Scattering Amplitudes as Differential Forms

Chi Zhang

Institute of Theoretical Physics

Background

$\mathcal{N} = 4$ Supersymmetric Yang-Mills theory

- AdS/CFT Duality
- Integrability
- Twistor strings
- Amplituhderon/Grassmaniann
- Lots of Math

The content of $\mathcal{N}=4$ SYM:

- 2 gauge bosons with $h=\pm1$: $|a\rangle^{+1}$, $|a\rangle^{-1}_{IJKL}$
- 8 fermions with $h=\pm\frac{1}{2}$: $|a\rangle^{+1/2}_{I}$,$|a\rangle^{-1/2}_{IJK}$
- 6 spinless scalars: $|a\rangle^{0}_{IJ}$

They are all related by supersymmetric generators $Q^{\alpha}_{I=1,2,3,4}$ and $\tilde{Q}^{\dot{\alpha}}_{I=1,2,3,4}$. For example, $|a\rangle^{+1/2}_{I}=\tilde{Q}_I|a\rangle^{+1}$
Thus, all on-shell states in $\mathcal{N}=4$ SYM can be grouped into a *single* supermultiplet:

$$\begin{aligned}|a\rangle &\equiv \exp(\tilde{Q}_I\cdot\tilde{\lambda}\cdot\tilde{\eta}^I)|a\rangle^{+1}\\ &=|a\rangle^{+1}+\tilde{\eta}^I|a\rangle^{1/2}_{I}+\cdots+\frac{1}{4!}\tilde{\eta}^I\tilde{\eta}^J\tilde{\eta}^K\tilde{\eta}^L|a\rangle^{-1}_{IJKL}\end{aligned}$$

Superamplitudes, not amplitudes!

$$\mathcal{A}_{n,\text{tree}}^{k=2} = \frac{\delta^4(P)\delta^8(Q)}{\langle 12\rangle \cdots \langle n1\rangle}$$

$$\mathcal{A}_{n,\text{tree}}^{k=3} = \mathcal{A}_{n,\text{tree}}^{k=2} \sum_{2<i<j<n-1} [1, i-1, i, j, j+1] \, \big(\textbf{Momentum Twistor come in!}\big)$$

$\mathcal{A}_{n,\text{tree}}^{k}$ are known for any n and any k

$\mathcal{A}_{n,L=1}^{k}$ are known for any n and any k

$\mathcal{A}_{n,L=2}^{k}$ are known for any n and k=2

$\ldots$

$\mathcal{A}_{6,5}^{2}$ are known.

[Bern, Caron-Huot, Dixon, Drummond, Duhr, Foster, Gürdo gan, He, Henn, von Hippel, Golden, Kosower, Papathanasiou, Pennington, Roiban, Smirnov, Spradlin, Vergu, Volovich, ...]

Amplituhderon & Positive geometry

Positive geometry is defined by its boundary which also is a *positive geometry*. For each positive geometry, we define a canonical form such that it only has logarithmic residues on its boundary. For example:

$$\Omega[\{0 < x < 1\}] = \frac{dx}{x} - \frac{dx}{x-1} = d\log\frac{x}{1-x}$$

$$\Omega[\{0 < x < y < 1\}] = d\log\frac{x}{x-y} \wedge d\log\frac{x-y}{1-y}$$

$$\Omega[\{x_{ij}z_{ij} + y_{ij}w_{ij} < 0 | 1 \le i < j \le L\}] = \text{4pt L-loop planar integrand!}$$

Amplituhderon is a positive geometry defined in Momentum twistor space.

[Arkani-Hamed, Bai, Thomas, Trnka,$\cdots$]

Motivation

- Motivation: Try to define the Amplituhderon in terms of usual spinor-helicity variables $\{\lambda_i, \tilde{\lambda}_i\}$

Motivation

- Motivation: Try to define the Amplituhderon in terms of usual spinor-helicity variables $\{\lambda_i, \tilde{\lambda}_i\}$
- It is too ambitious. Find the canonical form first.

Motivation

- Motivation: Try to define the Amplituhderon in terms of usual spinor-helicity variables $\{\lambda_i, \tilde{\lambda}_i\}$
- It is too ambitious. Find the canonical form first.
- Method: Combine different particles by $d\lambda, d\tilde{\lambda}$ in place of η_I

$$|a\rangle = (d\lambda)^2|a\rangle^{+1} + \cdots + (d\tilde{\lambda})^2|a\rangle^{-1}$$

Motivation

- Motivation: Try to define the Amplituhderon in terms of usual spinor-helicity variables $\{\lambda_i, \tilde{\lambda}_i\}$
- It is too ambitious. Find the canonical form first.
- Method: Combine different particles by $d\lambda, d\tilde{\lambda}$ in place of η_I

$$|a\rangle = (d\lambda)^2|a\rangle^{+1} + \cdots + (d\tilde{\lambda})^2|a\rangle^{-1}$$

- The result of this simple replacement is *every amplitude with n-point and L-loop turn out to be* $(2n-4+4L)$-$d\log$ form

MHV case

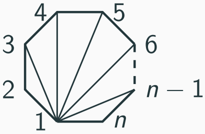
some simple examples:

$$\Omega_{3,\mathrm{MHV}} = d\log\frac{\langle 12\rangle}{\langle 23\rangle}\wedge d\log\frac{\langle 23\rangle}{\langle 13\rangle}$$

$$\Omega_{4,\mathrm{MHV}} = d\log\frac{\langle 12\rangle}{\langle 23\rangle}\wedge d\log\frac{\langle 23\rangle}{\langle 13\rangle}\wedge d\log\frac{\langle 13\rangle}{\langle 34\rangle}\wedge d\log\frac{\langle 34\rangle}{\langle 14\rangle}$$

The result for MHV n-pt corresponds to a triangulation of n-gon :

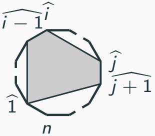
$$\Omega_{n,\mathrm{MHV}} = \bigwedge_{i=1}^{n-2} d\log\frac{\langle a_i b_i\rangle}{\langle b_i c_i\rangle}\wedge d\log\frac{\langle b_i c_i\rangle}{\langle a_i b_i\rangle}$$



$$= \bigwedge_{i=2}^{n-1} d\log\frac{\langle 1\, i\rangle}{\langle i+1\, 1\rangle}\wedge d\log\frac{\langle i\, i+1\rangle}{\langle i+1\, 1\rangle}\ .$$

NMHV case

$$\sum_{1<i<j<n}$$



$$= \sum_{1<i<j<n} \Omega_{\mathrm{MHV}}(1, \cdots, i-1) \wedge \Omega_{\mathrm{MHV}}(i, \cdots, j)$$

$$\wedge\, \Omega_{\mathrm{MHV}}(j+1, \cdots, n, 1) \wedge \Omega_{\overline{\mathrm{MHV}}}(\widehat{1}, \widehat{i-1}, \widehat{i}, \widehat{j}, \widehat{j+1}) \,,$$

where

$$\tilde{\lambda}_{\hat{1}} = \tilde{\lambda}_1 + \sum_{a=2}^{i-1} \frac{\langle i\, a\rangle}{\langle i\, 1\rangle} \tilde{\lambda}_a + \sum_{a=j+2}^{n} \frac{\langle a\, j+1\rangle}{\langle 1\, j+1\rangle} \tilde{\lambda}_a \,,$$

$$\tilde{\lambda}_{\hat{i}} = \tilde{\lambda}_i + \sum_{a=2}^{i-1} \frac{\langle a\, 1\rangle}{\langle i\, 1\rangle} \tilde{\lambda}_a \,, \qquad \tilde{\lambda}_{\widehat{i+1}} = \tilde{\lambda}_{i+1} + \sum_{a=i+2}^{j-1} \frac{\langle j\, a\rangle}{\langle j\, i+1\rangle} \tilde{\lambda}_a \,,$$

$$\tilde{\lambda}_{\hat{j}} = \tilde{\lambda}_j + \sum_{a=m+1}^{n-1} \frac{\langle a\, i+1\rangle}{\langle j\, i+1\rangle} \tilde{\lambda}_a \,, \qquad \tilde{\lambda}_{\widehat{j+1}} = \tilde{\lambda}_{j+1} + \sum_{a=m+1}^{n-1} \frac{\langle 1\, a\rangle}{\langle 1\, j+1\rangle} \tilde{\lambda}_a \,.$$

There is an algorithm to obtain $\Omega^{\mathrm{tree}}_{n,k}$

- The obstacle to its geometry is the dimension.

- The obstacle to its geometry is the dimension.
- Recently, Cachazo *et al.* find the canonical form for MHV case can be written as

$$\Omega_{n,\mathrm{MHV}} = \frac{1}{(n-2)!}\left(\sum_{i=2}^{n-1} d\log\frac{\langle 1\, i\rangle}{\langle i+1\, 1\rangle}\wedge d\log\frac{\langle i\, i+1\rangle}{\langle i+1\, 1\rangle}\right)^{n-2}$$
$$\sim \frac{1}{(n-2)!}\left(\text{canonical form for } n\text{-gon}\right)^{n-2}$$

Outlook

- The connection with amplituhderon $4k$-form.
- How to intrinsically define amplituehdron in momentum space.
- Generalization to loop level and more general theories.