

# Algebraic letters and NMHV last entry conditions from $\bar{Q}$ -equation

Based on ongoing and recent works with Song He and Zhenjie Li

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## Last entry conditions

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# N<sup>2</sup>MHV Yangian invariants

The N<sup>2</sup>MHV Yangian invariants have already been classified.

[Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka]

They are

$$Y_1^{(2)} = [1, 2, (23) \cap (456), (234) \cap (56), 6][2, 3, 4, 5, 6]$$

$$Y_2^{(2)} = [1, 2, (34) \cap (567), (345) \cap (67), 7][3, 4, 5, 6, 7]$$

$$Y_3^{(2)} = [1, 2, 3, (345) \cap (67), 7][3, 4, 5, 6, 7]$$

$$Y_4^{(2)} = [1, 2, 3, (456) \cap (78), 8][4, 5, 6, 7, 8]$$

$$Y_5^{(2)} = [1, 2, 3, 4, 8][4, 5, 6, 7, 8]$$

$$Y_6^{(2)} = [1, 2, 3, (45) \cap (678), 8][4, 5, 6, 7, 8]$$

$$Y_7^{(2)} = [1, 2, 3, (45) \cap (678), (456) \cap (78)][4, 5, 6, 7, 8]$$

$$Y_8^{(2)} = [1, 2, 3, 4, (456) \cap (78)][4, 5, 6, 7, 8]$$

$$Y_9^{(2)} = [1, 2, 3, 4, 9][5, 6, 7, 8, 9]$$

$$Y_{10}^{(2)} = [1, 2, 3, 4, (567) \cap (89)][5, 6, 7, 8, 9]$$

$$Y_{11}^{(2)} = [1, 2, 3, 4, (56) \cap (789)][5, 6, 7, 8, 9]$$

$$Y_{12}^{(2)} = \varphi[1, 2, 3, (45) \cap (789), (46) \cap (789)][(45) \cap (123), (46) \cap (123), 7, 8, 9]$$

$$Y_{13}^{(2)} = [1, 2, 3, 4, 5][6, 7, 8, 9, 10]$$

$$Y_{14}^{(2)} = \psi[A, 1, 2, 3, 4][B, 5, 6, 7, 8]$$

where

$$(ij) \cap (klm) = \mathcal{Z}_i \langle jklm \rangle - \mathcal{Z}_j \langle iklm \rangle$$

# NMHV last entry conditions

For  $N^2$ MHV yangian invariants, this operation gives three kinds of last entries

1.

$$[ijklm] \bar{Q} \log \frac{\langle \bar{n}a \rangle}{\langle \bar{n}b \rangle}$$

2.

$$[1i_1i_2i_3i_4] \bar{Q} \log \frac{\langle 1(n-1n)(i_1i_2)(i_3i_4) \rangle}{\langle \bar{n}i_1 \rangle \langle 1i_2i_3i_4 \rangle}, \quad [i_1i_2i_3i_4n-1] \bar{Q} \log \frac{\langle n-1(n1)(i_1i_2)(i_3i_4) \rangle}{\langle \bar{n}i_1 \rangle \langle n-1i_2i_3i_4 \rangle}$$

$$[i_1i_2i_3i_4n] \bar{Q} \log \frac{\langle n(1n-1)(i_1i_2)(i_3i_4) \rangle}{\langle \bar{n}i_1 \rangle \langle ni_2i_3i_4 \rangle} \quad \text{with } 1 < i_1 < i_2 < i_3 < i_4 < n-1$$

where  $\langle a(bc)(de)(fg) \rangle = \langle abc(de) \cap (fga) \rangle = \langle abde \rangle \langle acfg \rangle - \langle acde \rangle \langle abfg \rangle$

3.

$$[i_1i_2i_3i_4i_5] \bar{Q} \log \frac{\langle \bar{n}(i_1i_2) \cap (i_3i_4i_5) \rangle}{\langle \bar{n}i_1 \rangle \langle i_2i_3i_4i_5 \rangle}, \quad [i_1i_2i_3i_4i_5] \bar{Q} \log \frac{\langle \bar{n}(i_1i_2i_3) \cap (i_4i_5) \rangle}{\langle \bar{n}i_1 \rangle \langle i_2i_3i_4i_5 \rangle},$$

with  $1 < i_1 < i_2 < i_3 < i_4 < i_5 < n$

# Algebraic letters and words in two-loop NMHV amplitudes

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# Input

To compute the 2-loop NMHV  $n$  point BDS-normalized amplitude, we need the input of the one-loop  $N^2$ MHV BDS-normalized amplitude  $R_{n+1,2}^{(1)}$ , which can be obtained from the chiral/scalar box expansion [Bourjaily, Caron-Huot, Trnka]:

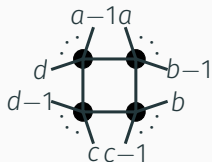
$$R_{n+1,2}^{(1)} = \sum_{a < b < c < d} (f_{a,b,c,d} - R_{n+1,2}^{\text{tree}} f_{a,b,c,d}^{\text{MHV}}) \mathcal{I}_{a,b,c,d}^{\text{fin}}$$

where

- $f_{a,b,c,d}$  are box coefficients for one-loop  $N^2$ MHV amplitudes
- $f_{a,b,c,d}^{\text{MHV}}$  are either 1 or 0
- $\mathcal{I}_{a,b,c,d}^{\text{fin}}$  denote the finite part of DCI-regulated box integrals

# Four-mass box

The most generic term in the scalar box expansion:



$$\left\{ \begin{aligned} u_{abcd} &= \frac{x_{ab}^2 x_{cd}^2}{x_{ac}^2 x_{bd}^2}, \quad v_{abcd} = \frac{x_{ad}^2 x_{bc}^2}{x_{ac}^2 x_{bd}^2}, \quad \Delta_{abcd} = \sqrt{(1-u-v)^2 - 4uv} \\ z_{abcd} &= \frac{1}{2}(1+u-v+\Delta), \quad \bar{z}_{a,b,c,d} = \frac{1}{2}(1+u-v-\Delta), \end{aligned} \right.$$

For such a box,

$$f_{a,b,c,d} = \sum_{\pm} \frac{1-u-v \pm \Delta}{2\Delta} [\alpha_{\pm}, b-1, b, c-1, c] [\delta_{\pm}, d-1, d, a-1, a]$$

$$\mathcal{I}_{a,b,c,d}^{\text{fin}} = \mathcal{I}_{a,b,c,d} = \text{Li}_2(z) - \text{Li}_2(\bar{z}) + \frac{1}{2} \log(z\bar{z}) \log \frac{1-z}{1-\bar{z}}$$

where  $\alpha_{\pm}$  and  $\delta_{\pm}$  are two solutions of Schubert problem

$$\alpha = (a-1a) \cap (dd-1\gamma), \quad \gamma = (c-1c) \cap (bb-1\alpha)$$

The square root will disappear when one mass corner become massless, e.g.  $b = a+1$

# Rationalize the square root $\Delta$

Only  $f_{a,b,c,n+1}$  and  $f_{1,b,c,n}$  survive under the  $d^{2|3} \mathcal{Z}_{n+1}$  integration, however only the first kind is an obstacle since

$$\begin{aligned}\Delta_{1,b,c,n} &\xrightarrow{\mathcal{Z}_{n+1}||\mathcal{Z}_n} 1 - u_{1,b,c,n} \\ \Delta_{a,b,c,n+1} &\xrightarrow{\mathcal{Z}_{n+1}||\mathcal{Z}_n} \frac{\sqrt{A\tau^2 + B\tau + C}}{\tau + u_{1,b,c,n}}\end{aligned}$$

$A, B, C$ : Some quadratic polynomials of cross ratios.

This is just the classic problem to find a rational parameterization of a quadratic curve  $y^2 = x^2 + ax + b$ .

This square root can be rationalized by a variable substitution  $t(\tau)$  :

$$\int_0^\infty R(\tau, \Delta(\tau)) d\tau \xrightarrow{t(\tau)} \int_{\Delta_{abcn}}^{\Delta_{1abc}} R'(t) dt$$



# Algebraic letters of two-loop NMHV amplitudes

1-D  $\tau$ -integrals for these four masses introduce new algebraic letters<sup>1</sup>

$$\mathcal{X}_{a,b,c,d}^* := \frac{(X_{a,b,c,d}^* + 1)^{-1} - \bar{Z}_{d,a,b,c}}{(X_{a,b,c,d}^* + 1)^{-1} - Z_{d,a,b,c}}, \quad \tilde{\mathcal{X}}_{a,b,c,d}^* := \frac{(X_{a,b,c,d-1}^* + 1)^{-1} - Z_{d,a,b,c}}{(X_{a,b,c,d-1}^* + 1)^{-1} - \bar{Z}_{d,a,b,c}}$$

with 6 choices  $a-1, a, b-1, b, c-1, c$  of the superscript, where

$$\begin{aligned} x_{a,b,c,d}^a &= \frac{\langle \bar{d}(c-1c) \cap (ab-1b) \rangle}{\langle \bar{d}a \rangle \langle b-1bc-1c \rangle}, & x_{a,b,c,d}^{a-1} &= x_{a,b,c,d}^a|_{a \leftrightarrow a-1} \\ x_{a,b,c,d}^b &= \frac{\langle \bar{d}(c-1c) \cap (a-1ab) \rangle}{\langle \bar{d}(a-1a) \cap (bc-1c) \rangle}, & x_{a,b,c,d}^{b-1} &= x_{a,b,c,d}^b|_{b \leftrightarrow b-1} \\ x_{a,b,c,d}^c &= \frac{\langle \bar{d}c \rangle \langle a-1ab-1b \rangle}{\langle \bar{d}(a-1a) \cap (b-1bc) \rangle}, & x_{a,b,c,d}^{c-1} &= x_{a,b,c,d}^c|_{c \leftrightarrow c-1} \end{aligned}$$

Note that  $\mathcal{X}_{a,b,c,d}^*$ ,  $\mathcal{X}_{b,c,d,a}^*$ ,  $\mathcal{X}_{c,d,a,b}^*$  and  $\mathcal{X}_{d,a,b,c}^*$  involve the same square root  $\Delta_{a,b,c,d}$

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<sup>1</sup>The algebraic letters always can and will be written in terms of  $(a + \Delta)/(a - \Delta)$  such that their multiplicative relations don't involve rational letters.

# Counting

In the most generic case (each corner with at least 3 particles), there would be  $12 \times 4 + 2 = 50$  letters associated with the same  $\Delta_{a,b,c,d}$ .

However, some degeneracy happens when some mass corners only involve 2 particles, for example

$$\mathcal{X}_{d+2,b,c,d}^{d+1} = \frac{\bar{Z}_{d,d+2,b,c}}{Z_{d,d+2,b,c}}, \quad \tilde{\mathcal{X}}_{a,b,d-2,d}^{d-2} = \frac{1 - Z_{d,a,b,d-2}}{1 - \bar{Z}_{d,a,b,d-2}}.$$

This leave us  $50 - 2m$  algebraic letters associated with the same  $\Delta$  where

$m =$  the number of corners that contain only two particles

These  $\mathcal{X}$ 's and  $\tilde{\mathcal{X}}$ 's, together with  $z/\bar{z}$  and  $(1 - z)/(1 - \bar{z})$  give a cyclic and reflection invariant set of algebraic letters.

# Multiplicative relations among algebraic letters

33 multiplicative relations:

$$\begin{aligned} \frac{\chi_{a,b,c,d}^{a-1}}{\chi_{a,b,c,d}^a} &= \frac{\chi_{d,a,b,c}^a}{\chi_{d,a,b,c}^{a-1}}, & \frac{\chi_{d,a,b,c}^{a-1}}{\chi_{d,a,b,c}^a} &= \frac{\chi_{c,d,a,b}^a}{\chi_{c,d,a,b}^{a-1}}, \\ \frac{\tilde{\chi}_{a,b,c,d}^{a-1}}{\tilde{\chi}_{a,b,c,d}^a} &= \frac{\tilde{\chi}_{d,a,b,c}^a}{\tilde{\chi}_{d,a,b,c}^{a-1}}, & \frac{\tilde{\chi}_{d,a,b,c}^{a-1}}{\tilde{\chi}_{d,a,b,c}^a} &= \frac{\tilde{\chi}_{c,d,a,b}^a}{\tilde{\chi}_{c,d,a,b}^{a-1}}, \\ \frac{\chi_{a,b,c,d}^{a-1}}{\chi_{a,b,c,d}^a} &= \frac{\tilde{\chi}_{c,d,a,b}^a}{\tilde{\chi}_{c,d,a,b}^{a-1}}, & \frac{\chi_{a,b,c,d}^a}{\chi_{a,b,c,d}^b} &= \frac{\tilde{\chi}_{a,b,c,d}^b}{\tilde{\chi}_{a,b,c,d}^a}, & \frac{\chi_{a,b,c,d}^b}{\chi_{a,b,c,d}^c} &= \frac{\tilde{\chi}_{a,b,c,d}^c}{\tilde{\chi}_{a,b,c,d}^b} \end{aligned}$$

and 21 images under the rotations of  $a \rightarrow b \rightarrow c \rightarrow d \rightarrow a$ ,

$$\begin{aligned} \frac{\chi_{a,b,c,d}^a \chi_{b,c,d,a}^d \chi_{c,d,a,b}^d \chi_{d,a,b,c}^a}{\chi_{a,b,c,d}^b \chi_{b,c,d,a}^c \chi_{c,d,a,b}^c \chi_{d,a,b,c}^b} &= 1, \\ \frac{\chi_{a,b,c,d}^a \chi_{b,c,d,a}^d \chi_{d,a,b,c}^a}{\chi_{a,b,c,d}^c \chi_{b,c,d,a}^c \chi_{d,a,b,c}^d} &= 1, & \frac{\chi_{b,c,d,a}^b \chi_{c,d,a,b}^a \chi_{d,a,b,c}^a}{\chi_{b,c,d,a}^c \chi_{c,d,a,b}^c \chi_{d,a,b,c}^b} &= 1, \\ \frac{\chi_{c,d,a,b}^a \chi_{d,a,b,c}^a}{\chi_{c,d,a,b}^d \chi_{d,a,b,c}^d} &= \frac{Z_{a,b,c,d}}{\bar{Z}_{a,b,c,d}}, & \frac{\chi_{b,c,d,a}^c \chi_{c,d,a,b}^c}{\chi_{b,c,d,a}^d \chi_{c,d,a,b}^d} &= \frac{1 - Z_{a,b,c,d}}{1 - \bar{Z}_{a,b,c,d}}. \end{aligned}$$

Taking these relations into account:

number of multiplicatively independent algebraic letters =  $17 - 2m$

## Two kinds of cuts

The algebraic letters can be rewritten as  $(a \pm \sqrt{a^2 - 4b})$ .

$(a, b)$  are polynomials of Plücker coordinates.

Such letters indicate two kinds of cuts:

- One arises from the discriminant  $a^2 - 4b$ .
- The other arises from  $b \rightarrow 0$  which is the same as the cut of  $\log b$ .

That is,  $b$  must belong to the alphabet of **rational letters**:

For example

$$|(x_{a,b,c,d}^c + 1)^{-1} - \bar{z}_{d,a,b,c}|^2 \propto \langle c(A)(B)(D) \rangle \langle (A) \cap (\bar{d})B(C) \cap (\bar{d}) \rangle \langle AB \rangle ,$$

$$|(x_{a,b,c,d}^b + 1)^{-1} - \bar{z}_{d,a,b,c}|^2 \propto \langle b(A)(C)(D) \rangle \langle (A) \cap (\bar{d})B(C) \cap (\bar{d}) \rangle ,$$

$$|(x_{a,b,c,d}^a + 1)^{-1} - \bar{z}_{d,a,b,c}|^2 \propto \langle a(B)(C)(D) \rangle \langle (A) \cap (\bar{d})B(C) \cap (\bar{d}) \rangle \langle BC \rangle ,$$

here  $A = (a-1 a)$ ,  $B = (b-1 b)$ ,  $C = (c-1 c)$ ,  $D = (d-1 d)$ .

For 9-point, the rational letters on R.H.S belongs to  $9 \times 58$  rational letters we found.

New class of rational letters in 9-point:  $\langle (A) \cap (\bar{d})B(C) \cap (\bar{d}) \rangle$

# Algebraic Words

The first two entries of algebraic words always can consist of the symbol of the four-mass box.

For non-degenerate  $\mathcal{X}_{a,b,c,d}^*$ 's as the third entry:

$$\begin{aligned}
 & \mathcal{S}(\mathcal{I}_{a,b,c,d}) \otimes \mathcal{X}_{a,b,c,d}^{c-1} \otimes x_{a,b,c,d}^{c-1} [a-1 a b-1 b c-1] \\
 & - \mathcal{S}(\mathcal{I}_{a,b,c,d}) \otimes \mathcal{X}_{a,b,c,d}^c \otimes x_{a,b,c,d}^c [a-1 a b-1 b c] \\
 & + \mathcal{S}(\mathcal{I}_{a,b,c,d}) \otimes \mathcal{X}_{a,b,c,d}^{b-1} \otimes x_{a,b,c,d}^{b-1} [a-1 a b-1 c-1 c] \\
 & - \mathcal{S}(\mathcal{I}_{a,b,c,d}) \otimes \mathcal{X}_{a,b,c,d}^b \otimes x_{a,b,c,d}^c [a-1 a b c-1 c] \\
 & + \mathcal{S}(\mathcal{I}_{a,b,c,d}) \otimes \mathcal{X}_{a,b,c,d}^{a-1} \otimes x_{a,b,c,d}^{a-1} [a-1 b-1 b c-1 c] \\
 & - \mathcal{S}(\mathcal{I}_{a,b,c,d}) \otimes \mathcal{X}_{a,b,c,d}^a \otimes x_{a,b,c,d}^a [a b-1 b c-1 c],
 \end{aligned}$$

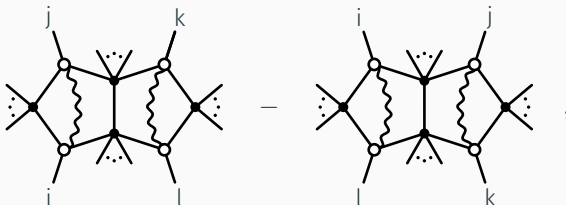
likewise for  $\tilde{\mathcal{X}}$ . Recall that  $\mathcal{X}^* := \frac{(x^*+1)^{-1}-\bar{z}}{(x^*+1)^{-1}-z}$ ,

The results for  $z/\bar{z}$  or  $(1-z)/(1-\bar{z})$  are slight lengthy, but the accompanied  $R$  invariants always are the form of  $[i i+1 j j+1 k]!$

# A class of special components

The  $\chi_i \chi_j \chi_k \chi_l$  components with  $i, j, k, l$  nonadjacent

- first show up in the two-loop amplitudes,
- **completely free** of algebraic letters,
- correspond to the difference of double-pentagon integrals  
[Arkani-Hamed, Bourjaily, Cachazo, Trnka]



each of which depend on **many** algebraic roots.[Bourjaily, McLeod, Vergu, Volk, von Hippel, Wilhelm]

- Octagons at three-loop MHV and two-loop  $N^2$ MHV.
- The connection to cluster algebra, tropical Grassmannian
- $\bar{Q}$  equations for individual integral and other theories.

Thank You

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# The kernel of $\bar{Q}$

When  $k=1$ ,

$$\begin{aligned}\bar{Q}\left([1, 2, 3, 4, 5] \log \frac{\langle 1234 \rangle}{\langle 2345 \rangle}\right) &= [1, 2, 3, 4, 5] \bar{Q} \log \frac{\langle 1234 \rangle}{\langle 2345 \rangle} \\ &= (\bar{3})_a [1, 2, 3, 4, 5] \frac{\langle 1234 \rangle \chi_5^A + \text{cyclic}}{\langle 2345 \rangle \langle 2341 \rangle}\end{aligned}$$

When  $k=2$ , it's easy to show

$$Y_1^{(2)} = \frac{\delta^{0|4} (\langle 1234 \rangle \chi_5 \chi_6 + \text{cyclic})}{\langle 1234 \rangle \cdots \langle 6123 \rangle} \propto \bar{Q} \log u \bar{Q} \log v \bar{Q} \log w$$

then

$$\bar{Q}(Y_1^{(2)} F(u, v, w)) = 0$$

# Outline of derivation of $\bar{Q}$ -equation

By using chiral Lagrangian insertion [Caron-Huot], one can show

$$\bar{Q}_{\dot{\alpha}}^A \langle W_{n,k} \rangle \propto \oint dx_{\dot{\alpha}\alpha} \langle (\psi^A + F\theta^A + \dots)^\alpha W_{n,k} \rangle$$

To obtain the  $\bar{Q}$ -equation, there are two powerful facts:

- The fermion insertion is the unique twist-one excitation with the quantum numbers of  $\bar{Q}$ .
- Its expectation value can be extracted from any object having a nonzero overlap with it in the OPE limit. [Alday, Gaiotto, Maldacena, Sever, Vieira]

$\langle W_{n+1,k+1} \rangle$  has a nonzero overlap with  $\bar{Q}_{\dot{\alpha}}^A \langle W_{n,k} \rangle$  under the collinear limit, while  $\int d^2|3 \mathcal{Z}_{n+1}$  has the same quantum number as  $\bar{Q}$ ,

$$\bar{Q}_{\dot{\alpha}}^A \langle W_{n,k} \rangle \propto \int d^2|3 \mathcal{Z}_{n+1} \langle W_{n+1,k+1} \rangle$$