Ch8 不定积分

总结及习题评讲

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QQ群、QQ、微信群、微信随时答疑解惑

P183/习题8.3/1(3) 求不定积分 $\int\frac{\mathrm{d}x}{1+x^3}$.

$$\int\frac{1}{(x-a)^2+b^2}\mathrm{d}x=\frac{1}{b}\arctan\frac{x-a}{b}+C$$

解 $\int\frac{\mathrm{d}x}{1+x^3}=\int\frac{\mathrm{d}x}{(x+1)(x^2-x+1)}$, 设 $\frac{1}{(x+1)(x^2-x+1)}=\frac{A}{x+1}+\frac{Bx+D}{x^2-x+1}$,

则 $1=A(x^2-x+1)+(Bx+D)(x+1)$, 从而 $\begin{cases}A+B=0\\-A+B+D=0,\\A+D=1\end{cases}$ 解得 $\begin{cases}A=\frac{1}{3}\\B=-\frac{1}{3}.\\D=\frac{2}{3}\end{cases}$

故

$$\int\frac{\mathrm{d}x}{1+x^3}=\frac{1}{3}\int\frac{1}{x+1}\mathrm{d}x-\frac{1}{3}\int\frac{x-2}{x^2-x+1}\mathrm{d}x=\frac{1}{3}\ln|x+1|-\frac{1}{3}\int\frac{\frac{1}{2}(x^2-x+1)'-\frac{3}{2}}{x^2-x+1}\mathrm{d}x$$

$$=\frac{1}{3}\ln|x+1|-\frac{1}{6}\ln(x^2-x+1)+\frac{1}{2}\int\frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\mathrm{d}x$$

$$=\frac{1}{3}\ln|x+1|-\frac{1}{6}\ln(x^2-x+1)+\frac{1}{2\cdot\frac{\sqrt{3}}{2}}\arctan\frac{\left(x-\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}}+C$$

$$=\frac{1}{3}\ln|x+1|-\frac{1}{6}\ln(x^2-x+1)+\frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}+C.$$

P183/习题8.3/1(4) 求不定积分$\displaystyle\int\frac{\mathrm{d}x}{1+x^4}$.

解 $$\int\frac{1}{1+x^4}\mathrm{d}x=\int\frac{1}{\left(x^2+1\right)^2-2x^2}\mathrm{d}x=\int\frac{1}{\left(x^2+\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right)}\mathrm{d}x,$$

设 $$\frac{1}{\left(x^2+\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right)}=\frac{Ax+B}{x^2+\sqrt{2}x+1}+\frac{Dx+E}{x^2-\sqrt{2}x+1}.$$

则 $$(Ax+B)\left(x^2-\sqrt{2}x+1\right)+(Dx+E)\left(x^2+\sqrt{2}x+1\right)=1,$$

从而 $$\begin{cases}A+D=0\\-\sqrt{2}A+B+\sqrt{2}D+E=0\\A-\sqrt{2}B+D+\sqrt{2}E=0\\B+E=0\end{cases},\quad \text{解得}\begin{cases}A=\dfrac{\sqrt{2}}{4}\\B=\dfrac{1}{2}\\D=-\dfrac{\sqrt{2}}{4}\\E=\dfrac{1}{2}\end{cases}.$$

故 $$\int\frac{1}{1+x^4}\mathrm{d}x=\int\frac{\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2+\sqrt{2}x+1}\mathrm{d}x+\int\frac{-\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2-\sqrt{2}x+1}\mathrm{d}x$$

P183/习题8.3/1(4) 求不定积分 $\int \frac{\mathrm{d}x}{1+x^4}$.

故
$$\int \frac{1}{1+x^4}\mathrm{d}x=\int \frac{\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2+\sqrt{2}x+1}\mathrm{d}x+\int \frac{-\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2-\sqrt{2}x+1}\mathrm{d}x$$

$$=\int \frac{\frac{\sqrt{2}}{4}\cdot\frac{1}{2}\left(x^2+\sqrt{2}x+1\right)'+\frac{1}{4}}{x^2+\sqrt{2}x+1}\mathrm{d}x+\int \frac{-\frac{\sqrt{2}}{4}\cdot\frac{1}{2}\left(x^2-\sqrt{2}x+1\right)'+\frac{1}{4}}{x^2-\sqrt{2}x+1}\mathrm{d}x$$

$$=\frac{\sqrt{2}}{8}\ln\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}+\frac{1}{4}\int \frac{1}{\left(x+\frac{\sqrt{2}}{2}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}\mathrm{d}x+\frac{1}{4}\int \frac{1}{\left(x-\frac{\sqrt{2}}{2}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}\mathrm{d}x$$

$$=\frac{\sqrt{2}}{8}\ln\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}+\frac{1}{4}\cdot\frac{1}{\frac{1}{\sqrt{2}}}\arctan\frac{x+\frac{\sqrt{2}}{2}}{\frac{1}{\sqrt{2}}}+\frac{1}{4}\cdot\frac{1}{\frac{1}{\sqrt{2}}}\arctan\frac{x-\frac{\sqrt{2}}{2}}{\frac{1}{\sqrt{2}}}+C$$

$$=\frac{\sqrt{2}}{8}\ln\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}+\frac{\sqrt{2}}{4}\arctan\left(\sqrt{2}x+1\right)+\frac{\sqrt{2}}{4}\arctan\left(\sqrt{2}x-1\right)+C.$$

P183/习题8.3/1(4) 求不定积分 $\int \frac{\mathrm{d}x}{1+x^4}$.

$$\int \frac{1}{(x-a)^2-b^2}\mathrm{d}x = \frac{1}{2b}\ln\left|\frac{x-a-b}{x-a+b}\right| + C$$

另解
$$\int \frac{1}{1+x^4}\mathrm{d}x = \frac{1}{2}\int\left(\frac{1+x^2}{1+x^4}+\frac{1-x^2}{1+x^4}\right)\mathrm{d}x = \frac{1}{2}\int\frac{1+x^2}{1+x^4}\mathrm{d}x + \frac{1}{2}\int\frac{1-x^2}{1+x^4}\mathrm{d}x$$

$$= \frac{1}{2}\int\frac{\frac{1}{x^2}+1}{\frac{1}{x^2}+x^2}\mathrm{d}x + \frac{1}{2}\int\frac{\frac{1}{x^2}-1}{\frac{1}{x^2}+x^2}\mathrm{d}x = \frac{1}{2}\int\frac{\frac{1}{x^2}+1}{\left(x-\frac{1}{x}\right)^2+\left(\sqrt{2}\right)^2}\mathrm{d}x + \frac{1}{2}\int\frac{\frac{1}{x^2}-1}{\left(x+\frac{1}{x}\right)^2-\left(\sqrt{2}\right)^2}\mathrm{d}x$$

$$= \frac{1}{2}\int\frac{\mathrm{d}\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+\left(\sqrt{2}\right)^2} - \frac{1}{2}\int\frac{\mathrm{d}\left(x+\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)^2-\left(\sqrt{2}\right)^2}$$

$$= \frac{1}{2\sqrt{2}}\arctan\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) - \frac{1}{4\sqrt{2}}\ln\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right| + C$$

$$= \frac{1}{2\sqrt{2}}\arctan\left(\frac{x^2-1}{\sqrt{2}x}\right) - \frac{1}{4\sqrt{2}}\ln\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1} + C.$$

此法有点瑕疵,能否指出?

如何修正?

可按如下方法修正上述解法中的不足：

当$x>0$时，记$F(x)=\dfrac{1}{2\sqrt{2}}\arctan\left(\dfrac{x^2-1}{\sqrt{2}x}\right)-\dfrac{1}{4\sqrt{2}}\ln\dfrac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}+C_1$;

当$x<0$时，记$F(x)=\dfrac{1}{2\sqrt{2}}\arctan\left(\dfrac{x^2-1}{\sqrt{2}x}\right)-\dfrac{1}{4\sqrt{2}}\ln\dfrac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}+C_2$.

由于$F(x)$在$x=0$处连续，令$C_1=0$，从而 $\lim\limits_{x\to0^+}F(x)=-\dfrac{\pi}{4\sqrt{2}}=\lim\limits_{x\to0^-}F(x)=\dfrac{\pi}{4\sqrt{2}}+C_2$,

解得 $C_2=-\dfrac{\pi}{2\sqrt{2}}$.

$$\text{故 } F(x)=\begin{cases}\dfrac{1}{2\sqrt{2}}\arctan\left(\dfrac{x^2-1}{\sqrt{2}x}\right)-\dfrac{1}{4\sqrt{2}}\ln\dfrac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}, & x>0\\ -\dfrac{\pi}{4\sqrt{2}}, & x=0.\\ \dfrac{1}{2\sqrt{2}}\arctan\left(\dfrac{x^2-1}{\sqrt{2}x}\right)-\dfrac{1}{4\sqrt{2}}\ln\dfrac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}-\dfrac{\pi}{2\sqrt{2}}, & x<0\end{cases}$$

所以 $\displaystyle\int\frac{1}{1+x^4}\mathrm{d}x=F(x)+C.$

注： $\tan\left(\arctan(\sqrt{2}x+1)+\arctan(\sqrt{2}x-1)\right)$

$$=\frac{\tan\left(\arctan(\sqrt{2}x+1)\right)+\tan\left(\arctan(\sqrt{2}x-1)\right)}{1-\tan\left(\arctan(\sqrt{2}x+1)\right)\tan\left(\arctan(\sqrt{2}x-1)\right)}=\frac{\left(\sqrt{2}x+1\right)+\left(\sqrt{2}x-1\right)}{1-\left(\sqrt{2}x+1\right)\left(\sqrt{2}x-1\right)}$$

$$=\frac{\sqrt{2}x}{1-x^2},$$

$$\tan\left(\arctan\left(\frac{x^2-1}{\sqrt{2}x}\right)\right)=\frac{x^2-1}{\sqrt{2}x}=-\cot\left(\arctan(\sqrt{2}x+1)+\arctan(\sqrt{2}x-1)\right).$$

所以

$$\arctan\left(\frac{x^2-1}{\sqrt{2}x}\right)=\frac{\pi}{2}+\left(\arctan(\sqrt{2}x+1)+\arctan(\sqrt{2}x-1)\right).$$

P183/习题8.3/1(5) 求不定积分 $\int \frac{\mathrm{d}x}{(x-1)(x^2+1)^2}$.

解 设 $\frac{1}{(x-1)(x^2+1)^2} = \frac{A}{x-1} + \frac{Bx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}$,

则 $1 = A(x^2+1)^2 + (Bx+D)(x-1)(x^2+1) + (Ex+F)(x-1)$,

从而 $\begin{cases} A+B=0 \\ -B+D=0 \\ 2A+B-D+E=0 \\ -B+D-E+F=0 \\ A-D-F=1 \end{cases}$, 解得 $\begin{cases} A=\frac{1}{4} \\ B=-\frac{1}{4} \\ D=-\frac{1}{4} \\ E=-\frac{1}{2} \\ F=-\frac{1}{2} \end{cases}$.

故 $\int \frac{\mathrm{d}x}{(x-1)(x^2+1)^2} = \frac{1}{4}\int \frac{1}{x-1}\mathrm{d}x - \frac{1}{4}\int \frac{x+1}{x^2+1}\mathrm{d}x - \frac{1}{2}\int \frac{x+1}{(x^2+1)^2}\mathrm{d}x$

P183/习题8.3/1(5) 求不定积分 $\int \frac{\mathrm{d}x}{(x-1)(x^2+1)^2}$.

$$\int \frac{\mathrm{d}x}{(x-1)\left(x^2+1\right)^2}=\frac{1}{4}\int\frac{1}{x-1}\mathrm{d}x-\frac{1}{4}\int\frac{x+1}{x^2+1}\mathrm{d}x-\frac{1}{2}\int\frac{x+1}{\left(x^2+1\right)^2}\mathrm{d}x$$

$$=\frac{1}{4}\ln|x-1|-\frac{1}{8}\ln\left(x^2+1\right)-\frac{1}{4}\arctan x-\frac{1}{2}\int\frac{\frac{1}{2}\left(x^2+1\right)'+1}{\left(x^2+1\right)^2}\mathrm{d}x$$

$$=\frac{1}{4}\ln|x-1|-\frac{1}{8}\ln\left(x^2+1\right)-\frac{1}{4}\arctan x+\frac{1}{4\left(x^2+1\right)}-\frac{1}{2}\int\frac{1+x^2-x^2}{\left(x^2+1\right)^2}\mathrm{d}x$$

$$=\frac{1}{4}\ln|x-1|-\frac{1}{8}\ln\left(x^2+1\right)-\frac{3}{4}\arctan x+\frac{1}{4\left(x^2+1\right)}-\frac{1}{4}\int x\,\mathrm{d}\frac{1}{x^2+1}$$

$$=\frac{1}{4}\ln|x-1|-\frac{1}{8}\ln\left(x^2+1\right)-\frac{3}{4}\arctan x+\frac{1}{4\left(x^2+1\right)}-\frac{x}{4\left(x^2+1\right)}+\frac{1}{4}\arctan x+C$$

$$=\frac{1}{4}\ln|x-1|-\frac{1}{8}\ln\left(x^2+1\right)-\frac{1}{2}\arctan x+\frac{1-x}{4\left(x^2+1\right)}+C.$$

P183/习题8.3/1(6) 求不定积分 $\displaystyle\int\frac{x-2}{\left(2x^2+2x+1\right)^2}\,\mathrm{d}x.$

解

$$\int\frac{x-2}{\left(2x^2+2x+1\right)^2}\,\mathrm{d}x=\int\frac{\frac{1}{4}\left(2x^2+2x+1\right)'-\frac{5}{2}}{\left(2x^2+2x+1\right)^2}\,\mathrm{d}x$$

$$=\frac{1}{4}\int\frac{1}{\left(2x^2+2x+1\right)^2}\,\mathrm{d}\left(2x^2+2x+1\right)-\frac{5}{2}\int\frac{1}{4\left(\left(x+\frac{1}{2}\right)^2+\frac{1}{4}\right)^2}\,\mathrm{d}x=-\frac{1}{4\left(2x^2+2x+1\right)}-\frac{5}{2}\int\frac{\left(x+\frac{1}{2}\right)^2+\frac{1}{4}-\left(x+\frac{1}{2}\right)^2}{\left(\left(x+\frac{1}{2}\right)^2+\frac{1}{4}\right)^2}\,\mathrm{d}x$$

$$=-\frac{1}{4\left(2x^2+2x+1\right)}-\frac{5}{2}\int\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{1}{4}}\,\mathrm{d}x-\frac{5}{4}\int\left(x+\frac{1}{2}\right)\mathrm{d}\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{1}{4}}$$

$$=-\frac{1}{4\left(2x^2+2x+1\right)}-5\arctan\left(2x+1\right)-\frac{5}{4}\frac{x+\frac{1}{2}}{\left(x+\frac{1}{2}\right)^2+\frac{1}{4}}+\frac{5}{4}\int\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{1}{4}}\,\mathrm{d}x$$

$$=-\frac{1}{4\left(2x^2+2x+1\right)}-\frac{5}{2}\arctan\left(2x+1\right)-\frac{5}{4}\frac{2x+1}{2x^2+2x+1}+C=-\frac{5x+3}{2\left(2x^2+2x+1\right)}-\frac{5}{2}\arctan\left(2x+1\right)+C.$$

P183/习题8.3/2(1) 求不定积分$\int \frac{\mathrm{d}x}{5-3\cos x}$.

解 令$t=\tan\frac{x}{2}$. 则 $\mathrm{d}x=\frac{2}{1+t^2}\mathrm{d}t$. 从而

$$\int \frac{\mathrm{d}x}{5-3\cos x}=\int \frac{1}{5-3\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\mathrm{d}t=\int \frac{\mathrm{d}t}{1+4t^2}$$

$$=\frac{1}{2}\int \frac{\mathrm{d}(2t)}{1+(2t)^2}=\frac{1}{2}\arctan(2t)+C$$

$$=\frac{1}{2}\arctan\left(2\tan\frac{x}{2}\right)+C.$$

P183/习题8.3/2(6) 求不定积分 $\int \frac{1}{x^2}\sqrt{\frac{1-x}{1+x}}\,\mathrm{d}x.$

解1 令 $t=\sqrt{\frac{1-x}{1+x}}$，即 $x=\frac{1-t^2}{1+t^2}$，则 $\mathrm{d}x=\frac{-4t}{\left(1+t^2\right)^2}\mathrm{d}t$. 从而

$$\int \frac{1}{x^2}\sqrt{\frac{1-x}{1+x}}\,\mathrm{d}x=\int \frac{1}{\left(\frac{1-t^2}{1+t^2}\right)^2}t\cdot\frac{-4t}{\left(1+t^2\right)^2}\mathrm{d}t=-4\int\frac{t^2}{\left(1-t^2\right)^2}\mathrm{d}t=2\int\frac{t}{\left(1-t^2\right)^2}\mathrm{d}\left(1-t^2\right)$$

$$=-2\int t\,\mathrm{d}\frac{1}{1-t^2}=-\frac{2t}{1-t^2}+2\int\frac{1}{1-t^2}\mathrm{d}t=-\frac{2t}{1-t^2}+\ln\left|\frac{1+t}{1-t}\right|+C$$

$$=-\frac{2\sqrt{\frac{1-x}{1+x}}}{1-\frac{1-x}{1+x}}+\ln\left|\frac{1+\sqrt{\frac{1-x}{1+x}}}{1-\sqrt{\frac{1-x}{1+x}}}\right|+C=-\frac{\sqrt{1-x^2}}{x}+\ln\left|\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right|+C$$

$$=-\frac{\sqrt{1-x^2}}{x}+\ln\left|\frac{1+\sqrt{1-x^2}}{x}\right|+C.$$

P183/习题8.3/2(6) 求不定积分 $\int \frac{1}{x^2}\sqrt{\frac{1-x}{1+x}}\,\mathrm{d}x.$

解2 令 $t=\sqrt{\frac{1-x}{1+x}}$，即 $x=\frac{1-t^2}{1+t^2}$，则 $\mathrm{d}x=\frac{-4t}{\left(1+t^2\right)^2}\mathrm{d}t$. 从而

$$\int \frac{1}{x^2}\sqrt{\frac{1-x}{1+x}}\,\mathrm{d}x=\int \frac{1}{\left(\frac{1-t^2}{1+t^2}\right)^2}t\cdot\frac{-4t}{\left(1+t^2\right)^2}\mathrm{d}t=-4\int\frac{t^2}{\left(1-t^2\right)^2}\mathrm{d}t$$

$$=4\int\frac{1-t^2-1}{\left(1-t^2\right)^2}\mathrm{d}t=4\int\frac{1}{1-t^2}\mathrm{d}t-4\int\left(\frac{1}{(1-t)(1+t)}\right)^2\mathrm{d}t$$

$$=2\ln\left|\frac{1+t}{1-t}\right|-\int\left(\frac{1}{1-t}+\frac{1}{1+t}\right)^2\mathrm{d}t=2\ln\left|\frac{1+t}{1-t}\right|-\int\left(\frac{1}{(1-t)^2}+\frac{2}{(1-t)(1+t)}+\frac{1}{(1+t)^2}\right)\mathrm{d}t$$

$$=2\ln\left|\frac{1+t}{1-t}\right|-\frac{1}{1-t}-\ln\left|\frac{1+t}{1-t}\right|+\frac{1}{1+t}+C=\ln\left|\frac{1+t}{1-t}\right|-\frac{2t}{1-t^2}+C$$

$$=\ln\left|\frac{1+\sqrt{\frac{1-x}{1+x}}}{1-\sqrt{\frac{1-x}{1+x}}}\right|-\frac{2\sqrt{\frac{1-x}{1+x}}}{1-\frac{1-x}{1+x}}+C=\ln\left|\frac{1+\sqrt{1-x^2}}{x}\right|-\frac{\sqrt{1-x^2}}{x}+C.$$

P183/习题8.3/2(6) 求不定积分 $\displaystyle\int \frac{1}{x^2}\sqrt{\frac{1-x}{1+x}}\,\mathrm{d}x.$

解3 令$t=\sqrt{\dfrac{1-x}{1+x}}$，即$x=\dfrac{1-t^2}{1+t^2}$，则 $\mathrm{d}x=\dfrac{-4t}{(1+t^2)^2}\mathrm{d}t$. 从而

$$\int \frac{1}{x^2}\sqrt{\frac{1-x}{1+x}}\,\mathrm{d}x=\int \frac{1}{\left(\dfrac{1-t^2}{1+t^2}\right)^2}t\cdot\frac{-4t}{(1+t^2)^2}\mathrm{d}t=-4\int\frac{t^2}{(1-t^2)^2}\mathrm{d}t,$$

令$t=\sin u$，则 $\mathrm{d}t=\cos u\,\mathrm{d}u$. 从而

$$\int \frac{1}{x^2}\sqrt{\frac{1-x}{1+x}}\,\mathrm{d}x=-4\int\frac{\sin^2 u}{\cos^4 u}\cdot\cos u\,\mathrm{d}u=4\int\sec u\,\mathrm{d}u-4\int\sec^3 u\,\mathrm{d}u,$$

其中$\displaystyle\int\sec^3 u\,\mathrm{d}u=\int\sec u\,\mathrm{d}\tan u=\sec u\tan u-\int\tan^2 u\sec u\,\mathrm{d}u=\sec u\tan u-\int\sec^3 u\,\mathrm{d}u+\int\sec u\,\mathrm{d}u$

$$=\frac{1}{2}\sec u\tan u+\frac{1}{2}\int\sec u\,\mathrm{d}u.$$

$$\int \frac{1}{x^2}\sqrt{\frac{1-x}{1+x}}\,\mathrm{d}x=2\ln|\sec u+\tan u|-2\sec u\tan u+C=-\frac{\sqrt{1-x^2}}{x}+\ln\left|\frac{1+\sqrt{1-x^2}}{x}\right|+C.$$

P183/习题8.3/2(6) 求不定积分 $\int \frac{1}{x^2}\sqrt{\frac{1-x}{1+x}}\,\mathrm{d}x.$

解4 $\int \frac{1}{x^2}\sqrt{\frac{1-x}{1+x}}\,\mathrm{d}x = -\int \sqrt{\frac{\frac{1}{x}-1}{\frac{1}{x}+1}}\,\mathrm{d}\frac{1}{x},$ 令 $t=\frac{1}{x}$, 从而

$$\int \frac{1}{x^2}\sqrt{\frac{1-x}{1+x}}\,\mathrm{d}x = -\int \sqrt{\frac{t-1}{t+1}}\,\mathrm{d}t = -\int \frac{t-1}{\sqrt{t^2-1}}\mathrm{d}t = -\int \frac{t}{\sqrt{t^2-1}}\mathrm{d}t + \int \frac{1}{\sqrt{t^2-1}}\mathrm{d}t$$

$$= -\sqrt{t^2-1} + \ln\left|t+\sqrt{t^2-1}\right| + C$$

$$= -\sqrt{\frac{1}{x^2}-1} + \ln\left|\frac{1}{x}+\sqrt{\frac{1}{x^2}-1}\right| + C$$

$$= -\frac{\sqrt{1-x^2}}{x} + \ln\left|\frac{1+\sqrt{1-x^2}}{x}\right| + C.$$

P183/习题8.3/2(6) 求不定积分$\int \frac{1}{x^2}\sqrt{\frac{1-x}{1+x}}\,\mathrm{d}x.$

解4 $\int \frac{1}{x^2}\sqrt{\frac{1-x}{1+x}}\,\mathrm{d}x = -\int \sqrt{\frac{\frac{1}{x}-1}{\frac{1}{x}+1}}\,\mathrm{d}\frac{1}{x},$ 令$t=\frac{1}{x}$，从而

$$\int \frac{1}{x^2}\sqrt{\frac{1-x}{1+x}}\,\mathrm{d}x = -\int \sqrt{\frac{t-1}{t+1}}\,\mathrm{d}t = -\int \frac{t-1}{\sqrt{t^2-1}}\,\mathrm{d}t = -\int \frac{t}{\sqrt{t^2-1}}\,\mathrm{d}t + \int \frac{1}{\sqrt{t^2-1}}\,\mathrm{d}t$$

$$= -\sqrt{t^2-1} + \ln\left|t+\sqrt{t^2-1}\right| + C$$

$$= -\sqrt{\frac{1}{x^2}-1} + \ln\left|\frac{1}{x}+\sqrt{\frac{1}{x^2}-1}\right| + C$$

$$= -\frac{\sqrt{1-x^2}}{x} + \ln\left|\frac{1+\sqrt{1-x^2}}{x}\right| + C.$$

P184/第八章总练习题/1(4) 求不定积分 $\int e^{\sin x}\sin 2x\,\mathrm{d}x$.

解 $\int e^{\sin x}\sin 2x\,\mathrm{d}x = 2\int e^{\sin x}\sin x\cos x\,\mathrm{d}x = 2\int e^{\sin x}\sin x\,\mathrm{d}\sin x$

$= 2\int \sin x\,\mathrm{d}e^{\sin x} = 2e^{\sin x}\sin x - 2\int e^{\sin x}\,\mathrm{d}\sin x$

$= 2e^{\sin x}\sin x - 2e^{\sin x} + C.$

P184/第八章总练习题/1(5) 求不定积分$\int e^{\sqrt{x}}\,dx$.

解1 令 $t=\sqrt{x}$, 即 $x=t^2$. 则 $dx=2t\,dt$. 从而

$$\int e^{\sqrt{x}}\,dx=2\int te^t\,dt=2\int t\,de^t=2te^t-2\int e^t\,dt$$

$$=2te^t-2e^t+C=2e^{\sqrt{x}}\left(\sqrt{x}-1\right)+C.$$

解2 $$\int e^{\sqrt{x}}\,dx=2\int\sqrt{x}e^{\sqrt{x}}\,d\sqrt{x}=2\int\sqrt{x}\,de^{\sqrt{x}}=2\sqrt{x}e^{\sqrt{x}}-2\int e^{\sqrt{x}}\,d\sqrt{x}$$

$$=2\sqrt{x}e^{\sqrt{x}}-2e^{\sqrt{x}}+C=2e^{\sqrt{x}}\left(\sqrt{x}-1\right)+C.$$

P184/第八章总练习题/1(7) 求不定积分 $\int \frac{1-\tan x}{1+\tan x}\mathrm{d}x.$

解1 $\int \frac{1-\tan x}{1+\tan x}\mathrm{d}x = \int \frac{\cos x-\sin x}{\cos x+\sin x}\mathrm{d}x = \int \frac{1}{\cos x+\sin x}\mathrm{d}(\cos x+\sin x)$

$= \ln|\cos x+\sin x| + C.$

解2 $\int \frac{1-\tan x}{1+\tan x}\mathrm{d}x = \int \frac{\cos x-\sin x}{\cos x+\sin x}\mathrm{d}x = \int \frac{(\cos x-\sin x)^2}{\cos^2 x-\sin^2 x}\mathrm{d}x$

$= \int \frac{1-\sin 2x}{\cos 2x}\mathrm{d}x = \int \sec 2x\,\mathrm{d}x - \int \frac{\sin 2x}{\cos 2x}\mathrm{d}x$

$= \frac{1}{2}\ln|\sec 2x+\tan 2x| + \frac{1}{2}\ln|\cos 2x| + C.$

P184/第八章总练习题/1(7) 求不定积分 $\int \frac{1-\tan x}{1+\tan x}\,\mathrm{d}x.$

解3 $$\int \frac{1-\tan x}{1+\tan x}\,\mathrm{d}x = \int \frac{\cos x-\sin x}{\cos x+\sin x}\,\mathrm{d}x = \int \frac{\cos^2 x-\sin^2 x}{(\cos x+\sin x)^2}\,\mathrm{d}x$$

$$= \int \frac{\cos 2x}{1+\sin 2x}\,\mathrm{d}x = \frac{1}{2}\int \frac{1}{1+\sin 2x}\,\mathrm{d}\sin 2x$$

$$= \frac{1}{2}\ln(1+\sin 2x)+C.$$

解4 $$\int \frac{1-\tan x}{1+\tan x}\,\mathrm{d}x = \int \frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\,\mathrm{d}x = \int \tan\left(\frac{\pi}{4}-x\right)\mathrm{d}x$$

$$= \ln\left|\cos\left(\frac{\pi}{4}-x\right)\right|+C.$$

P184/第八章总练习题/1(7) 求不定积分$\displaystyle\int\frac{1-\tan x}{1+\tan x}\mathrm{d}x.$

解5 令 $t=\tan x$, 即 $x=\arctan t$. 则 $\mathrm{d}x=\dfrac{1}{1+t^2}\mathrm{d}t$. 从而

$$\int\frac{1-\tan x}{1+\tan x}\mathrm{d}x=\int\frac{1-t}{1+t}\cdot\frac{1}{1+t^2}\mathrm{d}t,$$

设 $\dfrac{1-t}{(1+t)(1+t^2)}=\dfrac{A}{1+t}+\dfrac{Bt+D}{1+t^2}$,

则 $1-t=A\left(1+t^2\right)+\left(Bt+D\right)\left(1+t\right)$, 从而 $\begin{cases}A+B=0\\B+D=-1,\\A+D=1\end{cases}$ 解得 $\begin{cases}A=1\\B=-1.\\D=0\end{cases}$ 故

$$\int\frac{1-\tan x}{1+\tan x}\mathrm{d}x=\int\left(\frac{1}{1+t}+\frac{-t}{1+t^2}\right)\mathrm{d}t=\ln|1+t|-\frac{1}{2}\ln\left(1+t^2\right)+C$$

$$=\ln|1+\tan x|-\frac{1}{2}\ln\left(1+\tan^2x\right)+C.$$

P184/第八章总练习题/1(12) 求不定积分$\int \arctan\left(1+\sqrt{x}\right)\mathrm{d}x$.

解1 令 $t=1+\sqrt{x}$, 即 $x=(t-1)^2$. 则 $\mathrm{d}x=2(t-1)\mathrm{d}t$. 从而

$$\int \arctan\left(1+\sqrt{x}\right)\mathrm{d}x=2\int (t-1)\arctan t\,\mathrm{d}t=\int \arctan t\,\mathrm{d}(t-1)^2$$

$$=(t-1)^2\arctan t-\int \frac{(t-1)^2}{1+t^2}\mathrm{d}t=(t-1)^2\arctan t-\int \frac{1+t^2-2t}{1+t^2}\mathrm{d}t$$

$$=(t-1)^2\arctan t-\int \left(1-\frac{2t}{1+t^2}\right)\mathrm{d}t=(t-1)^2\arctan t-t+\ln\left(1+t^2\right)+C$$

$$=x\arctan\left(1+\sqrt{x}\right)-\left(1+\sqrt{x}\right)+\ln\left(1+\left(1+\sqrt{x}\right)^2\right)+C$$

$$=x\arctan\left(1+\sqrt{x}\right)-\sqrt{x}+\ln\left(2+x+2\sqrt{x}\right)+C_1.$$

P184/第八章总练习题/1(12) 求不定积分 $\int \arctan\left(1+\sqrt{x}\right)\mathrm{d}x$.

解2 令 $1+\sqrt{x}=\tan t$, 即 $x=\left(\tan t-1\right)^2$. 从而

$$\int \arctan\left(1+\sqrt{x}\right)\mathrm{d}x=\int t\,\mathrm{d}\left(\tan t-1\right)^2$$

$$=t\left(\tan t-1\right)^2-\int\left(\tan t-1\right)^2\mathrm{d}t=t\left(\tan t-1\right)^2-\int\left(\tan^2 t-2\tan t+1\right)\mathrm{d}t$$

$$=t\left(\tan t-1\right)^2-\int\left(\sec^2 t-2\tan t\right)\mathrm{d}t=t\left(\tan t-1\right)^2-\tan t-2\ln\left|\cos t\right|+C$$

$$=x\arctan\left(1+\sqrt{x}\right)-\left(1+\sqrt{x}\right)-2\ln\left(\frac{1}{\sqrt{1+\left(1+\sqrt{x}\right)^2}}\right)+C$$

$$=x\arctan\left(1+\sqrt{x}\right)-\sqrt{x}+\ln\left(2+x+2\sqrt{x}\right)+C_1.$$

P184/第八章总练习题/1(15) 求不定积分$\int \frac{x^2}{(1-x)^{100}}\mathrm{d}x.$

解1 令 $t=1-x$, 即 $x=1-t$. 则 $\mathrm{d}x=-\mathrm{d}t$. 从而

$$\int \frac{x^2}{(1-x)^{100}}\mathrm{d}x=-\int \frac{(1-t)^2}{t^{100}}\mathrm{d}t=-\int \frac{t^2-2t+1}{t^{100}}\mathrm{d}t$$

$$=-\int \left(t^{-98}-2t^{-99}+t^{-100}\right)\mathrm{d}t$$

$$=-\left(\frac{t^{-97}}{-97}-2\frac{t^{-98}}{-98}+\frac{t^{-99}}{-99}\right)+C$$

$$=\frac{(1-x)^{-97}}{97}-\frac{(1-x)^{-98}}{49}+\frac{(1-x)^{-99}}{99}+C$$

$$=\frac{1}{97(1-x)^{97}}-\frac{1}{49(1-x)^{98}}+\frac{1}{99(1-x)^{99}}+C.$$

P184/第八章总练习题/1(15) 求不定积分$\int \frac{x^2}{(1-x)^{100}}\mathrm{d}x.$

解2 $$\int \frac{x^2}{(1-x)^{100}}\mathrm{d}x=\int \frac{(1-x)^2-2(1-x)+1}{(1-x)^{100}}\mathrm{d}x$$

$$=\int\left((1-x)^{-98}-2(1-x)^{-99}+(1-x)^{-100}\right)\mathrm{d}x$$

$$=-\int\left((1-x)^{-98}-2(1-x)^{-99}+(1-x)^{-100}\right)\mathrm{d}(1-x)$$

$$=-\left(\frac{(1-x)^{-97}}{-97}-2\frac{(1-x)^{-98}}{-98}+\frac{(1-x)^{-99}}{-99}\right)+C$$

$$=\frac{1}{97(1-x)^{97}}-\frac{1}{49(1-x)^{98}}+\frac{1}{99(1-x)^{99}}+C.$$

P184/第八章总练习题/1(15) 求不定积分 $\displaystyle\int\frac{x^2}{(1-x)^{100}}\,\mathrm{d}x.$

解3 $$\int\frac{x^2}{(1-x)^{100}}\,\mathrm{d}x=-\int\frac{x^2}{(1-x)^{100}}\,\mathrm{d}(1-x)=\frac{1}{99}\int x^2\,\mathrm{d}\frac{1}{(1-x)^{99}}$$

$$=\frac{1}{99}\frac{x^2}{(1-x)^{99}}-\frac{2}{99}\int\frac{x}{(1-x)^{99}}\,\mathrm{d}x$$

$$=\frac{1}{99}\frac{x^2}{(1-x)^{99}}-\frac{2}{99\cdot 98}\int x\,\mathrm{d}\frac{1}{(1-x)^{98}}$$

$$=\frac{1}{99}\frac{x^2}{(1-x)^{99}}-\frac{1}{99\cdot 49}\frac{x}{(1-x)^{98}}+\frac{1}{99\cdot 49}\int\frac{1}{(1-x)^{98}}\,\mathrm{d}x$$

$$=\frac{1}{99}\frac{x^2}{(1-x)^{99}}-\frac{1}{99\cdot 49}\frac{x}{(1-x)^{98}}+\frac{1}{99\cdot 97\cdot 49}\frac{1}{(1-x)^{97}}+C.$$

P184/第八章总练习题/1(16) 求不定积分 $\int \frac{\arcsin x}{x^2}\mathrm{d}x$.

解1 $\int \frac{\arcsin x}{x^2}\mathrm{d}x = -\int \arcsin x\,\mathrm{d}\frac{1}{x} = -\frac{\arcsin x}{x} + \int \frac{1}{x}\mathrm{d}\arcsin x$

$$= -\frac{\arcsin x}{x} + \int \frac{1}{x\sqrt{1-x^2}}\mathrm{d}x,$$

对于 $\int \frac{1}{x\sqrt{1-x^2}}\mathrm{d}x$, 令 $x = \sin t, t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. 则

$$\int \frac{1}{x\sqrt{1-x^2}}\mathrm{d}x = \int \frac{1}{\sin t \cos t}\cdot \cos t\,\mathrm{d}t = \int \csc t\,\mathrm{d}t = \ln|\csc t - \cot t| + C$$

$$= \ln\left|\frac{1}{x} - \frac{\sqrt{1-x^2}}{x}\right| + C.$$

故 $$\int \frac{\arcsin x}{x^2}\mathrm{d}x = -\frac{\arcsin x}{x} + \ln\left|\frac{1}{x} - \frac{\sqrt{1-x^2}}{x}\right| + C.$$

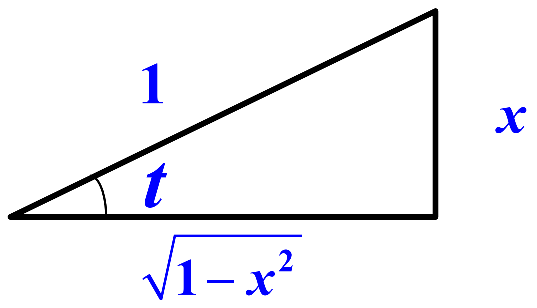
P184/第八章总练习题/1(16) 求不定积分 $\int \frac{\arcsin x}{x^2}\mathrm{d}x$.

解2 令 $t=\arcsin x$, 即 $x=\sin t$. 则 $\mathrm{d}x=\cos t\,\mathrm{d}t$. 从而

$$\int \frac{\arcsin x}{x^2}\mathrm{d}x=\int \frac{t}{\sin^2 t}\cdot\cos t\,\mathrm{d}t=\int t\cdot\csc t\cot t\,\mathrm{d}t=-\int t\,\mathrm{d}\csc t$$

$$=-t\csc t+\int \csc t\,\mathrm{d}t=-t\csc t+\ln\left|\csc t-\cot t\right|+C$$

$$=-\frac{\arcsin x}{x}+\ln\left|\frac{1}{x}-\frac{\sqrt{1-x^2}}{x}\right|+C.$$

$x=\sin t\Rightarrow$ 

$\Rightarrow$ $\csc t=\frac{1}{x}$, $\cot t=\frac{\sqrt{1-x^2}}{x}$

P184/第八章总练习题/1(16) 求不定积分$\int\frac{\arcsin x}{x^2}\mathrm{d}x.$

解3 $\int\frac{\arcsin x}{x^2}\mathrm{d}x=-\int\arcsin x\,\mathrm{d}\frac{1}{x}=-\frac{\arcsin x}{x}+\int\frac{1}{x}\mathrm{d}\arcsin x$

$$=-\frac{\arcsin x}{x}+\int\frac{1}{\sin(\arcsin x)}\mathrm{d}\arcsin x$$

$$=-\frac{\arcsin x}{x}+\int\csc(\arcsin x)\,\mathrm{d}\arcsin x$$

$$=-\frac{\arcsin x}{x}+\ln\left|\csc(\arcsin x)-\cot(\arcsin x)\right|+C$$

$$=-\frac{\arcsin x}{x}+\ln\left|\frac{1}{x}-\frac{\sqrt{1-x^2}}{x}\right|+C.$$

BY GYH

P184/第八章总练习题/1(17) 求不定积分 $\int x\ln\frac{1+x}{1-x}\mathrm{d}x.$

解

$$\int x\ln\frac{1+x}{1-x}\mathrm{d}x=\int\ln\frac{1+x}{1-x}\mathrm{d}\frac{x^2}{2}=\frac{x^2}{2}\ln\frac{1+x}{1-x}-\int\frac{x^2}{2}\mathrm{d}\ln\frac{1+x}{1-x}$$

$$=\frac{x^2}{2}\ln\frac{1+x}{1-x}-\int\frac{x^2}{2}\cdot\left(\ln(1+x)-\ln(1-x)\right)'\mathrm{d}x$$

$$=\frac{x^2}{2}\ln\frac{1+x}{1-x}-\int\frac{x^2}{2}\cdot\left(\frac{1}{1+x}+\frac{1}{1-x}\right)\mathrm{d}x=\frac{x^2}{2}\ln\frac{1+x}{1-x}-\int\frac{x^2}{1-x^2}\mathrm{d}x$$

$$=\frac{x^2}{2}\ln\frac{1+x}{1-x}+\int\frac{1-x^2-1}{1-x^2}\mathrm{d}x=\frac{x^2}{2}\ln\frac{1+x}{1-x}+\int\left(1-\frac{1}{1-x^2}\right)\mathrm{d}x$$

$$=\frac{x^2}{2}\ln\frac{1+x}{1-x}+x-\frac{1}{2}\ln\frac{1+x}{1-x}+C=\frac{x^2-1}{2}\ln\frac{1+x}{1-x}+x+C.$$

P184/第八章总练习题/1(19) 求不定积分$\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx.$

解1 $\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx=\int e^x\frac{1+x^2-2x}{(1+x^2)^2}dx$

$$=\int\frac{e^x}{1+x^2}dx-2\int e^x\frac{x}{(1+x^2)^2}dx=\int\frac{e^x}{1+x^2}dx-\int e^x\frac{1}{(1+x^2)^2}d(1+x^2)$$

$$=\int\frac{e^x}{1+x^2}dx+\int e^x d\frac{1}{1+x^2}=\int\frac{e^x}{1+x^2}dx+\frac{e^x}{1+x^2}-\int\frac{e^x}{1+x^2}dx$$

$$=\frac{e^x}{1+x^2}+C.$$

P184/第八章总练习题/1(19) 求不定积分 $\int e^x\left(\dfrac{1-x}{1+x^2}\right)^2 dx.$

解2 $$\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx=\int e^x\frac{1+x^2-2x}{\left(1+x^2\right)^2}dx$$

$$=\int\frac{\left(e^x\right)'\left(1+x^2\right)-e^x\left(1+x^2\right)'}{\left(1+x^2\right)^2}dx$$

$$=\int\left(\frac{e^x}{1+x^2}\right)'dx=\frac{e^x}{1+x^2}+C.$$

P184/第八章总练习题/2(1) 求不定积分 $\displaystyle\int\frac{\mathrm{d}x}{x^4+x^2+1}$.

解 $$\int\frac{\mathrm{d}x}{x^4+x^2+1}=\int\frac{1}{\left(x^2+1\right)^2-x^2}\mathrm{d}x=\int\frac{1}{\left(x^2-x+1\right)\left(x^2+x+1\right)}\mathrm{d}x,$$

设 $$\frac{1}{\left(x^2-x+1\right)\left(x^2+x+1\right)}=\frac{Ax+B}{x^2-x+1}+\frac{Dx+E}{x^2+x+1},$$

则 $1=(Ax+B)\left(x^2+x+1\right)+(Dx+E)\left(x^2-x+1\right)$，从而 $\begin{cases}A+D=0\\A+B-D+E=0\\A+B+D-E=0\\B+E=1\end{cases}$，解得 $\begin{cases}A=-\frac{1}{2}\\B=\frac{1}{2}\\D=\frac{1}{2}\\E=\frac{1}{2}\end{cases}$.

故

$$\int\frac{\mathrm{d}x}{x^4+x^2+1}=\frac{1}{2}\int\left(\frac{x+1}{x^2+x+1}-\frac{x-1}{x^2-x+1}\right)\mathrm{d}x=\frac{1}{2}\int\frac{x+1}{x^2+x+1}\mathrm{d}x-\frac{1}{2}\int\frac{x-1}{x^2-x+1}\mathrm{d}x$$

$$=\frac{1}{2}\int\frac{\frac{1}{2}\left(x^2+x+1\right)'+\frac{1}{2}}{x^2+x+1}\mathrm{d}x-\frac{1}{2}\int\frac{\frac{1}{2}\left(x^2-x+1\right)'-\frac{1}{2}}{x^2-x+1}\mathrm{d}x$$

$$=\frac{1}{4}\ln\frac{x^2+x+1}{x^2-x+1}+\frac{1}{4}\int\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\mathrm{d}x+\frac{1}{4}\int\frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\mathrm{d}x$$

$$=\frac{1}{4}\ln\frac{x^2+x+1}{x^2-x+1}+\frac{1}{2\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}+\frac{1}{2\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}+C.$$

P184/第八章总练习题/2(2) 求不定积分 $\displaystyle\int\frac{x^9}{\left(x^{10}+2x^5+2\right)^2}\mathrm{d}x.$

解 $\displaystyle\int\frac{x^9}{\left(x^{10}+2x^5+2\right)^2}\mathrm{d}x=\frac{1}{5}\int\frac{x^5}{\left(\left(x^5+1\right)^2+1\right)^2}\mathrm{d}x^5\overset{t=x^5+1}{=}\frac{1}{5}\int\frac{t-1}{\left(t^2+1\right)^2}\mathrm{d}t$

$$=\frac{1}{5}\int\frac{t}{\left(t^2+1\right)^2}\mathrm{d}t-\frac{1}{5}\int\frac{1}{\left(t^2+1\right)^2}\mathrm{d}t=\frac{1}{10}\int\frac{1}{\left(t^2+1\right)^2}\mathrm{d}\left(t^2+1\right)-\frac{1}{5}\int\frac{t^2+1-t^2}{\left(t^2+1\right)^2}\mathrm{d}t$$

$$=-\frac{1}{10\left(t^2+1\right)}-\frac{1}{5}\arctan t+\frac{1}{5}\int t\frac{t}{\left(t^2+1\right)^2}\mathrm{d}t=-\frac{1}{10\left(t^2+1\right)}-\frac{1}{5}\arctan t-\frac{1}{10}\int t\,\mathrm{d}\frac{1}{t^2+1}$$

$$=-\frac{1}{10\left(t^2+1\right)}-\frac{1}{5}\arctan t-\frac{t}{10\left(t^2+1\right)}+\frac{1}{10}\int\frac{1}{t^2+1}\mathrm{d}t=-\frac{1+t}{10\left(t^2+1\right)}-\frac{1}{10}\arctan t+C$$

$$=-\frac{2+x^5}{10\left(x^{10}+2x^5+2\right)}-\frac{1}{10}\arctan\left(x^5+1\right)+C.$$

P184/第八章总练习题/2(3)求不定积分$\displaystyle\int\frac{x^{3n-1}}{\left(x^{2n}+1\right)^2}\mathrm{d}x$.

解 $$\int\frac{x^{3n-1}}{\left(x^{2n}+1\right)^2}\mathrm{d}x=\int\frac{x^{2n}\cdot x^{n-1}}{\left(x^{2n}+1\right)^2}\mathrm{d}x=\frac{1}{n}\int\frac{x^{2n}}{\left(x^{2n}+1\right)^2}\mathrm{d}x^n\overset{t=x^n}{=}\frac{1}{n}\int\frac{t^2}{\left(t^2+1\right)^2}\mathrm{d}t$$

$$=\frac{1}{n}\int t\cdot\frac{t}{\left(t^2+1\right)^2}\mathrm{d}t=-\frac{1}{2n}\int t\,\mathrm{d}\frac{1}{t^2+1}=-\frac{1}{2n}\frac{t}{t^2+1}+\frac{1}{2n}\int\frac{1}{t^2+1}\mathrm{d}t$$

$$=-\frac{1}{2n}\frac{t}{t^2+1}+\frac{1}{2n}\arctan t+C=-\frac{x^n}{2n\left(x^{2n}+1\right)}+\frac{1}{2n}\arctan x^n+C.$$

P184/第八章总练习题/2(4) 求不定积分 $\displaystyle\int\frac{\cos^3 x}{\cos x+\sin x}\mathrm{d}x.$

解1 $\displaystyle\int\frac{\cos^3 x}{\cos x+\sin x}\mathrm{d}x=\int\frac{\cos^2 x}{1+\tan x}\mathrm{d}x=\int\frac{\sec^2 x}{\sec^4 x+\sec^4 x\tan x}\mathrm{d}x=\int\frac{\mathrm{d}\tan x}{(1+\tan^2 x)^2(1+\tan x)}\overset{t=\tan x}{=}\int\frac{\mathrm{d}t}{(1+t)(1+t^2)^2},$

设 $\displaystyle\frac{1}{(1+t)(1+t^2)^2}=\frac{A}{1+t}+\frac{Bt+D}{1+t^2}+\frac{Et+F}{(1+t^2)^2},$

则 $1=A\left(1+t^2\right)^2+\left(Bt+D\right)\left(1+t\right)\left(1+t^2\right)+\left(Et+F\right)\left(1+t\right),$

从而 $\begin{cases}A+B=0\\B+D=0\\2A+B+E=0\\B+D+E+F=0\\A+D+F=1\end{cases}$，解得 $\begin{cases}A=\dfrac{1}{4}\\B=-\dfrac{1}{4}\\D=\dfrac{1}{4}\\E=-\dfrac{1}{2}\\F=\dfrac{1}{2}\end{cases}.$

故 $\displaystyle\int\frac{\cos^3 x}{\cos x+\sin x}\mathrm{d}x=\frac{1}{4}\int\left(\frac{1}{1+t}+\frac{-t+1}{1+t^2}+\frac{-2t+2}{(1+t^2)^2}\right)\mathrm{d}t$

P184/第八章总练习题/2(4) 求不定积分 $\displaystyle\int\frac{\cos^3 x}{\cos x+\sin x}\mathrm{d}x.$

$$\int\frac{\cos^3 x}{\cos x+\sin x}\mathrm{d}x=\frac{1}{4}\int\left(\frac{1}{1+t}+\frac{-t+1}{1+t^2}+\frac{-2t+2}{(1+t^2)^2}\right)\mathrm{d}t$$

$$=\frac{1}{4}\ln|1+t|-\frac{1}{8}\ln|1+t^2|+\frac{1}{4}\arctan t-\frac{1}{4}\int\frac{(1+t^2)'-2}{(1+t^2)^2}\mathrm{d}t$$

$$=\frac{1}{4}\ln|1+t|-\frac{1}{8}\ln|1+t^2|+\frac{1}{4}\arctan t+\frac{1}{4(1+t^2)}+\frac{1}{2}\int\frac{1}{(1+t^2)^2}\mathrm{d}t$$

$$=\frac{1}{4}\ln|1+t|-\frac{1}{8}\ln|1+t^2|+\frac{1}{4}\arctan t+\frac{1}{4(1+t^2)}+\frac{1}{2}\int\frac{1+t^2-t^2}{(1+t^2)^2}\mathrm{d}t$$

$$=\frac{1}{4}\ln|1+t|-\frac{1}{8}\ln|1+t^2|+\frac{1}{4}\arctan t+\frac{1}{4(1+t^2)}+\frac{1}{2}\arctan t-\frac{1}{2}\int t\cdot\frac{t}{(1+t^2)^2}\mathrm{d}t$$

$$=\frac{1}{4}\ln|1+t|-\frac{1}{8}\ln|1+t^2|+\frac{3}{4}\arctan t+\frac{1}{4(1+t^2)}+\frac{1}{4}\int t\cdot\mathrm{d}\frac{1}{1+t^2}$$

$$=\frac{1}{4}\ln|1+t|-\frac{1}{8}\ln|1+t^2|+\frac{3}{4}\arctan t+\frac{1}{4(1+t^2)}+\frac{t}{4(1+t^2)}-\frac{1}{4}\arctan t+C$$

$$=\frac{1}{8}\ln\frac{(1+t)^2}{1+t^2}+\frac{1}{2}\arctan t+\frac{1+t}{4(1+t^2)}+C=\frac{1}{8}\ln\frac{(1+\tan x)^2}{1+\tan^2 x}+\frac{1}{2}x+\frac{1+\tan x}{4(1+\tan^2 x)}+C.$$

P184/第八章总练习题/2(4) 求不定积分 $\int\frac{\cos^3 x}{\cos x+\sin x}\mathrm{d}x.$

解2 $\int\frac{\cos^3 x}{\cos x+\sin x}\mathrm{d}x=\int\frac{\cos^2 x\cos x}{\cos x+\sin x}\mathrm{d}x=\int\frac{(\cos 2x+1)\cos x}{2(\cos x+\sin x)}\mathrm{d}x$

$$=\int\frac{(\cos^2 x-\sin^2 x+1)\cos x}{2(\cos x+\sin x)}\mathrm{d}x=\frac{1}{2}\int(\cos x-\sin x)\cos x\,\mathrm{d}x+\frac{1}{2}\int\frac{\cos x}{\cos x+\sin x}\mathrm{d}x$$

$$=\frac{1}{2}\int\cos^2 x\,\mathrm{d}x-\frac{1}{2}\int\sin x\cos x\,\mathrm{d}x+\frac{1}{2}\int\frac{1}{1+\tan x}\mathrm{d}x$$

$$=\frac{1}{4}\int(1+\cos 2x)\mathrm{d}x-\frac{1}{2}\int\sin x\,\mathrm{d}\sin x+\frac{1}{2}\int\frac{1}{1+\tan x}\mathrm{d}x$$

$$=\frac{1}{4}\left(x+\frac{\sin 2x}{2}\right)-\frac{\sin^2 x}{4}+\frac{1}{2}\int\frac{1}{1+\tan x}\mathrm{d}x,$$ 其中 $\int\frac{1}{1+\tan x}\mathrm{d}x$，令 $t=\tan x$，则

$$\int\frac{1}{1+\tan x}\mathrm{d}x=\int\frac{1}{(1+t)(1+t^2)}\mathrm{d}t=\frac{1}{2}\int\left(\frac{1}{1+t}-\frac{t-1}{1+t^2}\right)\mathrm{d}t$$

$$=\frac{1}{2}\ln|1+t|-\frac{1}{4}\ln(1+t^2)+\frac{1}{2}\arctan t+C=\frac{1}{2}\ln|1+\tan x|-\frac{1}{4}\ln(1+\tan^2 x)+\frac{1}{2}x+C.$$

所以

$$\int\frac{\cos^3 x}{\cos x+\sin x}\mathrm{d}x=\frac{1}{4}\left(x+\frac{\sin 2x}{2}\right)-\frac{\sin^2 x}{4}+\frac{1}{4}\ln|1+\tan x|-\frac{1}{8}\ln(1+\tan^2 x)+\frac{1}{4}x+C.$$

P184/第八章总练习题/2(4) 求不定积分 $\int \frac{\cos^3 x}{\cos x+\sin x}\mathrm{d}x.$

解3 $\int \frac{\cos^3 x}{\cos x+\sin x}\mathrm{d}x=\int \frac{\cos^2 x\cos x}{\cos x+\sin x}\mathrm{d}x=\int \frac{(\cos 2x+1)\cos x}{2(\cos x+\sin x)}\mathrm{d}x$

$$=\int \frac{(\cos^2 x-\sin^2 x+1)\cos x}{2(\cos x+\sin x)}\mathrm{d}x=\frac{1}{2}\int(\cos x-\sin x)\cos x\,\mathrm{d}x+\frac{1}{2}\int \frac{\cos x}{\cos x+\sin x}\mathrm{d}x$$

$$=\frac{1}{2}\int \cos^2 x\,\mathrm{d}x-\frac{1}{2}\int \sin x\cos x\,\mathrm{d}x+\frac{1}{2}\int \frac{A(\cos x+\sin x)'+B(\cos x+\sin x)}{\cos x+\sin x}\mathrm{d}x$$

$$=\frac{1}{4}\int(1+\cos 2x)\mathrm{d}x-\frac{1}{2}\int \sin x\,\mathrm{d}\sin x+\frac{1}{4}\int \frac{(\cos x+\sin x)'+(\cos x+\sin x)}{\cos x+\sin x}\mathrm{d}x$$

$$=\frac{1}{4}\left(x+\frac{\sin 2x}{2}\right)-\frac{\sin^2 x}{4}+\frac{1}{4}\ln|\cos x+\sin x|+\frac{1}{4}x+C$$

$$=\frac{\sin 2x}{8}+\frac{x}{2}-\frac{\sin^2 x}{4}+\frac{1}{4}\ln|\cos x+\sin x|+C.$$

P184/第八章总练习题/2(4) 求不定积分 $\displaystyle\int \frac{\cos^3 x}{\cos x+\sin x}\,\mathrm{d}x.$

解4 $\displaystyle\int \frac{\cos^3 x}{\cos x+\sin x}\,\mathrm{d}x=\int \frac{\cos^3 x(\cos x-\sin x)}{(\cos x+\sin x)(\cos x-\sin x)}\,\mathrm{d}x=\int \frac{\cos^4 x-\cos^3 x\sin x}{\cos^2 x-\sin^2 x}\,\mathrm{d}x$

$$=\int \frac{\left(\frac{1+\cos 2x}{2}\right)^2-\frac{1}{2}\sin 2x\cdot\frac{1+\cos 2x}{2}}{\cos 2x}\,\mathrm{d}x$$

$$=\frac{1}{4}\int \frac{1+2\cos 2x+\cos^2 2x-\sin 2x-\sin 2x\cos 2x}{\cos 2x}\,\mathrm{d}x$$

$$=\frac{1}{4}\int(\sec 2x+2+\cos 2x-\tan 2x-\sin 2x)\,\mathrm{d}x$$

$$=\frac{1}{8}\ln|\sec 2x+\tan 2x|+\frac{1}{2}x+\frac{1}{8}\sin 2x+\frac{1}{8}\ln|\cos 2x|+\frac{1}{8}\cos 2x+C$$

$$=\frac{1}{8}\ln(1+\sin 2x)+\frac{1}{2}x+\frac{1}{8}\sin 2x+\frac{1}{8}\cos 2x+C.$$

P184/第八章总练习题/2(4) 求不定积分 $\displaystyle\int \frac{\cos^3 x}{\cos x+\sin x}\mathrm{d}x.$

解5 $$\int \frac{\cos^3 x}{\cos x+\sin x}\mathrm{d}x=\frac{1}{\sqrt{2}}\int \frac{\cos^2 x\cos x}{\sin\left(x+\frac{\pi}{4}\right)}\mathrm{d}x=\frac{1}{2\sqrt{2}}\int \frac{(\cos 2x+1)\cos x}{\sin\left(x+\frac{\pi}{4}\right)}\mathrm{d}x$$

$$=\frac{1}{2\sqrt{2}}\int \frac{\left(\sin 2\left(x+\frac{\pi}{4}\right)+1\right)\cos x}{\sin\left(x+\frac{\pi}{4}\right)}\mathrm{d}x=\frac{1}{2\sqrt{2}}\int \frac{\sin 2\left(x+\frac{\pi}{4}\right)}{\sin\left(x+\frac{\pi}{4}\right)}\mathrm{d}x+\frac{1}{2\sqrt{2}}\int \frac{\cos x}{\sin\left(x+\frac{\pi}{4}\right)}\mathrm{d}x$$

$$=\frac{1}{\sqrt{2}}\int \frac{\sin\left(x+\frac{\pi}{4}\right)\cos\left(x+\frac{\pi}{4}\right)\cos\left(x+\frac{\pi}{4}-\frac{\pi}{4}\right)}{\sin\left(x+\frac{\pi}{4}\right)}\mathrm{d}x+\frac{1}{2\sqrt{2}}\int \frac{\cos\left(x+\frac{\pi}{4}-\frac{\pi}{4}\right)}{\sin\left(x+\frac{\pi}{4}\right)}\mathrm{d}x$$

$$=\frac{1}{2}\int \cos\left(x+\frac{\pi}{4}\right)\left(\cos\left(x+\frac{\pi}{4}\right)+\sin\left(x+\frac{\pi}{4}\right)\right)\mathrm{d}x+\frac{1}{4}\int \frac{\cos\left(x+\frac{\pi}{4}\right)+\sin\left(x+\frac{\pi}{4}\right)}{\sin\left(x+\frac{\pi}{4}\right)}\mathrm{d}x$$

$$=\frac{1}{4}\int\left(\cos 2\left(x+\frac{\pi}{4}\right)+1\right)\mathrm{d}x+\frac{1}{4}\sin^2\left(x+\frac{\pi}{4}\right)+\frac{1}{4}\ln\left|\sin\left(x+\frac{\pi}{4}\right)\right|+\frac{1}{4}x$$

$$=\frac{1}{8}\sin 2\left(x+\frac{\pi}{4}\right)+\frac{1}{4}\sin^2\left(x+\frac{\pi}{4}\right)+\frac{1}{4}\ln\left|\sin\left(x+\frac{\pi}{4}\right)\right|+\frac{1}{2}x+C.$$

P184/第八章总练习题/3(1) 求不定积分 $\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\,\mathrm{d}x.$

解1 令 $t=\sqrt[4]{x}$, 即 $x=t^4$. 则 $\mathrm{d}x=4t^3\,\mathrm{d}t$. 从而

$$\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\,\mathrm{d}x=\int \frac{\sqrt[3]{1+t}}{t^2}\cdot 4t^3\,\mathrm{d}t=4\int t\sqrt[3]{1+t}\,\mathrm{d}t,$$

令 $u=\sqrt[3]{1+t}$, 即 $t=u^3-1$. 则 $\mathrm{d}t=3u^2\,\mathrm{d}u$. 从而

$$\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\,\mathrm{d}x=4\int (u^3-1)u\cdot 3u^2\,\mathrm{d}u=12\int (u^6-u^3)\,\mathrm{d}u$$

$$=12\left(\frac{u^7}{7}-\frac{u^4}{4}\right)+C=\frac{12}{7}\left(1+\sqrt[4]{x}\right)^{\frac{7}{3}}-3\left(1+\sqrt[4]{x}\right)^{\frac{4}{3}}+C.$$

P184/第八章总练习题/3(1) 求不定积分 $\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\mathrm{d}x.$

解2 令 $t=\sqrt[4]{x}$, 即 $x=t^4$. 则 $\mathrm{d}x=4t^3\mathrm{d}t$. 从而

$$\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\mathrm{d}x=\int \frac{\sqrt[3]{1+t}}{t^2}\cdot 4t^3\mathrm{d}t=4\int t\sqrt[3]{1+t}\,\mathrm{d}t$$

$$=4\int (1+t-1)\sqrt[3]{1+t}\,\mathrm{d}t$$

$$=4\int\left((1+t)^{\frac{4}{3}}-(1+t)^{\frac{1}{3}}\right)\mathrm{d}(1+t)$$

$$=4\left(\frac{3}{7}(1+t)^{\frac{7}{3}}-\frac{3}{4}(1+t)^{\frac{4}{3}}\right)+C$$

$$=\frac{12}{7}\left(1+\sqrt[4]{x}\right)^{\frac{7}{3}}-3\left(1+\sqrt[4]{x}\right)^{\frac{4}{3}}+C.$$

P184/第八章总练习题/3(1) 求不定积分 $\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\mathrm{d}x.$

解3 令 $t=\sqrt[3]{1+\sqrt[4]{x}}$，即 $x=\left(t^3-1\right)^4$. 则 $\mathrm{d}x=12t^2\left(t^3-1\right)^3\mathrm{d}t$. 从而

$$\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\mathrm{d}x=\int \frac{t}{\left(t^3-1\right)^2}\cdot 12t^2\left(t^3-1\right)^3\mathrm{d}t=12\int t^3\left(t^3-1\right)\mathrm{d}t$$

$$=12\int\left(t^6-t^3\right)\mathrm{d}t=12\left(\frac{t^7}{7}-\frac{t^4}{4}\right)+C$$

$$=\frac{12}{7}\left(1+\sqrt[4]{x}\right)^{\frac{7}{3}}-3\left(1+\sqrt[4]{x}\right)^{\frac{4}{3}}+C.$$

P184/第八章总练习题/3(2) 求不定积分$\int \frac{\mathrm{d}x}{\sqrt[4]{1+x^4}}$.

解 当$x>0$时，$\int \frac{\mathrm{d}x}{\sqrt[4]{1+x^4}} = \int \frac{\mathrm{d}x}{x\sqrt[4]{\frac{1}{x^4}+1}} = \int \frac{x^3}{x^4\sqrt[4]{\frac{1}{x^4}+1}}\mathrm{d}x = \frac{1}{4}\int \frac{1}{x^4\sqrt[4]{\frac{1}{x^4}+1}}\mathrm{d}\left(1+x^4\right)$,

令 $t=\sqrt[4]{\frac{1}{x^4}+1}$,即 $x^4=\frac{1}{t^4-1}$, 则 $\mathrm{d}\left(1+x^4\right) = -\frac{4t^3}{\left(t^4-1\right)^2}\mathrm{d}t$. 从而

$$\int \frac{\mathrm{d}x}{\sqrt[4]{1+x^4}} = \frac{1}{4}\int \frac{t^4-1}{t}\cdot\frac{-4t^3}{\left(t^4-1\right)^2}\mathrm{d}t = -\int \frac{t^2}{t^4-1}\mathrm{d}t = -\int \frac{t^2}{(t-1)(t+1)\left(t^2+1\right)}\mathrm{d}t,$$

设 $\frac{t^2}{(t-1)(t+1)\left(t^2+1\right)} = \frac{A}{t-1}+\frac{B}{t+1}+\frac{Dt+E}{t^2+1}$, 则 $t^2 = A(t+1)\left(t^2+1\right)+B(t-1)\left(t^2+1\right)+(Dt+E)\left(t^2-1\right)$,

从而 $\begin{cases} A+B+D=0 \\ A-B+E=1 \\ A+B-D=0 \\ A-B-E=0 \end{cases}$, 解得 $\begin{cases} A=\frac{1}{4} \\ B=-\frac{1}{4} \\ D=0 \\ E=\frac{1}{2} \end{cases}$.

故 $\int \frac{\mathrm{d}x}{\sqrt[4]{1+x^4}} = -\frac{1}{4}\int\left(\frac{1}{t-1}-\frac{1}{t+1}+\frac{2}{t^2+1}\right)\mathrm{d}t$

P184/第八章总练习题/3(2) 求不定积分 $\int \frac{\mathrm{d}x}{\sqrt[4]{1+x^4}}$.

$$\int \frac{\mathrm{d}x}{\sqrt[4]{1+x^4}} = -\frac{1}{4}\int\left(\frac{1}{t-1}-\frac{1}{t+1}+\frac{2}{t^2+1}\right)\mathrm{d}t = \frac{1}{4}\ln\frac{t+1}{t-1}-\frac{1}{2}\arctan t + C = \frac{1}{4}\ln\frac{\sqrt[4]{\frac{1}{x^4}+1}+1}{\sqrt[4]{\frac{1}{x^4}+1}-1}-\frac{1}{2}\arctan\sqrt[4]{\frac{1}{x^4}+1}+C$$

$$=\frac{1}{4}\ln\frac{\sqrt[4]{1+x^4}+x}{\sqrt[4]{1+x^4}-x}-\frac{1}{2}\arctan\frac{\sqrt[4]{1+x^4}}{x}+C.$$

当$x<0$时，令 $t=-x$，即 $x=-t$，则 $\mathrm{d}x=-\mathrm{d}t$. 从而

$$\int \frac{\mathrm{d}x}{\sqrt[4]{1+x^4}} = -\int \frac{\mathrm{d}t}{\sqrt[4]{1+t^4}} = -\left(\frac{1}{4}\ln\frac{\sqrt[4]{1+t^4}+t}{\sqrt[4]{1+t^4}-t}-\frac{1}{2}\arctan\frac{\sqrt[4]{1+t^4}}{t}\right)+C = -\left(\frac{1}{4}\ln\frac{\sqrt[4]{1+x^4}-x}{\sqrt[4]{1+x^4}+x}-\frac{1}{2}\arctan\frac{\sqrt[4]{1+x^4}}{-x}\right)+C$$

$$=\frac{1}{4}\ln\frac{\sqrt[4]{1+x^4}+x}{\sqrt[4]{1+x^4}-x}-\frac{1}{2}\arctan\frac{\sqrt[4]{1+x^4}}{x}+C.$$

因此 $\int \frac{\mathrm{d}x}{\sqrt[4]{1+x^4}} = \frac{1}{4}\ln\frac{\sqrt[4]{1+x^4}+x}{\sqrt[4]{1+x^4}-x}-\frac{1}{2}\arctan\frac{\sqrt[4]{1+x^4}}{x}+C.$ 更为完善的结论应该是

$$F(x)=\begin{cases}\frac{1}{4}\ln\frac{\sqrt[4]{1+x^4}+x}{\sqrt[4]{1+x^4}-x}-\frac{1}{2}\arctan\frac{\sqrt[4]{1+x^4}}{x}, & x>0\\ -\frac{\pi}{4}, & x=0,\\ \frac{1}{4}\ln\frac{\sqrt[4]{1+x^4}+x}{\sqrt[4]{1+x^4}-x}-\frac{1}{2}\arctan\frac{\sqrt[4]{1+x^4}}{x}-\frac{\pi}{2}, & x<0\end{cases} \qquad \int \frac{\mathrm{d}x}{\sqrt[4]{1+x^4}} = F(x)+C.$$

P184/第八章总练习题/3(3) 求不定积分 $\int \frac{dx}{x+\sqrt{x^2-x+1}}$.

解1 令 $\sqrt{x^2-x+1}=t-x$，即 $x=\frac{t^2-1}{2t-1}$. 则 $\sqrt{x^2-x+1}=\frac{-t^2+t-1}{2t-1}$，$dx=\frac{2t^2-2t+2}{(2t-1)^2}dt$.

从而

$$\int \frac{dx}{x+\sqrt{x^2-x+1}}=\int \frac{1}{t}\cdot\frac{2t^2-2t+2}{(2t-1)^2}dt=\int\left(\frac{A}{t}+\frac{B}{2t-1}+\frac{D}{(2t-1)^2}\right)dt$$

$$=\int\left(\frac{2}{t}-\frac{3}{2t-1}+\frac{3}{(2t-1)^2}\right)dt=2\ln|t|-\frac{3}{2}\ln|2t-1|-\frac{3}{2(2t-1)}+C$$

$$=2\ln\left|x+\sqrt{x^2-x+1}\right|-\frac{3}{2}\ln\left|2x+2\sqrt{x^2-x+1}-1\right|-\frac{3}{2\left(2x+2\sqrt{x^2-x+1}-1\right)}+C.$$

P184/第八章总练习题/3(3) 求不定积分 $\int\frac{\mathrm{d}x}{x+\sqrt{x^2-x+1}}$.

解2 令 $\sqrt{x^2-x+1}=x+t$, 即 $x=\frac{1-t^2}{1+2t}$. 则 $\sqrt{x^2-x+1}=\frac{t^2+t+1}{1+2t}$, $\mathrm{d}x=-\frac{2t^2+2t+2}{(1+2t)^2}\mathrm{d}t$.

从而

$$\int\frac{\mathrm{d}x}{x+\sqrt{x^2-x+1}}=\int\frac{1}{\frac{1-t^2}{1+2t}+\frac{t^2+t+1}{1+2t}}\cdot\left(-\frac{2t^2+2t+2}{(1+2t)^2}\right)\mathrm{d}t=-\int\frac{2t^2+2t+2}{2t^2+5t+2}\mathrm{d}t$$

$$=-\int\frac{2t^2+5t+2-3t}{2t^2+5t+2}\mathrm{d}t=-\int 1\,\mathrm{d}t+\int\frac{3t}{2t^2+5t+2}\mathrm{d}t=-t+\int\frac{3t}{(t+2)(2t+1)}\mathrm{d}t$$

$$=-t+\int\left(\frac{2}{t+2}-\frac{1}{2t+1}\right)\mathrm{d}t=-t+2\ln|t+2|-\frac{1}{2}\ln|2t+1|+C$$

$$=-\sqrt{x^2-x+1}+x+2\ln\left|\sqrt{x^2-x+1}-x+2\right|-\frac{1}{2}\ln\left|2\sqrt{x^2-x+1}-2x+1\right|+C.$$

P184/第八章总练习题/3(3) 求不定积分 $\int \frac{\mathrm{d}x}{x+\sqrt{x^2-x+1}}$.

解3 令 $\sqrt{x^2-x+1}=x-t$, 即 $x=\frac{t^2-1}{2t-1}$. 则 $\sqrt{x^2-x+1}=\frac{-t^2+t-1}{2t-1}$, $\mathrm{d}x=\frac{2t^2-2t+2}{(2t-1)^2}\mathrm{d}t$.

从而

$$\int \frac{\mathrm{d}x}{x+\sqrt{x^2-x+1}}=\int \frac{1}{\frac{t^2-1}{2t-1}+\frac{-t^2+t-1}{2t-1}}\cdot\frac{2t^2-2t+2}{(2t-1)^2}\mathrm{d}t=\int \frac{2t^2-2t+2}{2t^2-5t+2}\mathrm{d}t$$

$$=\int \frac{2t^2-5t+2+3t}{2t^2-5t+2}\mathrm{d}t=\int 1\,\mathrm{d}t+\int \frac{3t}{2t^2-5t+2}\mathrm{d}t \quad =t+\int \frac{3t}{(t-2)(2t-1)}\mathrm{d}t$$

$$=t+\int\left(\frac{2}{t-2}-\frac{1}{2t-1}\right)\mathrm{d}t \quad =t+2\ln|t-2|-\frac{1}{2}\ln|2t-1|+C$$

$$=x-\sqrt{x^2-x+1}+2\ln\left|x-\sqrt{x^2-x+1}-2\right|-\frac{1}{2}\ln\left|2x-2\sqrt{x^2-x+1}-1\right|+C.$$

P184/第八章总练习题/3(3) 求不定积分 $\displaystyle\int\frac{\mathrm{d}x}{x+\sqrt{x^2-x+1}}$.

解4 令 $\sqrt{x^2-x+1}=tx-1$, 即 $x=\dfrac{2t-1}{t^2-1}$. 则 $\sqrt{x^2-x+1}=\dfrac{t^2-t+1}{t^2-1}$, $\mathrm{d}x=\dfrac{-2t^2+2t-2}{(t^2-1)^2}\mathrm{d}t$.

从而

$$\int\frac{\mathrm{d}x}{x+\sqrt{x^2-x+1}}=\int\frac{1}{\dfrac{2t-1}{t^2-1}+\dfrac{t^2-t+1}{t^2-1}}\cdot\frac{-2t^2+2t-2}{(t^2-1)^2}\mathrm{d}t=\int\frac{-2t^2+2t-2}{(t^2+t)(t^2-1)}\mathrm{d}t$$

$$=\int\left(\frac{A}{t}+\frac{B}{t-1}+\frac{D}{t+1}+\frac{E}{(t+1)^2}\right)\mathrm{d}t=\int\left(\frac{2}{t}+\frac{-\frac{1}{2}}{t-1}+\frac{-\frac{3}{2}}{t+1}+\frac{-3}{(t+1)^2}\right)\mathrm{d}t$$

$$=2\ln|t|-\frac{1}{2}\ln|t-1|-\frac{3}{2}\ln|t+1|+\frac{3}{t+1}+C$$

$$=2\ln\left|\frac{\sqrt{x^2-x+1}+1}{x}\right|-\frac{1}{2}\ln\left|\frac{\sqrt{x^2-x+1}-x+1}{x}\right|-\frac{3}{2}\ln\left|\frac{\sqrt{x^2-x+1}+x+1}{x}\right|+\frac{3x}{\sqrt{x^2-x+1}+x+1}+C.$$

P184/第八章总练习题/3(3) 求不定积分 $\displaystyle\int\frac{\mathrm{d}x}{x+\sqrt{x^2-x+1}}$.

解5 令 $\sqrt{x^2-x+1}=tx+1$, 即 $x=\dfrac{2t+1}{1-t^2}$. 则 $\sqrt{x^2-x+1}=\dfrac{t^2+t+1}{1-t^2}$, $\mathrm{d}x=\dfrac{2t^2+2t+2}{(1-t^2)^2}\mathrm{d}t$.

从而

$$\int\frac{\mathrm{d}x}{x+\sqrt{x^2-x+1}}=\int\frac{1}{\frac{2t+1}{1-t^2}+\frac{t^2+t+1}{1-t^2}}\cdot\frac{2t^2+2t+2}{(1-t^2)^2}\mathrm{d}t=\int\frac{2t^2+2t+2}{(t^2+3t+2)(1-t^2)}\mathrm{d}t$$

$$=\int\frac{2t^2+2t+2}{(t+2)(1-t)(t+1)^2}\mathrm{d}t=\int\left(\frac{A}{t+2}+\frac{B}{1-t}+\frac{D}{t+1}+\frac{E}{(t+1)^2}\right)\mathrm{d}t$$

$$=\int\left(\frac{2}{t+2}+\frac{\frac{1}{2}}{1-t}+\frac{-\frac{3}{2}}{t+1}+\frac{1}{(t+1)^2}\right)\mathrm{d}t=2\ln|t+2|-\frac{1}{2}\ln|1-t|-\frac{3}{2}\ln|t+1|-\frac{1}{t+1}+C$$

$$=2\ln\left|\frac{\sqrt{x^2-x+1}+2x-1}{x}\right|-\frac{1}{2}\ln\left|\frac{x-\sqrt{x^2-x+1}+1}{x}\right|-\frac{3}{2}\ln\left|\frac{\sqrt{x^2-x+1}+x-1}{x}\right|-\frac{x}{\sqrt{x^2-x+1}+x-1}+C.$$

P184/第八章总练习题/3(3) 求不定积分 $\int \frac{\mathrm{d}x}{x+\sqrt{x^2-x+1}}$.

解6 $\int \frac{\mathrm{d}x}{x+\sqrt{x^2-x+1}} = \int \frac{1}{x+\sqrt{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}}\mathrm{d}x$，令 $t=x-\frac{1}{2}$，即 $x=t+\frac{1}{2}$. 则 $\mathrm{d}x=-\mathrm{d}t$.

从而 $\int \frac{\mathrm{d}x}{x+\sqrt{x^2-x+1}} = \int \frac{1}{t+\frac{1}{2}+\sqrt{t^2+\frac{3}{4}}}\mathrm{d}t$，令 $\sqrt{t^2+\frac{3}{4}}=u-t$，即 $t=\frac{u^2-\frac{3}{4}}{2u}$. 则 $\mathrm{d}t=\left(\frac{1}{2}+\frac{3}{8u^2}\right)\mathrm{d}u$.

$$\int \frac{\mathrm{d}x}{x+\sqrt{x^2-x+1}} = \int \frac{1}{u+\frac{1}{2}}\cdot\left(\frac{1}{2}+\frac{3}{8u^2}\right)\mathrm{d}u \quad = \frac{1}{2}\int \frac{1}{u+\frac{1}{2}}\mathrm{d}u + \frac{3}{8}\int \frac{1}{u^2\left(u+\frac{1}{2}\right)}\mathrm{d}u$$

$$= \frac{1}{2}\ln\left|u+\frac{1}{2}\right| + \frac{3}{8}\int\left(\frac{A}{u}+\frac{B}{u^2}+\frac{D}{u+\frac{1}{2}}\right)\mathrm{d}u = \frac{1}{2}\ln\left|u+\frac{1}{2}\right| + \frac{3}{8}\int\left(\frac{-4}{u}+\frac{2}{u^2}+\frac{4}{u+\frac{1}{2}}\right)\mathrm{d}u$$

$$= 2\ln\left|u+\frac{1}{2}\right| - \frac{3}{2}\ln|u| - \frac{3}{4u} + C$$

$$= 2\ln\left|\sqrt{x^2-x+1}+x\right| - \frac{3}{2}\ln\left|\sqrt{x^2-x+1}+x-\frac{1}{2}\right| - \frac{3}{4\left(\sqrt{x^2-x+1}+x-\frac{1}{2}\right)} + C.$$

P184/第八章总练习题/3(4) 求不定积分 $\int \frac{1+x^4}{\left(1-x^4\right)^{\frac{3}{2}}}\mathrm{d}x.$

解 $\int \frac{1+x^4}{\left(1-x^4\right)^{\frac{3}{2}}}\mathrm{d}x = \int \frac{1+x^4}{\left(1-x^4\right)\left(1-x^4\right)^{\frac{1}{2}}}\mathrm{d}x = \int \frac{\left(1+x^4\right)\left(1-x^4\right)^{-\frac{1}{2}}}{1-x^4}\mathrm{d}x$

$$= \int \frac{\left(1-x^4+2x^4\right)\left(1-x^4\right)^{-\frac{1}{2}}}{\left(\left(1-x^4\right)^{\frac{1}{2}}\right)^2}\mathrm{d}x = \int \frac{\left(1-x^4\right)^{\frac{1}{2}}+2x^4\left(1-x^4\right)^{-\frac{1}{2}}}{\left(\left(1-x^4\right)^{\frac{1}{2}}\right)^2}\mathrm{d}x$$

$$= \int \frac{(x)'\left(1-x^4\right)^{\frac{1}{2}} - x\left(\left(1-x^4\right)^{\frac{1}{2}}\right)'}{\left(\left(1-x^4\right)^{\frac{1}{2}}\right)^2}\mathrm{d}x = \int \left(\frac{x}{\left(1-x^4\right)^{\frac{1}{2}}}\right)'\mathrm{d}x = \frac{x}{\left(1-x^4\right)^{\frac{1}{2}}} + C.$$

P184/第八章总练习题/5(2)

导出不定积分$\int \frac{\sin nx}{\sin x}\mathrm{d}\,x$对于正整数$n$的递推公式.

解1 记$I_n = \int \frac{\sin nx}{\sin x}\mathrm{d}\,x.$

$$I_n = \int \frac{\sin\left(\left(n-1\right)+1\right)x}{\sin x}\mathrm{d}\,x = \int \frac{\sin\left(\left(n-1\right)x+x\right)}{\sin x}\mathrm{d}\,x$$

$$= \int \frac{\sin\left(n-1\right)x\cos x+\cos\left(n-1\right)x\sin x}{\sin x}\mathrm{d}\,x = \int \cos\left(n-1\right)x\,\mathrm{d}\,x + \int \frac{\sin\left(n-1\right)x\cos x}{\sin x}\mathrm{d}\,x$$

$$= \frac{\sin\left(n-1\right)x}{n-1} + \int \frac{\sin nx+\sin\left(n-2\right)x}{2\sin x}\mathrm{d}\,x = \frac{\sin\left(n-1\right)x}{n-1} + \frac{1}{2}I_n + \frac{1}{2}I_{n-2},$$

从而 $I_n = \frac{2\sin\left(n-1\right)x}{n-1} + I_{n-2}$, $n \geq 2$, $I_0 = C$, $I_1 = x + C$.

P184/第八章总练习题/5(2)

导出不定积分$\int \frac{\sin nx}{\sin x}\mathrm{d}x$对于正整数$n$的递推公式.

解2 记$I_n = \int \frac{\sin nx}{\sin x}\mathrm{d}x.$

$$I_n = \int \frac{\sin\left(\left(n-2\right)+2\right)x}{\sin x}\mathrm{d}x = \int \frac{\sin\left(\left(n-2\right)x+2x\right)}{\sin x}\mathrm{d}x$$

$$= \int \frac{\sin\left(n-2\right)x\cos 2x + \cos\left(n-2\right)x\sin 2x}{\sin x}\mathrm{d}x$$

$$= \int \frac{\sin\left(n-2\right)x\left(1-2\sin^2 x\right) + 2\cos\left(n-2\right)x\sin x\cos x}{\sin x}\mathrm{d}x$$

$$= \int \frac{\sin\left(n-2\right)x}{\sin x}\mathrm{d}x + 2\int\left(\cos\left(n-2\right)x\cos x - \sin\left(n-2\right)x\sin x\right)\mathrm{d}x$$

$$= I_{n-1} + 2\int \cos\left(n-1\right)x\,\mathrm{d}x = I_{n-1} + \frac{2\sin\left(n-1\right)x}{n-1},\quad n \geq 2,$$

$$I_0 = C,\quad I_1 = x + C.$$