Ch9 定积分

总结及习题评讲(2)

主讲教师：顾燕红

办公室：汇星楼409

办公室答疑时间：每周四下午2点至4点

微信号：18926511820 QQ号：58105217

Email: yhgu@szu.edu.cn

(添加好友时请备注 学号 姓名 数学分析2)

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BY GYH

P213/习题9.5/2 变限积分求导问题

设f在$[a,b]$上连续，$F(x)=\int_a^x f(t)(x-t)\,\mathrm{d}t$.

证明 $F''(x)=f(x), x\in[a,b]$.

证 由于 $F(x)=x\int_a^x f(t)\,\mathrm{d}t-\int_a^x tf(t)\,\mathrm{d}t$,

从而

$$F'(x)=\left(x\int_a^x f(t)\,\mathrm{d}t-\int_a^x tf(t)\,\mathrm{d}t\right)'$$

$$=\int_a^x f(t)\,\mathrm{d}t+xf(x)-xf(x)=\int_a^x f(t)\,\mathrm{d}t,$$

$$F''(x)=\left(\int_a^x f(t)\,\mathrm{d}t\right)'=f(x).$$

变限积分求导问题

P213/习题9.5/3(1) 求极限 $\lim\limits_{x\to 0}\dfrac{1}{x}\int_0^x \cos t^2\,\mathrm{d}t.$

解

$$\lim_{x\to 0}\frac{1}{x}\int_0^x \cos t^2\,\mathrm{d}t=\lim_{x\to 0}\frac{\int_0^x \cos t^2\,\mathrm{d}t}{x}$$

$$=\lim_{x\to 0}\frac{\left(\int_0^x \cos t^2\,\mathrm{d}t\right)'}{(x)'}$$

$$=\lim_{x\to 0}\frac{\cos x^2}{1}$$

$$=1.$$

变限积分求导问题

P213/习题9.5/3(2) 求极限 $\lim\limits_{x\to\infty}\dfrac{\left(\int_0^x e^{t^2}\,dt\right)^2}{\int_0^x e^{2t^2}\,dt}$.

解

$$\lim_{x\to\infty}\frac{\left(\int_0^x e^{t^2}\,dt\right)^2}{\int_0^x e^{2t^2}\,dt}=\lim_{x\to\infty}\frac{\left(\left(\int_0^x e^{t^2}\,dt\right)^2\right)'}{\left(\int_0^x e^{2t^2}\,dt\right)'}=\lim_{x\to\infty}\frac{2\int_0^x e^{t^2}\,dt\cdot e^{x^2}}{e^{2x^2}}$$

$$=2\lim_{x\to\infty}\frac{\int_0^x e^{t^2}\,dt}{e^{x^2}}=2\lim_{x\to\infty}\frac{\left(\int_0^x e^{t^2}\,dt\right)'}{\left(e^{x^2}\right)'}$$

$$=2\lim_{x\to\infty}\frac{e^{x^2}}{2xe^{x^2}}=\lim_{x\to\infty}\frac{1}{x}=0.$$

P213/习题9.5/4(4) 计算$\int_0^1 \frac{\mathrm{d}x}{(x^2-x+1)^{\frac{3}{2}}}$.

解 由于 $\int_0^1 \frac{\mathrm{d}x}{(x^2-x+1)^{\frac{3}{2}}} = \int_0^1 \frac{\mathrm{d}x}{\left(\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2\right)^{\frac{3}{2}}}$,

令 $x-\frac{1}{2}=\frac{\sqrt{3}}{2}\tan t$, 则 $\mathrm{d}x=\frac{\sqrt{3}}{2}\sec^2 t\,\mathrm{d}t$,

当$x=0$时,$t=-\frac{\pi}{6}$; 当$x=1$时,$t=\frac{\pi}{6}$. 从而

$$\int_0^1 \frac{\mathrm{d}x}{(x^2-x+1)^{\frac{3}{2}}} = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{\frac{\sqrt{3}}{2}\sec^2 t\,\mathrm{d}t}{\left(\left(\frac{\sqrt{3}}{2}\right)^2\sec^2 t\right)^{\frac{3}{2}}} = \frac{4}{3}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\cos t\,\mathrm{d}t = \frac{4}{3}\sin t\Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}} = \frac{4}{3}.$$

★★

P213/习题9.5/4(9) 计算$\int_{\frac{1}{e}}^{e}|\ln x|\,dx$.

解

$$\int_{\frac{1}{e}}^{e}|\ln x|\,dx=-\int_{\frac{1}{e}}^{1}\ln x\,dx+\int_{1}^{e}\ln x\,dx$$

$$=-\left(x\ln x\Big|_{\frac{1}{e}}^{1}-\int_{\frac{1}{e}}^{1}x\,d\ln x\right)+\left(x\ln x\Big|_{1}^{e}-\int_{1}^{e}x\,d\ln x\right)$$

$$=-\left(-\frac{1}{e}\ln\frac{1}{e}-\int_{\frac{1}{e}}^{1}1\,dx\right)+\left(e\ln e-\int_{1}^{e}1\,dx\right)$$

$$=-\left(\frac{1}{e}-\left(1-\frac{1}{e}\right)\right)+\left(e-(e-1)\right)$$

$$=2-\frac{2}{e}.$$

★★

P213/习题9.5/4(10) 计算$\int_0^1 e^{\sqrt{x}}\,dx$.

解 令 $t=\sqrt{x}$, 即 $x=t^2$, 则 $dx=2t\,dt$,

当$x=0$时, $t=0$; 当$x=1$时, $t=1$.

从而 $\int_0^1 e^{\sqrt{x}}\,dx = 2\int_0^1 te^t\,dt = 2\int_0^1 t\,de^t = 2\left(te^t\Big|_0^1 - \int_0^1 e^t\,dt\right)$

$$= 2\left(e - e^t\Big|_0^1\right) = 2.$$

P213/习题9.5/4(12) 计算$\displaystyle\int_0^{\frac{\pi}{2}}\frac{\cos\theta}{\sin\theta+\cos\theta}\mathrm{d}\theta.$

解1 令$t=\dfrac{\pi}{2}-\theta$, 即$\theta=\dfrac{\pi}{2}-t$,则$\mathrm{d}\theta=-\mathrm{d}t$,当$\theta=0$时,$t=\dfrac{\pi}{2}$;当$\theta=\dfrac{\pi}{2}$时,$t=0$.

从而

$$\int_0^{\frac{\pi}{2}}\frac{\cos\theta}{\sin\theta+\cos\theta}\mathrm{d}\theta=\int_{\frac{\pi}{2}}^{0}\frac{\cos\left(\frac{\pi}{2}-t\right)}{\sin\left(\frac{\pi}{2}-t\right)+\cos\left(\frac{\pi}{2}-t\right)}\mathrm{d}\left(\frac{\pi}{2}-t\right)$$

$$=\int_0^{\frac{\pi}{2}}\frac{\sin t}{\cos t+\sin t}\mathrm{d}t=\int_0^{\frac{\pi}{2}}\frac{\sin\theta}{\sin\theta+\cos\theta}\mathrm{d}\theta.$$

所以

$$\int_0^{\frac{\pi}{2}}\frac{\cos\theta}{\sin\theta+\cos\theta}\mathrm{d}\theta=\frac{1}{2}\left(\int_0^{\frac{\pi}{2}}\frac{\cos\theta}{\sin\theta+\cos\theta}\mathrm{d}\theta+\int_0^{\frac{\pi}{2}}\frac{\sin\theta}{\sin\theta+\cos\theta}\mathrm{d}\theta\right)$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}1\mathrm{d}\theta=\frac{\pi}{4}.$$

P213/习题9.5/4(12) 计算$\int_0^{\frac{\pi}{2}}\frac{\cos\theta}{\sin\theta+\cos\theta}\mathrm{d}\theta.$

解2 $$\int_0^{\frac{\pi}{2}}\frac{\cos\theta}{\sin\theta+\cos\theta}\mathrm{d}\theta=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\cos\theta+\sin\theta-\sin\theta+\cos\theta}{\sin\theta+\cos\theta}\mathrm{d}\theta$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\left(1-\frac{\sin\theta-\cos\theta}{\sin\theta+\cos\theta}\right)\mathrm{d}\theta$$

$$=\frac{1}{2}\cdot\frac{\pi}{2}+\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{1}{\sin\theta+\cos\theta}\mathrm{d}(\sin\theta+\cos\theta)$$

$$=\frac{\pi}{4}+\frac{1}{2}\ln|\sin\theta+\cos\theta|\Big|_0^{\frac{\pi}{2}}$$

$$=\frac{\pi}{4}.$$

P213/习题9.5/4(12) 计算$\int_0^{\frac{\pi}{2}}\frac{\cos\theta}{\sin\theta+\cos\theta}\mathrm{d}\theta.$

解3 记$I_1=\int_0^{\frac{\pi}{2}}\frac{\cos\theta}{\sin\theta+\cos\theta}\mathrm{d}\theta$, $I_2=\int_0^{\frac{\pi}{2}}\frac{\sin\theta}{\sin\theta+\cos\theta}\mathrm{d}\theta.$

从而

$$I_1+I_2=\int_0^{\frac{\pi}{2}}\frac{\cos\theta}{\sin\theta+\cos\theta}\mathrm{d}\theta+\int_0^{\frac{\pi}{2}}\frac{\sin\theta}{\sin\theta+\cos\theta}\mathrm{d}\theta=\int_0^{\frac{\pi}{2}}\mathrm{d}\theta=\frac{\pi}{2}.$$

$$I_1-I_2=\int_0^{\frac{\pi}{2}}\frac{\cos\theta}{\sin\theta+\cos\theta}\mathrm{d}\theta-\int_0^{\frac{\pi}{2}}\frac{\sin\theta}{\sin\theta+\cos\theta}\mathrm{d}\theta=\int_0^{\frac{\pi}{2}}\frac{\cos\theta-\sin\theta}{\sin\theta+\cos\theta}\mathrm{d}\theta$$

$$=\int_0^{\frac{\pi}{2}}\frac{1}{\sin\theta+\cos\theta}\mathrm{d}(\sin\theta+\cos\theta)=\ln\left|\sin\theta+\cos\theta\right|\Big|_0^{\frac{\pi}{2}}=0.$$

所以

$$I_1=\int_0^{\frac{\pi}{2}}\frac{\cos\theta}{\sin\theta+\cos\theta}\mathrm{d}\theta=\frac{\pi}{4}.$$

P213/习题9.5/4(12) 计算$\int_0^{\frac{\pi}{2}}\frac{\cos\theta}{\sin\theta+\cos\theta}\mathrm{d}\theta.$

解4 令$t=\tan\frac{\theta}{2}$, 即$\theta=2\arctan t$,则$\mathrm{d}\theta=\frac{2}{1+t^2}\mathrm{d}t$,

当$\theta=0$时,$t=0$; 当$\theta=\frac{\pi}{2}$时,$t=1$. 从而

$$\int_0^{\frac{\pi}{2}}\frac{\cos\theta}{\sin\theta+\cos\theta}\mathrm{d}\theta=\int_0^1\frac{\frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\mathrm{d}t=2\int_0^1\frac{1-t^2}{(1+2t-t^2)(1+t^2)}\mathrm{d}t$$

$$=2\int_0^1\left(\frac{At+B}{1+2t-t^2}+\frac{Dt+E}{1+t^2}\right)\mathrm{d}t=\int_0^1\left(\frac{-t+1}{1+2t-t^2}+\frac{-t+1}{1+t^2}\right)\mathrm{d}t$$

$$=\frac{1}{2}\int_0^1\frac{1}{1+2t-t^2}\mathrm{d}\left(1+2t-t^2\right)-\frac{1}{2}\ln\left(1+t^2\right)\Big|_0^1+\arctan t\Big|_0^1$$

$$=\frac{1}{2}\ln\left(1+2t-t^2\right)\Big|_0^1-\frac{1}{2}\ln 2+\frac{\pi}{4}=\frac{\pi}{4}.$$

P213/习题9.5/4(12) 计算$\int_0^{\frac{\pi}{2}}\frac{\cos\theta}{\sin\theta+\cos\theta}\mathrm{d}\theta$.

解5 令$t=\tan\theta$, 即$\theta=\arctan t$,则$\mathrm{d}\theta=\frac{1}{1+t^2}\mathrm{d}t$,

当$\theta=0$时,$t=0$; 当$\theta=\frac{\pi}{2}$时,$t=+\infty$. 从而

$$\int_0^{\frac{\pi}{2}}\frac{\cos\theta}{\sin\theta+\cos\theta}\mathrm{d}\theta=\int_0^{\frac{\pi}{2}}\frac{1}{\tan\theta+1}\mathrm{d}\theta=\int_0^{+\infty}\frac{1}{t+1}\cdot\frac{1}{1+t^2}\mathrm{d}t$$

$$=\lim_{u\to+\infty}\int_0^u\frac{1}{(1+t)(1+t^2)}\mathrm{d}t=\lim_{u\to+\infty}\int_0^u\left(\frac{A}{1+t}+\frac{Bt+D}{1+t^2}\right)\mathrm{d}t$$

$$=\lim_{u\to+\infty}\left(-\frac{1}{2}\int_0^u\left(\frac{-1}{1+t}+\frac{t-1}{1+t^2}\right)\mathrm{d}t\right)=-\frac{1}{2}\lim_{u\to+\infty}\left(-\ln(1+t)+\frac{1}{2}\ln(1+t^2)-\arctan t\right)\Big|_0^u$$

$$=-\frac{1}{2}\lim_{u\to+\infty}\left(\ln\frac{(1+u^2)^{\frac{1}{2}}}{1+u}-\arctan u\right)=\frac{\pi}{4}.$$

P213/习题9.5/4(12) 计算$\displaystyle\int_0^{\frac{\pi}{2}}\frac{\cos\theta}{\sin\theta+\cos\theta}\mathrm{d}\theta.$

解6 $$\int_0^{\frac{\pi}{2}}\frac{\cos\theta}{\sin\theta+\cos\theta}\mathrm{d}\theta=\int_0^{\frac{\pi}{2}}\frac{\cos\theta(\sin\theta-\cos\theta)}{(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)}\mathrm{d}\theta$$

$$=\int_0^{\frac{\pi}{2}}\frac{\sin\theta\cos\theta-\cos^2\theta}{\sin^2\theta-\cos^2\theta}\mathrm{d}\theta=-\int_0^{\frac{\pi}{2}}\frac{\frac{1}{2}\sin2\theta-\frac{1+\cos2\theta}{2}}{\cos2\theta}\mathrm{d}\theta$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{1+\cos2\theta-\sin2\theta}{\cos2\theta}\mathrm{d}\theta=\frac{1}{4}\int_0^{\frac{\pi}{2}}\frac{1+\cos2\theta-\sin2\theta}{\cos2\theta}\mathrm{d}2\theta$$

$$=\frac{1}{4}\int_0^{\frac{\pi}{2}}(\sec2\theta+1-\tan2\theta)\mathrm{d}2\theta$$

$$=\frac{1}{4}\left(\ln|\sec2\theta+\tan2\theta|+2\theta+\ln|\cos2\theta|\right)\Big|_0^{\frac{\pi}{2}}=\frac{\pi}{4}.$$

P213/习题9.5/5

设f在$[-a,a]$上可积. 证明:(1) 若f为奇函数, 则$\int_{-a}^{a} f(x)\,\mathrm{d}x = 0.$

(2) 若f为偶函数, 则$\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_{0}^{a} f(x)\,\mathrm{d}x.$

证 由于 $\int_{-a}^{a} f(x)\,\mathrm{d}x = \int_{-a}^{0} f(x)\,\mathrm{d}x + \int_{0}^{a} f(x)\,\mathrm{d}x$, 其中对于 $\int_{-a}^{0} f(x)\,\mathrm{d}x$,

令$x=-t$, 则 $\mathrm{d}x = -\mathrm{d}t$, 当$x=-a$时, $t=a$; 当$x=0$时, $t=0$.

从而

$$\int_{-a}^{0} f(x)\,\mathrm{d}x = -\int_{a}^{0} f(-t)\,\mathrm{d}t = \int_{0}^{a} f(-t)\,\mathrm{d}t = \int_{0}^{a} f(-x)\,\mathrm{d}x,$$

于是 $\int_{-a}^{a} f(x)\,\mathrm{d}x = \int_{0}^{a} f(-x)\,\mathrm{d}x + \int_{0}^{a} f(x)\,\mathrm{d}x$

$$= \int_{0}^{a} \big(f(-x) + f(x)\big)\,\mathrm{d}x = \begin{cases} 0, & f(-x) = -f(x) \\ 2\int_{0}^{a} f(x)\,\mathrm{d}x, & f(-x) = f(x) \end{cases}.$$

P213/习题9.5/6

设f为$(-\infty,+\infty)$上以p为周期的周期函数.证明对任何实数a,恒有

$$\int_a^{a+p} f(x)\mathrm{d}x=\int_0^p f(x)\mathrm{d}x.$$

证1 由于 $\int_a^{a+p} f(x)\mathrm{d}x=\int_a^0 f(x)\mathrm{d}x+\int_0^p f(x)\mathrm{d}x+\int_p^{a+p} f(x)\mathrm{d}x,$

其中对于 $\int_p^{a+p} f(x)\mathrm{d}x$, 令$t=x-p$, 即$x=t+p$, 则$\mathrm{d}x=\mathrm{d}t$,

当$x=p$时, $t=0$; 当$x=a+p$时, $t=a$.

从而 $\int_p^{a+p} f(x)\mathrm{d}x=\int_0^a f(t+p)\mathrm{d}t=\int_0^a f(t)\mathrm{d}t=\int_0^a f(x)\mathrm{d}x,$

于是 $\int_a^{a+p} f(x)\mathrm{d}x=\int_a^0 f(x)\mathrm{d}x+\int_0^p f(x)\mathrm{d}x+\int_0^a f(x)\mathrm{d}x$

$$=\int_0^p f(x)\mathrm{d}x.$$

P213/习题9.5/6

设f为$(-\infty,+\infty)$上以p为周期的周期函数.证明对任何实数a,恒有

$$\int_a^{a+p} f(x)\mathrm{d}x=\int_0^p f(x)\mathrm{d}x.$$

证2 由于f是连续函数,故存在原函数, 设 $F(x)=\int_0^x f(t)\mathrm{d}t,\ x\in(-\infty,+\infty).$

从而 $\int_a^{a+p} f(x)\mathrm{d}x=F(a+p)-F(a).$

上式关于a求导, 又 $f(a+p)=f(a)$,

$$\big(F(a+p)-F(a)\big)'=F'(a+p)-F'(a)=f(a+p)-f(a)=0.$$

根据拉格朗日中值定理推论知, $F(a+p)-F(a)=C.$

令$a=0$, 得 $C=F(p)-F(0)=\int_0^p f(x)\mathrm{d}x.$

于是

$$\int_a^{a+p} f(x)\mathrm{d}x=F(a+p)-F(a)=C=F(p)-F(0)=\int_0^p f(x)\mathrm{d}x.$$

P213/习题9.5/6

设f为$(-\infty,+\infty)$上以p为周期的周期函数. 证明对任何实数a, 恒有

$$\int_a^{a+p} f(x)\mathrm{d}x=\int_0^p f(x)\mathrm{d}x.$$

证3 $\int_a^{a+p} f(x)\mathrm{d}x-\int_0^p f(x)\mathrm{d}x=\int_a^{a+p} f(x)\mathrm{d}x+\int_p^a f(x)\mathrm{d}x-\int_0^p f(x)\mathrm{d}x-\int_p^a f(x)\mathrm{d}x$

$$=\int_p^{a+p} f(x)\mathrm{d}x-\int_0^a f(x)\mathrm{d}x,$$

其中对于 $\int_p^{a+p} f(x)\mathrm{d}x$, 令$t=x-p$, 即$x=t+p$, 则$\mathrm{d}x=\mathrm{d}t$,

当$x=p$时, $t=0$; 当$x=a+p$时, $t=a$. 从而

$$\int_p^{a+p} f(x)\mathrm{d}x=\int_0^a f(t+p)\mathrm{d}t=\int_0^a f(t)\mathrm{d}t=\int_0^a f(x)\mathrm{d}x,$$

于是

$$\int_a^{a+p} f(x)\mathrm{d}x-\int_0^p f(x)\mathrm{d}x=0.$$

所以

$$\int_a^{a+p} f(x)\mathrm{d}x=\int_0^p f(x)\mathrm{d}x.$$

P213/习题9.5/6

设f为$(-\infty,+\infty)$上以p为周期的周期函数. 证明对任何实数a, 恒有

$$\int_a^{a+p} f(x)\,\mathrm{d}x = \int_0^p f(x)\,\mathrm{d}x.$$

证4 记$G(a)=\int_a^{a+p} f(x)\,\mathrm{d}x.$

由于f是连续函数, 故G可导, 又$f(a+p)=f(a)$, 从而

$$G'(a)=\left(\int_a^{a+p} f(x)\,\mathrm{d}x\right)' = f(a+p)-f(a)=0.$$

根据拉格朗日中值定理推论知, $G(a)=C.$

令$a=0$, 得$C=G(0)=\int_0^p f(x)\,\mathrm{d}x.$

于是

$$\int_a^{a+p} f(x)\,\mathrm{d}x = G(a)=C=G(0)=\int_0^p f(x)\,\mathrm{d}x.$$

P213/习题9.5/7

设f为连续函数. 证明: (1) $\int_0^{\frac{\pi}{2}} f(\sin x)\mathrm{d}x = \int_0^{\frac{\pi}{2}} f(\cos x)\mathrm{d}x.$

(2) $\int_0^{\pi} xf(\sin x)\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\mathrm{d}x.$

证1 (1) 令$x=\frac{\pi}{2}-t$, 则$\mathrm{d}x=-\mathrm{d}t$, 当$x=0, t=\frac{\pi}{2}$; 当$x=\frac{\pi}{2}, t=0$. 从而

$$\int_0^{\frac{\pi}{2}} f(\sin x)\mathrm{d}x = -\int_{\frac{\pi}{2}}^{0} f\left(\sin\left(\frac{\pi}{2}-t\right)\right)\mathrm{d}t = \int_0^{\frac{\pi}{2}} f(\cos t)\mathrm{d}t = \int_0^{\frac{\pi}{2}} f(\cos x)\mathrm{d}x.$$

(2) 令$x=\pi-t$, 则$\mathrm{d}x=-\mathrm{d}t$, 当$x=0, t=\pi$; 当$x=\pi, t=0$. 从而

$$\int_0^{\pi} xf(\sin x)\mathrm{d}x = -\int_{\pi}^{0}(\pi-t)f(\sin(\pi-t))\mathrm{d}t = \pi\int_0^{\pi} f(\sin t)\mathrm{d}t - \int_0^{\pi} tf(\sin t)\mathrm{d}t$$

$$= \pi\int_0^{\pi} f(\sin x)\mathrm{d}x - \int_0^{\pi} xf(\sin x)\mathrm{d}x,$$

于是

$$\int_0^{\pi} xf(\sin x)\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\mathrm{d}x.$$

P213/习题9.5/7

设f为连续函数. 证明: (1) $\int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x=\int_0^{\frac{\pi}{2}} f(\cos x)\,\mathrm{d}x.$

(2) $\int_0^{\pi} xf(\sin x)\,\mathrm{d}x=\dfrac{\pi}{2}\int_0^{\pi} f(\sin x)\,\mathrm{d}x.$

证2 (1)令$t=\sin x$,即$x=\arcsin t$,则$\mathrm{d}x=\dfrac{1}{\sqrt{1-t^2}}\mathrm{d}t$, 当$x=0,t=0$;当$x=\dfrac{\pi}{2},t=1$. 从而

$$\int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x=\int_0^1\frac{f(t)}{\sqrt{1-t^2}}\mathrm{d}t=\int_0^1\frac{f(x)}{\sqrt{1-x^2}}\mathrm{d}x.$$

令$u=\cos x$,即$x=\arccos u$,则$\mathrm{d}x=-\dfrac{1}{\sqrt{1-u^2}}\mathrm{d}u$, 当$x=0,t=1$;当$x=\dfrac{\pi}{2},t=0$.从而

$$\int_0^{\frac{\pi}{2}} f(\cos x)\,\mathrm{d}x=-\int_1^0\frac{f(t)}{\sqrt{1-t^2}}\mathrm{d}t=\int_0^1\frac{f(t)}{\sqrt{1-t^2}}\mathrm{d}t=\int_0^1\frac{f(x)}{\sqrt{1-x^2}}\mathrm{d}x.$$

所以

$$\int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x=\int_0^1\frac{f(x)}{\sqrt{1-x^2}}\mathrm{d}x=\int_0^{\frac{\pi}{2}} f(\cos x)\,\mathrm{d}x.$$

P213/习题9.5/7

设f为连续函数. 证明: (1) $\int_0^{\frac{\pi}{2}} f(\sin x)\mathrm{d}x = \int_0^{\frac{\pi}{2}} f(\cos x)\mathrm{d}x.$

(2) $\int_0^{\pi} xf(\sin x)\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\mathrm{d}x.$

证2 (2) $\int_0^{\pi} xf(\sin x)\mathrm{d}x - \frac{\pi}{2}\int_0^{\pi} f(\sin x)\mathrm{d}x = \int_0^{\pi}\left(x-\frac{\pi}{2}\right)f(\sin x)\mathrm{d}x,$

令$t = x - \frac{\pi}{2}$,即$x = t + \frac{\pi}{2}$,则$\mathrm{d}x = \mathrm{d}t$, 当$x = 0, t = -\frac{\pi}{2}$;当$x = \pi, t = \frac{\pi}{2}$. 从而

$$\int_0^{\pi} xf(\sin x)\mathrm{d}x - \frac{\pi}{2}\int_0^{\pi} f(\sin x)\mathrm{d}x = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} tf\left(\sin\left(t+\frac{\pi}{2}\right)\right)\mathrm{d}t$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} tf(\cos t)\mathrm{d}t.$$

由于$tf(\cos t)$是奇函数, 所以 $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} tf(\cos t)\mathrm{d}t = 0.$

于是 $\int_0^{\pi} xf(\sin x)\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\mathrm{d}x.$

P213/习题9.5/9

证明：若在$(0,+\infty)$上f为连续函数，且对任何$a>0$有

$$g(x)=\int_x^{ax} f(t)\,\mathrm{d}t \equiv \text{常数},\ x\in(0,+\infty),$$

则$f(x)=\dfrac{c}{x}$, $x\in(0,+\infty)$, c为常数.

证1 由于 $g'(x)=\left(\int_x^{ax} f(t)\,\mathrm{d}t\right)'=af(ax)-f(x)=0,$

当$a=\dfrac{1}{x}$时，恒有 $f(x)=\dfrac{1}{x}f(1)$成立，记$f(1)=c$，则c为常数.

从而 $f(x)=\dfrac{c}{x}$, $x\in(0,+\infty)$.

证2 由于 $g'(x)=\left(\int_x^{ax} f(t)\,\mathrm{d}t\right)'=af(ax)-f(x)=0,$

令$x=1$，则对任意的$a>0$，有 $af(a)=f(1)$，记$f(1)=c$,则c为常数.

从而 $f(x)=\dfrac{c}{x}$, $x\in(0,+\infty)$.

P213/习题9.5/12 利用积分中值定理

设f为$[0,2\pi]$上的单调递减函数. 证明：对任何正整数n, 恒有

$$\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x \geq 0.$$

证1 由于$f(x)$在$[0,2\pi]$上单调, $\sin nx$在$[0,2\pi]$上连续,

根据积分第二中值定理推论知, $\exists \xi \in [0,2\pi]$, 使得

$$\begin{aligned}
\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x &= f(0)\int_0^{\xi}\sin nx\,\mathrm{d}x + f(2\pi)\int_{\xi}^{2\pi}\sin nx\,\mathrm{d}x \\
&= f(0)\left(-\frac{\cos nx}{n}\right)\Bigg|_0^{\xi} + f(2\pi)\left(-\frac{\cos nx}{n}\right)\Bigg|_{\xi}^{2\pi} \\
&= \frac{1}{n}f(0)(1-\cos n\xi) + \frac{1}{n}f(2\pi)(\cos n\xi - 1) \\
&= \frac{1}{n}(1-\cos n\xi)\big(f(0)-f(2\pi)\big) \geq 0.
\end{aligned}$$

P213/习题9.5/12 利用积分中值定理

设f为$[0,2\pi]$上的单调递减函数. 证明：对任何正整数n, 恒有

$$\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x \geq 0.$$

证2 设 $g(x)=f(x)-f(2\pi)$, 则$g(x)$在$[0,2\pi]$上非负、递减.

又$\sin nx$在$[0,2\pi]$上连续, 根据积分第二中值定理知, $\exists\xi\in[0,2\pi]$, 使得

$$\begin{aligned}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x &= \int_0^{2\pi}\big(g(x)+f(2\pi)\big)\sin nx\,\mathrm{d}x \\ &= \int_0^{2\pi} g(x)\sin nx\,\mathrm{d}x + \int_0^{2\pi} f(2\pi)\sin nx\,\mathrm{d}x \\ &= g(0)\int_0^{\xi}\sin nx\,\mathrm{d}x + f(2\pi)\left(-\frac{\cos nx}{n}\right)\Bigg|_0^{2\pi} \\ &= g(0)\left(-\frac{\cos nx}{n}\right)\Bigg|_0^{\xi} + 0 \\ &= \frac{1}{n}g(0)(1-\cos n\xi) \geq 0.\end{aligned}$$

P213/习题9.5/13 利用积分中值定理

证明：当$x>0$时有不等式$\left|\int_x^{x+c}\sin t^2\,\mathrm{d}t\right|\le\frac{1}{x}\ (c>0)$.

证 令$u=t^2$，即$t=\sqrt{u}$，则 $\mathrm{d}t=\frac{1}{2\sqrt{u}}\mathrm{d}u$，当$t=x,u=x^2$;当$t=x+c,u=(x+c)^2$.

从而 $\int_x^{x+c}\sin t^2\,\mathrm{d}t=\int_{x^2}^{(x+c)^2}\sin u\cdot\frac{1}{2\sqrt{u}}\mathrm{d}u.$

由于$\sin u$在$\left[x^2,(x+c)^2\right]$上连续，$\frac{1}{2\sqrt{u}}$在$\left[x^2,(x+c)^2\right]$上单调减且非负，

根据积分第二中值定理知，$\exists\xi\in\left[x^2,(x+c)^2\right]$，使得

$$\int_x^{x+c}\sin t^2\,\mathrm{d}t=\frac{1}{2\sqrt{x^2}}\int_{x^2}^{\xi}\sin u\,\mathrm{d}u=\frac{1}{2x}\left(-\cos u\right)\Big|_{x^2}^{\xi}=\frac{1}{2x}\left(\cos x^2-\cos\xi\right),$$

于是

$$\left|\int_x^{x+c}\sin t^2\,\mathrm{d}t\right|=\frac{1}{2x}\left|\cos x^2-\cos\xi\right|\le\frac{1}{2x}\left(\left|\cos x^2\right|+\left|\cos\xi\right|\right)\le\frac{2}{2x}=\frac{1}{x}.$$

P213/习题9.5/16 利用积分中值定理

证明：若在$[a,b]$上f为连续函数，g为连续可微的单调函数，则存在$\xi\in[a,b]$，使得

$$\int_a^b f(x)g(x)\,\mathrm{d}x=g(a)\int_a^{\xi} f(x)\,\mathrm{d}x+g(b)\int_{\xi}^b f(x)\,\mathrm{d}x.$$

证1 设 $F(x)=\int_a^x f(x)\,\mathrm{d}x$，从而$F'(x)=f(x)$. 于是

$$\int_a^b f(x)g(x)\,\mathrm{d}x=\int_a^b g(x)\,\mathrm{d}F(x)=F(x)g(x)\Big|_a^b-\int_a^b F(x)g'(x)\,\mathrm{d}x$$

$$=F(b)g(b)-\int_a^b F(x)g'(x)\,\mathrm{d}x,$$

由于$g(x)$为连续可微单调函数，故$g'(x)$在$[a,b]$上连续且不变号.

又$f(x)$在$[a,b]$上连续，根据推广的积分第一中值定理知，$\exists\xi\in[a,b]$，使得

$$\begin{aligned}\int_a^b f(x)g(x)\,\mathrm{d}x&=g(b)\int_a^b f(x)\,\mathrm{d}x-F(\xi)\int_a^b g'(x)\,\mathrm{d}x\\&=g(b)\int_a^b f(x)\,\mathrm{d}x-\big(g(b)-g(a)\big)\int_a^{\xi} f(x)\,\mathrm{d}x\\&=g(a)\int_a^{\xi} f(x)\,\mathrm{d}x+g(b)\int_{\xi}^b f(x)\,\mathrm{d}x.\end{aligned}$$

P213/习题9.5/16 利用积分中值定理

证明：若在$[a,b]$上f为连续函数，g为连续可微的单调函数，则存在$\xi\in[a,b]$，使得

$$\int_a^b f(x)g(x)\,\mathrm{d}x = g(a)\int_a^\xi f(x)\,\mathrm{d}x + g(b)\int_\xi^b f(x)\,\mathrm{d}x.$$

证2 不妨设g单调递减. 设$H(x)=g(a)-g(x)$. 从而H单调递增且非负.

又$f(x)$在$[a,b]$上连续，根据积分第二中值定理知，$\exists\xi\in[a,b]$，使得

$$\int_a^b f(x)H(x)\,\mathrm{d}x = H(b)\int_\xi^b f(x)\,\mathrm{d}x,$$

即
$$\begin{aligned}\int_a^b f(x)g(x)\,\mathrm{d}x &= \int_a^b f(x)\big(g(a)-H(x)\big)\,\mathrm{d}x\\ &= g(a)\int_a^b f(x)\,\mathrm{d}x - \int_a^b f(x)H(x)\,\mathrm{d}x\\ &= g(a)\int_a^b f(x)\,\mathrm{d}x - \big(g(a)-g(b)\big)\int_\xi^b f(x)\,\mathrm{d}x\\ &= g(a)\int_a^\xi f(x)\,\mathrm{d}x + g(b)\int_\xi^b f(x)\,\mathrm{d}x.\end{aligned}$$

利用积分中值定理

补充 设f在$[0,1]$上连续. 计算$\lim\limits_{n\to\infty}\int_0^1 \frac{nf(x)}{1+n^2x^2}\mathrm{d}x$.

解 由于f在$[0,1]$上连续, 因此f在$[0,1]$上有界, 即$\exists M>0,\forall x\in[0,1]$, 有$|f(x)|\le M$.

$$\int_0^1 \frac{nf(x)}{1+n^2x^2}\mathrm{d}x=\int_0^{n^{-\frac14}} \frac{nf(x)}{1+n^2x^2}\mathrm{d}x+\int_{n^{-\frac14}}^1 \frac{nf(x)}{1+n^2x^2}\mathrm{d}x,$$

其中 $\left|\int_{n^{-\frac14}}^1 \frac{nf(x)}{1+n^2x^2}\mathrm{d}x\right|\le\int_{n^{-\frac14}}^1 \left|\frac{nf(x)}{1+n^2x^2}\right|\mathrm{d}x\le M\int_{n^{-\frac14}}^1 \frac{n}{1+n^2x^2}\mathrm{d}x$

$$\le M\int_{n^{-\frac14}}^1 \frac{n}{1+n^2\left(n^{-\frac14}\right)^2}\mathrm{d}x=M\frac{n\left(1-n^{-\frac14}\right)}{1+n^{\frac32}}\to 0\ (n\to\infty).$$

由于$\frac{n}{1+n^2x^2}$在$\left[0,n^{-\frac14}\right]$上不变号, 根据推广的积分第一中值定理知, $\exists\xi\in\left[0,n^{-\frac14}\right]$, 使得

$$\int_0^{n^{-\frac14}} \frac{nf(x)}{1+n^2x^2}\mathrm{d}x=f(\xi)\int_0^{n^{-\frac14}} \frac{n}{1+n^2x^2}\mathrm{d}x=f(\xi)\arctan(nx)\Big|_0^{n^{-\frac14}}=f(\xi)\arctan n^{\frac34},$$

当$n\to\infty$时, $\xi\to0$, $\arctan n^{\frac34}\to\frac{\pi}{2}$. 于是

$$\lim_{n\to\infty}\int_0^1 \frac{nf(x)}{1+n^2x^2}\mathrm{d}x=\lim_{n\to\infty}\int_0^{n^{-\frac14}} \frac{nf(x)}{1+n^2x^2}\mathrm{d}x+\lim_{n\to\infty}\int_{n^{-\frac14}}^1 \frac{nf(x)}{1+n^2x^2}\mathrm{d}x=\frac{\pi}{2}f(0).$$

利用积分中值定理

补充 设f在$[0,\pi]$上连续. 证明：$\lim\limits_{n\to\infty}\int_0^\pi f(x)|\sin nx|\,\mathrm{d}x=\dfrac{2}{\pi}\int_0^\pi f(x)\,\mathrm{d}x.$

证 对区间$[0,\pi]$ n等分，得分点$x_i=\dfrac{i}{n}\pi,\ i=0,1,2,\cdots,n.$

于是 $$\int_{x_{i-1}}^{x_i}|\sin nx|\,\mathrm{d}x=\int_{\frac{i-1}{n}\pi}^{\frac{i}{n}\pi}|\sin nx|\,\mathrm{d}x\overset{t=nx}{=}\frac{1}{n}\int_{(i-1)\pi}^{i\pi}|\sin t|\,\mathrm{d}t=\frac{1}{n}\int_0^\pi\sin t\,\mathrm{d}t=\frac{2}{n}.$$

因为f在$[0,\pi]$连续，$|\sin nx|$不变号，根据推广的积分第一中值定理，

$$\int_0^\pi f(x)|\sin nx|\,\mathrm{d}x=\sum_{i=1}^n\int_{x_{i-1}}^{x_i}f(x)|\sin nx|\,\mathrm{d}x=\sum_{i=1}^n f(\xi_i)\int_{x_{i-1}}^{x_i}|\sin nx|\,\mathrm{d}x$$
$$=\sum_{i=1}^n f(\xi_i)\cdot\frac{2}{n}=\frac{2}{\pi}\sum_{i=1}^n f(\xi_i)\cdot\frac{\pi}{n},\quad \xi_i\in[x_{i-1},x_i].$$

所以 $$\lim_{n\to\infty}\int_0^\pi f(x)|\sin nx|\,\mathrm{d}x=\lim_{n\to\infty}\frac{2}{\pi}\sum_{i=1}^n f(\xi_i)\cdot\frac{\pi}{n}=\frac{2}{\pi}\int_0^\pi f(x)\,\mathrm{d}x.$$

利用积分中值定理

补充 设f在$[a,b]$上连续且单调增加. 证明: $\int_a^b xf(x)\mathrm{d}x \geq \frac{a+b}{2}\int_a^b f(x)\mathrm{d}x.$

证 $\int_a^b xf(x)\mathrm{d}x - \frac{a+b}{2}\int_a^b f(x)\mathrm{d}x = \int_a^b \left(x-\frac{a+b}{2}\right)f(x)\mathrm{d}x$

$$=\int_a^{\frac{a+b}{2}}\left(x-\frac{a+b}{2}\right)f(x)\mathrm{d}x+\int_{\frac{a+b}{2}}^b\left(x-\frac{a+b}{2}\right)f(x)\mathrm{d}x,$$

由于$x-\frac{a+b}{2}$在$\left[a,\frac{a+b}{2}\right]$不变号, $x-\frac{a+b}{2}$在$\left[\frac{a+b}{2},b\right]$不变号,

根据推广的积分第一中值定理, $\exists \xi_1\in\left[a,\frac{a+b}{2}\right], \xi_2\in\left[\frac{a+b}{2},b\right]$, 使得

$$\int_a^b xf(x)\mathrm{d}x-\frac{a+b}{2}\int_a^b f(x)\mathrm{d}x = f(\xi_1)\int_a^{\frac{a+b}{2}}\left(x-\frac{a+b}{2}\right)\mathrm{d}x+f(\xi_2)\int_{\frac{a+b}{2}}^b\left(x-\frac{a+b}{2}\right)\mathrm{d}x$$

$$=f(\xi_1)\frac{\left(x-\frac{a+b}{2}\right)^2}{2}\Bigg|_a^{\frac{a+b}{2}}+f(\xi_2)\frac{\left(x-\frac{a+b}{2}\right)^2}{2}\Bigg|_{\frac{a+b}{2}}^b=\frac{(b-a)^2}{8}\left(f(\xi_2)-f(\xi_1)\right)\geq 0.$$

所以 $\int_a^b xf(x)\mathrm{d}x \geq \frac{a+b}{2}\int_a^b f(x)\mathrm{d}x.$