Algebraic letters and NMHV last entry conditions from $\bar{Q}$-equation

Based on oncoming and recent works with Song He and Zhenjie Li

Chi Zhang

September 13, 2020

Institute of Theoretical Physics, CAS

Last entry conditions

N^2MHV Yangian invariants

The N^2MHV Yangian invariants have already been classified.
[Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka]

They are

$$Y_1^{(2)} = [1, 2, (23) \cap (456), (234) \cap (56), 6][2, 3, 4, 5, 6]$$
$$Y_2^{(2)} = [1, 2, (34) \cap (567), (345) \cap (67), 7][3, 4, 5, 6, 7]$$
$$Y_3^{(2)} = [1, 2, 3, (345) \cap (67), 7][3, 4, 5, 6, 7]$$
$$Y_4^{(2)} = [1, 2, 3, (456) \cap (78), 8][4, 5, 6, 7, 8]$$
$$Y_5^{(2)} = [1, 2, 3, 4, 8][4, 5, 6, 7, 8]$$
$$Y_6^{(2)} = [1, 2, 3, (45) \cap (678), 8][4, 5, 6, 7, 8]$$
$$Y_7^{(2)} = [1, 2, 3, (45) \cap (678), (456) \cap (78)][4, 5, 6, 7, 8]$$
$$Y_8^{(2)} = [1, 2, 3, 4, (456) \cap (78)][4, 5, 6, 7, 8]$$
$$Y_9^{(2)} = [1, 2, 3, 4, 9][5, 6, 7, 8, 9]$$
$$Y_{10}^{(2)} = [1, 2, 3, 4, (567) \cap (89)][5, 6, 7, 8, 9]$$
$$Y_{11}^{(2)} = [1, 2, 3, 4, (56) \cap (789)][5, 6, 7, 8, 9]$$
$$Y_{12}^{(2)} = \varphi[1, 2, 3, (45) \cap (789), (46) \cap (789)][(45) \cap (123), (46) \cap (123), 7, 8, 9]$$
$$Y_{13}^{(2)} = [1, 2, 3, 4, 5][6, 7, 8, 9, 10]$$
$$Y_{14}^{(2)} = \psi[A, 1, 2, 3, 4][B, 5, 6, 7, 8]$$

where

$$(ij) \cap (klm) = \mathcal{Z}_i \langle j\,k\,l\,m \rangle - \mathcal{Z}_j \langle i\,k\,l\,m \rangle$$

NMHV last entry conditions

For N^{2}MHV yangian invariants, this operation gives three kinds of last entries

1.

$$[i\,j\,k\,l\,m]\bar{Q}\log\frac{\langle\bar{n}a\rangle}{\langle\bar{n}b\rangle}$$

2.

$$[1\,i_1\,i_2\,i_3\,i_4]\,\bar{Q}\log\frac{\langle 1(n{-}1\,n)(i_1\,i_2)(i_3\,i_4)\rangle}{\langle\bar{n}i_1\rangle\langle 1i_2i_3i_4\rangle}\,,\quad [i_1\,i_2\,i_3\,i_4\,n{-}1]\,\bar{Q}\log\frac{\langle n{-}1(n\,1)(i_1\,i_2)(i_3\,i_4)\rangle}{\langle\bar{n}i_1\rangle\langle n{-}1\,i_2i_3i_4\rangle}$$

$$[i_1\,i_2\,i_3\,i_4\,n]\,\bar{Q}\log\frac{\langle n(1\,n{-}1)(i_1\,i_2)(i_3\,i_4)\rangle}{\langle\bar{n}i_1\rangle\langle n\,i_2i_3i_4\rangle}\quad\text{with } 1<i_1<i_2<i_3<i_4<n{-}1$$

where $\langle a(bc)(de)(fg)\rangle := \langle abde\rangle\langle acfg\rangle - \langle acde\rangle\langle abfg\rangle$

3.

$$[i_1\,i_2\,i_3\,i_4\,i_5]\,\bar{Q}\log\frac{\langle\bar{n}(i_1i_2)\cap(i_3i_4i_5)\rangle}{\langle\bar{n}i_1\rangle\langle i_2i_3i_4i_5\rangle}\,,\;\; [i_1\,i_2\,i_3\,i_4\,i_5]\,\bar{Q}\log\frac{\langle\bar{n}(i_1i_2i_3)\cap(i_4i_5)\rangle}{\langle\bar{n}i_1\rangle\langle i_2i_3i_4i_5\rangle}\,,$$

with $1<i_1<i_2<i_3<i_4<i_5<n$

Algebraic letters and words in two-loop NMHV amplitudes

Input

To compute the 2-loop NMHV n point BDS-normalized amplitude, we need the input of the one-loop N^{2}MHV BDS-normalized amplitude $R^{(1)}_{n+1,2}$, which can be obtained from the chiral/scalar box expansion [Bourjaily, Caron-Huot, Trnka]:

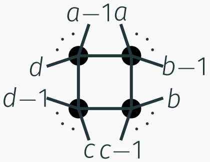
$$R^{(1)}_{n+1,2} = \sum_{a<b<c<d} (f_{a,b,c,d} - R^{\text{tree}}_{n+1,2} f^{\text{MHV}}_{a,b,c,d}) \mathcal{I}^{\text{fin}}_{a,b,c,d}$$

where

- $f_{a,b,c,d}$ are linear combinations of N^{2}MHV yangian invariants
- $f^{\text{MHV}}_{a,b,c,d}$ are either 1 or 0
- $\mathcal{I}^{fin}_{a,b,c,d}$ denote the finite part of DCI-regulated box integrals

Four-mass box

The most generic term in the scalar box expansion:



$$\begin{cases} u_{abcd} = \frac{x_{ab}^2 x_{cd}^2}{x_{ac}^2 x_{bd}^2},\ v_{abcd} = \frac{x_{ad}^2 x_{bc}^2}{x_{ac}^2 x_{bd}^2},\ \Delta_{abcd} = \sqrt{(1-u-v)^2 - 4uv} \\ z_{abcd} = \frac{1}{2}(1+u-v+\Delta),\ \bar{z}_{a,b,c,d} = \frac{1}{2}(1+u-v+\Delta), \end{cases}$$

For such a box,

$$f_{a,b,c,d} = \sum_{\pm} \frac{1-u-v\pm\Delta}{2\Delta}[\alpha_\pm, b-1, b, c-1, c][\delta_\pm, d-1, d, a-1, a]$$

$$\mathcal{I}^{\text{fin}}_{a,b,c,d} = \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \frac{1}{2}\log(z\bar{z})\log\frac{1-z}{1-\bar{z}}$$

where $\alpha_\pm$ and $\delta_\pm$ are two solutions of Schubert problem
$\alpha = (a-1\,a)\cap(d\,d-1\,\gamma)$, $\gamma = (c-1\,c)\cap(b\,b-1\,\alpha)$

The square root will disappear when one mass corner become massless, *e.g.* $b = a+1$

Rationalize the square root Δ

Only $f_{a,b,c,n+1}$ and $f_{1,b,c,n}$ survive under the $\mathrm{d}^{2|3}Z_{n+1}$ integration, however only the first kind is an obstacle since

$$\Delta_{1,b,c,n} \xrightarrow{\mathcal{Z}_{n+1}||\mathcal{Z}_n} 1 - u_{1,b,c,n}$$
$$\Delta_{a,b,c,n+1} \xrightarrow{\mathcal{Z}_{n+1}||\mathcal{Z}_n} \frac{\sqrt{A\tau^2 + B\tau + C}}{\tau + u_{1,b,c,n}}$$

A, B, C: Some quadratic polynomials of cross ratios.

This is just the classic problem to find a rational parameterization of a quadratic curve $y^2 = x^2 + ax + b$.
This square root can be rationalized by a variable substitution $t(\tau)$:

$$\int_0^\infty R(\tau, \Delta(\tau))\,\mathrm{d}\tau \xrightarrow{t(\tau)} \int_{\Delta_{abcn}}^{\Delta_{1abc}} R'(t)\,\mathrm{d}t$$

Algebraic letters of two-loop NMHV amplitudes

1-D τ-integrals for these four masses introduce new algebraic letters[1]

$$\mathcal{X}^*_{a,b,c,d} := \frac{(x^*_{a,b,c,d}+1)^{-1}-\bar{z}_{d,a,b,c}}{(x^*_{a,b,c,d}+1)^{-1}-z_{d,a,b,c}}\,, \qquad \widetilde{\mathcal{X}}^*_{a,b,c,d} := \frac{(x^*_{a,b,c,d-1}+1)^{-1}-z_{d,a,b,c}}{(x^*_{a,b,c,d-1}+1)^{-1}-\bar{z}_{d,a,b,c}}$$

with 6 choices $a-1, a, b-1, b, c-1, c$ of the superscript, where

$$x^a_{a,b,c,d} = \frac{\langle \bar{d}(c-1\,c)\cap(a\,b-1\,b)\rangle}{\langle \bar{d}\,a\rangle\langle b-1\,b\,c-1\,c\rangle}\,, \qquad x^{a-1}_{a,b,c,d} = x^a_{a,b,c,d}|_{a\leftrightarrow a-1}$$

$$x^b_{a,b,c,d} = \frac{\langle \bar{d}(c-1\,c)\cap(a-1\,a\,b)\rangle}{\langle \bar{d}(a-1\,a)\cap(b\,c-1\,c)\rangle}\,, \qquad x^{b-1}_{a,b,c,d} = x^b_{a,b,c,d}|_{b\leftrightarrow b-1}$$

$$x^c_{a,b,c,d} = \frac{\langle \bar{d}\,c\rangle\langle a-1\,a\,b-1\,b\rangle}{\langle \bar{d}(a-1\,a)\cap(b-1\,b\,c)\rangle}\,, \qquad x^{c-1}_{a,b,c,d} = x^c_{a,b,c,d}|_{c\leftrightarrow c-1}$$

Note that $\mathcal{X}^*_{a,b,c,d}$, $\mathcal{X}^*_{b,c,d,a}$, $\mathcal{X}^*_{c,d,a,b}$ and $\mathcal{X}^*_{d,a,b,c}$ involve the same square root $\Delta_{a,b,c,d}$

[1]The algebraic letters always can and will be written in terms of $(a+\Delta)/(a-\Delta)$ such that their multiplicative relations don't involve rational letters.

Counting

Naïvely, there would be $12 \times 4 + 2 = 50$ letters associated with the same $\Delta_{a,b,c,d}$.

However, some degeneracy happens when some mass corners only involve 2 particles, for example

$$\mathcal{X}^{d+1}_{d+2,b,c,d} = \frac{\bar{z}_{d,d+2,b,c}}{z_{d,d+2,b,c}}\,, \qquad \widetilde{\mathcal{X}}^{d-2}_{a,b,d-2,d} = \frac{1 - z_{d,a,b,d-2}}{1 - \bar{z}_{d,a,b,d-2}}\,.$$

This leave us $50 - 2m$ algebraic letters associated with the same Δ where

$$m = \text{ the number of corners that contain only two particles}$$

These $\mathcal{X}$'s and $\widetilde{\mathcal{X}}$'s, together with $z/\bar{z}$ and $(1-z)/(1-\bar{z})$ give a cyclic and reflection invariant set of algebraic letters.

Multiplicative relations among algebraic letters

33 multiplicative relations:

$$\frac{\mathcal{X}^{a-1}_{a,b,c,d}}{\mathcal{X}^{a}_{a,b,c,d}} = \frac{\mathcal{X}^{a}_{d,a,b,c}}{\mathcal{X}^{a-1}_{d,a,b,c}}\,, \quad \frac{\mathcal{X}^{a-1}_{d,a,b,c}}{\mathcal{X}^{a}_{d,a,b,c}} = \frac{\mathcal{X}^{a}_{c,d,a,b}}{\mathcal{X}^{a-1}_{c,d,a,b}}\,,$$

$$\frac{\widetilde{\mathcal{X}}^{a-1}_{a,b,c,d}}{\widetilde{\mathcal{X}}^{a}_{a,b,c,d}} = \frac{\widetilde{\mathcal{X}}^{a}_{d,a,b,c}}{\widetilde{\mathcal{X}}^{a-1}_{d,a,b,c}}\,, \quad \frac{\widetilde{\mathcal{X}}^{a-1}_{d,a,b,c}}{\widetilde{\mathcal{X}}^{a}_{d,a,b,c}} = \frac{\widetilde{\mathcal{X}}^{a}_{c,d,a,b}}{\widetilde{\mathcal{X}}^{a-1}_{c,d,a,b}}\,,$$

$$\frac{\mathcal{X}^{a-1}_{a,b,c,d}}{\mathcal{X}^{a}_{a,b,c,d}} = \frac{\widetilde{\mathcal{X}}^{a}_{c,d,a,b}}{\widetilde{\mathcal{X}}^{a-1}_{c,d,a,b}}\,, \quad \frac{\mathcal{X}^{a}_{a,b,c,d}}{\mathcal{X}^{b}_{a,b,c,d}} = \frac{\widetilde{\mathcal{X}}^{b}_{a,b,c,d}}{\widetilde{\mathcal{X}}^{a}_{a,b,c,d}}\,, \qquad \frac{\mathcal{X}^{b}_{a,b,c,d}}{\mathcal{X}^{c}_{a,b,c,d}} = \frac{\widetilde{\mathcal{X}}^{c}_{a,b,c,d}}{\widetilde{\mathcal{X}}^{b}_{a,b,c,d}}$$

and 21 images under the rotations of $a \to b \to c \to d \to a$,

$$\frac{\mathcal{X}^{a}_{a,b,c,d}\mathcal{X}^{d}_{b,c,d,a}\mathcal{X}^{d}_{c,d,a,b}\mathcal{X}^{a}_{d,a,b,c}}{\mathcal{X}^{b}_{a,b,c,d}\mathcal{X}^{c}_{b,c,d,a}\mathcal{X}^{c}_{c,d,a,b}\mathcal{X}^{b}_{d,a,b,c}} = 1\,,$$

$$\frac{\mathcal{X}^{a}_{a,b,c,d}\mathcal{X}^{d}_{b,c,d,a}\mathcal{X}^{a}_{d,a,b,c}}{\mathcal{X}^{c}_{a,b,c,d}\mathcal{X}^{c}_{b,c,d,a}\mathcal{X}^{d}_{d,a,b,c}} = 1\,, \quad \frac{\mathcal{X}^{b}_{b,c,d,a}\mathcal{X}^{a}_{c,d,a,b}\mathcal{X}^{a}_{d,a,b,c}}{\mathcal{X}^{c}_{b,c,d,a}\mathcal{X}^{c}_{c,d,a,d}\mathcal{X}^{b}_{d,a,b,c}} = 1\,,$$

$$\frac{\mathcal{X}^{a}_{c,d,a,b}\mathcal{X}^{a}_{d,a,b,c}}{\mathcal{X}^{d}_{c,d,a,b}\mathcal{X}^{d}_{d,a,b,c}} = \frac{z_{a,b,c,d}}{\bar{z}_{a,b,c,d}}\,, \quad \frac{\mathcal{X}^{c}_{b,c,d,a}\mathcal{X}^{c}_{c,d,a,b}}{\mathcal{X}^{d}_{b,c,d,a}\mathcal{X}^{d}_{c,d,a,b}} = \frac{1-z_{a,b,c,d}}{1-\bar{z}_{a,b,c,d}}\,.$$

Taking these relations into account:

number of multiplicatively independent algebraic letters $= 17 - 2m$

Two kinds of cuts

The algebraic letters can be rewritten as $(a \pm \sqrt{a^2 - 4b})$.
(a, b) are polynomials of Plücker coordinates.
Such letters indicate two kinds of cuts:

- One arises from the discriminant $a^2 - 4b$.
- The other arises from $b \to 0$ which is the same as the cut of $\log b$.

That is, b must belong to the alphabet of rational letters:
For example

$$\left|(x^c_{a,b,c,d}+1)^{-1} - \bar{z}_{d,a,b,c}\right|^2 \propto \langle c(A)(B)(D)\rangle\langle (A)\cap(\bar{d})B(C)\cap(\bar{d})\rangle\langle AB\rangle\,,$$
$$\left|(x^b_{a,b,c,d}+1)^{-1} - \bar{z}_{d,a,b,c}\right|^2 \propto \langle b(A)(C)(D)\rangle\langle (A)\cap(\bar{d})B(C)\cap(\bar{d})\rangle\,,$$
$$\left|(x^a_{a,b,c,d}+1)^{-1} - \bar{z}_{d,a,b,c}\right|^2 \propto \langle a(B)(C)(D)\rangle\langle (A)\cap(\bar{d})B(C)\cap(\bar{d})\rangle\langle BC\rangle\,,$$

here $A = (a{-}1\,a)$, $B = (b{-}1\,b)$, $C = (c{-}1\,c)$, $D = (d{-}1\,d)$.

New class of rational letters first appearing in 9-point:

$$\langle (A)\cap(\bar{d})B(C)\cap(\bar{d})\rangle$$

Algebraic Words

The first two entries of algebraic words always can consist of the symbol of the four-mass box.
For non-degenerate $\mathcal{X}^*_{a,b,c,d}$'s as the third entry:

$$\begin{aligned}
&\mathcal{S}(I_{a,b,c,d}) \otimes \mathcal{X}^{c-1}_{a,b,c,d} \otimes x^{c-1}_{a,b,c,d}\,[a{-}1\,a\,b{-}1\,b\,c-1] \\
-&\mathcal{S}(I_{a,b,c,d}) \otimes \mathcal{X}^{c}_{a,b,c,d} \otimes x^{c}_{a,b,c,d}\,[a{-}1\,a\,b{-}1\,b\,c] \\
+&\mathcal{S}(I_{a,b,c,d}) \otimes \mathcal{X}^{b-1}_{a,b,c,d} \otimes x^{b-1}_{a,b,c,d}\,[a{-}1\,a\,b{-}1\,c-1\,c] \\
-&\mathcal{S}(I_{a,b,c,d}) \otimes \mathcal{X}^{b}_{a,b,c,d} \otimes x^{c}_{a,b,c,d}\,[a{-}1\,a\,b\,c{-}1\,c] \\
+&\mathcal{S}(I_{a,b,c,d}) \otimes \mathcal{X}^{a-1}_{a,b,c,d} \otimes x^{a-1}_{a,b,c,d}\,[a{-}1\,b{-}1\,b\,c-1\,c] \\
-&\mathcal{S}(I_{a,b,c,d}) \otimes \mathcal{X}^{a}_{a,b,c,d} \otimes x^{a}_{a,b,c,d}\,[a\,b{-}1\,b\,c{-}1\,c]\,,
\end{aligned}$$

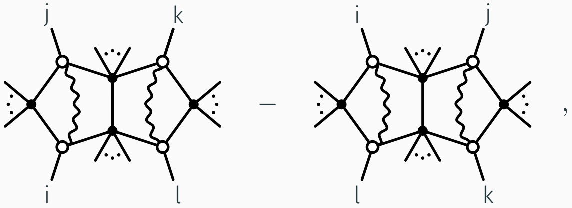
likewise for $\widetilde{\mathcal{X}}$. Recall that $\mathcal{X}^* := \frac{(x^*+1)^{-1}-\bar{z}}{(x^*+1)^{-1}-z}$,

The results for $z/\bar{z}$ or $(1-z)/(1-\bar{z})$ are slight lengthy, but the accompanied R invariants always are the form of $[i\,i{+}1\,j\,j{+}1\,k]$!

A class of special components

The $\chi_i\chi_j\chi_k\chi_l$ components with i, j, k, l nonadjacent

- first show up in the two-loop amplitudes,
- completely free of algebraic letters,
- correspond to the difference of double-pentagon integrals [Arkani-Hamed, Bourjaily, Cachazo, Trnka]



each of which depend on many algebraic roots.[Bourjaily, McLeod, Vergu, Volk, von Hippel, Wilhelm]

Outlook

- Octagons at three-loop MHV and two-loop N^2MHV.
- The connection to cluster algebra, tropical Grassmannian
- $\bar{Q}$ equations for individual integral and other theories.

Thank You

The kernel of $\bar{Q}$

When k=1,

$$\bar{Q}\left([1,2,3,4,5]\log\frac{\langle 1234\rangle}{\langle 2345\rangle}\right) = [1,2,3,4,5]\bar{Q}\log\frac{\langle 1234\rangle}{\langle 2345\rangle}$$
$$= (\bar{3})_a[1,2,3,4,5]\frac{\langle 1234\rangle\chi_5^A + \text{cyclic}}{\langle 2345\rangle\langle 2341\rangle}$$

When k=2, it's easy to show

$$Y_1^{(2)} = \frac{\delta^{0|4}(\langle 1234\rangle\chi_5\chi_6 + \text{cyclic})}{\langle 1234\rangle\cdots\langle 6123\rangle} \propto \bar{Q}\log u\bar{Q}\log v\bar{Q}\log w$$

then

$$\bar{Q}\big(Y_1^{(2)}F(u,v,w)\big) = 0$$

Outline of derivation of $\bar{Q}$-equation

By using chiral Lagrangian insertion [Caron-Huot], one can show

$$\bar{Q}^A_{\dot{\alpha}}\langle W_{n,k}\rangle \propto \oint \mathrm{d}x_{\dot{\alpha}\alpha}\langle(\psi^A + F\theta^A + \cdots)^\alpha W_{n,k}\rangle$$

To obtain the $\bar{Q}$-equation, there are two powerful facts:

- The fermion insertion is the unique twist-one excitation with the quantum numbers of $\bar{Q}$.
- Its expectation value can be extracted from any object having a nonzero overlap with it in the OPE limit. [Alday, Gaiotto, Maldacena, Sever, Vieira]

$\langle W_{n+1,k+1}\rangle$ has a nonzero overlap with $\bar{Q}^A_{\dot{\alpha}}\langle W_{n,k}\rangle$ under the colliner limit, while $\int \mathrm{d}^{2|3}\mathcal{Z}_{n+1}$ has the same quantum number as $\bar{Q}$,

$$\bar{Q}^A_{\dot{\alpha}}\langle W_{n,k}\rangle \propto \int \mathrm{d}^{2|3}\mathcal{Z}_{n+1}\langle W_{n+1,k+1}\rangle$$