Algebraic letters and NMHV last entry conditions from $\bar{Q}$-equation

Based on works with Song He and Zhenjie Li

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Review of $\mathcal{N} = 4$ sYM and its amplitudes

planar $\mathcal{N} = 4$ sYM: Harmonic oscillator of QFT

1. Solvable 4-dimensional QFT
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4. SUSY cousin of QCD

Field Content and Superamplitude

Simplicity of field content:

- 2 gauge bosons with $h = \pm 1$: $|a\rangle^{+1}, |a\rangle^{-1}_{ABCD}$,
- 8 fermions with $h = \pm 1/2$: $|a\rangle^{+1/2}_{A}, |a\rangle^{-1/2}_{BCD}$,
- 6 scalars: $|a\rangle^{0}_{AB}$.

Related by SUSY generators Q^{α}_{A} and $\widetilde{Q}^{\dot{\alpha}}_{A}$,

grouped into a single supermultiplet:

$$\begin{aligned} |a\rangle &:= \exp(\widetilde{Q}_A \cdot \widetilde{\lambda} \cdot \widetilde{\eta}^A)|a\rangle^+ \\ &= |a\rangle^+ + \widetilde{\eta}^A |a\rangle^{1/2}_A + \cdots + \frac{1}{4!}\widetilde{\eta}^A\widetilde{\eta}^B\widetilde{\eta}^C\widetilde{\eta}^D |a\rangle^-_{ABCD} \end{aligned}$$

We are considering the scattering of n supermultiplets:

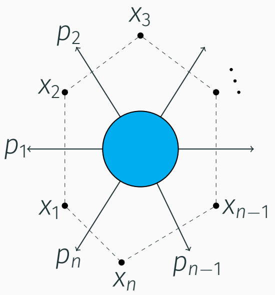
$$A_n(\{p_i, \widetilde{\eta}_i\}) = \delta^4(P)\delta^8(Q)\big(\mathcal{A}_{n,0}(\{p_i\}) + \mathcal{A}_{n,1}(\{p_i, \widetilde{\eta}_i\}) + \cdots\big)$$

Scattering Amplitude/Wilson loop duality

Amplitudes in planar $\mathcal{N}=4$ sYM enjoy not only superconformal symmetries, but also dual superconformal symmetries, [Drummond,Henn,Smirnov,Sokatchev] which is manifest in a chiral superspace cooidnates (x,θ)

$$x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}} = \lambda_i^{\alpha}\widetilde{\lambda}_i^{\dot{\alpha}} = p_i^{\mu}\sigma_{\mu}^{\alpha\dot{\alpha}}, \qquad \theta_i^{\alpha A} - \theta_{i+1}^{\alpha A} = \lambda_i^{\alpha}\widetilde{\eta}_i^{A}$$

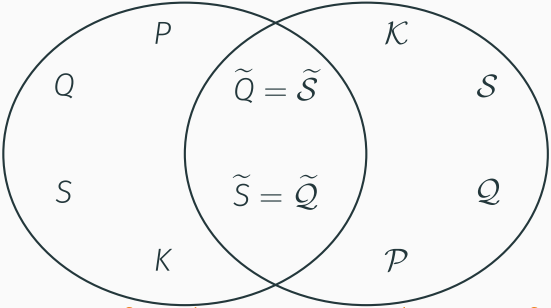
$$\text{planar poles:} \quad (p_i + p_{i+1} + \cdots + p_{j-1})^2 = x_{ij}^2$$



In dual space, an amplitude become a light-like polygonal Wilson loop.

$$\mathcal{A}_n(p_1, p_2, \cdots, p_n) \Leftrightarrow W_n(x_1, \cdots, x_n)$$

Superconformal and dual superconformal symmetries



Q: Supercharge
P: Translation
S: Superconformal
K: SCFT

Superconformal Generator $+$ Dual Superconformal Generator $=$ Yangian!

In terms of λ, $\widetilde{\lambda}$, x, the generators are not linear realized:

$$K_{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \frac{\partial^2}{\partial\lambda_i^{\alpha}\partial\widetilde{\lambda}_i^{\dot{\alpha}}}\,, \qquad \mathcal{K}^{\alpha\dot{\alpha}} = \sum_{i=1}^{n}\left[x_i^{\alpha\dot{\beta}}x_i^{\beta\dot{\alpha}}\frac{\partial}{\partial x_i^{\beta\dot{\beta}}} + \cdots\right]$$

Yangian and Grassmannian

(Dual) superconformal symmetry $\mathrm{SL}(4|4)$ is linearly realized in terms of super (momentum) twistor

$$\mathcal{W}_i^I = (W_i^a|\widetilde{\eta}_i^A) := (\widetilde{\mu}_i^\alpha, \widetilde{\lambda}_i|\widetilde{\eta}_i^A) \quad \text{by Fourier trans. } \int \exp(-\lambda_i \widetilde{\mu}_i)$$

$$\mathcal{Z}_i^I = (Z_i^a|\chi_i^A) := (\lambda_i^\alpha, x_i^{\alpha\dot{\alpha}}\lambda_{i\alpha}|\theta_i^{\alpha A}\lambda_{i\alpha}). \qquad \text{[Hodges]}$$

For funture convenience, we introduce two basic invariants:

$$\text{Plücker coordinate :} \quad \langle ijkl\rangle := \varepsilon_{abcd} Z_i^a Z_j^b Z_k^c Z_l^d, \; \left(x_{ij}^2 = \frac{\langle i{-}1\, i\, j{-}1\, j\rangle}{\langle i{-}1\, i\rangle\langle j{-}1\, j\rangle} \right)$$

$$\text{R invariant :} \quad [i\,j\,k\,l\,m] := \frac{\delta^{0|4}(\chi_i^A\langle jklm\rangle + \text{cyclic})}{\langle ijkl\rangle\langle jklm\rangle\langle klmi\rangle\langle lmij\rangle\langle mijk\rangle}\,,$$

where R invariants are the first kind of non-trivial Yangian invariant!

In terms of (momentum) twistor, Yangian symmetries are generated by

$$\text{level 0:} \quad \sum_{i=1}^{n} G^I_{iJ} \,,$$

$$\text{level 1:} \quad \sum_{i<j}^{n} (-1)^{|K|} [G^I_{iK} G^K_{jJ} - (i \leftrightarrow j)] \,, \cdots$$

where

$$G^I_{iJ} = \mathcal{Z}^I_i \frac{\partial}{\partial \mathcal{Z}^J_i} \quad \text{or} \quad \mathcal{W}^I_i \frac{\partial}{\partial \mathcal{W}^J_i}$$

Then the Yangian invariants can be proven to be Grassmannian integrals [Arkani-Hamed, Drummond,$\cdots$]

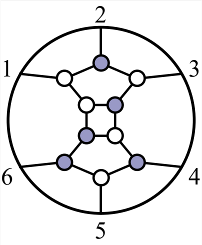
$$Y^{(\gamma)}_{n,k}(\mathcal{Z}) = \int_\gamma \frac{\mathrm{d}^{k\times n} C}{\text{vol}\,\mathrm{GL}(k)} \frac{\delta^{4k|4k}(C\cdot \mathcal{Z})}{(1\cdots k)\cdots(n\cdots k-1)}$$

or

$$\mathcal{L}^{(\gamma)}_{n,k}(\lambda,\widetilde{\lambda},\widetilde{\eta}) = \int_\gamma \frac{\mathrm{d}^{K\times n} C}{\text{vol}\,\mathrm{GL}(K)} \frac{\delta^{2K}(C\cdot \widetilde{\lambda})\delta^{2(n-K)}(C^\perp \cdot \lambda)\delta^{4K}(C\cdot \widetilde{\eta})}{(1\cdots K)\cdots(n\cdots K{-}1)} \quad (K := k+2)\,.$$

Yangian invariant as leading singularities

When Yangian invariants are expressed as Grassmannian integrals, they are associated with a diagrammatic representation [Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka]:



black/white vertex $\rightarrow$ 3-pt MHV/$\overline{\text{MHV}}$

propagator $\rightarrow \int \frac{\mathrm{d}^2\lambda_I \mathrm{d}^2\widetilde{\lambda}_I \mathrm{d}^4\widetilde{\eta}_I}{\mathrm{GL}(1)}$

Yangian invariants are leading singularities of loop integrand! By generalized unitarity method,

$$\mathcal{A}_{n,k,L} = \sum Y_{n,k} \times \text{scalar integrals}\,.$$

General structure of loop amplitudes in planar $\mathcal{N} = 4$ sYM

Recall that $A_{n,k} = \sum Y_{n,k} \times$ scalar integral.

- The dual conformal invariance of amplitudes is broken at the loop-level due to the infrared divergence.
- This symmtery can be restored by subtracting the infrared part A_n^{BDS} [Bern, Dixon, Smirnov].

$$A_n = \underbrace{A_n^{\mathrm{BDS}}}_{IR} \times \overbrace{\underbrace{\exp(R_n)}_{\substack{\text{Remainder}\\ \text{function}}} \times \underbrace{\left(1 + \mathcal{P}_n^{\mathrm{NMHV}} + \cdots + \mathcal{P}_n^{\overline{\mathrm{MHV}}}\right)}_{\text{Ratio functions}}}^{\text{finite functions of dual conformal invariants}}$$

For example, R_6 will be a function of $u = \frac{\langle 1234\rangle\langle 4561\rangle}{\langle 1245\rangle\langle 3461\rangle}$, $v = \frac{\langle 3456\rangle\langle 6123\rangle}{\langle 3461\rangle\langle 5623\rangle}$, $w = \frac{\langle 5612\rangle\langle 2345\rangle}{\langle 5623\rangle\langle 1245\rangle}$.

We are interested in the function $R_{n,1}^{(2)} = \left(\exp(R_n)\mathcal{P}_n^{\mathrm{NMHV}}\right)^{(2)}$

In the following, we will denote $\exp(R_n)\mathcal{P}_n^{\mathrm{N}^k\mathrm{MHV}}$ by $R_{n,k}$

Poles and Cuts

In general, the BDS-normalized amplitudes $R_{n,k}$ can be written as

$$R_{n,k}^{(L)} = \sum_{\alpha} Y_{n,k}^{\alpha} F_{\alpha}^{(2L)}$$

where $Y_{n,k}$ are Yangian invariants

- $Y_{n,k}$ bear the pole structure of amplitudes,
- $Y_{n,k}$ are independent of loop-integral,
- F are transcendental functions of cross-ratios bearing the cut structure of amplitudes.

For MHV and NMHV amplitudes, F are believe to be just polylogarithms of weight $2L$ [Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka].

Polylogarithms and Symbol

Polylogarithms [Goncharov] of weight $2L$ are $2L$-fold iterated integrals.

$$F^{(2L)} = \int_\gamma \mathrm{d}\log s_1 \circ \cdots \circ \mathrm{d}\log s_{2L}$$

This define the symbol of F:

$$\mathcal{S}(F^{(2L)}) := s_1 \otimes \cdots \otimes s_{2L}$$

where s_i are called symbol letters.

Some example:

$$\mathcal{S}(\log x \log y) = x \otimes y + y \otimes x, \quad \mathcal{S}(\mathrm{Li}_2(z)) = -\big((1-z) \otimes z\big)$$

The first entries of symbol indicate the locus of cuts of F

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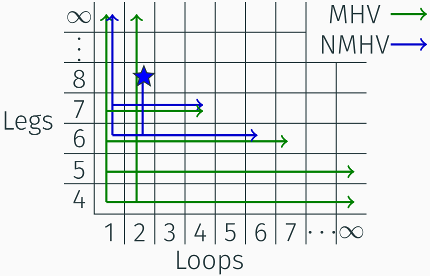
Last entries of Symbol:

$$\mathcal{S}(\mathrm{d}F^{(2L)}) = s_1 \otimes \cdots \otimes s_{2L-1} \mathrm{d}\log s_{2L}$$

Boostrap



The alphabets (collection of all possible letters) for hexagon and heptagon are constrained by finite-type cluster algebras $G(4,6)$ and $G(4,7)$

[Bern, Caron-Huot, Dixon, Drummond,...]

For more than seven particles, symbol alphabets are not well understood

- $G(4, n \geq 8)$ are infinite-type cluster algebras.
- Square roots appear in symbol letters even at one-loop in N^2MHV amplitudes

We will call the letters involving square roots as algebraic letters.

$\bar{Q}$-equation and last entry conditions

Dual superconformal anomaly and $\bar{Q}$ equations

BDS-normalized amplitudes $R_{n,k}$ are dual conformal invariants, but $R_{n,k}$ are not dual superconformal invariants, they have anomalies under the symmetries generated by

$$\bar{Q}_a^A = \sum_i \chi_i^A \frac{\partial}{\partial Z_i^a}$$

An OPE analysis tell us the action of $\bar{Q}$ on $R_{n,k}$ can be given by an integral over higher-point amplitudes [Caron-Huot, He]

$$\bar{Q}_a^A R_{n,k} = \frac{\Gamma_{\text{cusp}}}{4} \int_{\tau=0}^{\tau=\infty} \oint_{\epsilon=0} \left(\mathrm{d}^{2|3} \mathcal{Z}_{n+1}\right)_a^A \left[R_{n+1,k+1} - R_{n,k} R_{n+1,1}^{\text{tree}}\right] + \text{cyclic}$$

where the particle $n+1$ is added in a collinear limit

$$\mathcal{Z}_{n+1} = \mathcal{Z}_n - \epsilon \mathcal{Z}_{n-1} + \frac{\langle n{-}1\, n\, 2\, 3\rangle}{\langle n\, 1\, 2\, 3\rangle} \epsilon \tau \mathcal{Z}_1 + \frac{\langle n{-}2\, n{-}1\, n\, 1\rangle}{\langle n{-}2\, n{-}1\, 2\, 1\rangle} \epsilon^2 \mathcal{Z}_2$$

Dual superconformal anomaly and $\bar{Q}$ equations

BDS-normalized amplitudes $R_{n,k}$ are dual conformal invariants, but $R_{n,k}$ are not dual superconformal invariants, they have anomalies under the symmetries generated by

$$\bar{Q}^A_a = \sum_i \chi^A_i \frac{\partial}{\partial Z^a_i}$$

Perturbatively, this equation becomes

$$\bar{Q}^A_a R^{(L)}_{n,k} = \int_{\tau=0}^{\tau=\infty} \oint_{\epsilon=0} \left(\mathrm{d}^{2|3}\mathcal{Z}_{n+1}\right)^A_a [R^{(L-1)}_{n+1,k+1} - R^{(L-1)}_{n,k} R^{\text{tree}}_{n+1,1}] + \text{cyclic}$$

where the particle $n+1$ is added in a collinear limit

$$\mathcal{Z}_{n+1} = \mathcal{Z}_n - \epsilon \mathcal{Z}_{n-1} + \underbrace{\frac{\langle n{-}1\, n\, 2\, 3\rangle}{\langle n\, 1\, 2\, 3\rangle}}_{C} \epsilon\tau \mathcal{Z}_1 + \underbrace{\frac{\langle n{-}2\, n{-}1\, n\, 1\rangle}{\langle n{-}2\, n{-}1\, 2\, 1\rangle}}_{C'} \epsilon^2 \mathcal{Z}_2$$

The integral measure

The basic operation $\int (\mathrm{d}^{2|3}\mathcal{Z}_{n+1})^A_a$ consist of bosonic part and fermionic part:

$$(\mathrm{d}^{2|3}\mathcal{Z}_{n+1})^A_a \begin{cases} \varepsilon_{abcd} Z^b_{n+1} \mathrm{d}Z^c_{n+1} \mathrm{d}Z^d_{n+1} = C(\bar{n})_a \epsilon \mathrm{d}\epsilon \mathrm{d}\tau & \text{(Bosonic Part)} \\ \\ (\mathrm{d}^3\chi_{n+1})^A & \text{(Fermionic Part)} \end{cases}$$

where $(\bar{n})_a := \varepsilon_{abcd} Z^b_{n-1} Z^c_n Z^d_1$

The order of performing integral:

- Fermionic integral $(\mathrm{d}^3\chi_{n+1})^A$
- The substitution $\mathcal{Z}_{n+1} \to \mathcal{Z}_n - \epsilon \mathcal{Z}_{n-1} + C\epsilon\tau \mathcal{Z}_1 + C' \epsilon^2 \mathcal{Z}_2$
- Take the residue $\oint_{\epsilon=0} \mathrm{d}\epsilon$ (Collinear limit)
- 1-D integral $\int_0^\infty \mathrm{d}\tau$ (Real integral)

$\bar{Q}$ as Differenial

Due to dual conformal invariance(DCI), F are functions of cross ratios of Plücker coordinates. Since F are expected to be polylogarithms, they satisfy

$$\mathrm{d}F^{(2L)} = \sum_{\beta} F_{\beta}^{(2L-1)} \mathrm{d}\log s_{\beta} \qquad \left(\mathrm{d} := \sum_i \mathrm{d}Z_i \frac{\partial}{\partial Z_i}\right)$$

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$$\mathrm{d}F^{(2L)} = \sum_{\beta} F_{\beta}^{(2L-1)} \mathrm{d}\log s_{\beta} \qquad \left(\mathrm{d} := \sum_{i} \mathrm{d}Z_i \frac{\partial}{\partial Z_i}\right)$$

Thus, the action of $\bar{Q}$ on $R_{n,k}^{(L)}$ gives

$$\bar{Q}R_{n,k}^{(L)} = \sum_{\alpha,\beta} Y_{n,k}^{\alpha} F_{\alpha,\beta}^{(2L-1)} \bar{Q}\log s_{\alpha,\beta} \qquad \left(\bar{Q} := \sum_{i} \chi_i \frac{\partial}{\partial Z_i}\right)$$

where $s_{\alpha,\beta}$ are some DCI of Plücker coordinates and referred to the last entries of amplitudes

Kernel of $\bar{Q}$

$\bar{Q}$-equation can not determine N^2MHV amplitudes on its own due to the non-trivial dependence of its kernel on k:

- For $k = 0$, the kernel of $\bar{Q}$ is trivial
- For $k = 1$, it's non-trivial, but has no space of DCI functions
- For $k \geq 2$, it's non-trivial, and it indeed contains DCI functions.

By also considering $Q^{(1)}$ equations, $\mathrm{N}^{k\geq 2}$MHV amplitudes can be fixed uniquely up to some Yangian invariants.

RHS of $\bar{Q}$ equations

Now let us consider the RHS of $\bar{Q}$ equation:

$$\int_{\tau=0}^{\tau=\infty} \oint_{\epsilon=0} \left(\mathrm{d}^{2|3}\mathcal{Z}_{n+1}\right)_a^A [R_{n+1,k+1}^{(L-1)} - \underbrace{R_{n,k}^{(L-1)} R_{n+1,1}^{\text{tree}}]}_{\text{trivial}} + \text{cyclic}$$

The first step:

$$R_{n+1,k+1}^{(L-1)} \begin{cases} Y_{n+1,k+1} & \xrightarrow{C(\bar{n})_a \oint_{\epsilon=0} \epsilon \mathrm{d}\epsilon \mathrm{d}^3 \chi_{n+1}} \sum_{I,J} Y_{n,k}^{I,J} \bar{Q} \log \frac{\langle \bar{n} I \rangle}{\langle \bar{n} J \rangle} \mathrm{d} \log f_{I,J}(\tau) \\ F^{(2L-2)} & \xrightarrow{Z_{n+1} \to Z_n - \epsilon Z_{n-1} + \cdots} F^{(2L-2)}(\tau, \epsilon \to 0) \end{cases}$$

The second step:

$$\int_0^{\infty} \mathrm{d} \log f_{I,J}(\tau) F^{(2L-2)}(\tau, \epsilon \to 0) = F^{(2L-1)}$$

where $f_{I,J}(\tau)$ are rational functions of τ (with some exceptions discussed later)

RHS of $\bar{Q}$ equations

Now let us consider the RHS of $\bar{Q}$ equation:

$$\int_{\tau=0}^{\tau=\infty} \oint_{\epsilon=0} \left(\mathrm{d}^{2|3}\mathcal{Z}_{n+1}\right)_a^A [R^{(L-1)}_{n+1,k+1} - \underbrace{R^{(L-1)}_{n,k} R^{\text{tree}}_{n+1,1}}_{\text{trivial}}] + \text{cyclic}$$

The first step:

$$R^{(L-1)}_{n+1,k+1} \begin{cases} Y_{n+1,k+1} & \xrightarrow{C(\bar{n})_a \oint_{\epsilon=0} \epsilon \mathrm{d}\epsilon \mathrm{d}^3\chi_{n+1}} \sum_{I,J} Y^{I,J}_{n,k} \bar{Q} \log \frac{\langle \bar{n} I\rangle}{\langle \bar{n} J\rangle} \mathrm{d}\log f_{I,J}(\tau) \\ F^{(2L-2)} & \xrightarrow{Z_{n+1} \to Z_n - \epsilon Z_{n-1} + \cdots} F^{(2L-2)}(\tau, \epsilon \to 0) \end{cases}$$

The operation $C(\bar{n})_a \oint_{\epsilon=0} \epsilon \mathrm{d}\epsilon \mathrm{d}^3\chi_{n+1}$ is independent of loop order which gives last entry conditions on amplitudes.

For R invariants, the operation gives the well known [Caron-Huot]

MHV last entries : $\quad \bar{Q} \log \frac{\langle \bar{n} i\rangle}{\langle \bar{n} j\rangle}$

N^2MHV Yangian invariants

The N^2MHV Yangian invariants have already been classified.

[Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka]

They are

$$Y_1^{(2)} = [1, 2, (23) \cap (456), (234) \cap (56), 6][2, 3, 4, 5, 6]$$
$$Y_2^{(2)} = [1, 2, (34) \cap (567), (345) \cap (67), 7][3, 4, 5, 6, 7]$$
$$Y_3^{(2)} = [1, 2, 3, (345) \cap (67), 7][3, 4, 5, 6, 7]$$
$$Y_4^{(2)} = [1, 2, 3, (456) \cap (78), 8][4, 5, 6, 7, 8]$$
$$Y_5^{(2)} = [1, 2, 3, 4, 8][4, 5, 6, 7, 8]$$
$$Y_6^{(2)} = [1, 2, 3, (45) \cap (678), 8][4, 5, 6, 7, 8]$$
$$Y_7^{(2)} = [1, 2, 3, (45) \cap (678), (456) \cap (78)][4, 5, 6, 7, 8]$$
$$Y_8^{(2)} = [1, 2, 3, 4, (456) \cap (78)][4, 5, 6, 7, 8]$$
$$Y_9^{(2)} = [1, 2, 3, 4, 9][5, 6, 7, 8, 9]$$
$$Y_{10}^{(2)} = [1, 2, 3, 4, (567) \cap (89)][5, 6, 7, 8, 9]$$
$$Y_{11}^{(2)} = [1, 2, 3, 4, (56) \cap (789)][5, 6, 7, 8, 9]$$
$$Y_{12}^{(2)} = \varphi[1, 2, 3, (45) \cap (789), (46) \cap (789)][(45) \cap (123), (46) \cap (123), 7, 8, 9]$$
$$Y_{13}^{(2)} = [1, 2, 3, 4, 5][6, 7, 8, 9, 10]$$
$$Y_{14}^{(2)} = \psi[A, 1, 2, 3, 4][B, 5, 6, 7, 8]$$

where

$$(ij) \cap (klm) = \mathcal{Z}_i \langle j\, k\, l\, m \rangle - \mathcal{Z}_j \langle i\, k\, l\, m \rangle$$

NMHV last entry conditions

For N^{2}MHV yangian invariants, this operation gives three kinds of last entries

1.
$$[i\,j\,k\,l\,m]\bar{Q}\log\frac{\langle\bar{n}a\rangle}{\langle\bar{n}b\rangle}$$

2.
$$[1\,i_1\,i_2\,i_3\,i_4]\,\bar{Q}\log\frac{\langle 1(n-1\,n)(i_1\,i_2)(i_3\,i_4)\rangle}{\langle\bar{n}i_1\rangle\langle 1i_2i_3i_4\rangle}\,,\quad [i_1\,i_2\,i_3\,i_4\,n-1]\,\bar{Q}\log\frac{\langle n-1(n\,1)(i_1\,i_2)(i_3\,i_4)\rangle}{\langle\bar{n}i_1\rangle\langle n-1\,i_2i_3i_4\rangle}$$
$$[i_1\,i_2\,i_3\,i_4\,n]\,\bar{Q}\log\frac{\langle n(1\,n-1)(i_1\,i_2)(i_3\,i_4)\rangle}{\langle\bar{n}i_1\rangle\langle n\,i_2i_3i_4\rangle}\quad\text{with } 1<i_1<i_2<i_3<i_4<n-1$$

where $\langle a(bc)(de)(fg)\rangle := \langle abde\rangle\langle acfg\rangle - \langle acde\rangle\langle abfg\rangle$

3.
$$[i_1\,i_2\,i_3\,i_4\,i_5]\,\bar{Q}\log\frac{\langle\bar{n}(i_1i_2)\cap(i_3i_4i_5)\rangle}{\langle\bar{n}i_1\rangle\langle i_2i_3i_4i_5\rangle}\,,\;[i_1\,i_2\,i_3\,i_4\,i_5]\,\bar{Q}\log\frac{\langle\bar{n}(i_1i_2i_3)\cap(i_4i_5)\rangle}{\langle\bar{n}i_1\rangle\langle i_2i_3i_4i_5\rangle}\,,$$

with $1<i_1<i_2<i_3<i_4<i_5<n$

Algebraic letters in two-loop NMHV amplitudes

Input

To compute the 2-loop NMHV n point BDS-normalized amplitude, we need the input of the one-loop N^{2}MHV BDS-normalized amplitude $R^{(1)}_{n+1,2}$, which can be obtained from the chiral/scalar box expansion [Bourjaily, Caron-Huot, Trnka]:

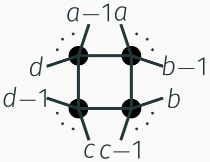
$$R^{(1)}_{n+1,2} = \sum_{a<b<c<d} (f_{a,b,c,d} - R^{\text{tree}}_{n+1,2} f^{\text{MHV}}_{a,b,c,d}) \mathcal{I}^{\text{fin}}_{a,b,c,d}$$

where

- $f_{a,b,c,d}$ are linear combinations of N^{2}MHV yangian invariants
- $f^{\text{MHV}}_{a,b,c,d}$ are either 1 or 0
- $\mathcal{I}^{fin}_{a,b,c,d}$ denote the finite part of DCI-regulated box integrals

Four-mass box

The most generic term in box expansion:



$$\begin{cases} x_{ab}^2 := \dfrac{\langle a{-}1\,a\,b{-}1\,b\rangle}{\langle a{-}1\,a\rangle\langle b{-}1\,b\rangle} = (p_a + \cdots + p_{b-1})^2, \\[2ex] u = \dfrac{x_{ad}^2 x_{bc}^2}{x_{ac}^2 x_{bd}^2} = z\bar{z}, \quad v = \dfrac{x_{ab}^2 x_{cd}^2}{x_{ac}^2 x_{bd}^2} = (1-z)(1-\bar{z}), \\[2ex] \Delta_{abcd} = \sqrt{(1{-}u{-}v)^2 - 4uv} \end{cases}$$

For such a box,

$$f_{a,b,c,d} = \frac{1-u-v\pm\Delta}{2\Delta}[\alpha_\pm, b{-}1, b, c-1, c][\delta_\pm, d{-}1, d, a{-}1, a]$$

$$\mathcal{I}^{\text{fin}}_{a,b,c,d} = \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \frac{1}{2}\log(z\bar{z})\log\frac{1-z}{1-\bar{z}}$$

where $\alpha_\pm$ and $\delta_\pm$ are two solutions of Schubert problem $\alpha = (a{-}1\,a)\cap(d\,d{-}1\,\gamma)$, $\gamma = (c{-}1\,c)\cap(b\,b{-}1\,\alpha)$

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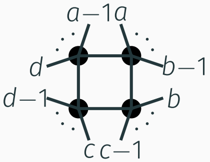


$$
\begin{cases}
x_{ab}^2 := \dfrac{\langle a{-}1\, a\, b{-}1\, b\rangle}{\langle a{-}1\, a\rangle \langle b{-}1\, b\rangle} = (p_a + \cdots + p_{b-1})^2, \\[2ex]
u = \dfrac{x_{ad}^2 x_{bc}^2}{x_{ac}^2 x_{bd}^2} = z\bar{z}, \quad v = \dfrac{x_{ab}^2 x_{cd}^2}{x_{ac}^2 x_{bd}^2} = (1-z)(1-\bar{z}), \\[2ex]
\Delta_{abcd} = \sqrt{(1{-}u{-}v)^2 - 4uv}
\end{cases}
$$

For such a box,

$$
f_{a,b,c,d} = \frac{1-u-v \pm \Delta}{2\Delta}[\alpha_\pm, b{-}1, b, c-1, c][\delta_\pm, d{-}1, d, a{-}1, a]
$$

$$
\mathcal{I}^{\text{fin}}_{a,b,c,d} = \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \frac{1}{2}\log(z\bar{z})\log\frac{1-z}{1-\bar{z}}
$$

where $\alpha_\pm$ and $\delta_\pm$ are two solutions of Schubert problem
$\alpha = (a{-}1\, a) \cap (d\, d{-}1\, \gamma)$, $\gamma = (c{-}1\, c) \cap (b\, b{-}1\, \alpha)$

The square root will disappear when one mass corner become massless, *e.g.* $b = a{+}1$

Rationalize the square root Δ

Under the collinear limit of $\mathcal{Z}_{n+1}||\mathcal{Z}_n$, some Δ's become algebraic functions $\Delta(\tau)$ of τ.

- Perform τ-integral for four-mass box coefficients $f_{a,b,c,d}$ is difficult due to the appearance of square root Δ.

- However, $\Delta(\tau)$ can be rationlized by a variable substitution, since Δ^2 is only a quadratic polynomial of τ.

- This is just the classic problem to find a rational parameterization of a quadratic curve $y^2 = x^2 + ax + b$.

Algebraic letters of two-loop NMHV amplitudes

1-D τ-integrals for these four masses introduce new algebraic letters in the symbol of two-loop NMHV amplitudes of the form

$$\frac{(x_*+1)^{-1}-z_{1,a,b,c}}{(x_*+1)^{-1}-\bar{z}_{1,a,b,c}} \quad \text{and} \quad \frac{(x_*^{-1}+1)^{-1}-z_{a,b,c,n}}{(x_*^{-1}+1)^{-1}-\bar{z}_{a,b,c,n}}$$

with

$$x_a = \frac{\langle \bar{n}(c{-}1\,c)\cap(a\,b{-}1\,b)\rangle}{\langle \bar{n}\,a\rangle\langle b{-}1\,b\,c{-}1\,c\rangle}\,, \quad x_{a-1} = x_a|_{a\leftrightarrow a-1}\,,$$

$$x_b = \frac{\langle \bar{n}(c{-}1\,c)\cap(a{-}1\,a\,b)\rangle}{\langle \bar{n}(a{-}1\,a)\cap(b\,c{-}1\,c)\rangle}\,, \quad x_{b-1} = x_b|_{b\leftrightarrow b-1}\,,$$

$$x_c = \frac{\langle \bar{n}\,c\rangle\langle a{-}1\,a\,b{-}1\,b\rangle}{\langle \bar{n}(a{-}1\,a)\cap(b{-}1\,b\,c)\rangle}\,, \quad x_{c-1} = x_c|_{c\leftrightarrow c-1}\,,$$

Further more, all new algebraic letters always enter the symbol in the following combinations

$$\left(u\otimes\frac{1-z}{1-\bar{z}}+v\otimes\frac{\bar{z}}{z}\right)\otimes\frac{(x_*+1)^{-1}-z}{(x_*+1)^{-1}-\bar{z}}\otimes x_*$$

$$\underbrace{\left(u\otimes\frac{1-z}{1-\bar{z}}+v\otimes\frac{\bar{z}}{z}\right)}_{\text{symbol of 4-mass box}}\otimes\frac{(x_*^{-1}+1)^{-1}-z}{(x_*^{-1}+1)^{-1}-\bar{z}}\otimes x_*$$

Alphabet for 2-loop NMHV octagon

For the two-loop NMHV octagon [He, Li, CZ], we find 44 algebraic letters generated by cyclic permutations of the following 7 seeds

$$\frac{\bar{x}_* - z}{\bar{x}_* - \bar{z}} \begin{cases} \bar{x}_a = \frac{\langle 1(52)(34)(78)\rangle \langle 3456\rangle}{\langle 1345\rangle \langle 1256\rangle \langle 3478\rangle} \,, \quad \bar{x}_b = x_a|_{5\leftrightarrow 6} \,, \\ \bar{x}_c = \frac{\langle 1378\rangle \langle 3456\rangle}{\langle 1356\rangle \langle 3478\rangle} \,, \; \bar{x}_d = \bar{x}_c|_{3\leftrightarrow 4} \,, \; \bar{x}_e = \frac{\langle 187(34)\cap(256)\rangle}{\langle 1256\rangle \langle 3478\rangle} \\ \bar{x}_f = 1, \quad \bar{x}_g = 0, \quad z = z_{2,4,6,8} \end{cases}$$

and 180 rational letters which are contained in the prediction from Laudau equations [Prlina, Spradlin, Stankowicz, Stanojevic].

Only 18 of 44 algebraic letters are multiplicatively independent.

Two kinds of cuts

The algebraic letters can be rewritten as $(a \pm \sqrt{a^2 - 4b})$.
(a, b) are polynomials of Plücker coordinates.
Such letters indicate two kinds of cuts:

- One arise from the discriminant $a^2 - 4b$.
- The other arise from $b \to 0$ which is the same as the cut of $\log b$.

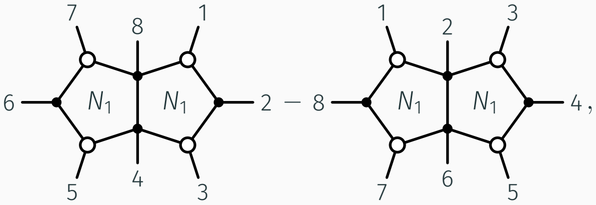
That is, b must belong to the alphabet of rational letters:
For example

$$(\bar{x}_a - z)(\bar{x}_a - \bar{z}) \propto \langle 1(34)(56)(78)\rangle\langle 5(12)(34)(78)\rangle\,,$$
$$(\bar{x}_c - z)(\bar{x}_c - \bar{z}) \propto \langle 1(34)(56)(78)\rangle\langle 3(12)(56)(78)\rangle\,,$$
$$(\bar{x}_e - z)(\bar{x}_e - \bar{z}) \propto \langle 1(34)(56)(78)\rangle\langle 2(34)(56)(78)\rangle\,,$$

Comparison with Feynman integral computation

The simplest component of two-loop NMHV octagon: $\chi_1\chi_3\chi_5\chi_7$

- completely free of algebraic letters.
- correspond to the difference of two Feynman integrals [Bourjaily, McLeod, Vergu, Volk, von Hippel, Wilhelm]



each of which depend on many algebraic roots.

Outlook

- Octagon and algebraic letters at 3-loop MHV.
- The connection to cluster algebra, tropical Grassmannian
- $\bar{Q}$ equations for individual integral and other theories.

Thank You

The kernel of $\bar{Q}$

When k=1,

$$\bar{Q}\left([1,2,3,4,5]\log\frac{\langle 1234\rangle}{\langle 2345\rangle}\right) = [1,2,3,4,5]\bar{Q}\log\frac{\langle 1234\rangle}{\langle 2345\rangle}$$
$$= (\bar{3})_a[1,2,3,4,5]\frac{\langle 1234\rangle\chi_5^A + \text{cyclic}}{\langle 2345\rangle\langle 2341\rangle}$$

When k=2, it's easy to show

$$Y_1^{(2)} = \frac{\delta^{0|4}(\langle 1234\rangle\chi_5\chi_6 + \text{cyclic})}{\langle 1234\rangle\cdots\langle 6123\rangle} \propto \bar{Q}\log u\bar{Q}\log v\bar{Q}\log w$$

then

$$\bar{Q}\big(Y_1^{(2)}F(u,v,w)\big) = 0$$

Outline of derivation of $\bar{Q}$-equation

By using chiral Lagrangian insertion [Caron-Huot], one can show

$$\bar{Q}^A_{\dot\alpha}\langle W_{n,k}\rangle \propto \oint \mathrm{d}x_{\dot\alpha\alpha}\langle(\psi^A + F\theta^A + \cdots)^\alpha W_{n,k}\rangle$$

To obtain the $\bar{Q}$-equation, there are two powerful facts:

- The fermion insertion is the unique twist-one excitation with the quantum numbers of $\bar{Q}$.
- Its expectation value can be extracted from any object having a nonzero overlap with it in the OPE limit. [Alday, Gaiotto, Maldacena, Sever, Vieira]

$\langle W_{n+1,k+1}\rangle$ has a nonzero overlap with $\bar{Q}^A_{\dot\alpha}\langle W_{n,k}\rangle$ under the colliner limit, while $\int \mathrm{d}^{2|3}\mathcal{Z}_{n+1}$ has the same quantum number as $\bar{Q}$,

$$\bar{Q}^A_{\dot\alpha}\langle W_{n,k}\rangle \propto \int \mathrm{d}^{2|3}\mathcal{Z}_{n+1}\langle W_{n+1,k+1}\rangle$$