Nonlinear Control Theory

Lecture 5. Lyapunov Direct Method II.

Last time

- General Lyapunov function methods (time-varying systems)
  - Class K functions
  - Locally positive definite functions : $\alpha(\|x\|) \le V(t,x)$.
  - Decrescent functions : $V(t,x) \le \beta(\|x\|)$
- Stability theorems
  - Critical stability. $(\dot{V}(t,x) \le 0)$
  - Instability.
  - Asymptotic stability
  - Domain of attraction
  - Exponential stability $(a\|x\|^2 \le V(t,x) \le b\|x\|^2,\ \dot{V}(t,x) \le -c\|x\|^2)$.
  - Lasalle's invariance principle. (Start in a compact invariant set, $\dot{V}(x) \le 0$ converge to the largest invariant set in $\dot{V}(x) = 0$)

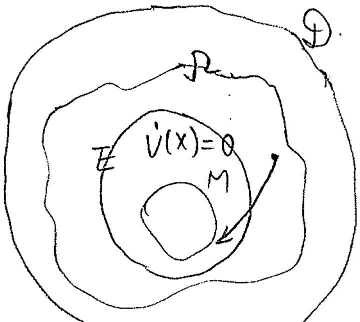
Today

- Proof of Lasalle's invariance principle.
- The Lure's problem
- Global stability
- Converse theorems $\begin{cases} \text{local asymptotic} \\ \text{global asymptotic.} \end{cases}$

Consider the autonomon system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \quad f: \mathcal{D} \mapsto \mathbb{R}^n \text{ locally Lipschitz on } \mathcal{D} \qquad (*)$$

Thm (Lasalle)
Let $\Omega \subset \mathcal{D}$ be a compact set that is positively invariant with respect to $(*)$, Let $V: \mathcal{D} \mapsto \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(x) \le 0$. Let $E \doteq \{x \in \Omega \mid \dot{V}(x) = 0\}$. Let M be the largest invariant set in E, then every solution in Ω approaches M as $t \to \infty$.



Lemma. If a solution $x(t)$ of $(*)$ is bounded and belongs to $\mathcal{D}$ for $t \geq 0$, then its ω-limit set $\mathcal{L}^+$ is a nonempty, compact, invariant set. Moreover, $x(t)$ approaches $\mathcal{L}^+$ as $t \to \infty$.

<u>proof</u> : <u>Nonempty</u> : (Bolzano-Weierstrass: each bounded sequence in $\mathbb{R}^n$ has a <u>convergent subsequence</u>)

since $x(t)$ is bounded $\Rightarrow$ has accumulation pts. as $t \to \infty$.

$\Rightarrow \mathcal{L}^+$ is nonempty.

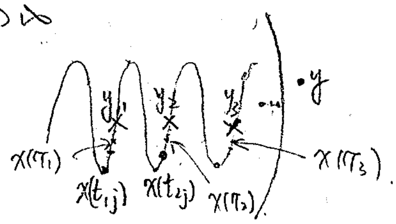
<u>compact</u> : <u>Boundness</u> : $\forall y \in \mathcal{L}^+$, $\exists \{t_n\}$, where $t_n \to \infty$ as $n \to \infty$, such that $x(t_n) \to y$ as $n \to \infty$.

Since $x(t)$ is bounded $\Rightarrow$ $x(t_n)$ is bounded uniformly in n

$\Rightarrow$ the limit y is also bounded $\Rightarrow$ $\mathcal{L}^+$ bounded.

<u>closedness</u> : Let $\{y_n\} \in \mathcal{L}^+$, we want to show $y \in \mathcal{L}^+$ if $y_n \to y$, as $n \to \infty$,

For every n, $\exists$ subsequence $\{t_{nj}\}$ with $t_{nj} \to \infty$. as $j \to \infty$, such that $x(t_{nj}) \to y_i$ as $j \to \infty$



construct a particular $\{T_i\}$, s.t.

Given the sequence $\{t_{nj}\}$,

$T_2 > t_{12}$ $\quad \|x(T_2) - y_2\| < 1/2$

$T_3 > t_{13}$ $\quad \|x(T_3) - y_3\| < 1/3$

$\vdots$

$T_j \to \infty$ as $j \to \infty$. and $\|x(T_j) - y_j\| < \frac{1}{j}$

For $\forall \varepsilon > 0$, there exists $N_1, N_2 > 0$, s.t. $\quad \lim_{j \to \infty} y_j = y$.

$\|x(T_j) - y_j\| < \frac{\varepsilon}{2}$, $\forall j > N_1$, $\|y_j - y\| < \frac{\varepsilon}{2}$, $\forall j > N_2$.

$\Rightarrow \|x(T_j) - y\| \leq \|x(T_j) - y_j\| + \|y_j - y\| < \varepsilon$, $\forall i > N := \max\{N_1, N_2\}$

$\Rightarrow$ <u>$x(T_j) \to y$ as $j \to \infty$</u>. $\to$ $y \in \mathcal{L}^+$. is an element in ω-limit set

$\Rightarrow \mathcal{L}^+$ is closed.

Invariant Let $y \in \mathcal{L}^+$, and we want to show $x(t;y) \in \mathcal{L}^+, \forall t$.

Since y is an ω-limit point, $\exists \{t_i\}$ with $t_i \to \infty$ as $i \to \infty$ s.t.

$x(t_i) \to y$ as $i \to \infty$.

Since the solution of ODE is unique,

$$x(t+t_i; x_0) = x(t; x(t_i; x_0)) = x(t; x(t_i))$$

For sufficiently large t_i, $t+t_i > 0$. Since the solution of ODE is continous w.r.t initial value,

$$\Rightarrow \underbrace{\lim_{i\to\infty} x(t+t_i; x_0)}_{\in \mathcal{L}^+} = \lim_{i\to\infty} x(t; x(t_i, x_0)) = \lim_{i\to\infty} x(t; x(t_i)) = x(t;y)$$

$x(t) \to \mathcal{L}^+$ as $t \to \infty$ **Contradiction proof.**

Suppose not the case, then $\exists\, \varepsilon > 0$ and $\{t_i\}$ with $t_i \to \infty$ as $i \to \infty$, s.t. $\mathrm{dist}\,(x(t_i), \mathcal{L}^+) > \varepsilon$. $\forall i$ (Bolzano–Weierstrass)

Since $\{x(t_i)\}$ is bounded sequence, it contains a convergent subsequence. $\{x(t_{ij})\} \to x^*$. $x^* \in \mathcal{L}^+$. However, there must be some distance ε between $x(t_{ij})$ and $\mathcal{L}^+$. Contradiction.

proof for Lasalle

Suppose $x(t; x_0)$ is a solution, $x_0 \in \Omega$,

$\dot{V}(x) \le 0$, $x \in \Omega \Rightarrow V(x(t))$ decreasing w.r.t t.

$\left.\begin{array}{l} V(x) \in C^1 \\ \Omega \text{ is compact} \end{array}\right\} \Rightarrow V(x(t))$ is bounded from below $\Big\} \Rightarrow V(x(t))$ has a limit as $t \to \infty$. $\boxed{\lim_{t\to\infty} V(x(t)) := a}$

$\left.\begin{array}{l} \text{Lemma, } \mathcal{L}^+ \text{ is compact} \\ \Omega \text{ is compact/invariant} \end{array}\right\} \Rightarrow \mathcal{L}^+ \subset \Omega$.

$\forall p \in \mathcal{L}^+$, $\exists \{t_n\}$ with $t_n \to \infty$, as $n \to \infty$ s.t. $x(t_n) \to p$ as $n \to \infty$ (Def. of ω-limit set)

$\boxed{V(x) \text{ is continuous}}$ $V(p) \overset{\downarrow}{=} \lim_{n\to\infty} V(x(t_n)) := a$

$\Rightarrow V(x) = a$, $\forall x \in \mathcal{L}^+$. (since p is arbitrary in $\mathcal{L}^+$)

$\mathcal{L}^+$ is invariant set (Lemma), $\dot{V}(x) = 0$ holds on $\mathcal{L}^+$.

$\Rightarrow \mathcal{L}^+ \subseteq M \subseteq E \subseteq \Omega$ $\quad \dot{V}(x) = 0$

largest invariant set

Total stability

Suppose $\dot{x} = f(t,x)$ is affected by some disturbance $\dot{x} = f(t,x) + g(t,x)$ $(*)$

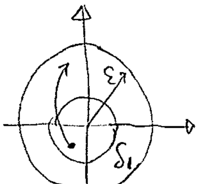
g is ANY function.

Denote the solution of $(*)$ as $x_g(t; x_0, t_0)$

Def. $x=0$ is said to be totally stable (stable under persistent disturbances) if for all $\varepsilon > 0$, there exists two positive numbers $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$ such that

$|x_g(t; x_0, t_0)| < \varepsilon, \ \forall t \geq t_0 \geq 0$

if $|x_0| < \delta_1$, and $|g(t,x)| < \delta_2, \forall x \in B_\varepsilon, \forall t \geq t_0 \geq 0$



Thm If $x=0$ of $\dot{x} = f(t,x)$ is uniformly asymptotically stable, it is also totally stable.

(Make controller design possible. since we always have unknown disturbances and modeling errors)

The Lure's problem

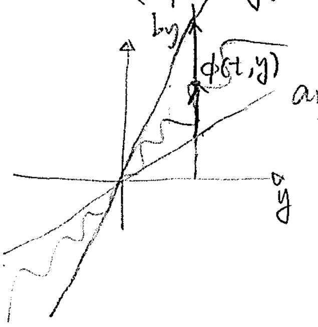
Consider $\dot{x} = Ax + Bu$, $y = Cx + Du$ $(**)$, where $x \in \mathbb{R}^n$, $y, u \in \mathbb{R}^m$.

The feedback is defined by $u = -\phi(t,y)$ ← nonlinearity induced by the actuators.

Def Suppose $\phi: \mathbb{R}^m \times \mathbb{R}_+ \longmapsto \mathbb{R}^m$. then ϕ is said to be belong to the sector $[a,b]$ (where $a<b$) if

(1) $\phi(t,0) = 0, \ \forall t \geq 0$

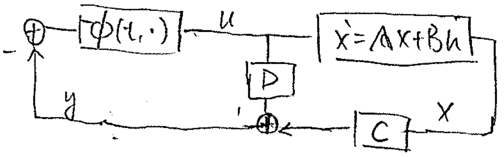
(2) $(\phi(t,y) - ay)^T (by - \phi(t,y)) \geq 0, \ \forall t \geq 0, \forall y \in \mathbb{R}^m$



absolute stability problem:

Suppose the pair (A,B) is controllable and the pair (C,A) is observable and let $G(s) = C(sI-A)^{-1}B + D$ be the transfer function.

Derive conditions involves only $G(s)$ and a, b, such that $x=0$ is globally uniformly asymptotically stable for EVERY ϕ belonging to the sector $[a,b]$.



Lemma (Kalman - Yakubovich - Popov)

Consider system (***). Suppose A is Hurwitz and (A,B) controllable, (C,A) observable, and

$$\inf_{\omega\in\mathbb{R}} \lambda_{min}(G(j\omega)+G^*(j\omega)) > 0 \quad \text{(strictly positive real).}$$

Then there exist a positive definite matrix $P\in\mathbb{R}^{n\times n}$, and $Q\in\mathbb{R}^{m\times n}$, $W\in\mathbb{R}^{m\times m}$, and $\varepsilon>0$ such that

$$A^TP+PA=-\varepsilon P-Q^TQ$$
$$B^TP+W^TQ=C$$
$$W^TW=D+D^T$$

Thm (Passivity) Suppose in system (***), A is Hurwitz (A,B) is controllable (C,A) is observable, $G(s)$ is strictly positive real, and ϕ belongs to sector $[0,\infty)$, i.e. $y^T\phi(t,y)\geq 0$, $\phi(t,0)=0$, $\forall t\geq 0, \forall y\in\mathbb{R}^m$

Then the feedback system is globally exponentially stable

Global Stability = Asymptotically stable + domain of attraction is $\mathbb{R}^n$

Thm: The equilibrium 0 is globally exponentially stable if there exists a pdf $V(t,x)$ such that

$$a\|x\|^2 \leq V(t,x) \leq b\|x\|^2, \ \forall t\geq 0, \ \forall x\in\mathbb{R}^n$$

and $\quad \dot{V}(t,x)\leq -c\|x\|^2, \forall t\geq 0, \ \forall x\in\mathbb{R}^n. \ (a,b,c>0)$

Thm. Given autonomous system $\dot{x}=f(x)$, $x=0$ is globally asymptotically stable, if there exists a radially unbounded, decresent pdf $V(x)$, such that $-\dot{V}$ is pdf

Ex 1) $\dot{x}_1=-x_1^3+x_2^2$, $\dot{x}_2=-x_2^3-x_1x_2$ Let $V(x)=\frac{1}{2}(x_1^2+x_2^2)$, $\dot{V}(x)=-x_1^4-x_2^4$

2) $\dot{x}_1=-x_1^3+x_2^2x_1^3$, $\dot{x}_2=-x_2$ Let $V(x)=\frac{x_1^2}{1+x_1^2}+2x_2^2$, $\dot{V}(x)\leq -\frac{2x_1^4}{(1+x_1^2)^2}-2x_2^2$

↑ pdf but not radially unbounded   ↑ $-\dot{V}(x)$ pdf

locally asymptotically stable, but not global.

Question: Given a stable system, can I always find such Lyapunov functions?

Here we only consider autonomous systems $\dot{x} = f(x)$, $x \in R^n$, $f \in C^1$.

$x = 0$ is an equilibrium.

Thm. Suppose $x=0$ is asymptotically stable, then there exists a C^1 function $V(x)$ and α, β, γ of class K such that

$$\alpha(\|x\|) \leq V(x) \leq \beta(\|x\|), \forall x \in B_r.$$

$$\dot{V}(x) \leq -\gamma(\|x\|), \forall x \in B_r.$$

Thm Suppose $x=0$ is exponentially stable, then there exists a C^1 function $V(x)$ and positive constants a, b and c such that

$$a\|x\|^2 \leq V(x) \leq b\|x\|^2, \forall x \in B_r$$

and $\dot{V}(x) \leq -c\|x\|^2$, $\forall x \in B_r$,

or $\frac{\partial V}{\partial x} f(x) \leq -c\|x\|^2$, $\|\frac{\partial V}{\partial x}\| \leq \mu\|x\|^2$, $\forall x \in B_r$.

proof: Suppose $\forall x_0 \in B_r$, $\|x(t; x_0)\| \leq \alpha\|x_0\|e^{-\gamma t}$, $\alpha, \gamma > 0$.

Let $V(\underline{\underline{x_0}}) = \int_0^\infty \|x(\tau; x_0)\|^2 d\tau$

$\Rightarrow V(x_0) \leq \int_0^\infty \alpha^2\|x_0\|^2 e^{-2\gamma\tau} d\tau \leq \frac{\alpha^2}{2\gamma}\|x_0\|^2$

On the other hand, $\frac{d}{dt}(\frac{1}{2}\|x(t;x_0)\|^2) = x(t;x_0)^T \dot{x}(t;x_0)$

$= x(t;x_0)^T f(x(t;x_0))$

f is C^1, $\Rightarrow$ f is Lipschitz in B_r.

$\Rightarrow \|x(t;x_0)^T f(x(t;x_0)) - x(t;x_0)^T \underset{0}{\underset{\|}{f(0)}}\| = \|x(t;x_0)^T(f(x(t;x_0)) - 0)\|$

$\leq \|x(t;x_0)\| \cdot \|f(x(t;x_0))\| \leq L\|x(t;x_0)\|^2$

$\Rightarrow -L\|x(t;x_0)\|^2 \leq \|\frac{d}{dt}(\frac{1}{2}\|x(t;x_0)\|^2) \leq L\|x(t;x_0)\|^2$

$\Rightarrow \frac{1}{2}\|x(t;x_0)\|^2 \geq \frac{1}{2}\|x_0\|^2 \cdot e^{-2Lt} \Rightarrow V(x) \geq \frac{\|x_0\|^2}{2L}$

$\dot{V}(x) = \frac{dV(x(t;x_0))}{dt} = \frac{d}{dt}\int_0^\infty \|x(\tau, x(t;x_0))\|^2 d\tau = \frac{d}{dt}\int_0^\infty \|x(\tau+t, x_0)\|^2 d\tau$

Let $\tau + t = s \Rightarrow \dot{V}(x) = \frac{d}{dt}\int_t^\infty \|x(s, x_0)\|^2 ds = -\|x(t, x_0)\|^2$

Converse theorems to global asymptotic stability.

Thm Suppose $x=0$ is globally asymptotically stable, then there exists a C^1 function $V(x)$ and α, β, γ of class $\mathcal{K}_\infty$, such that.

$$\alpha(\|x\|) \le V(x) \le \beta(\|x\|), \ \forall x \in \mathbb{R}^n$$

$$\dot{V}(x) \le -\gamma(\|x\|), \ \forall x \in \mathbb{R}^n.$$

class $\mathcal{K}$: $\phi(r)$ strictly increasing, $\phi(0) = 0$, continuous

class $\mathcal{K}_\infty$: class $\mathcal{K}$, $\lim_{r\to\infty}\phi(r) = \infty$

Remark: In general, we can not show the global version of "exponentially stable", unless we assume that $f(x)$ satisfies a linear growth condition.

$$\left|\frac{\partial f(x)}{\partial x}\right| < k, \quad \forall x \in \mathbb{R}^n.$$

We now give results as an application of the converse theorems.

Thm Let $f(x)$ be C^2 and $A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$, then $x=0$ of $\dot{x} = f(x)$ is exponentially stable iff $z=0$ of the linearized system $\dot{z} = Az$ is exponentially stable.

proof: sufficiency: we have covered this in earlier lectures.

Necessity: Suppose $x=0$ is exponentially stable, then by the converse Lyapunov Thm. there $\exists$ $V(x) \in C^1$, s.t.

$$\alpha\|x\|^2 \le V(x) \le \beta\|x\|^2, \ \forall x \in B_r.$$

$$\dot{V}(x) \le -\gamma\|x\|^2, \quad \forall x \in B_r$$

$$\left\|\frac{\partial V}{\partial x}\right\| \le \mu\|x\|, \quad \forall x \in B_r$$

Rewrite $\dot{x} = f(x) = \underbrace{Ax}_{A=\left.\frac{\partial f}{\partial x}\right|_{x=0}} + \underbrace{f(x) - Ax}_{\mathcal{F}(x)}$

(Recall that $\left\|f(x) - \left.\frac{\partial f}{\partial x}\right|_{x=0} x\right\| \le \|f(x)\| + \|A\|_F\|x\|.$
$\le L\|x\| + \|A\|_F\|x\|$
$\Rightarrow \|f(x) - Ax\| = O(\|x\|^2)$)

$$\dot{V} = \underbrace{\frac{\partial V(x)}{\partial x}}_{\le \mu\|x\|}(Ax + \mathcal{F}(x)) = \frac{\partial V(x)}{\partial x}Ax + O(\|x\|^3)$$

Ansatz: $V(x) = P_1x + x^TP_2x + O(\|x\|^3)$

$V(x) \ge 0 \Rightarrow P_2 > 0, \ P_1 = 0$

$$\Rightarrow \frac{\partial V(x)}{\partial x}Ax = \left(2x^TP_2 + O(\|x\|^2)\right)Ax = x^T(A^TP_2 + P_2A)x + O(\|x\|^3)$$

since $\dot{V}(x) \le -\gamma\|x\|^2$, $\Rightarrow A^TP_2 + P_2A < 0 \Rightarrow A$ is Hurwitz