Nonlinear Control Theory

Lecture 14. Feedback stabilization III

Last time

- Passivity approach
- Arstein – Sontag's theorem. ⤳ control Lyapunov function: iff condition for the existence of "almost smooth" control that globally asymptotically stabilizes the system.
- Back stepping. $\begin{cases} \dot{\eta} = f(\eta) + g(\eta)\xi \\ \dot{\xi} = u \end{cases}$ ⤳ "pretend" ξ is the control input of the η-system, design asymptotic stabilizing control and the corresponding Lyapunov function.

Today

- Sliding mode control:

Recall in the first lecture, we discussed the solution definition of ODE.

$\dot{x} = f(x)$ classical solution: $x(t) \in C^1$ (Continuously diffentiable)
f: locally Lipschitz. $\Rightarrow$ existenence of unique classical solution

Some systems can not be stabilized using C^1 feedback controller.

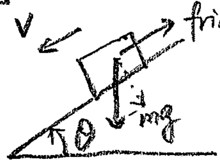
Ex| Recall in Lecture 12, by Brockett necessary condition, we know for unicycle:

$$\begin{aligned} \dot{x} &= v\cos\theta \\ \dot{y} &= v\sin\theta \\ \dot{\theta} &= \omega \end{aligned}$$

there does not exist C^1 feedback controller that asymptotically stabilizes it.

For some systems, classical solution just does not exist.

Ex| Brick on a frictional ramp.



$$\dot{V} = g\sin\theta - \alpha g\cos\theta \cdot \mathrm{sgn}(V)$$

$\mathrm{sgn}(V)$: discontinuous

$$\mathrm{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

α: friction coefficient.

The brick would stop and stays stopped, namely, the brick attains $V=0$ in finite-time and maintains $V=0$.

But this means $\left.\begin{array}{l} V = 0 \\ \dot{V} = 0 \end{array}\right\} \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$. Contradiction.

Hence there is no classical solutions to this system.

We need extensions of solution concepts!

Def (Absolute continuous functions)

The function $\gamma: [a,b] \mapsto \mathbb{R}$ is <u>absolute continuous</u> if, $\forall \varepsilon > 0$, $\exists \delta > 0$, such that, for each finite collection $\{(a_1, b_1), \cdots, (a_n, b_n)\}$ of disjoint open intervals contained in $[a,b]$ with $\sum_{i=1}^{n} (b_i - a_i) < \delta$, it holds $\sum_{i=1}^{n} |\gamma(b_i) - \gamma(a_i)| < \varepsilon$

continuous differentiable $\Longrightarrow$ Lipschitz continuous $\Rightarrow$ Absolute continuous

<u>Caratheodory solution</u>

<u>Roughly speaking</u>, are absolutely continuous curves that satisfy

$$x(t) = x(t_0) + \underbrace{\int_{t_0}^{t} f(x(\tau))d\tau}_{\text{Lebesgue integral}}, \quad t > t_0$$

It relax the requirement that the solution must follow the vector field at all times, namely, the differential equation $\dot{x} = f(x)$ <u>need not to be satisfied on a set of measure zero</u>.

* <u>Fillipov solutions</u>

Relax $\dot{x} = f(x)$ into a diffential inclusion $\dot{x} \in \mathcal{F}(x)$, where $\mathcal{F}: \mathbb{R}^n \mapsto \underbrace{\mathcal{B}(\mathbb{R}^n)}_{\text{collection of all subsets of } \mathbb{R}^n.}$ is a <u>set-valued map</u>. It maps <u>a point</u> to <u>a set</u>.

Def (Fillipov set-valued map)

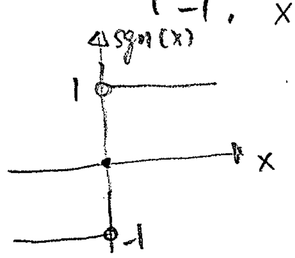
Let $\mathcal{B}(\mathbb{R}^n)$ denote the collection of subsets in $\mathbb{R}^n$. For $f: \mathbb{R}^n \mapsto \mathbb{R}^n$, the Fillipov set-valued map $\mathcal{F}[f]: \mathbb{R}^n \mapsto \mathcal{B}(\mathbb{R}^n)$ is defined by:

$$\mathcal{F}[f](x) := \bigcap_{\delta > 0} \bigcap_{\mu(S) = 0} \overline{co}\{f(B_\delta(x) \setminus S)\}, \quad x \in \mathbb{R}^n.$$

μ: Lebesgue measure "length", "area", "volume"

$\overline{co}$: convex closure/convex hull

Ex $\mathcal{F}(-\text{sgn})(x) = \begin{cases} -1, & x > 0 \\ [-1, 1], & x = 0 \\ 1, & x < 0 \end{cases}$, $\text{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$



Thm (Existence of Fillipov Solution)

If $f: \mathbb{R}^n \mapsto \mathbb{R}^n$ is measurable and locally essentially bounded,

$f^{-1}(E) \in \mathcal{B}(\mathbb{R}^n)$. $\forall E \subset \mathbb{R}^n$ $\qquad \exists C$ s.t $\mu(\{x \in I ; |f(x)| > C\}) = 0$.

then for all $x_0 \in \mathbb{R}^n$, there exists a Fillipov solution with initial condition $x(0) = x_0$

Thm (Uniqueness of Fillipov Solution)

Let $f(x)$ be measurable and locally essentially bounded. Assume $\forall x \in \mathbb{R}^n$, $\exists \gamma_x$ and $\varepsilon > 0$, such that for almost every $x_1, x_2 \in B_\varepsilon(x)$,

$(\xi_1 - \xi_2)^T (x_1 - x_2) \le \gamma_x \|x_1 - x_2\|^2$ holds for all $\xi_1 \in \mathcal{F}[f](x_1)$ and $\xi_2 \in \mathcal{F}[f](x_2)$

Then, $\forall x_0 \in \mathbb{R}^n$, $\dot{x} = f(x), x(0) = x_0$ has a unique Fillipov solution with the initial condition $x(t_0) = x_0$.

Sliding mode control.

Ex | Consider $\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= h(x) + g(x) u \end{aligned}$, where h and g are unknown linear functions.

$g(x) \ge g_0 > 0, \forall x$.

Idea: design a control law that constrains the system motion on the manifold

$s = a_1 x_1 + x_2 = 0$.

On the manifold, the motion reads $\dot{x}_1 = -a_1 x_1$. Hence if we choose $a_1 > 0$, this would guarantee $x_1(t) \to 0 \Rightarrow x_2(t) = -a_1 x_1(t) \to 0$. as $t \to \infty$.

Q: How can we bring the trajectory to the manifold s and remains in there?

$\dot{s} = a_1 \dot{x}_1 + \dot{x}_2 = a_1 x_2 + h(x) + g(x) \cdot u$

Suppose the unknown h and g function satisfies $\left| \frac{a_1 x_2 + h(x)}{g(x)} \right| \le \underbrace{\rho(x)}_{\text{known}}, \forall x \in \mathbb{R}^2$

then choose $V = \frac{1}{2} s^2$

and $\dot{V} = s \cdot \dot{s} = s \cdot (a_1 x_2 + h(x) + g(x) \cdot u) = s[a_1 x_2 + h(x)] + s \cdot g(x) \cdot u$

$\underbrace{\le}_{\text{Cauchy-Swartz}} |s| \cdot |a_1 x_2 + h(x)| + s g(x) u \le |s| \cdot g(x) \rho(x) + s g(x) u$

Taking $u = -\beta(x) \operatorname{sgn}(s)$ (This control will only be used for $s \neq 0$, otherwise we will have to analyze using Fillipov's framework)

and let $\beta(x) \ge \rho(x) + \beta_0$, $\beta_0 > 0$.

This yields: $\dot{V} \le g(x)|s|\rho(x) + g(x)s \cdot u = g(x)|s|\rho(x) + g(x)\cdot s[-\beta(x)\,\mathrm{sgn}(s)]$

$$\le g(x)|s|\rho(x) - g(x)\cdot s[\rho(x)+\beta_0]\,\mathrm{sgn}(s).$$

$$= -g(x)|s|\cdot\beta_0 \le -g_0\cdot\beta_0|s|$$

$\{ s\,\mathrm{sgn}(s) = |s|,\ g(x) \ge g_0 > 0 \}$

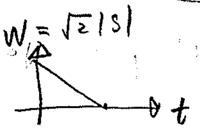
Denote $W = \sqrt{2V} \Rightarrow D^+W = \sqrt{2}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{V}}\cdot\dot{V} = \frac{1}{|s|}\dot{V}$

$$\le -g_0\beta_0$$

$\{ D^+W = \limsup_{h\to 0^+} \frac{W(t+h)-W(t)}{h} \}$

By comparison Lemma, $W(s(t)) \le W(s(0)) - g_0\beta_0 t$.

Not very rigorously speaking, the above inequality implies,



1. the trajectory reaches the manifold $s=0$ in finite time
2. Once the trajectory reaches the manifold $s=0$, it cannot leave it. ($\dot{V} \le -g_0\beta_0|s|$)

Hence, the entire motion consists:

1. reaching the manifold $s=0$ in finite time.
2. sliding on the manifold, whose dynamics is governed by $\dot{x}_1 = -a_1x_1$

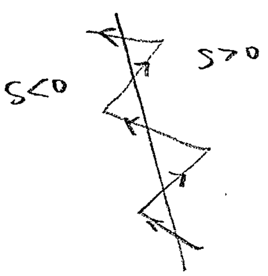
"$s=0$" is called "sliding manifold"

"$u = -\beta(x)\,\mathrm{sgn}(x)$" is called "sliding mode control"

! Note! The above argument is not entirely rigorous. It is only used for illustration of the idea behind sliding mode control. In the following, we will still use similar presentation to illustrate the ideas. If one want to be rigorous, one should use Lyapunov stability theorems as well as Lasalle's invariance principle for Fillipov framework variations. We will not cover this in this lecture.

Chattering problems

Due to imperfect switching devices and delays, chattering is a problem that sliding mode control suffers.



"zig-zag" motion. yields
1. low control accuracy
2. high heat loss
3. constant "switching" would wear out the endurance of the physical system

Ex | sliding mode control for inverted pendulum.

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \overbrace{-(g_0/l)\sin(x_1+\delta_1) - (k_0/m)x_2}^{h(x)} + \overbrace{(1/ml^2)}^{g(x)}u$$

$$u = -k\,\mathrm{sgn}(\underbrace{a_1x_1 + x_2}_{\text{sliding manifold}})$$

Goal: stabilizes the inverted pendulum at $\delta_1 = \pi/2$, and $x_1 = \theta - \delta_1$, $x_2 = \dot{\theta}$

Suppose we know it holds for the physical system that

$0.05 \le m \le 0.2$, $0.9 \le l \le 1.1$, $0 \le k_0 \le 0.05$.

If we chose $a_1 = 1$, and consider the solution in the area $|x_1| \le \pi$, $|s| = |x_1 + x_2| \le \pi$

$$\left|\frac{a_1 x_2 + h(x)}{g(x)}\right| = \left|\frac{x_2 - g_0/l \sin(x_1+\delta_1) - k_0/m\, x_2}{1/ml^2}\right| = \left| l^2 (m-k_0) x_2 - m g_0 l \cos x_1 \right|$$

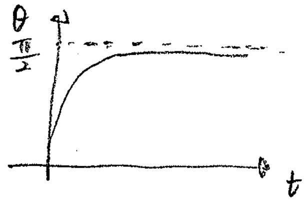
$$\le l^2 |m-k_0|\,|x_2| + m g_0 l\,|\cos x_1| \le l^2 |m-k_0| \cdot (2\pi) + m g_0 l \le 3.68$$

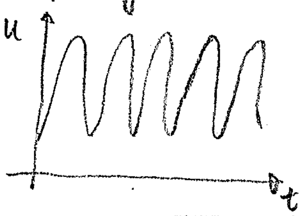
$|x_1 + x_2| \le \pi$
$\Rightarrow |x_2| - |x_1| \le \pi \Rightarrow |x_2| \le |x_1| + \pi \le 2\pi$

This is the $\rho(x)$ in the previous example.

Therefore, if we choose $k = 4 > 3.68$, we should be able to stabilizes the system using the sliding mode control.

But we will still be facing the problem of chattering!





To mitigate the chattering, we divide the control into two parts: continuous & switching components and reduce the amplitude of "switching"

Design the control as $u = -\dfrac{a_1 x_2 + \hat{h}(x)}{\hat{g}(x)} + v$

The sliding manifold would hold that

$$\dot{s} = a_1 \dot{x}_1 + \dot{x}_2 = a_1 x_2 + h(x) + g(x)\left[-\frac{a_1 x_2 + \hat{h}(x)}{\hat{g}(x)} + v\right]$$

$$= \underbrace{a_1\left[1 - \frac{g(x)}{\hat{g}(x)}\right] x_2 + h(x) - \frac{g(x)}{\hat{g}(x)} \hat{h}(x)}_{\delta(x)} + g(x) v$$

Before:
$\dot{s} = a_1 \dot{x}_1 + \dot{x}_2$
$= a_1 x_2 + h(x) + g(x) \cdot u$
and we were looking at the bound $\left|\dfrac{a_1 x_2 + h(x)}{g(x)}\right| \le \rho_1(x)$

Looking at the bound $\left|\dfrac{\delta(x)}{g(x)}\right| \le \rho_2(x)$

and design the "true" control as the old way.

Since it is foreseenable that $|\delta(x)|$ is small, (it is seen as error), the upper bound $\rho_2(x)$ is much smaller than the old upper bound $\rho_1(x)$.

Note that if we design the "true" controller $v = -\beta(x)\,\mathrm{sgn}(s)$ the old way, $\beta(x)$ is chosen such that $\beta(x) \ge \rho_2(x) + \beta_0$ $\quad \beta_0 > 0$

This means that $\beta(x)$ does not have to be that large since $\rho_2(x)$ is smaller compared to $\rho_1(x)$. This actually suppresses the amplitude of "switching".

Ex] In the previous pendulum example, if we choose the "norminal" parameters as:

$\hat{m} = 0.125$, $\hat{l} = 1$, $\hat{k}_0 = 0.025$. then

$$\left|\frac{\delta(x)}{g(x)}\right| = \left|\left\{a_1\left[1-\frac{\hat{m}\hat{l}^2}{ml^2}\right]x_2 - g_0/l \cos x_1 - k_0/m\, x_2 + \frac{\hat{m}\hat{l}^2}{ml^2}\left(g_0/\hat{l} \cos x_1 + \hat{k}_0/\hat{m}\, x_2\right)\right\}\cdot ml^2\right|$$

$$= \left|(a_1 ml^2 - a_1 \hat{m}\hat{l}^2)x_2 - g_0 m l \cos x_1 - k_0 l^2 x_2 + g_0 \hat{m}\hat{l} \cos x_1 + \hat{k}_0 \hat{l}^2 x_2\right|$$

$$= \left|(a_1 ml^2 - a_1 \hat{m}\hat{l}^2 - k_0 l^2 + \hat{k}_0\hat{l}^2) x_2 - g_0(ml - \hat{m}\hat{l}) \cos x_1\right|$$

We choose $a_1 = 1$ and consider the things that happen within $|x_1| \le \pi$, $|S| = |x_1 + x_2| \le \pi$ and $0.05 \le m \le 0.2$, $0.9 \le l \le 1.1$, $0 \le k_0 \le 0.05$, we can estimate that $\left|\frac{\delta(x)}{g(x)}\right| \le 1.83$.

Hence we can choose $\beta(x) = 2 \geq 1.83$.

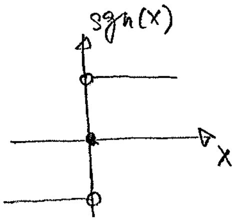
The overall control $u = -\frac{x_2 + \hat{h}(x)}{\hat{g}(x)} + v = -0.1 x_2 + 1.2263 \cos x_1 - 2\,\mathrm{sgn}(S)$

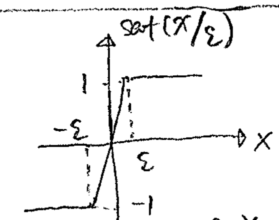
this is smaller than the old control $-4\,\mathrm{sgn}(S)$

If you simulate this, the oscillation magnitude in u due to chattering would be much smaller than the old control.

Idea: Replace $\mathrm{sgn}(\cdot)$ by $\mathrm{sat}(\cdot)$. ← saturation

But the slope is very steep.





Take $u = -\beta(x)\,\mathrm{sgn}(S) \Longrightarrow u = -\beta(x)\,\mathrm{sat}\left(\frac{S}{\varepsilon}\right)$,

"Continuous" sliding mode control.

$$\mathrm{sat}(x) = \begin{cases} x, & \text{if } |x| \le 1 \\ \mathrm{sgn}(x), & \text{if } |x| > 1 \end{cases}$$

$V = \frac{1}{2}S^2 \Rightarrow \dot{V} = S\cdot\dot{S} = S(a_1 x_1 + h(x) + g(x)\cdot u)$

Since $u = -\beta(x)\,\mathrm{sat}\left(\frac{S}{\varepsilon}\right) = -\beta(x)\cdot\mathrm{sgn}\left(\frac{S}{\varepsilon}\right) = -\beta(x)\,\mathrm{sgn}(S)$ for $|S| \ge \varepsilon$ → "reaching" phase.

Therefore, the old analysis using $V(x) = \frac{1}{2}S^2$ still holds,

namely, $\dot{V} \le -g_0\beta_0|S|$ holds for $|S| \ge \varepsilon$.

This means that if the trajectory starts at a point such that $|S(0)| = |a_1 x_1(0) + x_2(0)| \ge \varepsilon$, $|S(t)|$ would still goes down since $\dot{V} \le -g_0\beta_0\cdot\varepsilon < 0$ until it hits the set $|S| \le \varepsilon$ in finite time. and remains in there.

Within $|S| \le \varepsilon$, we have $\dot{x}_1 = x_2 = -a_1 x_1 + S$. ($S = a_1 x_1 + x_2$)

Let $V_1 = \frac{1}{2}x_1^2 \Rightarrow \dot{V}_1 = x_1\cdot\dot{x}_1 = x_1(-a_1 x_1 + S) = -a_1 x_1^2 + S\cdot x_1 \le -a_1 x_1^2 + |S|\,|x_1|$ ($|S| \le \varepsilon$)

$$\le -a_1 x_1^2 + \varepsilon|x_1| \le -(1-\theta_1)a_1 x_1^2, \quad \forall |x_1| \ge \frac{\varepsilon}{a_1\theta_1}, \quad 0 < \theta_1 < 1$$

$$\Rightarrow \dot{V}_1 \le -(1-\theta_1)a_1\frac{\varepsilon^2}{a_1^2\theta_1^2} = -\frac{1-\theta_1}{a_1\theta_1^2}\varepsilon^2$$

This means that the trajectory would reach the set $\Omega_\varepsilon = \left\{|x_1| \le \frac{\varepsilon}{a_1\theta_1}, |S| \le \varepsilon\right\}$ in finite-time.

The idea is not stabilize the origin, but let the trajectory to be bounded in a small set. (the bound could be reduced by reducing ε).

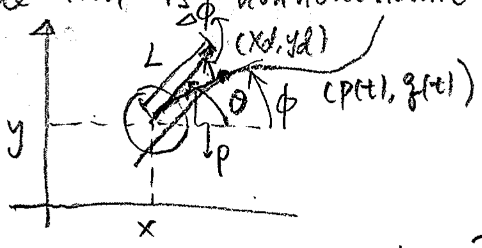
But the "price to pay" to reduce ε is that $sat(\frac{s}{\varepsilon})$ would be more and more like $sgn(\cdot)$ function, and cause the chattering problem again.

So it's a trade-off between control accuracy and chattering.

Trajectory tracking for nonholonomic system.

Consider the unicycle that is nonholonomic.

$$\dot{x} = v\cos\theta$$
$$\dot{y} = v\sin\theta$$
$$\dot{\theta} = \omega$$



The reference trajectory to track is given by $\begin{matrix} x_d(t) = p(t) \\ y_d(t) = q(t) \end{matrix}$, $0 \le t \le T$.

Suppose $\dot{p}(t)^2 + \dot{q}(t)^2 \neq 0$, $\forall t \in [0, T]$ (this means the reference trajectory stays still at some time t).

In order to track it, we need

$$\begin{matrix} \dot{x}_d = V_d \cos\theta_d \\ \dot{y}_d = V_d \sin\theta_d \end{matrix} \Rightarrow V_d = \sqrt{\dot{p}(t)^2 + \dot{q}(t)^2}, \quad \theta_d = \operatorname{atan2}(\dot{p}(t), \dot{q}(t)).$$

$$\Rightarrow \omega_d = \dot{\theta}_d = \frac{\ddot{q}(t)\dot{p}(t) - \ddot{p}(t)\dot{q}(t)}{V_d(t)^2}$$

This is an open-loop control. We want a closed-loop control.

Find a reference point (X_L, Y_L) which is in front of the unicycle's orientation with a distance of L, namely,

$$\begin{cases} X_L = x + L\cos\theta \\ Y_L = y + L\sin\theta \end{cases} \Rightarrow \begin{matrix} \dot{X}_L = v\cos\theta - L\sin\theta \cdot \omega \\ \dot{Y}_L = v\sin\theta + L\cos\theta \cdot \omega \end{matrix} \Rightarrow \begin{bmatrix} \dot{X}_L \\ \dot{Y}_L \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -L\sin\theta \\ \sin\theta & L\cos\theta \end{bmatrix}}_{\text{is always invertible.}} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\frac{1}{L}\sin\theta & \frac{1}{L}\cos\theta \end{bmatrix} \begin{bmatrix} \dot{X}_L \\ \dot{Y}_L \end{bmatrix}$$

If we choose $\dot{X}_L$ and $\dot{Y}_L$ as

$$\begin{matrix} \dot{X}_L = -k(X_L - X_d) + \dot{X}_d \\ \dot{Y}_L = -k(Y_L - Y_d) + \dot{Y}_d \end{matrix} \Rightarrow \begin{matrix} \dot{X}_L - \dot{X}_d = -k(X_L - X_d) \\ \dot{Y}_L - \dot{Y}_d = -k(Y_L - Y_d) \end{matrix} \Rightarrow \text{the error } \begin{bmatrix} X_L - X_d \\ Y_L - Y_d \end{bmatrix} \text{ converges to zero.}$$

$$\Rightarrow \begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\frac{1}{L}\sin\theta & \frac{1}{L}\cos\theta \end{bmatrix} \cdot \begin{bmatrix} -k(X_L - X_d) + \dot{X}_d \\ -k(Y_L - Y_d) + \dot{Y}_d \end{bmatrix}$$

$$\begin{aligned} \Rightarrow v &= \cos\theta[-k(X_L - X_d) + \dot{X}_d] + \sin\theta[-k(Y_L - Y_d) + \dot{Y}_d] \\ &= \cos\theta[-k(X + L\cos\theta - X_d) + V_d\cos\theta_d] + \sin\theta[-k(Y + L\sin\theta - Y_d) + V_d\sin\theta_d] \\ &= -k[\cos\theta(X + L\cos\theta - X_d) + \sin\theta(Y + L\sin\theta - Y_d)] + V_d\cos(\theta_d - \theta) \end{aligned}$$

$$= -k\left[L\cos^2\theta + L\sin^2\theta + \cos\theta(x - x_d) + \sin\theta(y - y_d)\right] + V_d\cos(\theta_d - \theta)$$

$$= -k(L + \cos\theta \cdot \rho \cdot \cos\phi + \sin\theta \cdot \rho\sin\phi) + V_d\cos(\theta_d - \theta)$$

$$= -k(L - \rho\cos\Delta\phi) + V_d\cos(\theta_d - \theta)$$

$$\omega = -\frac{1}{L}\sin\theta\left[-k(x_L - x_d) + \dot{x}_d\right] + \frac{1}{L}\cos\theta\left[-k(y_L - y_d) + \dot{y}_d\right]$$

$$= -\frac{1}{L}\sin\theta\left[-k(x + L\cos\theta - x_d) + V_d\cos\theta_d\right] + \frac{1}{L}\cos\theta\left[-k(y + L\sin\theta - y_d) + V_d\sin\theta_d\right]$$

$$= -\frac{1}{L}\sin\theta\left[-k(x + L\cos\theta - x_d)\right] + \frac{1}{L}\cos\theta\left[-k(y + L\sin\theta - y_d)\right] + \frac{V_d}{L}\sin(\theta_d - \theta)$$

$$= \frac{k}{L}\sin\theta \cdot x + \cancel{k\sin\theta\cos\theta} - \frac{kx_d}{L}\sin\theta - \frac{k}{L}\cos\theta \cdot y - \cancel{k\sin\theta\cos\theta} + \frac{ky_d}{L}\cos\theta + \frac{V_d}{L}\sin(\theta_d - \theta)$$

$$= \frac{k}{L}\sin\theta \cdot \underbrace{(x - x_d)}_{\rho\cos\phi} - \frac{k}{L}\cos\theta\,\underbrace{(y - y_d)}_{\rho\sin\phi} + \frac{V_d}{L}\sin(\theta_d - \theta)$$

$$= \frac{k\rho}{L}\sin(\underbrace{\theta - \phi}_{\Delta\phi}) + \frac{V_d}{L}\sin(\theta_d - \theta)$$