Nonlinear Control Theory

Lecture 3. Lyapunov Stability

Last time

- Second-order systems (phase-plots) } Qualitive behaviour of dynamical systems
- Limit cycles

Today

- Concepts of stability
- Anaysis via linearization

Consider a time-varying dynamic nonlinear system

$$\dot{x} = f(t, x), \quad x \in \mathbb{R}^n, \ t \geq 0 \qquad (*)$$

Recall the definition of equilibrium pt :
if $f(t, x^*) = 0, \ \forall t \geq 0$, then x^* is an equilibrium.

Suppose f is piecewise continuous in t, and "Lipschitz" in x.
$\Rightarrow$ the solution $x(t; x_0, t_0)$ is "properly" defined.

! The stability of system is always w.r.t. an equilibrium pt!

Conventionally, we work with equilibrium at the origin.

If for a system $\dot{x} = f(t, x)$, $x^* \neq 0$ is the origin.
do a coordinate change $z = x - x^*$, $\Rightarrow$ $\dot{z} = \dot{x} = f(t, x)$
$= f(t, z + x^*) := \tilde{f}(t, z)$
then $z = 0$ is the equilibrium of $\dot{z} = \tilde{f}(t, z)$.

Def (stability)

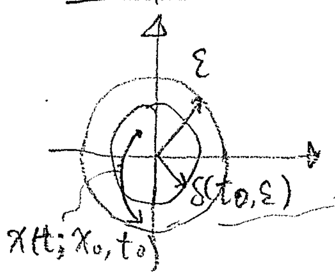
Consider system $(*)$, and assume $x = 0$ is an equilibrium.

(1). $x = 0$ is stable, if for any $\varepsilon > 0$ and $t_0 \geq 0$, there exists a $\delta(t_0, \varepsilon) > 0$, such that $\|x_0\| < \delta(t_0, \varepsilon) \Rightarrow \|x(t; x_0, t_0)\| < \varepsilon, \ \forall t \geq t_0 \geq 0$

(2) $x = 0$ is uniformly stable, if it is stable, and the above δ is indepentent of t_0, i.e. $\delta(t_0, \varepsilon) = \delta(\varepsilon)$

(3) $x = 0$ is unstable if it is not stable.

Interpretation



if this $S(t_0,\varepsilon)$ does not depend on the starting time t_0, then it is uniform.

Next, consider the convergence of solutions.

Def (Attractive)

Consider the system (*) and assume $x=0$ is an equilibrium.

(1) $x=0$ is attractive, if for each $t_0 \geq 0$, and $\varepsilon > 0$, there exists a $\eta(t_0) > 0$ and $T(\varepsilon, x_0, t_0) > 0$, such that

$$\|x_0\| < \eta(t_0), \Rightarrow \|x(t;x_0,t_0)\| < \varepsilon, \forall t > T(\varepsilon, x_0, t_0)$$

(2) $x=0$ is uniformly attractive, if for each $\varepsilon > 0$, there exists a $\eta > 0$ and $T(\varepsilon) > 0$, such that

$$\|x_0\| < \eta \Rightarrow \|x(t;x_0,t_0)\| < \varepsilon, \forall t > T(\varepsilon)$$

☆ uniform convergence on x_0 and t_0.

! Does "stability" implies "attractive"? NO!

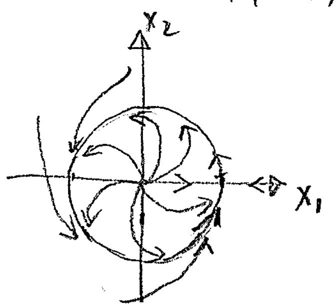
Does "attractive" implies "stability"? NO!

Ex ① $\dot{x}_1 = x_2$, $\dot{x}_2 = -w^2 x_1$, $x(0) = \begin{bmatrix} x_{0,1} \\ 0 \end{bmatrix}$ $\Rightarrow \begin{cases} x_1(t) = x_{0,1}\cos wt \\ x_2(t) = -x_{0,2}\sin wt \end{cases}$ $\Rightarrow$ Not attractive. the limit when $t \to \infty$ does not even exist.

② $\dot{x}_1 = x_1 + x_1 x_2 - (x_1 + x_2)(x_1^2 + x_2^2)^{\frac{1}{2}}$

$\dot{x}_2 = x_2 - x_1^2 + (x_1 - x_2)(x_1^2 + x_2^2)^{\frac{1}{2}}$ $\xrightarrow{\text{polar coordinates}}$ $\dot{\rho} = \rho(1-\rho)$, $\dot{\theta} = \rho(1-\cos\theta)$

$x_1 = 1$, $x_2 = 0$ is an equilibrium



$\Rightarrow$ Attractive but not stable!

Finally, we consider asymptotic stability.

Def: Consider system $(*)$ and assume $x=0$ is an equilibrium.

(1) $x=0$ is asymptotically stable if $x=0$ is stable and attractive.

(2) $x=0$ is uniformly asymptotically stable if $x=0$ is uniformly stable & uniformly attractive.

Ex (1) Consider $\dot{x} = -(2+\sin(t))x$, $x(t_0)=x_0$, $t_0 \geq 0$

$$x(t) = x_0 e^{2t_0 - \cos(t_0)} e^{-2t+\cos(t)}$$

$$\Rightarrow |x(t)| \leq \left| x_0 e^{2t_0-2t} e^{-\cos(t_0)+\cos(t)} \right| \leq |x_0| e^2 \cdot e^{-2(t-t_0)}$$

Stable? Yes, since $e^{-2(t-t_0)}$ is monotonously decreasing w.r.t t.

For any ε, you can let $|x_0| < \varepsilon / e^2$

uniformly stable? Yes.

attractive? Yes, $x(t) \to 0$ as $t \to \infty$.

uniformly attractive? Yes, η can be arbitrary positive number.

$\Rightarrow$ uniformly asymptotically stable

(2) Consider $\dot{x} = -\frac{1}{1+t} x$.

The solution is $x(t) = \frac{1+t_0}{1+t_0+t} x_0$

Asymptotically stable. But not uniformly asymptotically stable.
because it is not asymptotically attractive.

Def: The system $(*)$ is said to be exponentially stable at $x^* = 0$ if there exists positive numbers $a>0$, $b>0$ and a neighbourhood N_0 of the origin, such that $\|x(t; x_0, t_0)\| \leq a \|x_0\| e^{-b(t-t_0)}$

$$t \geq t_0 \geq 0, \quad x_0 \in N_0.$$

Q: Does exponentially stability implies uniform asymptotic stability?

(Yes).

For autonomous systems $\dot{x} = f(x)$. $(**)$, $\boxed{f: \text{Lipschitz}}$ stable = uniformly stable
attractive = uniformly attractive.

Stability analysis for autonomous system is much simpler. We start with the simple case.

We have the "well known" simplified version of stability theorem regarding Lyapunov functions:

Thm: Let $x=0$ be an equilibrium point for $(**)$ and $D \subset \mathbb{R}^n$ be a domain containing $x=0$. Let $V: D \to \mathbb{R}$ be continuously differentiable function such that

① continuous.
② derivative is also continuous.

$$V(0) = 0 \ \& \ V(x) > 0 \text{ in } D \setminus \{0\}$$

$$\dot{V}(x) \le 0 \text{ in } D.$$

then, $x=0$ is stable. Moreover, if $\dot{V}(x) < 0$ in $D \setminus \{0\}$, then $x=0$ is asymptotically stable.

proof: Given any $\varepsilon > 0$, choose $0 < r \le \varepsilon$, s.t. $B_r = \{x \in \mathbb{R}^n \mid \|x\| \le r\} \subseteq D$.

Let $\alpha := \min_{x \in \partial B_r} V(x)$. Since $V(x) > 0$ in $D \setminus \{0\}$. $\alpha > 0$.

Take $0 < \beta < \alpha$, and define $\Omega_\beta = \{x \in B_r \mid V(x) \le \beta\}$.

It is clear that $\Omega_\beta \subseteq B_r$, and suppose $x^* \in \Omega_\beta$ is on ∂B_r, then $V(x^*) \ge \alpha$ but $\beta < \alpha$, hence Ω_β is in the interior of B_r.

Since $\dot{V}(x(t)) \le 0$ in D, and $\Omega_\beta \subset B_r \subseteq D$, if $x(0) \in \Omega_\beta$, then

$$\Rightarrow V(x(t)) = V(x(0)) + \int_0^t \underbrace{\dot{V}(x(s))}_{\le 0} ds \le V(x(0)) \le \beta$$

Hence Ω_β is an invariant set (if a trajectory starts there, it would never leave)

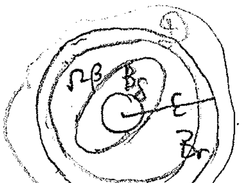
As $V(x)$ is continuous, there exists $\delta > 0$, s.t. $\|x\| < \delta \Rightarrow V(x) < \beta$.

Thus $B_\delta \subseteq \Omega_\beta \subset B_r$.

$$\Rightarrow x(0) \in B_\delta \Rightarrow \underbrace{x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta}_{\text{invariant}} \Rightarrow x(t) \in B_r.$$

Hence $\|x(0)\| < \delta \Rightarrow \|x(t)\| < r \le \varepsilon$. $\forall t \ge 0$ Stability proved.

If $\dot{V}(x) < 0$, we want to show asymptotic stability. What is left to show is the attractive.



Namely, $x(t) \to 0$ as $t \to \infty$.

Since $V(0) = 0$ and $V(x) > 0$ in $D \setminus \{0\}$, it is sufficient to show.

$$V(x(t)) \to 0 \text{ as } t \to \infty$$

Since $V(x(t))$ is bounded from below by 0, monotonically decreasing,

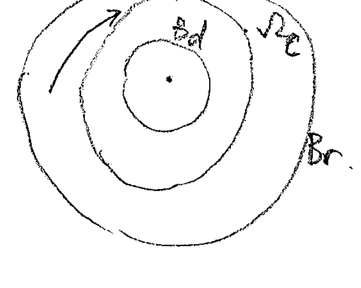
$$V(x(t)) \to c \text{ as } t \to \infty.$$

(We want to show this c is actually zero, contradiction proof)

$\boxed{\text{Suppose } c > 0}$

Since $V(x)$ is continuous, there exists a $d > 0$ such that $B_d \subseteq \Omega_c$, where $\Omega_c = \{x \in B_r \mid V(x) \leq c\}$

Therefore, $\lim_{t \to \infty} V(x(t)) = c \Rightarrow$ the trajectory lies out of B_d $\forall t \geq 0$.



Let $-\gamma = \underbrace{\max_{d \leq \|x\| \leq r}}_{\text{compact}} \dot{V}(x)$, where $-\gamma < 0$.

compact, $\dot{V}(x)$ is continuous. hence $-\gamma$ exists.

$$V(x(t)) = V(x(0)) + \int_0^t \underbrace{\dot{V}(x(s))}_{\leq -\gamma} ds \leq V(x(0)) - \gamma t$$

$\to -\infty$ as $t \to \infty$. contradicts to $V(x) > 0$ $\forall x \in D \setminus \{0\}$

- Level set $\Omega_c = \{x \in \mathbb{R}^n \mid V(x) \leq c\}$. The idea is if the trajectory go inside the level set, it never goes out, c is shrinking to zero.
- The above theorem is a sufficient condition for stability.
- Domain of attraction: $\{x_0 \in \mathbb{R} \mid \lim_{t \to \infty} x(t; x_0) = 0\}$

Finding domain of attraction is analytically difficult, but can be approximated by level sets Ω_c if the system is asymptotically stable. But it is a <u>conservative approximation</u>.

<u>Thm</u> (Barbashin–Krasovskii)

Let $x = 0$ be an equilibrium for (**). Let $V: \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function, such that

$$V(0) = 0 \text{ and } V(x) > 0, \ \forall x \neq 0.$$

$\|x\| \to \infty \Rightarrow V(x) \to \infty$. $\leftarrow$ simple case of <u>radially unbounded</u>

$$\dot{V}(x) < 0, \ \forall x \neq 0$$

then $x = 0$ is <u>globally</u> asymptotically stable.

Thm The equilibrium pt $x=0$ of $\dot{x}=Ax$ is stable if and only if all eigen values satisfy $Re(\lambda_i)\leq 0$, and for every eigen value on the imaginary axis, the algebraic multiplicity is equal to the geometric multiplicity.

The equilibrium pt is (globally) asymptotically stable iff all eigen values satisfy $Re(\lambda_i)<0$.

(Hint: Consider the Jordan block. $e^{At}=Pe^{Jt}P^{-1}=\sum_{i=1}^{r}\sum_{k=1}^{m_i} t^{k-1}e^{\lambda_i t}R_{ik}$).

Consider the non-linear system $\dot{x}=f(x)$, $f:\mathcal{D}\mapsto\mathbb{R}^n$ is a continuously differentiable map.

Let $x=0$ be the equilibrium. By mean value theorem,

$$\dot{x}_i=f_i(x)=\underbrace{f_i(0)}_{=0}+\frac{\partial f_i}{\partial x}(z_i)x,\quad z_i\in[0,x]$$

$$\Rightarrow \dot{x}_i=\frac{\partial f_i}{\partial x}(0)\,x+\underbrace{\left[\frac{\partial f_i}{\partial x}(z_i)-\frac{\partial f_i}{\partial x}(0)\right]x}_{g_i(x)}$$

$$\Rightarrow \dot{x}=Ax+g(x)$$

$A = \frac{\partial f}{\partial x}(0)$

Stability analysis by linearization

We hope the systems would behave similarly around the origin.

Cauchy-Schwarz

Note that $|g_i(x)|\leq\left|\frac{\partial f_i}{\partial x}(z_i)-\frac{\partial f_i}{\partial x}(0)\right|\|x\|$

$\Rightarrow \frac{\|g(x)\|}{\|x\|}\to 0$ as $\|x\|\to 0$ since $\frac{\partial f}{\partial x}$ is continuous, and $0\leq z_i\leq x$

Thm Let $x=0$ be an equilibrium point for the nonlinear system $\dot{x}=f(x)$, $f:\mathcal{D}\mapsto\mathbb{R}^n$ is continuously differentiable and $\mathcal{D}$ is a neighbourhood of the origin.

Let $A=\left.\frac{\partial f}{\partial x}(x)\right|_{x=0}$.

Then. ① The origin is asymptotically stable if $Re(\lambda_i(A))<0$.

② The origin is unstable if $Re(\lambda_i(A))>0$ holds for at least one i.

Lemma A matrix A is Hurwitz, iff for any given positive definite symmetric matrix Q, there exists a positive definite matrix P that satisfies $A^TP + PA = -Q$. Moreover, the solution P is unique.

proof. Sufficiency: Construct Lyapunov function $V(x) = x^TPx$.

$$\dot V(x) = \dot x^T P x + x^T P \dot x = x^T A^T P x + x^T P A x = -x^T Q x$$

Since Q is positive definite, $\dot V(x) = -x^TQx < 0 \quad \forall x$.

$\Rightarrow$ A is Hurwitz.

Necessity: Assume $Re(\lambda_i(A)) < 0 \ \forall i$, define $P = \int_0^\infty e^{A^Tt} Q e^{At} dt$.

the integrand is a sum of terms like $t^{k-1} e^{\lambda_i t}$, $Re(\lambda_i) < 0$,

Hence the integral does not blow-up. and $P \succ 0$.

Plug P into the Lyapunov equation:

$$LHS: A^TP + PA = \int_0^\infty A^T e^{A^Tt} Q e^{At} dt + \int_0^\infty e^{A^Tt} Q e^{At} A \, dt$$

$$= \int_0^\infty \frac{d}{dt}(e^{A^Tt}) Q e^{At} dt + \int_0^\infty e^{A^Tt} Q \frac{d}{dt}[e^{At}] dt$$

$$= \int_0^\infty \frac{d}{dt}[e^{A^Tt} Q e^{At}] dt = e^{A^Tt} Q e^{At} \Big|_0^\infty = -Q = RHS$$

Hence P is the solution to Lyapunov equation.

what remains to show is the uniqueness. Suppose $P' \neq P$,

then $A^T(P-P') + (P-P')A = 0$

$$\Rightarrow e^{A^Tt} A^T (P-P') e^{At} + e^{A^Tt}(P-P') A e^{At} = 0$$

$$\Rightarrow \frac{d}{dt}\{e^{A^Tt}(P-P')e^{At}\} = 0 \Rightarrow e^{A^Tt}(P-P')e^{At} \equiv \text{const} \ \forall t.$$

Since $e^{A0} = I \Rightarrow P - P' = e^{A^Tt}(P-P')e^{At} \to 0$ as $t \to \infty$

$$\Rightarrow P - P' = 0$$

proof for linearization (We only show the first part).

Suppose A is Hurwitz, $\Rightarrow \exists P>0 \ \forall Q$ that satisfies the Lyap. eq.

$V(x) = x^T P x \Rightarrow \dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = f(x)^T P x + x^T P f(x)$.

Since $\dot{x} = f(x) = Ax + g(x)$

$$\dot{V}(x) = (Ax+g(x))^T P x + x^T P(Ax+g(x)) = x^T(A^T P + PA)x + 2x^T P g(x)$$
$$= -x^T Q x + 2x^T P g(x)$$

Recall that $\frac{\|g(x)\|}{\|x\|} \to 0$ as $\|x\| \to 0$.

Hence for any $\gamma > 0$, $\exists r > 0$, s.t. $\|g(x)\| < \gamma \|x\| \ \forall \|x\| < r$.

Therefore, $\dot{V}(x) = -x^T Q x + 2x^T P g(x) \leq -x^T Q x + 2\|x\| \cdot \|P\| \cdot \|g(x)\|$
$$< -x^T Q x + 2\gamma \|x\|^2 \cdot \|P\|, \ \forall \|x\| < r$$

Note that $x^T Q x \geq \lambda_{min}(Q) \|x\|^2$

$\Rightarrow \dot{V}(x) < -\lambda_{min}(Q)\|x\|^2 + 2\gamma \|x\|^2 \|P\|, \ \forall \|x\| < r$

$\to$ choose γ such that $-\lambda_{min}(Q) + 2\gamma\|P\| < 0 \Rightarrow \dot{V}(x) < 0$

$\Rightarrow \dot{x} = f(x)$ is stable.