Amortizing Division and Exponentiation

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Inscrypt 2022

Outline

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Multi-Party Computation

- n Parties $P_1, \ldots, P_n$.
- Party P_i holds private input x_i.
- Party P_i obtains output $f_i(x_1, x_2, \ldots, x_n)$.
- Goal: Construct a protocol Π securely implement some functionality f.
- Correctness: P_i learns $f_i(x_1, x_2, \ldots, x_n)$.
- Privacy: P_i only learns $f_i(x_1, x_2, \ldots, x_n)$.

Multi-Party Computation

The general framework of MPC:

Addition $+$

Turing-complete $\Rightarrow$ Any functionality f

Multiplication $\times$

Multi-Party Computation

The general framework of MPC:

Addition $+$

Turing-complete

Any functionality f

Multiplication $\times$

Problem: the commonly used operations such as division, and exponentiation over the integers and/or the finite fields are relatively complex to express by addition and multiplication. More efficient protocols for these basic operations can result in a more efficient protocol for f.

Division and Expoentiation

Our focus in this work:

- Division: the parties have shares of a and b and want to compute the share of $a \cdot b^{-1}$ over a finite field.
- Private Exponentiation: the parties have shares of a and b and want to compute the share of b^a over a finite field.

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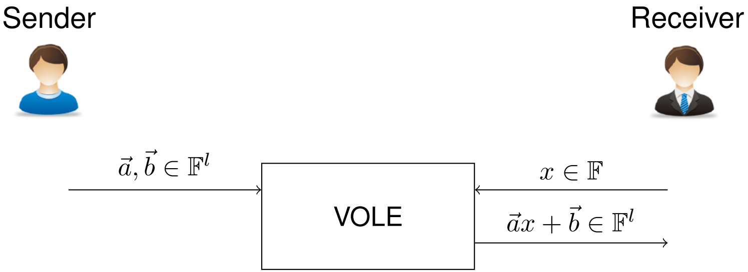
Vector Oblivious Linear-Function Evaluation



Cost: To generate length l VOLE, the communication cost is only $O(\kappa \log l)$, using recent works on pseudorandom correlation generator (PCG) [BCGI18, BCG$^+$19a, BCG$^+$19b, SGRR19, WYKW21, CRR21].

Generating Random Shares and Random Coins

Generating Random Shares functionality $\mathcal{F}_{\mathsf{rand}}$

- Input: None.
- Output: A random share $[r]$.
- Implement: Each party select a random value locally.
- Communication cost: 0.

Generating Random Coins functionality $\mathcal{F}_{\mathsf{coin}}$

- Input: None.
- Output: A random value $r \in \mathbb{F}$.
- Implement: Invoke $\mathcal{F}_{\mathsf{rand}}$ to generate $[r]$ and then open it.
- Communication cost: $O(n^2\kappa)$.

- $[r] = (r_1, \ldots, r_n)$ is additive sharing, $r = r_1 + \cdots + r_n$. Party P_i holds r_i.
- Linear: $[r] + [s] = [r + s]$.

Secure Multiplication

Secure Multiplication functionality $\mathcal{F}_{\mathsf{mult}}$

- Input: $[x]$ and $[y]$.
- Output: $[z] = [xy]$.
- Implement: Using Beaver triple [Bea91] $([a], [b], [c])$, where $c = ab$:
  1. The parties compute and open shares of $\sigma = x - a$ and $\rho = y - b$.
  2. The parties compute $[z] := \sigma[b] + \rho[a] + [c] + \sigma\rho$.
- Communication cost: $O(n^2\kappa)$.

Secure Inversion

Secure Inversion functionality $\mathcal{F}_{\text{inver}}$

- Input: $[x]$.
- Output: $[x^{-1}]$.
- Implement: Using the method of [BB89]:
  1. The parties invoke $\mathcal{F}_{\text{rand}}$ functionality to generate a random share $[r]$.
  2. The parties invoke the $\mathcal{F}_{\text{mult}}$ functionality with inputs $[x]$ and $[r]$, and obtain $[xr]$.
  3. All parties open the xr and compute $[x^{-1}] := (xr)^{-1}[r]$.
- Communication cost: $O(n^2\kappa)$.

Unbounded Fan-In Multiplication

Unbounded Fan-In Multiplication functionality $\mathcal{F}^{l}_{\text{unbounded-mult}}$

- Input: $[x_1], \ldots, [x_l]$.
- Output: $[x] = [\Pi_{i=1}^{l} x_i]$.
- Implement: Using the method of [BB89]:
  1. The parties invoke $\mathcal{F}_{\text{rand}}$ to generate $l+1$ random shares $[r_0], [r_1], \ldots, [r_l]$.
  2. The parties invoke $\mathcal{F}_{\text{inver}}$ with inputs $[r_0], \ldots, [r_l]$ and obtain $[r_0^{-1}], \ldots, [r_l^{-1}]$.
  3. For $i = 1, \ldots, l$, the parties invoke $2\ \mathcal{F}_{\text{mult}}$ to compute $[d_i] := [r_{i-1}] \cdot [x_i] \cdot [r_i^{-1}]$. Then the parties open d_i to each other.
  4. The parties invoke $\mathcal{F}_{\text{mult}}$ with input $[r_0^{-1}]$ and $[r_l]$, and compute $[x] = \Pi_{i=1}^{l}[x_i] := \Pi_{i=1}^{l} d_i \cdot [r_0^{-1}] \cdot [r_l]$.
- Communication cost: $O(n^2 \kappa l)$.

Public Base Exponentiation

Public Base Exponentiation functionality $\mathcal{F}_{\text{pbexp}}$

- Input: $[a]$ and b.
- Output: $[b^a]$.
- Implement: Using the method of [AAN18]:
  1. Each party P_i locally computes $c_i := b^{a_i}$, where a_i is the share of a.
  2. P_i shares $[c_i]$ to all parties.
  3. The parties invoke $\mathcal{F}^n_{\text{unbounded-mult}}$ to compute $[\Pi^n_{i=1} c_i] = [b^a]$.
- Communication cost: $O(n^3\kappa)$.

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Correlated Multiplication Triple Generation

The Correlated Multiplication Triple Generation functionality $\mathcal{F}^{l}_{\text{ctriple}}$ is defined as follows:

- Input: None.
- Output: Random shares $[a_1], \ldots, [a_l], [b], [c_1], \ldots, [c_l]$, where $c_i = a_i b$ for $i = 1, \ldots, l$.

Protocol

The main idea is to use VOLE to generate correlated multiplication triples. The formal description of our protocol is as follows:

1. The parties invoke $\mathcal{F}_{\text{rand}}$ to generate $[a_1], \ldots, [a_l], [b]$, where $[a_k] = (a_1^k, \ldots, a_n^k), [b] = (b_1, \ldots, b_n)$ conditioned on $\sum_{j=1}^n a_j^k = a_k, \sum_{j=1}^n b_j = b$ for $k = 1, \ldots, l$.
2. For every distinct $i, j = 1, \ldots, n$, P_j picks $v_{i,j}^k \xleftarrow{R} \mathbb{F}$ for $k = 1, \ldots, l$ and defines $\vec{a_j} := (a_j^1, \ldots, a_j^l) \in \mathbb{F}^l, \vec{v_{i,j}} := (v_{i,j}^1, \ldots, v_{i,j}^l)$. Then P_i and P_j invoke $\mathcal{F}_{\text{vole}}$, where P_i acts as receiver with input b_i and P_j acts as sender with input $(\vec{a_j}, -\vec{v}_{i,j})$. As a result, P_i receives $\vec{u_{i,j}} = \vec{a_j} b_i - \vec{v_{i,j}}$.
3. For $i = 1, \ldots, n, k = 1, \ldots, l$, P_i computes $c_i^k := a_i^k b_i + \sum_{j \neq i} (u_{i,j}^k + v_{j,i}^k)$

Communication Cost: The total communication is $O(n^2 \kappa \log l)$.

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Single Division Case

The secure division functionality $\mathcal{F}_{\text{div}}$ is defined as follows:

- Input: $[x], [y]$.
- Output: $[x^{-1}y]$.

Single Division Case

Recall the inversion protocol [BB89]:

1. The parties invoke $\mathcal{F}_{\mathsf{rand}}$ functionality to generate a random share $[a]$.
2. The parties invoke the $\mathcal{F}_{\mathsf{mult}}$ functionality with inputs $[x]$ and $[a]$, and obtain $[ax]$.
3. All parties open the ax and compute $[x^{-1}] := (ax)^{-1}[a]$.

Main observation: If we compute $[ax]$ and $[ay]$ *at the same time*, then open ax as before, the division $[x^{-1}y]$ can be obtained directly from $(ax)^{-1}[ay]$.

Single Division Case

Our single division protocol:

1. The parties invoke $\mathcal{F}^2_{\text{ctriple}}$ to generate $[a_1], [a_2], [b], [c_1], [c_2]$, where $a_1 b = c_1, a_2 b = c_2$.
2. The parties compute $[\rho] := [x - a_1], [\sigma] := [y - a_2]$ locally and open them.
3. The parties compute $[r] = [bx] := \rho[b] + [c_1]$ and $[s] = [by] := \sigma[b] + [c_2]$.
4. The parties open r and compute $[x^{-1}y] := r^{-1}[s]$.

Correctness. The correctness of π_{div} follows from:
$r^{-1}s = (\rho b + c_1)^{-1}(\sigma b + c_2) = ((x - a_1)b + c_1)^{-1}((y - a_2)b + c_2) = x^{-1}y$.
Cost. The cost is 3 openings and one instance of length 2 correlated multiplication triple generation, which is strictly less than 2 independent multiplication triple generation.

Batch Division Case

The secure division functionality in batch setting $\mathcal{F}_{\mathsf{bdiv}}$ is defined as follows:

- Input: $[x]$ and $[y_1], \ldots, [y_l]$.
- Output: $[x^{-1} y_i]$ for $i = 1, \ldots, l$.

Batch Division Case

Our batch division protocol:

1. The parties invoke $\mathcal{F}^{l+1}_{\text{ctriple}}$ to generate $[a_0], [a_1], \ldots, [a_l], [b], [c_0], [c_1], \ldots, [c_l]$, where $a_i b = c_i$ for $i = 0, 1, \ldots, l$.
2. The parties compute $[\rho] := [x - a_0], [\sigma_i] := [y_i - a_i]$ locally for $i = 1, \ldots, l$ and open them.
3. The parties compute $[r] = [bx] := \rho[b] + [c_0]$ and $[s_i] = [by_i] := \sigma_i[b] + [c_i]$ for $i = 1, \ldots, l$.
4. The parties open r and compute $[x^{-1}y_i] := r^{-1}[s_i]$ for $i = 1, \ldots, l$.

Correctness. Similar to single division case.
Cost. The cost is $l + 2$ openings and one instance of length $l + 1$ correlated multiplication triple generation. Since the open step does not need computation, and the cost of $l + 1$ correlated multiplication triple generation is only about $O(\log l)$ multiplication triple generation. The cost of our protocol is almost the same as a single division instance (except for a logarithmic factor).

Batch Exponentiation Case

The batch private exponentiation functionality $\mathcal{F}_{\mathsf{bexp}}$ is defined as follows:

- Input: $[y], [x_1], \ldots, [x_l]$.
- Output: $[y^{x_1}], \ldots, [y^{x_l}]$.

Batch Exponentiation Case

Our batch exponentiation protocol:

1. The parties invoke $\mathcal{F}_{\text{coin}}$ to generate a random generator $g \in \mathbb{F}$.
2. The parties invoke $\mathcal{F}^{l}_{\text{ctriple}}$ to generate $[a_1], \ldots, [a_l], [b], [c_1], \ldots, [c_l]$.
3. The parties invoke $\mathcal{F}_{\text{pbexp}}$ with inputs $g, [b]$ to obtain $[t] = [g^b]$.
4. The parties invoke $\mathcal{F}_{\text{mult}}$ with inputs $[y], [t]$ to obtain $[p] = [t \cdot y]$.
5. The parties open p.
6. For $i = 1, \ldots, l$, the parties invoke $\mathcal{F}_{\text{pbexp}}$ with inputs $p, [x_i]$ to obtain $[q_i] = [p^{x_i}]$.
7. For $i = 1, \ldots, l$, the parties compute $[\sigma_i] := [x_i - a_i]$ locally and open σ_i. Then the parties compute $[r_i] := -\sigma_i[b] - [c_i]$.
8. For $i = 1, \ldots, l$, the parties invoke $\mathcal{F}_{\text{pbexp}}$ with input $g, [r_i]$ to obtain $[s_i] = [g^{r_i}]$.
9. For $i = 1, \ldots, l$, the parties invoke $\mathcal{F}_{\text{mult}}$ with input $[q_i]$ and $[s_i]$ to obtain $[y^{x_i}] := [q_i s_i]$.

Correctness. The correctness of π_{bexp} follows from:
$q_i s_i = p^{x_i} \cdot g^{r_i} = (t \cdot y)^{x_i} \cdot g^{-\sigma_i b - c_i} = g^{b x_i} y^{x_i} \cdot g^{-b x_i} = y^{x_i}$

Batch Exponentiation Case

Cost. The cost of π_{bexp} is as follows:

- In step 1, one opening is needed in $\mathcal{F}_{\mathsf{coin}}$.
- In step 2, length l correlated multiplication triple generation.
- In step 3, one public base exponentition is needed, including $3n+2$ multiplications and $2n+2$ openings.
- In step 4, one multiplication is needed.
- In step 5, one opening is needed.
- In step 6, l public base exponentition is needed, including $(3n+2)l$ multiplications and $(2n+2)l$ openings.
- In step 7, l openings are needed.
- In step 8, l public base exponentitions are needed, including $(3n+2)l$ multiplications and $(2n+2)l$ openings.
- In step 9, l multiplications are needed.

Batch Exponentiation Case

The total costs are $2n + 4 + (4n + 5)l$ openings, a length l correlated multiplication triple generation, $3n + 3 + (6n + 5)l$ multiplications. While in the original single instance protocol of [AAN18], the costs are $6n + 9$ openings and $9n + 9$ multiplications. If we use l instance of [AAN18] to implement the batch private exponentiation, the costs are $(6n + 9)l$ openings and $(9n + 9)l$ multiplications. Since the main costs is multiplication, the costs saved by our protocol is about 33%.

THANK YOU!

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