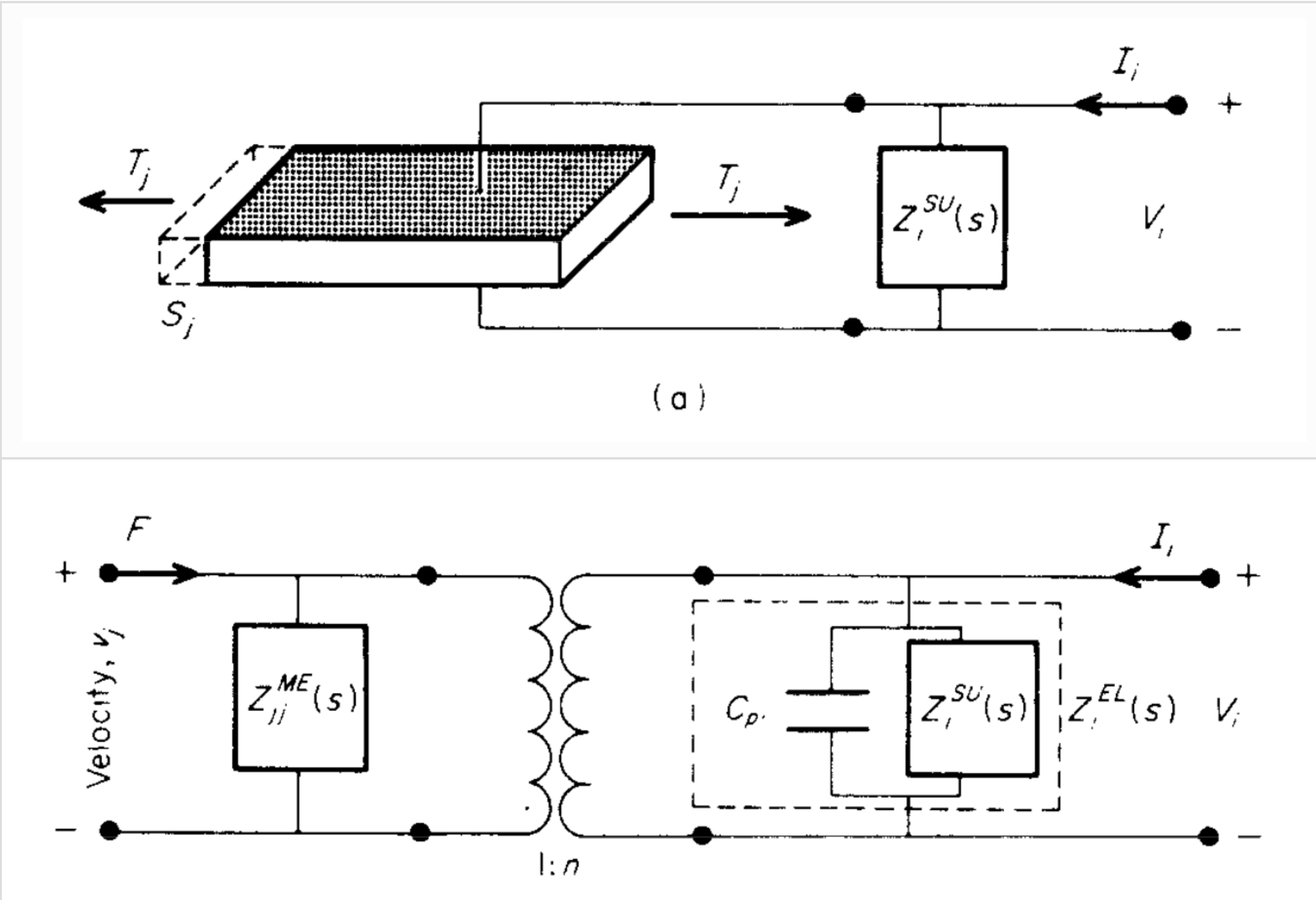
1. 1 Beam with connecting circuit



The constitutive law is

$$\begin{bmatrix}\mathbf{D}\\ \mathbf{S}\end{bmatrix}=\begin{bmatrix}\mathbf{e}^T & \mathbf{d}\\ \mathbf{d}_t & \mathbf{s}^E\end{bmatrix}\begin{bmatrix}\mathbf{E}\\ \mathbf{T}\end{bmatrix} \tag{1}$$

Assume the field in the piezo patch is uniform, we have the relation after Laplace transform

$$\mathbf{V}(s)=\mathbf{L}\cdot\mathbf{E}(s),\quad \mathbf{I}(s)=s\mathbf{A}\cdot\mathbf{D}(s) \tag{2}$$

Then the constitutive law can be transformed as

$$\begin{bmatrix}\mathbf{I}\\ \mathbf{S}\end{bmatrix}=\begin{bmatrix}s\mathbf{A}\varepsilon^T\mathbf{L}^{-1} & s\mathbf{A}\mathbf{d}\\ \mathbf{d}_t\mathbf{L}^{-1} & \mathbf{s}^{\mathrm{E}}\end{bmatrix}\begin{bmatrix}\mathbf{V}\\ \mathbf{T}\end{bmatrix} \tag{3}$$

We define the capacity of piezo patch as

$$A_i\varepsilon_i^T/L_i=C_{pi}^T \tag{4}$$

Then Eq. (3) can be rewritten as

$$\begin{bmatrix} \mathbf{I} \\ \mathbf{S} \end{bmatrix} = \begin{bmatrix} s\mathbf{C}_p^T & s\mathbf{Ad} \\ \mathbf{d}_t\mathbf{L}^{-1} & \mathbf{s}^E \end{bmatrix} \begin{bmatrix} \mathbf{V} \\ \mathbf{T} \end{bmatrix} = \begin{bmatrix} \mathbf{Y}^D(s) & s\mathbf{Ad} \\ \mathbf{d}_t\mathbf{L}^{-1} & \mathbf{s}^E \end{bmatrix} \begin{bmatrix} \mathbf{V} \\ \mathbf{T} \end{bmatrix}, \tag{5}$$

where $\mathbf{Y}^D(s)$ is the open circuit admittance of the piezoelectric patch.
For shunted piezoelectric applications,

$$\begin{bmatrix} \mathbf{I} \\ \mathbf{S} \end{bmatrix} = \begin{bmatrix} \mathbf{Y}^{EL} & s\mathbf{Ad} \\ \mathbf{d_t}\mathbf{L}^{-1} & \mathbf{S}^E \end{bmatrix} \begin{bmatrix} \mathbf{V} \\ \mathbf{T} \end{bmatrix} \text{where} \quad \mathbf{Y}^{EL} = \mathbf{Y}^D + \mathbf{Y}^{SU} \tag{6}$$

The first equation gives

$$\mathbf{V} = \left(\mathbf{Z}^{EL}\right)\mathbf{I} - \left(\mathbf{Z}^{EL} s\mathbf{Ad}\right)\mathbf{T} \tag{7}$$

and inserting into second equation gives

$$\mathbf{S} = \left[\mathbf{s}^E - \mathbf{d}_t\mathbf{L}^{-1}\mathbf{Z}^{EL} s\mathbf{Ad}\right]\mathbf{T} + \left[\mathbf{d}_t\mathbf{L}^{-1}\mathbf{Z}^{EL}\right]\mathbf{I}. \tag{8}$$

The mechanical compliance is

$$\mathbf{s}^{SU} = \left[\mathbf{s}^E - \mathbf{d}_t\mathbf{L}^{-1}\mathbf{Z}^{EL}\boldsymbol{sAd}\right]. \tag{9}$$

Upon noting that with constant stress

$$\begin{gathered} \mathbf{Z}^E(\mathrm{~s}) = \mathbf{0} = \text{ short circuit electrical impedance,} \\ \mathbf{Z}^D(s) = \left(\mathbf{C}_p^T s\right)^{-1} = \text{ open circuit electrical impedance,} \end{gathered} \tag{10}$$

and that

$$s\mathbf{L}^{-1}\boldsymbol{\varepsilon}^T\mathbf{A} = \mathbf{C}_p^T s, \tag{11}$$

equation (16) can be put in the form

$$\mathbf{s}^{SU} = \left[\mathbf{s}^E - \mathbf{d_t}\overline{\mathbf{Z}}^{EL}\left(\boldsymbol{\varepsilon}^T\right)^{-1}\mathbf{d}\right], \tag{12}$$

where the matrix of non-dimensional electrical impedances is defined as

$$\overline{\mathbf{Z}}^{EL} = \mathbf{Z}^{EL}\left(\mathbf{Z}^D\right)^{-1} = \left(s\mathbf{C}_p^T + \mathbf{Y}^{SU}\right)^{-1} s\mathbf{C}_p^T \tag{13}$$

Here, we introduce the electromechanical coupling coefficients

$$\begin{gathered} \text{shear, } k_{15} = d_{15}/\sqrt{s_{55}^E\varepsilon_1^T} = k_{24}, \quad \text{ transverse, } k_{31} = d_{31}/\sqrt{s_{11}^E\varepsilon_3^T} = k_{32}, \\ \text{longitudinal, } k_{33} = d_{33}/\sqrt{s_{33}^E\varepsilon_3^T}, \end{gathered} \tag{14}$$

or, in the notation used, for force in the j th direction and field in the i th direction

$$k_{ij} = d_{ij}/\sqrt{s_{jj}^E\varepsilon_i^T}. \tag{15}$$

Substituting equation (25) into (23) gives

$$s_{ji}^{SU} = s_{jj}^{E}\left[1 - k_{ij}^{2}\bar{Z}_{i}^{EL}\right]. \tag{16}$$

1.1. 1.1 Shunted with negative capacity

For our problem, the electric field is along z direction and stress is along x direction. So we have

$$s_{33}^{SU} = s_{33}^{E}\left[1 - k_{31}^{2}\bar{Z}_{1}^{EL}\right] = s_{33}^{E}\left[1 - \frac{sC_p^T k_{31}^2}{sC_p^T + Y^{SU}}\right] = s_{33}^{E}\left[\frac{sC_p^T(1-k_{31}^2) + Y^{SU}}{sC_p^T + Y^{SU}}\right] \tag{17}$$

Then we have

$$E_{\mathrm{p}}^{\mathrm{SU}}(\omega) = E_{\mathrm{p}}^{E}\frac{\mathrm{i}\omega C_{\mathrm{p}}^{T} + Y^{\mathrm{SU}}}{\mathrm{i}\omega C_{\mathrm{p}}^{T}\left(1-k_{31}^{2}\right) + Y^{\mathrm{SU}}} \tag{18}$$

where $s = i\omega$.

If shunted with a negative capacity, we have

$$Y^{SU} = -i\omega C' = -i\omega H C_0 \tag{19}$$

So the shunting Young's modulus is (see reference [1])

$$E_{\mathrm{p}}^{\mathrm{SU}}(\omega) = E_{\mathrm{p}}^{E}\frac{C' - C_{\mathrm{p}}^{T}}{C' - C_{\mathrm{p}}^{T}\left(1-k_{31}^{2}\right)} \tag{20}$$

or

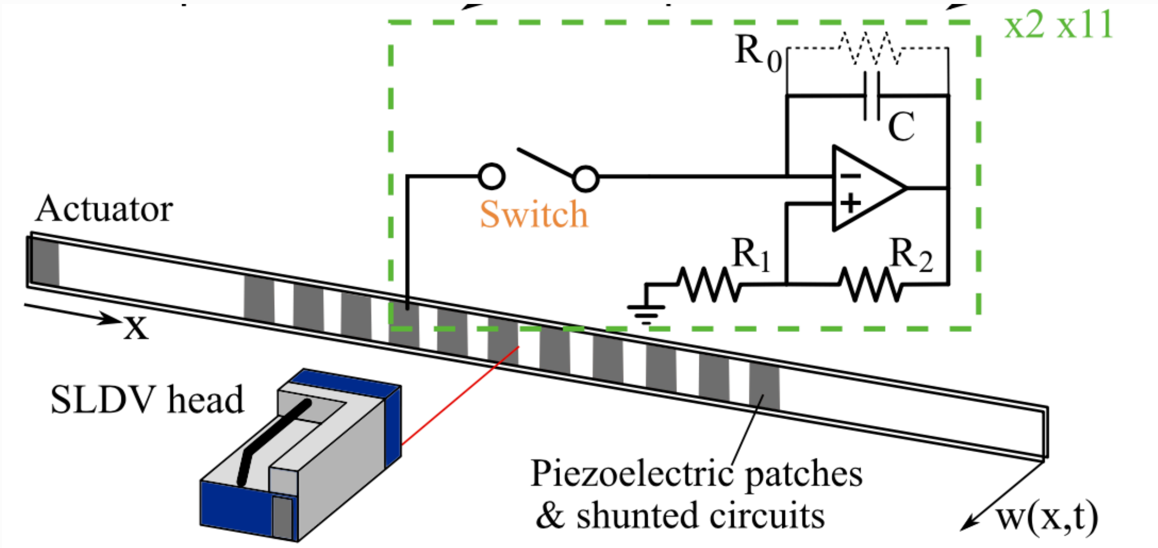
$$E_{\mathrm{p}}^{\mathrm{SU}}(\omega) = E_{\mathrm{p}}^{E}\frac{C' - C_{\mathrm{p}}^{T}}{C' - C_{\mathrm{p}}^{S}} \tag{21}$$

where $C_p^S = (1-k_{31}^2)C_p^T$, $C_p^T = A\varepsilon_{33}^T/d$, which is the result from reference [3,4].

Finally,

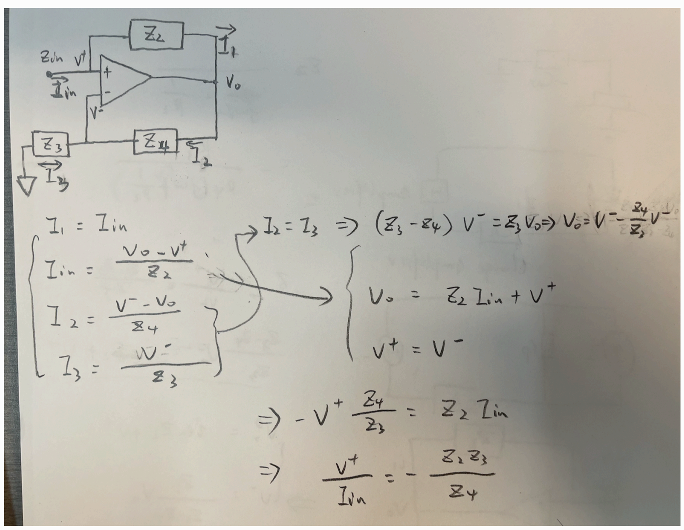
$$\begin{aligned} E_{eq} &= \frac{E_b I_b + 2E_p^{SU} I_p}{I_b + 2I_p} \\ I_b &= \frac{bH^3}{12}, \quad I_p = \frac{bh_p^3}{12} + bh_p\left(\frac{H}{2} + \frac{h_p}{2}\right)^2 \end{aligned} \tag{22}$$

1.2. 1.2 Shunting circuit



Circuit diagram

The impedance is derived in the following picture.



Derivation of impedance

Now we have

$$Z_2=\frac{1}{j\omega C+1/R_0},\quad Z_3=R_1,\quad Z_4=R_2 \tag{23}$$

Then we have impedance

$$Z_{\text{in}}=-\frac{R_1}{R_2\left(j\omega C+1/R_0\right)}=\frac{1}{j\omega C_N} \tag{24}$$

where R_0 is very big and $1/R_0$ can be neglected. The negative capacity is

$$C_N = -\frac{R_2}{R_1}C \tag{25}$$

2. 2 Reference

[1] Hagood, N. W., & Von Flotow, A. (1991). Damping of structural vibrations with piezoelectric materials and passive electrical networks. *Journal of sound and vibration*, *146*(2), 243-268.

[2] Beck, B. S., Cunefare, K. A., & Collet, M. (2013). The power output and efficiency of a negative capacitance shunt for vibration control of a flexural system. *Smart Materials and Structures*, *22*(6), 065009.

[3] Trainiti, G., Xia, Y., Marconi, J., Cazzulani, G., Erturk, A., & Ruzzene, M. (2019). Time-periodic stiffness modulation in elastic metamaterials for selective wave filtering: Theory and experiment. *Physical review letters*, *122*(12), 124301.