Nonlinear Control Theory

Lecture 2. Periodic Solutions

Today (Second Order Systems)

- Qualitative behaviour
  - Linear Systems
  - Nonlinear Systems near equilibria.
- Limit cycles & its existence.

Def A point $x = x^*$ is said to be equilibrium of $\dot{x} = f(t, x)$ if it has the property that whenever $x(t)$ starts at x^*, it remains at x^* for all future time.

For autonomous system $\dot{x} = f(x)$, the equilibrium is the real roots of $f(x) = 0$

Consider Linear time-invariant system $\dot{x} = Ax$.

Suppose $A = P \Lambda P^T$, where $P = [V_1, V_2]$ is unitary, $\Lambda = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}$ ① or $\begin{bmatrix} \lambda & k \\ & \lambda \end{bmatrix}$ ③ (either 0 or 1) or $\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$ ②

Do coordinate change $z = P^T x$,

$$\dot{x} = P \Lambda P^T x \Leftrightarrow P^T \dot{x} = \Lambda P^T x \Leftrightarrow \dot{z} = \Lambda z$$

① $\lambda_1 \neq \lambda_2 \neq 0$

$$\dot{z}_1 = \lambda_1 z_1, \quad \dot{z}_2 = \lambda_2 z_2 \Rightarrow z_1(t) = z_{10} e^{\lambda_1 t}, \quad z_2(t) = z_{20} e^{\lambda_2 t}$$

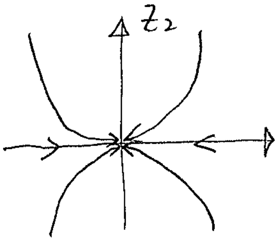
$$t = \frac{1}{\lambda_1} \ln \frac{z_1}{z_{10}} \Rightarrow z_2 = z_{20} e^{\frac{\lambda_2}{\lambda_1} \ln \frac{z_1}{z_{10}}} = z_{20} \left(e^{\ln \frac{z_1}{z_{10}}}\right)^{\frac{\lambda_2}{\lambda_1}}$$

$$= \underbrace{\frac{z_{20}}{z_{10}^{\lambda_2/\lambda_1}}}_{c} (z_1)^{\lambda_2/\lambda_1} \Rightarrow z_2 = c z_1^{\lambda_2/\lambda_1}$$

i) $\lambda_2 < \lambda_1 < 0$.

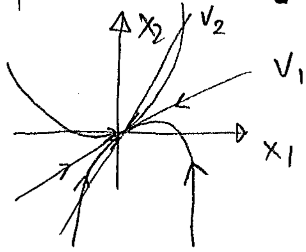
slope of curve $\frac{dz_2}{dz_1} = c \frac{\lambda_2}{\lambda_1} z_1^{\overbrace{[\lambda_2/\lambda_1 - 1]}^{>0}}$

$\Rightarrow \frac{dz_2}{dz_1} \to 0$ as $|z_1| \to 0$, and $\frac{dz_2}{dz_1} \to \infty$ as $|z_1| \to \infty$



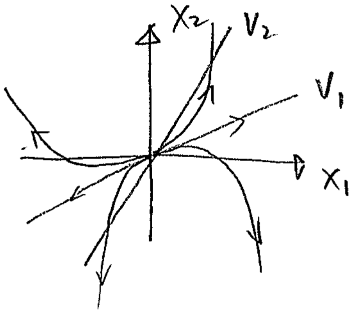
$$x = Pz = [V_1 \ \ V_2]\, z$$

$\Rightarrow$



stable node.

ii) $\lambda_2 > \lambda_1 > 0$

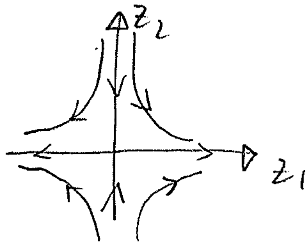


unstable node.

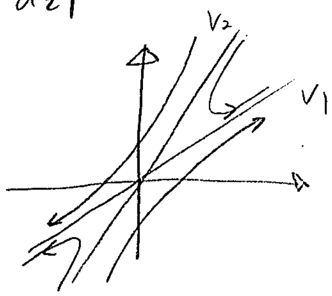
iii) $\lambda_2 < 0 < \lambda_1$

$e^{\lambda_1 t} \to \infty, \quad e^{\lambda_2 t} \to 0 \quad$ as $t \to \infty$,

$$\frac{dz_1}{dz_1} = c\, e^{\underbrace{[\lambda_2/\lambda_1 - 1]}_{<0}} \Rightarrow \frac{dz_2}{dz_1} \to 0 \text{ as } |z_1| \to \infty, \quad \frac{dz_2}{dz_1} \to \infty \text{ as } |z_1| \to 0$$



$\Rightarrow$



② $\lambda_{1,2} = \alpha + \beta j$.

$\dot{z}_1 = \alpha z_1 - \beta z_2, \quad \dot{z}_2 = \beta z_1 + \alpha z_2$

Change into polar coordinates: $\rho = \sqrt{z_1^2 + z_2^2} \quad \theta = \arctan\left(\frac{z_2}{z_1}\right)$

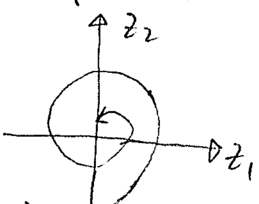
$$\dot{\rho} = \frac{1}{2}(z_1^2 + z_2^2)^{-\frac{1}{2}} [2z_1 \cdot \dot{z}_1 + 2z_2 \cdot \dot{z}_2] = \rho^{-1} \cdot [z_1(\alpha z_1 - \beta z_2) + z_2(\beta z_1 + \alpha z_2)]$$

$$= \rho^{-1}[\alpha z_1^2 + \alpha z_2^2] = \alpha \rho.$$

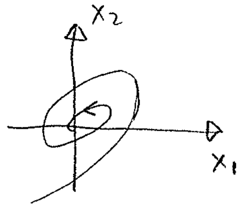
$$\dot{\theta} = \frac{1}{1 + \left(\frac{z_2}{z_1}\right)^2} \cdot \frac{\dot{z}_2 \cdot z_1 - z_2 \cdot \dot{z}_1}{z_1^2} = \frac{z_1^2}{z_1^2 + z_2^2} \frac{(\beta z_1 + \alpha z_2) \cdot z_1 - z_2 \cdot (\alpha z_1 - \beta z_2)}{z_1^2}$$

$$= \beta$$

$$\Rightarrow \rho(t) = \rho_0 e^{\alpha t}, \quad \theta(t) = \theta_0 + \beta t.$$

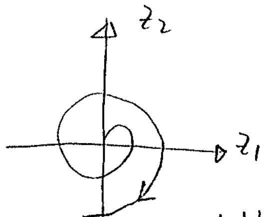


$\Rightarrow$

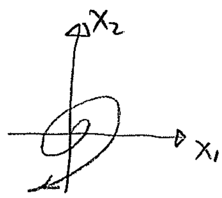


stable focus

$\alpha < 0$

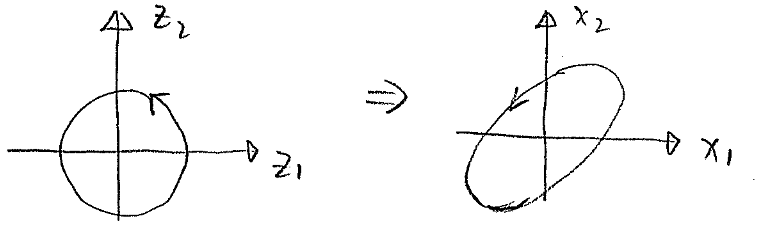


$\Rightarrow$



unstable focus

$\alpha > 0$

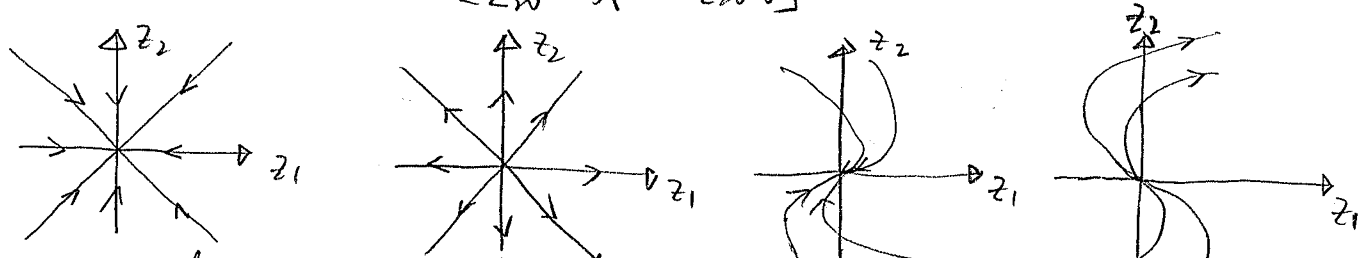


center. $\alpha = 0$

③ $\lambda_1 = \lambda_2 = \lambda \neq 0$

$$\dot{z}_1 = \lambda z_1 + k z_2, \qquad \dot{z}_2 = \lambda z_2 \Rightarrow z_1 = e^{\lambda t}(z_{10} + k z_{20} t)$$
$$z_2 = e^{\lambda t} z_{20}$$

Eliminate t, $\Rightarrow z_1 = z_2 \left[\frac{z_{10}}{z_{20}} + \frac{k}{\lambda} \ln\left(\frac{z_2}{z_{20}}\right)\right]$



stable node $k=0$, $\lambda<0$ | unstable node $k=0$, $\lambda>0$ | stable node $k=1$, $\lambda<0$ | unstable node $\lambda>0$, $k=1$

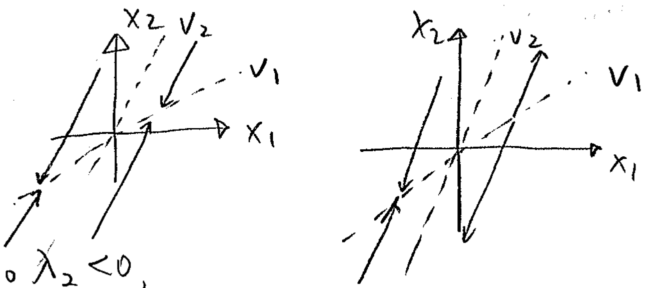
④ One or both λ's are zero.

The system has an equilibrium subspace.

i) $\lambda_1 = 0$, $\lambda_2 \neq 0$, $P = [v_1 \quad v_2]$

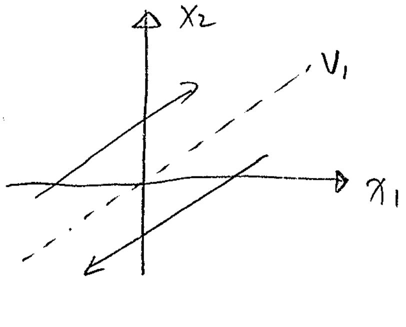
v_1 spans the nullspace of A.

$$\dot{z}_1 = 0, \quad \dot{z}_2 = \lambda_2 z_2 \Rightarrow z_1(t) = z_{10}, \quad z_2(t) = z_{20} e^{\lambda_2 t}$$



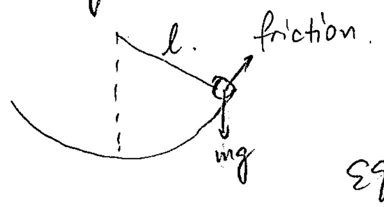
$\lambda_1 = 0, \lambda_2 < 0,$

ii) $\lambda_1 = \lambda_2 = 0$, $\quad \dot{z}_1 = z_2$, $\dot{z}_2 = 0 \Rightarrow z_1(t) = z_{10} + z_{20} t$
$z_2(t) = z_{20}$

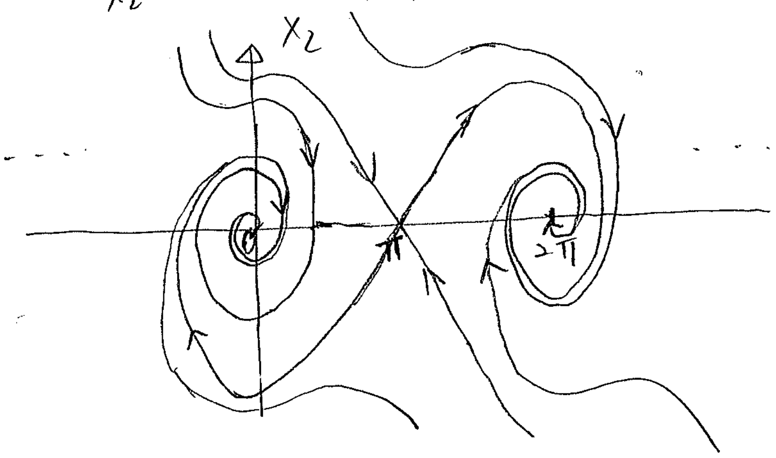


Consider the following nonlinear system.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -10\sin x_1 - x_2.$$



Equilibria:
$x_1 = k\pi$
$x_2 = 0$



Analyze by linearization

Let $p = (p_1, p_2)$ be an equilibrium pt of $\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases}$

Linearization:

$$\dot{x}_1 = \overset{\overset{0}{\parallel}}{f_1(p_1,p_2)} + \overbrace{\frac{\partial f_1(x_1,x_2)}{\partial x_1}\Big|_{x=p}}^{a_{11}}(x_1 - p_1) + \overbrace{\frac{\partial f_1(x_1,x_2)}{\partial x_2}\Big|_{x=p}}^{a_{12}}(x_2 - p_2) + O(x_1, x_2)$$

$$\dot{x}_2 = \underset{\underset{0}{\parallel}}{f_2(p_1,p_2)} + \underbrace{\frac{\partial f_2(x_1,x_2)}{\partial x_1}\Big|_{x=p}}_{a_{21}}(x_1 - p_1) + \underbrace{\frac{\partial f_2(x_1,x_2)}{\partial x_2}\Big|_{x=p}}_{a_{22}}(x_2 - p_2) + O(x_1, x_2)$$

$$\Rightarrow \quad \dot{x} \approx A x = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} x$$

Intuitively, we would expect the vector field (the right hand side of ODE) in the vicinity of the equilibrium, act close to the linearized version

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The "pendulum with friction":

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -10\cos x_1 & -1 \end{bmatrix} \Rightarrow A_1 = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix} \quad x^* = (2k\pi, 0), \quad \underline{\lambda_{1,2} = -0.5 \pm j3.12} \text{ stable focus}$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ 10 & -1 \end{bmatrix} \quad x^* = ((2k+1)\pi, 0), \quad \underline{\lambda_{1,2} = -3.7, 2.7} \text{ saddle.}$$

☆ Linearization can only deal with cases when the linearized state equation has no eigenvalue on the imaginary axis.

In particular, if the equilibrium is either a node with distinct eigenvalues, a focus, or a saddle, then in a vicinity of the equilibrium pt, the behaviour is preserved by linearization.

Moreover, if f_1, f_2 are analytic in the vicinity of the equilibrium pt, then the trajectory of the nonlinear system would behave like a node in a vicinity of the equilibrium whether or not the eigenvalues of the linearization are distinct.

! If the Jacobian has eigenvalues on imaginary axis, the system can behave very different from the linearized one. $\rightarrow$ center manifold theorem

Ex
$$\dot{x}_1 = -x_2 - \mu x_1 (x_1^2 + x_2^2)$$
$$\dot{x}_2 = x_1 - \mu x_2 (x_1^2 + x_2^2)$$

has an equilibrium at the origin.

Linearization:
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \left.\begin{bmatrix} -3\mu x_1^2 - \mu x_2^2 & -2\mu x_1 x_2 - 1 \\ 1 - 2\mu x_1 x_2 & -3\mu x_2^2 \end{bmatrix}\right|_{x=0} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

eigenvalues $\pm j$.

The origin is a center pt.

However, change into polar coordinates: $x_1 = \rho\cos\theta$, $x_2 = \rho\sin\theta$.

$\Rightarrow \dot{\rho} = -\mu\rho^3 \quad \dot{\theta} = 1$

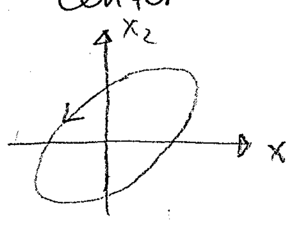
- $\mu > 0$ it is a stable focus
- $\mu < 0$, it is an unstable focus.

Def Nontrivial periodic solution (periodic / closed orbit)

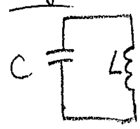
$x(t+T) = x(t)$, $\forall t \geq 0$ for some $T > 0$.

and $x(t)$ is NOT an equilibrium.

Recall the linear system $\dot{x} = Ax$, where A has eigenvalues on the imaginary axis. "Center"



e.g.

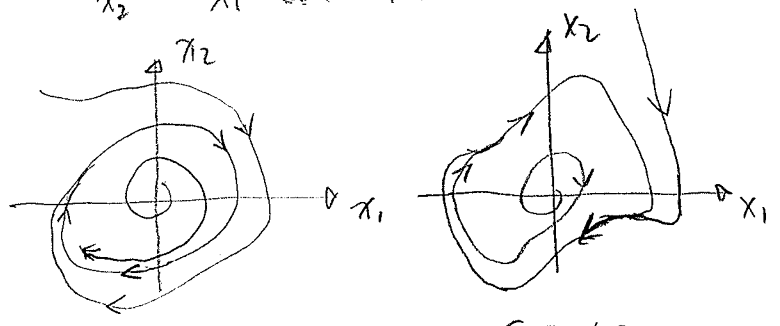


Fact:
- Not robust, infintesimally small perturbation would destroy the oscillation.
- Oscillation amplitude depends on initial value.

Def An isolated periodic solution is called a <u>limit cycle</u>.

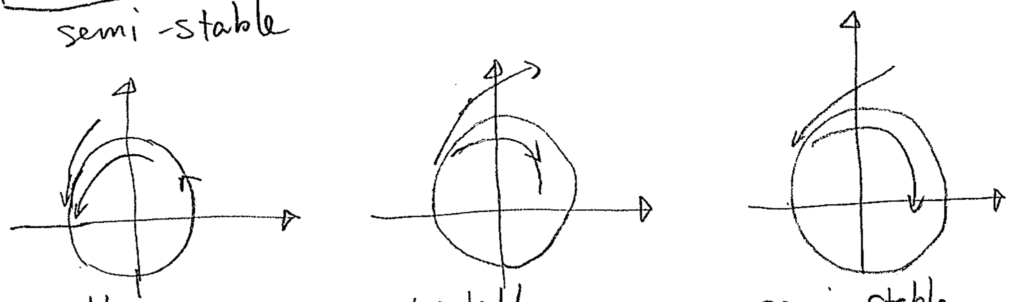
Ex Van der Pol oscillator

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2$$



$\varepsilon = 0.2$ $\qquad\qquad$ $\varepsilon = 1.0$

- Stable limit cycles (attractive)
  Trajectories that starts in the vicinity tends ultimately to the limit cycle as $t \to \infty$
- unstable limit cycles
  Trajectories that starts ——— || ———
  $\boxed{\text{as } t \to -\infty}$
- semi-stable



Stable $\qquad$ unstable $\qquad$ semi-stable

! The "stability" here for the limit cycle should not be messed with the Lyapunov stability to-come. The "stability" here is "attractive" compared to "Lyapunov" definition.

Existence of Periodic Solutions

Consider $\dot{x} = f(x)$. Let $x(t; x_0)$ be the solution whose initial value is x_0.

<u>ω-limit point</u>. A point p is said to be an ω-limit point of $x(t; x_0)$ if $\exists \{t_n\}$ with $t_n \to \infty$ as $n \to \infty$, such that $\lim_{n\to\infty} x(t_n; x_0) = p$

<u>ω-limit set</u>. The set of all limit points is called ω-limit set.

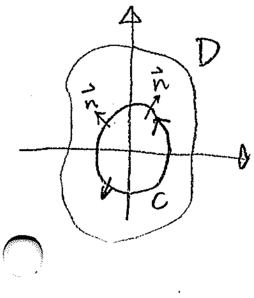
Thm (Bendixon Criterion)

Suppose $\mathcal{D}$ is a simply connected domain such that (no "holes")

$$\operatorname{div} f(x) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$$

is not identically zero over any solution in $\mathcal{D}$ and does not change sign over $\mathcal{D}$, then there does not exist any periodic solutions in $\mathcal{D}$.

proof: On any orbit C in $\mathcal{D}$, it holds for the vector field f that



$$\oint_S f \cdot \vec{n}\, ds = 0$$ since the vector field f is tangential to C, where $\vec{n}$ is the normal vector of C.

By divergence theorem, $\oint_S f \cdot \vec{n}\, ds = 0 = \iint \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right) dx_1 dx_2$

If $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \neq 0$ or does not change sign, the above equation won't hold.

Thm (Bendixon Theorem) Given a differentiable real dynamical system defined on an open subset of the plane, every non-empty compact ω-limit set of an orbit, which contains only finitely many equilibria, is either

- an equilibrium
- periodic solution (Cycle)
- phase polygon (consists of phase curves connecting several equilibria)

(guaranteed by $\gamma^+(x_0) = \{x(t) \mid t \geq 0\} \in \mathcal{D}$, where $\mathcal{D}$ is a bounded region)

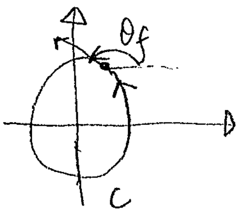
Thm (Poincaré Bendixon Theorem)

For a trajectory $\gamma^+(x_0) = \{x(t) \mid t \geq 0\}$, let $\mathcal{L}$ denote its ω-limit set. If $\mathcal{L}$ is contained in a closed-bounded region $\mathcal{D}$, and if $\mathcal{D}$ contains no equilibria, then either

1. $\gamma^+(x_0)$ is a periodic solution ($\gamma^+ = \mathcal{L}$)
2. $\mathcal{L}$ is a periodic solution, but $\gamma^+(x_0)$ is not ($\gamma^+ \cap \mathcal{L} = \phi$)

Index of a curve

$$I_f(C) = \frac{1}{2\pi}\int_C d\theta_f$$



If C is chosen that only encircle an isolated equilibrium, then $I_f(C)$ is the index of that equilibrium.

- The index of a node, focus, center, cycle is +1
- The index of a saddle is −1

☆ The index of a closed curve not encircling any equilibrium point is 0.

☆ Suppose $f(x)$ has no singular points on a simple, closed curve S. If S only encloses a finite number of singular points, then $I_f(S)$ is equal to the sum of the indices of the equilibria within it.