Nonlinear Control Theory

Lecture 12. Feedback stabilization I.

Last time

- Feedback linearization for MIMO systems.
  - Relative degree definition extension.
  - Exact feedback linearization → distributions $G_i = \text{span}\{ad_f^k g_j, 0 \le k \le i, 1 \le j \le m\}$ being ① nonsingular near x^0, $\forall 0 \le i \le n-1$
    ② $\dim(G_{n-1}) = n$
    ③ G_i involutive $\forall 0 \le i \le n-2$
- Idea of extending to MIMO
- Global vs local.

Today

- Zero dynamics
- Feedback stabilizability
- Local asymptotic stabilization

Recall:

SISO affine system

$$\dot{x} = f(x) + g(x)u, \quad x \in \mathbb{R}^n, u \in \mathbb{R}, y \in \mathbb{R},$$
$$y = h(x)$$

Relative degree r at x^0:

$$L_g L_f^k h(x) = 0, \ \forall k < r_i - 1, \ \forall x \in N(x^0)$$
$$L_g L_f^{r_i - 1} h(x) \ne 0.$$

Normal form

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = z_3$$
$$\vdots$$
$$\dot{z}_{r-1} = z_r$$
$$\dot{z}_r = b(\xi, \eta) + a(\xi, \eta) u$$
$$\dot{\eta} = q(\xi, \eta)$$
$$y = z_1$$

$b(\xi,\eta) = L_f^{r-1} h(\phi^{-1}(\xi, \eta))$, $a(\xi,\eta) = L_g L_f^{r-1}(\phi^{-1}(\xi, \eta)) \ne 0$.

$$\eta = \begin{bmatrix} z_{r+1} \\ \vdots \\ z_n \end{bmatrix}, \quad \xi = \begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix}$$

MIMO affine "square" system

$$\dot{x} = f(x) + g(x) \cdot u, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^m$$
$$y = h(x).$$

Relative degree $\{r_1, \cdots, r_m\}$ at x^0:

$$L_{g_j} L_f^k h_i(x) = 0, \ \forall 1 \le j \le m, \forall k < r_i - 1,$$
$$\forall 1 \le i \le m, \forall x \in N(x^0)$$

and
$$A(x^0) = \begin{bmatrix} L_{g_1} L_f^{r_1 - 1} h_1(x^0), & \cdots & L_{g_m} L_f^{r_1 - 1} h_1(x^0) \\ \vdots & & \\ L_{g_1} L_f^{r_m - 1} h_m(x^0), & \cdots & L_{g_m} L_f^{r_m - 1} h_m(x^0) \end{bmatrix}$$
is nonsingular.

Normal form

$$\dot{\xi}_1^i = \xi_2^i$$
$$\vdots$$
$$\dot{\xi}_{r_i - 1}^i = \xi_{r_i}^i$$
$$\dot{\xi}_{r_i}^i = b_i(\xi, \eta) + \sum_{j=1}^m a_{ij}(\xi, \eta) u_j$$
$$\dot{\eta} = q(\xi, \eta) + \sum_{i=1}^m p_i(\xi, \eta) u_i$$
$$= q(\xi, \eta) + p(\xi, \eta) u$$
$$y_i = \xi_1^i, \ i = 1, \cdots, m.$$

$b_i(\xi,\eta) = L_f^{r_i} h_i(\phi^{-1}(\xi, \eta))$, $a_{ij}(\xi,\eta) = L_{g_j} L_f^{r_i - 1} h_i(\phi^{-1}(\xi, \eta))$

$p(\xi,\eta)u$ can be made equals to zero if $\text{span}\{g_1, \cdots, g_m\}$ is involutive.

zero dynamics

$\dot{\eta} = q(0, \eta(t))$

Zero dynamics

$\dot{\eta} = q(0,\eta) - p(0,\eta) A^{-1}(0,\eta) b(0,\eta)$

Interpretation of the zero dynamics.

For SISO system, find initial state and input function $(x^0, u^0(t))$, defined in a neighbourhood of $t=0$, such that $y(t) \equiv 0$, for all t in the neighbourhood of $t=0$.

Since $y(t) = z_1(t) \equiv 0 \Rightarrow \dot{z}_1 = \dot{z}_2 = \cdots = \dot{z}_r = 0 \Rightarrow \xi(t) \equiv 0$

$$\Rightarrow 0 = b(0,\eta(t)) + a(0,\eta(t))\, u(t)$$

$$\Rightarrow u(t) = -\frac{b(0,\eta(t))}{a(0,\eta(t))}, \quad \dot{\eta}(t) = q(0,\eta(t))$$

zero dynamics of SISO systems.

The zero dynamics describes the "internal" behaviour of the system when the input and initial conditions have been chosen such that the output is identically zero.

Suppose for the original SISO affine nonlinear system $\begin{cases} \dot{x} = f(x) + g(x) u \\ y = h(x) \end{cases}$, it holds that $f(0) = 0$.

$\Rightarrow$ Linearization around $x^0 = 0$,

$$f(x) = Ax + f_1(x), \quad \text{where } A = \left[\frac{\partial f}{\partial x}\right]_{x=0}, \quad \left[\frac{\partial f_1}{\partial x}\right]_{x=0} = 0$$

$$g(x) = Bx + g_1(x) \qquad B = g(0)$$

$$h(x) = Cx + h_1(x), \quad C = \left[\frac{\partial h}{\partial x}\right]_{x=0}, \quad \left[\frac{\partial h_1}{\partial x}\right]_{x=0} = 0$$

therefore, $L_f h(x) = \frac{\partial h}{\partial x} \cdot f = (C + \frac{\partial h_1}{\partial x})(Ax + f_1(x))$

$$= CAx + d_1(x) \qquad d_1(x) = \frac{\partial h_1}{\partial x} Ax + (C + \frac{\partial h_1}{\partial x}) f_1(x)$$

$$\Rightarrow \left[\frac{\partial d_1}{\partial x}\right]_{x=0} = 0$$

$$L_f^2 h(x) = \frac{\partial L_f h(x)}{\partial x} \cdot f$$

$$= [CA + \frac{\partial d_1}{\partial x}][Ax + f_1(x)] = CA^2 x + d_2(x) \qquad d_2(x) = \frac{\partial d_1}{\partial x} Ax + (CA + \frac{\partial d_1}{\partial x}) f_1(x)$$

$$\Rightarrow \left[\frac{\partial d_2}{\partial x}\right]_{x=0} = 0$$

....

$$\Rightarrow L_f^k h(x) = CA^k x + d_k(x), \quad \text{where } \left[\frac{\partial d_k}{\partial x}\right]_{x=0} = 0$$

$$\Rightarrow \left.\begin{array}{l} CA^k B = L_g L_f^k h(0) = 0, \quad \forall k < r-1 \\ CA^{r-1} B = L_g L_f^{r-1} h(0) \neq 0. \end{array}\right\} \text{The relative degree of the linear approximated system at } x=0 \text{ remains } r.$$

☆

Without losing generality, assume

$$\left.\begin{array}{l} A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ & & \ddots & & & \\ 0 & \cdots & & & 0 & 1 \\ -a_0 & -a_1 & \cdots & & & -a_{n-1} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ k \end{bmatrix} \\ C = [b_0, b_1, \cdots, b_{n-r-1}, 1, 0 \cdots 0] \end{array}\right\}$$

Canonical form.

Transfer function

$$G(s) = K \frac{b_0 + b_1 s + \cdots + b_{n-r-1} s^{n-r-1} + s^{n-r}}{a_0 + a_1 s + \cdots + a_{n-1} s^{n-1} + s^n}$$

Hence in view of the normal form

$$z_1 = h(x) \Rightarrow \dot{z}_1 = \frac{\partial h}{\partial x}\cdot \dot{x} = \frac{\partial h}{\partial x}\cdot (f(x) + g(x)\cdot u) = \overbrace{\frac{\partial h}{\partial x} f(x)}^{L_f h(x)} + \overbrace{\frac{\partial h}{\partial x}\cdot g(x)}^{L_g L_f^0 h(x) \,=\, 0}\cdot u$$

$$= L_f h(x) := z_2$$

$$\dot{z}_2 = \frac{\partial L_f h(x)}{\partial x}(f(x) + g(x)\cdot u) = L_f^2 h(x) + \overset{0}{\overset{\parallel}{L_g L_f h(x)}} := z_3$$

$$\vdots$$

$$\dot{z}_{r-1} = \frac{\partial L_f^{r-2} h(x)}{\partial x}\cdot (f(x) + g(x)u) = L_f^{r-1} h(x) + \overset{0}{\overset{\parallel}{L_g L_f^{r-2} h(x)}}\, u := z_r.$$

$$\dot{z}_r = \frac{\partial L_f^{r-1} h(x)}{\partial x}(f(x) + g(x)\cdot u) = \underbrace{L_f^r h(x)}_{b(\xi,\eta)} + \underbrace{L_g L_f^{r-1} h(x)}_{a(\xi,\eta)}\cdot u$$

Choose the last n-z coordinates as:

$$\eta \sim \boxed{\begin{aligned} z_{r+1} &= x_1 \\ &\vdots \\ z_n &= x_{n-r} \end{aligned}}$$

Note that it holds for the above transform $z = \phi(x)$

that $\frac{\partial \phi}{\partial x} = \begin{bmatrix} (\cdots) & * \\ & * \\ & * \;\; * \\ 1 & \\ & \ddots & 0 \\ & & 1 \end{bmatrix}$, the Jacobian is of full-rank, $\phi(x)$ is indeed a valid transformation.

since $A = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ -a_0 & \cdots & & -a_{n-1} \end{bmatrix}$

$$\dot{z}_{r+1} = \dot{x}_1 = x_2 = z_{r+2}$$

$$\vdots$$

$$\dot{z}_{n-1} = \dot{x}_{n-r-1} = x_{n-r} = z_n$$

$$y = z_1 = Cx + h_1(x) = [b_0, b_1 \cdots; b_{n-r-1}\; 1\; 0 \cdots 0]\begin{bmatrix} x_1 \\ \vdots \\ x_{n-r} \\ x_{n-r+1} \\ \vdots \\ x_n \end{bmatrix} + h_1(x)$$

$$\dot{z}_n = \dot{x}_{n-r} = x_{n-r+1} = -b_0 x_1 - \cdots - b_{n-r-1} x_{n-r} + z_1 - h_1(x)$$

$$= -b_0 z_{r+1} - \cdots - b_{n-r-1} z_n + z_1 - h_1(x)$$

$$\Rightarrow \underbrace{\begin{bmatrix} \dot{z}_{r+1} \\ \vdots \\ \dot{z}_n \end{bmatrix}}_{\dot{\eta}} = \underbrace{\begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ -b_0 & \cdots & & -b_{n-r-1} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} z_{r+2} \\ \vdots \\ z_n \end{bmatrix}}_{\eta} + \left.\begin{bmatrix} 0 \\ \vdots \\ 0 \\ z_1 - h_1(x) \end{bmatrix}\right\} := q(\xi, \eta).$$

$$G(s) = k\frac{b_0 + b_1 s + \cdots + b_{n-r-1}s^{n-r-1} + s^{n-r}}{a_0 + a_1 s + \cdots + a_{n-1}s^{n-1} + s^n}$$

$\Rightarrow$ since $y = Cx + h_1(x)$ and $h(0) = 0$, $\Rightarrow h_1(0) = 0$

$\Rightarrow$ zero dynamics $\dot{\eta} = Q\eta$

eigenvalues are zeros of the transfer function of the linearized model.

Zero dynamics describes the system's "internal behavior" when the output $y(t)$ is forced to be zero.

This would play a very important role in stabilization of the nonlinear affine systems.

Ex] $\dot{x} = \begin{bmatrix} x_3 - x_2^3 \\ -x_2 \\ x_1^2 - x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} u$

$y = x_1$

$L_g h(x) = \frac{\partial h}{\partial x} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = [1\ 0\ 0] \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = 0, \quad L_f h(x) = \frac{\partial h}{\partial x} \cdot f = [1\ 0\ 0] \begin{bmatrix} x_3 - x_2^3 \\ -x_2 \\ x_1^2 - x_3 \end{bmatrix} = x_3 - x_2^3$

$L_g L_f h(x) = \frac{\partial L_f h(x)}{\partial x} \cdot g = [0\ \ -3x_2^2\ \ 1] \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = 3x_2^2 + 1.$

Relative degree $r = 2$

Take $z_1 = h(x) = x_1$, $z_2 = L_f h(x) = x_3 - x_2^3$, $z_3 = x_2$

$\Rightarrow \dot{z}_1 = \dot{x}_1 = x_3 - x_2^3 = z_2$

$\dot{z}_2 = -3x_2^2 \cdot \dot{x}_2 + \dot{x}_3 = -3x_2^2(-x_2 - u) + (x_1^2 - x_3 + u) = \underbrace{z_1^2 - z_2 + 2z_3^3}_{b(\xi,\eta)} + \underbrace{(3z_3^2 + 1)}_{a(\xi,\eta)} u$

$\dot{z}_3 = \dot{x}_2 = -x_2 - u = -z_3 - u$

$y = x_1 = z_1$

$\Rightarrow y \equiv 0 \Rightarrow z_1 \equiv 0 \Rightarrow u = -\frac{b(\xi,\eta)}{a(\xi,\eta)} = -\frac{z_1^2 - z_2 + 2z_3^3}{3z_3^2 + 1}$

$\Rightarrow$ Zero dynamics: $\dot{z}_3 = \left[-z_3 + \frac{z_1^2 - z_2 + 2z_3^3}{3z_3^2 + 1} \right]_{\substack{z_1 = 0 \\ z_2 = 0}} = -z_3 + \frac{2z_3^3}{3z_3^2 + 1}$

Feedback Stabilizability

Before we start to design stabilizing controllers, we would first ask the question: Is it possible to design such control input?

Consider nonlinear affine system (*) $\dot{x} = f(x) + g(x) \cdot u$, $y = h(x)$. $x \in N(0) \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $f(0) = 0$, $h(0) = 0$, $f \in C^1$, $g \in C^1$

Stabilizable: all non-stable modes are controllable, all non-controllable modes are stable.

Linearize the system around origin: $A = \left[\frac{\partial f(x)}{\partial x}\right]_{x=0}$, $B = g(0)$

Recall that in the previous lecture, we know that: if (A, B) is controllable, then $x = 0$ is "locally" controllable

Fact if the pair (A, B) is stabilizable, then (*) is locally stabilizable.

$u = k(x)$, $k \in C^1$, s.t $\dot{x} = f(x, k(x))$ is $x = 0$ is asym. stable

Proposition (Brockett's necessary condition)

A necessary condition for (*) to be stabilizable by a C^1 feedback control is

i) The linearization pair (A, B) does not have uncontrollable modes which are associated with unstable eigenvalues. (stabilizable)

ii) The map $(x, u) \mapsto f(x) + g(x) u$ is onto a neighbourhood of 0. namely, $f(x) + g(x) u = \varepsilon$ has a solution $x(\varepsilon), u(\varepsilon)$, $\forall \varepsilon \in N(0)$. and $\lim_{\varepsilon \to 0} (x(\varepsilon), u(\varepsilon)) = (0, 0)$

Ex) Consider the model of unicycle.

$$\begin{cases} \dot{x} = v\cos\theta \\ \dot{y} = v\sin\theta \\ \dot{\theta} = w. \end{cases}$$

suppose I want the mapping $f(x)+g(x)u = \begin{bmatrix} \varepsilon \\ \varepsilon \\ \varepsilon \end{bmatrix}$, namely,

$$\begin{bmatrix} v\cos\theta = \varepsilon \\ v\sin\theta = \varepsilon \\ w = \varepsilon \end{bmatrix} \quad \left.\begin{matrix} v\cos\theta = \varepsilon \\ v\sin\theta = \varepsilon \end{matrix}\right\} \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$$

$\varepsilon \to 0$, $\theta \not\to 0$

$\Rightarrow$ There does not exist C^1 feedback controller $K(x)$ to asymptotically stabilize the unicycle.

Now we show how the idea of zero dynamics is useful when designing asymptotically stable controllers.

Consider the normal form:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_{r-1} &= z_r \\ \dot{z}_r &= b(\xi,\eta) + a(\xi,\eta)u \\ \dot{\eta} &= q(\xi,\eta). \end{aligned}$$

where $\xi = \begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix}$ and without loss of generality, assume $(\xi,\eta) = (0,0)$ is an equilibrium.

Impose a feedback of the form:

$$(**) \quad u = \frac{1}{a(\xi,\eta)}(-b(\xi,\eta) - c_0 z_1 - c_1 z_2 - \cdots - c_{r-1} z_r)$$

The closed-loop system is $\begin{aligned} \dot{\xi} &= A\xi \\ \dot{\eta} &= q(\xi,\eta) \end{aligned}$, with $A = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ -c_0 & -c_1 & \cdots & -c_{r-1} \end{bmatrix}$

<u>proposition</u>. Suppose the equilibrium $\eta = 0$ of the <u>zero dynamics</u> of the system is <u>locally asymptotically stable</u>, and all of the roots of the polynomial $p(s) = c_0 + c_1 s + \cdots + c_{r-1}s^{r-1} + s^r$ have negative real part. Then the feedback law $(**)$ locally asymptotically stabilizes the equilibrium $(\xi,\eta) = (0,0)$.

<u>proof</u> Note that it is possible to transform the normal form $\begin{aligned} \dot{\xi} &= A\xi \\ \dot{\eta} &= q(\xi,\eta) \end{aligned}$ into

$$\dot{\xi} = A\xi$$

$$\begin{cases} \dot{\eta}_1 = F_1\eta_1 + q_1(\xi,\eta_1,\eta_2) \\ \dot{\eta}_2 = F_2\eta_2 + q_2(\xi,\eta_1,\eta_2) \end{cases}$$

by "linearization" and linear coordinate transformation, where F_1 has <u>zero eigenvalues</u> and <u>F_2 is Hurwitz</u>.

Since by assumption, the zero dynamics is locally asymptotically stable, namely,

$$\begin{aligned} \dot{\eta}_1 &= F_1\eta_1 + q_1(0,\eta_1,\eta_2) \\ \dot{\eta}_2 &= F_2\eta_2 + q_2(0,\eta_1,\eta_2) \end{aligned}$$

is asymptotically stable.

$\Rightarrow$ On the center manifold $\eta_2 = \pi(\eta_1)$ that satisfies

$$\frac{\partial\pi}{\partial\eta_1}(F_1\eta_1 + q_1(0,\eta_1,\eta_2)) = F_2\pi(\eta_1) + q_2(0,\eta_1,\pi(\eta_1)).$$ and

$$\dot{\eta}_1 = F_1\eta_1 + q_1(0,\eta_1,\pi(\eta_1))$$ is asymptotically stable.

Consider the full normal system $\dot{\xi} = A\xi$

$\dot{\eta}_1 = F_1\eta_1 + g_1(\xi, \eta_1, \eta_2)$

$\dot{\eta}_2 = F_2\eta_2 + g_2(\xi, \eta_1, \eta_2)$

A center manifold would be:

$\xi = \pi_1(\eta_1)$, $\eta_2 = \pi_2(\eta_1)$

satisfying: $\frac{\partial \pi_1}{\partial \eta_1}(F_1\eta_1 + g_1(\pi_1(\eta_1), \eta_1, \pi_2(\eta_1))) = A\pi_1(\eta_1)$

$$\frac{\partial \pi_2}{\partial \eta_1}(F_1\eta_1 + g_1(\pi_1(\eta_1), \eta_1, \pi_2(\eta_1))) = F_2\pi_2(\eta_1) + g_2(\pi_1(\eta_1), \eta_1, \pi_2(\eta_1))$$

Note that the above two equations are solved by $\pi_1(\eta_1) = 0$ and $\pi_2(\eta_1) = \pi(\eta_1)$.

$\Rightarrow$ By Reduction principle, the original systems stability is determined by

$\dot{\eta}_1 = F_1\eta_1 + g_1(0, \eta_1, \pi(\eta_1))$, which is asymptotically stable by assumption.

$\Rightarrow$ The statement is proved.

This proposition and the proof of it, <u>does not requre</u> the linearized matrix $\left[\frac{\partial g(\xi,\eta)}{\partial \eta}\right]_{(\xi,\eta)=(0,0)}$ to be Hurwitz.

Ex

$$\dot{x} = \begin{bmatrix} x_1x_2 - x_1^3 \\ x_1 \\ -x_3 \\ x_1^2 + x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2+2x_3 \\ 1 \\ 0 \end{bmatrix} u, \quad y = h(x) = x_4$$

$\frac{\partial h}{\partial x} = [0 \ 0 \ 0 \ 1]$, $L_g h(x) = 0$, $L_f h(x) = x_1^2 + x_2$ $\rightsquigarrow$ relative degree $r = 2$

$\frac{\partial L_f h}{\partial x} = [2x_1 \ 1 \ 0 \ 0]$, $L_g L_f h(x) = 2(1+x_3) \neq 0$, if $x_3 \neq -1$.

Hence it is possible to find a normal form away from $x_3 = -1$

Let $z_1 = \phi_1(x) = h(x) = x_4$, $z_2 = \phi_2(x) = L_f h(x) = x_2 + x_1^2$.

We need to find $\phi_3(x)$ and $\phi_4(x)$ to complete the transformation.

<u>The best choice</u> is $L_g\phi_3(x) = L_g\phi_4(x) = 0$ since

$$\dot{z}_i = \frac{\partial \phi_i(x)}{\partial x}[f(x) + g(x)\cdot u] = L_f\phi_i(x) + \underbrace{L_g\phi_i(x)}_{=0}\cdot u = \underbrace{L_f\phi_i(\phi^{-1}(z))}_{\text{has nothing to do with } u}$$

Although it is possible to compute $\phi_3(x) = x_2 - 2x_3 - x_3^2$ that makes $L_g\phi_3(x) = 0$, <u>sometimes it is not easy to do this</u>.

We just let $z_3 = \phi_3(x) = x_3$, $z_4 = \phi_4(x) = x_1$.

The Jacobian is $\frac{\partial \phi}{\partial x} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 2x_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ is nonsingular.

The inverse transformation is
$$\begin{aligned} x_1 &= z_4 \\ x_2 &= z_2 - z_4^2 \\ x_3 &= z_3 \\ x_4 &= z_1 \end{aligned} \Rightarrow \begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_4 + 2z_4(z_4(z_2 - z_4^2) - z_4^3) \\ &\quad + (2+2z_3)u \\ \dot{z}_3 &= -z_3 + u \\ \dot{z}_4 &= -2z_4^3 + z_2z_4 \end{aligned}$$

The above transformed system is not in normal form, since there is a term involving u in the equation of z_3, but this will not prevent us from getting the zero dynamics.

Setting $z_1 = z_2 = 0$, yields $u = -\frac{z_4 + 2z_4(-z_4^3 - z_4^3)}{2+2z_3} = -\frac{z_4 - 4z_4^4}{2+2z_3}$

$\Rightarrow$ zero dynamics
$$\dot{z}_3 = -z_3 - \frac{z_4 - 4z_4^4}{2+2z_3}$$
$$\dot{z}_4 = -2z_4^3$$
is asymptotically stable at $(z_3 = 0, z_4 = 0)$ $\rightsquigarrow$ verify this

Hence $u = \frac{1}{L_g L_f h(x)}[-L_f^2 h(x) - C_0 h(x) - C_1 L_f h(x)]$ locally stabilizes the equilibrium, where the root of $p(s) = C_0 + C_1 s + s^2$ shall have negative real parts.

MIMO case ?
$\rightarrow$ Decoupling.

$$\dot{\xi}_1^i = \xi_2^i$$
$$\vdots$$
$$\dot{\xi}_{r_i-1}^i = \xi_{r_i}^i$$
$$\dot{\xi}_{r_i}^i = b_i(\xi,\eta) + a_i(\xi,\eta)u$$
$$\dot{\eta} = g(\xi,\eta) + p(\xi,\eta)u.$$

Non interacting control

Choosing $u = -A^{-1}(\xi,\eta)[b(\xi,\eta) + v]$

$\Rightarrow$ Zero Dynamics $\quad \dot{\eta} = g(\xi,\eta) - p(\xi,\eta)A^{-1}(\xi,\eta)b(\xi,\eta) - p(\xi,\eta)A^{-1}(\xi,\eta)v.$

letting $\xi = 0$.

$v_i = -C_0^i \xi_1^i - \cdots - C_{r_i-1}^i \xi_{r_i-1}^i$ $\leftarrow$ with roots all have negative real parts.

$\rightsquigarrow$ The control stabilizes the system locally if the zero dynamics is asymptotically stable.

Global vs local

We use an example to illustrate there are nonlinear systems which are locally stabilizable but are never globally stabilizable.

Consider
$$\dot{\xi}_1 = \xi_2$$
$$\dot{\xi}_2 = u$$
$$\dot{\eta} = -\eta + (\xi_1 - \xi_2)\eta^2$$

If we chose $y = \xi_1$, then the system has relative degree $r = 2$, zero dynamics : $\dot{\eta} = -\eta$ which is asymptotically stable. The system is locally stabilizable.

However, consider $\Omega = \{(\xi_1, \xi_2, \eta) : w = \eta\xi_1 = 1\}$.

$$\dot{w} = \dot{\eta}\xi_1 + \eta\dot{\xi}_1 = (-\eta + (\xi_1 - \xi_2)\eta^2)\cdot\xi_1 + \eta\xi_2 = -\eta\xi_1 + \xi_1^2\eta^2 - \xi_1\xi_2\eta^2 + \eta\xi_2$$
$$= -1 + 1 - \xi_2\eta + \eta\xi_2 = 0$$

This implies that Ω is <u>invariant</u>! This means if you start with an initial condition $[\xi_1(0), \xi_2(0), \eta(0)]$ such that $\xi_1(0)\,\eta(0) = 1$, it would remain in Ω.
But $0 \notin \Omega$, $\Rightarrow$ The origin is thus not globally stabilizable by any control.