## Monlinear Control Theory Lecture 3. Lyapunor Stability.

) Last time

· Second-order systems (phase-pluts)} Qualitive behaviour . Limit cycles

Today

· Concepts of stability.
· Anaysis via linearization

Consider a time-varying dynamic nonlinear system  $\vec{x} = f(t, x)$ .  $x \in \mathbb{R}^n$ , t > 0 (\*)

Recall the definition of equilibrium pt ? if f(t, x\*)=0, yt>0, then x\* is an equilibrium.

Suppose of is piecewise continuous in t, and Lipschitz" in A. =) the solution X(t; xo,to)is properly defined.

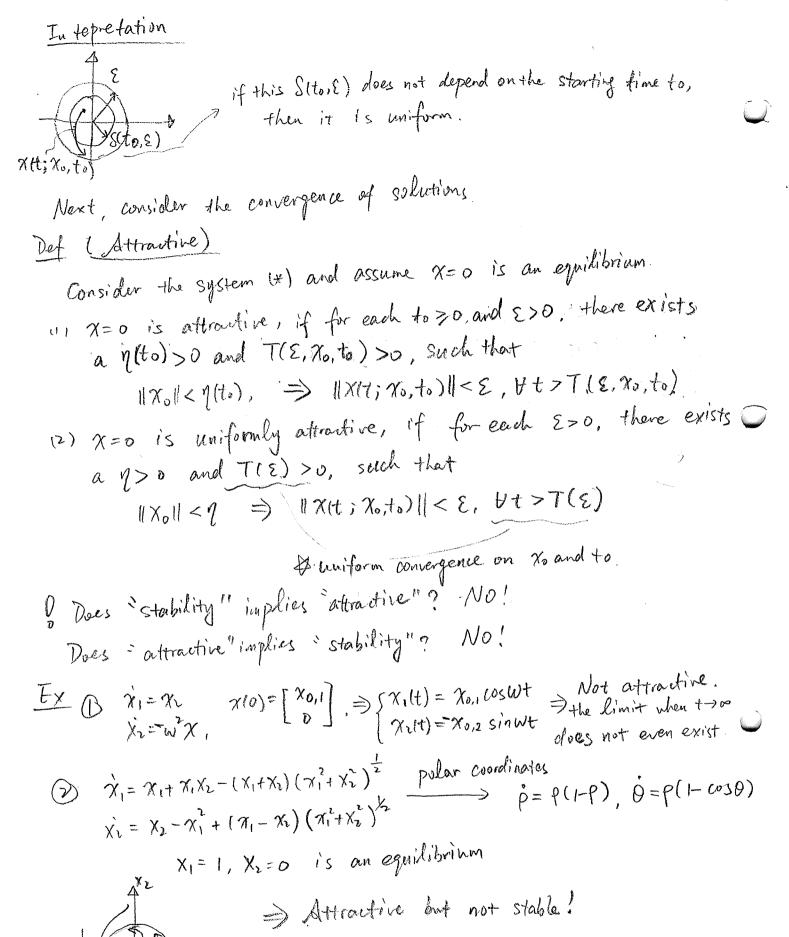
of the stability of system is always w.r.t. an equilibrium pt! Conventionally, we work with equilibrium at the origin.

If for a system  $\hat{\chi} = f(t, x)$ ,  $\chi \neq 0$  is the origin. do a coordinate change  $Z = \chi - \chi^*$ ,  $\Rightarrow \dot{z} = \dot{\chi} = f(t, \chi)$ then z=0 is the equilibrium of  $z=f(t,z+x^*):=f(t,z)$ 

Def (Stability)

Consider system (\*), and assume x=0 is an equilibrium.

- (1)  $\chi=0$  is stable, if for any  $\epsilon>0$  and  $t_0>0$ , there exists a  $S(t_0,\epsilon)>0$ , such that  $||\chi_0||< S(t_0,\epsilon)=$   $||\chi(t;\chi_0,t_0)||<\epsilon$ ,  $\forall t>t>>0$
- (2) X=0 is uniformly stable, if it is stable, and the above & is independent of to, i.e. S(to, E) = S(E)
- (3)  $\gamma = 0$  is unstable if it is not stable.



Finally, we consider asymptotic stability. Def: Consider system (\*) and assume x=0 is an equilibrium. (1) x=0 is asymptotically stable if x=0 is stable and attractive (2) X=0 is uniformly asymptotically stable if X=0 is uniformly Stable & uniformly attractive. Ex (1) Consider  $\dot{\chi} = -(2+\sin(t))\chi$ ,  $\chi(t) = \chi_0$ ,  $t_0 \ge 0$   $\chi(t) = \chi_0 e^{2t_0 - \cos(t_0)} e^{-2t + \cos(t)}$   $\Rightarrow |\chi(t)| \le |\chi_0 e^{2t_0 - 2t} e^{-\cos(t_0) + \cos(t)}| \le |\chi_0| e^{2t_0 - 2(t_0 - t_0)}$ Stable? Yes, since e2(t-to) is monotonously decreasing wirtt. For any E, you can let 1Xol < E/e2 uniformly stable? Yes. attractive? Tes, x(t) -> 0 ors +> 00. uniformly attractive? Yes I can be arbitrary positive number.

=) uniformly asymptotically stable (2) Consider  $\dot{x} = -\frac{1}{1+t} \chi$ . The solution is  $\chi(t) = \frac{1+t_0}{1+t_0+t} \chi_0$ Asymptotically stable. But not uniformly asymptotically stable because it is not asymptotically attractive. Det: The system (x) is said to be exponentially stable at xx=0 if there exists positive numbers a>0, b>0 and a neighbourhood No of the origin, such that 11x(t; 70, to) | all xolle b(+-to) t > to > 0, 90 ∈ No. Q = Does exponentially stability implies uniform asymptotic stability?

For antonomous systems  $\chi = f(x)$ , stable = uniformly stable Stability analysis for autonomous system is much simpler. We start with the simple rase the simple case. We have the "well known" simplified version of stability theorem negarding Lyapunov functions: This : Let X=0 be an equilibrium point for (\*\*) and DCR" be a domain containing x=0, Let V: D > R be continuously differentiable function such that @ alenvative is also continuous. V(0) = 0 & . V(x)>0 in \$\)\{0\}  $V(X) \leq 0$  in  $\mathcal{D}$ . then, x=0 is stable. More over, if V(x) <0 in D\{0}, then 7=0 is asymptotically stable. Proof: Given any 2 >0, choose or 18, s.t. Br= [x+R" | lix11 srj = D Let d:= min V(x). Since V(x) >0 in D/fof. d>0. Taloeo<β < d. and define Do = {x ∈ Br | V(x) ≤ β}. It is clear that  $\Omega_{\beta} \subseteq B_r$ , and suppose  $x \in \Omega_{\beta}$  is on  $\partial B_r$ , then  $V(x^*) \ge d$ but B Kd, hence Op is in the interior of Br. Since V(MA)) So in D, and D&CBr SD, if X10) & DB, then > V(x(+)) = V(x(0))+ f+ v(x(0)) ols = V(x(0)) = B Hence DB is an invariant set ( if a trajectory starts there, it would never leave) As V(x) is continuous, there exists S>0, s.t.  $||x||< S \Rightarrow V(x)< \beta$ . Thus Bs = NB CBr.. => XIO) EBS => XIO) EDB => XH) EBr. Hona. 11×10) 11<8 => 11×11) 11< r < E. VIZO Stability proved. If V(X)<0, we want to show asymptotic stability. What is left to show is the attractive.

<ul> <li>Level set Rc = {X+R^  V(X) ≤ C}. The idea is if the trajectory go inside the level set, it never goes out, C is shrinking to zero.</li> <li>The above theorem is a sufficient condition for stability.</li> <li>Domain of attraction: {X₀∈R  lim X(t; N₀) = 0}</li> <li>Finding domain of attraction is analytically difficult, but can be approximated by level sets Rc if the system is asymptotically stable. But it is a conservative approximation.</li> <li>Them (Barbashin-Krasovskii)</li> <li>Lt X=0 be an equilibrium for (*x). Let V:R→R be a continuou differentiable function. Such that</li> <li>V(v) = 0 and V(x) &gt; 0. ∀x ≠ 0.</li> <li>Simple case of vacinty unbounded</li> </ul>		
Since V(0) = 0 and V(x) > 0 in £)(10), it is sufficient to show the V(xxxx) > 0 as (-> 0).  Since V(xxxxx) = 0 and from below by 0, monotonically decreasing,  Witt) -> 0 c Bx +> 00.  (We want to show this c is actually term, Contradiction proof).  Suppose c>0)  Since V(x) is continuous, there exists as d>0 such that Bd \in Dec.  Where \(\Omega_c = \in \text{V(x)} \in C\) the trajectory lies out of Bd \$100.  Le -\gamma = \text{max } \(\var{V(x)}\), where -\gamma < 0.  Le -\gamma = \text{max } \(\var{V(x)}\), where -\gamma < 0.  Le -\gamma = \text{max } \(\var{V(x)}\), where -\gamma < 0.  Le -\gamma = \text{max } \(\var{V(x)}\), where -\gamma < 0.  Le -\gamma = \text{max } \(\var{V(x)}\), where -\gamma < 0.  Le -\gamma = \text{max } \(\var{V(x)}\)) + \int \(\var{V(x)}\) \(\var{V(x)}\) > 0 & \text{vectory}  For inside the level cet, it never goes out, c is shrinking to zero.  The above theorem is a sefficient condition for stability.  Domain of attraction: \(\var{Y(x)}\) \(\var{V(x)}\) = 0 \(\var{V(x)}\)  Tindity domain \(\var{Y(x)}\) dethermal sets \(\var{V(x)}\) is a conservative approximation.  Then (Barbashin - Krasovskii)  Let \(\var{V(x)}\) > 0 and \(\var{V(x)}\) > 0. \(\var{V(x)}\) det \(\var{V(x)}\) is a continuous differentiable function. Such that  \(\var{V(v)}\) = 0 and \(\var{V(x)}\) > 0. \(\var{V(x)}\) is continuous of rediably unbounded.	N	lamely, XIt) -> 0 as t-> 0.
V(XII) > 0 as t > 00  Since V(XII) is bounded from below by 0, monotonically decreasing,  V(XII) >> c Bs t >> 00.  (We want to show this c is actually zero, contradiction proof)  [Suppose C>0]  Since V(X) is continuous, there exists at d > 0 south that Bd \le le,  where \(\Omega_c = \int \text{Pr} \colored V(X) \le c\right\)  Therefore, \(\left(\text{int} \vert V(X) \right) = \colored \rightarrow \text{the trajectory lies out of Bd btzo.}  Lt \(-\gamma = \text{max } \vert V(X)\), where \(-\gamma < \colored \text{constraintons.}\)  Lt \(-\gamma = \text{max } \vert V(X)\), where \(-\gamma < \colored \text{constraintons.}\)  Lt \(-\gamma = \text{max } \vert V(X)\), where \(-\gamma < \colored \text{constraintons.}\)  V(X(\gamma)) = V(X(\gamma)) + \int \vert \(\frac{\gamma \text{V}}{\gamma \text{V}}\), where \(-\gamma \text{constraintons.}\)  V(X(\gamma)) = V(X(\gamma)) + \int \(\frac{\gamma \text{V}}{\gamma \text{V}}\), where \(\gamma \text{visions.}\)  I have set \(\omega_c = \int \frac{\gamma \text{V}}{\gamma \text{V}}\) \(\gamma \text{V}(X) \right) = 0 \text{ is shrinking } + \sigma \text{ zero.}\)  V(X(\gamma)) = V(\gamma \text{V}(\gamma) \right) \(\gamma \text{V}(\gamma) \right) \text{V}(\gamma) \right) \(\gamma \text{V}(\gamma) \right) \right) \(\gamma \text{V}(\gamma) \right) \text{V}(\gamma) \right) \(\gamma \text{V}(\gamma) \right) \\  V(X(\gamma)) = 0 \text{V} \(\gamma \text{V} \text{V} \right) \(\gamma \text{V}(\gamma) \right) \\  \(\gamma \text{V}(\gamma) \right) \(\gamma \text{V} \text{V} \right) \right) \(\gamma \text{V}(\gamma) \right) \\  \(\gamma \text{V}(\gamma) \right) \(\gamma \text{V} \right) \\  \(\gamma \text{V}(\gamma) \right) \\(\gamma \text{V}(\gamma) \right) \\  \(\gamma \text{V}(\gamma) \r	Si	ince V(0) = 0 and V(x) >0 in D/80}, it is sufficient to show.
Since V(X(t)) is bounded from bidous by 0, wonotonically decreasif,  V(X(t)) \rightarrow C & S + \rightarrow D.  (We want to show this C is actually zero, Contradiction proof)  Suppose C > D.  Since V(X) is continuous, there exist a d > 0 such that \$\frac{1}{2} \in \text{V}(X)\$ = C \rightarrow \text{The exists}  Where \$\int_C = \frac{1}{2} \text{K} \text{R} \cdot \frac{1}{2} \text{V}(X) \leq C \rightarrow \text{The exists}  V(X(t)) = \frac{1}{2} \text{V}(X), where \$-\text{V}(X)\$ is continuous.  W(X(t)) = \frac{1}{2} \text{V}(X(t)) + \frac{1}{2} \text{V}(X(t)) \rightarrow \text{V}(X(t)) - \text{V}(X(t)) \rightarrow \text{V}(X(t)) - \text{V}(X(t)) \rightarrow \te	1	VITVEIL > A AL LIVE
(We want to show this C is actually zero, Contradiction proof)  Suppose C>0]  Sina V(x) is continuous, there exists a d>0 such that Bd S. Dc,  where Dc = {xtBr   V(x) < c}  Therefore, Lim V(x(x)) = C => the trajectory lies and of Bd 4t70  Lt -x = max v(x), where -x < 0.  Lt -x = max v(x), where -x < 0.  Lt -x = max v(x), where -x < 0.  Lt -x = max v(x), where -x < 0.  Lt -x = max v(x), where -x < 0.  Level set Dc = {xtR^1   V(x) > c} < v(x(0)) - 7t  - o as t > w  contradicts to  V(x) > 0 & xeD   fo  level set Dc = {xtR^1   V(x) < c}. The idea is if the trajectory  go inside the level set, it never goes out, c is shrinking to zero.  The above theorem is a sofficient condition for stability.  Donain of attraction: {xcR  fin x(t; xo) = 0}  Tinding donain of attraction is analytically difficult, but can be approximated by level sets Dc if the system is asymptotically stable  But it is a conservative approximation.  Them (Parbashin-Krasovckii)  Lt x = 0 be an equilibrium for (xx). Let V: R -> R be a continuous differentiable function. Such that  V(0) = 0 and V(x) > 0. 4x to  11x11 > \omega = v(x) - \omega case of vodially unbounded.	Si	nce V(X(t)) is bounded from below by 0, monotonically alcereasing,
(We want to show this C is actually zero, Contradiction proof)  Suppose C>0]  Sina VIX) is continuous, there exists a d>0 sach that \$\mathbb{B}d \le \mathbb{I}_C, \$\text{Where } \mathbb{I}_C \circ \mathbb{S}(\text{RP} \mathbb{V}(\text{X}) \le C \rightarrow \text{the fore}, \limbdr{Immuns}, \text{Where } \mathbb{V}(\text{X}) \le C \rightarrow \text{The fore}, \text{Immuns}, \text{Where} \text{V(X)} \\  \text{Lt -Y = max } \text{V(X)}, \text{where -Y < 0.} \\  del		$\mathcal{N}(x t) \longrightarrow c  \mathcal{W} \leftarrow \mathcal{W}.$
where $\Omega_{c} = \{ \chi \in fr \mid V(\chi) \leq C \}$ Therefore, $\lim_{t \to \infty} V(\chi) = C \Rightarrow \text{th} + \text{rajectory lies out of Bod. } \forall 1 \geq 0 \}$ Let $Y = \max_{t \to \infty} V(\chi)$ , where $-Y < 0$ .  Let $Y = \max_{t \to \infty} V(\chi)$ , where $-Y < 0$ .  Let $Y = \max_{t \to \infty} V(\chi)$ , where $-Y < 0$ .  Let $Y = \max_{t \to \infty} V(\chi)$ , where $-Y < 0$ .  Let $Y = \max_{t \to \infty} V(\chi)$ , where $-Y < 0$ .  Let $Y = \max_{t \to \infty} V(\chi)$ , where $-Y < 0$ .  Let $Y = \max_{t \to \infty} V(\chi) = 0$ .  Let $Y = \max_{t \to \infty} V(\chi) = 0$ .  Let $Y = \max_{t \to \infty} V(\chi) = 0$ .  Let $Y = \min_{t \to \infty} V(\chi) = 0$ .  The above theorem is a sufficient condition for stability.  Domain of attraction: $\{ \chi \in F \} = \min_{t \to \infty} V(\chi) = 0 \}$ Thinding domain of attraction is analytically difficult, but can be approximated by level sets $\{ \chi \in F \} = 0 \}$ Thus $\{ \chi \in F \} = \min_{t \to \infty} V(\chi) = 0 \}$ Thus $\{ \chi \in F \} = \min_{t \to \infty} V(\chi) = 0 \}$ Let $\chi \in F = 0$ be an equilibrium for $\{ \chi \in F \} = 0 \}$ be a continuor differentiable function. Such that $\{ \chi \in F \} = 0 \}$ $\{ \chi \in F \} = 0 \}$ $\{ \chi \in F \} = 0 \}$ Let $\{$		(We want to show this c is actually zero, Contradiction proof)
where $\Omega_{c} = \{ \chi \in fr \mid V(\chi) \leq C \}$ Therefore, $\lim_{t \to \infty} V(\chi) = C \Rightarrow \text{th} + \text{rajectory lies out of Bod. } \forall 1 \geq 0 \}$ Let $Y = \max_{t \to \infty} V(\chi)$ , where $-Y < 0$ .  Let $Y = \max_{t \to \infty} V(\chi)$ , where $-Y < 0$ .  Let $Y = \max_{t \to \infty} V(\chi)$ , where $-Y < 0$ .  Let $Y = \max_{t \to \infty} V(\chi)$ , where $-Y < 0$ .  Let $Y = \max_{t \to \infty} V(\chi)$ , where $-Y < 0$ .  Let $Y = \max_{t \to \infty} V(\chi)$ , where $-Y < 0$ .  Let $Y = \max_{t \to \infty} V(\chi) = 0$ .  Let $Y = \max_{t \to \infty} V(\chi) = 0$ .  Let $Y = \max_{t \to \infty} V(\chi) = 0$ .  Let $Y = \min_{t \to \infty} V(\chi) = 0$ .  The above theorem is a sufficient condition for stability.  Domain of attraction: $\{ \chi \in F \} = \min_{t \to \infty} V(\chi) = 0 \}$ Thinding domain of attraction is analytically difficult, but can be approximated by level sets $\{ \chi \in F \} = 0 \}$ Thus $\{ \chi \in F \} = \min_{t \to \infty} V(\chi) = 0 \}$ Thus $\{ \chi \in F \} = \min_{t \to \infty} V(\chi) = 0 \}$ Let $\chi \in F = 0$ be an equilibrium for $\{ \chi \in F \} = 0 \}$ be a continuor differentiable function. Such that $\{ \chi \in F \} = 0 \}$ $\{ \chi \in F \} = 0 \}$ $\{ \chi \in F \} = 0 \}$ Let $\{$	Sul Su	prose $C>0$ noe $V(x)$ is continuous, there exists a $d>0$ such that $Bd \leq Rc$ ,
Therefore, lim V(x1x) = C => the trajectory has any of the trajectory has all the trajectory has a continuous.  Lt - Y = max V(x), where - Y < 0.  d		$(A \cap A \cap$
Lt -Y = Max $V(x)$ , where -Y < 0.  delixiky compact, $V(x)$ is continuous.  hence -Y exists. $V(x x) = V(x x) + \int_0^x V(x x) dx \le V(x x) - Yt$ $= -x$ Level set $R_c = \int_0^x x + \int_0^x V(x) \le C$ . The idea is if the trajectory go inside the level set, it never goes out, $C$ is shrinking to zero.  The above theorem is a sufficient condition for stability.  Domain of attraction: $\int_0^x x + \int_0^x x + \int_0^x$		Therefore, lim VIXItI) = C => the trajectory has an of partition
V(XH)) = V(X10))+ (+ V(X15)) els < V(X10)) - It	0 1	$A-Y=\max_{d\leq 11\times 11\times Y} v(x)$ , where $-Y<0$ .
V(NO)+). V(NO)ds ≤ V(NO)-0t		
contradicts to $V(\pi) > 0$ $\forall \pi \in \mathbb{D} \setminus \mathbb{C}$ . The idea is if the trajectory go inside the level set, it never goes out, $C$ is shrinking to zero.  The above theorem is a sufficient condition for stability,  Domain of attraction: $\{\chi_0 \in \mathbb{R} \mid \lim_{t \to \infty} \chi(t; Y_0) = 0\}$ Finding domain of attraction is analytically difficult, but can be approximated by level sets $\Re c$ if the system is asymptotically stable But it is a conservative approximation.  Thum (Barbashin-Krasovskii)  Let $\chi = 0$ be an equilibrium for $(\chi_0)$ . Let $\chi_0 \in \mathbb{R}$ be a continuou differentiable function. Such that $\chi(0) = 0$ and $\chi(0) > 0$ . Hat $\chi(0) = 0$ simple case of variably unbounded	. V	$V(X(t)) = V(X(0)) + \int_0^t \dot{V}(X(S)) dS \leq V(X(0)) - \gamma t$
V(x) >0 ∀ x∈D\so  • Level set Ω <sub>c</sub> = {xt R^n   V(x) ≤ c}. The idea is if the trajectory go inside the level set, it never goes out, c is shrinking to zero.  • The above theorem is a sufficient condition for stability.  • Domain of attraction: {x₀∈R  lim x(t; x₀) =0 }  Tinding domain of attraction is analytically difficult, but can be approximated by level sets Ω <sub>c</sub> if the system is asymptotically stable  But it is a conservative approximation.  Thum (Parbashin-Krasovskii)  Lt x=0 be an equilibrium for (**). Let V:R→R be a continuou differentiable function. Such that  V(v) = 0 and V(x) > 0. ∀x to simple case of vadially unbounded  1 x 1→∞ ⇒ V(x)→∞. ∈ simple case of vadially unbounded		$\leq -1$ $\longrightarrow -\infty$ as $t \to \infty$
The above theorem is a sufficient condition for stability,  Domain of attraction: { XoER   Lim X(t; Xo) = 0 }  Finding domain of attraction is analytically difficult, but can be approximated by level sets \$\frac{1}{2}c \text{ if the system is asymptotically stable}  But it is a conservatione approximation.  Them (Barbashin-Krasovskii)  Let X=0 be an equilibrium for (**). Let V: R \rightarrow R be a continuou differentiable function, such that  V(0) = 0 and V(X) > 0. \text{ YA \rightarrow 0}  N(X) \rightarrow \in \text{ simple case of variably unbounded}		V(x)>0 4 XED/50
The above theorem is a sufficient condition for stability,  Domain of attraction: { XoER   Lim X(t; Xo) = 0 }  Finding domain of attraction is analytically difficult, but can be approximated by level sets \$\frac{1}{2}c \text{ if the system is asymptotically stable}  But it is a conservatione approximation.  Them (Barbashin-Krasovskii)  Let X=0 be an equilibrium for (**). Let V: R \rightarrow R be a continuou differentiable function, such that  V(0) = 0 and V(X) > 0. \text{ YA \rightarrow 0}  N(X) \rightarrow \in \text{ simple case of variably unbounded}	a Ler	vel set $\Omega_c = \{x \in \mathbb{R}^n   V(x) \leq c \}$ . The idea is if the trajectory
The above theorem is a sufficient condition for stability,  Domain of attraction: { XoER   Lim X(t; Xo) = 0 }  Finding domain of attraction is analytically difficult, but can be approximated by level sets \$\frac{1}{2}c \text{ if the system is asymptotically stable}  But it is a conservatione approximation.  Them (Barbashin-Krasovskii)  Let X=0 be an equilibrium for (**). Let V: R \rightarrow R be a continuou differentiable function, such that  V(0) = 0 and V(X) > 0. \text{ YA \rightarrow 0}  N(X) \rightarrow \in \text{ simple case of variably unbounded}	8	o inside the level set, it never goes out, c is shrinking to zero.
Finding domain of attraction is analytically difficult, but can be approximated by level sets Ri if the system is asymptotically stable But it is a conservatione. approximation.  Them (Barbashin-Krasovskii)  Let X=0 be an equilibrium for (***). Let V=R > R be a continuou differentiable function. Such that  V(0)=0 and V(X)>0. 47+0.  IIXII > W => V(X) > W. C simple couse of vadially unbounded	00 1	he above theorem is a sufficient condition for stability,
Finding domain of attraction is analytically difficult, but can be approximated by level sets Ri if the system is asymptotically stable But it is a conservative approximation.  Thum (Barbashin-Krasovskii)  Let X=0 be an equilibrium for (**). Let V: R > R be a continuou differentiable function, such that  V(0) = 0 and V(X) > 0. 4x to simple case of radially unbounded  11X11 > 11X111 > 11X11 > 11X	• 2	anain of attraction: { XoER   lim X(t; Xe) = 0 }
Them (Borbashin-Krasovskii)  Let $X=0$ be an equilibrium for $(**)$ . Let $V:R\to R$ be a continuou differentiable function. Such that $V(0)=0$ and $V(X)>0$ . $\forall x\neq 0$ . $V(0)=0$ and $V(X)>0$ . $\forall x\neq 0$ . Simple case of radially unbundled $V(X)\to \infty$ . $\in$ Simple case of radially unbundled	7 a	Finding domain of attraction is analytically difficult, but can be approximated by level sets Ri if the system is asymptotically stable Bt it is a conservative approximation.
Let $x=0$ be an equilibrium for $(*x)$ . Let $V = \mathbb{R}$ be a continuou differentiable function. Such that $V(v) = 0 \text{ and } V(x) > 0 \text{ . } \forall x \neq 0.$ $V(v) = 0 \text{ and } V(x) > 0 \text{ . } \forall x \neq 0.$ $V(v) = 0 \text{ and } V(x) > 0 \text{ . } \forall x \neq 0.$ $V(v) = 0 \text{ and } V(x) > 0 \text{ . } \forall x \neq 0.$ $V(v) = 0 \text{ and } V(x) > 0 \text{ . } \forall x \neq 0.$ $V(v) = 0 \text{ and } V(v) = 0 \text{ . } \forall x \neq 0.$		
$V(0) = 0$ and $V(x) > 0$ . $\forall x \neq 0$ . $  X   \to  w  \Rightarrow V(x) \to  w $ . E simple case of radially unbundled		Let X=0 be an equilibrium for (+x). Let V The a continuou
		V(v) = 0 and V(x)>0. 4x+0.
V(V) CO F V A T O		$V(x) < 0$ , $\forall x \neq 0$
then x=0 is globally asymptotically stable.	,	then x=0 is globally asymptotically stable.

This The equilibrium pt x=0 of x=Ax is stable if ond only if all eigen values satisfy  $Re(Ai) \le 0$ , and for every eigen value on the imaginary axis, the algebraic multiplicity is equal to the geometric multiplicity.

The equilibrium pt is (globally) asymptotically stable iff all eigen values satisfy Re(Ai) < 0.

(Hint: Consider the Jordan block.  $e^{At} = Pe^{Tt} P^{-1} = \sum_{i=1}^{\infty} t^{k-i} Ait Rin)$ .

Consider the non-linear system x=f(x),  $f:D \mapsto \mathbb{R}^{M}$  is a continuously

Consider the non-linear system  $\hat{x} = f(x)$ ,  $f : D \mapsto \mathbb{R}^{d}$ , is a continuously differentiable map.

Let x = 0 be the equilibrium. By meanwalue theorem,  $\hat{x}_{i} = f_{i}(x) = f_{i}(0) + \frac{\partial f_{i}}{\partial x}(Z_{i})X$ ,  $Z_{i} \in [0, X]$   $\Rightarrow \hat{x}_{i} = \frac{\partial f_{i}}{\partial x}(0) + \frac{\partial f_{i}}{\partial x}(Z_{i}) + \frac{\partial f_{i}}{\partial x}(Z_{i}) + \frac{\partial f_{i}}{\partial x}(Z_{i}) = \frac{\partial f_{i}}{\partial x}(Z_{i}) + \frac{\partial f_{i}}{\partial x}(Z_{i}) + \frac{\partial f_{i}}{\partial x}(Z_{i}) = \frac{\partial f_{i}}{\partial x}(Z_{i}) + \frac{\partial f_{i}}{\partial x}(Z_{i}) + \frac{\partial f_{i}}{\partial x}(Z_{i}) = \frac{\partial f_{i}}{\partial x}(Z_{i}) + \frac{\partial f_{i}}{\partial x}(Z_{i}) + \frac{\partial f_{i}}{\partial x}(Z_{i}) = \frac{\partial f_{i}}{\partial x}(Z_{i}) + \frac{\partial f_{i}}{\partial x}(Z_{i}) + \frac{\partial f_{i}}{\partial x}(Z_{i}) = \frac{\partial f_{i}}{\partial x}(Z_{i}) + \frac{\partial f_{i}}{\partial x}(Z_{i}) + \frac{\partial f_{i}}{\partial x}(Z_{i}) = \frac{\partial f_{i}}{\partial x}(Z_{i}) + \frac{\partial f_{i}}{\partial x}(Z_{i}) + \frac{\partial f_{i}}{\partial x}(Z_{i}) + \frac{\partial f_{i}}{\partial x}(Z_{i}) = \frac{\partial f_{i}}{\partial x}(Z_{i}) + \frac{\partial f_{i}}{\partial x}(Z_{i})$ 

 $\begin{array}{c} \Rightarrow \chi = A \times + f(x) & f_i(x) \\ \hline \frac{\partial f}{\partial x}(0) & \text{Note that } 18_i(x) | \leq |\frac{\partial f_i'}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0)| || \times || \\ \hline \Rightarrow \frac{|| f(x) ||}{|| \times ||} \rightarrow 0 \text{ as } || \times || \rightarrow 0 \text{ since} \\ \hline \text{Stability analysis by linearization} & \frac{\partial f}{\partial x} \text{ is continuous, and } 0 \leq z_i \leq \chi \\ \hline \text{We hope the Systems would behave} \\ \hline \text{Similarly around the origin.} \end{array}$ 

This Let x=0 be an equilibrium point for the nonlinear system  $\dot{x}=f(x)$ ,  $f: \mathcal{D} \mapsto_{\mathbb{R}^n}$  is continuously differentiable and  $\mathcal{D}$  is a neighbourhood of the origin.

Let  $d=\frac{\partial f}{\partial x}(x)|_{x=0}$ .

Then. O the origin is asymptotically stable if Re(Ai(A)) < 0.

The origin is unstable if Re(Ai(A)) > 0 holds for at least one i.

Lemna A matrix d is Hurwitz, iff for any given positive definite symmetric matrix Q there exists a positive definite matrix P that satisfies d'P+PA = -Q. Moreover, the solution P is unique. proof Sufficiency: Construct Lyaquinou Runction VIX) = XTPX.

 $\dot{V}(x) = \dot{x}^{\mathsf{T}} p x + x^{\mathsf{T}} p \dot{x} = x^{\mathsf{T}} A^{\mathsf{T}} p x + x^{\mathsf{T}} p A x = -x^{\mathsf{T}} Q x$ Since Q is positive definite,  $V(x) = -x^{T}Qx < 0 \quad \forall x$ .

=) A is Hur witz.

Necessity: Assume Re(Ai(A)) < 0 42, define P= 50 e At Q e At. the integrand is a sum of terms like  $t^{k-1}e^{\lambda it}$ ,  $Re(\lambda_i)<0$ , Hence the integral does not blow-up and  $P \geqslant 0$ .

Plug P into the Lyapunov equation: LHS: ATP+PA = SouteAt QeAtd+ SoetaetAdt = Southert) Qe At alt + So e dit Q of [e At] alt = sod [extacht] dt = extacht | = -0=Rris

Hence P is the solution to Lyapunov equation.
What remains to show is the uniqueness. Suppose P'+P,

then A'(P-P')+ (P-P') A=0 => e^{A^{T}}(P-P')e^{At} + e^{A^{T}}(P-P')Ae^{A+} = 0

 $\Rightarrow \frac{d}{dt} \left\{ e^{At} (P - P') e^{At} \right\} = 0 \Rightarrow e^{A't} (P - P') e^{At} = const \ \forall t.$ Since  $e^{AO} = I \Rightarrow P - P' = e^{A't}(P - P')e^{At} \rightarrow 0$  as  $t \rightarrow \infty$  $\Rightarrow$  P-P'=0

proof for linearization (We only show the first pourt). Suppose A is Hurwitz, => IP>O VQ that satisfies the Lyap. ef.  $V(x) = x^{T}PX \Rightarrow \dot{V}(x) = \dot{x}^{T}Px + x^{T}P\dot{x} = f(x)^{T}Px + x^{T}Pf(x)$ Since  $\hat{x} = f(x) = Ax + f(x)$  $\dot{y}(x) = (x^T + x^T +$  $= -\chi^T Q X + 2\chi^T P g(\chi)$ Recall that  $\frac{\|9(x)\|}{\|x\|} \rightarrow 0$  as  $\|x\| \rightarrow 0$ . thence for any 8>0, 7 r>0, sit. 118(x) 11 < x ||x|| + 11x11 < r. Therefore,  $\dot{V}(x) = -x^TQx + 2x^TPg(x) \leq -x^TQx + 2||x|| \cdot ||P|| \cdot ||g(x)||$ < - xTQ X + 2 x ||x112. ||P| , A ||x11< r Note that  $x^TQX \ge \lambda_{min}(Q) \|x\|^2$ 

=) V(X) < ->min(Q) ||X||2+28||X||2||P|| , y ||X||< r

 $\rightarrow$  choose  $\forall$  such that  $-\lambda_{min}(Q) + 2\forall ||P|| < 0 \Rightarrow \hat{V}(X) < 0$ 

=) x=fix) is stable.