Introduction to Theoretical Computer Science — Zhang Chi
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Homework 1

Problem 1. Decide whether the following is a λ term (or an abbreviation of a λ term). If it is not, explain the reason.

(a) $\lambda x.xxx$

(b) $\lambda\lambda x.x$

(c) $\lambda y.(\lambda x.x)$

(d) $\lambda uv.(((xxy)xy)y)$

Solution. (a) Yes.

(b) No. two λ in a row unseparated isn't allowed by the definition of λ term.

(c) Yes.

(d) Yes.

Problem 2. Compute the terms represented by the following substitutions:

(a) $(xyz)[y/z]$.

(b) $(\lambda x.x)[y/z]$.

(c) $(\lambda y.xy)[yy/x]$.

Solution. (a) $(xyz)[y/z] = xyy$.

(b) $(\lambda x.x)[y/z] = \lambda x.x$.

(c) $(\lambda y.xy)[yy/x] = \lambda z.xz[yy/x] = \lambda z.yyz$.

Problem 3. Prove the following equalities in the theory of $\lambda\beta$. You need to draw the "proof tree" using the rules we defined in the lecture.

(a) $\lambda uv.v = \lambda u.(\lambda x.x)(\lambda u.u)$.

(b) $\lambda uv.(\lambda xy.y)uv = \lambda ab.b$.

Solution. (a) To make sure that the width of the proof tree does not exceed the paper's edge, we split the proof tree into two parts to complete it.

$$\text{(abs)}\ \dfrac{\text{(trans)}\ \dfrac{\text{(appl)}\ \dfrac{\text{(refl)}\ \dfrac{}{\lambda x.x = \lambda x.x} \qquad (\alpha)\ \dfrac{}{\lambda u.u = \lambda v.v}}{(\lambda x.x)(\lambda u.u) = (\lambda x.x)(\lambda v.v)} \qquad (\beta)\ \dfrac{}{(\lambda x.x)(\lambda v.v) = \lambda v.v}}{(\lambda x.x)(\lambda u.u) = \lambda v.v}}{\lambda u.(\lambda x.x)(\lambda u.u) = \lambda u.(\lambda v.v)}$$

$$\text{(sym)}\ \dfrac{\text{(trans)}\ \dfrac{\text{(abs)}\ \dfrac{\text{[above]}}{\lambda u.(\lambda x.x)(\lambda u.u) = \lambda u.(\lambda v.v)} \qquad \text{(abbr)}\ \dfrac{}{\lambda u.(\lambda v.v) = \lambda uv.v}}{\lambda u.(\lambda x.x)(\lambda u.u) = \lambda uv.v}}{\lambda uv.v = \lambda u.(\lambda x.x)(\lambda u.u)}$$

(b)

$$\text{(trans)}\ \dfrac{\text{(abs)}\ \dfrac{\text{(trans)}\ \dfrac{\text{(appl)}\ \dfrac{(\beta)\ \dfrac{}{(\lambda xy.y)u = \lambda y.y} \qquad \text{(refl)}\ \dfrac{}{v = v}}{(\lambda xy.y)uv = (\lambda y.y)v} \qquad (\beta)\ \dfrac{}{(\lambda y.y)v = v}}{(\lambda xy.y)uv = v}}{\lambda uv.(\lambda xy.y)uv = \lambda uv.v} \qquad (\alpha)\ \dfrac{}{\lambda uv.v = \lambda ab.b}}{\lambda uv.(\lambda xy.y)uv = \lambda ab.b}$$

Problem 4.

(a) Find a λ term s such that the equality $stu = ut$ holds in $\lambda\beta$ for all terms t and u.

(b) Show that there is a λ term s such that for all term t, $\lambda\beta \vdash st = ss$.

Solution. (a) Consider $s = \lambda xy.yx$, then $\lambda\beta \vdash stu = (\lambda xy.yx)tu = \lambda x.(\lambda y.yx)tu = (\lambda y.yt)u = ut, \forall t, u \in \Lambda$.

(b) Consider $s = \lambda x.y$, then $\lambda\beta \vdash st = (\lambda x.y)t = y, \forall t \in \Lambda$. Meanwhile $\lambda\beta \vdash ss = (\lambda x.y)s = y$, so $\lambda\beta \vdash st = ss, \forall t \in \Lambda$.

Problem 5. Show that there is a term G such that all fixed-point combinators can be *characterized* as the fixed points of G. That is, s is a fixed-point combinator if and only if $\lambda\beta \vdash Gs = s$.

Solution. Consider $G = \lambda fr.r(fr)$. We'll first prove that all the fixed points of G are FPCs, and then prove the converse. Say s is G's fixed point, then it satisfies $\lambda\beta \vdash s = Gs = (\lambda fr.r(fr))s = \lambda r.r(sr)$. So $\forall t \in \Lambda$, $\lambda\beta \vdash st = (\lambda r.r(sr))t = t(st)$, which means s is a FPC.

Conversely, if s is a FPC, giving $\lambda\beta \vdash st = (\lambda r.r(sr))t = t(st), \forall t \in \Lambda$. We suppose s has the form $\lambda x.w$. Then $\lambda\beta \vdash Gs = (\lambda fr.r(fr))s = \lambda r.r(sr) =_{FPC} \lambda r.sr = \lambda r.(\lambda x.w)r =_\beta \lambda r.(w[r/x]) =_\alpha^* \lambda x.(w[x/x]) = \lambda x.w = s$. In the * step, the α conversion can be applied because x exists no more in $w[r/x]$ and thus is not a free variable, so renaming the bounded variable to x raises no conflict.