Ch5 导数和微分

主讲教师：顾燕红

办公室：汇星楼409

办公室答疑时间：每周二15点至17点

微信号：18926511820 QQ号：58105217

Email: yhgu@szu.edu.cn

(添加好友、加群请备注 学号 姓名 数学分析1)

QQ群、QQ、微信群、微信随时答疑解惑

§1 导数的概念

§2 求导法则

§3 参变量函数的导数

§4 高阶导数

§5 微分

将学习：

高阶导数

高阶导数

如果$f(x)$的导函数$f'(x)$在点x_0可导，则称$f'(x)$在点x_0的导数为$f(x)$在点x_0的二阶导数，记作$f''(x_0)$，

$$f''(x_0)=\lim_{x\to x_0}\frac{f'(x)-f'(x_0)}{x-x_0}.$$

此时也称$f(x)$在点x_0的二阶可导.

如果$f(x)$在区间I上每一点都二阶可导,则得到一个定义在I上的函数，这个函数称为$f(x)$的二阶导函数，记作$f''(x), x\in I$. 可由f的$n-1$阶导函数定义f的n阶导数.

二阶以及二阶以上的导数称为高阶导数.

注： 函数 f 在点 x_0 处的 n 阶导数记作

$$f^{(n)}(x_0),\quad y^{(n)}\Big|_{x=x_0},\quad \frac{\mathrm{d}^n y}{\mathrm{d}x^n}\Big|_{x=x_0},\quad \frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}\Big|_{x=x_0}.$$

n阶导函数记作

$$f^{(n)}(x)\left(\text{或} f^{(n)}\right),\quad y^{(n)},\quad \frac{\mathrm{d}^n y}{\mathrm{d}x^n},\quad \frac{\mathrm{d}^n}{\mathrm{d}x^n}f(x).$$

这里 $\dfrac{\mathrm{d}^n y}{\mathrm{d}x^n}$ 也可写作 $\left(\dfrac{\mathrm{d}}{\mathrm{d}x}\right)^n y$,即对y进行了n次求导

运算“$\dfrac{\mathrm{d}}{\mathrm{d}x}$”(看作一个算符).

例1 求下列函数的各阶导数:

(1) $y = x^n$(n为正整数);(2) $y = \mathrm{e}^{ax}$(a为常数);

(3) $y = \sin x, y = \cos x$;(4) $y = \ln x$.

解 (1) $y' = nx^{n-1}$, $y'' = n(n-1)x^{n-2}, \cdots$, $y^{(n)} = n!$, $y^{(m)} = 0\ (m > n)$.

(2) $y' = a\mathrm{e}^{ax}$, $y'' = a^2\mathrm{e}^{ax}$, 对一切 $n \in \mathrm{N}_+$, $(\mathrm{e}^{ax})^{(n)} = a^n\mathrm{e}^{ax}$.

(3) 对 $y = \sin x$, 有 $y' = \cos x = \sin\left(x + \dfrac{\pi}{2}\right)$,

$$y'' = \cos\left(x + \frac{\pi}{2}\right) = \sin\left(x + 2\cdot\frac{\pi}{2}\right), \cdots, \quad y^{(n)} = \sin\left(x + n\cdot\frac{\pi}{2}\right), \ n \in \mathbb{N}_+.$$

同理 $(\cos x)^{(n)} = \cos\left(x + n\cdot\dfrac{\pi}{2}\right)$, $n \in \mathrm{N}_+$.

(4) $$y' = \frac{1}{x}, \ y'' = -\frac{1}{x^2}, \ y''' = \frac{1\cdot 2}{x^3}, y^{(4)} = -\frac{1\cdot 2\cdot 3}{x^4}, \ \cdots, \ y^{(n)} = (-1)^{n-1}\frac{(n-1)!}{x^n}.$$

$$\left(\frac{1}{x}\right)^{(n)} = (\ln x)^{(n+1)} = (-1)^n\frac{n!}{x^{n+1}}.$$

高阶导数运算法则

加法 $\left(u(x)\pm v(x)\right)^{(n)}=u^{(n)}(x)+v^{(n)}(x).$

乘法 $\left(u(x)v(x)\right)^{(n)}=u^{(n)}(x)v^{(0)}(x)+C_n^1u^{(n-1)}(x)v^{(1)}(x)+\cdots+$

$$C_n^ku^{(n-k)}(x)v^{(k)}(x)+\cdots+u^{(0)}(x)v^{(n)}(x)$$

$=\sum\limits_{k=0}^{n}C_n^ku^{(n-k)}(x)v^{(k)}(x)$, 莱布尼茨公式

其中 $u^{(0)}(x)=u(x)$, $v^{(0)}(x)=v(x)$,

$$C_n^k=\frac{n!}{k!(n-k)!}=\frac{n(n-1)\cdots(n-k+1)}{k!}.$$

证莱布尼茨公式 利用数学归纳法证明.

当$n=1$时,上式为 $\big(u(x)v(x)\big)'=u'(x)v(x)+u(x)v'(x)$

设当$n=m$时,莱布尼茨公式成立,即有

$$\big(u(x)v(x)\big)^{(m)}=\sum_{k=0}^{m}\mathrm{C}_m^k u^{(m-k)}(x)v^{(k)}(x),$$

则当$n=m+1$时,有

$$\big(u(x)v(x)\big)^{(m+1)}=\Big(\big(u(x)v(x)\big)^{(m)}\Big)'=\left(\sum_{k=0}^{m}\mathrm{C}_m^k u^{(m-k)}(x)v^{(k)}(x)\right)'$$

$$=\sum_{k=0}^{m}\mathrm{C}_m^k\left(u^{(m-k)}(x)v^{(k)}(x)\right)'=\sum_{k=0}^{m}\mathrm{C}_m^k\left(u^{(m-k+1)}(x)v^{(k)}(x)+u^{(m-k)}(x)v^{(k+1)}(x)\right)$$

$$=\sum_{k=0}^{m}\mathrm{C}_m^k u^{(m-k+1)}(x)v^{(k)}(x)+\sum_{k=0}^{m}\mathrm{C}_m^k u^{(m-k)}(x)v^{(k+1)}(x)$$

$$=u^{(m+1)}(x)v^{(0)}(x)+\sum_{k=1}^{m}\mathrm{C}_m^k u^{(m+1-k)}(x)v^{(k)}(x)+\sum_{k=1}^{m}\mathrm{C}_m^{k-1}u^{(m+1-k)}(x)v^{(k)}(x)+u^{(0)}(x)v^{(m+1)}(x)$$

$$=u^{(m+1)}(x)v^{(0)}(x)+\sum_{k=1}^{m}\mathrm{C}_{m+1}^k u^{(m+1-k)}(x)v^{(k)}(x)+u^{(0)}(x)v^{(m+1)}(x)=\sum_{k=0}^{m+1}\mathrm{C}_{m+1}^k u^{(m+1-k)}(x)v^{(k)}(x).$$

所以对任意正整数成立.

例2 求$y=\ln(1+x)$的n阶导数.

解 $y'=\dfrac{1}{1+x}$, $y''=-\dfrac{1}{(1+x)^2}$,

$y'''=\dfrac{2}{(1+x)^3}$, $y^{(4)}=-\dfrac{2\cdot 3}{(1+x)^4}$,

$\cdots$

$$y^{(n)}=(-1)^{n-1}\frac{(n-1)!}{(1+x)^n}.$$

例3 求$y=(1+x)^{\alpha}\,(\alpha\in\mathbb{R})$的$n$阶导数.

解 $y'=\alpha(1+x)^{\alpha-1},\quad y''=\alpha(\alpha-1)(1+x)^{\alpha-2},$

$\cdots$

$$y^{(n)}=\alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n}.$$

例4 求n次多项式$P_n(x)=a_0+a_1x+a_2x^2+a_3x^3+\cdots+a_nx^n$的各阶导数.

解 $P_n'(x)=a_1+2a_2x+3a_3x^2+\cdots+na_nx^{n-1}$,

$P_n''(x)=2a_2+3\cdot 2a_3x+\cdots+n(n-1)a_nx^{n-2}$,

$\cdots$

$P_n^{(n)}(x)=n(n-1)\cdots 2\cdot 1\cdot a_n=n!a_n$,

$P_n^{(k)}(x)=0, k>n.$

例5 求$y=(3x^2-2)\sin 2x$的100阶导数.

解 令$u(x)=\sin 2x, v(x)=3x^2-2.$

$$u^{(n)}(x)=2^n\sin(2x+n\cdot\frac{\pi}{2}); v'(x)=6x, v''(x)=6, v^{(n)}(x)=0(n\ge 3).$$

根据莱布尼茨公式，有

$$y^{(100)}=\sum_{k=0}^{100}\mathrm{C}_{100}^{k}u^{(100-k)}(x)v^{(k)}(x)$$

$$=(\sin 2x)^{(100)}(3x^2-2)+\mathrm{C}_{100}^{1}(\sin 2x)^{(99)}(3x^2-2)'+\mathrm{C}_{100}^{2}(\sin 2x)^{(98)}(3x^2-2)''$$

$$=2^{100}(3x^2-2)\sin(2x+100\cdot\frac{\pi}{2})+100\cdot 6x\cdot 2^{99}\sin(2x+99\cdot\frac{\pi}{2})$$
$$+\frac{100\cdot 99}{2}\cdot 6\cdot 2^{98}\sin(2x+98\cdot\frac{\pi}{2})$$

$$=2^{100}(3x^2-2)\sin 2x-600\cdot 2^{99}x\cos 2x-4950\cdot 6\cdot 2^{98}\sin 2x$$

$$=2^{98}\Big((12x^2-29708)\sin 2x-1200x\cos 2x\Big).$$

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例6 求$y = x^2 e^{2x}$ 的20阶导数.

解 令$u(x) = e^{2x}, v(x) = x^2$.

$$u^{(n)}(x) = 2^n e^{2x}; v'(x) = 2x, v''(x) = 2, v^{(n)}(x) = 0 (n \geq 3).$$

根据莱布尼茨公式，有

$$y^{(20)} = \sum_{k=0}^{20} C_{20}^k u^{(20-k)}(x) v^{(k)}(x)$$

$$= (e^{2x})^{(20)} \cdot x^2 + C_{20}^1 (e^{2x})^{(19)} \cdot 2x + C_{20}^2 (e^{2x})^{(18)} \cdot 2$$

$$= 2^{20} \cdot e^{2x} \cdot x^2 + 20 \cdot 2^{19} \cdot e^{2x} \cdot 2x + 190 \cdot 2^{18} \cdot e^{2x} \cdot 2$$

$$= 2^{20} e^{2x} \left(x^2 + 20x + 95 \right).$$

例7 求$y = x^2 \cos x$ 的50阶导数.

解 令$u(x) = \cos x, v(x) = x^2$.

$$u^{(n)}(x) = \cos\left(x + n \cdot \frac{\pi}{2}\right); v'(x) = 2x, v''(x) = 2, v^{(n)}(x) = 0 (n \geq 3).$$

根据莱布尼茨公式，有

$$y^{(50)} = \sum_{k=0}^{50} C_{50}^{k} u^{(50-k)}(x) v^{(k)}(x)$$

$$= (\cos x)^{(50)} \cdot x^2 + C_{50}^{1} (\cos x)^{(19)} \cdot 2x + C_{50}^{2} (\cos x)^{(18)} \cdot 2$$

$$= \cos(x + 25\pi) \cdot x^2 + 50 \cdot \cos\left(x + 49\frac{\pi}{2}\right) \cdot 2x + 1225 \cdot \cos(x + 24\pi) \cdot 2$$

$$= -x^2 \cos x - 100 x \sin x + 2450 \cos x.$$

例8 求$y=\mathrm{e}^x\cos x$ 的三阶导数.

解1 $y'=\mathrm{e}^x\cos x-\mathrm{e}^x\sin x,$

$$y''=\mathrm{e}^x\cos x-\mathrm{e}^x\sin x-\mathrm{e}^x\sin x-\mathrm{e}^x\cos x=-2\mathrm{e}^x\sin x,$$

$$y'''=-2\left(\mathrm{e}^x\sin x+\mathrm{e}^x\cos x\right)=-2\mathrm{e}^x\left(\sin x+\cos x\right).$$

例8 求$y=\mathrm{e}^x\cos x$ 的三阶导数.

解2 令$u(x)=e^x, v(x)=\cos x$. 根据莱布尼茨公式，有

$$y'''=\sum_{k=0}^{3}\mathrm{C}_3^k u^{(3-k)}(x)v^{(k)}(x)$$

$$=e^x\cos x+\mathrm{C}_3^1 e^x(\cos x)'+\mathrm{C}_3^2 e^x(\cos x)''+\mathrm{C}_3^3 e^x(\cos x)'''$$

$$=e^x\cos x+3e^x\cos\left(x+\frac{\pi}{2}\right)+3e^x\cos\left(x+\frac{2\pi}{2}\right)+e^x\cos\left(x+\frac{3\pi}{2}\right)$$

$$=e^x\cos x-3e^x\sin x-3e^x\cos x+e^x\sin x$$

$$=-2\mathrm{e}^x(\sin x+\cos x).$$

例8 求$y = e^x \cos x$ 的三阶导数.

解3 $y' = e^x \cos x - e^x \sin x = \sqrt{2}e^x \cos(x + \frac{\pi}{4})$;

$$y'' = \sqrt{2}e^x \left(\cos(x + \frac{\pi}{4}) - \sin(x + \frac{\pi}{4}) \right) = 2e^x \cos(x + 2 \cdot \frac{\pi}{4})$$

$$y''' = 2\sqrt{2}\, e^x \cos(x + 3 \cdot \frac{\pi}{4}).$$

例9 多项式$P_n(x)=\dfrac{1}{2^n n!}\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n$称为勒让德$n$次多项式,求$P_n(1)$与$P_n(-1)$.

解 $$P_n(x)=\frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n=\frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x+1)^n(x-1)^n.$$

设$u(x)=(x+1)^n, v(x)=(x-1)^n$.

$$\frac{\mathrm{d}u}{\mathrm{d}x}=n(x+1)^{n-1},\frac{\mathrm{d}v}{\mathrm{d}x}=n(x-1)^{n-1};$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2}=n(n-1)(x+1)^{n-2},\frac{\mathrm{d}^2v}{\mathrm{d}x^2}=n(n-1)(x-1)^{n-2};$$

$$\cdots$$

$$\frac{\mathrm{d}^{n-1}u}{\mathrm{d}x^{n-1}}=n(n-1)\cdots2(x+1),\frac{\mathrm{d}^{n-1}v}{\mathrm{d}x^{n-1}}=n(n-1)\cdots2(x-1);$$

$$\frac{\mathrm{d}^n u}{\mathrm{d}x^n}=n!,\frac{\mathrm{d}^n v}{\mathrm{d}x^n}=n!.$$

$$P_n(x)=\frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}(x^2-1)^n$$

根据Leibniz公式，有

$$=\frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}(x+1)^n(x-1)^n.$$

$$P_n(x)=\frac{1}{2^n n!}\sum_{k=0}^{n}\mathrm{C}_n^k u^{(n-k)}(x)v^{(k)}(x)$$

$$=\frac{1}{2^n n!}\left(u^{(n)}(x)v^{(0)}(x)+\mathrm{C}_n^1 u^{(n-1)}(x)v'(x)+\cdots+\mathrm{C}_n^{n-1}u'(x)v^{(n-1)}(x)+u(x)v^{(n)}(x)\right)$$

$$=\frac{1}{2^n n!}\Big(n!\cdot(x-1)^n+n\cdot n(n-1)\cdots 2(x+1)\cdot n(x-1)^{n-1}+\cdots$$

$$+n\cdot n(x+1)^{n-1}n(n-1)\cdots 2(x-1)+(x+1)^n\cdot n!\Big)$$

于是

$$P_n(1)=\frac{1}{2^n n!}\cdot 2^n n!=1,$$

$$P_n(-1)=\frac{1}{2^n n!}\cdot n!(-2)^n=(-1)^n.$$

例10 讨论分段函数$f(x)=\begin{cases} x^2, & x\ge 0, \\ -x^2, & x<0 \end{cases}$的高阶导数.

解 分段函数要分段讨论：

当$x>0$时，$f'(x)=2x$，$f''(x)=2$，$f^{(n)}(x)\equiv 0(n\ge 3)$;

当$x<0$时，$f'(x)=-2x$，$f''(x)=-2$，$f^{(n)}(x)\equiv 0(n\ge 3)$;

当$x=0$时,用左、右导数定义可得

$$f_+'(0)=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{x^2}{x}=\lim_{x\to 0^+}x=0,\quad f_-'(0)=\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{-x^2}{x}=\lim_{x\to 0^-}(-x)=0,$$

由于$f_-'(0)=f_+'(0)=0$,所以$f'(0)=0$. 从而 $f'(x)=\begin{cases} 2x, & x>0 \\ 0, & x=0 \\ -2x, & x<0 \end{cases}$,

$$f_+''(0)=\lim_{x\to 0^+}\frac{f'(x)-f'(0)}{x-0}=\lim_{x\to 0^+}\frac{2x}{x}=2,\quad f_-''(0)=\lim_{x\to 0^-}\frac{f'(x)-f'(0)}{x-0}=\lim_{x\to 0^-}\frac{-2x}{x}=-2,$$

由于$f_+''(0)=2$，$f_-''(0)=-2$，因此在$x=0$处$f''(0)$不存在. 从而

$f''(x)=\begin{cases} 2, & x>0 \\ -2, & x<0 \end{cases}$，故当$n\ge 2$时，$f^{(n)}(0)$不存在.

当$n\ge 3$时，$f^{(n)}(x)=0(x\ne 0)$，$f^{(n)}(0)$不存在.

注：复合函数$y=f\big(g(x)\big)$的二阶导数：$y=f(u),u=g(x)$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x}\right)$$

$$=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}u}\right)\cdot\frac{\mathrm{d}u}{\mathrm{d}x}+\frac{\mathrm{d}y}{\mathrm{d}u}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)$$

$$=\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\mathrm{d}y}{\mathrm{d}u}\right)\cdot\frac{\mathrm{d}u}{\mathrm{d}x}\cdot\frac{\mathrm{d}u}{\mathrm{d}x}+\frac{\mathrm{d}y}{\mathrm{d}u}\cdot\frac{\mathrm{d}^2u}{\mathrm{d}x^2}$$

$$=\frac{\mathrm{d}^2y}{\mathrm{d}u^2}\cdot\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2+\frac{\mathrm{d}y}{\mathrm{d}u}\cdot\frac{\mathrm{d}^2u}{\mathrm{d}x^2}.$$

注：$y=f(x)$是$x=g(y)$的反函数：$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{\dfrac{\mathrm{d}x}{\mathrm{d}y}}=\dfrac{1}{g'(y)}$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{g'(y)}\right)=-\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(g'(y)\right)}{\left(g'(y)\right)^2}=-\frac{\frac{\mathrm{d}}{\mathrm{d}y}\left(g'(y)\right)\cdot\frac{\mathrm{d}y}{\mathrm{d}x}}{\left(g'(y)\right)^2}$$

$$=-\frac{g''(y)\cdot\frac{1}{g'(y)}}{\left(g'(y)\right)^2}=-\frac{g''(y)}{\left(g'(y)\right)^3}.$$

注：$y=f(x)$是$x=g(y)$的反函数：$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}=\dfrac{1}{g'(y)}$　$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\dfrac{g''(y)}{(g'(y))^3}$

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{g''(y)}{(g'(y))^3}\right)=-\frac{\frac{\mathrm{d}}{\mathrm{d}x}(g''(y))\cdot(g'(y))^3-g''(y)\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left((g'(y))^3\right)}{(g'(y))^6}$$

$$=-\frac{\frac{\mathrm{d}}{\mathrm{d}y}(g''(y))\cdot\frac{\mathrm{d}y}{\mathrm{d}x}\cdot(g'(y))^3-g''(y)\cdot 3(g'(y))^2\cdot\frac{\mathrm{d}}{\mathrm{d}x}(g'(y))}{(g'(y))^6}$$

$$=-\frac{g'''(y)\cdot\frac{1}{g'(y)}\cdot(g'(y))^3-g''(y)\cdot 3(g'(y))^2\cdot\frac{\mathrm{d}}{\mathrm{d}y}(g'(y))\cdot\frac{\mathrm{d}y}{\mathrm{d}x}}{(g'(y))^6}$$

$$=-\frac{g'''(y)\cdot(g'(y))^2-g''(y)\cdot 3(g'(y))^2\cdot g''(y)\cdot\frac{1}{g'(y)}}{(g'(y))^6}=\frac{3(g''(y))^2-g'(y)\cdot g'''(y)}{(g'(y))^5}.$$

例11 求$y = \mathrm{e}^{\sin x}$ 的二阶导数.

解

$$y' = \cos x e^{\sin x},$$

$$y'' = \left(\cos x e^{\sin x}\right)'$$

$$= \left(\cos x\right)' e^{\sin x} + \cos x\left(e^{\sin x}\right)'$$

$$= -\sin x e^{\sin x} + \cos x e^{\sin x} \cdot \cos x$$

$$= e^{\sin x}(\cos^2 x - \sin x).$$

例12 求$y = f\left(x^2\right)$的三阶导数.

解 $$y' = 2xf'\left(x^2\right),$$

$$y'' = \left(2xf'\left(x^2\right)\right)' = 2f'\left(x^2\right) + 4x^2 f''\left(x^2\right),$$

$$y''' = \left(2f'\left(x^2\right) + 4x^2 f''\left(x^2\right)\right)' = 12xf''\left(x^2\right) + 8x^3 f'''\left(x^2\right).$$

注： 参变量函数 $\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}$ 的二阶导数.

一阶导数为 $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\psi'(t)}{\varphi'(t)}$,

把它写成参数方程: $\begin{cases} x = \varphi(t), \\ \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\psi'(t)}{\varphi'(t)}. \end{cases}$

由此求得
$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\psi'(t)}{\varphi'(t)}\right)}{\dfrac{\mathrm{d}x}{\mathrm{d}t}} = \frac{\left(\dfrac{\psi'(t)}{\varphi'(t)}\right)'}{\varphi'(t)},$$

即
$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\psi''(t)\varphi'(t) - \psi'(t)\varphi''(t)}{[\varphi'(t)]^3}.$$

例12 求参变量函数(摆线)$\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}$ 的二阶导数.

解1 公式法(不建议)

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{1}{[a(t-\sin t)']^3}[\,a(1-\cos t)''a(t-\sin t)' - a(1-\cos t)'a(t-\sin t)''\,]$$

$$= \frac{a^2\left[\cos t(1-\cos t) - \sin^2 t\right]}{a^3(1-\cos t)^3}$$

$$= \frac{-1}{a(1-\cos t)^2} = \frac{-1}{4a\sin^4\frac{t}{2}} = -\frac{1}{4a}\csc^4\frac{t}{2}.$$

例12 求参变量函数(摆线)$\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}$ 的二阶导数.

解2 直接计算法

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dfrac{\mathrm{d}y}{\mathrm{d}t}}{\dfrac{\mathrm{d}x}{\mathrm{d}t}} = \frac{a\sin t}{a(1-\cos t)} = \frac{\sin t}{1-\cos t}.$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)}{\dfrac{\mathrm{d}x}{\mathrm{d}t}} = \frac{\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\sin t}{1-\cos t}\right)}{\dfrac{\mathrm{d}}{\mathrm{d}t}\big(a(t-\sin t)\big)}$$

$$= \frac{\cos t\,(1-\cos t) - \sin^2 t}{(1-\cos t)^2} \cdot \frac{1}{a(1-\cos t)} = \frac{\cos t - 1}{a(1-\cos t)^3} = -\frac{1}{a(1-\cos t)^2}.$$

你应该：

会求高阶导数