Some Notes on Polylogarithm

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1 Two ways of defining (Multiple) Polylogarithm

1. MPL's can be defined by iterated integrals (Goncharov G-function)

$$G(\sigma_1, \sigma_2, \ldots, \sigma_n; z) = \int_0^z \frac{\mathrm{d}t}{t-\sigma_1} G(\sigma_2, \ldots, \sigma_n; t) \tag{1.1}$$

with

$$G(z) := 1\,, \qquad G(\vec{0}_n; z) := \frac{1}{n!} \log^n z\,. \tag{1.2}$$

In Panzer's notation(arXiv: 1403.3385), this function is denoted by $L_{\sigma_1,\cdots,\sigma_n}$ or $L_{\omega_{\sigma_1},\cdots,\omega_{\sigma_n}}$

Furthermore, we introduce the iterated integrals with general endpoints

$$I(a_0; a_1, \ldots, a_n; a_{n+1}) = \int_{a_0<t_1<\cdots<t_n<a_{n+1}} \frac{\mathrm{d}t_1 \cdots \mathrm{d}t_n}{(t_1-a_1)\cdots(t_n-a_n)} \tag{1.3}$$

2. MPL's can be defined by nested sums

$$\mathrm{Li}_{m_1,\ldots,m_k}(z_1, \ldots, z_k) := \sum_{0<n_1<n_2<\cdots<n_k} \frac{z_1^{n_1} z_2^{n_2} \cdots z_k^{n_k}}{n_1^{m_1} n_2^{m_2} \cdots n_k^{m_k}} \qquad \text{for } |z_i| < 1 \tag{1.4}$$

This can be viewd as a generation of classical polylogarithm:

$$\mathrm{Li}_1(z) = \sum_{n=1}^{\infty} \frac{z^n}{n} = -\log(1-z) = \int_0^z \frac{\mathrm{d}t}{1-t} \tag{1.5}$$

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k} = \int_0^z \frac{\mathrm{d}t}{t} \mathrm{Li}_{k-1}(t) \tag{1.6}$$

This two kinds of MPL's are related by

$$G\left(\vec{0}_{m_1-1}, \sigma_1, \ldots, \vec{0}_{m_k-1}, \sigma_k; z\right) = (-1)^k \,\mathrm{Li}_{m_k,\ldots,m_1}\left(\frac{\sigma_{k-1}}{\sigma_k}, \ldots, \frac{\sigma_1}{\sigma_2}, \frac{z}{\sigma_1}\right) \tag{1.7}$$

Several special examples of G-functions

$$G(\vec{\sigma}_n; z) = \frac{1}{n!} \log^n\left(1 - \frac{z}{\sigma}\right) \qquad G(\vec{0}_{n-1}, \sigma; z) = -\mathrm{Li}_n\left(\frac{z}{\sigma}\right). \tag{1.8}$$

In this note, we will focus on G-functions and denote $G(\sigma_1, \sigma_2, \ldots, \sigma_n; z)$ by $G(\vec{\sigma}; z)$

2 Shuffle algebras, derivatives, integrals of MPL's

According to the definition of G-function (1.1), the differentiation is trivial

$$\partial_z G(\sigma_1, \ldots, \sigma_n; z) = \frac{1}{z - \sigma_1} G(\sigma_2, \ldots, \sigma_n; z) \,. \tag{2.1}$$

The integral of $R(z)G(\vec{\sigma}; z)$ with a rational function $R(z)$ is also easy due to

$$\int \mathrm{d}z \, \frac{G(\vec{\sigma}; z)}{(z-\rho)^{n+1}} = -\frac{G(\vec{\sigma}; z)}{n(z-\rho)^n} + \int \mathrm{d}z \, \frac{\partial_z G(\vec{\sigma}; z)}{n(z-\rho)^n} \,, \tag{2.2}$$

$$\int \mathrm{d}z \, z^n G(\vec{\sigma}; z) = \frac{z^{n+1} G(\vec{\sigma}; z)}{n+1} - \int \mathrm{d}z \, \frac{z^n \partial_z G(\vec{\sigma}; z)}{n+1} \tag{2.3}$$

Now let us consider a product of two G-functions

$$G(\vec{\sigma}; z) G(\vec{\rho}; z) = \int_{0<t_n<t_{n-1}<\cdots<t_1<z} \frac{\mathrm{d}t_1 \cdots \mathrm{d}t_n}{(t_1 - \sigma_1) \cdots (t_n - \sigma_n)} \int_{0<t'_m<t'_{m-1}<\cdots<t'_1<z} \frac{\mathrm{d}t'_1 \cdots \mathrm{d}t'_m}{(t'_1 - \rho_1) \cdots (t'_n - \rho_n)} \,.$$

This integral can be expressed as a sum of iterated integral by decomposed the integration region into simplexes. Obviously, The only request on t and t' in each simplex is that they should preserve their origin orders, respectively. Then all simplexes correspond to all possible relative orders between t and t'. Thus, we end up with

$$G(\vec{\sigma}; z) G(\vec{\rho}; z) = \sum_{\vec{\omega} \in \vec{\sigma} \sqcup\!\sqcup \vec{\rho}} G(\vec{\omega}; z) \tag{2.4}$$

Example:

$$\begin{aligned} G(a_1, a_2; z) G(b_1, b_2; z) = {} & G(a_1, a_2, b_1, b_2; z) + G(a_1, b_1, a_2, b_2; z) + G(b_1, a_1, a_2, b_2; z) \\ & + G(a_1, b_1, b_2, a_2; z) + G(b_1, a_1, b_2, a_2; z) + G(b_1, b_2, a_1, a_2; z) \end{aligned}$$

3 Symbol of MPL

There are two approaches to define the symbol for a MPL (iterated integral). One is coproduct, the other is differential.

1. Coproduct

There is a Hopf algebra on the space of iterated integrals(arXiv:math/0208144), that is, we can define coproduct for an iterated integral

$$\begin{aligned} & \big(\Delta(I(a_0; a_1, \ldots, a_n; a_{n+1}))\big) \\ & = \sum_{0=i_0<i_1<\cdots i_k<i_{k+1}=n+1} I(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1}) \otimes \left(\prod_{p=0}^{k} I(a_{i_p}; a_{i_p+1}, \ldots, a_{i_{p+1}-1}; a_{i_{p+1}}) \right) \end{aligned}$$

The symbol (map) can be defined as the maximal iteration of the coproduct (modulo iπ)

$$\mathcal{S}(F) = \Delta_{1,\dots,1}(F) \bmod \mathrm{i}\pi \tag{3.1}$$

(For a clear review of Hopf algebra, see arXiv:1203.0454)

2. Differential

The total differential of a MPL's $F^{(n)}(\sigma_1, \dots, \sigma_m)$ can be written as

$$\mathrm{d}F^{(n)} = \sum_i F_i^{(n-1)} \mathrm{d}\log R_i \tag{3.2}$$

where R_i are rational functions of $\sigma_1, \dots, \sigma_m$, then the symbol of $F^{(n)}$ can be computed recursively by

$$\mathcal{S}(F^{(n)}) = \sum_i \mathcal{S}(F_i^{(n-1)}) \otimes R_i \,. \tag{3.3}$$

Indeed, the differential of a iterated integral is

$$\mathrm{d}I(a_0; a_1, \dots, a_n; a_{n+1}) = \sum_{i=1}^{n} I(a_0; a_1, \dots, \hat{a}_i, \dots, a_n; a_{n+1}) \mathrm{d}\log \frac{a_{i+1} - a_i}{a_i - a_{i-1}} \,. \tag{3.4}$$

Example:

$$\mathcal{S}\left(\prod_{i=1}^{n} \log a_i\right) = \sum_{\sigma \in S_n} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)} \,, \qquad \mathcal{S}\Big(\mathrm{Li}_n(z)\Big) = -\Big((1-z) \otimes \underbrace{z \otimes \cdots \otimes z}_{n-1}\Big) \tag{3.5}$$

It is easy to see that

$$A \otimes a \otimes B + A \otimes b \otimes B = A \otimes ab \otimes B \tag{3.6}$$

$$A \otimes \rho_n \otimes B = 0 \tag{3.7}$$

$$\mathcal{S}\big(FG\big) = \mathcal{S}(F) \sqcup\!\sqcup\, \mathcal{S}(G) \tag{3.8}$$

where ρ_n is an n-th root of unity.

The introduction of symbol trivialize the functional relation of MPL, for example

$$\mathrm{Li}_2(1-z) = \zeta_2 - \mathrm{Li}_2(z) - \log(z)\log(1-z) \,. \tag{3.9}$$

To integrate a symbol

$$S = \sum_{I=(i_1,\dots,i_n)} c_I a_{i_1} \otimes \cdots \otimes a_{i_n} \tag{3.10}$$

to a function, the symbol must satisfy the integrability condition

$$\sum c_I a_{i_1} \otimes \cdots \otimes a_{i_{p-1}} \otimes a_{i_{p+2}} \otimes \cdots \otimes a_{i_n} \, \mathrm{d}\log a_{i_p} \wedge \mathrm{d}\log a_{i_{p+1}} = 0 \tag{3.11}$$

for every consecutive pair of indices.

4 Symbol Integration

(This part is the same as the argument in arXiv:1806.06072)

Consider the following integral where the integrand converges both at zero and infinity (more generally one can consider an integral with finite integration endpoints),

$$I(y, x_i) = \int_0^\infty \mathrm{d}\log(x+y) F^{(n)}(x, x_i) \equiv \int_0^\infty \frac{\mathrm{d}x}{x+y} F^{(n)}(x, x_i)\,. \tag{4.1}$$

The first step in the symbol integration involves taking the total differential of the integral $I(y, x_i)$,

$$\begin{aligned} \mathrm{d}I(y, x_i) &= \left[\mathrm{d}y\partial_y + \mathrm{d}x_i\partial_{x_i}\right] I(y, x_i) \\ &= -\mathrm{d}y \int_0^\infty \frac{\mathrm{d}x}{(x+y)^2} F^{(n)}(x, x_i) + \mathrm{d}x_i \int_0^\infty \frac{\mathrm{d}x}{(x+y)} \partial_{x_i} F^{(n)}(x, x_i)\,. \end{aligned} \tag{4.2}$$

The first term we write as a derivative with respect to x and integrate by parts, we find

$$\mathrm{d}y \int_0^\infty \mathrm{d}x \left[\partial_x \frac{1}{(x+y)}\right] F^{(n)}(x, x_i) = \mathrm{d}y \frac{F^{(n)}(x, x_i)}{x+y}\Big|_{x=0}^{x=\infty} - \mathrm{d}y \int_0^\infty \frac{\mathrm{d}x}{(x+y)} \partial_x F^{(n)}(x, x_i)\,. \tag{4.3}$$

Since the $F^{(n)}(x, x_i)$ are MPL's,

$$\begin{aligned} \partial_x F^{(n)}(x, x_i) &= \sum_j F_j^{(n-1)}(x, x_i) \frac{\partial \log(x+\beta_j)}{\partial x} \\ \partial_{x_i} F^{(n)}(x, x_i) &= \sum_j F_j^{(n-1)}(x, x_i) \frac{\partial \log(x+\beta_j)}{\partial \beta_j} \left(\frac{\partial \beta_j}{\partial x_i}\right) + \sum_{j'} H_{j'}^{(n-1)}(x, x_i) \frac{\partial \log f_{j'}}{\partial x_i} \end{aligned} \tag{4.4}$$

where $f_{j'}$ are functions of x_i's only.

Taking the boundary term at $x = \infty$ to vanish, the differential of the integral becomes,

$$\begin{aligned} \mathrm{d}I(y, x_i) = -\mathrm{d}\log y F^{(n)}(0, x_i) &- \sum_j \mathrm{d}y \int_0^\infty \frac{\mathrm{d}x}{(x+y)(x+\beta_j)} F_j^{(n-1)} \\ &+ \sum_j \underbrace{\mathrm{d}x_i \left(\frac{\partial \beta_j}{\partial x_i}\right)}_{=\mathrm{d}\beta_j} \int_0^\infty \frac{\mathrm{d}x}{(x+y)(x+\beta_j)} F_j^{(n-1)} \\ &+ \sum_{j'} \underbrace{\mathrm{d}x_i \frac{\partial \log f_{j'}}{\partial x_i}}_{=\mathrm{d}\log f_{j'}} \int_0^\infty \frac{\mathrm{d}x}{(x+y)} H_{j'}^{(n-1)} \end{aligned} \tag{4.5}$$

We use partial fraction for the first two terms,

$$\frac{1}{(x+y)(x+\beta_j)} = \frac{1}{(y-\beta_j)} \left[\frac{1}{(x+\beta_j)} - \frac{1}{(x+y)}\right]. \tag{4.6}$$

Putting everything together, we find,

$$
\begin{aligned}
\mathrm{d}I =& -\mathrm{d}\log y\, F^{(n)}(0,x_i) + \sum_{j'} \mathrm{d}\log f_{j'} \int_0^\infty \frac{\mathrm{d}x}{(x+y)} H_{j'}^{(n-1)} \\
& - \sum_j \left[\frac{\mathrm{d}y}{(y-\beta_j)} - \frac{\mathrm{d}\beta_j}{(y-\beta_j)}\right] \int_0^\infty \left[\frac{\mathrm{d}x}{(x+\beta_j)} - \frac{\mathrm{d}x}{(x+y)}\right] F_j^{(n-1)} .
\end{aligned} \tag{4.7}
$$

Combining the respective terms in the brackets back into d log-forms, we recover the expressions

$$
\begin{aligned}
\mathrm{d}I(y,x_i) =& -\mathrm{d}\log y\, F^{(n)}(0,x_i) + \sum_{j'} \mathrm{d}\log f_{j'} \int_0^\infty \mathrm{d}\log(x+y) H_{j'}^{(n-1)}(x,x_i) \\
& + \sum_j \mathrm{d}\log(y-\beta_j) \int_0^\infty \mathrm{d}\log\left(\frac{x+y}{x+\beta_j}\right) F_j^{(n-1)}(x,x_i)
\end{aligned} \tag{4.8}
$$

5 MPL Integration

(This part follows from arXiv: 0804.1660 and arXiv:1403.3385, for more details, see arXiv:1407.0074)

In the calculation of Feynman integrals or the α'-expansions of string integrals or other cases, we will usually encounter integrals involving MPL's, but the integral is of form

$$
\int \mathrm{d}z\; R(z) G(\vec{\sigma}(z); z'(z)) \tag{5.1}
$$

rather simply $\int \mathrm{d}R(z)G(\vec{\sigma};z)$ we have discussed in sec.2. To preform the integration in (5.1), we need to express $G(\vec{\sigma}(z);z'(z))$ as

$$
G(\vec{\sigma}(z); z'(z)) = \sum_{\vec{\tau}} c_{\vec{\tau}} G(\vec{\tau}; z) \tag{5.2}
$$

In the practical cases, the integral of form (5.1) usually appear in the intermediary steps of calculation of

$$
f_n = \int_0^\infty \mathrm{d}z_n\, f_{n-1}(z_n) = \int_0^\infty \mathrm{d}z_1 \cdots \int_0^\infty \mathrm{d}z_n\, f_0 \tag{5.3}
$$

with certain polylogarithms $f_0(\vec{z})$. Suppose $f_{k-1}(z_k)$ has a primitive $F(z_k)$, then

$$
\int_0^\infty f_{k-1}(z_k)\,\mathrm{d}z_k = \lim_{z_k\to\infty} F(z_k) - \lim_{z_k\to 0} F(z_k) \tag{5.4}
$$

There are divergences in each limit, although these divergences must cancel at the end. Thus, it would be convenient define a regularized limit for MPL $G(\vec{\sigma};z)$. The singularities of $G(\vec{\sigma};z)$ at $z\to\tau$ are logarithmic at worst, so we have the expansion

$$
G(\vec{\sigma}(z)) = \sum_{i=0}^{|\vec{\sigma}|} f^{(i)}_{\vec{\sigma},\tau}(z) \log^i(z-\tau) . \tag{5.5}
$$

Then regularized limits are defined as

$$\mathrm{Reg}_{z\to\tau}\, G(\vec{\sigma};z) := f^{(0)}_{\vec{\sigma},\tau}(\tau) \tag{5.6}$$

(For more details of regularized limits, see arXiv:1407.0074)

Then our problem becomes to express $\mathrm{Reg}_{z_k\to\infty}\, G(\vec{\sigma};z_k)$ in terms of $G(*;z_{k+1})$. We can approach this problem by taking partial differential of $G(\vec{\sigma};z_k)$ with respect to $z_{k+1} := t$, then we find

$$\begin{aligned}\partial_t G(\vec{\sigma};z_k) = &\sum_{i=1}^{k-1} \partial_t \log\big(\sigma_i(t)-\sigma_{i+1}(t)\big)\Big(G(\sigma_1,\dots,\hat{\sigma}_{i+1},\dots,\sigma_k;z_k) - G(\sigma_1,\dots,\hat{\sigma}_i,\dots,\sigma_k;z_k)\Big)\\ &+ \partial_t \log(\sigma_1 - z_k) G(\sigma_2,\dots,\sigma_k;z_k) - \partial_t \log\sigma_k G(\sigma_1,\dots,\sigma_{k-1};z_k)\end{aligned} \tag{5.7}$$

Using $\mathrm{Reg}_{z_k\to\infty}\,\partial_t = \partial_t\, \mathrm{Reg}_{z_k\to\infty}$ yields

$$\begin{aligned}&\partial_t\, \mathrm{Reg}_{z_k\to\infty}\, G(\vec{\sigma};z_k)\\ &= -\partial_t \log\sigma_k\, \mathrm{Reg}_{z_k\to\infty}\, G(\sigma_1,\dots,\sigma_{k-1};z_k)\\ &+ \sum_{i=1}^{k-1} \partial_t \log\big(\sigma_i(t)-\sigma_{i+1}(t)\big)\, \mathrm{Reg}_{z_k\to\infty}\Big(G(\sigma_1,\dots,\hat{\sigma}_{i+1},\dots,\sigma_k;z_k) - G(\sigma_1,\dots,\hat{\sigma}_i,\dots,\sigma_k;z_k)\Big)\end{aligned} \tag{5.8}$$

To make sure $\mathrm{Reg}_{z_k\to\infty}\, G(\vec{\sigma};z_k)$ is polylogarithms of t, We assume that $\sigma_1,\dots,\sigma_k \in \mathbb{Q}(t)$ such that any $\sigma_i - \sigma_j = c\prod_\tau (t-\tau)^{\lambda_\tau}$. We can use the same trick recursively in the right hand side, and end up with

$$\partial_t\, \mathrm{Reg}_{z_k\to\infty}\, G(\vec{\sigma};z_k) = \sum_{\vec{\rho}\in\Sigma_t^\times,\tau\in\Sigma_t} \frac{\lambda_{\tau,\vec{\rho}}}{t-\tau} G(\vec{\rho};t)\cdot c_{\vec{\rho}} \tag{5.9}$$

where $\lambda_{\tau,\vec{\rho}} \in \mathbb{Z}$, $c_{\vec{\rho}}$ is some constants,

$$\Sigma_t := \left\{ \text{zeros of } \prod_{i<j} [\sigma_i(t) - \sigma_j(t)] \right\}. \tag{5.10}$$

So

$$\mathrm{Reg}_{z_k\to\infty}\, G(\vec{\sigma};z_k) = C + \sum_{\vec{\rho}\in\Sigma_t^\times,\tau\in\Sigma_t} \lambda_{\tau,\vec{\rho}} G(\vec{\rho};t) c_{\vec{\rho}} \tag{5.11}$$

where the integration constant C is determined by

$$C = \mathrm{Reg}_{t\to 0}\, \mathrm{Reg}_{z_k\to\infty}\, G(\vec{\sigma};z_k). \tag{5.12}$$

This procedure is known as “expansion on a fibration basis”.