1. 1 Questions

How to make equivalence from real structures?

2. 2 Vibration mode

2.1. 2.1 Longitudinal vibration

Area: A, Length: L.

Governing equation:

$$\rho u_{tt} = E u_{xx} \tag{1}$$

Frequency of lowest mode under fixed-fixed, free-free condition:

$$\omega = \sqrt{\frac{E}{\rho}}\frac{\pi}{L} \tag{2}$$

Table 9.1 Boundary conditions of a bar in longitudinal vibration.

End conditions of the bar	Boundary conditions	Frequency equation	Mode shapes (normal functions)	Natural frequencies of vibration
1. Fixed–free	$u(0,t)=0$; $\frac{\partial u}{\partial x}(l,t)=0$	$\cos\frac{\omega l}{c}=0$	$U_n(x) = C_n \sin\frac{(2n+1)\pi x}{2l}$	$\omega_n = \frac{(2n+1)\pi c}{2l}$, $n=0,1,2,\ldots$
2. Fixed–fixed	$u(0,t)=0$; $u(l,t)=0$	$\sin\frac{\omega l}{c}=0$	$U_n(x) = C_n \sin\frac{n\pi x}{l}$	$\omega_n = \frac{n\pi c}{l}$, $n=1,2,3,\ldots$
3. Fixed–attached mass	$u(0,t)=0$; $EA\frac{\partial u}{\partial x}(l,t) = -M\frac{\partial^2 u}{\partial t^2}(l,t)$	$\alpha\tan\alpha=\beta$; $\alpha=\frac{\omega l}{c}$; $\beta=\frac{m}{M}$	$U_n(x) = C_n \sin\frac{\omega_n x}{c}$	$\omega_n = \frac{\alpha_n c}{l}$, $n=1,2,3,\ldots$
4. Fixed–attached spring	$u(0,t)=0$; $EA\frac{\partial u}{\partial x}(l,t) = -ku(l,t)$	$\alpha\tan\alpha=-\gamma$; $\alpha=\frac{\omega l}{c}$; $\gamma=\frac{m\omega^2}{k}$	$U_n(x) = C_n \sin\frac{\omega_n x}{c}$	$\omega_n = \frac{\alpha_n c}{l}$, $n=1,2,3,\ldots$
5. Fixed–attached spring and mass	$u(0,t)=0$; $EA\frac{\partial u}{\partial x}(l,t) = -M\frac{\partial^2 u}{\partial t^2}(l,t) - ku(l,t)$	$\alpha\cot\alpha=\frac{\alpha^2}{\beta}-\frac{k}{k_0}$; $\beta=\frac{m}{M}$; $k_0=\frac{AE}{l}$	$U_n(x) = C_n \sin\frac{\omega_n x}{c}$	$\omega_n = \frac{\alpha_n c}{l}$, $n=1,2,3,\ldots$
6. Free–free	$\frac{\partial u}{\partial x}(0,t)=0$; $\frac{\partial u}{\partial x}(l,t)=0$	$\sin\frac{\omega l}{c}=0$	$U_n(x) = C_n \cos\frac{n\pi x}{l}$	$\omega_n = \frac{n\pi c}{l}$, $n=0,1,2,\ldots$
7. Free–attached mass	$\frac{\partial u}{\partial x}(0,t)=0$; $EA\frac{\partial u}{\partial x}(l,t) = -M\frac{\partial^2 u}{\partial t^2}(l,t)$	$\tan\alpha=-\alpha\beta$	$U_n(x) = C_n \cos\frac{\omega_n x}{c}$	$\omega_n = \frac{\alpha_n c}{l}$, $n=1,2,3,\ldots$
8. Free–attached spring	$\frac{\partial u}{\partial x}(0,t)=0$; $EA\frac{\partial u}{\partial x}(l,t) = -ku(l,t)$	$\alpha\cot\alpha=\delta$; $\delta=\frac{AE}{lk}$	$U_n(x) = C_n \cos\frac{\omega_n x}{c}$	$\omega_n = \frac{\alpha_n c}{l}$, $n=1,2,3,\ldots$

Frequency of lowest mode under fixed-free condition:

$$\omega = \sqrt{\frac{E}{\rho}}\frac{\pi}{2L} \tag{3}$$

> Rao, S. S. (2019). *Vibration of continuous systems*. John Wiley & Sons.

2.2. 2.2 Flexural vibration

The governing equation is

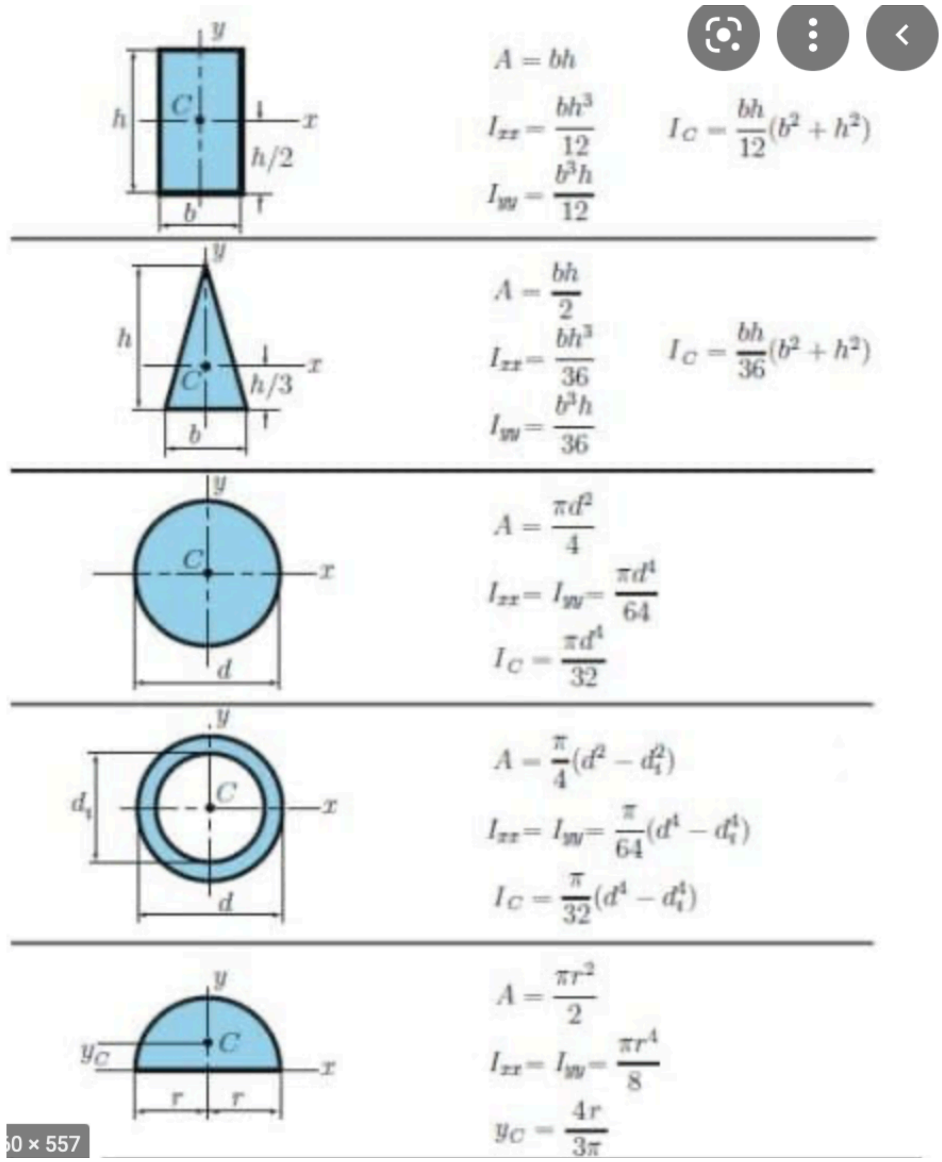
$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) + \rho A\frac{\partial^2 w}{\partial t^2} = f(x,t) \tag{4}$$

where

$$I = I_y = \iint_A z^2 dA \tag{5}$$

Lowest mode for beam simply supported at both ends

$$\omega = (\beta_1 l)^2\left(\frac{EI}{\rho A l^4}\right)^{1/2} = \pi^2\left(\frac{EI}{\rho A L^4}\right)^{1/2}. \tag{6}$$



2.3. 2.3 Torsional Vibration of Shafts

The governing equation is

$$\frac{\partial}{\partial x}\left(GJ\frac{\partial \theta(x,t)}{\partial x}\right) + m_t(x,t) = I_0\frac{\partial^2 \theta(x,t)}{\partial t^2} \tag{7}$$

where $I_0 = \rho J$, $J = \int_A r^2 dA$. Uniform and without attached mass m_t, we have

$$G\frac{\partial^2 \theta(x,t)}{\partial x^2} = \rho\frac{\partial^2 \theta(x,t)}{\partial t^2} \tag{8}$$

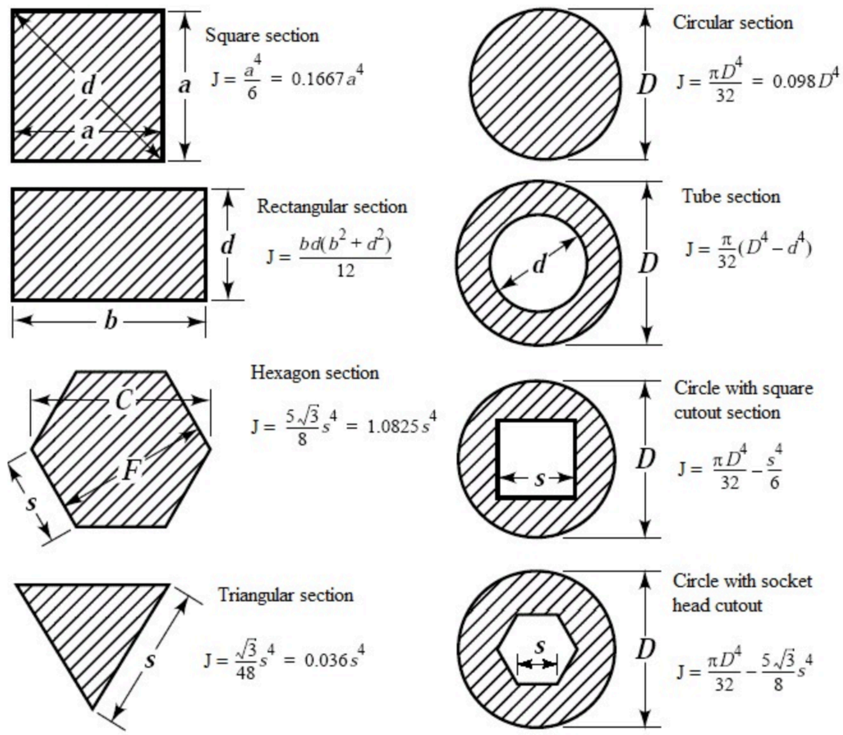
Frequency of lowest mode under fixed-fixed, free-free condition:

$$\omega = \sqrt{\frac{G}{\rho}}\frac{\pi}{L} \tag{9}$$

Frequency of lowest mode under fixed-free condition:

$$\omega = \sqrt{\frac{G}{\rho}}\frac{\pi}{2L} \tag{10}$$

where



2.4. 2.4 Shear vibration

Lowest frequency for shear mode is

$$\omega = \frac{\pi}{2b}\sqrt{\frac{\mu}{\rho}} \tag{11}$$

where b is half of the thickness d. Or

$$\omega = \frac{\pi}{4d}\sqrt{\frac{\mu}{\rho}} \tag{12}$$

where d is the thickness.

2.5. 2.5 Bending vibration of the plate

Governing equation is

$$D\nabla^4 w + \rho h\ddot{w} - f = 0 \tag{13}$$

in which D represents the flexural rigidity of the plate:

$$D = \frac{Eh^3}{12\left(1-\nu^2\right)} \tag{14}$$

Solution for a simply supported rectangular plate is

$$\omega_{mn} = \lambda_{mn}^2 \left(\frac{D}{\rho h}\right)^{1/2} = \pi^2 \left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right] \left(\frac{D}{\rho h}\right)^{1/2} \tag{15}$$

The lowest mode (if $a > b$) is

$$\omega = \pi^2 \left(\frac{1}{a}\right)^2 \left(\frac{D}{\rho h}\right)^{1/2} \tag{16}$$

Solution for circular plate is

$$\omega_{mn} = \lambda_{mn}^2 \left(\frac{D}{\rho h}\right)^{1/2} \tag{17}$$

where λ_{mn} is the solution of the following equation

$$I_m(\lambda a) J_{m-1}(\lambda a) - J_m(\lambda a) I_{m-1}(\lambda a) = 0, \quad m = 0, 1, 2, \ldots \tag{18}$$

The lowest root is $\lambda_{01} a = 3.196$. Therefore, the frequency of lowest mode is

$$\omega = \left(\frac{3.196}{a}\right)^2 \left(\frac{D}{\rho h}\right)^{1/2} \tag{19}$$