Ch10 定积分的应用

总结及习题评讲

主讲教师: 顾燕红

办公室: 汇星楼409

办公室答疑时间: 每周四下午2点至4点

微信号: 18926511820 QQ号: 58105217

Email: yhgu@szu.edu.cn

(添加好友时请备注 学号 姓名 数学分析2)

QQ群、QQ、微信群、微信随时答疑解惑

2024年04月25日

BY GYH

- 定积分的应用
  - 元素法(微元法)
  - 平面图形的面积
  - 由平行截面面积求体积
    - 平行截面面积已知的立体体积
    - 旋转体的体积
  - 平面曲线的弧长与曲率
  - 旋转曲面的面积
  - 定积分在物理中的应用
    - 液体侧压力
    - 引力
    - 变力作功

P225/习题10.1/2

求由曲线$y=|\ln x|$与直线$x=\dfrac{1}{10},x=10,y=0$所围平面图形的面积.

解 面积元素为 $\mathrm{d}S=|\ln x|\mathrm{d}x,$

故所求平面图形的面积为

$$S=\int_{\frac{1}{10}}^{10}\mathrm{d}S=\int_{\frac{1}{10}}^{10}|\ln x|\mathrm{d}x=-\int_{\frac{1}{10}}^{1}\ln x\mathrm{d}x+\int_{1}^{10}\ln x\mathrm{d}x$$

$$=-(x\ln x-x)\Big|_{\frac{1}{10}}^{1}+(x\ln x-x)\Big|_{1}^{10}$$

$$=\left(1-\frac{1}{10}\ln 10-\frac{1}{10}\right)+(10\ln 10-10+1)=\frac{99}{10}\ln 10-\frac{81}{10}.$$

注:被积函数带有绝对值,需要先去掉绝对值再计算.

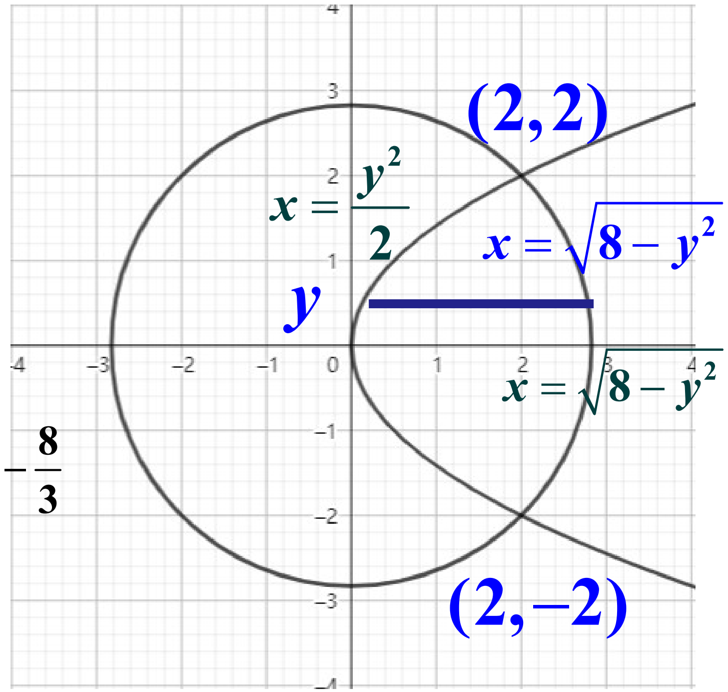
注:利用牛顿-莱布尼茨公式计算定积分不要缺原函数那一步.

P225/习题10.1/3

抛物线$y^2=2x$把圆$x^2+y^2\le 8$分成两部分,求这两部分面积之比.

解 已知圆的面积为 8π, 只需求出其中一部分的面积即可.

抛物线与圆的交点为 $(2,2),(2,-2)$.



右半部分面积元素为 $\mathrm{d}A=\left(\sqrt{8-y^2}-\frac{y^2}{2}\right)\mathrm{d}y.$

右半部分面积为

$$A=\int_{-2}^{2}\mathrm{d}A=\int_{-2}^{2}\left(\sqrt{8-y^2}-\frac{y^2}{2}\right)\mathrm{d}y$$

$$=\int_{-2}^{2}\sqrt{8-y^2}\,\mathrm{d}y-\left(\frac{y^3}{6}\right)\Bigg|_{-2}^{2}\overset{y=2\sqrt{2}\sin t}{=}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}2\sqrt{2}\cos t\cdot 2\sqrt{2}\cos t\,\mathrm{d}t-\frac{8}{3}$$

$$=4\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(1+\cos 2t)\,\mathrm{d}t-\frac{8}{3}=4\left(t+\frac{\sin 2t}{2}\right)\Bigg|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}-\frac{8}{3}=2\pi+\frac{4}{3}.$$

所以两部分面积之比为 $\dfrac{2\pi+\frac{4}{3}}{8\pi-\left(2\pi+\frac{4}{3}\right)}=\dfrac{3\pi+2}{9\pi-2}.$

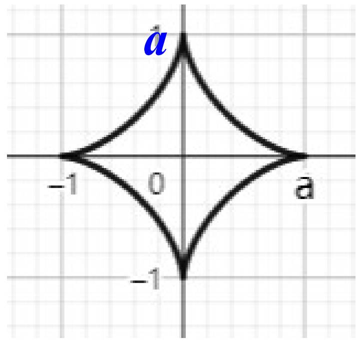
注:根据图形来确定用哪个积分变量,目的是便于定积分的计算.

P225/习题10.1/4 求内摆线$x=a\cos^3 t, y=a\sin^3 t(a>0)$所围图形的面积.

解1 根据对称性,所求图形的面积为

$$A=4\int_0^a y\,\mathrm{d}x=4\int_{\frac{\pi}{2}}^0 a\sin^3 t\,\mathrm{d}\left(a\cos^3 t\right)=-12a^2\int_{\frac{\pi}{2}}^0 \sin^4 t\cos^2 t\,\mathrm{d}t=12a^2\int_0^{\frac{\pi}{2}}\sin^4 t(1-\sin^2 t)\,\mathrm{d}t$$

$$=12a^2\left(\int_0^{\frac{\pi}{2}}\sin^4 t\,\mathrm{d}t-\int_0^{\frac{\pi}{2}}\sin^6 t\,\mathrm{d}t\right)=12a^2\left(\frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2}-\frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2}\right)=\frac{3}{8}\pi a^2.$$



解2 根据对称性,所求图形的面积为

$$A=4\int_0^a x\,\mathrm{d}y=4\int_0^{\frac{\pi}{2}} a\cos^3 t\,\mathrm{d}\left(a\sin^3 t\right)=12a^2\int_0^{\frac{\pi}{2}}\sin^2 t\cos^4 t\,\mathrm{d}t$$

$$=12a^2\left(\int_0^{\frac{\pi}{2}}\cos^4 t\,\mathrm{d}t-\int_0^{\frac{\pi}{2}}\cos^6 t\,\mathrm{d}t\right)=12a^2\left(\frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2}-\frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2}\right)=\frac{3}{8}\pi a^2.$$

解3 根据对称性,所求图形的面积为

平面图形的边界曲线由参数方程表示并且不封闭时的公式

$$A=4\int_0^{\frac{\pi}{2}}\left|y(t)x'(t)\right|\mathrm{d}t=12a^2\int_0^{\frac{\pi}{2}}\sin^4 t\cos^2 t\,\mathrm{d}t=12a^2\int_0^{\frac{\pi}{2}}\sin^4 t(1-\sin^2 t)\,\mathrm{d}t$$

$$=12a^2\left(\int_0^{\frac{\pi}{2}}\sin^4 t\,\mathrm{d}t-\int_0^{\frac{\pi}{2}}\sin^6 t\,\mathrm{d}t\right)=12a^2\left(\frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2}-\frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2}\right)=\frac{3}{8}\pi a^2.$$

解4 所求图形的面积为

平面图形的边界曲线由参数方程表示并且为封闭时的公式

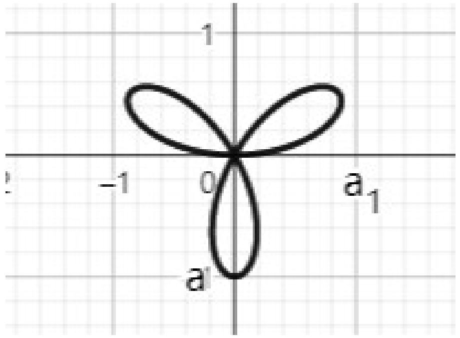
$$A=\left|\int_0^{2\pi}y(t)x'(t)\,\mathrm{d}t\right|=3a^2\left|\int_0^{2\pi}\sin^4 t\cos^2 t\,\mathrm{d}t\right|=3a^2\left|\int_0^{2\pi}(\sin^4 t-\sin^6 t)\,\mathrm{d}t\right|$$

$$=12a^2\left(\int_0^{\frac{\pi}{2}}\sin^4 t\,\mathrm{d}t-\int_0^{\frac{\pi}{2}}\sin^6 t\,\mathrm{d}t\right)=12a^2\left(\frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2}-\frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2}\right)=\frac{3}{8}\pi a^2.$$

注：公式要正确使用.

P225/习题10.1/6 求三叶形曲线$r = a\sin 3\theta (a > 0)$所围图形的面积.

解 面积元素为 $\mathrm{d}A = \frac{1}{2}r^2(\theta)\mathrm{d}\theta$,



根据对称性,所求图形的面积为

$$A = 3\cdot\int_0^{\frac{\pi}{3}}\mathrm{d}A = 3\cdot\frac{1}{2}\int_0^{\frac{\pi}{3}}r^2(\theta)\mathrm{d}\theta = \frac{3}{2}\int_0^{\frac{\pi}{3}}a^2\sin^2 3\theta\,\mathrm{d}\theta$$

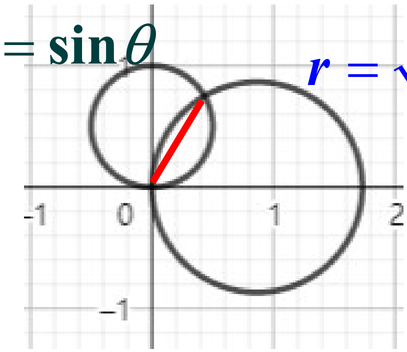
$$= \frac{3}{4}a^2\int_0^{\frac{\pi}{3}}(1-\cos 6\theta)\mathrm{d}\theta = \frac{3}{4}a^2\left(\theta - \frac{\sin 6\theta}{6}\right)\Bigg|_0^{\frac{\pi}{3}} = \frac{1}{4}\pi a^2.$$

注:极坐标下正确画出图形,进一步确定极角的范围.

BY GYH

P225/习题10.1/9 求二曲线 $r=\sin\theta$ 与 $r=\sqrt{3}\cos\theta$ 所围公共部分的面积.

解 两圆相交于 $\theta=\dfrac{\pi}{3}$. 所求图形的面积为



$$A=\frac{1}{2}\int_0^{\frac{\pi}{3}}\sin^2\theta\,\mathrm{d}\theta+\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\left(\sqrt{3}\cos\theta\right)^2\mathrm{d}\theta$$

$$=\frac{1}{4}\int_0^{\frac{\pi}{3}}(1-\cos 2\theta)\,\mathrm{d}\theta+\frac{3}{4}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}(1+\cos 2\theta)\,\mathrm{d}\theta$$

$$=\frac{1}{4}\left(\theta-\frac{\sin 2\theta}{2}\right)\Bigg|_0^{\frac{\pi}{3}}+\frac{3}{4}\left(\theta+\frac{\sin 2\theta}{2}\right)\Bigg|_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$=\frac{1}{4}\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right)+\frac{3}{4}\left(\frac{\pi}{2}-\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right)=\frac{5}{24}\pi-\frac{\sqrt{3}}{4}.$$

$$r=\sin\theta\Rightarrow r^2=r\sin\theta$$

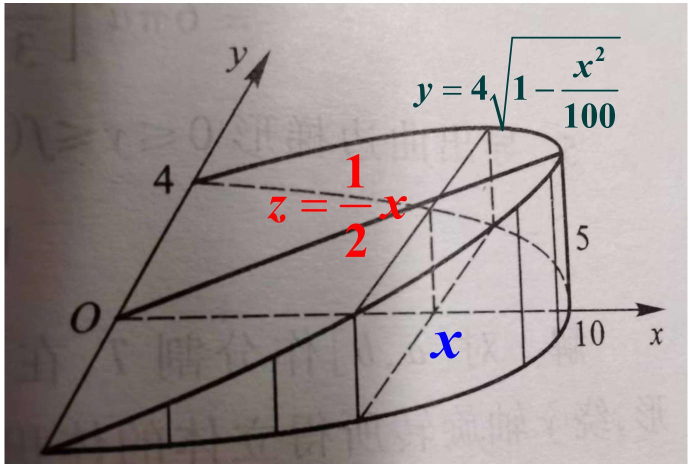
$$\Rightarrow x^2+y^2=y\Rightarrow x^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{4}$$

$$r=\sqrt{3}\cos\theta\Rightarrow r^2=r\sqrt{3}\cos\theta$$

$$\Rightarrow x^2+y^2=\sqrt{3}x\Rightarrow\left(x-\frac{\sqrt{3}}{2}\right)^2+y^2=\frac{3}{4}$$

P228/习题10.2/1

如图直椭圆柱体被通过底面短轴的斜平面所截,试求截得楔形体的体 积.



解1 如图建立坐标系.

直椭圆柱体的底面是椭圆面 $\frac{x^2}{100}+\frac{y^2}{16}\le 1.$

楔形体的垂直于x轴的矩形截面面积为

$$A(x) = 2\cdot 4\sqrt{1-\frac{x^2}{100}}\cdot\frac{x}{2} = \frac{2}{5}x\sqrt{100-x^2}.$$

截得的楔形体的体积为

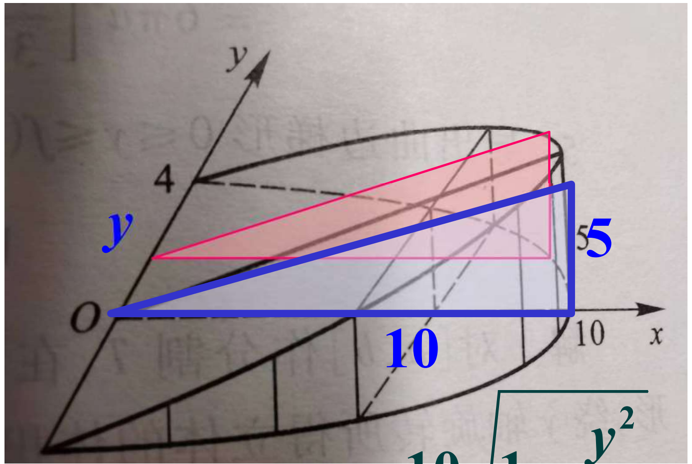
$$V = \int_0^{10} A(x)\,\mathrm{d}x = \frac{2}{5}\int_0^{10} x\sqrt{100-x^2}\,\mathrm{d}x = -\frac{1}{5}\int_0^{10}\sqrt{100-x^2}\,\mathrm{d}(100-x^2)$$

$$= -\frac{1}{5}\cdot\frac{2}{3}(100-x^2)^{\frac{3}{2}}\Big|_0^{10} = \frac{400}{3}.$$

P228/习题10.2/1

如图直椭圆柱体被通过底面短轴的斜平面所截,试求截得楔形体的体积.

解2 如图建立坐标系.



$$x = 10\sqrt{1-\frac{y^2}{16}}$$

直椭圆柱体的底面是椭圆面 $\dfrac{x^2}{100}+\dfrac{y^2}{16}\le 1.$

楔形体的垂直于y轴的矩形截面面积为

$$A(y) = \frac{1}{2}\cdot 10\sqrt{1-\frac{y^2}{16}}\cdot\frac{10}{2}\sqrt{1-\frac{y^2}{16}} = \frac{25}{16}(16-y^2).$$

截得的楔形体的体积为

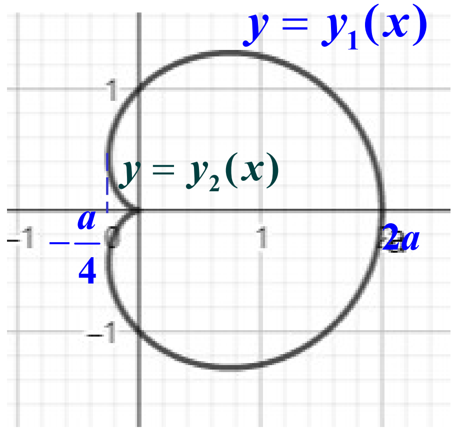
$$V = \int_{-4}^{4} A(y)\,\mathrm{d}y = \frac{25}{16}\int_{-4}^{4}(16-y^2)\,\mathrm{d}y = \frac{25}{8}\int_{0}^{4}(16-y^2)\,\mathrm{d}y$$

$$= \frac{25}{8}\left(16y-\frac{y^3}{3}\right)\Bigg|_0^4 = \frac{400}{3}.$$

注:先确定截面是什么,进一步求出截面的面积.

P228/习题10.2/2(3)

求曲线$r=a(1+\cos\theta)(a>0)$绕极轴旋转所得立体的体积.



解1 由于$x=a(1+\cos\theta)\cos\theta=a\left(\left(\cos\theta+\frac{1}{2}\right)^2-\frac{1}{4}\right)$,

因此当$\cos\theta=-\frac{1}{2}$,即$\theta=\frac{2\pi}{3}$时,x取得最小值$x=-\frac{a}{4}$.

旋转所得立体的体积为

$$V=\int_{-\frac{a}{4}}^{2a}\pi y_1^2(x)\mathrm{d}x-\int_{-\frac{a}{4}}^{0}\pi y_2^2(x)\mathrm{d}x=\pi\int_{\frac{2}{3}\pi}^{0}a^2(1+\cos\theta)^2\sin^2\theta\,\mathrm{d}\big(a(1+\cos\theta)\cos\theta\big)$$

$$-\pi\int_{\frac{2}{3}\pi}^{\pi}a^2(1+\cos\theta)^2\sin^2\theta\,\mathrm{d}\big(a(1+\cos\theta)\cos\theta\big)=\pi a^3\int_0^{\pi}(1+\cos\theta)^2\sin^3\theta(1+2\cos\theta)\mathrm{d}\theta$$

$$\overset{t=\frac{\pi}{2}-\theta}{=}\pi a^3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+\sin t)^2\cos^3t(1+2\sin t)\mathrm{d}t=\pi a^3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos^3t+4\sin t\cos^3t+5\cos^3t\sin^2t+2\sin^3t\cos^3t)\mathrm{d}t$$

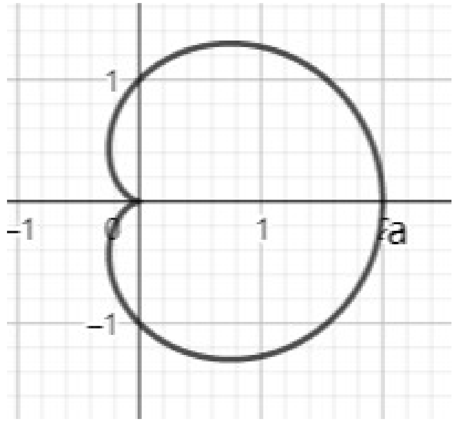
$$=2\pi a^3\int_0^{\frac{\pi}{2}}(\cos^3t+5\cos^3t\sin^2t)\mathrm{d}t=2\pi a^3\int_0^{\frac{\pi}{2}}(\cos^3t+5\cos^3t-5\cos^5t)\mathrm{d}t=2\pi a^3\int_0^{\frac{\pi}{2}}(6\cos^3t-5\cos^5t)\mathrm{d}t$$

$$=2\pi a^3\left(6\cdot\frac{2}{3}-5\cdot\frac{4\cdot2}{5\cdot3}\right)=\frac{8}{3}\pi a^3.$$

注:将极坐标方程转换为参数方程,先在边界曲线为显函数表示下得到计算公 式,然后再换成参数为积分变量,注意参数的上下限.

P228/习题10.2/2(3)

求曲线$r=a(1+\cos\theta)(a>0)$绕极轴旋转所得立体的体积.



解2 $V_{ox}=\dfrac{2\pi}{3}\int_0^{\pi} r^3(\theta)\sin\theta\,\mathrm{d}\theta$

$$=\frac{2\pi}{3}a^3\int_0^{\pi}(1+\cos\theta)^3\sin\theta\,\mathrm{d}\theta$$

$$=-\frac{2\pi}{3}a^3\int_0^{\pi}(1+\cos\theta)^3\,\mathrm{d}\cos\theta$$

$$=-\frac{2\pi}{3}a^3\left.\frac{(1+\cos\theta)^4}{4}\right|_0^{\pi}=\frac{8}{3}\pi a^3.$$

注：直接利用极坐标下平面图形绕极轴旋转的计算公式.

例8 证明曲边扇形$0\le\alpha\le\theta\le\beta, 0\le r\le r(\theta)$绕极轴旋转而成的体积为 $V_{ox}=\dfrac{2\pi}{3}\int_\alpha^\beta r^3(\theta)\sin\theta\,\mathrm{d}\theta.$

证 先求$[\theta,\theta+\mathrm{d}\theta]$上微曲边扇形绕极轴旋转而成的体积$\mathrm{d}V_{ox}$.

绿色面积元素 $\mathrm{d}A=r\,\mathrm{d}\theta\cdot\mathrm{d}r.$

该面积元素绕极轴旋转一周的体积元素

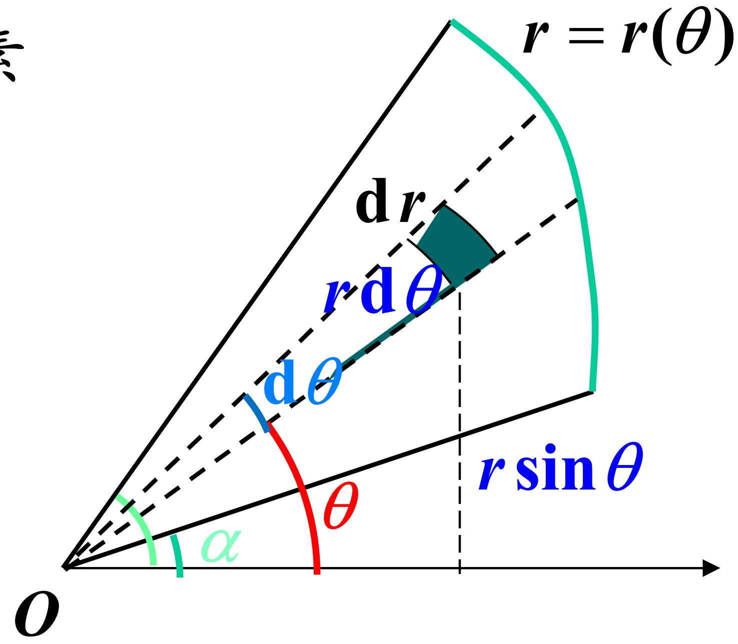
$$2\pi r\sin\theta\cdot r\,\mathrm{d}\theta\cdot\mathrm{d}r.$$

所以 $\mathrm{d}V_{ox}=2\pi\sin\theta\,\mathrm{d}\theta\int_0^{r(\theta)}r^2\,\mathrm{d}r$

$$=\frac{2\pi}{3}r^3(\theta)\sin\theta\,\mathrm{d}\theta.$$

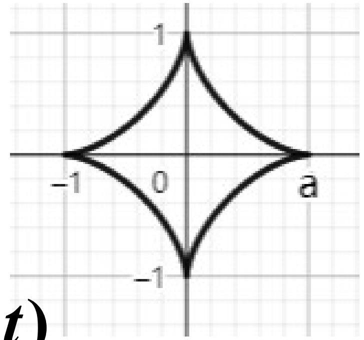
故

$$V_{ox}=\int_\alpha^\beta \mathrm{d}V_{ox}=\frac{2\pi}{3}\int_\alpha^\beta r^3(\theta)\sin\theta\,\mathrm{d}\theta.$$



P228/ 习题10.2/ 4

求曲线$x = a\cos^3 t, y = a\sin^3 t$所围平面图形绕$x$轴旋转所得立体的体 积.



解 绕x轴的旋转体的体积元素为 $\mathrm{d}V_x = \pi y^2 \,\mathrm{d}x,$

绕x轴的旋转体的体积为

$$V_x = 2\int_0^a \mathrm{d}V_x = 2\pi\int_0^a y^2\,\mathrm{d}x = 2\pi\int_{\frac{\pi}{2}}^{0}(a\sin^3 t)^2\,\mathrm{d}(a\cos^3 t)$$

$$= -6a^3\pi\int_{\frac{\pi}{2}}^{0}\sin^7 t\cdot\cos^2 t\,\mathrm{d}t = 6a^3\pi\int_0^{\frac{\pi}{2}}\left(\sin^7 t - \sin^9 t\right)\mathrm{d}t$$

$$= 6a^3\pi\left(\frac{6\cdot4\cdot2}{7\cdot5\cdot3} - \frac{8\cdot6\cdot4\cdot2}{9\cdot7\cdot5\cdot3}\right) = \frac{32}{105}\pi a^3.$$

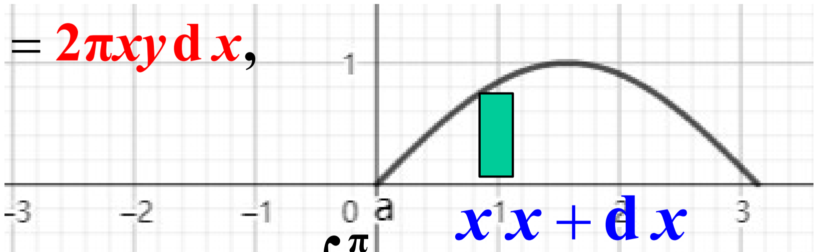
注:先在边界曲线为显函数表示下得到计算公式,

然后再换成参数为积分变量,注意参数的上下限.

P228/习题10.2/6

求$0\le y\le\sin x, 0\le x\le\pi$所示平面图形绕$y$轴旋转所得立体的体积.

解1 利用柱壳法. 绕y轴的旋转体的体积元素为 $\mathrm{d}V_y=2\pi xy\,\mathrm{d}x$,

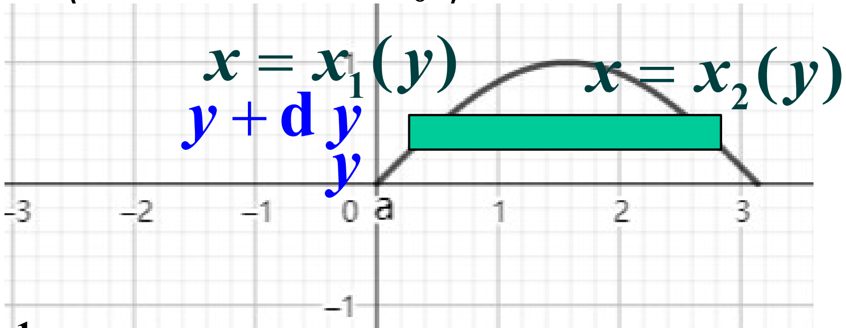


绕y轴的旋转体的体积为

$$V_y=\int_0^\pi \mathrm{d}V_y=2\pi\int_0^\pi xy\,\mathrm{d}x=2\pi\int_0^\pi x\sin x\,\mathrm{d}x=-2\pi\int_0^\pi x\,\mathrm{d}\cos x$$

$$=-2\pi\left((x\cos x)\Big|_0^\pi-\int_0^\pi\cos x\,\mathrm{d}x\right)=-2\pi\left(-\pi-(\sin x)\Big|_0^\pi\right)=2\pi^2.$$

解2 绕y轴的旋转体的体积元素为



$$\mathrm{d}V_y=\pi\left(x_2^2(y)-x_1^2(y)\right)\mathrm{d}y,$$

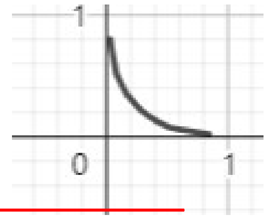
绕y轴的旋转体的体积为

$$V_y=\int_0^1\mathrm{d}V_y=\pi\int_0^1\left(x_2^2(y)-x_1^2(y)\right)\mathrm{d}y=\pi\int_0^1\left((\pi-\arcsin y)^2-\arcsin^2 y\right)\mathrm{d}y$$

$$=\pi\int_0^1\left(\pi^2-2\pi\arcsin y\right)\mathrm{d}y=\pi^3-2\pi^2\left(y\arcsin y\Big|_0^1-\int_0^1\frac{y}{\sqrt{1-y^2}}\mathrm{d}y\right)$$

$$=\pi^3-2\pi^2\left(\frac{\pi}{2}+\sqrt{1-y^2}\Big|_0^1\right)=2\pi^2.$$

P235/习题10.3/1(2) 求曲线$\sqrt{x}+\sqrt{y}=1$的弧长.



解1 $\sqrt{x}+\sqrt{y}=1$的显式方程为 $y=\left(1-\sqrt{x}\right)^2, 0\le x\le 1.$

则 $y'=\dfrac{\sqrt{x}-1}{\sqrt{x}}$.弧长元素为 $\mathrm{d}s=\sqrt{1+y'^2}\,\mathrm{d}x=\sqrt{1+\dfrac{(\sqrt{x}-1)^2}{x}}\,\mathrm{d}x,$

所求弧长为

$$s=\int_0^1\mathrm{d}s=\int_0^1\sqrt{1+\frac{(\sqrt{x}-1)^2}{x}}\,\mathrm{d}x=\int_0^1\sqrt{\frac{2x-2\sqrt{x}+1}{x}}\,\mathrm{d}x=\sqrt{2}\int_0^1\frac{\sqrt{\left(\sqrt{x}-\frac{1}{2}\right)^2+\frac{1}{4}}}{\sqrt{x}}\,\mathrm{d}x$$

$$=2\sqrt{2}\int_0^1\sqrt{\left(\sqrt{x}-\frac{1}{2}\right)^2+\frac{1}{4}}\,\mathrm{d}\left(\sqrt{x}-\frac{1}{2}\right)\overset{t=\sqrt{x}-\frac{1}{2}}{=}2\sqrt{2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\sqrt{t^2+\frac{1}{4}}\,\mathrm{d}t\overset{t=\frac{1}{2}\tan s}{=}4\sqrt{2}\int_0^{\frac{\pi}{4}}\frac{1}{2}\sec t\cdot\frac{1}{2}\sec^2 t\,\mathrm{d}t$$

$$=\sqrt{2}\int_0^{\frac{\pi}{4}}\sec^3 t\,\mathrm{d}t=\sqrt{2}\int_0^{\frac{\pi}{4}}\sec t\,\mathrm{d}\tan t=\sqrt{2}\left(\sec t\tan t\Big|_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\tan^2 t\sec t\,\mathrm{d}t\right)$$

$$=\sqrt{2}\left(\sqrt{2}+\int_0^{\frac{\pi}{4}}\sec t\,\mathrm{d}t-\int_0^{\frac{\pi}{4}}\sec^3 t\,\mathrm{d}t\right)=\sqrt{2}\left(\sqrt{2}+\ln(\sec t+\tan t)\Big|_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\sec^3 t\,\mathrm{d}t\right)$$

$$=2+\sqrt{2}\ln(\sqrt{2}+1)-\sqrt{2}\int_0^{\frac{\pi}{4}}\sec^3 t\,\mathrm{d}t=1+\frac{\sqrt{2}}{2}\ln(\sqrt{2}+1).$$

P235/习题10.3/1(2) 求曲线$\sqrt{x}+\sqrt{y}=1$的弧长.

解2 $\sqrt{x}+\sqrt{y}=1$的参数方程为 $\begin{cases} x=\cos^4 t \\ y=\sin^4 t \end{cases}, 0\le t\le \dfrac{\pi}{2}.$

弧长元素为 $\mathrm{d}s=\sqrt{x'^2(t)+y'^2(t)}\,\mathrm{d}t=4\sin t\cos t\sqrt{\sin^4 t+\cos^4 t}\,\mathrm{d}t,$

所求弧长为

$$s=\int_0^{\frac{\pi}{2}}\mathrm{d}s=\int_0^{\frac{\pi}{2}}4\sin t\cos t\sqrt{\sin^4 t+\cos^4 t}\,\mathrm{d}t=2\int_0^{\frac{\pi}{2}}\sin 2t\sqrt{(\sin^2 t+\cos^2 t)^2-2\sin^2 t\cos^2 t}\,\mathrm{d}t$$

$$=2\int_0^{\frac{\pi}{2}}\sin 2t\sqrt{1-\frac{1}{2}\sin^2 2t}\,\mathrm{d}t=-\int_0^{\frac{\pi}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\cos^2 2t}\,\mathrm{d}\cos 2t\overset{u=\cos 2t}{=}-\frac{\sqrt{2}}{2}\int_1^{-1}\sqrt{1+u^2}\,\mathrm{d}u$$

$$=\sqrt{2}\int_0^1\sqrt{1+u^2}\,\mathrm{d}u=\sqrt{2}\left(u\sqrt{1+u^2}\Big|_0^1-\int_0^1\frac{u^2}{\sqrt{1+u^2}}\mathrm{d}u\right)=\sqrt{2}\left(\sqrt{2}-\int_0^1\frac{u^2+1-1}{\sqrt{1+u^2}}\mathrm{d}u\right)$$

$$=\sqrt{2}\left(\sqrt{2}-\int_0^1\sqrt{1+u^2}\,\mathrm{d}u+\int_0^1\frac{1}{\sqrt{1+u^2}}\mathrm{d}u\right)=\sqrt{2}\left(\sqrt{2}-\int_0^1\sqrt{1+u^2}\,\mathrm{d}u+\ln(u+\sqrt{1+u^2})\Big|_0^1\right)$$

$$=\sqrt{2}\left(\sqrt{2}-\int_0^1\sqrt{1+u^2}\,\mathrm{d}u+\ln(1+\sqrt{2})\right)=1+\frac{\sqrt{2}}{2}\ln(1+\sqrt{2}).$$

P235/习题10.3/1(4)

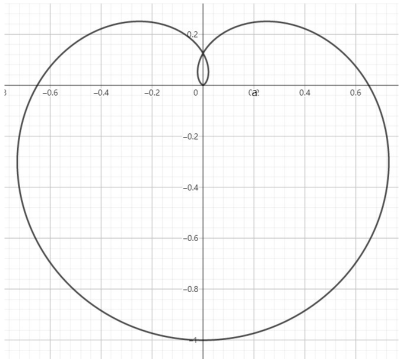
求曲线$x=a(\cos t+t\sin t), y=a(\sin t-t\cos t)(a>0), 0\le t\le 2\pi$的弧长.

解 弧长元素为

$$\mathrm{d}s=\sqrt{x'^2(t)+y'^2(t)}\,\mathrm{d}t=\sqrt{a^2t^2\cos^2 t+a^2t^2\sin^2 t}\,\mathrm{d}t=at\,\mathrm{d}t,$$

所求弧长为

$$s=\int_0^{2\pi}\mathrm{d}s=\int_0^{2\pi}at\,\mathrm{d}t=a\frac{t^2}{2}\bigg|_0^{2\pi}=2\pi^2a.$$



P235/习题10.3/1(5)

求曲线$r = a\sin^3\dfrac{\theta}{3}(a>0), 0\le\theta\le 3\pi$的弧长.

解 弧长元素为

$$\mathrm{d}s=\sqrt{r^2(\theta)+r'^2(\theta)}\,\mathrm{d}\theta=\sqrt{a^2\sin^6\frac{\theta}{3}+a^2\sin^4\frac{\theta}{3}\cos^2\frac{\theta}{3}}\,\mathrm{d}\theta=a\sin^2\frac{\theta}{3}\mathrm{d}\theta,$$

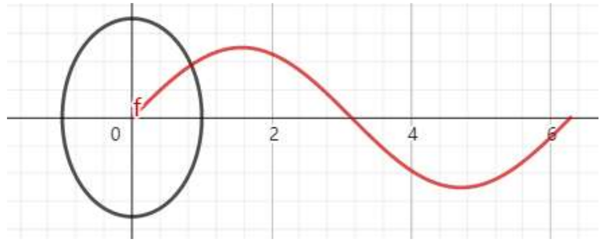
所求弧长为

$$s=\int_0^{3\pi}\mathrm{d}s=\int_0^{3\pi}a\sin^2\frac{\theta}{3}\mathrm{d}\theta=\frac{a}{2}\int_0^{3\pi}\left(1-\cos\frac{2\theta}{3}\right)\mathrm{d}\theta$$

$$=\frac{a}{2}\left(\theta-\frac{3}{2}\sin\frac{2\theta}{3}\right)\Bigg|_0^{3\pi}=\frac{3}{2}\pi a.$$

P235/习题10.3/3

求a,b的值,使椭圆$x=a\cos t, y=b\sin t$的周长等于正弦曲线$y=\sin x$在$0\le x\le 2\pi$上一段的长.



解 利用对称性,椭圆的周长为

$$s_1=4\int_0^{\frac{\pi}{2}}\sqrt{x'^2(t)+y'^2(t)}\,\mathrm{d}t=4\int_0^{\frac{\pi}{2}}\sqrt{a^2\sin^2 t+b^2\cos^2 t}\,\mathrm{d}t.$$

利用对称性,正弦曲线的弧长为

$$s_2=4\int_0^{\frac{\pi}{2}}\sqrt{1+y'^2(x)}\,\mathrm{d}x=4\int_0^{\frac{\pi}{2}}\sqrt{1+\cos^2 x}\,\mathrm{d}x$$

$$=4\int_0^{\frac{\pi}{2}}\sqrt{1+\cos^2 t}\,\mathrm{d}t=4\int_0^{\frac{\pi}{2}}\sqrt{1+\sin^2 t}\,\mathrm{d}t.$$

要使$s_1=s_2$,当$a>b$时,有$\int_0^{\frac{\pi}{2}}\sqrt{b^2+(a^2-b^2)\sin^2 t}\,\mathrm{d}t=\int_0^{\frac{\pi}{2}}\sqrt{1+\sin^2 t}\,\mathrm{d}t$,

从而 $a=\sqrt{2}, b=1$.

当$a\le b$时,有$\int_0^{\frac{\pi}{2}}\sqrt{a^2+(b^2-a^2)\cos^2 t}\,\mathrm{d}t=\int_0^{\frac{\pi}{2}}\sqrt{1+\cos^2 t}\,\mathrm{d}t$,

从而 $a=1, b=\sqrt{2}$.

P238/习题10.4/1(1) 求$y=\sin x, 0\le x\le \pi$绕x轴旋转所得旋转曲面的面积.

解 旋转曲面的面积元素为

$$\mathrm{d}S = 2\pi y(x)\sqrt{1+y'^2(x)}\,\mathrm{d}x = 2\pi\sin x\sqrt{1+\cos^2 x}\,\mathrm{d}x,$$

所求旋转曲面的面积为

$$S=\int_0^{\pi}\mathrm{d}S = \int_0^{\pi}2\pi\sin x\sqrt{1+\cos^2 x}\,\mathrm{d}x = -2\pi\int_0^{\pi}\sqrt{1+\cos^2 x}\,\mathrm{d}\cos x$$

$$\overset{t=\cos x}{=} -2\pi\int_1^{-1}\sqrt{1+t^2}\,\mathrm{d}t = 4\pi\int_0^1\sqrt{1+t^2}\,\mathrm{d}t \overset{t=\tan s}{=} 4\pi\int_0^{\frac{\pi}{4}}\sec^3 s\,\mathrm{d}s$$

$$=4\pi\int_0^{\frac{\pi}{4}}\sec s\,\mathrm{d}\tan s = 4\pi\left(\sec s\tan s\Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}}\tan^2 s\sec s\,\mathrm{d}s\right)$$

$$=4\pi\left(\sqrt{2}-\int_0^{\frac{\pi}{4}}\sec^3 s\,\mathrm{d}s+\int_0^{\frac{\pi}{4}}\sec s\,\mathrm{d}s\right) = 4\pi\left(\sqrt{2}-\int_0^{\frac{\pi}{4}}\sec^3 s\,\mathrm{d}s+\ln(\sqrt{2}+1)\right)$$

$$=2\pi\left(\sqrt{2}+\ln(\sqrt{2}+1)\right).$$

P238/习题10.4/1(2)

求$x=a(t-\sin t), y=a(1-\cos t)(a>0), 0\le t\le 2\pi$绕$x$轴旋转所得旋转曲面的面积.

解 旋转曲面的面积元素为

$$\mathrm{d}S=2\pi y(t)\sqrt{x'^2(t)+y'^2(t)}\,\mathrm{d}t=2\pi a(1-\cos t)\sqrt{a^2(1-\cos t)^2+a^2\sin^2 t}\,\mathrm{d}t$$

$$=2\sqrt{2}\pi a^2(1-\cos t)\sqrt{1-\cos t}\,\mathrm{d}t=2\sqrt{2}\pi a^2(1-\cos t)^{\frac{3}{2}}\,\mathrm{d}t,$$

所求旋转曲面的面积为

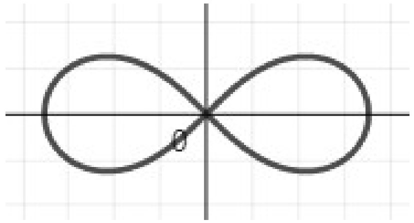
$$S=\int_0^{2\pi}\mathrm{d}S=2\sqrt{2}\pi a^2\int_0^{2\pi}(1-\cos t)^{\frac{3}{2}}\,\mathrm{d}t=8\pi a^2\int_0^{2\pi}\sin^3\frac{t}{2}\,\mathrm{d}t$$

$$\overset{u=\frac{t}{2}}{=}16\pi a^2\int_0^{\pi}\sin^3 u\,\mathrm{d}u=-16\pi a^2\int_0^{\pi}(1-\cos^2 u)\,\mathrm{d}\cos u$$

$$=-16\pi a^2\left(\cos u-\frac{\cos^3 u}{3}\right)\Bigg|_0^{\pi}=\frac{64}{3}\pi a^2.$$

P238/习题10.4/3(2)

试求双纽线$r^2=2a^2\cos 2\theta (a>0)$绕极轴旋转所得旋转曲面的面 积.



解 旋转曲面的面积元素为

$$\mathrm{d}S=2\pi r(\theta)\sin\theta\sqrt{r^2(\theta)+r'^2(\theta)}\,\mathrm{d}\theta$$

$$=2\pi\sqrt{2\cos 2\theta}\,a\sin\theta\sqrt{2a^2\cos 2\theta+2a^2\frac{\sin^2 2\theta}{\cos 2\theta}}\,\mathrm{d}\theta\ =4\pi a^2\sin\theta\,\mathrm{d}\theta,$$

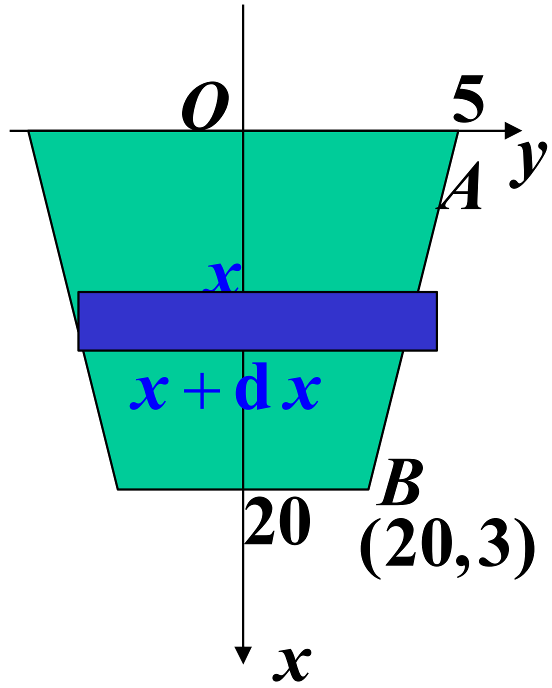
所求旋转曲面的面积为

$$S=2\int_0^{\frac{\pi}{4}}\mathrm{d}S\ =8\pi a^2\int_0^{\frac{\pi}{4}}\sin\theta\,\mathrm{d}\theta\ =8\pi a^2\left(-\cos\theta\right)\Big|_0^{\frac{\pi}{4}}\ =8\pi a^2\left(1-\frac{\sqrt{2}}{2}\right).$$

P242/习题10.5/1

有一等腰梯形的闸门,它的上、下两条底边各长为10m和6m,高为20m.计算当水面与上底边相齐时闸门一侧所受的静压力.

解 如图建立直角坐标系.



直线AB的方程为 $y=-\frac{1}{10}x+5.$

静压力元素为

$$\mathrm{d}P=\rho gx\cdot 2\left(-\frac{1}{10}x+5\right)\mathrm{d}x,$$

所求静压力为

$$P=\int_0^{20}\mathrm{d}P=2\rho g\int_0^{20}x\left(-\frac{1}{10}x+5\right)\mathrm{d}x$$

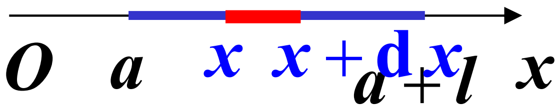
$$=2\rho g\left(-\frac{x^3}{30}+\frac{5x^2}{2}\right)\Bigg|_0^{20}=\frac{4400}{3}\rho g.$$

注:要建立合适的坐标系.

P242/习题10.5/4

设在坐标轴的原点有一质量为m的质点,在区间$[a,a+l](a>0)$上有一质量为M的均匀细杆. 试求质点与细杆之间的万有引力.

解 如图建立直角坐标系.



引力元素为

$$\mathrm{d}F=\frac{Gm\cdot\dfrac{M}{l}\mathrm{d}x}{x^2}=\frac{GmM}{x^2l}\mathrm{d}x,$$

质点与细杆之间的万有引力为

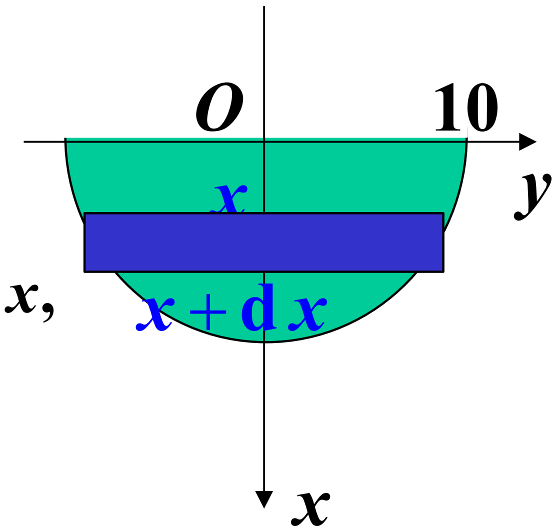
$$F=\int_a^{a+l}\mathrm{d}F=\frac{GmM}{l}\int_a^{a+l}\frac{1}{x^2}\mathrm{d}x=\frac{GmM}{l}\left(-\frac{1}{x}\right)\Bigg|_a^{a+l}$$

$$=\frac{GmM}{l}\left(\frac{1}{a}-\frac{1}{a+l}\right)=\frac{GmM}{a(a+l)}.$$

P242/习题10.5/7

一个半球形(直径为20m)的容器内盛满了水.试问把水抽尽需做多少功?

解 如图建立直角坐标系.



功元素为

$$\mathrm{d}W = \rho g \cdot \pi(100 - x^2)\,\mathrm{d}x \cdot x = \pi\rho g(100x - x^3)\,\mathrm{d}x,$$

把水抽尽所需功为

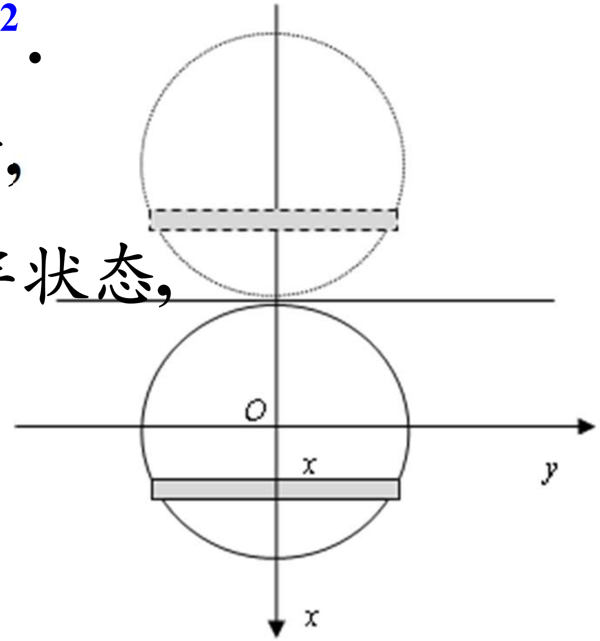
$$W = \int_0^{10} \mathrm{d}W = \pi\rho g \int_0^{10} (100x - x^3)\,\mathrm{d}x$$

$$= \pi\rho g\left(50x^2 - \frac{x^4}{4}\right)\Bigg|_0^{10} = 2500\pi\rho g.$$

注:要建立合适的坐标系.

P242/习题10.5/10

半径为r的球体沉入水中,其比重与水相同.则将球体从水中捞出需做的功.



解 如图建立直角坐标系. 圆的方程为$x^2+y^2=r^2$.

将阴影部分的水提升至水面以上相应位置时,

由于球体比重与水相同,在水面以下处于悬浮状态,

因此需要做功的距离为$r-x$,故功元素

$$\mathrm{d}W=\rho g(r-x)\cdot\pi\left(\sqrt{r^2-x^2}\right)^2\mathrm{d}x$$

$$=\rho g\pi(r-x)\cdot(r^2-x^2)\mathrm{d}x,\quad x\in[-r,r].$$

于是,将球体从水中捞出需做的功

$$W=\int_{-r}^{r}\mathrm{d}W=\rho g\pi\int_{-r}^{r}(r-x)\cdot(r^2-x^2)\mathrm{d}x$$

$$=\rho g\pi\int_{-r}^{r}(r^3-r^2x-rx^2+x^3)\mathrm{d}x=2\rho g\pi\left(r^3x-\frac{r}{3}x^3\right)\Big|_0^r=\frac{4}{3}\rho g\pi r^4.$$