Problem 1

Let X has singular value decomposition $X = UDV^T$, where U and V are $N \times N$ and $p \times p$ orthogonal matrices, and $D = diag\{d_1, \cdots, d_p\}$ with $d_1 \geq d_2 \geq \cdots \geq d_p \geq 0$.
Since $\lambda > 0$,

$$\begin{aligned} X^TX + \lambda I &= V(D^TD + \lambda I)V^T \\ &= V \begin{bmatrix} d_1^2+\lambda & & \\ & \ddots & \\ & & d_p^2+\lambda \end{bmatrix} V^T \\ (X^TX+\lambda I)^{-1} &= V \begin{bmatrix} \frac{1}{d_1^2+\lambda} & & \\ & \ddots & \\ & & \frac{1}{d_p^2+\lambda} \end{bmatrix} V^T \end{aligned}$$

So $X^TX + \lambda I$ is invertable.

Problem 2

According to Problem 1, let X has SVD $X = UDV^T$ and $(V)_{ij} = v_{ij}$.

$$\begin{aligned} Cov(\hat{\beta}^{ridge}) &= \sigma^2(X^TX+\lambda I)^{-1}X^TX(X^TX+\lambda I)^{-1} \\ &= \sigma^2 V \begin{bmatrix} \frac{d_1^2}{(d_1^2+\lambda)^2} & & \\ & \ddots & \\ & & \frac{d_p^2}{(d_p^2+\lambda)^2} \end{bmatrix} V^T \\ Var(\hat{\beta}_i^{ridge}) &= \sigma^2 \sum_{k=1}^{p} \frac{d_k^2 v_{ik}^2}{(d_k^2+\lambda)^2} \\ Cov(\hat{\beta}^{OLS}) &= \sigma^2(X^TX)^{-1} \\ &= \sigma^2 V \begin{bmatrix} \frac{1}{d_1^2} & & \\ & \ddots & \\ & & \frac{1}{d_p^2} \end{bmatrix} V^T \\ Var(\hat{\beta}_i^{OLS}) &= \sigma^2 \sum_{k=1}^{p} \frac{v_{ik}^2}{d_k^2} \end{aligned}$$

Since $\lambda > 0$, for $k = 1, \cdots, p$, $\dfrac{d_k^2}{(d_k^2+\lambda)^2} < \dfrac{1}{d_k^2}$. So

$$\sum_{k=1}^{p} \frac{d_k^2 v_{ik}^2}{(d_k^2+\lambda)^2} < \sum_{k=1}^{p} \frac{v_{ik}^2}{d_k^2}$$

That is to say, $Var(\hat{\beta}_i^{ridge}) < Var(\hat{\beta}_i^{OLS})$ for $j = 1, \cdots, p$.

Problem 3

See attachment.