(\lambda X_k))}{\exp(\lambda t)} = \frac{\mathbb{E}(e^{\lambda X_k})^n}{e^{\lambda t}} \leq \frac{e^{\frac{\lambda^2}{2} n}}{e^{\lambda t}} = e^{\frac{\lambda^2}{2} n - \lambda t} = e^{-\frac{t^2}{2n}}.
\end{aligned}$$

When we choose $\lambda = \frac{t}{n}$.

By taking $t = \delta n$, and by symmetry, we have:

$$\begin{aligned}
&\Pr\Big(\Big|\Big\langle \mathbf{w}^{(i)}, \mathbf{w}^{(j)} \Big\rangle\Big| \geq \delta\Big) = 2\Pr\Big(\Big\langle \mathbf{w}^{(i)}, \mathbf{w}^{(j)} \Big\rangle \geq \delta\Big) \\
&= 2\Pr(\sum_k X_k \geq \delta n) \leq 2e^{-\frac{1}{2}\delta^2 n} = \exp\Big(-\Omega\Big(\delta^2 n\Big)\Big).
\end{aligned}$$

(b) Take $m = \exp(k\delta^2 n) = \exp(\Omega(\delta^2 n))$, then by union bound we have:

$$\begin{aligned}
&\Pr\Big(\Big|\Big\langle \mathbf{w}^{(i)}, \mathbf{w}^{(j)} \Big\rangle\Big| \le \delta \text{ for all pairs } i \neq j\Big) \\
=&1 - \Pr\Big(\Big|\Big\langle \mathbf{w}^{(i)}, \mathbf{w}^{(j)} \Big\rangle\Big| > \delta \text{ for some pairs } i \neq j\Big) \\
=&1 - \Pr\Big(\bigcup_{i\neq j} \Big|\Big\langle \mathbf{w}^{(i)}, \mathbf{w}^{(j)} \Big\rangle\Big| > \delta\Big) \\
\ge&1 - \sum_{i\neq j} \Pr\Big(\Big|\Big\langle \mathbf{w}^{(i)}, \mathbf{w}^{(j)} \Big\rangle\Big| > \delta\Big) \\
=&1 - 2\exp\Big(-\frac{1}{2}\delta^2 n\Big)\binom{m}{2} \\
\ge&1 - \exp\Big(-\frac{1}{2}\delta^2 n\Big)m^2 \\
\ge&1 - \exp\Big(-\frac{1}{2}\delta^2 n + 2k\delta^2 n\Big).
\end{aligned}$$

Take $k = 0.1$ for instance, then for $\delta^2 n > \frac{\log 100}{0.3}$ we have $1 - \exp\Big(-\frac{1}{2}\delta^2 n + 2k\delta^2 n\Big) \ge 0.99$. which means for large enough $\delta^2 n > \frac{\log 100}{0.3}$, $m = \exp(0.1 \times \delta^2 n) = \exp(\Omega(\delta^2 n))$ satisfies demands.

Problem 3. Let RandomGraph(n, p) be the distribution of random graphs of n vertices where, for each pair of vertices u, v, $\{u, v\}$ is chosen as an edge of the graph independently with probability p. Prove the following for such a random graph $G \sim$ RandomGraph(n, p).

(a) If $p = o(n^{-2/3})$,

$$\lim_{n\to\infty} \Pr(G \text{ constains a 4-clique}) = 0.$$

(b) If $p = \omega(n^{-2/3})$,

$$\lim_{n\to\infty} \Pr(G \text{ does not constain a 4-clique}) = 0.$$

(Hint: Use Part (b) of Problem 1 and you need to carefully calculate the probability of 4-cliques occurring simultaneously on vertex sets A and B when $|A \cap B| \ge 2$.)

Solution. (a) The probability that a 4-clique is present is

$$\begin{aligned}\Pr(G \text{ contains a 4-clique}) &= \Pr(\bigcup_{a\neq b\neq c\neq d\in V} abcd \text{ is a four clique}) \\ &= \Pr(\bigcup_{a\neq b\neq c\neq d\in V} ab, ac, ad, bc, bd, cd \text{ are edges}) \\ &\leq \sum_{a\neq b\neq c\neq d\in V} \Pr(ab, ac, ad, bc, bd, cd \text{ are edges}) \\ &= \binom{n}{4} p^6 \leq n^4(o(n^{-2/3}))^6 = o(n^4 n^{-4}) = o(1).\end{aligned}$$

So:

$$\lim_{n\to\infty} \Pr(G \text{ contains a 4-clique}) \leq \lim_{n\to\infty} o(1) = 0.$$

As the probability is non-negative, so the limit is 0.

(b) Consider all subsets with four vertices $S \subset V, |S| = 4$. Let X_S be an indicator valuing 1 when S is a 4-clique else 0. Say the total number of 4-cliques is X, which means "G does not contain a 4-clique" $\iff$ $X = 0$. Then we have:

$$\mathbb{E}(X) = \mathbb{E}(\sum_{S\subset V, |S|=4} X_S) = \sum_{S\subset V, |S|=4} \mathbb{E}(X_S) = \binom{n}{4} p^6.$$

Meanwhile, the variance of X is:

$$\mathrm{Var}(X) = \sum_S \mathrm{Var}(X_S) + \sum_{S_1\neq S_2} \mathrm{Cov}(X_{S_1}, X_{S_2})$$

When $|S_1 \cap S_2| \leq 1$, X_{S_1} and X_{S_2} are independent because the cliques have no common edges, so the covariance is 0. When $|S_1 \cap S_2| = 2$, then there are 11 edges in $S_1 \cup S_2$ if they are both cliques. Thus the covariance is:

$$\mathrm{Cov}(X_{S_1}, X_{S_2}) = \mathbb{E}(X_{S_1} X_{S_2}) - \mathbb{E}(X_{S_1})\,\mathbb{E}(X_{S_2}) = p^{11} - p^{12}.$$

Similarly, if $|S_1 \cap S_2| = 3$, then there are 9 edges in $S_1 \cup S_2$ if they are both cliques. Thus:

$$\mathrm{Cov}(X_{S_1}, X_{S_2}) = \mathbb{E}(X_{S_1} X_{S_2}) - \mathbb{E}(X_{S_1})\,\mathbb{E}(X_{S_2}) = p^9 - p^{12}.$$

And $\mathrm{Var}(X_S) = \mathbb{E}(X_S^2) - (\mathbb{E}(X_S))^2 = p^6 - p^{12}$. So:

$$\begin{aligned}\mathrm{Var}(X) &= \sum_S (p^6 - p^{12}) + \sum_{|S_1\cap S_2|=2} (p^{11} - p^{12}) + \sum_{|S_1\cap S_2|=3} (p^9 - p^{12}) \\ &= \binom{n}{4}(p^6 - p^{12}) + \binom{n}{6}(p^{11} - p^{12}) + \binom{n}{5}(p^9 - p^{12})\end{aligned}$$

Therefore, by the result of 1(b), we have:

$$\begin{aligned}
\lim_{n\to\infty} \Pr(X=0) &\le \lim_{n\to\infty} \frac{\operatorname{Var}(X)}{(\mathbb{E}(X))^2} \\
&= \lim_{n\to\infty} \frac{\binom{n}{4}(p^6 - p^{12}) + \binom{n}{6}(p^{11} - p^{12}) + \binom{n}{5}(p^9 - p^{12})}{\binom{n}{4}^2 p^{12}} \\
&= \lim_{n\to\infty} C\frac{n^4(p^6 - p^{12}) + n^6(p^{11} - p^{12}) + n^5(p^9 - p^{12})}{n^8 p^{12}} \\
&= \lim_{n\to\infty} C\Big(n^{-4}p^{-6} - n^{-4} + n^{-2}p^{-1} - n^{-2} + n^{-3}p^{-3} - n^{-3}\Big)
\end{aligned}$$

Where C is constant.

Since $p = \omega(n^{-2/3})$, we know that $p^{-1} = o(n^{2/3})$. So we have:

$$\begin{aligned}
\lim_{n\to\infty} \Pr(X=0) &\le \lim_{n\to\infty} C\Big(n^{-4}p^{-6} - n^{-4} + n^{-2}p^{-1} - n^{-2} + n^{-3}p^{-3} - n^{-3}\Big) \\
&= \lim_{n\to\infty} C\Big(o(1) - n^{-4} + o(n^{-4/3}) - n^{-2} + o(n^{-1}) - n^{-3}\Big) \\
&= \lim_{n\to\infty} Co(1) \\
&= 0.
\end{aligned}$$

As the probability is non-negative, so the limit is 0.