Nonlinear Control Theory

Lecture 9. Linearization of Nonlinear Systems, I

Last time

- observability (Can the system generate the same output using the same control input but with different initial values?)
  → Not interesting, even the system is observable, observer may still not exist.
- Exponential observer

Today

- Linear equivalence of affine nonlinear systems.

Consider an affine nonlinear system.

$$(*) \quad \begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x) u_i \\ &:= f(x) + g(x) u, \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m, \ f(x_0) = 0 \end{aligned}$$

equilibrium pt.

Def. The system $(*)$ is said to be equivalent to a linear system (locally around an equilibrium pt x_0), iff there exists a (local) coordinate chart $(\mathcal{U}, z)$ ($z(x_0) = 0$) such that the system $(*)$ is expressed in z coordinate frame as

$$(**) \quad \dot{z} = Az + \sum_{i=1}^{m} b_i u_i := Az + Bu, \quad z \in \mathbb{R}^n \ (z \in \mathcal{U}), \ u \in \mathbb{R}^m,$$

where (A, B) is a completely controllable pair.

$\mathcal{U}$: open set, neighbourhood around x_0

z: homeomorphism, function $z: \mathcal{U} \mapsto \mathbb{R}^n$

- bijection
- continuous
- z^{-1} is continous

Idea. You do a "coordinate change" around the equilibrium pt, and your system becomes "linear" in the local coordinate.

"pushes the vector in T_pM to $T_{F(p)}N$"

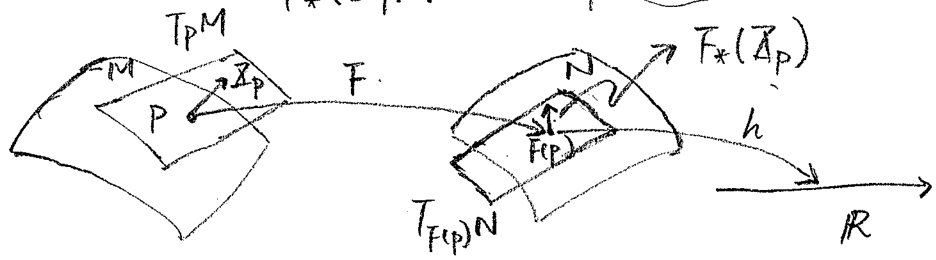
Def. (Push-forward).

Let M, N be two smooth manifolds, $F: M \mapsto N$ be a smooth mapping.

For a point $p \in M$ and $F(p) \in N$, we define a mapping $F_*: T_pM \mapsto T_{F(p)}N$,

such that $F_*(X_p)\,h = X_p(\underbrace{h \circ F}_{\in C^r(M,p)})$, $X_p \in T_pM$, $h \in C^r(N, F(p))$.

The set of C^r functions defined around $F(p)$.



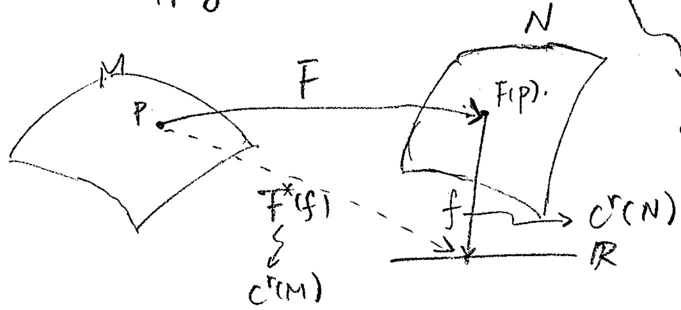
Def (Pull-back).

Let M, N be two smooth manifolds. $F: M \mapsto N$ be a diffeomorphism.

Let $f(x) \in C^r(N)$, an induced mapping $F^*: C^r(N) \mapsto C^r(M)$ is defined by

$F^*(f) = f \circ F \in C^r(M)$

"pulls the function defined on N back to M"



Lemma (Diffeomorphism does not affect the Lie bracket)

F: diffeomorphism, $M \mapsto N$, $X, Y \in TM$

$F_*[X,Y] = [F_*(X), F_*(Y)]$

vector in $T_{F(p)}N$

proof By definition, $F_*(X)_{F(p)}\, h = \boxed{F_*(X)}\, h\,(F(p)) = X(h \circ F)(p)$.

$\Rightarrow F_*(X)\,h \circ F = X(h \circ F)$

Hence $F_*[X,Y]_{F(p)}(h) = [X,Y]_p(h \circ F)$.

$= X_p(Y(h \circ F)) - Y_p(X(h \circ F)) = X_p(F_*(Y)\,h \circ F) - Y_p(F_*(X)\,h \circ F)$.

$= F_*(X)_{F(p)}\, F_*(Y)\,h - F_*(Y)_{F(p)}\, F_*(X)\,h$

$= [F_*(X), F_*(Y)]_{F(p)}\, h \Rightarrow F_*[X,Y] = [F_*(X), F_*(Y)]$.

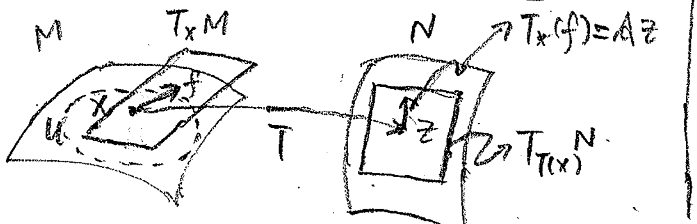
Thm. The nonlinear affine system (*) is equivalent to a linear system of the form (**) locally around an equilibrium x_0, iff

i) $\dim\{ad_f^k g_i(x_0) \mid 1\le i\le m,\ 0\le k\le n-1\} = n$

ii) there exists a neighbourhood U of x_0, such that

$$[ad_f^s g_i,\ ad_f^t g_j] = 0,\quad 1\le i,j\le m,\quad 0\le s,t\le n.$$

proof (Necessity). Suppose the coordinate change that realizes the linearization is $T: x\mapsto z$, then $T_*(f) = Az$, $T_*(g_i) = b_i$, $i=1,\cdots,m$.



$$\Rightarrow ad^k_{T_*(f)} T_*(g_i) = (-1)^k A^k b_i$$

$$\Rightarrow [ad^s_{T_*(f)} T_*(g_i),\ ad^t_{T_*(f)} T_*(g_j)] = [(-1)^s A^s b_i,\ (-1)^t A^t b_j] = 0$$

Since (**) is controllable, $\dim\{ad^k_{T_*(f)} T_*(g_i) \mid 1\le i\le m,\ 0\le k\le n-1\} = n$

By Lemma, $T_*[f, g_i] = [T_*(f), T_*(g_i)] = ad_{T_*(f)} T_*(g_i)$

$\Rightarrow$ i) ii) holds.

(Sufficiency) According to i), there exists n linearly independent vector fields $X^i_{k+1} = ad_f^k g_i$, $i=1,\cdots,m$, $k=0,1,\cdots,n_i-1$, where $\sum_{i=1}^m n_i = n$.

They are linearly independent on a neighbourhood U of x_0.

Choose a new local coordinate frame as $z = \{z_1^1,\cdots z_{n_1}^1,\cdots,z_1^m,\cdots,z_{n_m}^m\} \in \mathbb{R}^n$

and construct a mapping $F: z\mapsto U$ as

$$F(z) = \gamma^{X_1^1}_{z_1^1}\circ\cdots\circ\gamma^{X^1_{n_1}}_{z^1_{n_1}}\circ\cdots\circ\gamma^{X^m_1}_{z^m_1}\circ\cdots\circ\gamma^{X^m_{n_m}}_{z^m_{n_m}}(x_0).$$

integral curve / flow

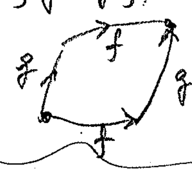
Let $p = F(z)$, since by ii), $[\underbrace{ad_f^s g_i}_{=X^i_{s+1}},\ \underbrace{ad_f^t g_j}_{=X^j_{t+1}}] = 0$, $1\le i,j\le m$, $0\le s,t\le n$, commute

$$\Rightarrow F(z) = \gamma^{X_1^1}_{z_1^1}\circ\cdots(x_0) = \gamma^{X_2^1}_{z_2^1}\circ\cdots(x_0) = \cdots$$

$$\Rightarrow \left.\frac{\partial F}{\partial z^i_k}\right|_{z=F^{-1}(p)} = X^i_k(p),\quad \left(\text{since } \frac{\partial}{\partial t}(\gamma^X_t)\Big|_p = X_p\right)$$

$i=1,\cdots,m$, $k=1,\cdots,n_i$.

$[f,g]=0$
$\Rightarrow fg - gf = 0$



Hence under the new coordinate z, we have:

$$(F_*^{-1}(X_1^1),\cdots F_*^{-1}(X^1_{n_1}),\cdots,F_*^{-1}(X_1^m),\cdots,F_*^{-1}(X^m_{n_m}))$$

$$= J_{F^{-1}}\circ(X_1^1,\cdots,X^1_{n_1},\cdots,X_1^m,\cdots X^m_{n_m})\circ F$$

$$= (J_F(z))^{-1}\circ F^{-1}\circ(X_1^1,\cdots X^1_{n_1},\cdots,X_1^m,\cdots X^m_{n_m})\circ F = I$$

$F(z) = p$
$J_{F^{-1}}(p) = (J_F(z))^{-1}$
$J_{F^{-1}}\circ F = (J_F(z))^{-1}$
$J_{F^{-1}} = (J_F(z))^{-1}\circ F^{-1}$

For the ease of notation, we still use f and g_i to represent their new forms under z coordinate, i.e, $\mathcal{T}_*^{-1}(f)$ and $\mathcal{T}_*^{-1}(g_i)$.

Let $n^i = n_1 + n_2 + \cdots + n_i$, δ_i denotes the ith column of I_n. (The ith element of δ_i is 1, other elements are zeros)

$$\underbrace{g_i}_{\mathcal{T}_*^{-1}(X_1^i)} = \delta_{(n^{i-1}+1)} \qquad i = 1, \cdots, m,$$

$$\underbrace{ad_f^k g_i}_{\mathcal{T}_*^{-1}(X_{k+1}^i)} = \delta_{(n^{i-1}+k+1)}, \qquad k = 1, \cdots, n_i - 1$$

$$[g_1\ ad_f g_1\ ad_f^2 g_1, g_2\ ad_f g_2, \cdots\cdots] = \begin{bmatrix} 1 & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{bmatrix}$$

Denote $f = (f_1^1, \cdots, f_{n_1}^1, \cdots; f_1^m, \cdots, f_{n_m}^m)^T$

Since $ad_f^k g_i = \delta_{(n^{i-1}+k+1)}$ $\Rightarrow$ $[f, ad_f^{k-1} g_i] = [f, \delta_{(n^{i-1}+k)}] = \delta_{(n^{i-1}+k+1)}$

$g_i = \delta_{(n^{i-1}+1)}$

$$\underbrace{\frac{\partial \delta_{(n^{i-1}+k)}}{\partial z}}_{=0} \cdot f - \frac{\partial f}{\partial z} \cdot \delta_{n^{i-1}+k} = \delta_{(n^{i-1}+k+1)}$$

$$\frac{\partial f}{\partial z} = \begin{bmatrix} \frac{\partial f_1^1}{\partial z_1^1} & \cdots & \frac{\partial f_1^1}{\partial z_{n_m}^m} \\ \vdots & & \vdots \\ \frac{\partial f_{n_m}^m}{\partial z_1^1} & \cdots & \frac{\partial f_{n_m}^m}{\partial z_{n_m}^m} \end{bmatrix}$$

$$\Rightarrow \frac{\partial f_t^s}{\partial z_j^i} = \begin{cases} -1, & s = i \text{ and } t = j+1, \\ 0, & \text{otherwise}, \end{cases} \qquad \begin{matrix} s, i = 1, \cdots, m, \\ t = 1, \cdots, n_i \\ j = 1, \cdots, n_i - 1 \end{matrix}$$

$$\Rightarrow f = (0, -z_1^1, \cdots, -z_{n_1-1}^1, \cdots, 0, z_1^m, \cdots, z_{n_m-1}^m) + \underbrace{Y(z_{n_1}^1, \cdots, z_{n_m}^m)}$$

The vector field only depend on $z_{n_i}^i$, $i = 1, \cdots, m$.

Calculating $ad_f^{n_i} g_i$, we have.

$$ad_f^{n_i} g_i = [f, ad_f^{n_i-1} g_i] = [f, \delta_{n^{i-1}+n_i}] = \underbrace{\frac{\partial \delta_{n^{i-1}+n_i}}{\partial z}}_{=0} \cdot f - \frac{\partial f}{\partial z} \cdot \delta_{n^{i-1}+n_i}$$

$$= -\left[\frac{\partial Y_1^1}{\partial z_{n_i}^i}, \cdots, \frac{\partial Y_{n_1}^1}{\partial z_{n_i}^i}, \cdots \frac{\partial Y_1^m}{\partial z_{n_i}^i}, \cdots \frac{\partial Y_{n_m}^m}{\partial z_{n_i}^i}\right]^T$$

Since $[ad_f^{n_i} g_i, ad_f^k g_i] = 0$, $i = 1, \cdots, m$, $k = 1, \cdots, n_i$

$$\Rightarrow F_*^{-1}[ad_f^{n_i} g_i, \underbrace{X_k^i}_{=ad_f^k g_i}] = [F_*^{-1} ad_f^{n_i} g_i, \underbrace{F_*^{-1} X_k^i}_{=\delta\text{'s}}] = 0.$$

$$\Rightarrow \underbrace{\frac{\partial Y_t^s}{\partial z_{n_i}^i}}_{\text{why?}} = \text{const.}\ \forall s, t, i \Rightarrow \frac{\partial f_t^s}{\partial z_{n_i}^i} = \text{const}\ \forall s, t, i$$

$$\Rightarrow \begin{bmatrix} \dot{z}^1 \\ \vdots \\ \dot{z}^m \end{bmatrix} = \begin{bmatrix} A_1^1 & \cdots & A_m^1 \\ \vdots & & \\ A_1^m & \cdots & A_m^m \end{bmatrix} \begin{bmatrix} z^1 \\ \vdots \\ z^m \end{bmatrix} + \begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_m \end{bmatrix} u$$

where $A_j^i = \begin{cases} \begin{bmatrix} 0 & \cdots & 0 & c_j^{i1} \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & c_j^{in_i} \end{bmatrix}, & i \neq j \\ \begin{bmatrix} 0 & \cdots & 0 & c_i^{i1} \\ 1 & 0 \cdots & 0 & c_i^{i2} \\ & \ddots & & \vdots \\ & & 1 & c_i^{in_i} \end{bmatrix} & i = j \end{cases}$ $\quad b_i = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; i = 1, \cdots m$

This is controllable canonical form.

Next, consider the affine system with outputs

$$(***) \quad \begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x) u_i, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p \\ y &= h(x) \end{aligned}$$

<u>Def</u> The system $(***)$ is said to be equivalent to the linear system locally around an equilibrium point x_0, if there exists a (local) coordinate chart (U, z) $(z(x_0) = 0)$ such that $(***)$ is expressed in z coordinate frame as:

$$\begin{aligned} \dot{z} &= Az + \sum_{i=1}^{m} b_i u_i \quad \text{where } z \in \mathbb{R}^n, (z \in U), u \in \mathbb{R}^m, \\ y &= Cz, \quad\quad (A, B, C) \text{ is minimal (controllable \& observable)}. \end{aligned}$$

<u>Thm</u> $(***)$ is equivalent to a linear system locally around an equilibrium point x_0, iff

i). $\dim\{ ad_f^k g_i(x_0) \mid 1 \le i \le m, \; 0 \le k \le n-1 \} = n$

ii) $\dim\{ dL_f^k h_j(x_0) \mid 1 \le j \le p, \; 0 \le k \le n-1 \} = n$.

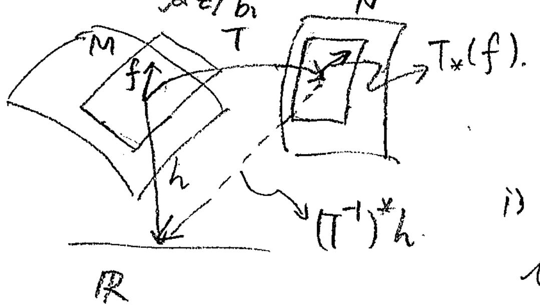
iii) there exists a neighbourhood, U of x_0, such that

$$L_{g_i} L_f^s L_{g_j} L_f^t h(x) = 0, \quad x \in U, \quad 1 \le i, j \le m, \quad s, t \ge 0$$

<u>proof</u> (Necessity) Let f be any vector field and h be any smooth function. Then, for any diffeomorphism T, it holds that

$$\underbrace{L_{T_*(f)}}_{Az/b_i} ((T^{-1})^* h) = \underbrace{(T^{-1})^* (L_f(h))}$$

$$\underbrace{L_f}_{f(x)/g_i}(h) \circ \underbrace{T^{-1}(z)}_{x}$$



i) ii) iii) holds since it holds for a minimal linear system.

(Sufficiency) We need the binomial formula of Lie derivative.

$$L_{ad_f^k g} h(x) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} L_f^{k-i} L_g L_f^i h(x).$$

$$L_{[f,g]} h = L_f L_g h - L_g L_f h$$

This implies $\underline{L_{[ad_f^s g_i,\, ad_f^t g_j]}} L_f^k h(x) = L_{ad_f^s g_i} L_{ad_f^t g_j} L_f^k h(x) - L_{ad_f^t g_j} L_{ad_f^s g_i} L_f^k h(x) = 0$

For a Lie derivative, $L_X f = \frac{\partial f}{\partial x} X = \sum_{i=1}^{n} X_i \frac{\partial f}{\partial x_i} = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i} \sum_{j=1}^{n} \frac{\partial h}{\partial x_j} dx_j = \langle df, X\rangle$

it follows that $\langle dL_f^k h(x), [ad_f^s g_i, ad_f^t g_j]\rangle = 0$ $\left(\frac{\partial}{\partial x_i} dx_j = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{if } i\neq j \end{cases} \right)$

By ii), $\Rightarrow [ad_f^s g_i, ad_f^t g_j] = 0$, $1 \le i,j \le m$, $0 \le s,t \le n-1$

By the above theorem, (***) can already be locally expressed as

$$\dot z = Az + Bu \quad \text{around } x_0.$$
$$y = h \circ T^{-1}(z) := \tilde h(z)$$

Now under z coordinates, the vector fields

$$\{X_1, \cdots X_n\} = \{g_1, \cdots, ad_f^{n_1-1} g_1, \cdots, g_m, \cdots, ad_f^{n_m-1} g_m\}$$

is transformed as $X_i = \delta_i = \frac{\partial}{\partial z_i}$, $i = 1, \cdots, n$.

From iii), we know $L_{X_i} L_{X_j} \tilde h_s(z) = 0$ (By using)

$$\Rightarrow L_{\delta_i}\left(\frac{\partial \tilde h_s}{\partial z}\cdot \delta_j\right) = \frac{\partial^2 \tilde h_s(z)}{\partial z_i \partial z_j} = 0$$

$\Rightarrow \tilde h_s(z)$ is only a linear function of z.

Since $h(x_0) = 0 \Rightarrow h \circ \underbrace{T^{-1}(0)}_{x_0} = \tilde h(0) = 0 \Rightarrow y = Cz$.

<u>Ex</u> Consider the following system

$$\dot x_1 = x_1 + u$$
$$\dot x_2 = \frac{x_1}{\cos(x_2)} \Rightarrow f(x) = \begin{bmatrix} x_1 \\ \frac{x_1}{\cos x_2} \end{bmatrix} \quad g = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad h = \sin x_2$$
$$y = \sin x_2$$

$$ad_f g = -\begin{bmatrix} 1 \\ \frac{1}{\cos x_2} \end{bmatrix}$$ i) is satisfied.

construct the mapping $F(z_1, z_2) = \gamma_{z_1}^{X_1} \circ \gamma_{z_2}^{X_2}(0)$

First, solve $\gamma_{z_2}^{X_2}(0)$, $\begin{bmatrix} \frac{dx_1}{dz_2} \\ \frac{dx_2}{dz_2} \end{bmatrix} = ad_f g = -\begin{bmatrix} 1 \\ \frac{1}{\cos x_2} \end{bmatrix}$, with $x_1(0) = 0$, $x_2(0) = 0$

$\Rightarrow x_1 = -z_2$
$x_2 = -\sin^{-1} z_2$

Next, to get $\mathcal{F}(z_1, z_2)$, solve $\begin{cases} \frac{dx_1}{dz_1} = 1 \\ \frac{dx_2}{dz_1} = 0, \end{cases}$

with the initial condition $x_1(0) = -z_2$
$x_2(0) = -\sin^{-1}(z_2)$

$$\Rightarrow \begin{array}{l} x_1 = z_1 - z_2 \\ x_2 = -\sin^{-1}(z_2) \end{array} \Rightarrow \mathcal{F}^{-1}(x_1, x_2) = \begin{cases} z_1 = x_1 - \sin x_2 \\ z_2 = -\sin x_2 \end{cases}$$

$$\Rightarrow \dot{z}_1 = \dot{x}_1 - (\sin x_2)' = x_1 + u - \cos x_2 \cdot \dot{x}_2 = x_1 + u - \cancel{\cos x_2} \frac{x_1}{\cancel{\cos x_2}} = u$$

$$\dot{z}_2 = -\cos x_2 \cdot \dot{x}_2 = -\cancel{\cos x_2} \cdot \frac{x_1}{\cancel{\cos x_2}} = -x_1 = -z_1 + z_2$$

$$y = \sin x_2 = -z_2$$

In fact, $dh = [0 . \cos x_2]$, $dL_f^k h = [1 \quad 0]$, $k \geq 0$

$\Rightarrow$ ii) is satisfied.

$L_g h = 0$. $L_g L_f^k h = 1$, $k \geq 1 \Rightarrow L_g L_f^s L_g L_f^t h = 0$, $s \geq 0$. $t \geq 0$.

$\Rightarrow$ iii) is satisfied.