Problem 4.4

(1).$X \sim N_3(\mu, \Sigma)$, $\mu = (2, -3, 1)^T, \Sigma = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}$.
记 $b = (3, -2, 1)^T$, $b^T X \sim N(b^T\mu, b^T\Sigma b)$,
因 $b^T\mu = 13, b^T\Sigma b = 9$, 故 $b^T X \sim N(13, 3^2)$.
(2). 令 $a = (a_1, a_2)^T$, X_2 与 $Y = X_2 - a^T \begin{bmatrix} X_1 \\ X_3 \end{bmatrix}$ 独立当且仅当 $Cov(X, Y) = 0$.

$$\begin{aligned} Cov(X,Y) &= VarX_2 - a_1 Cov(X_2, X_1) - a_2 Cov(X_2, X_3) \\ &= 3 - a_1 - 2a_2 \end{aligned}$$

所以可取 $a = (1, 1)$.

Problem 4.13

(a). 对 Σ 分块，$\Sigma = \begin{bmatrix} I & 0 \\ -\Sigma_{22}^{-1}\Sigma_{21} & I \end{bmatrix}$, 有 $\det \begin{bmatrix} I & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I \end{bmatrix} = \det \begin{bmatrix} I & 0 \\ -\Sigma_{22}^{-1}\Sigma_{21} & I \end{bmatrix} = 1$

$$\begin{bmatrix} I & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -\Sigma_{22}^{-1}\Sigma_{21} & I \end{bmatrix} = \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & 0 \\ 0 & \Sigma_{22} \end{bmatrix}$$

两边取行列式，得 $|\Sigma| = |\Sigma_{22}||\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}|$.
(b). 由 (a),

$$\begin{aligned} \Sigma^{-1} &= \begin{bmatrix} I & 0 \\ -\Sigma_{22}^{-1}\Sigma_{21} & I \end{bmatrix} \begin{bmatrix} (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1} & 0 \\ 0 & \Sigma_{22}^{-1} \end{bmatrix} \begin{bmatrix} I & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I \end{bmatrix} \\ (x-\mu)^T\Sigma^{-1}(x-\mu) &= \begin{bmatrix} x_1 - \mu_1 - \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2) \\ x_2 - \mu_2 \end{bmatrix}^T \begin{bmatrix} (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1} & 0 \\ 0 & \Sigma_{22}^{-1} \end{bmatrix} \\ &\quad \begin{bmatrix} x_1 - \mu_1 - \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2) \\ x_2 - \mu_2 \end{bmatrix} \\ &= (x_1 - \mu_1 - \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2))^T (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1} \\ &\quad (x_1 - \mu_1 - \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2)) + (x_2 - \mu_2)^T\Sigma_{22}^{-1}(x_2 - \mu_2) \end{aligned}$$

(c). 由 (a)(b)，$X_2 \sim N(\mu_2, \Sigma_{22}), X_1|X_2 \sim N(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$

Problem 4.16

$$\begin{aligned} \sum_{j=1}^{n}(x_j - \bar{x})(\bar{x} - \mu)^T &= \sum_{j=1}^{n}(x_j - n\bar{x})](\bar{x} - \mu)^T \\ &= (n\bar{x} - n\bar{x})(\bar{x} - \mu)^T \\ &= 0_{p\times p} \end{aligned}$$

同理，$\sum_{j=1}^{n}(\bar{x} - \mu)(x_j - \bar{x})^T = 0_{p\times p}$.

Problem P25

$$
\begin{aligned}
L(\mu, \Sigma) &= \frac{1}{(2\pi)^{np/2}|\Sigma|^{n/2}} exp[-\frac{\sum_{j=1}^{n}(x_j-\mu)^T\Sigma^{-1}(x_j-\mu)}{2}] \\
l(\mu, \Sigma) &= -\frac{np}{2}\ln 2\pi - \frac{n}{2}\ln|\Sigma| - \frac{\sum_{j=1}^{n}(x_j-\mu)^T\Sigma^{-1}(x_j-\mu)}{2} \\
\nabla_\mu l &= \frac{1}{2}\sum_{j=1}^{n}\{\Sigma^{-1}(x_j-\mu) + [(x_j-\mu)^T\Sigma^{-1}]^T\} \\
&= \Sigma^{-1}(\sum_{j=1}^{n} x_j - n\mu) \\
\nabla_\Sigma l &= \frac{n\Sigma^{-1}}{2} - \frac{1}{2}\sum_{j=1}^{n} x_j \nabla_\Sigma tr[(x_j-\mu)^T\Sigma^{-1}(x_j-\mu)] \\
&= \frac{n\Sigma^{-1}}{2} - \frac{1}{2}\sum_{j=1}^{n} x_j \nabla_\Sigma tr[\Sigma^{-1}(x_j-\mu)(x_j-\mu)^T] \\
&= \frac{\Sigma^{-1}}{2}[\sum_{j=1}^{n}(x_j-\mu)(x_j-\mu)^T\Sigma^{-1} - n] \\
&= \frac{n\Sigma^{-1}}{2}[\frac{1}{n}\sum_{j=1}^{n}(x_j-\mu)(x_j-\mu)^T - \Sigma]\Sigma^{-1}
\end{aligned}
$$

故 $\mu_{mle} = \bar{x}, \Sigma_{mle} = \frac{1}{n}\sum_{j=1}^{n}(x_j-\bar{x})(x_j-\bar{x})^T$.