1. # Reviewer NRCe

We thank you for recognizing the notable contributions of our paper. We appreciate your acknowledgment of our precise determination of the signal-to-noise ratio (SNR) threshold as a novel achievement. We appreciate your recognization of this novel achievement, as it is a key aspect of our work. Additionally, we are grateful for your positive remarks on our comprehensible presentation of the technical discussion, the construction of the Message Passing (MP) algorithm, the approximation method, and the connection between the distributional equations and the branching random walk process.

1.1. Update Table 1 (ICML Paper)

First, we update the Table 1 in our ICML paper and add error bars. We pick $n = 500$ and let $\mathbf{B}^{\natural} = \lambda \mathbf{I}_{m\times m}$. The phase transition point $\mathsf{snr}_{\mathsf{oracle}}$ is put in Table 1 (this report), which corresponds to the snr value when the error rate drops below 0.05. To prove our claim that the prediction gaps will be smaller with increasing m, we increase the m to the region [100 150].

Table 1. Comparison between the predicted value of the phase transition threshold $\mathsf{snr}_{\mathsf{oracle}}$ and its numerical value when $n = 500$. **P** denotes the predicted value while **N** denotes the numerical value (i.e., mean $\pm$ std). **N** value corresponds to the snr when the error rate drops below 0.05.

m	20	30	40	50	60	70
P	3.283	1.415	0.902	0.662	0.523	0.432
N	2.529 ± 0.079	1.290 ± 0.054	0.872 ± 0.034	0.649 ± 0.012	0.515 ± 0.016	0.429 ± 0.015
m	100	110	120	130	140	150
P	0.284	0.255	0.231	0.211	0.195	0.181
N	0.282 ± 0.008	0.256 ± 0.006	0.232 ± 0.006	0.212 ± 0.004	0.196 ± 0.006	0.183 ± 0.005

1.2. Update of Table 2 (ICML Paper)

In addition, we update the Table 2 in our ICML paper. Set $\lambda > 0$ as some positive value. From Table 2 (this report), we conclude that **the numerical values match to a good extent to the predicted values**.

- **Case I**: half of the eigenvalues are with λ while the other half are with $\lambda/2$;
- **Case II**: half of the eigenvalues are with λ while the other half are with $(3\cdot\lambda)/4$.

Table 2. Comparison between the predicted value of the phase transition threshold $\widetilde{\mathsf{snr}}_{\mathsf{oracle}}$ and its numerical value when $n = 600$. **Gauss** refers to $\mathbf{X}_{ij} \overset{\text{i.i.d}}{\sim} \mathsf{N}(0,1)$ while **Unif** refers to $\mathbf{X}_{ij} \overset{\text{i.i.d}}{\sim} \mathsf{Unif}[-1,1]$. **P** denotes the predicted value while **N** denotes the numerical value (i.e., mean $\pm$ std). **N** value corresponds to the snr when the error rate drops below 0.05. We average over 20 experiments.

m	100	110	120	130	140	150
(Case I) P	0.297	0.266	0.241	$.220$	0.203	0.188
(Gauss) N	0.307 ± 0.009	0.275 ± 0.005	0.246 ± 0.006	0.227 ± 0.007	0.210 ± 0.005	0.194 ± 0.004
(Unif) N	0.294 ± 0.008	0.266 ± 0.005	0.239 ± 0.008	0.216 ± 0.004	0.201 ± 0.005	0.189 ± 0.006
(Case II) P	0.310	0.276	0.249	0.227	0.209	0.193
(Gauss) N	0.294 ± 0.008	0.266 ± 0.006	0.241 ± 0.005	0.220 ± 0.004	0.204 ± 0.006	0.190 ± 0.003
(Unif) N	0.287 ± 0.007	$0.255 \pm .0043$	0.234 ± 0.007	0.213 ± 0.005	0.197 ± 0.003	0.185 ± 0.005

1.3. Update of Figure 2 (ICML Paper)

In addition to Figure 2 in our ICML paper, we add more experiments to evaluate the phase transition point τ_h in the non-oracle setting. We consider the case when $n = 700$ and $p = n/4$. The numerical experiment is shown in Figure 1 (this report), from which we can see the predicted phase transition τ_h matches to a good extent to the numerical experiments.

Then, we fix the p and study the cases for different n. The numerical results are shown in Figure 2 (this report).

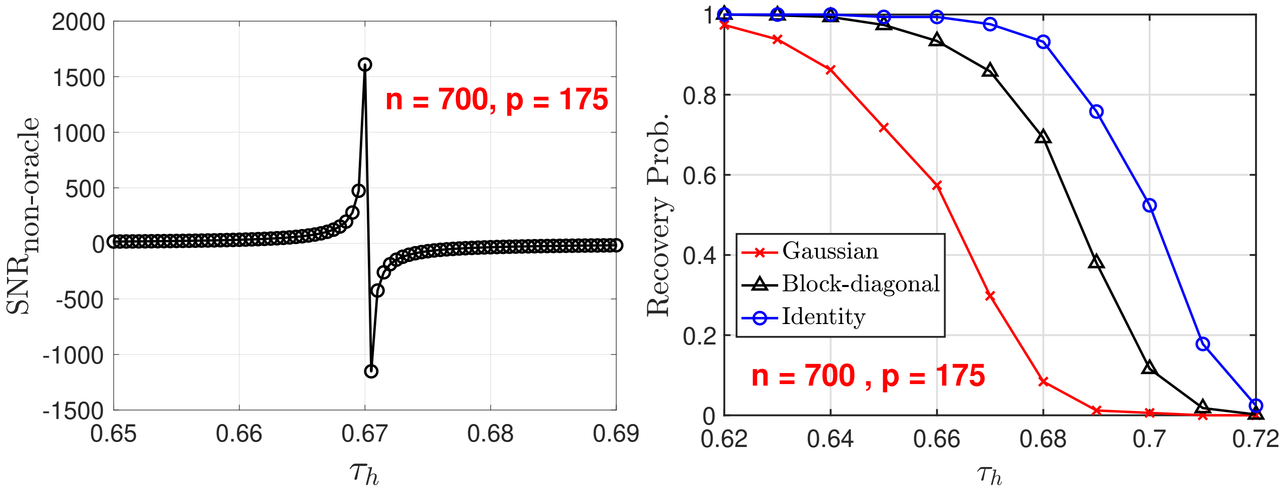

Figure 1. We pick $n = 700$ and $p = n/4 = 175$. (**Upper panel**) Predicted $\mathsf{snr}_{\text{non-oralce}}$. (**Lower panel**) Plot of recovery rate under the noiseless setting, i.e., $\mathsf{snr} = \infty$. **Gaussian**: $\mathbf{B}^{\natural}_{ij} \overset{\text{i.i.d}}{\sim} \mathsf{N}(0,1)$; **Identity**: $\mathbf{B}^{\natural} = \mathbf{I}_{p\times p}$; **Block-diagonal**: $\mathbf{B}^{\natural} = \mathrm{diag}\{1,\cdots,1,0.5,\cdots,0.5\}$. We observe the all correct recovery rates drop sharply within the regions of our predicted value.

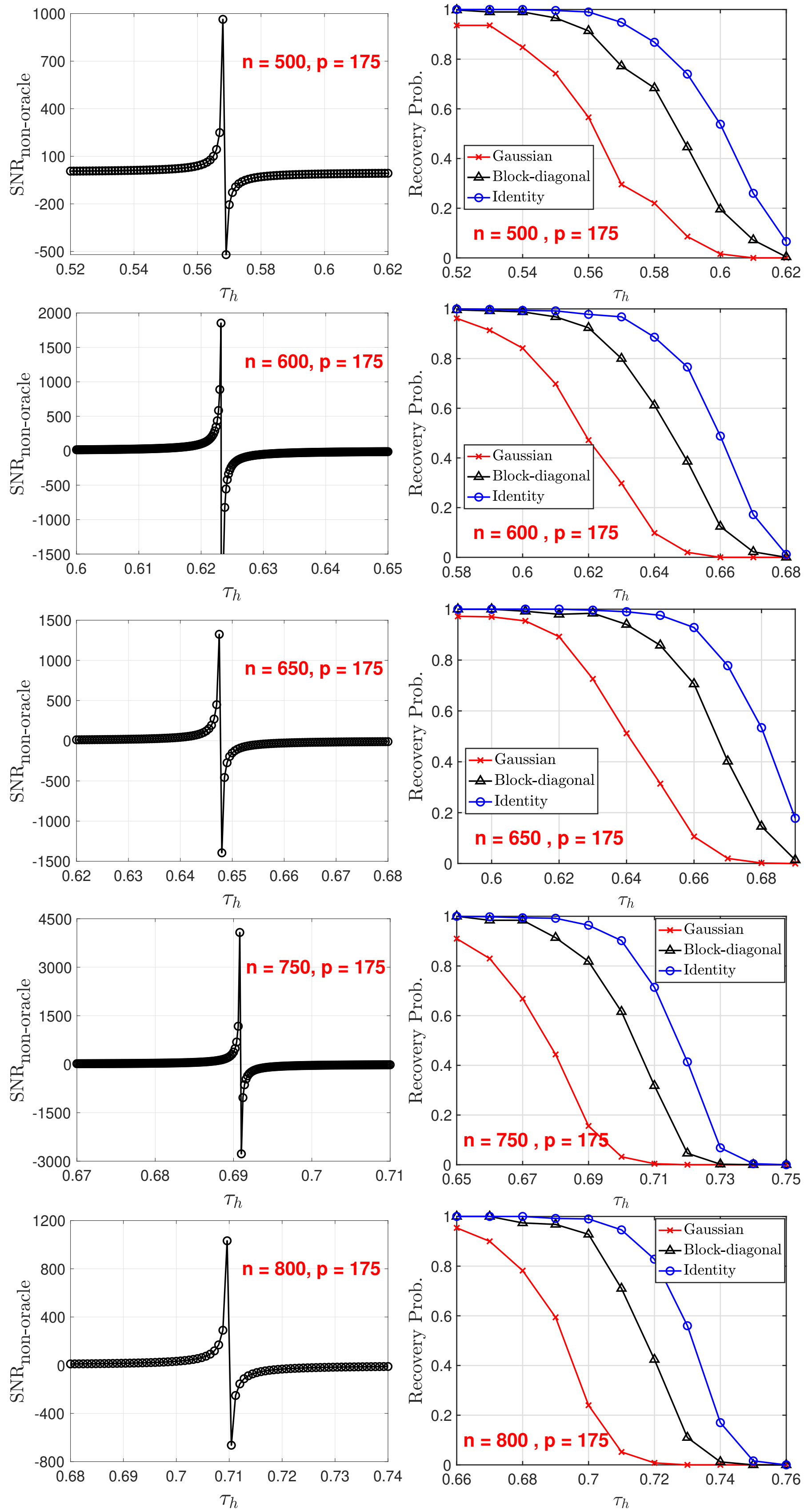

Figure 2. (**Upper panel**) Predicted $\mathsf{snr}_{\text{non-oralce}}$ for $n = \{500, 600, 650, 750, 800\}$. (**Lower panel**) Plot of recovery rate under the noiseless setting, i.e., $\mathsf{snr} = \infty$. **Gaussian**: $\mathbf{B}^{\natural}_{ij} \overset{\text{i.i.d}}{\sim} \mathsf{N}(0, 1)$; **Identity**: $\mathbf{B}^{\natural} = \mathbf{I}_{p\times p}$; **Block-diagonal**: $\mathbf{B}^{\natural} = \operatorname{diag}\{1, \cdots, 1, 0.5, \cdots, 0.5\}$. We observe the all correct recovery rates drop sharply within the regions of our predicted value.